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1. INTRODUCTION

1.1. Background

The problem of estimating the common mean of several normal populations with unknown and unequal variances was essentially initiated in the work of Yates (1940). The model for this problem as described by Meier (1953) is as follows. Let $X_1, X_2, \ldots, X_k$ be k estimators of a parameter μ, being independently and normally distributed with mean μ and variances $\sigma_1^2/n_1, \ldots, \sigma_k^2/n_k$ respectively. Now if the σ_i^2's are known, then the minimum variance unbiased estimator of μ is

$$\tilde{\mu} = \sum_{i=1}^{k}(n_i X_i/\sigma_i^2) / \sum_{i=1}^{k}(n_i/\sigma_i^2) \tag{1.1.1}$$

and the variance of this estimate is $1/\sum_{i=1}^{k}(n_i/\sigma_i^2)$. When, σ_i^2's are unknown, let $S_1^2, \ldots, S_k^2$ be independent unbiased estimators of $\sigma_1^2, \ldots, \sigma_k$ with $m_i S_i^2/\sigma_i^2$ distributed as chi-squared with m_i degrees of freedom, then analogous estimator $\hat{\mu}$ is proposed as follows.

$$\hat{\mu} = \sum_{i=1}^{k}(n_i X_i/S_i^2) / \sum_{i=1}^{k}(n_i/S_i^2) \tag{1.1.2}$$

Meier in his investigation gave the following basic results

(i) $$\operatorname{Var}(\hat{\mu}) = [\sum_{i=1}^{k}(1/\sigma_i^2)]^{-1}\{1+2\sum_{i=1}^{k} n_i^{-1}[\sigma_i^{-2}/\sum_{i=1}^{k}\sigma_i^{-2}][1-\sigma_i^{-2}/\sum_{i=1}^{k}\sigma_i^{-2}] + 0(\sum_{i=1}^{k} n_i^{-2})\} \tag{1.1.3}$$

(ii) An approximately unbiased estimate of $\mathrm{Var}(\hat{\mu})$ is given by

$$V^* = [\sum_{i=1}^{k} S_i^{-2}]^{-1}\{1+4\sum_{i=1}^{k} n_i^{-1}[S_i^{-2}/\sum_{i=1}^{k} S_i^{-2}][1-S_i^{-2}/\sum_{i=1}^{k} S_i^{-2}]\} \quad (1.1.4)$$

(iii) V^* is distributed approximately as a chi-square with f degrees of freedom where

$$f^{-1} = \sum_{i=1}^{k} n_i^{-1}(\sigma_i^{-2}/\sum_{i=1}^{k}\sigma_i^{-2})^2.$$

Later, Graybill and Deal (1959) proved the following theorem for combined estimator $\hat{\mu}$ when $k = 2$.

<u>Theorem 1.1.1</u> For the random variables X_1, X_2, S_1^2 and S_2^2, defined as above, a necessary and sufficient condition that

$$\hat{\mu} = (n_1S_2^2X_1+n_2S_1^2X_2)/(n_1S_2^2+n_2S_1^2) \quad (1.1.5)$$

is an unbiased estimator of μ which is uniformly better than either X_1 or X_2 is that m_1 and m_2 are both larger than nine.

<u>Proof</u> The proof given here is different from proofs given by Graybill and Deal, Norwood and Hinkelmann (1977).

Using the independence of (X_1,X_2) with (S_1^2,S_2^2), it follows that

$$E(\hat{\mu}) = EE(\hat{\mu}|S_1^2,S_2^2) = E[\frac{n_1\mu S_2^2+n_2\mu S_1^2}{n_1S_2^2 + n_2S_1^2}] = \mu \quad (1.1.6)$$

In order to prove the rest of the theorem, first note that

$$\mathrm{Var}(\hat{\mu}) = \mathrm{Var}[E(\hat{\mu} \mid S_1^2,S_2^2)] + E[\mathrm{Var}(\hat{\mu} \mid S_1^2,S_2^2)]$$

$$= \mathrm{Var}(\mu) + E[(\frac{n_1S_2^2}{n_1S_2^2+n_2S_1^2})^2 \frac{\sigma_1^2}{n_1} + (\frac{n_2S_1^2}{n_1S_2^2+n_2S_1^2})^2 \frac{\sigma_2^2}{n_2}]$$

$$= [\frac{\sigma_1^2}{n_1} E(1-W)^2 + \frac{\sigma_2^2}{n_2} E(W^2)] \tag{1.1.7}$$

where $W = \dfrac{n_2S_1^2}{n_1S_2^2+n_2S_1^2}$.

Thus, from (1.1.6)

$$\mathrm{Var}(\hat{\mu}) < \sigma_1^2/n_1 = \mathrm{Var}(X_1) \tag{1.1.8}$$

if and only if

$$(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2})\, E(W)^2 - \frac{2\sigma_1^2}{n_1}\, E(W) < 0 \tag{1.1.9}$$

$$\Longleftrightarrow (1+\rho)\, E(W^2) - 2\rho\, E(W) < 0$$

But,

$$W = \frac{n_2S_1^2}{n_1S_2^2+n_2S_1^2} = \frac{\frac{n_2}{n_1} \cdot \frac{S_1^2}{S_2^2}}{1+\frac{n_2}{n_1}\frac{S_1^2}{S_2^2}}$$

$$= \frac{\frac{n_2}{n_1} \cdot \frac{\sigma_1^2}{\sigma_2^2} (S_1^2/\sigma_1^2)/(S_2^2/\sigma_2^2)}{1+\frac{n_2}{n_1}\frac{\sigma_1^2}{\sigma_2^2} (S_1^2/\sigma_1^2 0/(S_2^2/\sigma_2^2)} = \frac{\rho F}{1+\rho F} \tag{1.1.10}$$

where $\rho = (\frac{\sigma_1^2}{n_1})/(\frac{\sigma_2^2}{n_2})$, $F = (S_1^2/\sigma_1^2)/(S_2^2/\sigma_2^2) \sim F_{m_1,m_2}$.

Hence,

$$(1+\rho)\ E(W^2) - 2\rho\ E(W) < 0 \iff$$

$$(1+\rho)\ E\{ \frac{\rho^2 F^2}{(1+\rho F)^2} \} - 2\rho^2\ E\{ \frac{F}{1+\rho F} \} < 0 \iff$$

$$E\{[(1+\rho)F^2 - 2F(1+\rho F)]/(1+\rho F)^2\} < 0 \iff$$

$$E\{[F^2-2F-\rho F^2]/(1+\rho F)^2\} < 0\ . \tag{1.1.11}$$

Now,

$$E\{ \frac{F^2-2F-\rho F^2}{(1+\rho F)^2} \} < E\{ \frac{F^2-2F}{(1+\rho F)^2} \}$$

$$= E[\frac{F^2-2F}{1+\rho F} I_{[F\leq 2]}] + E[\frac{F^2-2F}{(1+\rho F)^2} I_{[F>2]}]$$

$$\leq \frac{1}{(1+2\rho)^2} \{E[(F^2-2F)\ I_{[F\leq 2]} + (F^2-2F)\ I_{[F>2]}]\}$$

$$= \frac{1}{(1+2\rho)^2}\ E(F^2-2F) \tag{1.1.12}$$

Now, note that F is distributed as F_{m_1,m_2}. Hence, $\frac{1}{F} \sim F_{m_1,m_2}$.

From (1.1.12) it follows that $Var(\hat{\mu}) < Var(X_1)$ if $E(F^2-2F) \leq 0$.

Similarly, $Var(\hat{\mu}) < Var(X_2)$ if $E(\frac{1}{F^2} - \frac{2}{F}) < 0$. But for

$m_1 \geq 5$ and $m_2 \geq 5$,

$$E(F^2-2F) = \frac{m_2}{m_2-2} [\frac{m_2(m_1+2)}{m_1(m_2-4)} - 2] < 0$$

$$\iff -m_1m_2+2m_2+8m_1 < 0. \tag{1.1.13}$$

and

$$E(F^2-2F^{-1}) = \frac{m_1}{m_1-2}\left[\frac{m_1(m_2+2)}{m_2(m_1-4)} - 2\right] < 0$$

$$\iff -m_1m_2 + 2m_1 + 8m_2 < 0 \ . \tag{1.1.14}$$

Hence, (1.1.13) and (1.1.14) result $m_1 \geq 10$ and $m_2 \geq 10$.

$\therefore\ \mathrm{Var}(\hat{\mu}) < \min[\mathrm{Var}(X_1), \mathrm{Var}(X_2)]$ if $m_1 \geq 10$, $m_2 \geq 10$.

Necessary conditions. To prove that these conditions are also necessary, suppose that $E\{F^2-2F\} > 0$ or this expectation fails to exist. One can show that then $\mathrm{Var}(\hat{\mu}) > \sigma_1^2/n_1$ for sufficiently small value of ρ.

All previous studies concern the estimation problem for medium-sized or large samples. Zacks (1966) presented the result of his research for an efficient unbiased estimation procedure for very small samples.

As we have seen, an efficient large sample unbiased estimator of the common mean is the weighted average of the two samples means, weighted respectively by the reciprocals of the sample variances. This estimator is uniformly more efficient than either of the sample mean if, and only if, each sample size is larger than 10. The problem is to find its efficiency function for samples of a very small size and to try to improve the efficiency of the estimation procedure. To answer this problem, Zacks considered two classes of randomized unbiased estimation procedures (note that his study was confined to the case where the samples are of equal size) generated by two estimators $\hat{\mu}$ and $\bar{\mu}$ was described as follows.

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two independent simple random samples of equal size, n, from two normal distributions having a common mean μ, and variances σ_1^2 and σ_2^2 respectively. Then $(\bar{X}, \bar{Y}, S_1, S_2)$ is a minimal sufficient. This set is not, however, complete. Zacks discussed the properties of unbiased estimators of the common mean, μ, which are functions of this set of sufficient statistics. If the ratio of variances, $\rho = \sigma_2^2/\sigma_1^2$ is known, then it can be shown that the linear estimator

$$\hat{\mu}_0 = \frac{\rho\bar{X}+\bar{Y}}{1+\rho} \qquad (1.1.15)$$

is uniformly best unbiased. Zacks showed that, in a close neighborhood of $\rho = 1$ the best unbiased estimator is the grand mean

$$\bar{\mu} = \tfrac{1}{2}(\bar{X} + \bar{Y}) \qquad (1.1.16)$$

The efficiency of this estimator, which is the ratio of the variance of $\hat{\mu}_0$ to that of $\bar{\mu}$, is given by

$$\mathrm{eff}(\bar{\mu}|\rho) = 4\rho/(1+\rho)^2, \quad 0 < \rho < \infty. \qquad (1.1.17)$$

This efficiency function is independent of the sample size n, and has the symmetric property:

$$\mathrm{eff}(\bar{\mu}|\rho) = \mathrm{eff}(\bar{\mu}|1/\rho) \quad 0 < \rho < \infty \qquad (1.1.18)$$

Furthermore,

$$\mathrm{eff}(\bar{\mu}|\rho) \to 0 \text{ as } \rho \to \infty$$

Once again, an unbiased estimator of μ analogous to (1.1.15) is

$$\hat{\mu} = \frac{\frac{S_2}{S_1}\overline{X} + \overline{Y}}{\frac{S^2}{S_1} + 1} \tag{1.1.19}$$

which is a symmetric function of $\overline{X}$, $\overline{Y}$, S_1 and S_2. Since S_2/S_1 is a consistent estimator of ρ, the asymptotic distribution of $\hat{\mu}$ is the same as that of the maximum likelihood estimator $\hat{\mu}(\rho)$, when ρ is known.

The first class of randomized unbiased estimation procedure generated by $\overline{\mu}$ and $\hat{\mu}$ is essentially a preliminary test estimator. The estimator is derived as follows. An F-test of significance is carried out to decide whether the ratio of population variances $\rho = \sigma_2^2/\sigma_1^2$ is equal to 1 or different from 1. If the hypothesis $\rho = 1$ is accepted, the grand mean of the two samples is applied to estimate the common mean; otherwise, the average of the sample means weighted inversely by the sample s.s.d.'s provides the estimate. Also, Zacks characterized this class of estimators in the following manner:

$$\hat{\mu}(\rho^*) = I\left(\frac{S_2}{S_1};\rho^*\right)\overline{\mu} + \left[1 - I\left(\frac{S_2}{S_1};\rho^*\right)\right]\hat{\mu}, \quad 1 \le \rho^* \le \infty \tag{1.1.20}$$

where

$$I\left(\frac{S_2}{S_1};\rho^*\right) = \begin{cases} 1, & \text{if } \frac{1}{\rho^*} \le \frac{S_2}{S_1} \le \rho^* \\ 0, & \text{otherwise}\,. \end{cases} \tag{1.1.21}$$

Since $I\left(\frac{S_2}{S_1};1\right) = 0$ with probability one, $\hat{\mu}(1) = \hat{\mu}$ W·p·1. On the other hand, $I\left(\frac{S_2}{S_1};\infty\right)=1$ W·p·1. Thus, $\hat{\mu}(\infty) = \overline{\mu}$ W·p·1. Indeed, the estimators $\hat{\mu}(\rho^*)$ are unbiased, since

$$E[\hat{\mu}(\rho^*) \mid \frac{S_2}{S_1}] = I(\frac{S_2}{S_1};\rho^*)\ E[\bar{\mu} \mid \frac{S_2}{S_1}] + [1-I(\frac{S_2}{S_1};\rho^*)]\ E[\hat{\mu} \mid \frac{S_2}{S_1}] = \mu.$$

Another class of randomized unbiased estimators consists of estimators, designated by $\tilde{\mu}(\rho^*)$, which is defined for every $1 \leq \rho^* \leq \infty$ as follows:

$$\tilde{\mu}(\rho^*) = I(\frac{S_2}{S_1};\rho^*)\ \bar{\mu} + J_1(\frac{S_2}{S_1};\rho^*)\ \bar{X} + J_2(\frac{S_2}{S_1};\rho^*)\ \bar{Y} \tag{1.1.22}$$

where

$$J_1(\frac{S_2}{S_1};\rho^*) = \begin{cases} 1, & \text{if } S_2/S_1 > 1/\rho^* \\ 0, & \text{otherwise,} \end{cases}$$

$$J_2(\frac{S_2}{S_1};\rho^*) = \begin{cases} 1, & \text{if } S_2/S_1 < 1/\rho^* \\ 0, & \text{otherwise.} \end{cases} \tag{1.1.23}$$

The values ρ^* both in $\hat{\mu}(\rho^*)$ and in $\tilde{\mu}(\rho^*)$ are the critical values of the F-tests of significance, according to which one decides whether to apply the estimators $\bar{\mu}$, $\hat{\mu}$, $\bar{X}$ or $\bar{Y}$.

Next Zacks studied the efficiency of his estimators. He showed that for sample of size 3, $(n = 3)$, the efficiency values of the estimators $\hat{\mu}(\rho^*)$ are higher than those of $\tilde{\mu}(\rho^*)$, over the range of $1/6 \leq \rho^* \leq 6$, for all values of ρ^* the efficiency function of $\hat{\mu}$ is uniformly higher than that of $\tilde{\mu}$. In the neighborhood of $\rho = 1$, the difference in efficiency is about .25. The continuity of the efficiency functions leads one to conclude that for values of ρ^* close to 1 the efficiency of $\hat{\mu}(\rho^*)$ is uniformly higher than that of $\tilde{\mu}(\rho^*)$. Moreover, the efficiency values of $\hat{\mu}(\rho^*)$ are, for very large values of ρ, higher than those of $\tilde{\mu}(\rho^*)$, since the limit of $\mathrm{eff}(\hat{\mu}(\rho^*)|\rho,3)$ as $\rho \to \infty$ is 0.5 for all $1 \leq \rho^* < \infty$, while the limits of $\mathrm{eff}(\tilde{\mu}(\rho^*)|\rho,3)$ as $\rho \to \infty$ are smaller than 0.5 for every $\rho^* > 1$. Thus, the estimators $\hat{\mu}(\rho^*)$ are superior to the estimators $\tilde{\mu}(\rho^*)$. Also, at $\rho = 1$, $\mathrm{eff}(\hat{\mu}|1, 3) = 0.75$. This is the limit of $\mathrm{eff}(\hat{\mu}|\rho,3)$ as $\rho \to 1$. On the other hand, when $\rho \to \infty$ $\mathrm{eff}(\hat{\mu}|\rho,3)$ decreases monotonically to 0.5. The limit of $\mathrm{eff}(\hat{\mu}(\rho^*)|\rho,3)$, for every finite ρ^*, as $\rho \to \infty$ is. 0.5. Thus, for large values of ρ, the efficiency of $\hat{\mu}(\rho^*)$ converges to that of $\hat{\mu}$. Khatri and Shah (1974) improved the result of Graybill and Deal when it is required that the combined estimator should have smaller variance when compared with the corresponding estimator from the first sample alone for all values of the unknown variances. Khatri and Shah showed that uniform improvement over the mean of the first sample is possible provided that the size of second sample exceeds two. This is an improvement over the result of Graybill and Deal (1959) which required that this sample size should exceed ten. It is also shown that uniform improvement over each of the sample means is possible provided that the sample sizes n_1 and n_2 satisfy $(n_1-7)(n_2-7) \geq 16$.

The Graybill and Deal results were generalized to k populations by Norwood and Hinkelmann (1977). To be more specific, let $X_1, \ldots, X_k$ and $S_1^2, \ldots, S_k^2$ be mutually independent random variables such that

$X_i \sim N(\mu, \sigma_i^2)$ and $m_i S_i^2/\sigma_i^2 \sim \chi^2_{m_i}$ for $i = 1, \ldots, k$. The main theorem of Norwood and Hinkelmann (1977) is as follows.

<u>Theorem 1.1.2</u> (i) The estimator $\hat{\mu} = \sum_{i=1}^{k} (X_i/S_i^2)/ \sum_{i=1}^{k}(1/S_i^2)$ is unbiased for μ.

(ii) $\mathrm{Var}(\hat{\mu}) < \sigma_i^2$ for all values of σ_i^2 $(i = 1, \ldots, k)$ if and only if either

(A) $m_i > 9$ $(i = 1,\ldots,k)$

or

(B) $m_i = 9$ for some i and $m_j > 17$ $(j = 1,\ldots,k;\ j \neq 1)$.

<u>Proof</u> See Norwood and Hinkelmann (1977).

Sinha (1979) showed that for given $\sigma_1^2,\ldots,\sigma_k^2$, the maximum likelihood estimate (m.l.e.) of μ given by

$$\mu^* = \sum_{i=1}^{k} (X_i/\sigma_i^2)/ \sum_{i=1}^{k} (1/\sigma_i^2) \tag{1.1.24}$$

is admissible for μ in the class of all estimators under a general type of loss which includes any positive power of the absolute error loss. But in the case when σ_i^2's are unknown, the m.l.e. of μ is given by

$$\hat{\mu} = \sum_{i=1}^{k} (X_i/S_i^2)/ \sum_{i=1}^{k} (1/S_i^2) \tag{1.1.25}$$

is admissible for μ.

As we have seen, expressions and approximations for the variance of the Graybill and Deal estimator given by (1.1.25) for $k = 2$ are by Meier (1953). Nair (1980) also gave an expression for the variance of μ, which is given below.

Theorem 1.1.3

$$V(\hat{\mu}) = \sigma_1^2 \sum_{i=0}^{\infty} \frac{(i+1)(1-\ )^i}{B(n_1/2,n_2/2)} [B(\frac{n_1}{2} + i, \frac{n_2}{2} + 2)$$

$$+ \frac{n_2}{n_1} \alpha B(\frac{n_1}{2} + i + 2, \frac{n_2}{2})]$$

where $\alpha = n_2\sigma_1^2/n_1\sigma_2^2$ and it is assumed $0 < \alpha \leq 1$.

Proof Since $n_1S_1^2/\sigma_1^2$ and $n_2S_2^2/\sigma_2^2$ are independent chi-squares, $S_1^2/S_2^2 = \alpha W/(1-W)$ where W is a $B(n_1/2,n_2/2)$ variable. Hence,

$$\hat{\mu} = ((1-W)\ X_1 + \alpha W X_2)/(1 - (1-\alpha)W) \tag{1.1.26}$$

Using the facts that X_1 and X_2, and W are independent and $E(\hat{\mu}|W) = \mu$, we get

$$V(\hat{\mu}) = E\{V(\hat{\mu}|W)\} + V\{E(\hat{\mu}|W)\}$$

$$= E\{(1-W)^2\sigma_1^2 + \alpha^2\sigma_2^2W^2\}\{1-(1-\alpha)W\}^{-2}$$

$$= \sigma_1^2\ E\{(1-W)^2 + n_2n_1^{-1}\alpha W^2\} \sum_{i=0}^{\infty} (i+1)(1-\alpha)^i W^i$$

Since $0 \leq 1 - \alpha < 1$ and $0 \leq W < 1$, term-by-term expectations can be taken in the above. Using the fact that W is a $B(n_1/2,n_2/2)$ variable, the result follcws.

Also, Nair showed that for $n_1 = n_2$

$$0 \le \frac{V(\hat{\mu})}{\sigma_1^2} - \frac{1}{1+\alpha} \le \frac{1+\alpha}{4\alpha^2(n_1+1)} + \frac{(1+3\alpha)[1+\alpha-a(1-\alpha)](1-\alpha)}{8\alpha^2[1-a(1-\alpha)]^2(n_1+3)} \qquad (1.1.27)$$

where $a = 2^{-1+(e\ell n2)^{-1}}$.

Bounds for the distribution of $\hat{\mu}$ are given by Nair (1980) is as follows.

Theorem 1.1.4 If $n_1 = n_2$

$$0 \le \Phi(Z) - P[\sigma_1^{-1}(1+\alpha)^{\frac{1}{2}}(\hat{\mu}-\mu) < Z]$$

$$\le \frac{(1+\alpha)^{\frac{3}{2}}}{(1+\alpha)^{\frac{1}{2}}+\alpha^{\frac{1}{2}}} A(\alpha,n_1)\, Z\Phi[Z\alpha(1+\alpha)^{-\frac{1}{2}}], \quad Z > 0 \qquad (1.1.28)$$

and the inequalities are reversed for $Z < 0$ where Φ is the standard normal distribution, Φ is the standard normal density and $A(\alpha,n_1)$ is the upper bound given by (1.1.27).

Proof See Nair (1980.

Next, consider the point estimation of the parameter θ in the normal distribution $N(\theta,a\theta^2)$, $(\theta>0,\ a>0)$ where the coefficient of variation, $\sqrt{a}$ is assumed to be known. In the usual case $n(\theta,\sigma^2)$ where σ^2 does not depend on θ, the sample mean is known to be a UMVUE (uniformly minimum variance unbiased estimator) of θ. In the present case of known coefficient of variation, the minimal sufficient statistic for the normal mean is the sample mean together with the sample standard deviation. However, it turns out that the family of distributions induced by the minimal sufficient statistic is not complete. This leads one to suspect that there does not exist any

UMVUE of the normal mean. In fact, a stronger result is proved in the Ph.D. thesis of Unni (1977). He shows that for the normal $(\theta, a\theta^2)$ family of distributions with $\theta(>0)$ unknown, but $a(>0)$ known, there does not exist any UMVUE for any estimable parametric function except the trivial ones (namely the constants).

To be more specific, let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim N(\theta, a\theta^2)$, $i = 1,\ldots,n$. The problem is to find the best possible estimator of θ. Khan (1968) proved that

$$T_1 = \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \tag{1.1.29}$$

and

$$T_2 = \frac{c_n}{\sqrt{n}} \left[\sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \right]^{\frac{1}{2}} = c_n S_n \tag{1.1.30}$$

where

$$c_n = \frac{\sqrt{n}}{\sqrt{2a}} \Gamma\left(\frac{n-1}{2} \right) / \Gamma\left(\frac{n}{2} \right) \tag{1.1.31}$$

are two uncorrelated (in fact, independent) unbiased estimator of θ and in the class of linear unbiased estimators, $d = \alpha T_2 + (1-\alpha)T_1$, $0 \le \alpha \le 1$

$$\begin{aligned} \hat{\theta}_{LU} &= \alpha T_2 + (1-\alpha)T_1 \\ &= (d_n T_1 + n^{-1} a T_2)/(d_n + n^{-1} a) \end{aligned} \tag{1.1.32}$$

where

$$d_n = (n^{-1}(n-1)ac_n^2 - 1) \tag{1.1.33}$$

is uniformly better than T_1 and T_2 alone and more generally, uniformly better than any other unbiased estimator of the type $d = \alpha T_2 + (1-\alpha)T_1$. Gleser and Healy (1976) considered the class $\mathcal{C}$ of all estimators of θ which are linear in $\overline{X}$ and S_n, but not necessarily unbiased. Since $T_1 = \overline{X}$ and $T_2 = c_n S_n$, an estimator is in $\mathcal{C}$ if and only if it is linear in T_1 and T_2. The estimator in $\mathcal{C}$ with uniformly (in θ) minimum risk under squared-error loss is

$$\begin{aligned}\hat{\theta}_{LMMS} &= (d_n T_1 + n^{-1} a T_2)/(d_n + n^{-1} a + n^{-1} a d_n) \\ &= [(d_n + n^{-1} a)/(d_n + n^{-1} a + n^{-1} a d_n)]\,\hat{\theta}_{LU} \qquad (1.1.34)\end{aligned}$$

This fact follows from Lemma (1.1.1) and the fact that

$$ET_1 = ET_2 = \theta \qquad (1.1.35)$$

and

$$\mathrm{Var}_\theta(T_1) = n^{-1} a \theta^2, \; \mathrm{Var}_\theta(T_2) = d_n \theta^2 \qquad (1.1.36)$$

Lemma 1.1.1 Let T_1 and T_2 be any two uncorrelated and unbiased estimators of a parameter θ. Assume that the ratios

$$\nu_i = \theta^{-2}\,\mathrm{Var}_\theta(T_i) \qquad (1.1.37)$$

are independent of θ, $i = 1, 2$. Then the estimator

$$T_{LMMS} = (\nu_2 T_1 + \nu_1 T_2)/(\nu_1 + \nu_2 + \nu_1 \nu_2) \qquad (1.1.38)$$

has uniformly (in θ) minimum risk under squared-error loss among all estimators linear in T_1 and T_2.

<u>Proof</u> By the given conditions, the risk of any estimator $\alpha_1 T_1 + \alpha_2 T_2$ linear in T_1 and T_2 is

$$\begin{aligned}
R(\theta,\alpha_1T_1+\alpha_2T_2) &= E(\theta-\alpha_1T_1-\alpha_2T_2)^2 \\
&= E[\alpha_1(T_1-ET_1) + \alpha_2(T_2-ET_2) + \alpha_1\theta + \alpha_2\theta - \theta]^2 \\
&= \alpha_1^2 \operatorname{Var}_\theta(T_1) + \alpha_2^2 \operatorname{Var}_\theta(T_2) + \theta^2(\alpha_1+\alpha_2-1)^2 \\
&= \theta^2(\alpha_1^2\nu_1+\alpha_2^2\nu_2 + (\alpha_1+\alpha_2-1)^2) \qquad (1.1.39)
\end{aligned}$$

which is a quadratic in α_1 and α_2. The result now follows by standard techniques of partial differentiation.

We note from (1.1.34) that $\hat\theta_{LMMS} \neq \hat\theta_{LU}$. Since $\hat\theta_{LU}$ is a member of the class C, we conclude that $\hat\theta_{LMMS}$ dominates $\hat\theta_{LU}$ in risk, and thus $\hat\theta_{LU}$ is inadmissible. Also, $\hat\theta_{LMMS}$ is itself inadmissible under squared-error loss, because, it is dominated in risk by the positive part estimator

$$\begin{aligned}
\hat\theta_{LMMS}^{+} &= \hat\theta_{LMMS}, \quad \text{if } \hat\theta_{LMMS} > 0 \\
&= 0 \quad , \quad \text{otherwise.} \qquad (1.1.40)
\end{aligned}$$

Note that $\hat\theta_{LU}$ and $\hat\theta_{LMMS}$ are scale invariant estimators of θ. The maximum likelihood estimator

$$\hat\theta_{MLE} = [-\bar{X}+(4aS_n^2 + (1+4a)\ \bar{X}^2)^{\frac{1}{2}}]/2a \qquad (1.1.41)$$

is also scale invariant. So, it is of interest to find the scale invariant estimator of θ. Gleser and Healy (1976) find the unique scale invariant estimator

$$\hat{\theta}_I = \nu\varphi^*(b)$$

where

$$\nu = [a^{-1}n(\bar{X}^2+S_n^2)]^{\frac{1}{2}} = [a^{-1}\sum_{i=1}^{n} X_i^2]^{\frac{1}{2}}$$

$$b = a^{-1}\, n\bar{X}/\nu$$

and

$$\varphi^*(b) = \left(\int_0^\infty \nu^{n+1}\, e^{-\frac{1}{2}\nu^2+b\nu}\, d\nu\right)^{-1}\left(\int_0^\infty \nu^n\, e^{-\frac{1}{2}\nu^2+b\nu}\, d\nu\right)$$

which has minimum risk among all scale invariant estimators. Also, Gleser and Healy (1976) constructed a class of Bayes estimators against inverted-gamma priors and they showed that $\hat{\theta}_I$ is a limiting member of this class of estimators.

1.2. Outline

This thesis contains results in the following areas:

- Estimating the common location parameter of two exponentials,
- Estimating the common location parameter of several exponentials,
- Estimating the location parameter of an exponential distribution with known coefficient of variation.

As we have seen in section (1.1) while, the normal case has received considerable attention in the literature, virtually nothing seems to be known about the estimation of the common location parameter of distributions other than normal in the presence of unknown scale parameters. In Chapters II and III, we address this estimation problem when the underlying distributions are exponentials. This amounts to the estimation of the so-called common "guarantee time" in the face of unknown and possibly unequal failure rates when the assumed distributions are all exponentials. In Chapter II, we considered the two exponential case, and in Section 2.5, first show that in this case, there exist uniformly minimum variance unbiased estimator (UMVUE) of the common location parameter μ which is denoted by $\hat{\mu}_{UMV}$. In Section 2.8, we construct the UMVUE of μ which is unbiased on the restricted parameter space $\{(\sigma_1,\sigma_2): \sigma_2 = \rho\sigma_1\}$ and is shown by $\hat{\mu}_{MVE}$. In Section 2.9, two estimators, one the usual maximum likelihood estimator (MLE), and the other a modified MLE, are proposed and are denoted by $\hat{\mu}_{MLE}$ and $\hat{\mu}_{MLE*}$ respectively. In Section 2.10, two estimators $\hat{\mu}_{MLE}$ and $\hat{\mu}_{MLE*}$ are compared and is shown that for large sample sizes, use of modified MLE($\hat{\mu}_{MLE*}$) can result approximately 50% relative efficiency in terms of mean squared error and approximately 100% relative efficiency in terms of bias criterion. In Section 2.11, we show that $\hat{\mu}_{MLE}$ is dominated by $\hat{\mu}_{UMV}$, $\hat{\mu}_{MVU}$ and $\hat{\mu}_{MLE*}$. And also $\hat{\mu}_{MLE}$, $\hat{\mu}_{UMV}$ and $\hat{\mu}_{MLE*}$ are dominated by $\hat{\mu}_{MVE}$ in terms of mean squared error criterion.

