

A Condition Guaranteeing Commutativity

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Abstract

We give a simple characterization for a nonassociative algebra \mathcal{A} , having characteristic $\neq 2$, to be commutative. Namely, \mathcal{A} is commutative if and only if it is flexible with a commuting set of generators. A counterexample shows that characteristic $\neq 2$ is necessary. Both the characterization and the counterexample were discovered using the computer algebra system in [2].

Key words: commutative, flexible, nonassociative algebra.

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1 Introduction

Recently, an interactive computer program for building nonassociative algebras was developed [2]. With this system, the user specifies the generators and the identities that the algebra is to satisfy, as well as the underlying field of scalars. We observed that whenever the system was instructed to build an algebra generated by one element, over a field having characteristic $\neq 2$, it made no difference if the commutative law or the flexible law was employed; the resulting algebras were identical. This observation led us to investigate this phenomenon and eventually discover the main result of this paper.

As is customary in nonassociative algebra, we use $[x, y]$ to denote the commutator $xy - yx$ and (x, y, z) to denote the associator $(xy)z - x(yz)$. An algebra is called *flexible* if it satisfies the identity

$$(x, y, x) = 0. \tag{1}$$

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Flexible algebras have been well-studied (see [3] for references). Obviously (1) is satisfied by any associative algebra. But note that (1) is also satisfied by any commutative algebra since

$$(x, y, x) = [xy, x] + x[x, y].$$

It is straightforward to show that any flexible algebra must satisfy the identity

$$[pq, u] = p[q, u] + u[q, p] - [uq, p]. \quad (2)$$

2 Main result

The *commutative center* of an algebra \mathcal{A} is the set

$$\{x \in \mathcal{A} \mid [x, r] = 0 \ \forall r \in \mathcal{A}\}.$$

Equation (2) shows that the commutative center of a flexible algebra is closed under multiplication and therefore forms a commutative subalgebra. It follows that a flexible algebra whose generators are contained in the commutative center must be commutative. The following theorem is a strengthening of this observation, assuming characteristic $\neq 2$.

Theorem 1 *Let \mathcal{A} be an algebra over a field of characteristic $\neq 2$. Then \mathcal{A} is commutative if and only if it is flexible and generated by a set S whose elements pairwise commute.*

Proof: Since commutativity always implies flexibility, the “only if” statement is clear. Hence assume \mathcal{A} is flexible and generated by a set S whose elements pairwise commute. It suffices to show

$$[u, v] = 0$$

for all products u and v of the elements in S . We will prove this by induction on $\deg(u) + \deg(v)$. This number is minimized exactly when $u, v \in S$, and by assumption, $[u, v] = 0$ holds. Thus we can assume $n = \deg(u) + \deg(v) > 2$ and that $[r, s] = 0$ for words r, s where $\deg(r) + \deg(s) < n$. Assume without loss of generality that $\deg(v) \geq 2$ and write

$v = pq$. By our assumption, $[q, u] = 0 = [q, p]$, so from (2) we get $[pq, u] = -[uq, p]$. But $uq = qu$, so we get $[pq, u] = -[qu, p]$. Repeating the same argument two more times we get $-[qu, p] = [up, q] = -[pq, u]$. Hence $2[pq, u] = 0$ and by our characteristic assumption $0 = [pq, u] = [v, u]$, completing the induction and the proof. \square

Corollary 1 *Let \mathcal{A} be a flexible algebra over a field of characteristic $\neq 2$, and generated by one element. Then \mathcal{A} is commutative.*

Proof: In a one-element set, the elements pairwise commute. \square

We now obtain a simplified proof to a theorem of Albert ([1], p. 561).

Theorem 2 (Albert) *Let \mathcal{A} be a flexible algebra over a field of characteristic $\neq 2, 3, 5$, and satisfying $(x^2, x, x) = 0$ for all x . Then \mathcal{A} is power associative.*

Proof: Let S be a subalgebra of \mathcal{A} generated by one element. By Corollary 1, S is commutative. Albert showed ([1], pp. 554) that commutative algebras having characteristic $\neq 2, 3, 5$, and satisfying $(x^2, x, x) = 0$ are power associative. Hence S is power associative, and since it is generated by one element, it is associative. Thus \mathcal{A} is power associative. \square

3 Counterexample

We now give an example showing that the characteristic restriction in Theorem 1 is necessary. It is further noteworthy that this example, too, was constructed with the help of the same computer algebra program in [2]. Let \mathbf{F} be any field having characteristic 2, and let \mathcal{A} be the 8-dimensional algebra over \mathbf{F} having basis $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8$ in which

$$\begin{aligned} b_1 b_2 &= b_6 \\ b_2 b_1 &= b_6 \\ b_1 b_3 &= b_5 \\ b_3 b_1 &= b_5 \\ b_2 b_3 &= b_4 \\ b_3 b_2 &= b_4 \end{aligned}$$

$$\begin{aligned}
b_1 b_4 &= b_7 + b_8 \\
b_2 b_5 &= b_7 + b_8 \\
b_3 b_6 &= b_7 \\
b_6 b_3 &= b_8
\end{aligned}$$

and all other products are zero. Let

$$x = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 + \alpha_4 b_4 + \alpha_5 b_5 + \alpha_6 b_6 + \alpha_7 b_7 + \alpha_8 b_8$$

and let L_x, R_x be the linear transformations on \mathcal{A} where $L_x(y) = xy$, and $R_x(y) = yx$.

Relative to our basis, the matrix for L_x , suppressing zeros, is

$$M(L_x) = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \alpha_3 & \alpha_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_3 & \cdot & \alpha_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_2 & \alpha_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \alpha_1 & \alpha_2 & \alpha_3 & \cdot & \cdot \\ \cdot & \cdot & \alpha_6 & \alpha_1 & \alpha_2 & \cdot & \cdot & \cdot \end{bmatrix},$$

and the matrix for R_x is

$$M(R_x) = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \alpha_3 & \alpha_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_3 & \cdot & \alpha_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_2 & \alpha_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_4 & \alpha_5 & \alpha_6 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_4 & \alpha_5 & \cdot & \cdot & \cdot & \alpha_3 & \cdot & \cdot \end{bmatrix}.$$

Multiplying, we obtain

$$M(L_x)M(R_x) = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2\alpha_2\alpha_3 & 2\alpha_1\alpha_3 & 2\alpha_1\alpha_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_2\alpha_3 & \alpha_1\alpha_3 & 2\alpha_1\alpha_2 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad (3)$$

and

$$M(R_x)M(L_x) = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_2\alpha_3 & \alpha_1\alpha_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}. \quad (4)$$

By the characteristic of \mathbf{F} , the matrices in (3) and (4) are equal. Therefore $L_x R_x(y) = R_x L_x(y)$ for all x and y , and so \mathcal{A} is flexible. Note next that \mathcal{A} is generated by $S = \{b_1, b_2, b_3\}$, and these elements commute. However, \mathcal{A} is *not* commutative since $b_3 b_6 = b_7 \neq b_8 = b_6 b_3$.

References

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