In Chapter III, the results of Chapter II are generalized to the several exponential case.

In Chapter IV, we consider random samples from exponential distributions where scale parameters are known constant multiples of location parameters. As we shall see in Section 4.1, this really amounts to the consideration of exponential distributions with known coefficients of variation. The minimal sufficient statistic for the unknown location (or scale) parameter θ turns out to be the sample minimum, say T_1 together with the sum of deviations from the minimum, say T_2. It turns out the family of distributions induced by (T_1, T_2) is not complete.

In Section 4.1, we have first provided the MVUE of θ in the class of all unbiased estimators of the form $c_1T_1 + c_2T_2$. In Section 4.2, this estimator is improved (in the sense of smaller mean squared error (MSE)) by the use of a shrinking factor. The latter turns out to be the minimum MSE estimator of θ in the class of all estimators (not necessarily unbiased) of the form $d_1T_1 + d_2T_2$.

The last linear estimator proposed in Section 4.1 turns out to be a scale invariant estimator. The best scale invariant estimator of θ is obtained in Section 4.4, which turns out to be different from the best linear estimator as given in Section 4.1. Also, in Section 4.5, a class of Bayes estimators of θ is proposed, and the best scale invariant estimator turns out to be a limiting Bayes estimator with constant risk.

2. COMMON LOCATION PARAMETER OF TWO EXPONENTIAL DISTRIBUTION

2.1. The Basic Set Up

Let $X_{11}, \ldots, X_{1n_1}, X_{21}, \ldots, X_{2n_2}$ be independently distributed with X_{1j}'s $(1\le j\le n_1)$ iid with common probability density function (pdf)

$$f(x_1) = \sigma_1^{-1} \exp(-(x_1-\mu)/\sigma_1)\, I_{[x_1\ge\mu]} \tag{2.1.1}$$

and the X_{2j}'s $(1\le j\le n_2)$ iid with pdf

$$f(x_2) = \sigma_2^{-1} \exp(-(x_2-\mu)/\sigma_2)\, I_{[x_2\ge\mu]} \tag{2.1.2}$$

where I denotes the usual indicator function and μ (real), $\sigma_1(>0)$, $\sigma_2(>0)$ are all unknown. The joint pdf of the X_{1i}'s and X_{2j}'s can be written as

$$f(\underset{\sim}{x}_1,\underset{\sim}{x}_2) = \sigma_1^{-n_1}\sigma_2^{-n_2} \exp[-\{\sigma_1^{-1}\sum_{i=1}^{n_1}(x_{1i}-\mu) + \sigma_2^{-1}\sum_{j=1}^{n_2}(x_{2j}-\mu)\}] \cdot \tag{2.1.3}$$

$$I_{[\min(U(\underset{\sim}{x}_1),V(\underset{\sim}{x}_2)) \ge\mu]},$$

where $\underset{\sim}{x}_1 = (x_{11}, \ldots, x_{1n_1})$, $\underset{\sim}{x}_2 = (x_{21}, \ldots, x_{2n_2})$, $U(\underset{\sim}{x}_1) = \min_{1\le i\le n_1} x_{1i}$ and $V(\underset{\sim}{x}_2) = \min_{1\le j\le n_2} x_{2j}$.

2.2. Maximum Likelihood Estimators

From (2.1.3) likelihood function can be written as

$$L(\mu,\sigma_1,\sigma_2,\underset{\sim}{x}_1,\underset{\sim}{x}_2) = \sigma_1^{-n_1}\sigma_2^{-n_2} \exp[-\{\sigma_1^{-1}\sum_{i=1}^{n_1}(x_{1i}-\mu) + \sigma_2^{-1}\sum_{j=1}^{n_2}(x_{2j}-\mu)\}] \cdot$$

$$I_{[\min(U(x_1),V(x_2))\geq\mu]} \qquad (2.2.1)$$

Then from (2.2.1)

$$\max_{\mu} L(\mu,\sigma_1,\sigma_2,\underset{\sim}{x}_1,\underset{\sim}{x}_2) = \sigma_1^{-n_1}\sigma_2^{-n_2} \exp[-\{\sigma_1^{-1}\sum_{i=1}^{n_1}(x_{1i}-Z) + \sigma_2^{-1}\sum_{j=1}^{n_2}(x_{2j}-Z)\}]$$

where $Z = \min(U(\underset{\sim}{x}_1),V(\underset{\sim}{x}_2))$.

Now define $L^*(Z,\sigma_1,\sigma_2,\underset{\sim}{x}_1,\underset{\sim}{x}_2) = \max_{\mu} L(\mu,\sigma_1,\sigma_2,\underset{\sim}{x}_1,\underset{\sim}{x}_2)$. Then

$$\text{Log } L^*(Z,\sigma_1,\sigma_2,\underset{\sim}{x}_1,\underset{\sim}{x}_2) = -n_1\text{Log}\sigma_1-n_2\text{Log}\sigma_2 - \sigma_1^{-1}\sum_{i=1}^{n_1}(x_{1i}-Z) - \sigma_2^{-1}\sum_{j=1}^{n_2}(x_{2j}-Z)$$

and

$$\frac{\partial \text{Log}L^*(Z,\sigma_1,\sigma_2,\underset{\sim}{x}_1,\underset{\sim}{x}_2)}{\partial\sigma_1} = -\frac{n_1}{\sigma_1} + \frac{1}{\sigma_1^2}\sum_{i=1}^{n_1}(x_{1i}-Z) \qquad (2.2.2)$$

$$\gtrless 0 \quad \text{as} \quad \sigma_1 \lesseqgtr (\bar{x}_1-Z)$$

$$\frac{\partial \text{Log}L^*(Z,\sigma_1,\sigma_2,\underset{\sim}{x}_1,\underset{\sim}{x}_2)}{\partial\sigma_2} = -\frac{n_2}{\sigma_2} + \frac{1}{\sigma_2^2}\sum_{j=1}^{n_2}(x_{1i}-Z) \qquad (2.2.3)$$

$$\gtrless 0 \quad \text{as} \quad \sigma_2 \lesseqgtr (\bar{x}_2-Z)$$

From equations (2.2.2) and (2.2.3), the MLE's of σ_1 and σ_2 are

$\hat{\sigma}_1 = \bar{x}_1 - Z$ and $\hat{\sigma}_2 = \bar{x}_2 - Z$

where $\bar{x}_1$, $\bar{x}_2$ are sample averages of first and second populations respectively. And Z, $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are maximum likelihood estimators (MLE) of μ, σ_1 and σ_2 respectively.

Before proving the first main result in this section, we obtain the distribution of $Z = \min(U(\underset{\sim}{X}_1), V(\underset{\sim}{X}_2))$ and the joint distribution of $T_1 = n_1\hat{\sigma}_1 = \sum_{i=1}^{n_1} (X_{1i} - Z)$ and $T_2 = n_2\hat{\sigma}_2 = \sum_{i=1}^{n_2} (X_{2i} - Z)$.

2.3. Distribution Function of $Z = \min(U(\underset{\sim}{X}_1), V(\underset{\sim}{X}_2))$

Using the independence of $U(\underset{\sim}{X}_1)$ and $V(\underset{\sim}{X}_2)$, it follows that for $z \geq \mu$,

$$P(Z \leq z) = 1 - P(Z > z) = 1 - P(U(\underset{\sim}{X}_1) > z,\ V(\underset{\sim}{X}_2) > z)$$

$$= 1 - P(U(\underset{\sim}{X}_1) > z) \cdot P(V(\underset{\sim}{X}_2) > z)$$

$$= 1 - [1 - G_U(z)][1 - G_V^*(z)] \tag{2.3.1}$$

where G_U and G_V^* are the d.f. of $U(\underset{\sim}{X}_1) = \min_{1 \leq i \leq n_1} (\underset{\sim}{X}_1)$ and $V(\underset{\sim}{X}_2) = \min_{1 \leq j \leq n_2} (\underset{\sim}{X}_2)$ respectively.

From (2.3.1) it follows that the pdf of Z is

$$f_Z(z) = (n_1\sigma_1^{-1} + n_2\sigma_2^{-1}) \exp\{-(n_1\sigma_1^{-1} + n_2\sigma_2^{-1})(z - \mu)\}\ I_{[z \geq \mu]} \tag{2.3.2}$$

Thus, for all n_1 and n_2, $(n_1\sigma_1^{-1} + n_2\sigma_2^{-1})(Z - \mu)$ has a simple exponential distribution with scale parameter 1.

2.4. Joint Distribution of T_1 and T_2

Define $Z = \min(X_{1(1)}, X_{2(1)})$, $U_j = \sum_{i=1}^{n_j} (X_{ji} - X_{j(1)})$, $j = 1, 2$ and

$$T_j = \sum_{i=1}^{n_j} (X_{ji} - Z) = U_j + n_j(X_{j(1)} - Z), \quad j = 1, 2.$$

Then

$$P(T_1>t_1,T_2>t_2) = P(U_1+n_1(X_{1(1)}-Z)) > t_1,\ U_2 + n_2(X_{2(1)}-Z) > t_2)$$

$$= \sum_{k=1}^{2} P(U_1+n_1(X_{1(1)}-Z) > t_1,\ U_2 + n_2(X_{2(1)}-Z) > t_2,$$

$$X_{k(1)} = \min(X_{1(1)},X_{2(1)}))$$

$$= \sum_{k=1}^{2} P(U_1+n_1(X_{1(1)}-Z) > t_1,\ U_2 + n_2(X_{2(1)}-Z) > t_2 | X_{j(1)} > X_{k(1)}$$

$$\forall_j = 1,\ 2(\neq k)) \cdot P(X_{j(1)}>X_{k(1)}\ \forall_{j\neq k}) \tag{2.4.1}$$

Now,

$$P(X_{2(1)}>X_{1(1)}) = \int_0^\infty\{\int_{x_{1(1)}}^\infty \frac{n_2}{\sigma_2}\ e^{-\frac{n_2}{\sigma_2}x_{2(1)}} dx_{2(1)}\} \frac{n_1}{\sigma_1} e^{-\frac{n_1}{\sigma_1}x_{1(1)}} dx_{1(1)}$$

$$= \int_0^\infty \frac{n_1}{\sigma_1}\ e^{-(\frac{n_2}{\sigma_2}+\frac{n_1}{\sigma_1})x_{1(1)}} dx_{1(1)} = \frac{n_1\ /\ \sigma_1}{n_1/\sigma_1 + n_2/\sigma_2} \tag{2.4.2}$$

Similarly,

$$P(X_{1(1)}> X_{2(1)}) = \frac{n_2\ /\ \sigma_2}{n_1/\sigma_1 + n_2/\sigma_2} \tag{2.4.3}$$

Also,

$$P(U_1+n_1(X_{1(1)}-Z) > t_1,\ U_2 + n_2(X_{2(1)}-Z) > t_2 | X_{2(1)} > X_{1(1)})$$

$$= P(U_1>t_1,X_{2(1)}>X_{1(1)},\ U_2 + n_2(X_{2(1)}-X_{1(1)})>t_2)/P(X_{2(1)}>X_{1(1)}) \tag{2.4.4}$$

Using independence of U_j's and $X_{2(1)} - X_{1(1)}$,

rhs of (2.4.4) = $P(U_1>t_1)\cdot P(U_2+n_2(X_{2(1)}-Z) > t_2 | X_{2(1)}>X_{1(1)})$ (2.4.5)

Now

$$P(n_2(X_{2(1)}-Z) > w_2 | X_{2(1)}>X_{1(1)}) = \frac{P(X_{2(1)}-X_{1(1)} > w_2/n_2)}{P(X_{2(1)}>X_{1(1)})}$$

$$= \frac{\int_0^\infty\{\int_{w_2/n_2+x_{1(1)}}^\infty \frac{n_2}{\sigma_2} e^{-\frac{n_2}{\sigma_2}x_{2(1)}} dx_{2(1)}\} \frac{n_1}{\sigma_1} e^{-\frac{n_1}{\sigma_1}x_{1(1)}} dx_{1(1)}}{\frac{n_1}{\sigma_1} / (n_1/\sigma_1 + n_2/\sigma_2)}$$

$$= e^{-w_2/\sigma_2} \,. \tag{2.4.6}$$

Thus, conditional on $(X_{2(1)}-X_{1(1)})$, $n_2(X_{2(1)}-Z)$ is distributed as $G(\sigma_2^{-1},1)$. Hence, conditional on $X_{2(1)} > X_{1(1)}$, $U_2 + n_2(X_{2(1)}-Z)$ is distributed as $G(\sigma_2^{-1},n_2)$. Thus, (2.4.5) follows that

$$P(T_1>t_1,T_2>t_2|X_{2(1)}>X_{1(1)}) = (P(G(\sigma_1^{-1},n_1-1)>t_1)(P(G(\sigma_2^{-1},n_2)>t_2))$$

where $G(\sigma_i^{-1},n_i) = \text{Gamma}(\sigma_i^{-1},n_i)$.

Similarly,

$$P(T_1>t_1,T_2>t_2|X_{1(1)}>X_{2(1)}) = (P(G(\sigma_1^{-1},n_1>t_1))(p(G(\sigma_1^{-1},n_2-1)>t_2))$$

Hence,

$P(T_1 > t_1, T_2 > t_2)$

$$= \frac{n_1\sigma_1^{-1}}{n_1\sigma_1^{-1} + n_2\sigma_2^{-1}} P[G(\sigma_1^{-1}, n_1-1) > t_1]\, P[G(\sigma_2^{-1}, n_2) > t_2]$$

$$+ \frac{n_2\sigma_2^{-1}}{n_1\sigma_1^{-1} + n_2\sigma_2^{-1}} P[G(\sigma_1^{-1}, n_1) > t_1]\, P[G(\sigma_2^{-1}, n_2-1) > t_2] \qquad (2.4.7)$$

From (2.4.7), the joint pdf of T_1 and T_2 can be written as

$$h(t_1,t_2) = (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1}(\sigma_1^{-n_1}\sigma_2^{-n_2})\left[\frac{n_2 t_1^{n_1-1} t_2^{n_2-2}}{\Gamma(n_1)\Gamma(n_2-1)} + \frac{n_1 t_1^{n_1-2} t_2^{n_2-1}}{\Gamma(n_1-1)\Gamma(n_2)}\right] e^{-\sigma_1^{-1}t_1-\sigma_2^{-1}t_2} \qquad t_1 > 0,\ t_2 > 0 \qquad (2.4.8)$$

2.5. The UMVUE of the Common Location Parameter of Two Exponential Distributions

Theorem 2.5.1 If $X_{11}, \ldots, X_{1n_1}, X_{21}, \ldots, X_{2n_2}$ be independently distributed with X_{ij}'s $(1 \le j \le n_i,\ i = 1, 2)$ iid with common pdf

$$f(x_{i1}) = \sigma_i^{-1} \exp\{-(x_{i1}-\mu)/\sigma_i\}\, I_{[x_{i1}\ge\mu]},\ i = 1, 2$$

then the UMVUE of μ is given by $\hat{\mu}_{UMV} = Z-\{T_1T_2/[n_1(n_1-1)T_2+n_2(n_2-1)T_1]\}$.

Proof From (2.1.3), the joint pdf of $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ can be written as

$$f(\underset{\sim}{x}_1,\underset{\sim}{x}_2) = \sigma_1^{-n_1}\sigma_2^{-n_2} \exp\{-[\sigma_1^{-1}\sum_{i=1}^{n_1}(x_{1i}-Z)+\sigma_2^{-1}\sum_{j=1}^{n_2}(x_{2j}-Z)+(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})(Z-\mu)]\} \cdot I_{[Z\ge\mu]}$$

$$= \sigma_1^{-n_1}\sigma_2^{-n_2} \exp\{-[\sigma_1^{-1}t_1+\sigma_2^{-1}t_2+(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})(Z-\mu)]\} I_{[Z\geq\mu]}, \quad (2.5.1)$$

where $Z = \min(\min\limits_{1\leq i\leq n_1} x_{1i}, \min\limits_{1\leq j\leq n_2} x_{2j})$, $t_1 = \sum\limits_1^{n_1} (x_{1i}-Z)$, $t_2 = \sum\limits_1^{n_2} (x_{2j}-Z)$.

From (2.5.1), the minimal sufficient statistic for $(\mu, \sigma_1, \sigma_2)$ is given by $(Z, T_1 = \sum\limits_{i=1}^{n_1} (X_{1i}-Z), T_2 = \sum\limits_{i=1}^{n_2} (X_{2i}-Z))$.

To prove that the family of distributions induced by (Z, T_1, T_2) is complete, we need to show for any real valued function $g(z, t_1, t_2)$

$$E_{\mu,\sigma_1,\sigma_2}\, g(Z,T_1,T_2) = 0 \;\forall\, \mu(\text{real}),\ \sigma_i(>0) \quad i = 1, 2$$

$$\Rightarrow g(Z,T_1,T_2) = 0 \quad \text{a.s}$$

But

$$E_{\mu,\sigma_1,\sigma_2}\, g(Z,T_1,T_2) = 0 \;\forall\, \mu(\text{real}),\ \sigma_i(>0) \iff$$

$$\int_0^\infty\int_0^\infty\int_\mu^\infty g(z,t_1,t_2)(n_1\sigma_1^{-1}+n_2\sigma_2^{-1}) \exp\{-(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})(z-\mu)\} \cdot h(t_1,t_2)\,dz\,dt_1\,dt_2 = 0$$

Differentiating both sides with respect to μ, it follows that

$$\int_0^\infty\int_0^\infty g(\mu,t_1,t_2)\, h_{\sigma_1,\sigma_2}(t_1,t_2)\,dt_1\,dt_2 = 0 \quad \text{a.s}$$

Hence, it suffices to show that for each fixed μ, (T_1, T_2) is complete sufficient for (σ_1, σ_2) .

From (2.4.4), the joint pdf of T_1, T_2 can be written as

$$h(t_1,t_2) = C(\sigma_1,\sigma_2)Q(t_1,t_2)\ e^{-\sigma_1^{-1}t_1-\sigma_2^{-1}t_2},\ t_1 > 0,\ t_2 > 0 \qquad (2.5.2)$$

which belongs to the exponential family. Thus, (T_1, T_2) is jointly complete sufficient statistic for (σ_1, σ_2). Thus, the family of distributions induced by (Z, T_1, T_2) is complete.

Next note that, from (2.3.2) it follows that for each fixed σ_1 and σ_2, Z is complete sufficient statistic for μ. Also, the joint pdf of T_1 and T_2 is free from μ. Hence, by Basu's theorem T_1 and T_2 are jointly independent of Z.

In order to find the UMVUE of μ, let us first find $E(Z)$

$$E(Z) = \int_\mu^\infty Z(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})\ \exp\{-(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})(z-\mu)\}dz$$

$$= \mu + (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1} \qquad (2.5.3)$$

Thus, in order to find the <u>UMVUE</u> of μ, it suffices to find a real valued function of t_1 and t_2 say $g(t_1, t_2)$ which has expected value equal to $(n_1\sigma_1^{-1} + n_2\sigma_2^{-1})^{-1}$

$$\therefore\ E\ g(t_1,t_2) = \frac{1}{n_1/\sigma_1 + n_2/\sigma_2} \Rightarrow$$

$$\int_0^\infty\int_0^\infty g(t_1,t_2)h(t_1,t_2)dt_1dt_2 = \frac{1}{n_1/\sigma_1 + n_2/\sigma_2} \iff$$

$$\int_0^\infty\int_0^\infty g(t_1,t_2)\left[\frac{n_2}{t_2\Gamma(n_1)\Gamma(n_2-1)}+\frac{n_1}{t_1\Gamma(n_1-1)\Gamma(n_2)}\right]\frac{t_1^{n_1-1}\,t_2^{n_2-1}}{\sigma_1^{n_1}\,\sigma_2^{n_2}}\cdot e^{-t_1/\sigma_1-t_2/\sigma_2}\,dt_1dt_2 = 1$$

$$\Longrightarrow g(T_1,T_2) = \frac{T_1T_2}{n_1(n_1-1)T_2 + n_2(n_2-1)T_1}$$

Thus, the UMVUE of μ is given by

$$\hat{\mu}_{UMV} = Z - \frac{T_1T_2}{n_1(n_1-1)T_2 + n_2(n_2-1)T_1}$$

with expected value and variance as follows.

$$E\hat{\mu}_{UMV} = E\,Z - E\,\frac{T_1T_2}{n_1(n_1-1)T_2 + n_2(n_2-1)T_1} = \mu + \frac{1}{n_1/\sigma_1+n_2/\sigma_2} - \frac{1}{n_1/\sigma_1+n_2/\sigma_2} = \mu$$

and

$$\mathrm{Var}(\hat{\mu}_{UMV}) = \mathrm{Var}(Z) + \mathrm{Var}(g(T_1,T_2)) = \left(\frac{1}{n_1/\sigma_1+n_2/\sigma_2}\right)^2 + E\,g^2(T_1,T_2)-[E\,g(T_1,T_2)]^2$$

Thus,

$$\mathrm{Var}(\hat{\mu}_{UMV}) = E\,g^2(T_1,T_2) = E(T_1T_2/(n_1(n_1-1)T_2+n_2(n_2-1)T_1))^2$$

$$= \frac{1}{n_1/\sigma_1+n_2/\sigma_2}\int_0^\infty\int_0^\infty \frac{1}{n_2(n_2-1)t_1+n_1(n_1-1)t_2}\cdot\frac{t_1^{n_1}t_2^{n_2}\,e^{-t_1/\sigma_1-t_2/\sigma_2}}{\Gamma(n_1)\Gamma(n_2)\sigma_1^{n_1}\sigma_2^{n_2}}\,dt_1dt_2$$

$$= \frac{n_1 n_2 \sigma_1 \sigma_2}{n_1\sigma_1^{-1}+n_2\sigma_2^{-1}} \int_0^\infty\int_0^\infty \frac{1}{n_2(n_2-1)\sigma_1 u_1+n_1(n_1-1)\sigma_2 u_2} \cdot$$

$$\frac{u_1^{n_1} u_2^{n_2} e^{-u_1-u_2}}{\Gamma(n_1+1)\Gamma(n_2+1)}\ du_1 du_2$$

$$= \frac{n_1 n_2 \sigma_1 \sigma_2}{n_1\sigma_1^{-1}+n_2\sigma_2^{-1}}\ E\{\frac{1}{n_2(n_2-1)\sigma_1 U_1+n_1(n_1-1)\sigma_2 U_2}\}, \qquad (2.5.4)$$

where U_1 and U_2 are independently distributed; $U_j \sim \text{Gamma}(1,\ n_j+1)$.

To evaluate the rhs of (2.5.4), we need the following lemma.

<u>Lemma</u> (Cressie, N., et al., 1981). Let X be a random variable such that $P(X>0) = 1$. Then, $E(X^{-1}) = \int_0^\infty E(e^{-tX})\ dt$.

<u>Proof</u> Using Fubini's theorem,

$$E(X^{-1}) = \int_0^\infty \frac{1}{x}\ dF(x) = \int_0^\infty\int_0^\infty e^{-tx}\ dt\ dF(x)$$

$$= \int_0^\infty \left[\int_0^\infty e^{-tx}\ dF(x)\right]dt = \int_0^\infty E(e^{-tX})\ dt$$

Using the above lemma,

$$\text{rhs of } (2.5.4) = \frac{n_1 n_2 \sigma_1 \sigma_2}{n_1\sigma_1^{-1}+n_2\sigma_2^{-1}} \int_0^\infty E\{e^{-[n_2(n_2-1)\sigma_1 U_1+n_1(n_1-1)\sigma_2 U_2]t}\}\ dt$$

$$= \frac{n_1 n_2 \sigma_1 \sigma_2}{n_1\sigma_1^{-1}+n_2\sigma_2^{-1}} \int_0^\infty (1+tn_2(n_2-1)\sigma_1)^{-(n_1+1)} \cdot$$

$$(1+tn_1(n_1-1)\sigma_2)^{-(n_2+1)}\ dt$$

$$= \frac{(n_1 n_2 \sigma_1 \sigma_2)[n_2(n_2-1)\sigma_1]^{-(n_1+1)}[n_1(n_1-1)\sigma_2]^{-(n_2+1)}}{n_1\sigma_1^{-1} + n_2\sigma_2^{-1}} \times$$

$$\int_0^\infty \frac{dt}{(t-r_2)^{n_2+1}(t-r_1)^{n_1+1}}$$

where $r_1 = -(n_2(n_2-1)\sigma_1)^{-1}$, $r_2 = -(n_1(n_1-1)\sigma_2)^{-1}$.

After evaluating the above integral, the variance of $\hat{\mu}_{UMV}$ is as follows.

$$\mathrm{Var}(\hat{\mu}_{UMV}) = \frac{(n_1\sigma_2)^{-n_2}(n_2\sigma_1)^{-n_1}(n_1-1)^{-(n_2+1)}(n_2-1)^{-(n_1+1)}}{n_1\sigma_1^{-1}+n_2\sigma_2^{-1}} I_{(n_1+1)}$$

where

$$I_{(n_1+1)} = \frac{-1}{n_1(r_2-r_1)}\left[\frac{-1}{(-r_1)^{n_2}(-r_2)^{n_1}} + (n_1+n_2)\, I_{n_1}\right] \quad \text{for} \quad n_1 > 0$$

and

$$I_{(1)} = \frac{1}{r_2-r_1}\mathrm{Log}\left(\frac{r_1}{r_2}\right) + \sum_{k=2}^{n_2+1}(-1)^{n_2-k+1}\left(\left(\frac{1}{r_1-r_2}\right)\right)^{n_2-k+2}$$

<u>Lemma 2.5.1</u> The UMVUE of $\mathrm{Var}(\hat{\mu}_{UMV})$ is given by

$g^2(T_1,T_2) = [T_1T_2/(n_1(n_1-1)T_2+n_2(n_2-1)T_2)]^2$.

<u>Proof</u> Since (Z, T_1, T_2) is complete sufficient for $(\mu, \sigma_1, \sigma_2)$ and $E\, g^2(T_1T_2) = \mathrm{Var}(\hat{\mu}_{UMV})$, $g^2(T_1,T_2)$ is the UMVUE of $\mathrm{Var}(\hat{\mu}_{UMV})$.

2.6. Basic Set-up for the UMVUE of μ When the Ratio $\sigma_2/\sigma_1 = \rho(>0)$ is Known

Let $X_{11}, \ldots, X_{1n_1}, X_{21}, \ldots, X_{2n_2}$ be independently distributed with X_{ij}'s $(1 \le j \le n_i,\ i = 1, 2)$ iid (for every $i = 1, 2$) with common pdf given by (2.1.1) and (2.1.2) .

When $\sigma_2/\sigma_1 = \rho$, from (2.1.1), the joint pdf of $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ can be written as

$$f(\underset{\sim}{x}_1, \underset{\sim}{x}_2) = \sigma_1^{-(n_1+n_2)} \rho^{-n_2} \exp\{-\sigma_1^{-1}[\sum_{i=1}^{n_1} (x_{1i}-\mu) + \rho^{-1} \sum_{j=1}^{n_2} (x_{2j}-\mu)]\} I_{[z \ge \mu]} \tag{2.6.1}$$

or

$$f(\underset{\sim}{x}_1, \underset{\sim}{x}_2) = \sigma_1^{-(n_1+n_2)} \rho^{-n_2} \exp\{-\sigma_1^{-1}[\sum_{i=1}^{n_1} (x_{1i}) + \rho^{-1} \sum_{j=1}^{n_2} (x_{2j}) - (n_1+\rho^{-1} n_2)\mu]\} I_{[z \ge \mu]} \tag{2.6.2}$$

From factorization theorem, the sufficient statistic for (μ, σ_1) is given by $(Z, \sum_{i=1}^{n_1} X_{1i} + \rho^{-1} \sum_{j=1}^{n_2} X_{2j})$.

In order to prove that $(Z, \sum_{i=1}^{n_1} X_{1i} + \rho^{-1} \sum_{j=1}^{n_2} X_{2j})$ is minimal sufficient statistic for (μ, σ_1), define $\theta_1 = -\frac{1}{\sigma_1}$ and $\theta_2 = \mu$

$$\therefore f_{\theta_1, \theta_2}(\underset{\sim}{x}_1, \underset{\sim}{x}_2) = C(\theta_1, \theta_2) \exp\{\theta_1 [\sum_{i=1}^{n_1} x_{1i} + \rho^{-1} \sum_{j=1}^{n_2} x_{2j}]\} .$$

$$\cdot\, I_{[\min(x_{11},\ldots,x_{1n_1},x_{21},\ldots,x_{2n_2}) \geq \mu]} \,. \tag{2.6.3}$$

Now, for given $\underset{\sim}{x}' = (x'_{11},\ldots,x'_{1n_1},\ldots,x'_{2n_2})$ and $\underset{\sim}{x}'' = (x''_{11}, \ldots, x''_{1n_1}, x''_{21},\ldots,x''_{2n_2})$, let $\theta'_2 = \min(x'_{11},\ldots,x'_{1n_1},x'_{21},\ldots,x'_{2n_2})$ and

$\theta''_2 = \min(x''_{11},\ldots,x''_{1n_1},x''_{21},\ldots,x''_{2n_2})$

hence

$$f_{\theta'_1,\theta'_2}(\underset{\sim}{x}') \cdot f_{\theta''_1,\theta''_2}(\underset{\sim}{x}'') \neq f_{\theta'_1,\theta'_2}(\underset{\sim}{x}'') \cdot f_{\theta''_1,\theta''_2}(\underset{\sim}{x}')$$

$$\Longleftrightarrow C(\theta'_1,\theta'_2)\ \exp\{\theta'_1[\sum_{i=1}^{n_1} x'_{1i}+\rho^{-1}\sum_{j=1}^{n_2} x'_{2j}]\}\ I_{[\min(x'_{11},\ldots,x'_{2n_2}) \geq \min(x'_{11},\ldots,x'_{2n_2})]} \cdot$$

$$C(\theta''_1,\theta''_2)\ \exp\{\theta''_1[\sum_{i=1}^{n_1} x''_{1i}+\rho^{-1}\sum_{j=1}^{n_2} x''_{2j}]\}\ I_{[\min(x''_{11},\ldots,x''_{2n_2}) \geq \min(x''_{11},\ldots,x''_{2n_2})]}$$

$$\neq \quad C(\theta'_1,\theta'_2)\ \exp\{\theta'_1[\sum_{i=1}^{n_1} x''_{1i}+\rho^{-1}\sum_{j=1}^{n_2} X''_{2j}]\}\ I_{[\min(x''_{11},\ldots,x''_{2n_2}) \geq \min(x'_{11},\ldots,x'_{2n})]} \cdot$$

$$C(\theta''_1,\theta''_2)\ \exp\{\theta''_1[\sum_{i=1}^{n_1} x'_{1i}+\rho^{-1}\sum_{j=1}^{n_2} x'_{2j}]\}\ I_{[\min(x'_{11},\ldots,x'_{2n_2}) \geq \min(x''_{11},\ldots,x''_{2n})]}$$

$$\Longleftrightarrow \text{ either } \min(x'_{11},\ldots,x'_{1n_1},x'_{21},\ldots,x'_{2n_2})$$

$$\neq \min(x''_{11},\ldots,x''_{1n_1},x''_{21},\ldots,x''_{2n_2})$$

$$\text{or} \quad \min(x'_{11},\ldots,x'_{1n_1},x'_{21},\ldots,x'_{2n_2})$$

$$= \min(x''_{11},\ldots,x''_{1n_1},x''_{21},\ldots,x''_{2n_2})$$

$$\text{and} \quad \Sigma_1^{n_1} x'_{1i} + \rho^{-1} \Sigma_1^{n_2} x'_{2j} \neq \Sigma_1^{n_1} x''_{1i} + \rho^{-1} \Sigma_1^{n_2} x''_{2j}$$

$$\text{i.e.} \quad (\min(x'_{11},\ldots,x'_{2n_2}), \sum_{i=1}^{n_1} x'_{1i} + \rho^{-1} \sum_{j=1}^{n_2} x'_{2j})$$

$$\neq (\min(x''_{11},\ldots,x''_{2n_2}), \sum_{i=1}^{n_1} x''_{1i} + \rho^{-1} \sum_{j=1}^{n_2} x''_{2j}) .$$

Therefore, in this case the minimal sufficient statistic for (μ, σ_1) is given by $(Z, \sum_{i=1}^{n_1} X_{1i} + \rho^{-1} \sum_{j=1}^{n_2} X_{2j})$ or equivalently (Z, T), where $T = \sum_{j=1}^{n_1} (X_{1j}-Z) + \rho^{-1} \sum_{j=1}^{n_2} (X_{2j}-Z)$ and also Z and $(n_1-n_2)^{-1}T$ are MLE of μ and σ_1 respectively. When $\sigma_2/\sigma_1 = \rho$, from (2.3.2), the pdf of $Z = \min(U(\underset{\sim}{X}_1),V(\underset{\sim}{X}_2))$ can be written as

$$f_Z(z) = (n_1+\rho^{-1}n_2)\sigma_1^{-1} \exp\{-(n_1+\rho^{-1}n_2)\sigma_1^{-1}(z-\mu)\}\, I_{[z\geq\mu]} . \tag{2.6.4}$$

From (2.6.4), it follows that for each fixed σ_1, Z is complete sufficient statistic for μ.

2.7. Distribution of $T = \sum_{i=1}^{n_1}(X_{1i}-Z)+\rho^{-1}\sum_{j=1}^{n_2}(X_{2j}-Z)$

In order to obtain the distribution of T, we proceed as follows.

Define $Q_1 = \Sigma_{i=1}^{n_1}(X_{1i}-U(\underset{\sim}{X}_1))$, $Q_2 = \Sigma_{j=1}^{n_2}(X_{2j}-V(\underset{\sim}{X}_2))$ and $Q_3 = n_1(U(\underset{\sim}{X}_1)-Z) + n_2\rho^{-1}(V(\underset{\sim}{X}_2)-Z)$. It can be easily shown that Q_1, Q_2 and Q_3 are mutually independently distributed since $U(\underset{\sim}{X}_1)$, $V(\underset{\sim}{X}_2)$, $\sum_{i=1}^{n_1}(X_{1i}-U(\underset{\sim}{X}_1))$ and $\sum_{j=1}^{n_2}(X_{2j}-V(\underset{\sim}{X}_2))$ are mutually independent.

$$Q_1 = \sum_{i=1}^{n_1}(X_{1i}-U(\underset{\sim}{X}_1)) = \sum_{i=1}^{n_1}(X_{1i}-X_{1(1)}) = \sum_{i=2}^{n_1}(n_1-i+1)(X_{1(i)}-X_{1(i-1)})$$

where $(n_1-i+1)(X_{1(i)}-X_{1(i-1)})$, $i = 2, \ldots, n_1$ are iid $\text{Gamma}(\sigma_1^{-1}, 1)$ (see e.g. David (1981), p. 20). Thus, Q_1 has a $\text{Gamma}(\sigma_1^{-1}, n_1-1)$ pdf while Q_2 is distributed as $\text{Gamma}((\rho\sigma_1)^{-1}, n_2-1)$, since $\sigma_2 = \rho\sigma_1$.

In order to find the distribution of Q_3, first note that $X_{1(1)}$ and $X_{2(1)}$ are independently distributed with pdf's

$$f(x_{i(1)}) = n_i/\rho_i\sigma_1 \exp\{-\frac{n_i}{\rho_i\sigma_1}(x_{i(1)}-\mu)\}, \quad i = 1, 2 \tag{2.7.1}$$

(where $\rho_1 = 1$, $\rho_2 = \rho$).

Therefore, $Z = \min(X_{1(1)}, X_{2(1)})$ has pdf

$$f(z) = (n_1/\sigma_1 + n_2/\rho\sigma_1)\ \exp\{-(n_1/\sigma_1 + n_2/\rho\sigma_1)(z-\mu)\} \tag{2.7.2}$$

Note that from (2.7.2) for each fixed σ_1, Z is complete sufficient for μ, while $Q_3 = n_1(X_{1(1)}-Z) + n_2\rho^{-1}(X_{2(1)}-Z)$ has a distribution free from μ. Hence, by Basu's Theorem, Z and Q_3 are independently distributed.

Consider

$$\begin{aligned} Q_3 &= n_1(X_{1(1)}-Z) + n_2\rho^{-1}(X_{2(1)}-Z) \\ &= [n_1(X_{1(1)}-\mu)+n_2\rho^{-1}(X_{2(1)}-\mu)] - [(n_1+n_2\rho^{-1})(Z-\mu)] . \end{aligned}$$

$n_1(X_{1(1)}-\mu)$ and $n_2\rho^{-1}(X_{2(1)}-\mu)$ are iid $\text{Gamma}(\sigma_1^{-1}, 1)$, therefore, $[n_1(X_{1(1)}-\mu) + n_2\rho^{-1}(X_{2(1)}-\mu)]$ is distributed as $\text{Gamma}(\sigma_1^{-1}, 2)$. And from (2.7.2) $[(n_1+n_2\rho^{-1})(Z-\mu)]$ is distributed as $\text{Gamma}(\sigma_1^{-1}, 1)$. Now use the fact that, if X and Y are independent, $X + Y \sim \text{Gamma}(\alpha,p_1)$ and $X \sim \text{Gamma}(\alpha,p_2)$, $(p_1>p_2)$, then $Y \sim \text{Gamma}(\alpha,p_1-p_2)$. Hence, Q_3 is distributed as $\text{Gamma}(\sigma_1^{-1},1)$. Thus, using the independence of Q_1, Q_2 and Q_3, it follows that

$$T = Q_1 + Q_2\rho^{-1} + Q_3 \sim \text{Gamma}(\sigma_1^{-1}, n_1+n_2-1) \tag{2.7.3}$$

Since, the family of distributions induced by T belongs to the exponential family, T is complete sufficient for σ_1.

2.8. The UMVUE of μ When the Ratio $\sigma_2/\sigma_1 = \rho(>0)$ is Known

Theorem 2.8.1 If the ratio $\sigma_2/\sigma_1 = \rho(>0)$ is known, then the UMVUE of μ which is unbiased on the restricted parameter space $\{(\sigma_1,\sigma_2)\colon \sigma_2 = \rho\sigma_1\}$ is given by $\hat{\mu}_{MVE} = Z - [(n_1+n_2-1)(n_1+n_2\rho^{-1})]^{-1}T$.

Proof of Theorem 2.8.1 When $\sigma_2/\sigma_1 = \rho$, from (2.1.1), the joint pdf of $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ can be written as

$$f(\underset{\sim}{x}_1,\underset{\sim}{x}_2) = \sigma_1^{-(n_1+n_2)}\rho^{-n_2}\exp\{-\sigma_1^{-1}[\sum_{i=1}^{n_1}(x_{1i}-z)+\rho^{-1}\sum_{j=1}^{n_2}(x_{2j}-z)$$

$$+ (n_1+n_2\rho^{-1})(z-\mu)]\} \cdot I_{[z\geq\mu]} \qquad (2.8.1)$$

As we have shown in Section (2.6), the minimal sufficient statistic for (μ, σ_1) is given by (Z, T). Also from Section (2.6) with $\sigma_2 = \rho\sigma_1$, it follows that for each fixed σ_1, Z is complete sufficient for μ, while T has a distribution free from μ. Hence from Lemma 2, Z and T are independently distributed. In Section (2.7), we have proved $T \sim \text{Gamma}(\sigma_1^{-1}, n_1+n_2-1)$. Hence, $E_{\sigma_1}(T) = (n_1+n_2-1)\sigma_1$. Also, it follows from (2.6.4) that $E_{\mu,\sigma_1}(Z) = \mu + (n_1+n_2\rho^{-1})^{-1}\sigma_1$. Therefore, it follows that for the restricted parameter space $\{(\sigma_1,\sigma_2)\colon \sigma_2 = \rho\sigma_1\}$,

$$E_{\mu,\sigma_1}(\hat{\mu}_{MVE}) = \mu + (n_1+n_2\rho^{-1})^{-1}\sigma_1 - [(n_1+n_2-1)(n_1+n_2\rho^{-1})]^{-1}(n_1+n_2-1)\sigma_1$$

$$= \mu + (n_1+n_2\rho^{-1})^{-1}\sigma_1 - (n_1+n_2\rho^{-1})^{-1}\sigma_1 = \mu.$$

Now in order to prove the theorem, it suffices to prove that the family of distributions induced by Z and T is complete.

For $\rho = \rho_0$, Z and T are mutually independent. Now

$$E_{\mu,\sigma_1,\rho_0} h(Z,T) = 0 \quad \forall\, \mu(\text{real}),\ \sigma_1(>0)$$

$$\Longleftrightarrow \int_\mu^\infty\int_0^\infty h(z,t)\ g_1(t)\ f_{\mu,\sigma_1}(z)\ dt\ dz = 0$$

$$\Longleftrightarrow \int_\mu^\infty U_{\sigma_1}(z)\ f(z)\ dz = 0 \quad \forall\, \mu(\text{real}),\ \sigma_1(>0) \tag{2.8.2}$$

(where $U_{\sigma_1}(z) = \int_0^\infty h(z,t)\ g_{\sigma_1}(t)\ dt$).

Differentiating both sides with respect to μ, it follows that $U_{\sigma_1}(\mu) = 0$ a.e. for all real μ, i.e. $\int_0^\infty h(\mu,t)\ g_{\sigma_1}(t)\ dt = 0$ a.e. Lebesgue for all fixed μ.

Since for each fixed μ, T is complete for σ_1, thus

$$\int_0^\infty h(\mu,t)\ g_{\sigma_1}(t) = 0 \quad \forall\, \sigma_1(>0)$$

$\Longleftrightarrow$ $h(\mu,t) = 0$ for all real μ and $t > 0$.

Therefore, (Z, T) is jointly complete sufficient statistic for (μ, σ_1).

Now from the above discussion it follows that, for every fixed $\rho = \rho_0$,

$$\hat{\mu}_{MVE} = Z - [(n_1+n_2-1)(n_1+n_2\rho^{-1})]^{-1}T \tag{2.8.3}$$

is the UMVUE of μ when $\rho = \sigma_2/\sigma_1$ is known.

From (2.8.3), it follows that

$E(\hat{\mu}_{MVE}) = \mu$

and

$$\mathrm{Var}(\hat{\mu}_{MVE}) = \frac{(n_1+n_2)}{(n_1+n_2-1)(n_1+n_2\rho^{-1})^2}\,\sigma_1^2$$

2.9. Modified Maximum Likelihood Estimator (MLE*)

As we have seen in Section 2.2, the maximum likelihood estimator of common location parameter and scale parameters in the case of unknown and unequal scale parameters are given respectively by $\hat{\mu}_{MLE} = Z$, $\hat{\sigma}_1 = n_1^{-1}\Sigma_{i=1}^{n_1}(X_{1i}-Z)$ and $\hat{\sigma}_2 = n_2^{-1}\Sigma_{j=1}^{n_2}(X_{2j}-Z)$.

In the absence of any information on (σ_1,σ_2) except that $(\sigma_1,\sigma_2) \in (0,\infty)X(0,\infty)$, and motivated from the fact that for σ_1 and σ_2, the UMVUE of μ is $Z - (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1}$, a modified MLE is proposed as follows:

$\hat{\mu}_{MLE*} = Z - (n_1\hat{\sigma}_1^{-1}+n_2\hat{\sigma}_2^{-1})^{-1}$.

We next attempt to compare $\hat{\mu}_{MLE}$ and $\hat{\mu}_{MLE*}$ in terms of their distributions as well as the mean squared error criterion.

2.10. Comparison of MLE and MLE* in Terms of Their Distribution as Well as MSE

We have already observed that $(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})(Z-\mu)$ has a simple exponential distribution with scale parameter 1. Therefore,

$E(Z-\mu) = (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1}$. Thus, using Chebyshev's inequality $Z \longrightarrow \mu$ in probability as $\min(n_1,n_2) \longrightarrow \infty$. Also, using the law of large numbers, $\frac{1}{n_1}\sum_{i=1}^{n_1} X_{1i} \longrightarrow \mu + \sigma_1$ in probability as $n_1 \longrightarrow \infty$ and it follows that $\hat{\sigma}_1 \xrightarrow{P} \sigma_1$ as $n_1 \longrightarrow \infty$ and $\hat{\sigma}_2 \xrightarrow{P} \sigma_2$ as $n_2 \longrightarrow \infty$. Hence as $\min(n_1,n_2) \longrightarrow \infty$, $(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})/(n_1\hat{\sigma}_1^{-1}+n_2\hat{\sigma}_2^{-1}) \longrightarrow 1$ in probability.

Thus as $\min(n_1,n_2) \longrightarrow \infty$

$$(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})(\hat{\mu}_{MLE*}-\mu) = -\frac{(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})}{(n_1\hat{\sigma}_1^{-1}+n_2\hat{\sigma}_2^{-1})} + (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})(Z-\mu)$$

$\xrightarrow{L} Y - 1$, where Y is a simple exponential with scale parameter 1.

We next compare $\hat{\mu}_{MLE}$ and $\hat{\mu}_{MLE*}$ in terms of their MSE's. First note that since $\hat{\mu}_{MLE} = Z$, from (2.3.2)

$$E(\hat{\mu}_{MLE}-\mu)^2 = 2(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-2} \tag{2.10.1}$$

Next observe that

$$\begin{aligned} E(\hat{\mu}_{MLE*}-\mu)^2 &= MSE(\hat{\mu}_{MLE}) - 2(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})E(n_1\hat{\sigma}_1^{-1}+n_2\hat{\sigma}_2^{-1}) + E(n_1\hat{\sigma}_1^{-1}+n_2\hat{\sigma}_2^{-1})^{-2} \\ &= MSE(\hat{\mu}_{MLE}) - (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-2} + E\{(n_1\hat{\sigma}_1^{-1}+n_2\hat{\sigma}_2^{-1})^{-1} \\ &\qquad -(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1}\}^2 \end{aligned} \tag{2.10.2}$$

In order to prove theorem 2.10.1, the following assumption is made.

$$0 < d_1 = \liminf_{n\to\infty} n_i/n \leq \limsup_{n\to\infty} n_i/n = d_2 < \infty \tag{2.10.3}$$

where $n = n_1 + n_2$.

Theorem 2.10.1 Under the assumption (2.10.3),

$$E[(n_1\hat{\sigma}_1^{-1}+n_2\hat{\sigma}_2^{-1})^{-1} - (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1}]^2 = 0(n^{-3}) \tag{2.10.4}$$

Proof Let $g(\sigma_1,\sigma_2) = (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1}$. Use a Taylor expansion to get

$$\begin{aligned} g(\hat{\sigma}_1,\hat{\sigma}_2) - g(\sigma_1,\sigma_2) &= (\hat{\sigma}_1-\sigma_1)(\partial g(\sigma_1,\sigma_2)/\partial\sigma_1) + (\hat{\sigma}_2-\sigma_2)(\partial g(\sigma_1,\sigma_2)/\partial\sigma_2) \\ &+ \tfrac{1}{2}[(\hat{\sigma}_1-\sigma_1)^2(\partial^2 g(\sigma_1,\sigma_2)/\partial\sigma_1^2)_{\substack{\sigma_1=\sigma_1^* \\ \sigma_2=\sigma_2^*}} + (\hat{\sigma}_2-\sigma_2)^2(\partial^2 g(\sigma_1,\sigma_2)/\partial\sigma_2^2)_{\substack{\sigma_1=\sigma_1^* \\ \sigma_2=\sigma_2^*}} \\ &+ 2(\hat{\sigma}_1-\sigma_1)(\hat{\sigma}_2-\sigma_2)(\partial^2 g(\sigma_1,\sigma_2)/\partial\sigma_1\partial\sigma_2)_{\substack{\sigma_1=\sigma_1^* \\ \sigma_2=\sigma_2^*}}] \end{aligned} \tag{2.10.5}$$

with $|\sigma_i^* - \sigma_i| < |\hat{\sigma}_i - \sigma_i|$, $i = 1, 2$. (2.10.6)

Write $A_1 = (\hat{\sigma}_1-\sigma_1)(\partial g(\sigma_1,\sigma_2)/\partial\sigma_1) + (\hat{\sigma}_2-\sigma_2)(\partial g(\sigma_1,\sigma_2)/\partial\sigma_2)$ and A_2 for the remaining terms in (2.10.5). Thus,

$$\begin{aligned} E[(n_1\hat{\sigma}_1^{-1}+n_2\hat{\sigma}_2^{-1})^{-1} - (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1}]^2 &= E[A_1+A_2]^2 \\ &\leq 2[EA_1^2 + EA_2^2] . \end{aligned} \tag{2.10.7}$$

Next we obtain expressions for $E(\hat{\sigma}_j-\sigma_j)^2 (j = 1, 2)$ and $E(\hat{\sigma}_1-\sigma_1)(\hat{\sigma}_2-\sigma_2)$. First note that

$$E(\hat{\sigma}_1-\sigma_1)^2 = E\{n_1^{-1}\Sigma_{i=1}^{n_1}(X_{1i}-Z)-\sigma_1\}^2$$

$$= E[\{n_1^{-1}\Sigma_{i=1}^{n_1}(X_{1i}-U(\underset{\sim}{X}_1))-\sigma_1\}^2\ I_{[U(\underset{\sim}{X}_1)\le V(\underset{\sim}{X}_2)]}]$$

$$+ E[\{n_1^{-1}\Sigma_{i=1}^{n_1}(X_{1i}-V(\underset{\sim}{X}_2))-\sigma_1\}^2\ I_{[U(\underset{\sim}{X}_1)>V(\underset{\sim}{X}_2)]}] \qquad (2.10.8)$$

Using the independence of $\Sigma_{i=1}^{n_1}(X_{1i}-U(\underset{\sim}{X}_1))$ and $(U(\underset{\sim}{X}_1),\ V(\underset{\sim}{X}_2))$ and also, the independence of $\Sigma_{i=1}^{n_1}(X_{1i}-V(\underset{\sim}{X}_2))$ and $(U(\underset{\sim}{X}_1),V(\underset{\sim}{X}_2))$ one gets from (2.10.8),

$$E(\hat{\sigma}_1-\sigma_1)^2 = E[n_1^{-1}\Sigma_{i=1}^{n_1}(X_{1i}-U(\underset{\sim}{X}_1))-\sigma_1]^2\ P(U(\underset{\sim}{X}_1)\le V(\underset{\sim}{X}_2))$$

$$+ E[n_1^{-1}\Sigma_{i=1}^{n_1}(X_{1i}-U(\underset{\sim}{X}_1))-\sigma_1+U(\underset{\sim}{X}_1)-V(\underset{\sim}{X}_2)]^2\ P(U(\underset{\sim}{X}_1)>V(\underset{\sim}{X}_2)).$$

$$(2.10.9)$$

Using the fact that $\Sigma_{i=1}^{n_1}(X_{1i}-U(\underset{\sim}{X}_1))$ is distributed as the sum of n_1-1 iid Gamma$(\sigma_1^{-1},1)$ variables say $Y_1,\ldots,Y_{n_1-1}$. Accordingly,

$$E[n_1^{-1}\Sigma_{i=1}^{n_1}(X_{1i}-U(\underset{\sim}{X}_1))-\sigma_1]^2 = E[n_1^{-1}\Sigma_{i=1}^{n_1-1}(Y_i-\sigma_1)-n_1^{-1}\sigma_1]^2$$

$$= n_1^{-2}(n_1-1)\sigma_1^2+n_1^{-2}\sigma_1^2 = \sigma_1^2/n_1 \qquad (2.10.10)$$

Also,

$$E[n_1^{-1}\Sigma_{i=1}^{n_1}(X_{1i}-U(\underset{\sim}{X}_1))-\sigma_1+U(\underset{\sim}{X}_1)-V(\underset{\sim}{X}_2)]^2$$

$$= E[n_1^{-1}\Sigma_{i=1}^{n_1}(Y_i-\sigma_1)+(U(\underset{\sim}{X}_1)-\mu-n_1^{-1}\sigma_1)-(V(\underset{\sim}{X}_2)-\mu-n_2^{-1}\sigma_2)-n_2^{-1}\sigma_2]^2$$

$$= n_1^{-2}(n_1-1)\sigma_1^2+n_1^{-2}\sigma_1^2+n_2^{-2}\sigma_2^2+n_2^{-2}\sigma_2^2 = n_1^{-1}\sigma_1^2+2n_2^{-2}\sigma_2^2 \quad (2.10.11)$$

Noting also that

$$P(U(\underset{\sim}{X}_1)\leq V(\underset{\sim}{X}_2)) = n_1\sigma_1^{-1}/(n_1\sigma_1^{-1}+n_2\sigma_2^{-1}).$$

It follows from (2.10.8) - (2.10.11) that

$$E(\hat{\sigma}_1-\sigma_1)^2 = n_1^{-1}\sigma_1^2+2n_2^{-1}\sigma_2^2/(n_1\sigma_1^{-1}+n_2\sigma_2^{-1}). \quad (2.10.12)$$

Similarly, one can show that

$$E(\hat{\sigma}_2-\sigma_2)^2 = n_2^{-1}\sigma_2^2+2n_1^{-1}\sigma_1^2/(n_1\sigma_1^{-1}+n_2\sigma_2^{-1}). \quad (2.10.13)$$

Now consider

$$\begin{aligned}
E(\hat{\sigma}_1-\sigma_1)(\hat{\sigma}_2-\sigma_2) &= E[(n_1^{-1}\Sigma_{i=1}^{n_1}(X_{1i}-Z)-\sigma_1)(n_2^{-1}\Sigma_{i=1}^{n_2}(X_{2i}-Z)-\sigma_2)] \\
&= E[(n_1^{-1}\Sigma_{i=1}^{n_1}(X_{1i}-U(\underset{\sim}{X}_1))-\sigma_1)(n_2^{-1}\Sigma_{i=1}^{n_2}(X_{2i}-U(\underset{\sim}{X}_1))-\sigma_2)]\cdot \\
&\qquad P(U(\underset{\sim}{X}_1)<V(\underset{\sim}{X}_2)) \\
&\quad + E[(n_1^{-1}\Sigma_{i=1}^{n_1}(X_{1i}-V(\underset{\sim}{X}_2))-\sigma_1)(n_2^{-1}\Sigma_{i=1}^{n_2}(X_{2i}-V(\underset{\sim}{X}_2))-\sigma_2)]\cdot \\
&\qquad P(U(\underset{\sim}{X}_1)\geq V(\underset{\sim}{X}_2))
\end{aligned}$$

$$\begin{aligned}
&= E[(n_1^{-1}\Sigma_{i=1}^{n_1-1}(Y_i-\sigma_1)-n_1^{-1}\sigma_1)(n_2^{-1}\Sigma_{i=1}^{n_2-1}(Y_j^*-\sigma_2)-n_2^{-1}\sigma_2+(V(\underset{\sim}{X}_2)-U(\underset{\sim}{X}_1)]\cdot \\
&\qquad P(U(\underset{\sim}{X}_1)<V(\underset{\sim}{X}_2)) \\
&+ E[(n_1^{-1}\Sigma_{i=1}^{n_1-1}(Y_i-\sigma_1)-n_1^{-1}\sigma_1+(U(\underset{\sim}{X}_1)-V(\underset{\sim}{X}_2))(n_2^{-1}\Sigma_{j=1}^{n_2-1}(Y_j^*-\sigma_2)-n_2^{-1}\sigma_2)]\cdot \\
&\qquad P(U(\underset{\sim}{X}_1)>V(\underset{\sim}{X}_2))
\end{aligned}$$

$$= n_1^{-2}\sigma_1^2(n_1\sigma_1^{-1}/(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})) + n_2^{-2}\sigma_2^2(n_2\sigma_2^{-1}/(n_1\sigma_1^{-1}/(n_1\sigma_1^{-1}+n_2\sigma_2^{-1}))$$

$$= n_1^{-1}n_2^{-1}\sigma_1\sigma_2 \tag{2.10.14}$$

where Y_i, $(i = 1,2,\ldots,n_1-1)$ and Y_j^*, $(j=1,\ldots,n_2-1)$ are iid Gamma$(\sigma_1^{-1},1)$ and Gamma$(\sigma_2^{-1},1)$ respectively.

Next observe again that $\partial g/\partial\sigma_i = n_i\sigma_i^{-2}(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-2}$, $i = 1, 2$. So by using Minkowski's inequality,

$$E(A_1^2) \le 2E[(\hat{\sigma}_1-\sigma_1)^2(\partial g/\partial\sigma_1)^2 + (\hat{\sigma}_2-\sigma_2)^2(\partial g/\partial\sigma_2)^2]$$

$$= 2[\{n_1^{-1}\sigma_1^2+2n_2^{-1}\sigma_2(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1}\}n_1^2\sigma_1^{-4}(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-4}$$

$$+ \{n_2^{-1}\sigma_2^2+2n_1^{-1}\sigma_1(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1}\}n_2^2\sigma_2^{-4}(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-4}]$$

$$= O(n^{-3}) \tag{2.10.15}$$

under (2.10.3). Also,

$$E(A_2^2) \le E[(\hat{\sigma}_1-\sigma_1)^4(\partial^2 g/\partial\sigma_1^2)^2_{\substack{\sigma_1=\sigma_1^*\\ \sigma_2=\sigma_2^*}} + (\hat{\sigma}_2-\sigma_2)^4(\partial^2 g/\partial\sigma_2^2)^2_{\substack{\sigma_1=\sigma_1^*\\ \sigma_2=\sigma_2^*}}$$

$$+ 4(\hat{\sigma}_1-\sigma_1)^2(\hat{\sigma}_2-\sigma_2)^2(\partial^2 g/\partial\sigma\cdot\partial\sigma_2)^2_{\substack{\sigma_1=\sigma_1^*\\ \sigma_2=\sigma_2^*}}] \tag{2.10.16}$$

Now, $\partial^2 g/\partial\sigma_1^2 = -2n_1\sigma_1^{-3}(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-2}+2n_1^2\sigma_1^{-4}(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-3}$

$$= -2n_1n_2/[\sigma_1^3\sigma_2(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^3] \tag{2.10.17}$$

Hence,

$$E[(\hat{\sigma}_1-\sigma_1)^4(\partial^2 g(\sigma_1,\sigma_2)/\partial\sigma_1^2)^2_{\substack{\sigma_1=\sigma_1^* \\ \sigma_2=\sigma_2^*}}]$$

$$= 4n_1^2n_2^2E[(\hat{\sigma}_1-\sigma_1)^4/[\sigma_1^{*6}\sigma_2^{*2}(n_1\sigma_1^{*-1}+n_2\sigma_2^{*-1})^6]$$

$$= 4n_1^2n_2^2E[\sigma_2^{*4}(\hat{\sigma}_1-\sigma_1)^4/(n_1\sigma_2^*+n_2\sigma_1^*)^6]$$

$$= 4n_1^2n_2^2E[(\hat{\sigma}_1-\sigma_1)^4/[\sigma_2^{*2}(n_1+n_2\sigma_1^*/\sigma_2^*)^6]]$$

$$\leq 4n_1^{-4}n_2^2E[(\hat{\sigma}_1-\sigma_1)^4/\sigma_2^{*2}]$$

$$\leq 4n_1^{-4}n_2^2\{E(\hat{\sigma}_1-\sigma_1)^8E(\sigma_2^*)^{-4}\}^{\frac{1}{2}} \tag{2.10.18}$$

Note that

$$n_1^{-1}\Sigma_{i=1}^{n_1}(X_{1i}-Z)-\sigma_1 = [n_1^{-1}\Sigma_{i=1}^{n_1}(X_{1i}-U(\underset{\sim}{X}_1))-\sigma_1]\, I_{[U(\underset{\sim}{X}_1)<V(\underset{\sim}{X}_2)]}$$

$$+ [n_1^{-1}\Sigma_{i=1}^{n_1}(X_{1i}-U(\underset{\sim}{X}_1))-\sigma_1+(U(\underset{\sim}{X}_1)-V(\underset{\sim}{X}_2)]\, I_{[U(\underset{\sim}{X}_1)>V(\underset{\sim}{X}_2)]}$$

Thus,

$$E(\hat{\sigma}_1-\sigma_1)^8 = E(n_1^{-1}\Sigma_{i=1}^{n_1}(X_{1i}-Z)-\sigma_1)^8 \leq 2^7\{E[n_1^{-1}\Sigma_{i=1}^{n_1-1}(Y_i-\sigma_1)-n_1^{-1}\sigma_1]^8 + E[n_1^{-1}\Sigma_{i=1}^{n_1-1}(Y_i-\sigma_1)-n_1^{-1}\sigma_1 + (U(\underset{\sim}{X}_1)-V(\underset{\sim}{X}_2))]^8\}$$

$$\leq 2^7\{E[n_1^{-1}\Sigma_{i=1}^{n_1-1}(Y_i-EY_i)-n_1^{-1}\sigma_1]^8 + E[n_1^{-1}\Sigma_{i=1}^{n_1-1}(Y_i-EY_i) + (U(\underset{\sim}{X}_1)-\mu)-(V(\underset{\sim}{X}_2)-\mu)-n_1\sigma_1^{-1}]^8\}$$

$$\le 2^{14}\{E[n_1^{-1}\Sigma_{i=1}^{n_1-1}(Y_i-EY_i)]^8+(n_1^{-1}\sigma_1)^8\}+2^{28}\{E[n_1^{-1}\Sigma_{i=1}^{n_1-1}(Y_i-EY_i)]^8$$

$$+2E(U(\underset{\sim}{X}_1-\mu)^8+(-n_1^{-1}\sigma_1)^8\}$$

$$=2^{14}[O(n^{-4})+O(n^{-8})]+2^{28}[O(n^{-4})+O(n^{-8})+O(n^{-8})]=O(n^{-4}) \qquad (2.10.19)$$

Also, since $\sigma_2^* \ge \min(\hat{\sigma}_2,\sigma_2)$ or $(\sigma_2^*)^{-1} \le \max(\hat{\sigma}_2^{-1},\sigma_2^{-1}) \le \hat{\sigma}_2^{-1}+\sigma_2^{-1}$.

Thus,

$$E(\sigma_2^*)^{-4} \le E(\hat{\sigma}_2^{-1}+\sigma_2^{-1})^4 \le 8E(\hat{\sigma}_2^{-4}+\sigma_2^{-4})$$

But, $\hat{\sigma}_2 = n_2^{-1}\Sigma_{j=1}^{n_2}(X_{2j}-Z) \ge n_2^{-1}\Sigma_{j=1}^{n_2}(X_{2j}-V(\underset{\sim}{X}_2))$ and

$\Sigma_{j=1}^{n_2}(X_{2j}-V(\underset{\sim}{X}_2)) \sim \text{Gamma}(\sigma_2^{-1},n_2-1)$. Hence, for $n_2 \ge 6$,

$$E(\hat{\sigma}_2^{-4}) \le n_2^4 \int_0^\infty z^{-4}\exp(-z/\sigma_2)\, z^{n_2-2}/(\Gamma(n_2-1)\sigma_2^{n_2-1})\,dz$$

$$=\sigma_2^{-4}n_2^4/\{(n_2-2)(n_2-3)(n_2-4)(n_2-5)\} \qquad (2.10.20)$$

Combining (2.10.18) - (2.10.20), one gets

$$E[\hat{\sigma}_1-\sigma_1)^4(\partial^4 g(\sigma_1,\sigma_2)/\partial\sigma_1^2)^2_{\substack{\sigma_1=\sigma_1^*\\ \sigma_2=\sigma_1^*}}]=O(n^{-4}\cdot n^2\cdot n^{-2})=O(n^{-4}) \qquad (2.10.21)$$

Similarly,

$$E[(\hat{\sigma}_2-\sigma_2)^4(\partial^2 g(\sigma_1,\sigma_2)/\partial\sigma_2^2)^2_{\substack{\sigma_1=\sigma_1^*\\ \sigma_2=\sigma_2^*}}]=O(n^{-4}) \qquad (2.10.22)$$

Finally, since

$$\partial^2 g(\sigma_1,\sigma_2)/\partial\sigma_1\partial\sigma_2 = 2n_1n_2/[\sigma_1^2\sigma_2^2(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^3],$$

it follows that

$$E\{(\hat{\sigma}_1-\sigma_1)^2(\hat{\sigma}_2-\sigma_2)^2[4n_1^2n_2^2\sigma_1^{-4}\sigma_2^{-4}\sigma_1^6(n_1+n_2\sigma_1\sigma_2^{-1})^{-6}]_{\substack{\sigma_1=\sigma_1^* \\ \sigma_2=\sigma_2^*}}\}$$

$$\leq 4n_1^{-4}n_2^2\ E[(\hat{\sigma}_1-\sigma_1)^2(\hat{\sigma}_2-\sigma_2)^2\sigma_1^{*2}\sigma_2^{*-4}] \tag{2.10.23}$$

Repeated application of Schwarz's inequality gives

$$E[(\hat{\sigma}_1-\sigma_1)^2(\hat{\sigma}_2-\sigma_2)^2\sigma_1^{*2}\sigma_2^{*-4}]$$

$$\leq \{E(\hat{\sigma}_1-\sigma_1)^8\}^{\frac{1}{4}}\{E(\hat{\sigma}_2-\sigma_2)^8\}^{\frac{1}{4}}\{E(\sigma_1^{*8})\}^{\frac{1}{4}}\{E(\sigma_2^{*-16})\}^{\frac{1}{4}} \tag{2.10.24}$$

Once again, arguments similar as before lead to

$$E(\hat{\sigma}_i-\sigma_i)^8 = O(n^{-4}), \quad i = 1, 2 \tag{2.10.25}$$

Since $\sigma_1^* \leq \hat{\sigma}_1 + \sigma_1 \leq (\hat{\sigma}_1-\sigma_1+2\sigma_1)$. Thus,

$E(\sigma_1^{*8}) \leq 2^7[E(\hat{\sigma}_1-\sigma_1)^8+2^8\sigma_1^8] = 2^7[O(n^{-4})+O(1)] = O(1)$ and, also for

$n_2 \geq 17$, it can be shown that $E(\sigma_2^{*-16}) = O(1)$.

Using the above, it follows from (2.10.25) that

$$E[(\hat{\sigma}_1-\sigma_1)^2(\hat{\sigma}_2-\sigma_2)^2\sigma_1^{*2}\sigma_2^{*-16}] = O(n^{-2}) \tag{2.10.26}$$

Hence, from (2.10.23) and (2.10.26),

$$E[(\hat{\sigma}_1-\sigma_1)^2(\hat{\sigma}_2-\sigma_2)^2 4n_1^2n_2^2\sigma_1^{*-4}\sigma_2^{*-4}(n_1\sigma_1^{*-1}+n_2\sigma_2^{*-1})^{-6}] = O(n^{-4}) \tag{2.10.27}$$

Therefore, from (2.10.17), (2.10.21), (2.10.22) and (2.10.27), it follows that

$$E(A_2^2) = O(n^{-4}). \tag{2.10.28}$$

Combining (2.10.15) and (2.10.28), the result follows.

Remark 1 In view of (2.10.1) and (2.10.2), it follows that

$[\mathrm{MSE}(\hat{\mu}_{MLE})-\mathrm{MSE}(\hat{\mu}_{MLE*})]/[\mathrm{MSE}(\hat{\mu}_{MLE})] =$

$$\tfrac{1}{2} - \tfrac{1}{2}(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^2 E[(n_1\hat{\sigma}_1^{-1}+n_2\hat{\sigma}_2^{-1})^{-1} - (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1}]^2.$$

Now, in view of Theorem 2.10.1, it appears that for large n_1 and n_2, use of modified $\mathrm{MLE}(\hat{\mu}_{MLE*})$ can result approximately 50% relative efficiency in terms of the mean squared error.

We next compare $\hat{\mu}_{MLE}$ and $\hat{\mu}_{MLE*}$ in terms of their biases. First note that

$$\mathrm{Bias}(\hat{\mu}_{MLE}) = E(Z-\mu) = \mu + (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1}-\mu = (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1} \tag{2.10.29}$$

and

$$\begin{aligned}\mathrm{Bias}(\hat{\mu}_{MLE*}) &= E[(Z-(n_1\hat{\sigma}_1^{-1}+n_2\hat{\sigma}_2^{-1})^{-1})-\mu] \\ &= \mu + (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1} - E(n_1\hat{\sigma}_1^{-1}+n_2\hat{\sigma}_2^{-1})^{-1} - \mu \\ &= (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1} - E(n_1\hat{\sigma}_1^{-1}+n_2\hat{\sigma}_2^{-1})^{-1} \qquad (2.10.30)\end{aligned}$$

Lemma 2.10.1 Under the assumption (2.10.3)

$$E[(n_1\hat{\sigma}_1^{-1}+n_2\hat{\sigma}_2^{-1})^{-1} - (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1}] = O(n^{-2}) \tag{2.10.31}$$

Proof Once again considering $g(\sigma_1,\sigma_2) = (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1}$ and using a Taylor expansion, it follows that

$$g(\hat{\sigma}_1,\hat{\sigma}_2) - g(\sigma_1,\sigma_2) = (\hat{\sigma}_1-\sigma_1)(\partial g(\sigma_1,\sigma_2)/\partial\sigma_1) + (\hat{\sigma}_2-\sigma_2)(\partial g(\sigma_1,\sigma_2)/\partial\sigma_2)$$

$$+ \tfrac{1}{2}[(\hat{\sigma}_1-\sigma_1)^2(\partial^2 g(\sigma_1,\sigma_2)/\partial\sigma_1^2)_{\substack{\sigma_1=\sigma_1^* \\ \sigma_2=\sigma_2^*}}$$

$$+ (\hat{\sigma}_2-\hat{\sigma}_2)^2(\partial^2 g(\sigma_1,\sigma_2)/\partial\sigma_2^2)_{\substack{\sigma_1=\sigma_1^* \\ \sigma_2=\sigma_2^*}}$$

$$+ 2(\hat{\sigma}_1-\sigma_1)(\hat{\sigma}_2-\sigma_2)(\partial^2 g(\sigma_1,\sigma_2)/\partial\sigma_1\partial\sigma_2)_{\substack{\sigma_1=\sigma_1^* \\ \sigma_2=\sigma_2^*}}].$$

Next, we obtain expressions for $E(\hat{\sigma}_i-\sigma_i)$, $(i = 1, 2)$. First note that

$$E(\hat{\sigma}_1-\sigma_1) = E[n_1^{-1}\Sigma_{i=1}^{n_1-1}(Y_i-\sigma_1)-n_1^{-1}\sigma_1]\ P(U(\underset{\sim}{X}_1)<V(\underset{\sim}{X}_2)) +$$

$$E[n_1^{-1}\Sigma_{i=1}^{n_1-1}(Y_i-\sigma_1)+(U(\underset{\sim}{X}_1)-V(\underset{\sim}{X}_2))-n_1^{-1}\sigma_1]\ P(U(\underset{\sim}{X}_1)>V(\underset{\sim}{X}_2))$$

$$= -n_1^{-1}\sigma_1(n_1\sigma_1^{-1}/(n_1\sigma_1^{-1}+n_2\sigma_2^{-1}))-n_2^{-1}\sigma_2(n_2\sigma_2^{-1}/(n_1\sigma_1^{-1}+n_2\sigma_2^{-1}))$$

$$= -2(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1}.$$

Similarly,

$$E(\hat{\sigma}_2-\sigma_2) = -2(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1}.$$

Thus,

$$E(A_1) = (\partial g(\sigma_1,\sigma_2)/\partial\sigma_1)E(\hat{\sigma}_1-\sigma_1) + (\partial g(\sigma_1,\sigma_2)/\partial\sigma_2)E(\hat{\sigma}_2-\sigma_2)$$

$$= -2[n_1\sigma_1^{-2}+n_2\sigma_2^{-2}](n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-3}$$

$$= O(n^{-2}) \qquad (2.10.32)$$

And

$$E\{|(\hat{\sigma}_1-\sigma_1)^2(\partial^2 g(\sigma_1,\sigma_2)/\partial\sigma_1^2)_{\substack{\sigma_1=\sigma_1^* \\ \sigma_2=\sigma_2^*}}|\}$$

$$= E\{|(\hat{\sigma}_1-\sigma_1)^2[-2n_1n_2\sigma_1^{*-3}\sigma_2^{*-1}(n_1\sigma_1^{*-1}+n_2\sigma_2^{*-1})^{-3}|]$$

$$= E\{(\hat{\sigma}_1-\sigma_1)^2[2n_1n_2\sigma_2^{*-1}(n_1+n_2\sigma_1^*\sigma_2^{*-1})^{-3}]\}$$

$$\leq E\{(\hat{\sigma}_1,\sigma_1)^2(2n_1^{-2}n_2\sigma_2^{*-1})\} \leq 2n_1^{-2}n_2\{E(\hat{\sigma}_1-\sigma_1)^4E(\sigma_2^{*-2})\}^{\frac{1}{2}} .$$

Thus,

$$E(\hat{\sigma}_1-\sigma_1)^4 = E(n_1^{-1}\Sigma_{i=1}^{n_1}(X_{1i}-Z)-\sigma_1)^4$$

$$\leq 2^3\{E[n_1^{-1}\Sigma_{i=1}^{n_1-1}(Y_i-\sigma_1)-n_1^{-1}\sigma_1]^4 + E[n_1^{-1}\Sigma_{i=1}^{n_1-1}(Y_i-\sigma_1)-n_1^{-1}\sigma_1 + (U(\underset{\sim}{X}_1)-\mu)-(V(\underset{\sim}{X}_1)-\mu)]^4\}$$

$$\leq 2^6\{E[n_1^{-1}\Sigma_{i=1}^{n_1-1}(Y_i-\sigma_1)]^4+n_1^{-4}\sigma_1^4\} + 2^9\{E[n_1^{-1}\Sigma_{i=1}^{n_1-1}(Y_i-\sigma_1)]^4 +(n_1^{-1}\sigma_1)^4 + E(U(\underset{\sim}{X}_1)-\mu)^4+E(V(\underset{\sim}{X}_2)-\mu)^4\}$$

$$= 2^6[O(n^{-2})+O(n^{-4})] + 2^9[O(n^{-2})+O(n^{-4})+O(n^{-4})+O(n^{-4})] = O(n^{-2}) \tag{2.10.33}$$

Also, since $\sigma_2^* \geq \min(\hat{\sigma}_2,\sigma_2)$ or $(\sigma_2^*)^{-1} \leq \max(\hat{\sigma}_2^{-1},\sigma_2^{-1}) \leq \hat{\sigma}_2^{-1} + \sigma_2^{-1}$

thus,

$$E(\sigma_2^*)^{-2} \leq E(\hat{\sigma}_2^{-1}+\sigma_2^{-1})^2 \leq 2\{E(\hat{\sigma}_2^{-1})^2+(\sigma_2)^{-2}\}.$$

But,

$\hat{\sigma}_2 = n_2^{-1}\Sigma_{j=1}^{n_2}(X_{2j}-Z) \geq n_2^{-1}\Sigma_{j=1}^{n_2}(X_{2j}-V(\underset{\sim}{X}_2))$ and

$\Sigma_{j=1}^{n_2}(X_{2j}-V(\underset{\sim}{X}_2)) \sim \text{Gamma}(\sigma_2^{-1},n_2-1)$. Hence, for $n_2 > 3$,

$$E(\hat{\sigma}_2^{-2}) \leq n_2^2 \int_0^\infty z^{-2} \exp(-z/\sigma_2)\, z^{n_2-2}/(\Gamma(n_2-1)\sigma_2^{n_2-1})\, dz$$

$$= \sigma_2^{-2}n_2^2/\{(n_2-2)(n_2-3)\}.$$

Thus,

$$E(\sigma_2^*)^{-2} = O(1) \tag{2.10.34}$$

Combining (2.10.33) - (2.10.34), it follows that

$$E\{(\hat{\sigma}_1-\sigma_1)^2(\partial^2 g/\partial\sigma_1^2)_{\substack{\sigma_1=\sigma_1^*\\ \sigma_2=\sigma_2^*}}\} = O(n^{-2}\cdot n\cdot n^{-1}) = O(n^{-2}) \tag{2.10.35}$$

Similarly,

$$E\{(\hat{\sigma}_2-\sigma_2)(\partial^2 g/\partial\sigma_2^2)_{\substack{\sigma_1=\sigma_1^*\\ \sigma_2=\sigma_2^*}}\} = O(n^{-2}) \tag{2.10.36}$$

Finally, since

$$\partial^2 g(\sigma_1,\sigma_2)/\partial\sigma_1\partial\sigma_2 = 2n_1n_2/[\sigma_1^2\sigma_2^2(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^3],$$

it follows that

$$E\{(\hat{\sigma}_1-\sigma_1)(\hat{\sigma}_2-\sigma_2)[2n_1n_2\sigma_1^{-2}\sigma_2^{-2}(n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-3}]_{\substack{\sigma_1=\sigma_1^*\\ \sigma_2=\sigma_2^*}}\}$$

$$\leq 2n_1^{-2}n_2\, E[(\hat{\sigma}_1-\sigma_1)(\hat{\sigma}_2-\sigma_2)\sigma_1^*\sigma_2^{*-2}]$$

$$\leq 2n_1^{-2}n_2\{E(\hat{\sigma}_1-\sigma_1)^4\}^{1/4}\{E(\hat{\sigma}_2-\sigma_2)^4\}^{1/4}\{E(\sigma_1^*)^4\}^{1/4}\{E(\sigma_2^*)^{-8}\}^{1/4} \qquad (2.10.37)$$

from (2.10.33) $E(\hat{\sigma}_i-\sigma_i)^4 = 0(n^{-2})$ for $i = 1, 2$.

And since $\sigma_1^* \leq \hat{\sigma}_1 + \sigma_1 \leq (\hat{\sigma}_1-\sigma_1+2\sigma_1)$, thus

$$E(\sigma_1^*)^4 \leq 2^3[E(\hat{\sigma}_1-\sigma_1)^4+2^4\sigma_1^4] = 2^7[0(n^{-2})+0(1)] = 0(1) \qquad (2.10.38)$$

And, using the previous result, we can prove for $n_2 \geq 9$, $E(\sigma_2^*)^{-8} = 0(1)$.

Thus,

$$E\{(\hat{\sigma}_1-\sigma_1)(\hat{\sigma}_2-\sigma_2)(\partial^2 g(\sigma_1,\sigma_2)/\partial\sigma_1\partial\sigma_2)_{\substack{\sigma_1=\sigma_1^* \\ \sigma_2=\sigma_2^*}}\} = 0(n^{-2}) \qquad (2.10.39)$$

Now, from (2.10.33) - (2.10.38), it follows that

$$E(A_2) = 0(n^{-2}) \qquad (2.10.40)$$

Thus, (2.10.32) and (2.10.38) implies that

$$E[(n_1\hat{\sigma}_1^{-1}+n_2\hat{\sigma}_2^{-1})^{-1} - (n_1\sigma_1^{-1}+n_2\sigma_2^{-1})^{-1}] = E(A_1+A_2) = 0(n^{-2}) \qquad (2.10.41)$$

Thus, it follows that for large n_1 and n_2, use of the modified MLE can result approximately 100% relative efficiency in terms of bias criterion.

2.11. A Monte Carlo Study of the Bias and the Mean Squared Errors

Expressions for the biases of the maximum likelihood estimator and modified maximum likelihood estimator of the common location parameter of two exponential distributions have been derived. It is important to investigate the accuracy of the approximate expressions for the bias in finite samples. The behavior is investigated empirically by generating observations from exponential distributions. Modified MLE is compared with the MLE in terms of the bias, while the UMVUE, the MLE and the modified MLE are all compared in terms of the mean squared errors.

To generate two regular exponential random variables, two sequences of pseudo-random uniform (0,1) were generated and transformed to regular exponentials with location parameter $\mu = 0$ and given scale parameters σ_1 and σ_2, by inverting the regular exponential cumulative distributions functions. The simulation was performed using the latest verson of the IMSL package and all computations were performed using double precision arithmetic.

For each $(\mu = 0, \sigma_1, \sigma_2)$ combination, various point estimates were computed using the same set of observations. This was repeated each time. Sample biases and mean squared errors for some of the proposed estimators were obtained by averaging over the 1000 replications. The numerical results are reported in the following tables and graphs.

Tables 1 to 5 contain the Monte Carlo biases and percentage reduction in bias of modified MLE with respect to MLE for paired sample sizes (5,5), (5,10), (5,15), (5,20) and (10,15). For every $(\sigma_1, \sigma_2, n_1, n_2)$

Table 1. *Empirical biases of* $\hat{\mu}_{MLE}$ *and* $\hat{\mu}_{MLE*}$ for $n_1 = n_2 = 5$

True values of σ_1	σ_2	True value of μ	$\hat{\mu}_{MLE}$	$\hat{\mu}_{MLE*}$	P.R in Bias[a]
1	3	0	.15000	.02819	.81202
1	5	0	.16666	.03140	.81156
1	8	0	.17777	.03575	.81013
1	10	0	.18182	.03463	.80949
3	1	0	.15000	.02908	.80608
3	5	0	.37500	.07013	.80038
3	8	0	.43636	.08204	.81199
3	10	0	.46153	.08674	.81206
5	1	0	.16666	.03857	.80455
5	3	0	.37500	.07192	.80802
5	8	0	.61538	.11616	.81123
5	10	0	.66666	.12542	.81186
8	1	0	.17777	.03495	.80339
8	3	0	.43636	.08437	.80665
8	5	0	.61538	.11797	.80829
8	10	0	.88888	.16890	.80998
10	1	0	.18182	.13583	.80294
10	3	0	.46153	.08968	.80569
10	5	0	.66666	.12814	.80778
10	8	0	.88888	.16992	.80884

[a]P.R in Bias = Percentage Reduction in Bias

$$= \frac{\text{Bias}(\hat{\mu}_{MLE}) - \text{Bias}(\hat{\mu}_{MLE*})}{\text{Bias}(\hat{\mu}_{MLE})}$$

Table 2. Empirical biases of $\hat{\mu}_{MLE}$ and $\hat{\mu}_{MLE*}$ for $n_1 = 5$, $n_2 = 10$

True values of σ_1	σ_2	True value of μ	$\hat{\mu}_{MLE}$	$\hat{\mu}_{MLE*}$	P.R in Bias
1	3	0	.12000	.018704	.84413
1	5	0	.14285	.02352	.83535
1	8	0	.16000	.02751	.82803
1	10	0	.16666	.02914	.82515
3	1	0	.08571	.01019	.88107
3	5	0	.27272	.03943	.85540
3	8	0	.34285	.05267	.84635
3	10	0	.37500	.05919	.84227
5	1	0	.09091	.01033	.88632
5	3	0	.23077	.02922	.87338
5	8	0	.44444	.06392	.85619
5	10	0	.50000	.07400	.85198
8	1	0	.09412	.01035	.89004
8	3	0	.25263	.03037	.87978
8	5	0	.38095	.04848	.87273
8	10	0	.61538	.08568	.86077
10	1	0	.09524	.01035	.89134
10	3	0	.26086	.03075	.88214
10	5	0	.40000	.04957	.87607
10	8	0	.57143	.07506	.86864

Table 3. Empirical biases of $\hat{\mu}_{MLE}$ and $\hat{\mu}_{MLE*}$ for $n_1 = 5$, $n_2 = 15$

True values of σ_1	σ_2	True value of μ	$\hat{\mu}_{MLE}$	$\hat{\mu}_{MLE*}$	P.R in Bias
1	3	0	.10000	.01315	.86849
1	5	0	.12500	.01814	.85487
1	8	0	.14545	.02296	.84212
1	10	0	.15384	.02504	.83725
3	1	0	.06000	.00532	.91129
3	5	0	.21428	.02506	.88306
3	8	0	.28235	.03624	.87163
3	10	0	.31579	.04240	.86563
5	1	0	.06250	.00525	.91590
5	3	0	.16666	.01613	.90325
5	8	0	.23783	.04034	.88400
5	10	0	.40000	.04846	.87883
8	1	0	.06400	.00519	.91892
8	3	0	.17777	.01602	.90991
8	5	0	.27586	.02687	.90257
8	10	0	.47058	.05203	.88943
10	1	0	.06451	.00515	.92008
10	3	0	.18182	.01592	.91244
10	5	0	.28571	.02684	.90605
10	8	0	.42105	.04291	.89808

Table 4. Empirical biases of $\hat{\mu}_{MLE}$ and $\hat{\mu}_{MLE*}$ for $n_1 = 5$, $n_2 = 20$

True values of σ_1	σ_2	True value of μ	$\hat{\mu}_{MLE}$	$\hat{\mu}_{MLE*}$	P.R in Bias
1	3	0	.08571	.00847	.88947
1	5	0	.11111	.01419	.87225
1	8	0	.13333	.01926	.85552
1	10	0	.14286	.02158	.84894
3	1	0	.04615	.00291	.93689
3	5	0	.17647	.01651	.90644
3	8	0	.24000	.02565	.89314
3	10	0	.27272	.03106	.88609
5	1	0	.04762	.00278	.94146
5	3	0	.13043	.00930	.92869
5	8	0	.28571	.02642	.90751
5	10	0	.33333	.03281	.90155
8	1	0	.04848	.00209	.94432
8	3	0	.13714	.00884	.93551
8	5	0	.21622	.01557	.92797
8	10	0	.38095	.03290	.91362
10	1	0	.04878	.00266	.94539
10	3	0	.13953	.00865	.93803
10	5	0	.22222	.01520	.93160
10	8	0	.33333	.02559	.92323

Table 5. Empirical biases of $\hat{\mu}_{MLE}$ and $\hat{\mu}_{MLE*}$ for $n_1 = 10$, $n_2 = 15$

True values of σ_1	σ_2	True value of μ	$\hat{\mu}_{MLE}$	$\hat{\mu}_{MLE*}$	P.R in Bias
1	3	0	.06666	.00542	.91859
1	5	0	.07692	.00655	.91474
1	8	0	.08421	.00737	.91243
1	10	0	.08695	.00767	.91174
3	1	0	.05454	.00379	.93042
3	5	0	.15789	.01221	.92266
3	8	0	.19200	.01545	.91952
3	10	0	.20689	.01702	.91772
5	1	0	.05882	.00401	.93178
5	3	0	.14285	.01029	.92797
5	8	0	.25806	.01988	.92292
5	10	0	.28571	.02244	.92145
8	1	0	.06154	.00416	.93277
8	3	0	.16000	.01120	.92999
8	5	0	.23529	.016995	.92777
8	10	0	.36363	.02745	.92451
10	1	0	.06250	.00421	.93263
10	3	0	.16666	.01154	.93074
10	5	0	.25000	.01778	.92885
10	8	0	.34783	.02548	.92673

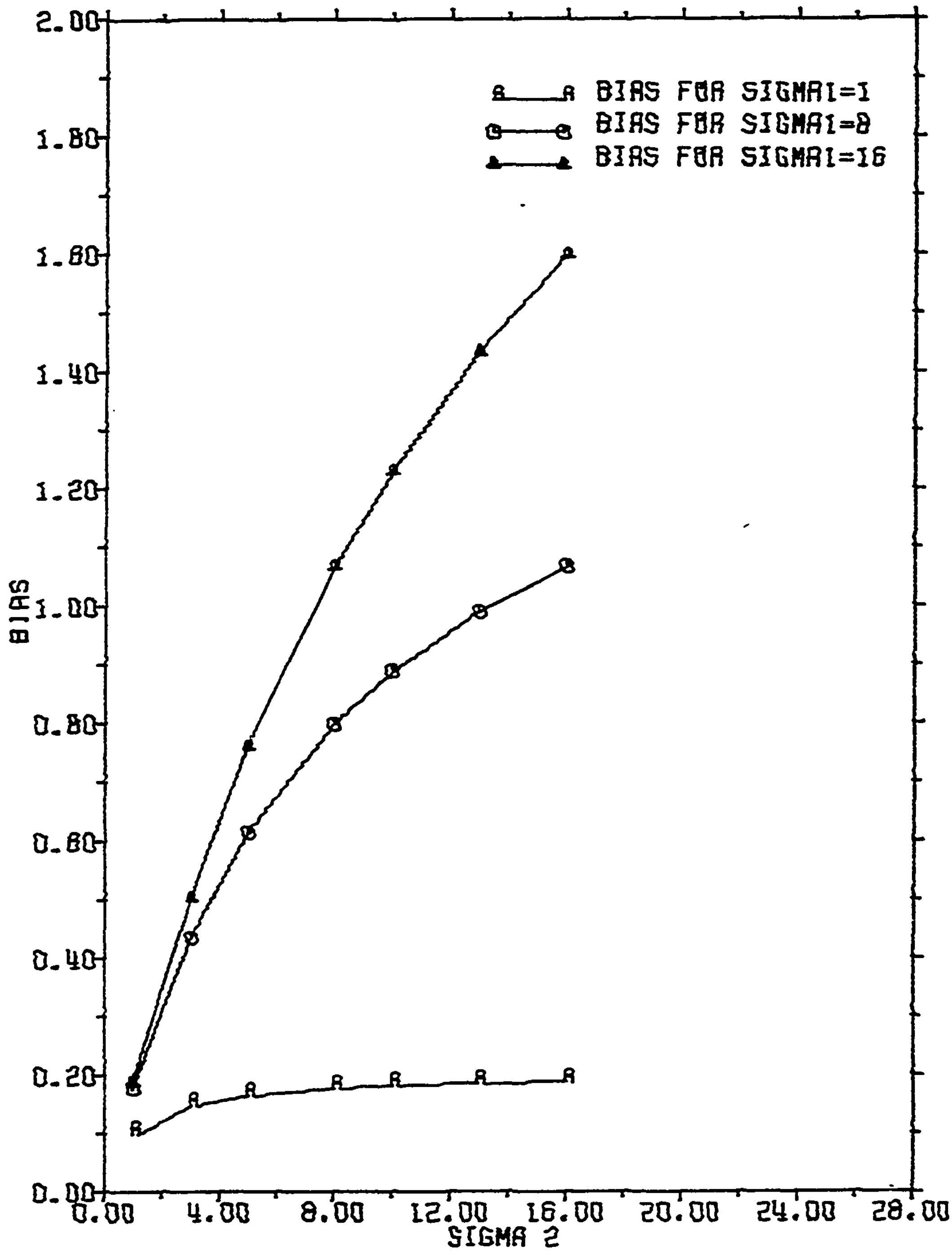

Figure 1. Empirical biases of MLE of μ for $n_1 = 5$ $n_2 = 5$

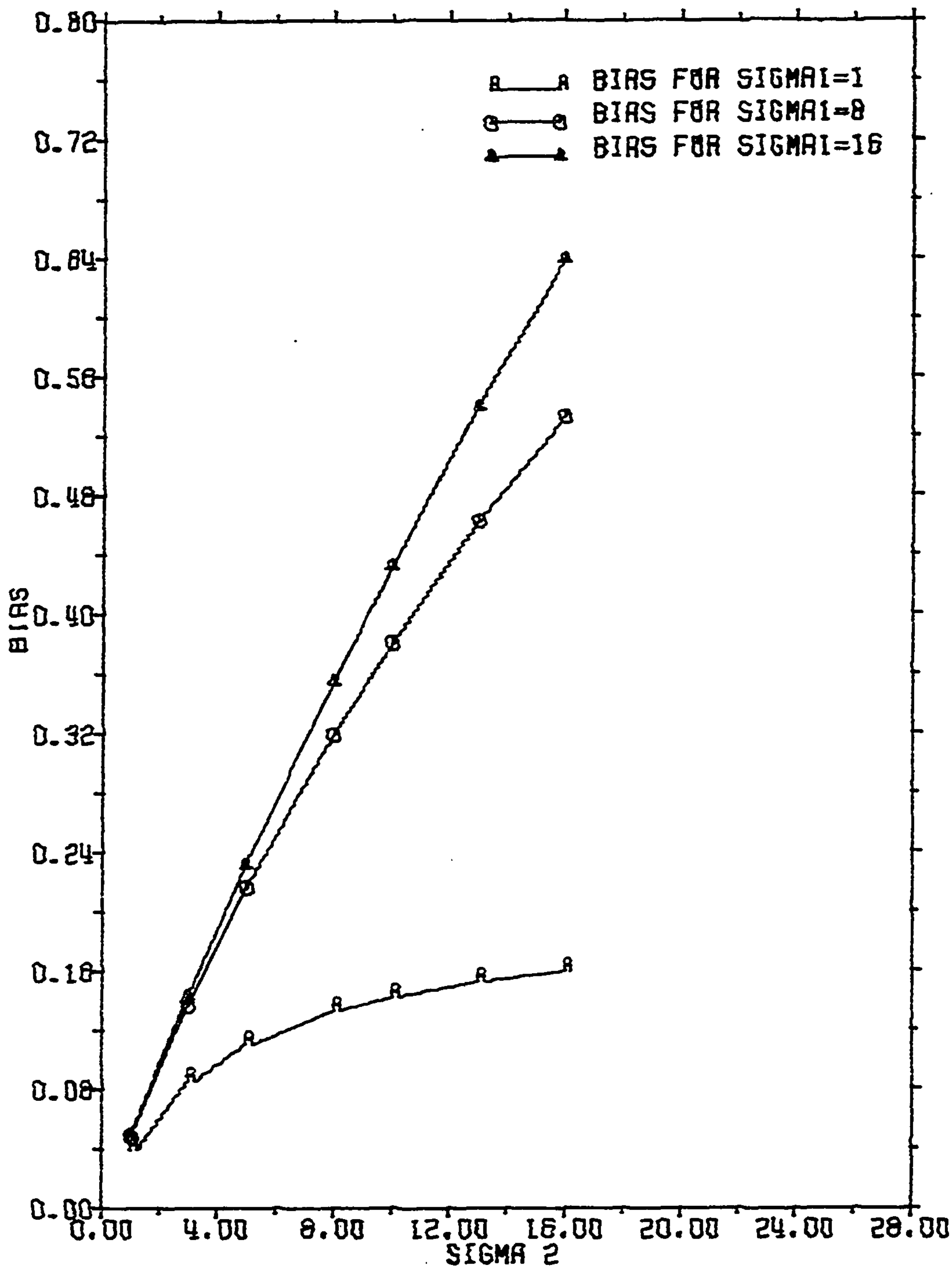

Figure 2. Empirical biases of MLE of μ for $n_1 = 5$ $n_2 = 20$

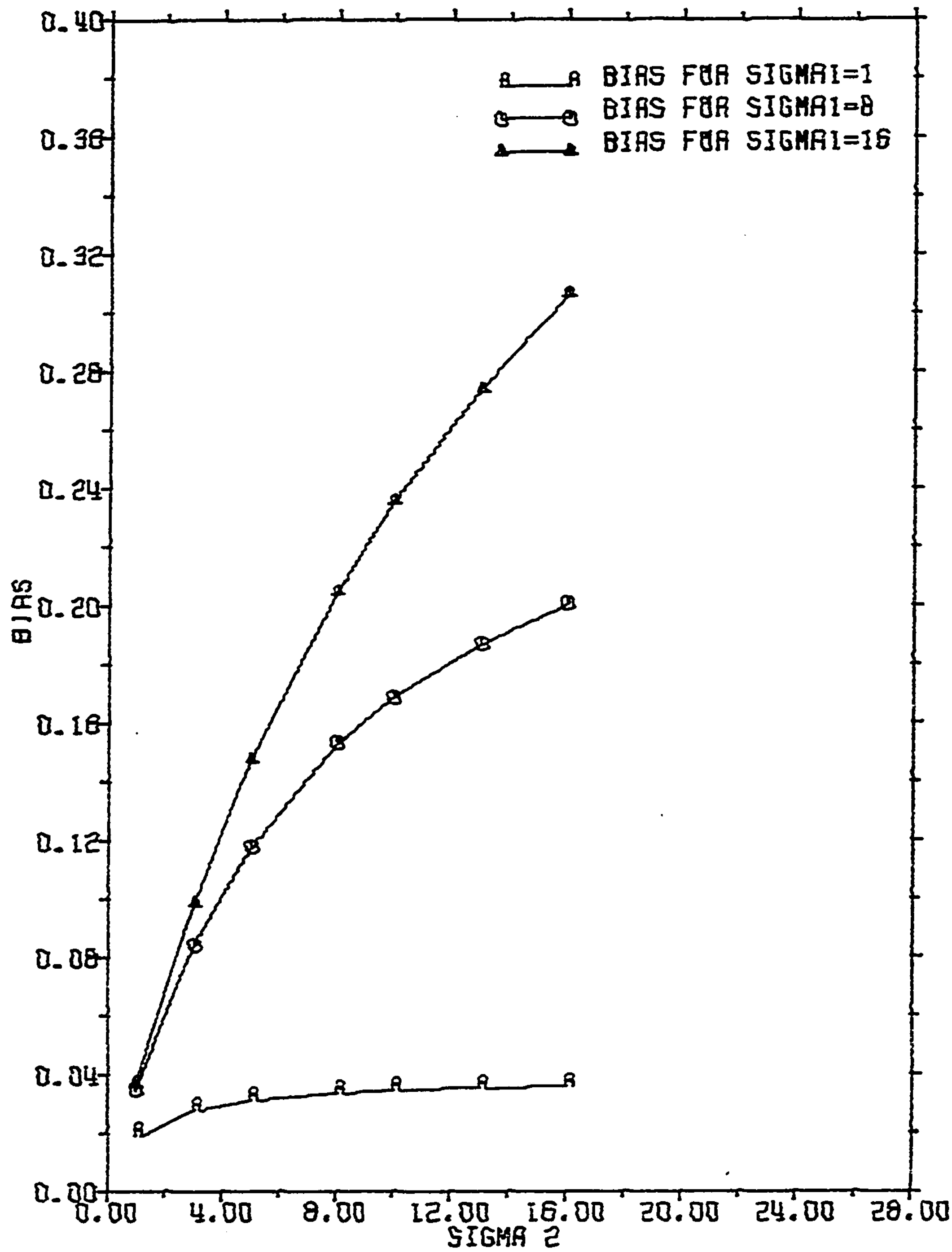

Figure 3. Empirical biases of modified MLE of μ for $n_1 = 5$ $n_2 = 5$

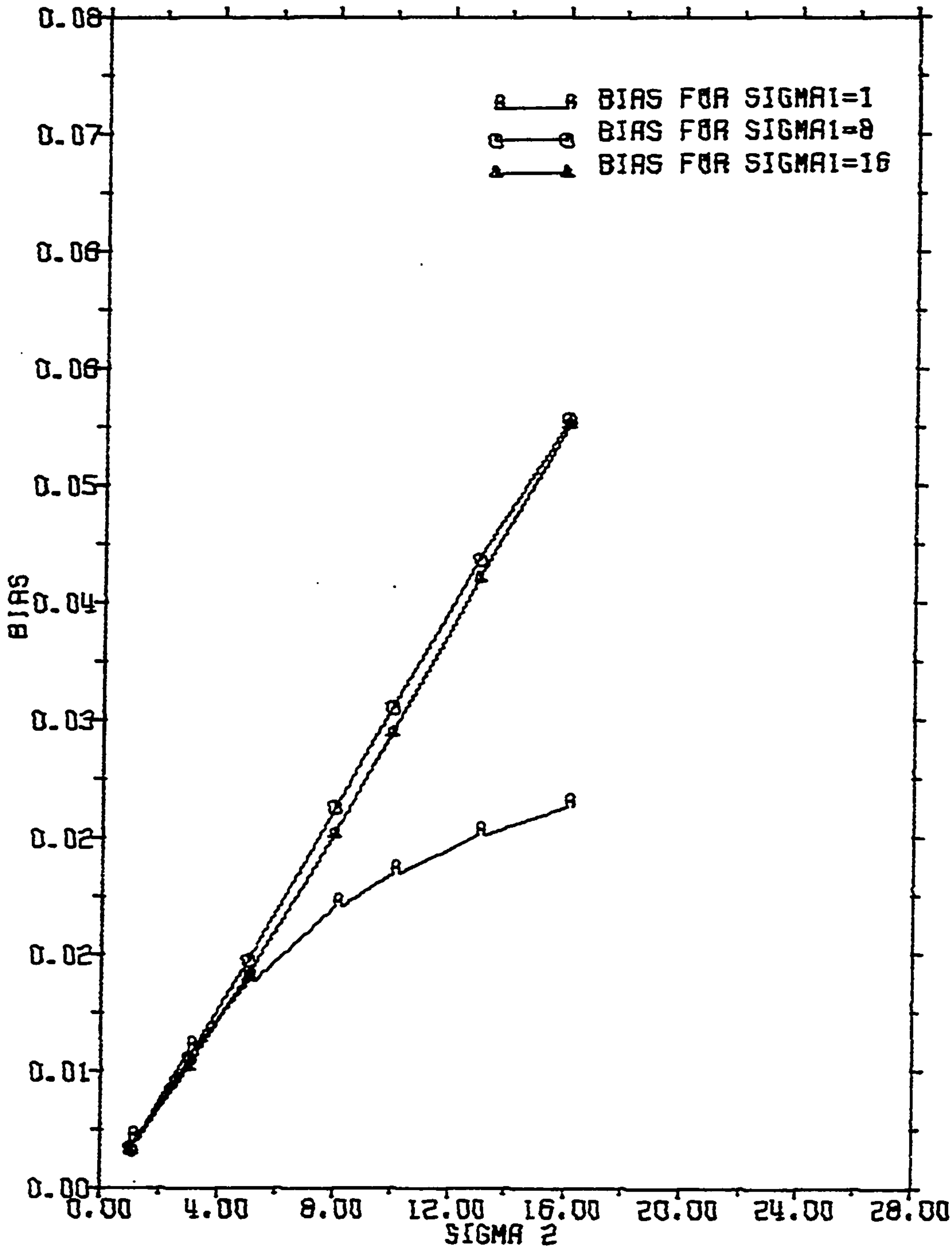

Figure 4. Empirical biases of modified MLE of μ for $n_1 = 5$ $n_2 = 20$

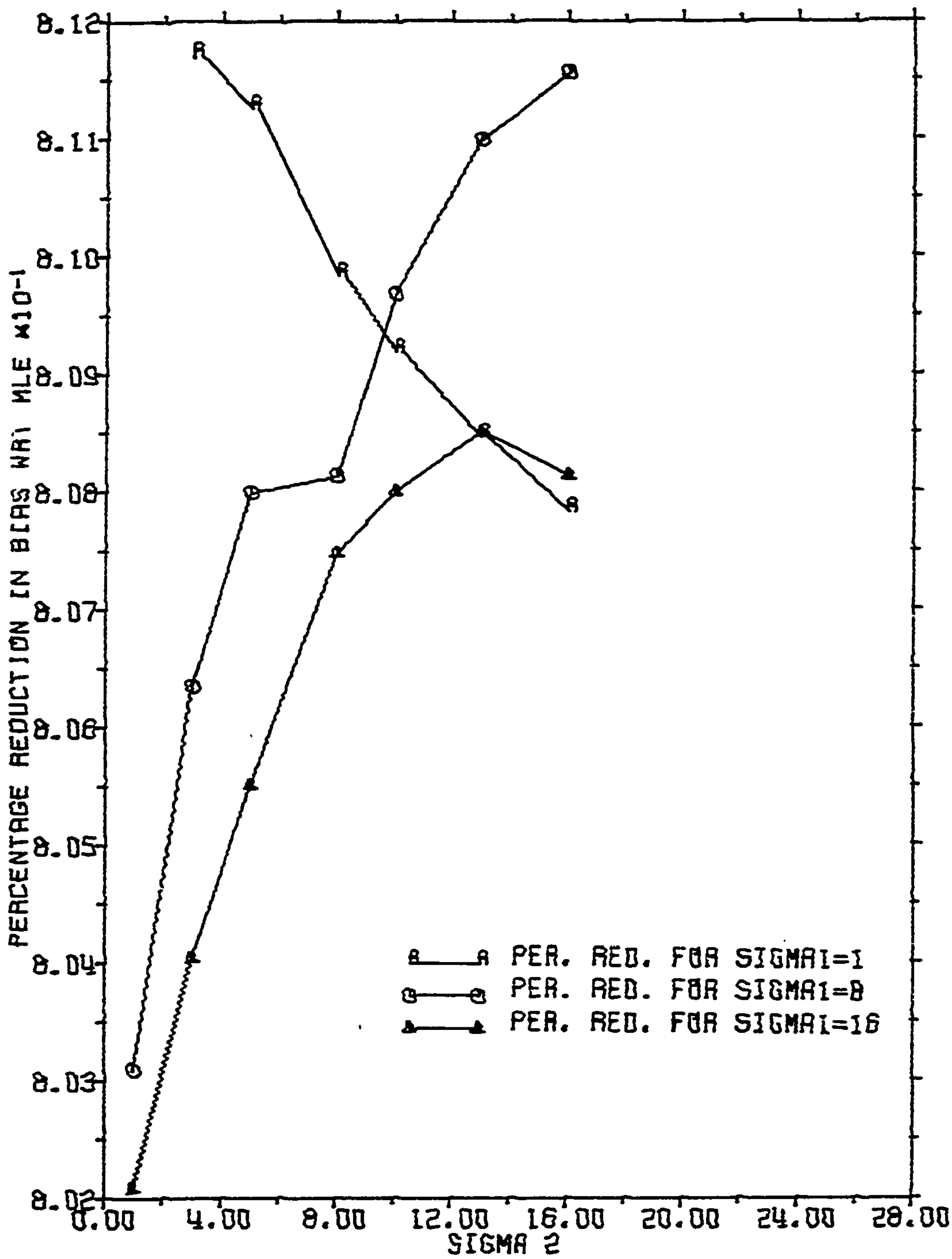

Figure 5. Empirical percentage reduction in bias of modified MLE WRT MLE for $n_1 = n_2 = 5$

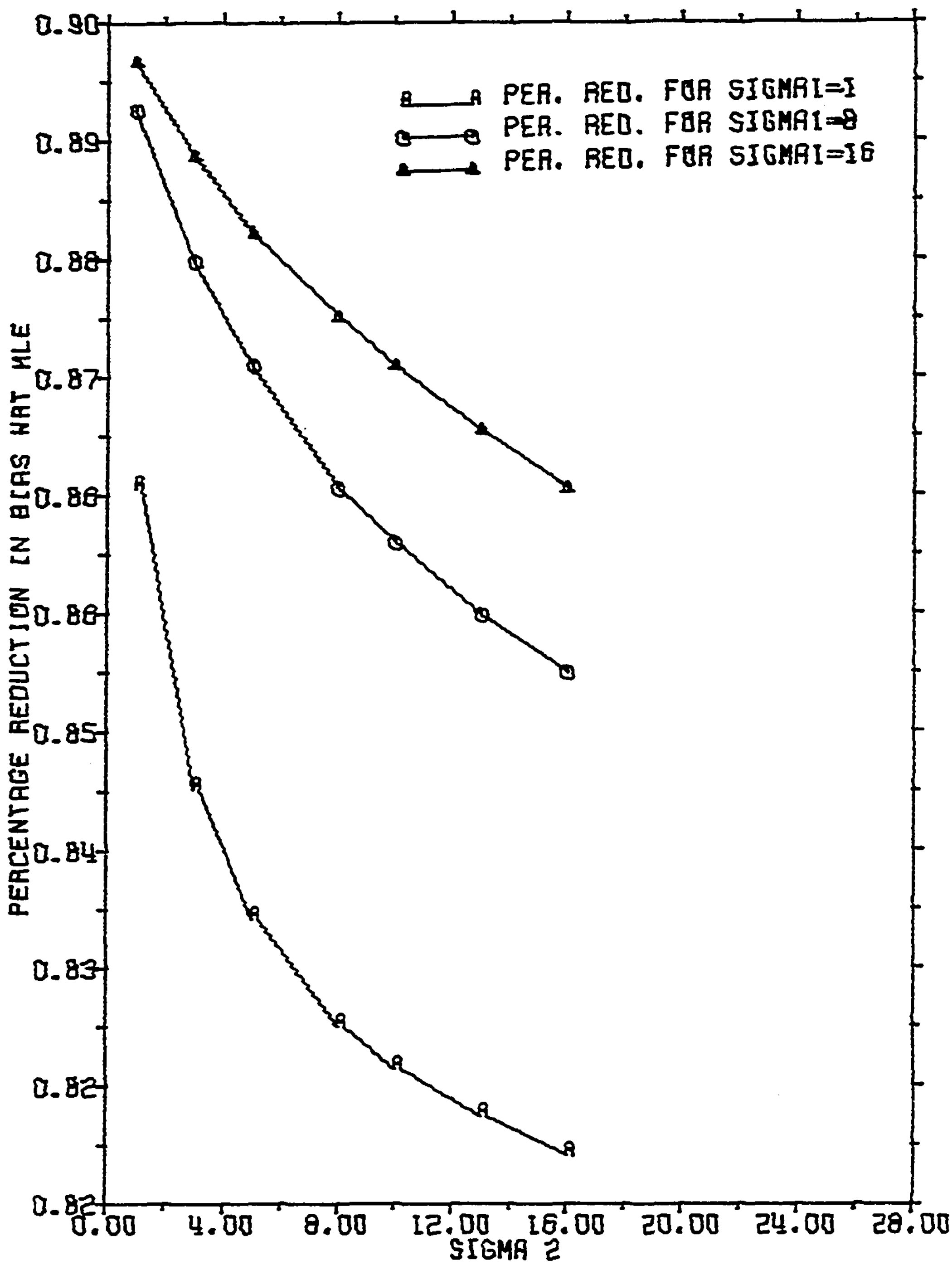

Figure 6. Empirical percentage reduction in bias of modified MLE WRT MLE for $n_1 = 5, n_2 = 10$

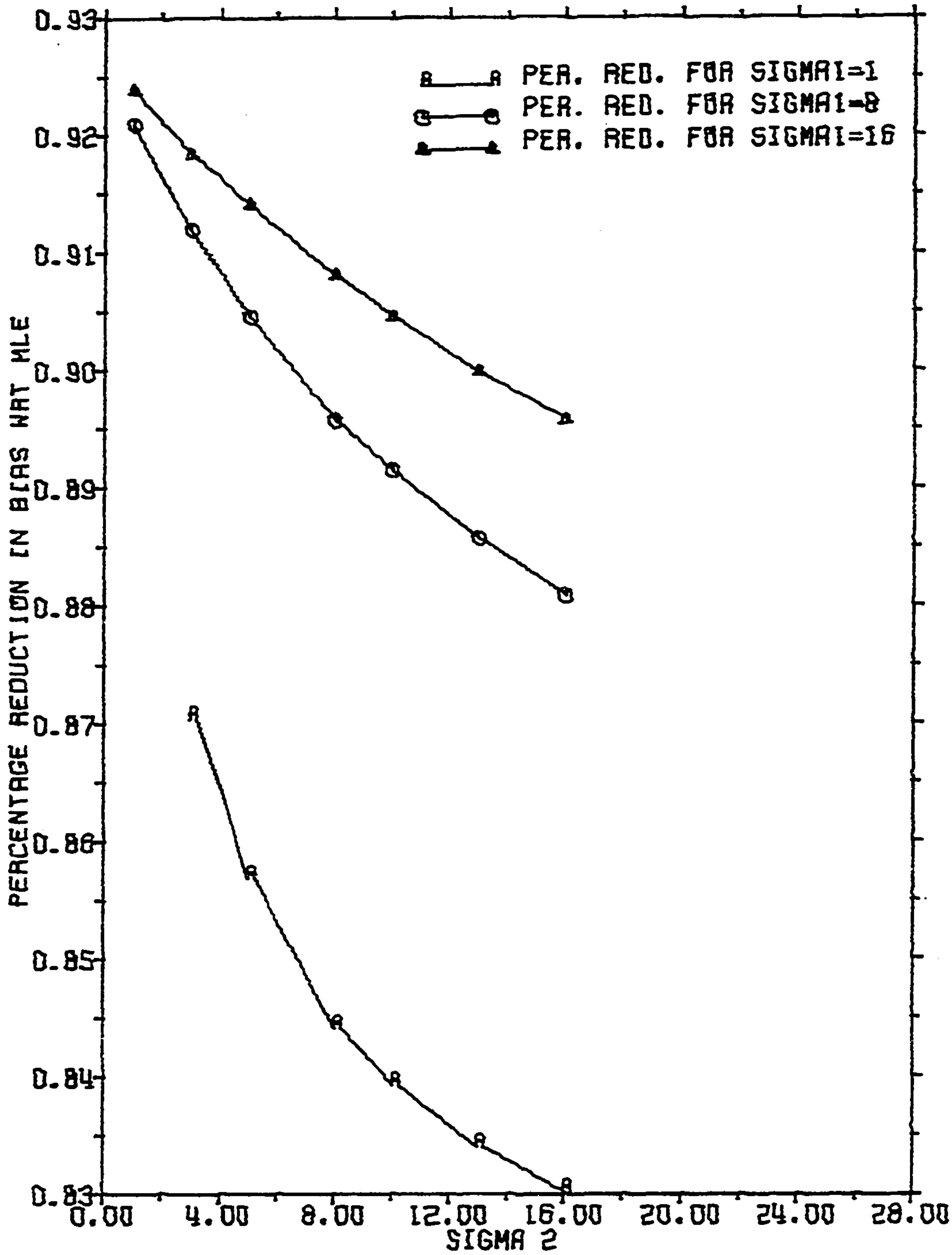

Figure 7. Empirical percentage reduction in bias of modified MLE WRT MLE for $n_1 = 5, n_2 = 15$

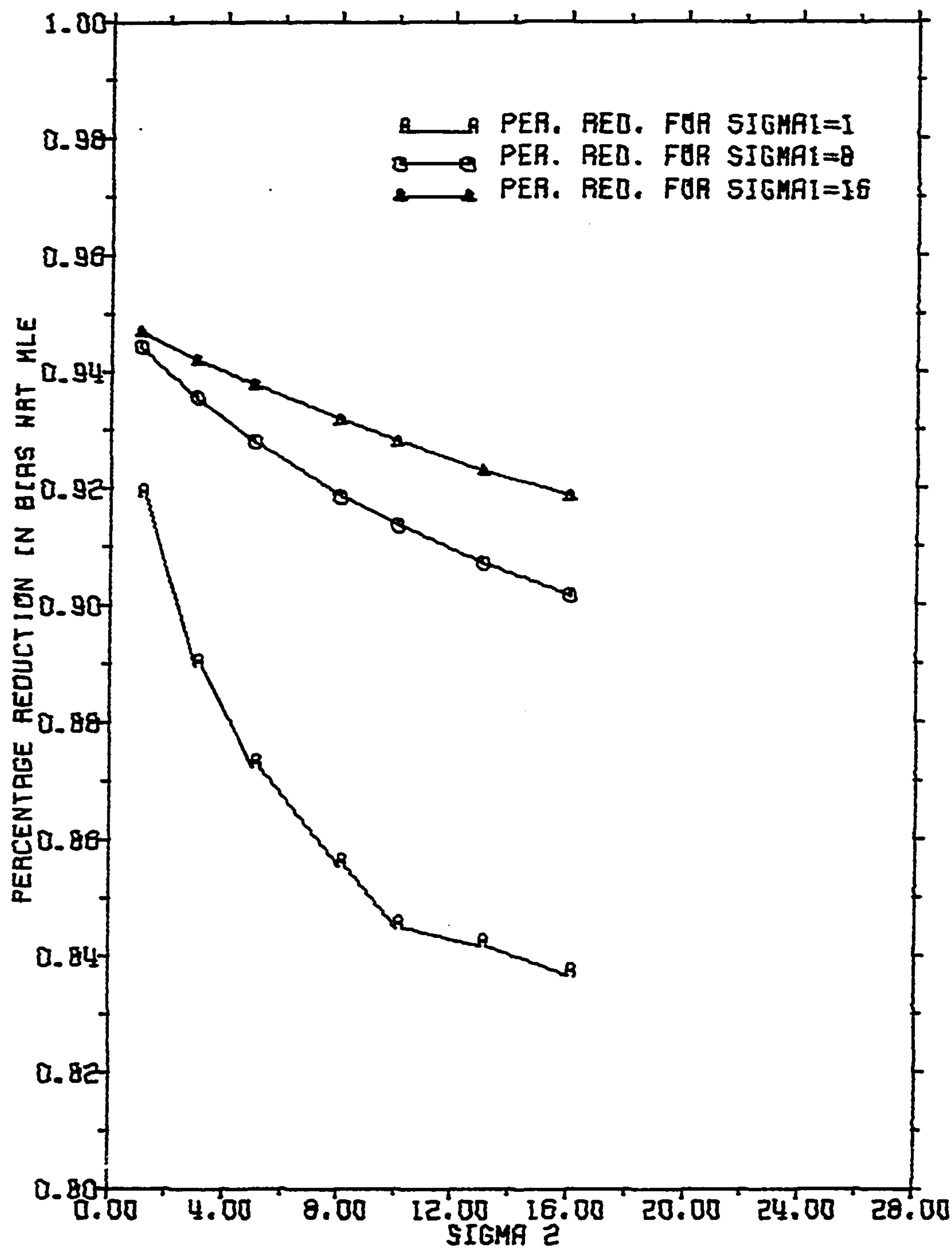

Figure 8. Empirical percentage reduction in bias of modified MLE WRT MLE for $n_1 = 5, n_2 = 20$

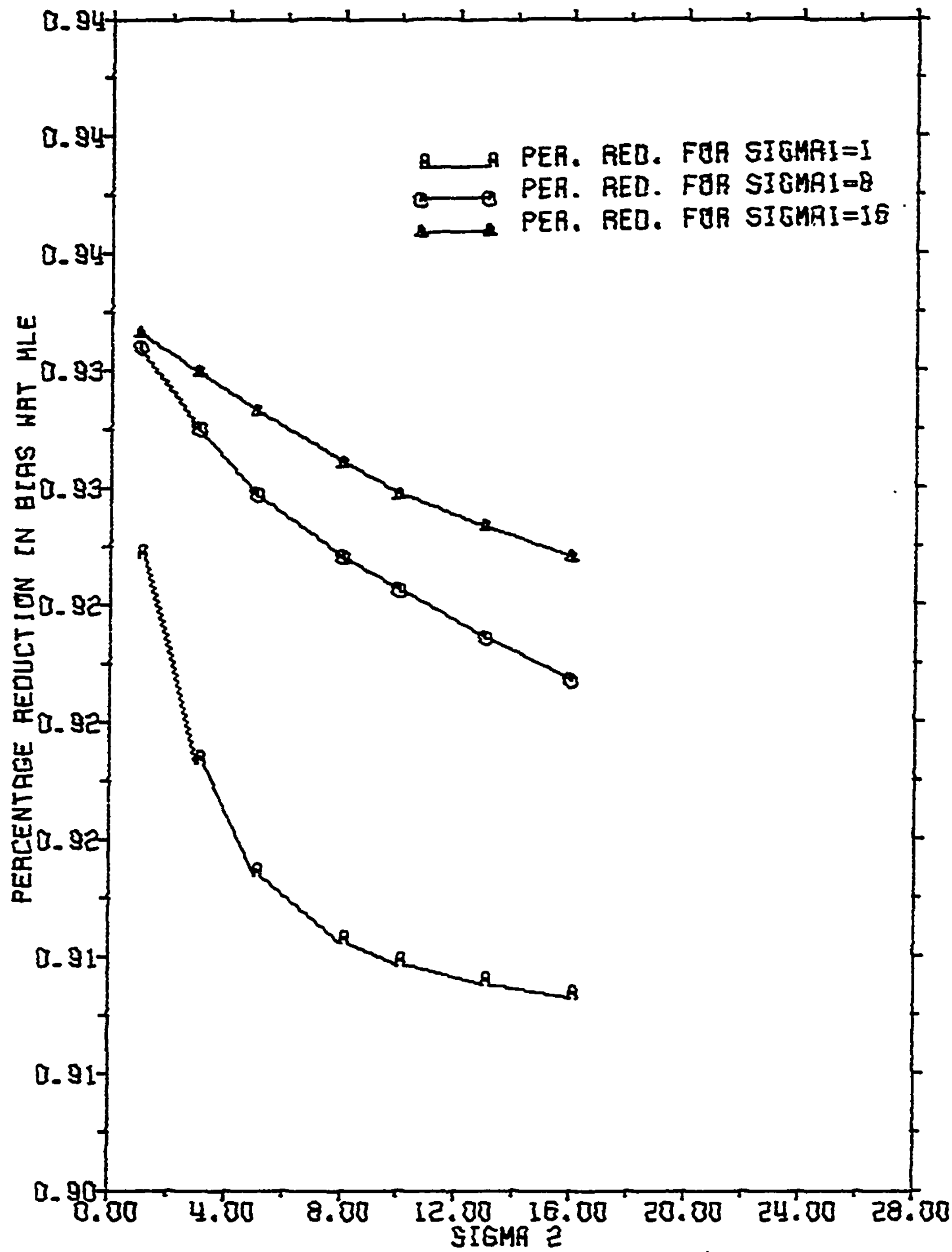

Figure 9. Empirical percentage reduction in bias of modified MLE WRT MLE for $n_1 = 10, n_2 = 15$

combination given in these tables, percentage reduction in bias of modified MLE with respect to MLE is at least 80%. This value increases for every (σ_1,σ_2) combination as at least one of the two sample sizes increases.

Figures 1 to 4 present the Monte Carlo biases of the MLE and of the modified MLE of μ for $\sigma_2 = 3, 5, 8, 10, 13, 16$ and different given values of σ_1 and paired sample sizes $(n_1 = 5, n_2 = 5)$ and $(n_1 = 5, n_2 = 20)$. As one can see from these figures, biases of MLE and modified MLE uniformly increase as both σ_1 and σ_2 increase. It is also noticed that as n_2 increases, the difference between biases corresponding to both the MLE and the modified MLE for $(\sigma_1 = 8, n_1 = 5)$ and $(\sigma_1 = 16, n_1 = 5)$ get smaller. Figures 5 to 9 present the Monte Carlo percentage reduction in bias with respect to MLE for three different values of σ_1 and for paired sample sizes (5,5), (5,10), (5,15), (5.20) and (10,15) respectively. With the exception of figure 5, these figures give reasonable representation of the shape of the percentage reduction functions for $(n_1 = 5, n_2 = 10)$ - $(n_1 = 5, n_2 = 20)$ and $(n_1 = 10, n_2 = 15)$. These four latest figures show that percentage reduction in bias increases as either σ_1 or n_2 or both increase. Tables 6 to 11 present the sample mean squared error (MSE) of the 4 estimators of common location parameter μ for paired sample sizes (5,5)-(5,20), (10,15) and (2,20), respectively. In general, the mean squared error of the 4 estimators of μ increases as scale parameters σ_1 and σ_2 increase and decreases as paired sample size (n_1,n_2) increases. Figure 10 represents graphically the Monte Carlo mean squared errors of the 4 estimators of μ for $3 \leq \sigma_2 \leq 13$, $\sigma_1 = 1$ and $(n_1 = 5, n_2 = 5)$. Figures 11 to 14 present empirical percentage reduction in MSE of UMVUE and modified MLE for $(n_1 = 5, n_2 = 5)$ and $(n_1 = 5, n_2 = 20)$, respectively. These figures and tables 6 to 11 show that percentage reduction in MSE of UMVUE and modified MLE decrease as the difference between σ_1 and σ_2 increases and the difference between n_1 and n_2 decreases. In this empirical study, as it is shown in figures 11 to 14, no matter what the differences between σ_1 and σ_2 is, percentage reduction in MSE WRT MLE for $\sigma_1 = 8$ dominates percentage reduction in MSE WRT MLE for $\sigma_1 = 1$ when the difference between n_1 and n_2 be large. Figures 15 to 20 present empirical percentage reduction in MSE of various estimators with respect to MLE for paired sample sizes $(n_1=5, n_2=5)-(n_1=5, n_2=20)$,$(n_1 = 10, n_2 = 15)$ and $(n_1 = 2, n_2 = 20)$. In general, percentage reduction in MSE of various estimators WRT MLE increases as either n_1 or n_2 or both increases. As one might expect, $\hat{\mu}_{MVU}$ dominates $\hat{\mu}_{UMV}$, $\hat{\mu}_{MLE*}$ and $\hat{\mu}_{MLE}$ and also $\hat{\mu}_{MLE}$ is dominated by $\hat{\mu}_{MVU}$, $\hat{\mu}_{UMV}$ and $\hat{\mu}_{MLE*}$.

As it is shown in Figures 15, 16 and 19, $\hat{\mu}_{UMV}$ is dominated by $\hat{\mu}_{MLE*}$ in terms of MSE for small values of differences between n_1 and n_2. But this domination changes as the differences between n_1 and n_2 increase. For example for $(n_1 = 2, n_2 = 20, \sigma_1 = 1)$ $\hat{\mu}_{UMV}$ dominates $\hat{\mu}_{MLE*}$ in terms of MSE. These results hold also for $\sigma_1 = 8$.

Table 6. Empirical mean square errors and efficiencies of various estimators of μ for $n_1 = n_2 = 5$

True values of σ_1	σ_2	$\hat{\mu}_{MLE}$	$\hat{\mu}_{UMV}$	$\hat{\mu}_{MLE*}$	$\hat{\mu}_{MVU}$	E_1 [a]	E_2 [b]	E_3 [c]
1	3	.0450	.0261	.0257	.0250	.4203	.4280	.4444
1	5	.0555	.0328	.0321	.0309	.4104	.4220	.4444
1	8	.0632	.0378	.0369	.0351	.4014	.4163	.4444
1	10	.0661	.0398	.0387	.0367	.3976	.4140	.4444
3	1	.0450	.0261	.0258	.025	.4203	.4272	.4444
3	5	.2813	.1606	.1595	.1563	.4291	.4330	.4444
3	8	.3808	.2200	.2174	.2116	.4224	.4292	.4444
3	10	.4260	.2478	.2442	.2367	.4183	.4268	.4444
5	1	.0555	.0328	.0321	.0309	.4104	.4218	.4444
5	3	.2813	.1606	.1597	.1563	.4291	.4321	.4444
5	8	.7574	.4321	.4293	.4208	.4295	.4332	.4444
5	10	.8888	.5095	.5050	.4938	.4269	.4319	.4444
8	1	.0632	.0378	.0369	.0351	.4014	.4168	.4444
8	3	.3808	.2200	.2177	.2116	.4224	.4284	.4444
8	5	.7574	.4321	.4299	.4208	.4295	.4323	.4444
8	10	1.5802	.8987	.8945	.8779	.4313	.4340	.4444
10	1	.0661	.0398	.0387	.0367	.3976	.4147	.4444
10	3	.4260	.2478	.2445	.2367	.4183	.4261	.4444
10	5	.8888	.5095	.5059	.4938	.4269	.4309	.4444
10	8	1.5802	.8987	.8952	.8779	.4313	.4335	.4444

[a] $E_1 = [MSE(\hat{\mu}_{MLE}) - MSE(\hat{\mu}_{UMV})]/MSE(\hat{\mu}_{MLE})$

[b] $E_2 = [MSE(\hat{\mu}_{MLE}) - MSE(\hat{\mu}_{MLE*})]/MSE(\hat{\mu}_{MLE})$

[c] $E_3 = [MSE(\hat{\mu}_{MLE}) - MSE(\hat{\mu}_{MVU})]/MSE(\hat{\mu}_{MVU})$

Table 7. Empirical mean square errors and efficencies of various estimators of μ for $n_1 = 5$, $n_2 = 10$

True values of σ_1	σ_2	$\hat{\mu}_{MLE}$	$\hat{\mu}_{UMV}$	$\hat{\mu}_{MLE*}$	$\hat{\mu}_{MVU}$	E_1	E_2	E_3
1	3	.0288	.0160	.0159	.0154	.4454	.4471	.4643
1	5	.0400	.0231	.0229	.0219	.4330	.4380	.4643
1	8	.0512	.0300	.0293	.0274	.4205	.4291	.4643
1	10	.0555	.0325	.0319	.0298	.4148	.4253	.4643
3	1	.0147	.0080	.0080	.0079	.4560	.4584	.4643
3	5	.1488	.0811	.0810	.07970	.4552	.4553	.4643
3	8	.2351	.1298	.1295	.1259	.4478	.4490	.4643
3	10	.2813	.1566	.1560	.1507	.4430	.4453	.4643
5	1	.0165	.0090	.0090	.0089	.4530	.4563	.4643
5	3	.1065	.0577	.0574	.0571	.4587	.4597	.4643
5	8	.3951	.2151	.2150	.2116	.4456	.4557	.4643
5	10	.5000	.2736	.2734	.2679	.4527	.4532	.4643
8	1	.0177	.0093	.0097	.0095	.4505	.4544	.4643
8	3	.1276	.0694	.0691	.0684	.4566	.4588	.4643
8	5	.2902	.1571	.1568	.1555	.4588	.4598	.4643
8	10	.7574	.4107	.4106	.4057	.4578	.4579	.4643
10	1	.0181	.0100	.0099	.0097	.4495	.4536	.4643
10	3	.1361	.0741	.0738	.0729	.4553	.4581	.4643
10	5	.3200	.1734	.1730	.1714	.4580	.4595	.4643
10	8	.6531	.3532	.3529	.3499	.4592	.4596	.4643

Table 8. Empirical mean square errors and efficiencies of various estimators of μ for $n_1 = 5$, $n_2 = 15$

True values of σ_1	σ_2	$\hat{\mu}_{MLE}$	$\hat{\mu}_{UMV}$	$\hat{\mu}_{MLE*}$	$\hat{\mu}_{MVU}$	E_1	E_2	E_3
1	3	.020	.0108	.0108	.0105	.4591	.4584	.4737
1	5	.0313	.0173	.0172	.0165	.4472	.4484	.4737
1	8	.0423	.0240	.0238	.0223	.4340	.4379	.4737
1	10	.0473	.0271	.0268	.0249	.4274	.4334	.4737
3	1	.0072	.0038	.0038	.0038	.4694	.4710	.4737
3	5	.0918	.0489	.0490	.0483	.4675	.4664	.4737
3	8	.1594	.0859	.0860	.0839	.4612	.4604	.4737
3	10	.1994	.1083	.1084	.1050	.4569	.4565	.4737
5	1	.0078	.0042	.0041+	.0041	.4680	.4700	.4737
5	3	.0555	.0294	.0294	.0292	.4708	.4713	.4737
5	8	.2420	.1288	.1290	.1274	.4679	.4668	.4737
5	10	.3200	.1710	.1714	.1684	.4655	.4644	.4737
8	1	.0082	.0044	.0044	.0043	.4669	.4690	.4737
8	3	.0632	.0335	.0334	.0333	.4678	.4711	.4737
8	5	.1522	.0805	.0805	.0801	.4708	.4713	.4737
8	10	.4429	.2349	.2353	.2331	.4696	.4688	.4737
10	1	.0083	.0044	.0044	.0044	.4664	.4686	.4737
10	3	.0661	.0351	.0350	.0348	.4692	.4708	.4737
10	5	.1633	.0865	.0863	.0859	.4705	.4713	.4737
10	8	.3546	.1876	.1877	.1866	.4709	.4708	.4737

Table 9. Empirical mean square errors and efficiencies of estimators of μ for $n_1 = 5$, $n_2 = 20$

True values of σ_1	σ_2	$\hat{\mu}_{MLE}$	$\hat{\mu}_{UMV}$	$\hat{\mu}_{MLE*}$	$\hat{\mu}_{MVU}$	E_1	E_2	E_3
1	3	.0147	.0078	.0079	.0077	.4676	.4653	.4792
1	5	.0247	.0134	.0134	.0128	.4569	.4555	.4792
1	8	.0356	.0198	.0198	.0185	.4439	.4445	.4792
1	10	.0408	.0230	.0229	.0213	.4372	.4395	.4792
3	1	.0043	.0022	.0022	.0022	.4766	.4768	.4792
3	5	.0623	.0330	.0328	.0324	.4747	.4726	.4792
3	8	.1152	.0611	.0614	.0600	.4695	.4672	.4792
3	10	.1488	.0795	.0798	.0775	.4657	.4635	.4792
5	1	.0045	.0024	.0024	.0024	.4759	.4763	.4792
5	3	.0340	.0178	.0178	.0177	.4774	.5769	.4792
5	8	.1633	.0857	.0860	.0850	.4750	.4730	.4792
5	10	.2222	.1171	.1176	.1157	.4731	.4709	.4792
8	1	.0047	.0025	.0025	.0024	.4751	.4792	.4792
8	3	.0376	.0197	.0197	.0196	.4768	.4769	.4792
8	5	.0935	.0489	.049	.049	.4774	.4768	.4792
8	10	.2903	.1520	.1525	.1512	.4764	.4747	.4792
10	1	.0048	.0025	.0025	.0025	.4749	.4792	.4792
10	3	.0389	.0204	.0204	.0203	.4746	.4746	.4792
1C	5	.0988	.0516	.0517	.0514	.4772	.4769	.4792
10	8	.2222	.1161	.1164	.1157	.4774	.4764	.4792

Table 10. Empirical mean square errors and efficiencies of estimators of μ for $n_1 = 10$, $n_2 = 15$

True values of σ_1	σ_2	$\hat{\mu}_{MLE}$	$\hat{\mu}_{UMV}$	$\hat{\mu}_{MLE*}$	$\hat{\mu}_{MVU}$	E_1	E_2	E_3
1	3	.0089	.0047	.0047	.0046	.4711	.4725	.4792
1	5	.0118	.0063	.0063	.0062	.4655	.4676	.4792
1	8	.0142	.0077+	.0076	.0074	.4602	.4630	.4792
1	10	.0151	.0082	.0082	.0079	.4580	.4611	.4792
3	1	.0060	.0031+	.0031	.0030	.4740	.4754	.4792
3	5	.0499	.0261	.0261	.0260	.4758	.4768	.4792
3	8	.0737	.0389	.0388	.0384	.4722	.4735	.4792
3	10	.0856	.0454	.0452	.0446	.4700	.4715	.4792
5	1	.0069	.0037	.0037	.0036	.4714	.4732	.4792
5	3	.0408	.0214	.0213	.0213	.4767	.4775	.4792
5	8	.1332	.0698	.0697	.0694	.4760	.4770	.4792
5	10	.1633	.0858	.0856	.0850	.4746	.4760	.4792
8	1	.0076	.0040	.0040	.0039	.4694	.4715	.4792
8	3	.0512	.0269	.0268	.0267	.4747	.4759.	.4792
8	5	.1107	.0579	.0578	.0577	.4767	.4775	.4792
8	10	.2645	.1383	.1381	.1377	.4770	.4778	.4792
10	1	.0078	.0042	.0041	.0040	.4685	.4708	.4792
10	3	.0556	.0293	.0292	.289	.4735	.4749	.4792
10	5	.1250	.0655	.0654	.0651	.4759	.4769	.4792
10	8	.2420	.1265	.1263	.1260	.4773	.4781	.4792

Table 11. Empirical mean square errors and efficiencies of estimators of μ for $n_1 = 2$, $n_2 = 20$

True values of σ_1	σ_2	$\hat{\mu}_{MLE}$	$\hat{\mu}_{UMV}$	$\hat{\mu}_{MLE*}$	$\hat{\mu}_{MVU}$	E_1	E_2	E_3
1	3	.0266	.0145	.0150	.0139	.4542	.4355	.4762
1	5	.0555	.0314	.0325	.0291	.4349	.4148	.4762
1	8	.0988	.0586	.0601	.0517	.4071	.3913	.4762
1	10	.1250	.0762	.0777	.0655	.3901	.3786	.4762
3	1	.0047	.0025	.0025$^+$	.0025	.4740	.4737	.4762
3	5	.0918	.0490	.0502	.0481	.4662	.4533	.4762
3	8	.1995	.1082	.1118	.1045	.4574	.4390	.4762
3	10	.2813	.1544	.1598	.1473	.4510	.4317	.4762
5	1	.0048	.0025	.0025$^+$	.0025	.4742	.4750	.4762
5	3	.0400	.0211	.0212	.0210	.4732	.4702	.4762
5	8	.2378	.1268	.1298	.1246	.4667	.4543	.4762
5	10	.3472	.1863	.1915	.1819	.4634	.4485	.4762
8	1	.0049	.0026	.0026	.0026	.4741	.4754	.4762
8	3	.0418	.0220	.0220	.6219	.4739	.4733	.4762
8	5	.1107	.0583	.0588	.0580	.4731	.4698	.4762
8	10	.3951	.2046	.2135	.2069	.4694	.4596	.4762
10	1	.0049	.0026	.0026	.0026	.4741	.4755	.4762
10	3	.0424	.0223	.0223	.0222	.4741	.4741	.4762
10	5	.1134	.0597	.0599	.0594	.4736	.4717	.4762
10	8	.2743	.1448	.1462	.1437	.4722	.4671	.4762

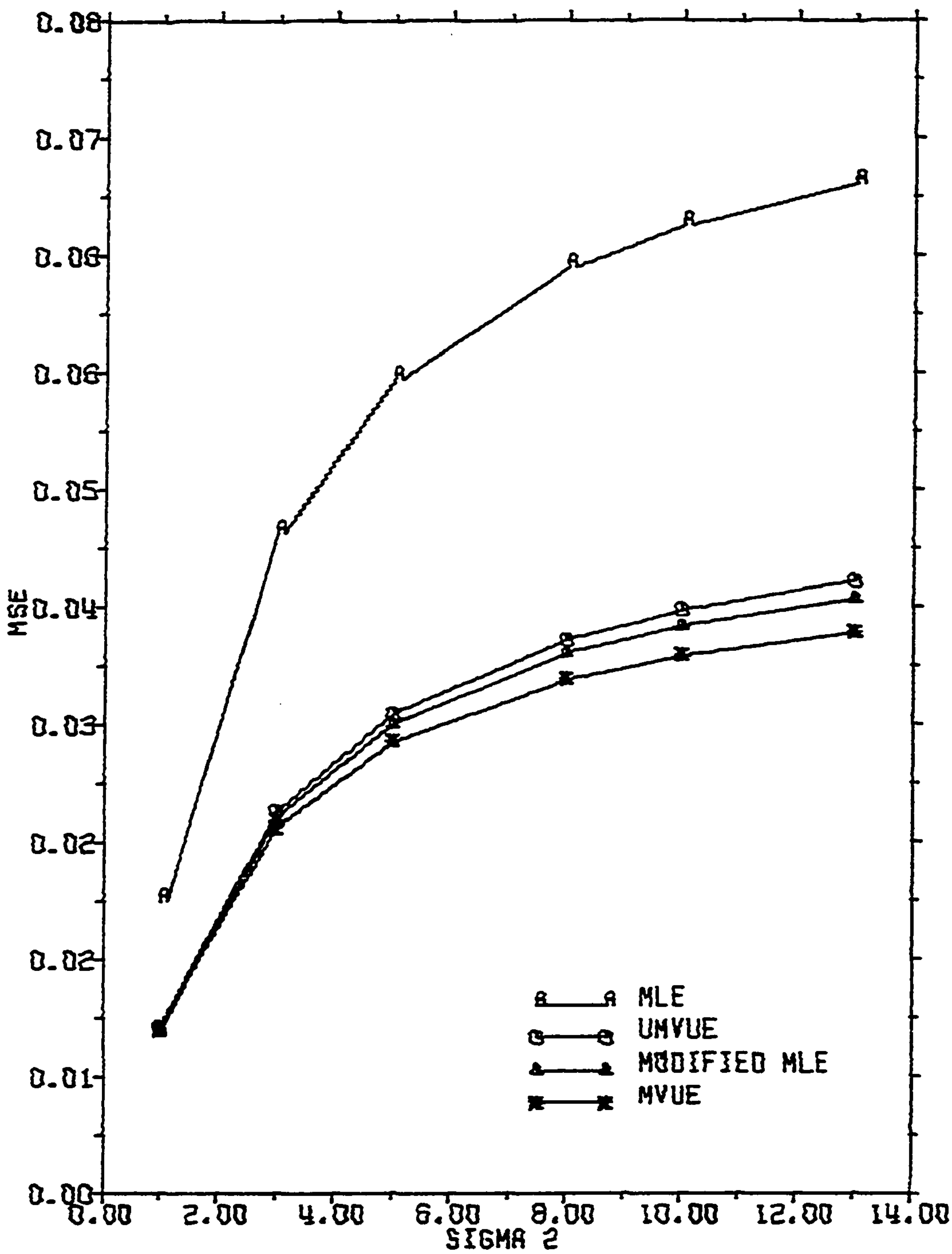

Figure 10. Empirical MSE of various estimators of μ for $n_1 = 5, n_2 = 5, \sigma_1 = 1$

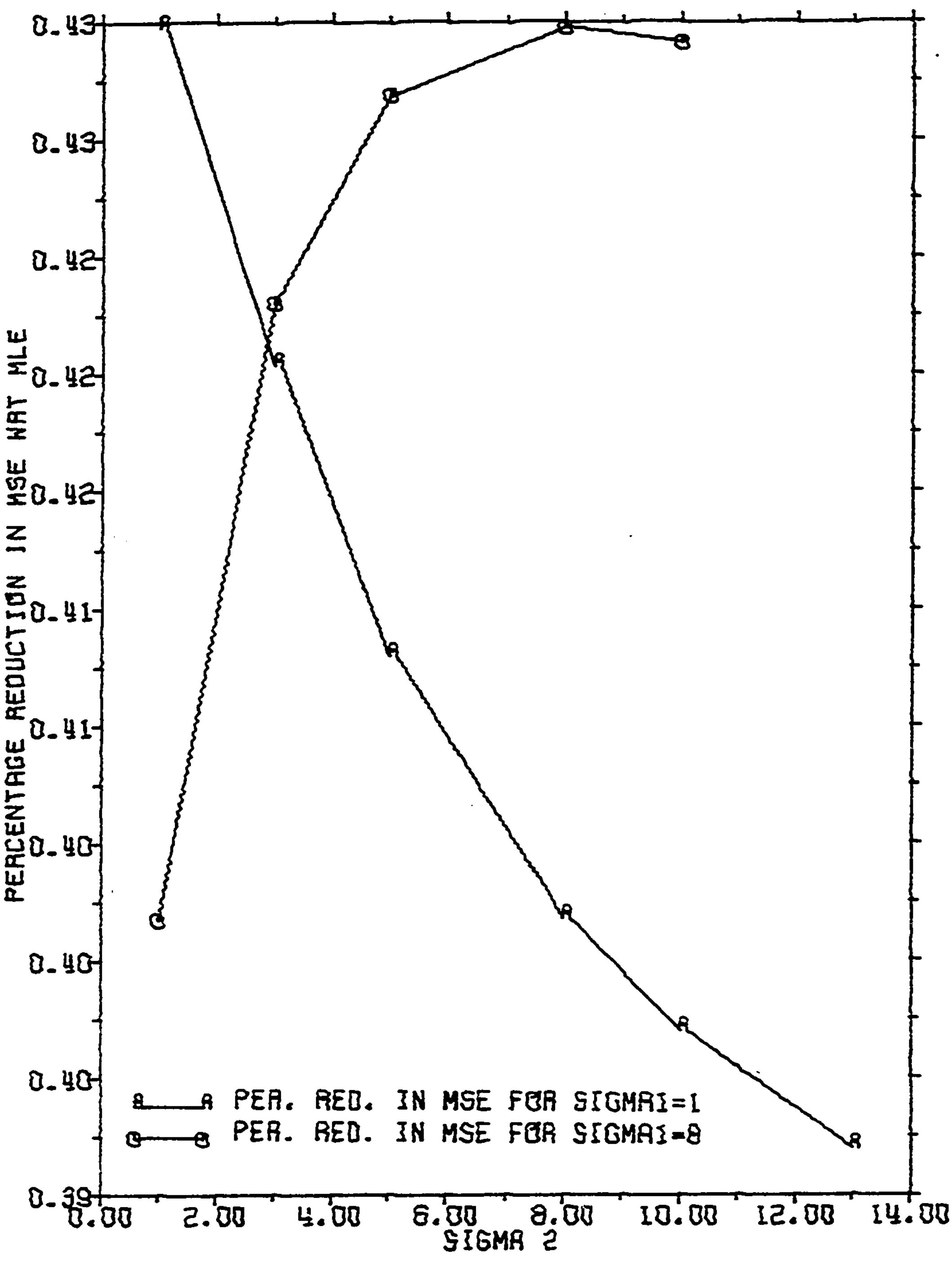

Figure 11. Empirical percentage reduction in MSE of UMVUE WRT MLE for $n_1 = 5, n_2 = 5$

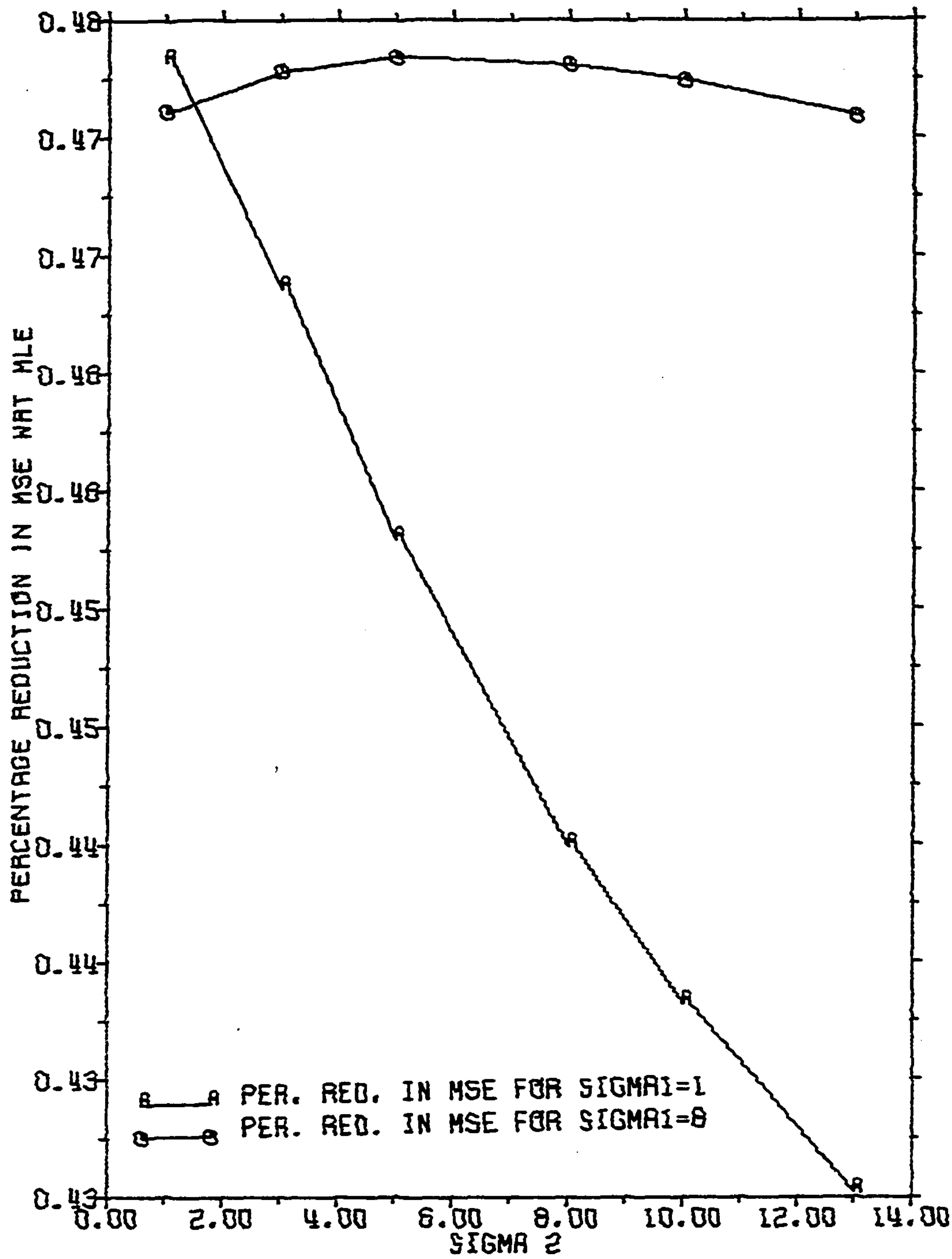

Figure 12. Empirical percentage reduction in MSE of UMVUE WRT MLE for $n_1 = 5, n_2 = 20$

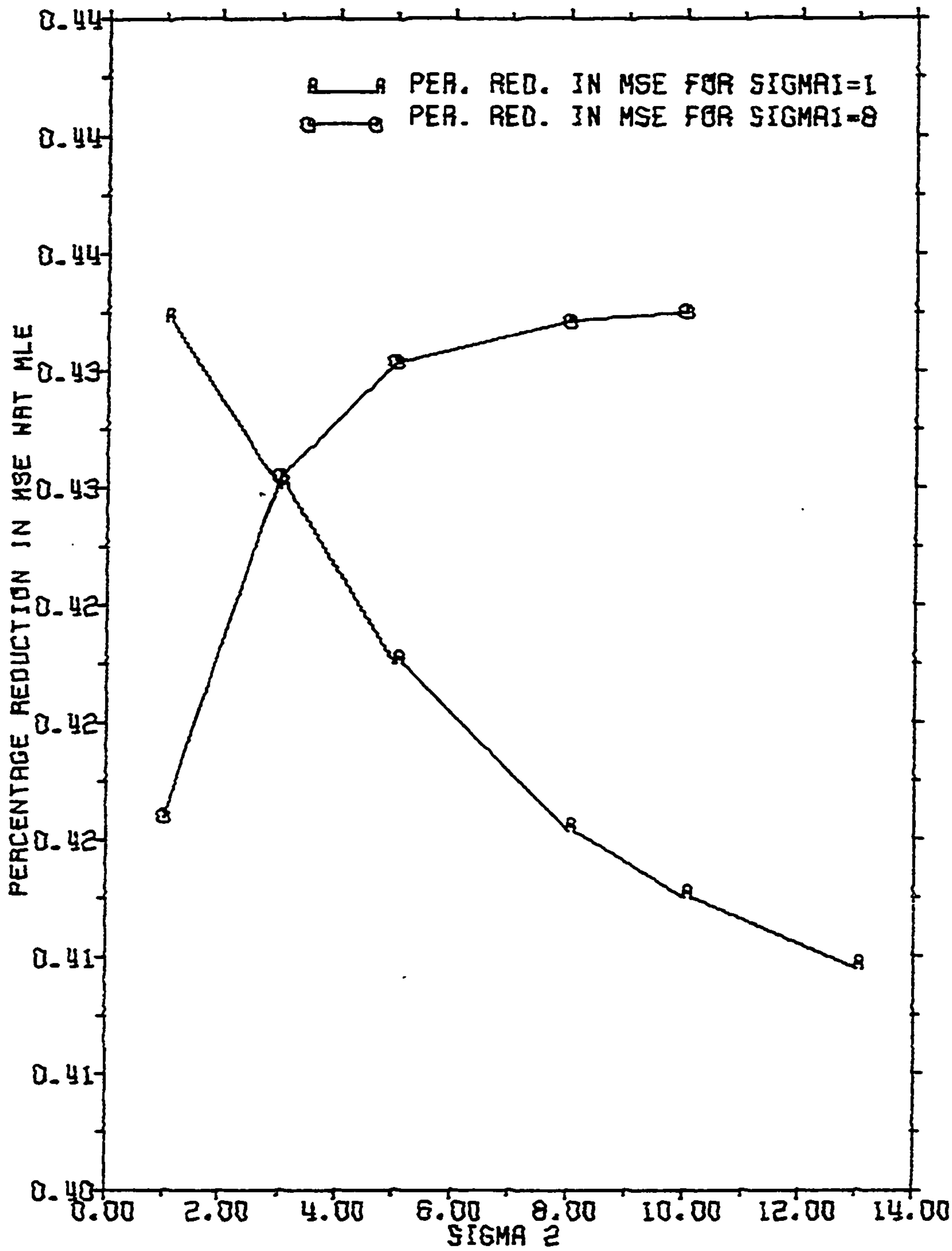

Figure 13. Empirical percentage reduction in MSE of modified MLE WRT for $n_1 = 5, n_2 = 5$

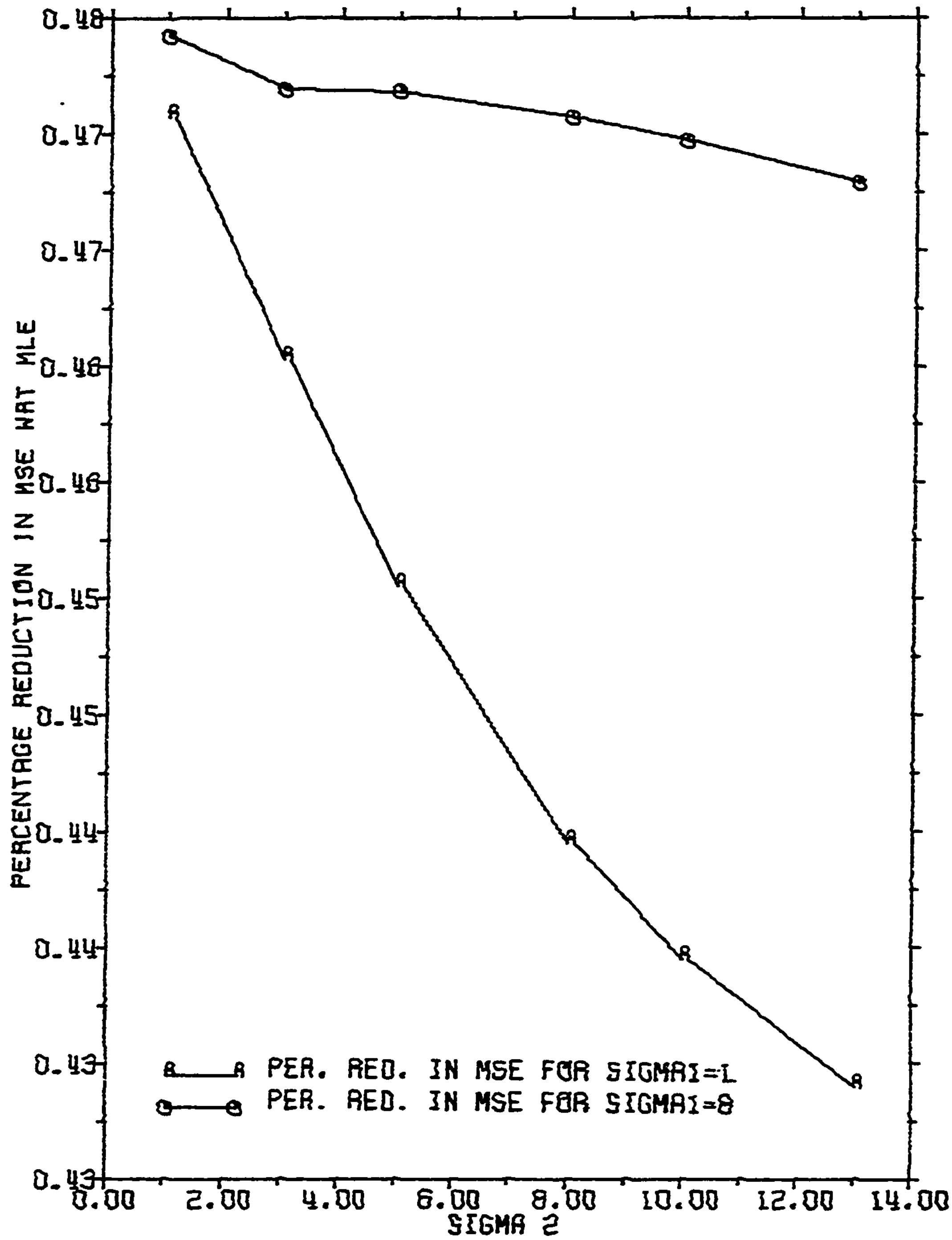

Figure 14. Empirical percentage reduction in MSE of modified MLE WRT MLE for $n_1 = 5$ $n_2 = 20$

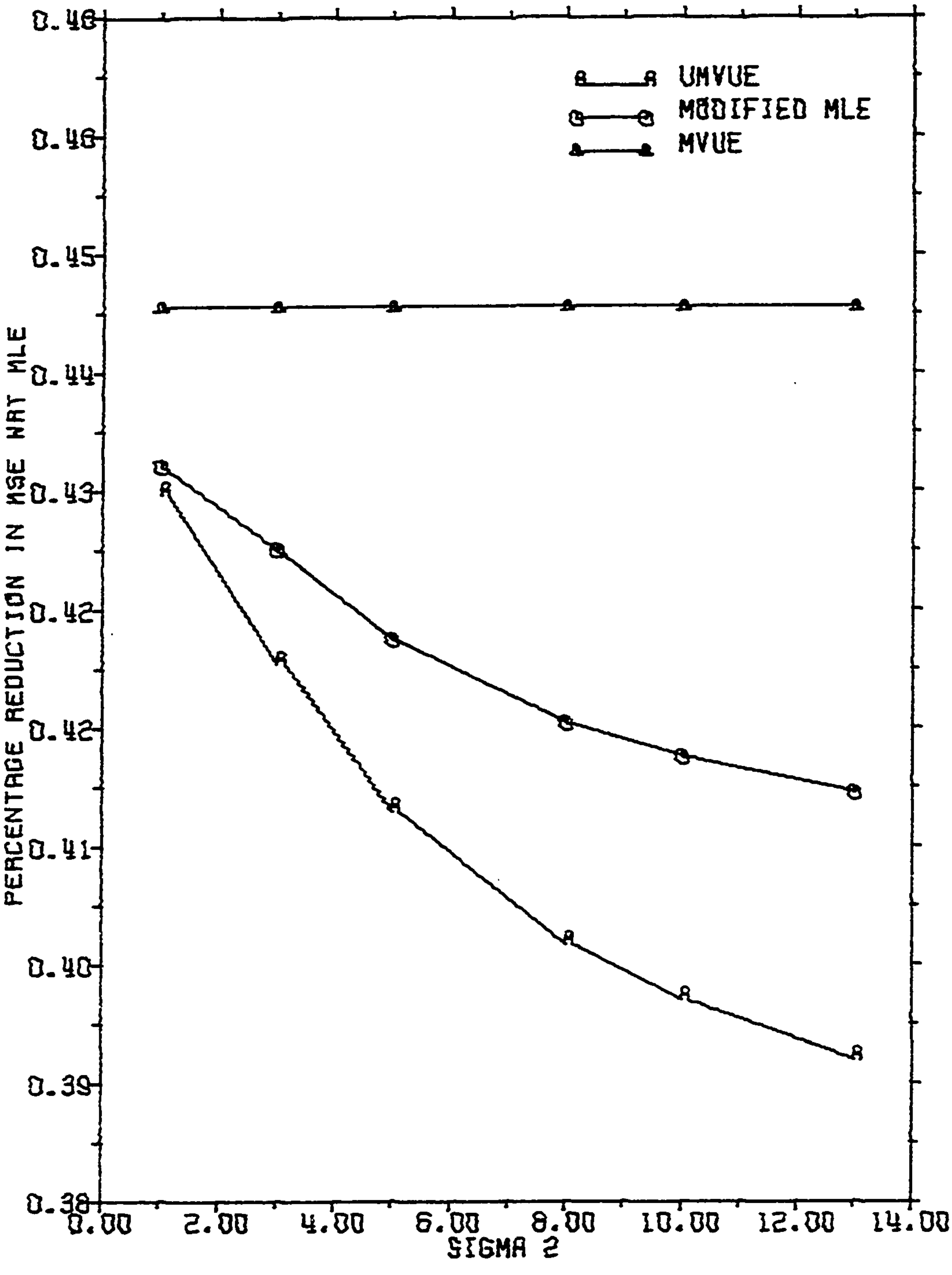

Figure 15. Empirical percentage reduction in MSE of various estimators WRT MLE for $n_1 = 5, n_2 = 5, \sigma_1 = 1$

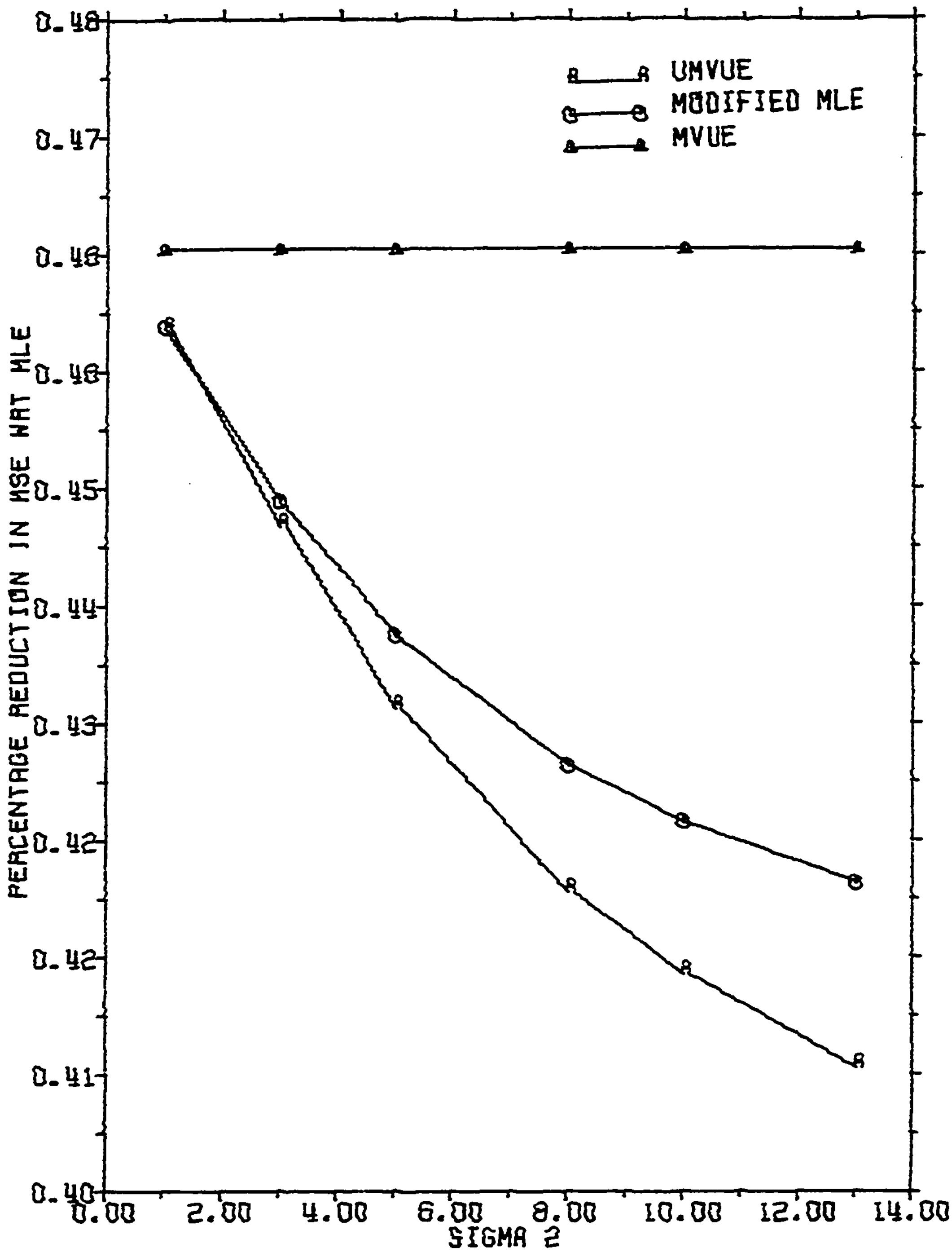

Figure 16. Empirical percentage reduction in MSE of various estimators WRT MLE for $n_1 = 5, n_2 = 10, \sigma_1 = 1$

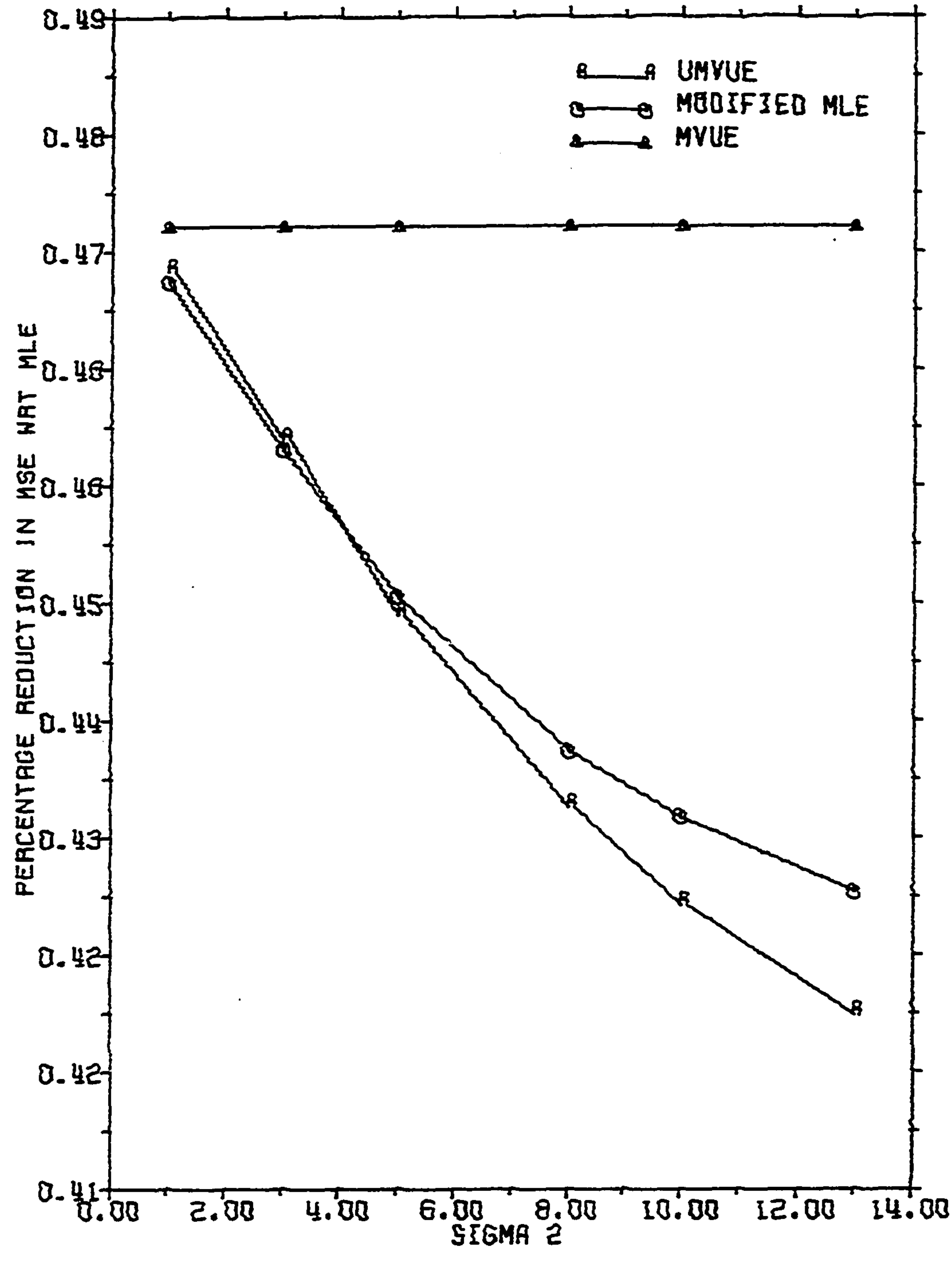

Figure 17. Empirical percentage reduction in MSE of various estimators WRT MLE for $n_1 = 5, n_2 = 15, \sigma_1 = 1$

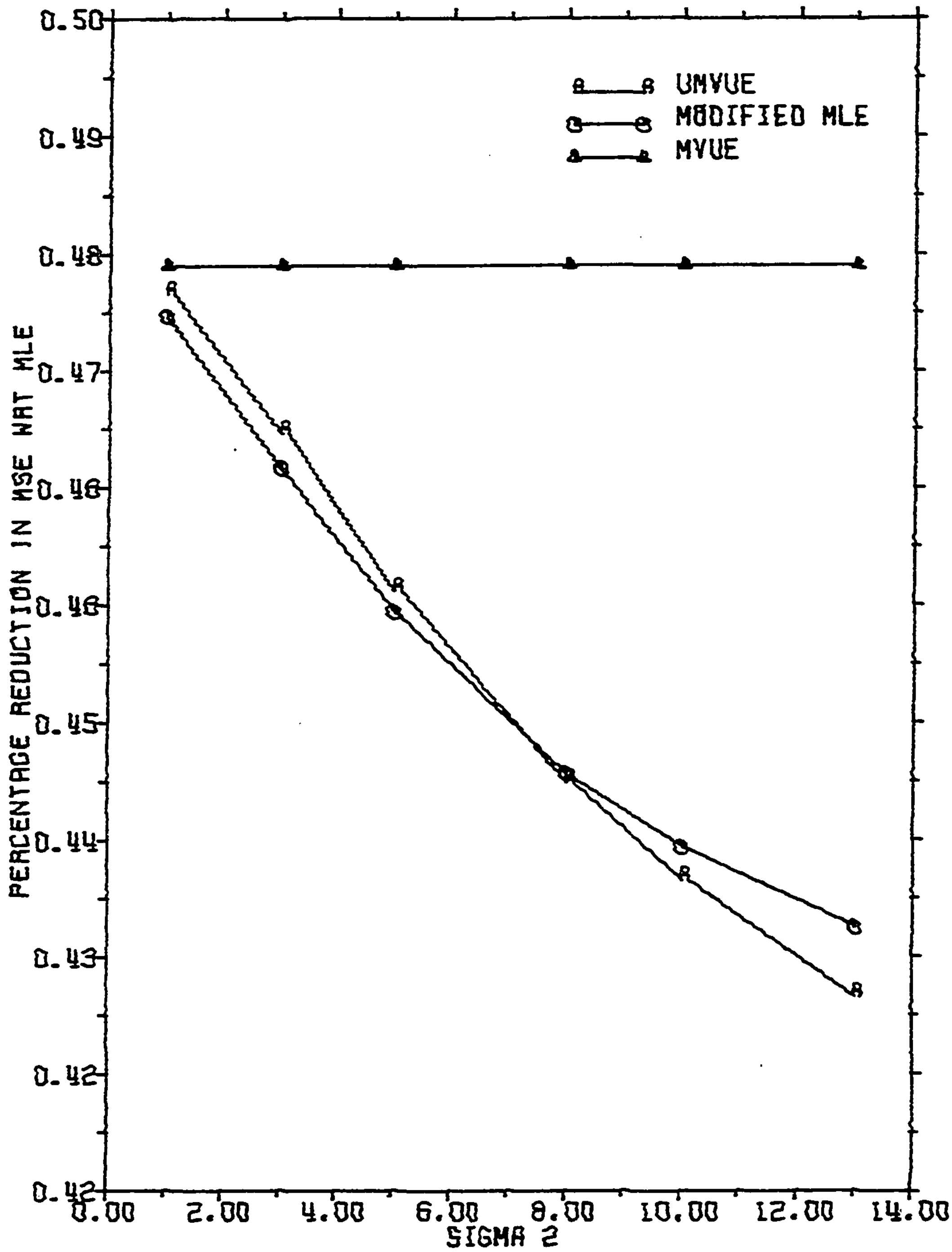

Figure 18. Empirical percentage reduction in MSE of various estimators WRT MLE for $n_1 = 5, n_2 = 20, \sigma_1 = 1$

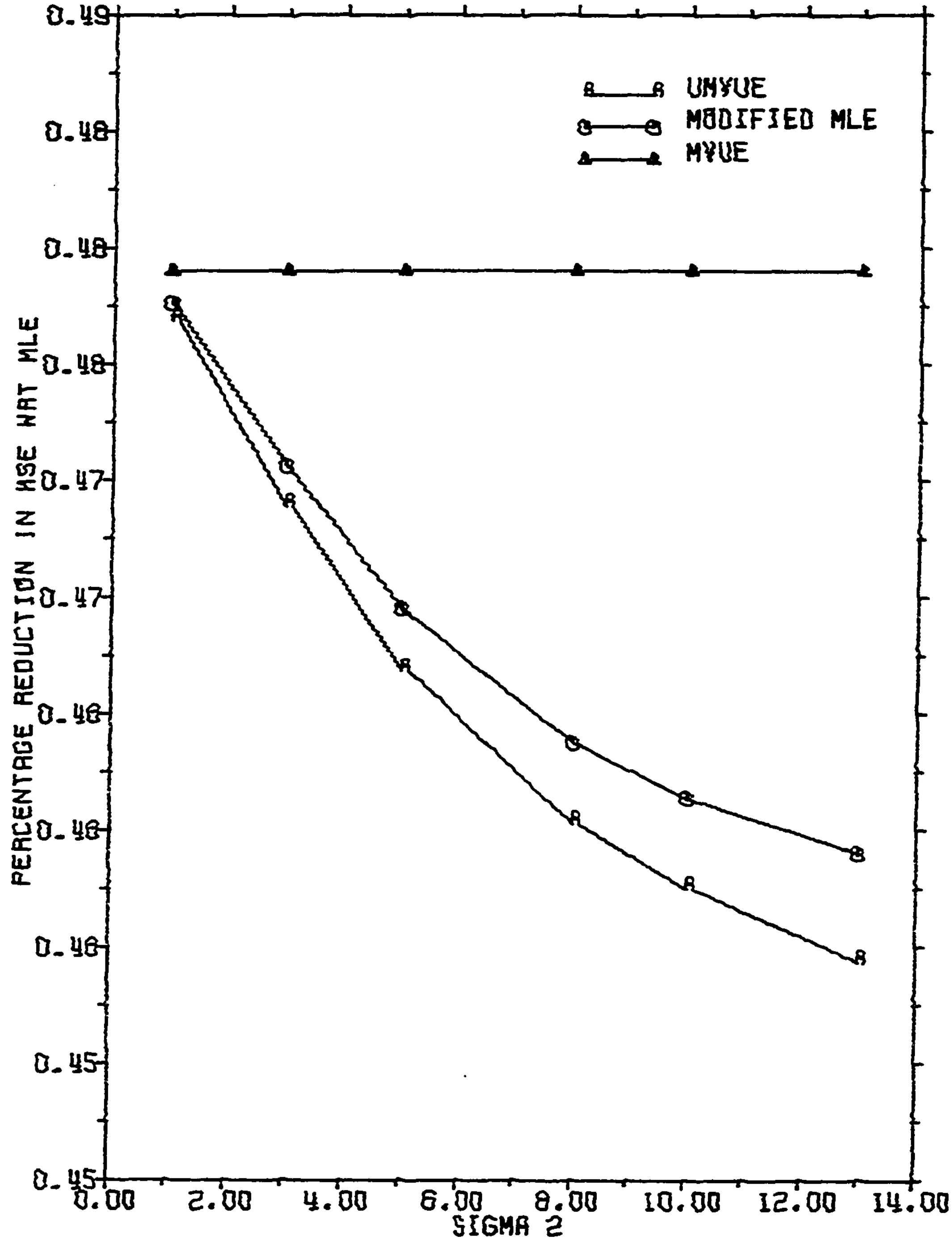

Figure 19. Empirical percentage reduction in MSE of various estimators WRT MLE for $n_1 = 10, n_2 = 15, \sigma_1 = 1$

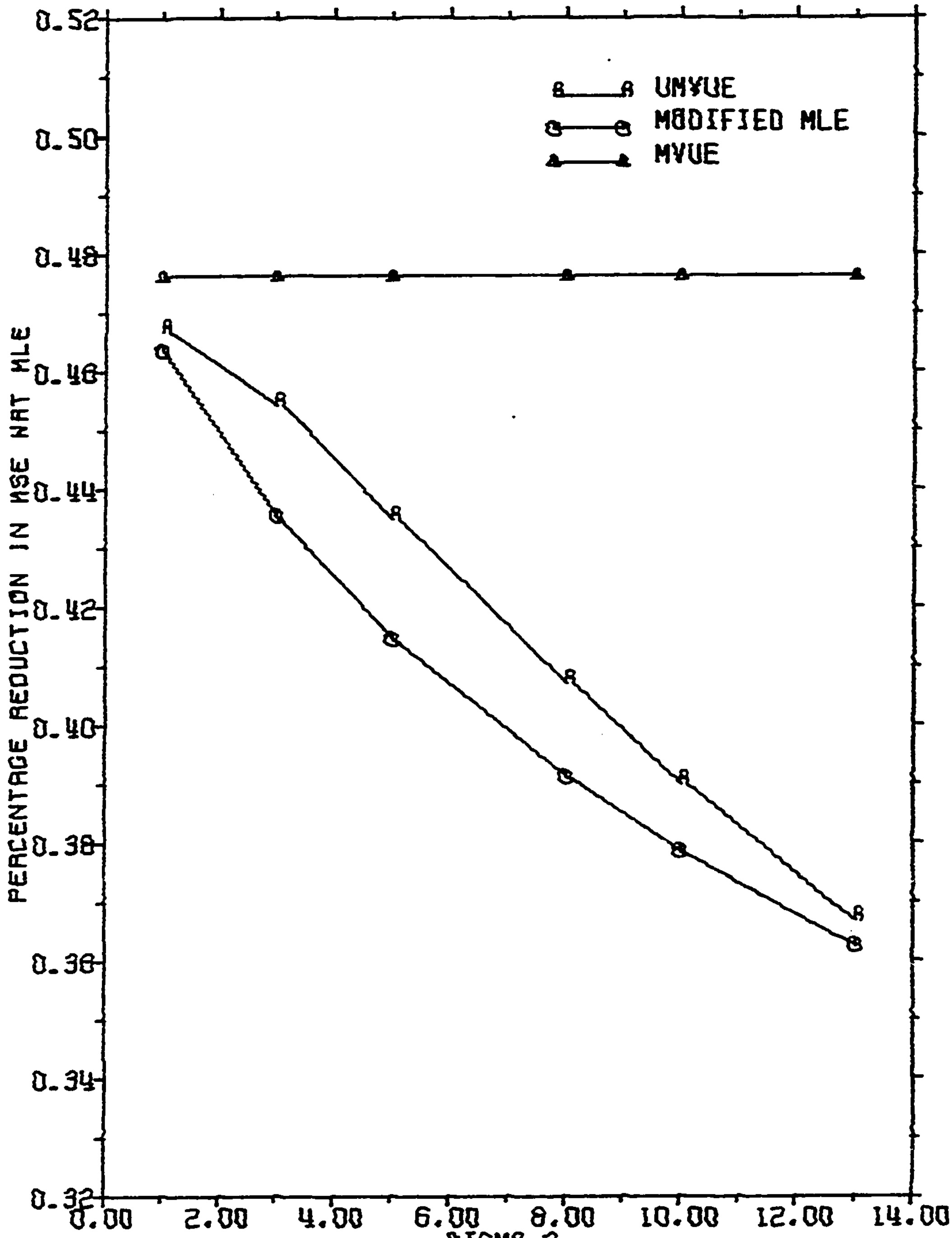

Figure 20. Empirical percentage reduction in MSE of various estimators WRT MLE for $n_1 = 2, n_2 = 20, \sigma_1 = 1$

3. THE SEVERAL EXPONENTIAL CASE

In this chapter, we will generalize the results of Chapter 2 to p populations.

3.1. Basic Set Up

Let $X_{i1}, X_{i2}, \ldots, X_{in_i}$, $(i = 1,2,\ldots,p)$ be independently distributed with X_{ij}'s, $(1 \leq j \leq n_j)$ iid with common probability density function

$$f(x_{i1}) = \sigma_i^{-1} \exp(-(x_{i1}-\mu)/\sigma_i)\, I_{[x_{i1} \geq \mu]}, \quad i = 1, \ldots, p, \tag{3.1.1}$$

where I denotes the usual indicator function and μ(real), $\sigma_i(>0)$, $i = 1, 2, \ldots, p$, are all unknown. The joint probability density function of the X_{ij}'s, $(i = 1\text{-}P$ and $j = 1,2,\ldots,n_i)$ can be written as

$$f(\underset{\sim}{x}_1,\underset{\sim}{x}_2,\ldots,\underset{\sim}{x}_p) = \left(\prod_{i=1}^{P} \sigma_i^{-n_i}\right) \exp\left[-\sum_{i=1}^{P} \sigma_i^{-1} \sum_{j=1}^{n_i} (x_{ij}-\mu)\right] \cdot I_{[\min\limits_{1 \leq i \leq p} (U_i(\underset{\sim}{x}_i)) \geq \mu]} \tag{3.1.2}$$

where $\underset{\sim}{x}_i = (x_{i1},x_{i2},\ldots,x_{in_i})$ and $U_i(\underset{\sim}{x}_i) = \min\limits_{1 \leq j \leq n_i} (x_{ij})$

3.2. Maximum Likelihood Estimators

From (3.1.2) the likelihood function can be written as

$$L(\mu,\sigma_i,\underset{\sim}{x}_i,\ i = 1,\ldots,p) = \left(\prod_{i=1}^{P} \sigma_i^{-n_i}\right) \exp\left[-\sum_{i=1}^{P} \sigma_i^{-1} \sum_{j=1}^{n_i} (x_{ij}-\mu)\right] \cdot I_{[\min\limits_{1 \leq i \leq p} (U_i(\underset{\sim}{x}_i)) \geq \mu]} \tag{3.2.3}$$

Then from (3.2.1)

$$\max_{\mu} L(\mu,\sigma_i,\underset{\sim}{x}_i) = \left(\prod_{i=1}^{p} \sigma_i^{-n_i}\right) \exp\left[-\sum_{i=1}^{p} \sigma_i^{-1} \sum_{j=1}^{n_i} (x_{ij}-z^*)\right] \quad i = 1 - p$$

where $z^* = \min\limits_{1\le i\le p} (U_i(\underset{\sim}{x}_i))$.

Now, define $L^*(z^*,\sigma_i,\underset{\sim}{x}_i) = \max\limits_{\mu} L(\mu,\sigma_i,\underset{\sim}{x}_i)$, $i = 1, \ldots, p$. Then

$$\text{Log } L^*(z^*,\sigma_i,\underset{\sim}{x}_i) = -\sum_{i=1}^{p} n_i \log\sigma_i - \sum_{i=1}^{p} \sigma_i^{-1}\left(\sum_{j=1}^{n_i} (x_{ij}-z^*)\right) \quad \text{and}$$

$$\frac{\partial \text{Log} L^*(z^*,\sigma_i,\underset{\sim}{x}_i)}{\partial\sigma_i} = -n_i\sigma_i^{-1} + \sigma_i^{-2} \sum_{j=1}^{n_i} (x_{ij}-z^*)$$

$$\gtreqless 0 \qquad \text{as} \quad \sigma_i \lesseqgtr (\bar{x}_i - z^*) \tag{3.2.2}$$

From (3.2.2), the MLE's of σ_i's are $\hat{\sigma}_i = \bar{x}_i - z^*$, where $\bar{x}_i$ is a sample average of i^{th} population.

3.3. Distribution Function of $Z^* = \min\limits_{1\le i\le p} (U_i(\underset{\sim}{X}_i))$

Using the independence of $U_i(\underset{\sim}{X}_i)$, $(i = 1,2,\ldots,p)$, it follows that for $Z^* \geq \mu$,

$$P(Z^*\le z) = 1 - P(Z^*>z) = 1 - \prod_{i=1}^{p} (1-G_{U_i}(z)) \tag{3.3.1}$$

where $G_{U_i}(z)$, $(i = 1,\ldots,p)$ are the d.f. of $U_i(\underset{\sim}{X}_i) = \min\limits_{1\le j\le n_i} (\underset{\sim}{x}_i)$ respectively. From (3.3.1) it follows that pdf of Z^* is

$$f_{Z*}(z) = (\sum_{i=1}^{p} n_i\sigma_i^{-1}) \exp[-(\sum_{i=1}^{p} n_i\sigma_i^{-1})(z-\mu)]\, I_{[z\geq\mu]} \qquad (3.3.2)$$

Thus, for all n_i's, $(\sum_{i=1}^{p} n_i\sigma_i^{-1})(Z-\mu)$ has a simple exponential distribution with scale parameter 1.

3.4. Joint Distribution Function of $T_i = n_i\hat{\sigma}_i$'s ($i = 1, \ldots, p$)

Define $Z^* = \min_{1\leq i\leq p} (X_{i(1)})$, $V_i = \sum_{j=1}^{n_i} (X_{ij}-X_{i(1)})$, $i = 1, \ldots, p$ and

$$T_i = \sum_{j=1}^{n_i} (X_{ij}-Z^*) = V_i + n_i(X_{i(1)}-Z^*), \quad i = 1, \ldots, p .$$

Then

$$P(T_1>t_1,\ldots,T_p>t_p) = P(V_1+n_1(X_{1(1)}-Z^*) > t_1, \ldots, V_p + n_p(X_{p(1)}-Z^* > t_p)$$

$$= \sum_{k=1}^{p} P(V_1+n_1(X_{1(1)}-Z^*) > t_1, \ldots, V_p + n_p(X_{p(1)}-Z^*) > t_p,$$

$$\underset{\sim}{X}_{k(1)} = \min_{1\leq i\leq p} (X_{i(1)}))$$

$$= \sum_{k=1}^{p} P(V_1+n_1(X_{1(1)}-Z^*) > t_1, \ldots, V_p + n_p(X_{p(1)}-Z^*) > t_p | \underset{\sim}{X}_{i(1)} > \underset{\sim}{X}_{k(1)}$$

$$\forall_i = 1,-P(\neq k)) \cdot P(X_{i(1)} > X_{k(1)}\ \forall_i \neq k) \qquad (3.4.1)$$

Now,

$$P(X_{i(1)}\ X_{k(1)} > \forall\ i = 1,\ldots,P(\neq k)) =$$

$$\int_0^\infty \{ \prod_{\substack{i=1 \\ i\neq k}}^{p} \int_{x_{k(1)}}^\infty n_i\sigma_i^{-1} \exp(-n_i\sigma_i^{-1}x_{i(1)})\, dx_{i(1)} \}\, n_k\sigma_k^{-1} \exp(-n_k\sigma_k^{-1}x_{k(1)})\, dx_{k(1)}$$

$$= \int_0^\infty \{ \prod_{\substack{i=1 \\ i\neq k}}^{p} [\exp(-n_i\sigma_i^{-1}x_{k(1)})\} \; n_k\sigma_k^{-1} \exp(-n_k\sigma_k^{-1}x_{k(1)}) \; dx_{k(1)}$$

$$= \int_0^\infty n_k\sigma_k^{-1} \exp[(-\sum_{\substack{i=1 \\ i\neq k}}^{p} n_i\sigma_i^{-1})x_{k(1)} - n_k\sigma_k^{-1}x_{k(1)}] \; dx_{k(1)}$$

$$= \int_0^\infty n_k\sigma_k^{-1} \exp\{-(\sum_{i=1}^{p} n_i\sigma_i^{-1})x_{k(1)}\} \; dx_{k(1)} = n_k\sigma_k^{-1}(\sum_{i=1}^{p} n_i\sigma_i^{-1})^{-1} \qquad (3.4.2)$$

for $k = 1, 2, \ldots, p$

Next consider

$$P[V_1+n_1(X_{1(1)}-Z^*) > t_1, \ldots, V_p + n_p(X_{p(1)}-Z^*) > t_p | X_{i(1)}>X_{k(1)} \;\; \forall i(\neq k)]$$

$$= P[(V_i+n_i(X_{1(1)}-Z^*)) > t_i \;\; \forall i(\neq k), \; V_k > t_k | X_{i(1)}>X_{k(1)} \;\; \forall i(\neq k)] \qquad (3.4.3)$$

and note that

$$P[n_i(X_{i(1)}-X_{k(1)} > w_i \;\; \forall i(\neq k) | X_{i(1)}>X_{k(1)} \;\; \forall \; i = 1,\ldots,p(\neq k)]$$

$$= \frac{P[n_i(X_{i(1)}-X_{k(1)})>w_i, \; \forall i(\neq k)]}{P[X_{i(1)}>X_{k(1)}, \; \forall \; i \; (\neq k)]}$$

$$= \frac{\int_0^\infty \{\sum_{i=1(\neq k)}^{p} \int_{n_i^{-1}w_i}^\infty n_i\sigma_i^{-1} \exp(-n_i\sigma_i^{-1}x_{i(1)})dx_{i(1)}\} \; n_k\sigma_k^{-1} \exp(-n_k\sigma_k^{-1}x_{k(1)})dx_{k(1)}}{n_k\sigma_k^{-1}(\sum_{i=1}^{p} n_i\sigma_i^{-1})^{-1}}$$

$$= \frac{\int_0^\infty (n_k\sigma_k^{-1}) \sum_{i=1(\neq k)}^{p} \exp[-n_i\sigma_i^{-1}(n_i^{-1}w_i+x_{k(1)})-n_k\sigma_k^{-1}x_{k(1)}] \; dx_{k(1)}}{n_k\sigma_k^{-1}(\sum_{i=1}^{p} n_i\sigma_i^{-1})^{-1}}$$

$$= [n_k\sigma_k^{-1}(\sum_{i=1}^{p} n_i\sigma_i^{-1})^{-1}]^{-1} \exp(-\sum_{i=1(\neq k)}^{p} w_i\sigma_i^{-1}) \int_0^\infty n_k\sigma_k^{-1}$$

$$\exp[(-\sum_{i=1}^{p} n_i\sigma_i^{-1})x_{k(1)}]\, dx_{k(1)}$$

$$= (n_k\sigma_k^{-1})(\sum_{i=1}^{p} n_i\sigma_i^{-1})^{-1}[(n_k\sigma_k^{-1})(\sum_{i=1}^{p} n_i\sigma_i^{-1})^{-1}]^{-1} \exp(-\sum_{i=1(\neq k)}^{p} w_i\sigma_i^{-1})$$

$$= \prod_{i=1(\neq k)}^{p} e^{-w_i\sigma_i^{-1}} \tag{3.4.4}$$

Thus, $n_i(X_{i(1)}-Z^*)$ $\forall i(\neq k)$, $(i = 1,2,\ldots,p)$ conditional on $X_{i(1)} > X_{k(1)}$ are independently distributed as $\text{Gamma}(\sigma_i^{-1}, 1)$.

Now, using independence of V_i's and $(X_{i(1)}-X_{k(1)})$ $\forall i(\neq k)$, therefore, $V_i + n_i(X_{i(1)}-Z^*)$, $1 \leq i(\neq k) \leq p$, conditional on $X_{i(1)}>X_{k(1)}$ are independently distributed as $\text{Gamma}(\sigma_i^{-1}, n_i)$. From (3.4.3) and (3.4.4) and the above,

$P[T_i>t_i\ \forall i = 1,2,\ldots,p|X_{i(1)}>X_{k(1)}\ \forall i(\neq k)] =$

$$\{\prod_{i=1(\neq k)}^{p} P(\text{Gamma}(\sigma_i^{-1},n_i)>t_i)\}\{P(\text{Gamma}(\sigma_k^{-1},n_k-1>t_k)\} \tag{3.4.5}$$

Thus, from (3.4.2) and (3.4.4), it follows that

$P(T_i>t_i\ \ \forall i = 1,2, \ldots,p) =$

$$\sum_{k=1}^{p} (n_k\sigma_k^{-1})(\sum_{j=1}^{p} n_j\sigma_j^{-1})^{-1} \{\prod_{i=1(\neq k)}^{p} P(\text{Gamma}(\sigma_i^{-1},n_i)>t_i)\}$$

$$\{P(\text{Gamma}(\sigma_k^{-1},n_k-1)>t_k)\} \tag{3.4.6}$$

From (3.4.5) it follows that the joint p.d.f of T_i's, $(i = 1,\ldots,p)$ is

$$h(t_1,t_2,\ldots,t_p) = \sum_{k=1}^{p}(n_k \sigma_k^{-1})(\sum_{j=1}^{p} n_j \sigma_j^{-1})^{-1}(\prod_{i=1(\neq k)}^{p} \frac{e^{-\sigma_i^{-1}t_i} t_i^{n_i-1}}{\Gamma(n_i)\,\sigma_i^{n_i}}) \left(\frac{e^{-\sigma_k^{-1}t_k} t_k^{n_k-2}}{\Gamma(n_k-1)\,\sigma_k^{n_k-1}}\right),\ k \neq i$$

$$= (\sum_{i=1}^{p} n_i\sigma_i^{-1})^{-1}(\prod_{i=1}^{p}\sigma_i^{-n_i})[\sum_{k=1}^{p} n_k(\prod_{i=1(\neq k)}^{p}\frac{t_i}{\Gamma(n_i)})(\frac{1}{\Gamma(n_k-1)})]\cdot$$

$$[\prod_{i=1}^{p}(t_i^{n_i-2}\exp(-\sigma_i^{-1}t_i))],\ t_i > 0\ \forall i = 1-p \qquad (3.4.7)$$

3.5. The UMVUE of Common Location Parameter of p-Exponential Distributions

<u>Theorem 3.5.1</u> If $X_{i1}, X_{i2}, \ldots, X_{in_i}$, $1 \leq i \leq p$, be independently distributed with X_{ij}'s $(1 \leq j \leq n_i)$ iid with common pdf

$$f(x_{i1}) = \sigma_i^{-1}\exp\{-(x_{i1}-\mu)/\sigma_i\}\ I_{[x_{i1}\geq\mu]},\quad \text{for } i = 1, \ldots, p .$$

Then the UMVUE of common location parameter (μ) is given by

$$\hat{\mu}_{UMV} = Z^* - (\prod_{i=1}^{p} T_i)\{(\prod_{k=1}^{p} n_k(n_k-1))(\prod_{i=1(\neq k)}^{p} T_i)\}^{-1}$$

<u>Proof of Theorem 3.5.1</u> From (3.1.2), the joint probability density function of X_{ij}'s, $(i = 1-p$ and $j = 1-n_i)$ can be written as

$$f(x_1,\ldots,x_p) = (\prod_{i=1}^{p}\sigma_i^{-n_i})\exp\{-\sum_{i=1}^{p}\sigma_i^{-1}(\sum_{i=1}^{n_i}(x_{ij}-z^*)) - (\sum_{i=1}^{p} n_i\sigma_i^{-1})(z^*-\mu)\}\cdot$$

$$I_{[z^*\geq\mu]}$$

$$= (\prod_{i=1}^{P} \sigma_i^{-n_i}) \exp\{-\sum_{i=1}^{P} \sigma_i^{-1} t_i - (\sum_{i=1}^{P} n_i\sigma_i^{-1})(z^*-\mu)\} I_{[z^*\geq\mu]}$$

(3.5.1)

From (3.5.1), the minimal sufficient statistic for $(\mu,\sigma_i,i=1,\ldots,p)$ is given by $(Z^*,T_i = \sum_{j=1}^{n_i} (X_{ij}-Z^*),\ i = 1\text{-}P)$.

To prove that the family of distributions induced by $(Z^*,T_i,i=1\text{-}P)$ is complete, proceed as follows.

We need to show

$$E_{\mu,\sigma_1,\ldots,\sigma_p}\, g(Z^*,T_1,\ldots,T_p) = 0 \cdot \forall\mu,\sigma_i(>0),\ i = 1\text{-}P$$

$$\Longrightarrow g(Z^*,T_1,\ldots,T_p) = 0 \quad \text{a}\cdot\text{s} \qquad (3.5.2)$$

But

$$E_{\mu,\sigma_1,\ldots,\sigma_p}\, g(Z^*,T_1,\ldots,T_p) = 0 \quad \forall\mu(\text{real}),\sigma_i(>0),\quad i = 1\text{-}P$$

$$\Longleftrightarrow \int_0^\infty\cdots\int_0^\infty \int_\mu^\infty g(z^*,t_1,\ldots,t_p)(\sum_{i=1}^{P} n_i\sigma_i^{-1}) \exp\{(-\sum_{i=1}^{P} n_i\sigma_i^{-1})(Z^*-\mu)\} \cdot$$

$$h(t_1,\ldots,t_p)\, dz^*dt_1\ldots dt_p = 0. \qquad (3.5.3)$$

Differentiating both sides with respect to μ, it follows that

$$\int_0^\infty\cdots\int_0^\infty g(\mu,t_1,\ldots,t_p)\, h_{\sigma_1,\ldots,\sigma_p}(t_1,\ldots,t_p)\, dt_1\cdot dt_2\ldots dt_p = 0 \ \text{a}\cdot\text{s}$$

(3.5.4)

Hence, it suffices to show that for each fixed μ, $(T_1,\ldots,T_p)$ is complete sufficient for $(\sigma_1,\ldots,\sigma_p)$.

From (3.4.6), the joint pdf of T_i's $(i = 1, \ldots, p)$ can be written

$$h(t_1,\ldots,t_p) = C(\ _1,\ldots,\ _p)\ Q(t_1,\ldots,t_p)\ \exp\{-\sum_{i=1}^{p} \sigma_i^{-1} t_i\}\quad t_i > 0,\ \forall i \tag{3.5.5}$$

which belongs to the exponential family. Thus, $(T_1,\ldots,T_p)$ is jointly complete sufficient statistic for $(\sigma_1,\ldots,\sigma_p)$. Thus, the family of distributions induced by $(Z^*,T_1,\ldots,T_p)$ is complete.

Next note that, from (3.3.2), it is easy to prove that for σ_i's fixed $(1 \leq i \leq p)$, Z^* is complete sufficent statistic for μ. Also, the joint distribution of T_i's are jointly independent of Z^*. And

$$EZ^* = \mu + (\sum_{i=1}^{p} n_i\sigma_i^{-1})^{-1} \tag{3.5.6}$$

Thus, in order to find the UMVUE of common location parameter (μ), it suffices to find a real valued function t_i's $(1 \leq i \leq p)$ say,

$g^*(t_1,\ldots,t_p)$ which has expected value equal to $(\sum_{i=1}^{p} n_i\sigma_i^{-1})^{-1}$.

Hence, $Eg^*(t_1,\ldots,t_p) = (\sum_{i=1}^{p} n_i\sigma_i^{-1})^{-1} \Longrightarrow$

$$\int_0^\infty\cdots\int_0^\infty g^*(t_1,\ldots,t_p)h(t_1,\ldots,t_p)dt_1\ldots dt_p = (\sum_{i=1}^{p} n_i\sigma_i^{-1})^{-1}$$

$$\Longleftrightarrow \int_0^\infty\cdots\int_0^\infty g^*(t_1,\ldots,t_p)[\sum_{i=1}^{p} \frac{n_k}{\Gamma(n_k-1)} \cdot (\prod_{\substack{i=1\\ \neq k}}^{p} \frac{t_i}{\Gamma(n_i)})]$$

$$\left[\prod_{i=1}^{p}\left(t_i^{n_i-2}\exp(-\sigma_i^{-1}t_i)\right)\right] = \left(\prod_{i=1}^{p}\sigma_i^{n_i}\right)$$

$$\Longrightarrow g^*(t_1,\ldots,t_p) = \left(\prod_{i=1}^{p} t_i\right)\left[\sum_{k=1}^{p} n_k(n_k-1)\left(\prod_{i=1(\neq k)}^{p} t_i\right)\right]^{-1} \qquad (3.5.7)$$

Thus, the UMVUE of μ is given by

$$\hat{\mu}^*_{UMV} = Z^* - \left\{\left(\prod_{i=1}^{p} T_i\right)\left(\sum_{k=1}^{p} n_k(n_k-1)\left(\prod_{i=1(\neq k)}^{p} T_i\right)\right)\right\}^{-1}$$

with expected value and variance as follows.

$$E\mu^*_{UMV} = EZ^* - Eg^*(T_1,\ldots,T_p)$$

$$= \mu + \left(\sum_{i=1}^{p} n_i\sigma_i^{-1}\right)^{-1} - \left(\sum_{i=1}^{p} n_i\sigma_i^{-1}\right)^{-1} = \mu$$

and

$$\mathrm{Var}(\mu^*_{UMV}) = \mathrm{Var}(Z^*) + \mathrm{Var}[g^*(T_1,\ldots,T_p)]$$

$$= \left(\sum_{i=1}^{p} n_i\sigma_i^{-1}\right)^{-2} + Eg^{*2}(T_1,\ldots,T_p) + [Eg(T_1,\ldots,T_p)]^2$$

$$= Eg^{*2}(T_1,\ldots,T_p)$$

<u>Lemma 3.5.1</u> The UMVUE of $\mathrm{Var}(\hat{\mu}^*_{UMV})$ is given by

$$g^{*2}(T_1,\ldots,T_p) = \left\{\left(\prod_{i=1}^{p} T_i\right)\left[\sum_{k=1}^{p} n_k(n_k-1)\left(\prod_{\substack{i=1\\ \neq k}}^{p} T_i\right)\right]^{-1}\right\}^2$$

3.6. The UMVUE of Common Location Parameter (μ), When the Ratios $\rho_i = \sigma_i \sigma_1^{-1}$ are Known

Let $X_{i1}, X_{i2}, \ldots, X_{in_i}$, $(i = 1,2,\ldots,P)$ be independently distributed with X_{ij}'s $(1 \le j \le n_i,\ i = 1,\ldots,P)$ iid (for every $i = 1,\ldots,P$) with common pdf given by (3.1.1). Thus, from (3.2.1) the joint probability density of $\underset{\sim}{X}_i$'s $(i = 1,2,\ldots,P)$ can be written as follows.

$$f(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_p) = (\sigma_1^{-\sum_{i=1}^{P} n_i} \prod_{i=2}^{P} \rho_i^{-n_i}) \exp\{-\sigma_1^{-1}[\sum_{j=1}^{n_1} (X_{1j}-\mu) + \sum_{i=2}^{P} [\rho_i^{-1} \sum_{j=1}^{n_i} (X_{ij}-\mu)]]\} \cdot I_{[\min_{1\le i\le p} (U_i(\underset{\sim}{x}_i)) \ge \mu]} \tag{3.6.1}$$

or

$$f(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_p) = (\sigma_1^{-\sum_{i=1}^{P} n_i})(\prod_{i=2}^{P} \rho_i^{-n_i}) \exp\{-\sigma_1^{-1}[\sum_{j=1}^{n_1} X_{1j} + \sum_{i=2}^{P} \rho_i^{-1} \sum_{j=1}^{n_i} (X_{ij}) - (n_1 + \sum_{i=2}^{P} n_i \rho_i^{-1})\mu]\} \cdot I_{[z^* \ge \mu]} \tag{3.6.2}$$

Now, by using the proof given in (2.6), it is easy to prove the minimal sufficient statistic for (μ, σ_1) is given by

$(Z^*, \sum_{j=1}^{n_1} X_{1j} + \sum_{i=2}^{P} \rho_i^{-1} (\sum_{j=1}^{n_i} X_{ij}))$ or equivalently (Z^*, T^*), where

$T^* = \sum_{j=1}^{n_1} (X_{1j} - Z^*) + \sum_{i=2}^{P} \rho_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - Z^*)$ and also Z^* and $(\sum_{i=1}^{P} n_i)^{-1} T^*$

are MLE of μ and σ_1 respectively.

3.7. Distribution Function of Z*

$$F_{Z*}(z) = P(Z^*\leq z) = 1 - P(Z^*>z)$$

$$= 1 - P(U_i(\underset{\sim}{X}_i) > z, \ \forall\, i = 1, \ldots, P)$$

$$= 1 - \prod_{i=1}^{P} [1 - G_{U_i(\underset{\sim}{X}_i)}(z)] \tag{3.7.1}$$

(where $U_i(\underset{\sim}{X}_i)$, $(i = 1,\ldots,P)$ are defined as before).

Thus, using the assumption that ρ_i's, $(i = 1,\ldots,P)$ are known and (3.3.2) and (3.7.1), the pdf of Z^* is as follows

$$f_{Z*}(z) = \sum_{i=1}^{P} g_{U_i(\underset{\sim}{X}_i)}(z) \Big[\prod_{\substack{j=1\\(\neq i)}}^{P} (1 - G_{U_j(\underset{\sim}{X}_j)}(z))\Big]$$

$$= \sigma_1^{-1}(n_1+\sum_{i=2}^{P} n_i\rho_i^{-1}) \{\exp[-\sigma_1^{-1}(n_1+\sum_{i=2}^{P} n_i\rho_i^{-1})(z-\mu)]\} I_{[z\geq\mu]} \tag{3.7.2}$$

From (3.7.2), it follows that for each fixed σ_1, Z^* is complete sufficient statistic for common location parameter (μ).

3.8. Distribution Function of $T^* = \sum_{j=1}^{n_1} (X_{1j}-Z^*)+\sum_{i=2}^{P} \rho_i^{-1} \sum_{j=1}^{n_i} (X_{ij}-Z^*)$

In order to obtain the distribution of T^*, we proceed as follows.

Define $Q_1^* = \sum_{j=1}^{n_1} (X_{1j}-U_1(\underset{\sim}{X}_1))$, $Q_2^* = \sum_{i=2}^{P} \rho_i^{-1} \sum_{j=1}^{n_i} (X_{ij}-U_i(\underset{\sim}{X}_i))$

and

$Q_3^* = n_1(U_1(\underset{\sim}{X}_1)-Z^*) + \sum_{i=2}^{P} n_i \rho_i^{-1}(U_i(\underset{\sim}{X}_i)-Z^*)$. It can be easily shown that Q_1, Q_2 and Q_3 are mutually independently distributed, since $U_i(\underset{\sim}{X}_1)$'s $(i = 1,2,\ldots,p)$ and $\sum_{j=1}^{n_i}(X_{ij}-U_i(\underset{\sim}{X}_i))$'s $(i = 1,\ldots,P)$ are mutually independent.

As we have proved in section (2.7)

$$Q_1^* \sim \text{Gamma}(\sigma_1^{-1}, n_1-1) \tag{3.8.1}$$

and similarly

$$Q_2^* \sim \text{Gamma}[\sigma_1^{-1}, (\sum_{i=2}^{P} n_i)-P+1] \tag{3.8.2}$$

since $\rho_i^{-1}(n_i-j+1)(X_{i(j)}-X_{i(j-1)})$ for $(i = 1,\ldots,P,\ j = 1,2,\ldots,n_i)$ are iid $\text{Gamma}(\sigma_1^{-1},1)$ (where $X_{i(0)} = 0$ and $X_{i(j)}$ is the jth order statistics from the ith population).

In order to find the distribution of Q_3^*, once again notice that from (3.7.2) for each fixed σ_1, Z^* is complete sufficient statistic for μ, while $Q_3^* = n_1(X_{1(1)}-Z^*)+\sum_{i=2}^{P} n_i\rho_i^{-1}(X_{i(1)}-Z^*)$ has a distribution free from μ. Hence, Z^* and Q_3^* are independently distributed (by Basu's theorem), (where $X_{i(1)} = U_i(\underset{\sim}{X}_i)$, $(i = 1,\ldots,P)$).

Next, observe that

$$Q_3^* = n_1(X_{1(1)}-Z^*) + \sum_{i=2}^{p} n_i\rho_i^{-1}(X_{i(1)}-Z^*)$$

$$= [n_1(X_{1(1)}-\mu) + \sum_{i=2}^{p} n_i\rho_i^{-1}(X_{i(1)}-\mu)] - [(n_1+\sum_{i=2}^{p} n_i\rho_i^{-1})(Z^*-\mu)]$$

But $n_1(X_{1(1)}-\mu)$ and $n_i\rho_i^{-1}(X_{i(1)}-\mu)$, $(i = 1,\ldots,p)$ are iid Gamma$(\sigma_1^{-1},1)$. And from (3.7.2) $[(n_1+\sum_{i=2}^{p} n_i\rho_i^{-1})(Z^*-\mu)]$ is distributed as Gamma$(\sigma_1^{-1},1)$. Now, use the lemma, if X and Y are independent, $X + Y \sim \text{Gamma}(\alpha,p_1)$ and $X \sim \text{Gamma}(\alpha,p_2)$, $(p_1 > p_2)$, then $Y \sim \text{Gamma}(\alpha,p_1-p_2)$. Hence, Q_3^* is distributed as Gamma$(\sigma_1^{-1},p-1)$.

Now, consider

$$T^* = \sum_{j=1}^{n_1}(X_{1j}-Z^*) + \sum_{i=2}^{p}\rho_i^{-1}\sum_{j=1}^{n_i}(X_{ij}-Z^*)$$

$$= \sum_{j=1}^{n_1}(X_{1j}-X_{1(1)}-Z^*) + \sum_{i=2}^{p}\rho_i^{-1}\sum_{j=1}^{n_i}(X_{ij}-X_{i(1)}+X_{i(1)}-Z^*)$$

$$= \sum_{j=1}^{n_1}(X_{1j}-X_{1(1)}) + \sum_{i=2}^{p}\rho_i^{-1}\sum_{j=1}^{n_i}(X_{ij}-X_{i(1)}) + [n_1(X_{1(1)}-Z^*)$$
$$+ \sum_{i=2}^{p} n_i\rho_i^{-1}(X_{i(1)}-Z^*)]$$

$$= Q_1^* + Q_2^* + Q_3^* .$$

Thus, using once again the independence of Q_1^*, Q_2^* and Q_3^*, it follows that

$$T^* \sim \text{Gamma}(\sigma_1^{-1}, \sum_{i=1}^{P} n_i - 1) \tag{3.8.3}$$

From (3.8.3) the MLE of σ_1 i.e. $\hat{\sigma}_1 = (\sum_{i=1}^{P} n_i)^{-1} T^*$ is distributed as $\text{Gamma}[(\sum_{i=1}^{P} n_i)^{-1}\sigma_1^{-1}, \sum_{i=1}^{P} n_i - 1]$.

Lemma 3.5.2 Z^* and T^* are independently distributed.

Proof As we have proved, for every fixed σ_1, Z^* is complete sufficient statistic for μ, while T^* is distributed free from μ, thus, by Basu's theorem, Z^* and T^* are independently distributed.

It is easy to prove, for each fixed μ, T^* is minimal sufficient statistic for σ_1. Now T^* is complete sufficient for σ_1, because the family of distributions induced by T^* belongs to the exponential family.

3.9. The UMVUE of Common Location Parameter When $\rho_i(=\sigma_i\sigma_1^{-1})$'s, $(i = 1,\ldots,P)$ are Known

Theorem 3.9.1 If the ratios $\rho_i(>0)$ are known, then the UMVUE of the common location parameter which is unbiased on the restricted parameter space $\{(\sigma_1,\sigma_2,\ldots,\sigma_p), \sigma_i = \rho_i\sigma_1\}$ is given by

$$\hat{\mu}^*_{MVE} = Z^* - [(\sum_{i=1}^{P} n_i - 1)(\sum_{i=1}^{P} n_i\rho_i^{-1})]^{-1}(\sum_{i=1}^{P} n_i)T^* .$$

Proof Under the given assumption the joint probability density of $\underset{\sim}{X}_i$'s, $(i = 1,2,\ldots,P)$ can be written as

$$f(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_p) = (\sigma_1^{-\sum_{i=1}^{P} n_i})(\prod_{i=2}^{P} \rho_i^{-n_i}) \exp\{-\sigma_1^{-1}[\sum_{j=1}^{n_1}(X_{1j}-Z^*)$$

$$+ \sum_{i=2}^{P} \rho_i^{-1} \sum_{j=1}^{n_i}(X_{ij}-Z^*)+(n_1-\sum_{i=2}^{P} n_i\rho_i^{-1})(z^*-\mu)]\} \cdot$$

$$I_{[z^*\geq\mu]} \tag{3.9.1}$$

As we have shown in Section (3.6), the minimal sufficient statistic for (μ,σ_1) is given by (Z^*,T^*). Also, we have proved in lemma 3.5.2 that Z^* and T^* are independently distributed. Hence, from (3.7.2) and (3.8.3) it follows that

$$E_{\mu,\sigma_1}(Z^*) = \mu + (\sum_{i=1}^{P} n_i\sigma_i^{-1})^{-1} \tag{3.9.2}$$

and

$$E_{\sigma_1}(T^*) = (\sum_{i=1}^{P} n_i-1)\sigma_1 \tag{3.9.3}$$

Thus, it follows that just on the restricted parameter space

$\{(\sigma_1,\ldots,\sigma_p),\ \rho_i = \sigma_i\sigma_1^{-1},\ (i = 1\text{-}P)$

$$E_{\mu,\sigma_1}(\hat{\mu}^*_{MVE}) = \mu + (n_1+\sum_{i=2}^{P} n_i\rho_i^{-1})^{-1}\sigma_1 - [(\sum_{i=1}^{P} n_i-1)(n_1+\sum_{i=2}^{P} n_i\rho_i^{-1})]^{-1} \cdot$$

$$(\sum_{i=1}^{P} n_i-1)\sigma_1$$

$$= \mu + (n_1+\sum_{i=2}^{P} n_i\rho_i^{-1})^{-1}\sigma_1 - (n_1+\sum_{i=2}^{P} n_i\rho_i^{-1})^{-1}\sigma_1 = \mu$$

Now, in order to prove the theorem, it suffices to prove that the family of distributions induced by Z^* and T^* is complete.

For $\rho_i = \rho_i^0$, $(i = 1,\ldots,p)$, we have proved Z^* and T^* are mutually independent. Now,

$$E_{\mu,\sigma_1;\rho_1^0,\ldots,\rho_p^0}\, h^*(Z^*,T^*) = 0 \quad \forall\, \mu(\text{real}),\ \sigma_1(>0) \Longleftrightarrow$$

$$\int_\mu^\infty \int_0^\infty h^*(z,t)\, g_{\sigma_1}(t)\, f_{\mu,\sigma_1}(z)\, dtdz = 0 \Longleftrightarrow$$

$$\int_\mu^\infty U_{\sigma_1}(z)\, f(z)\, dz = 0 \quad \forall\, \mu(\text{real}),\ \sigma_1(>0) \tag{3.9.4}$$

(where $g_{\sigma_1}(t)$ and $f_{\mu,\sigma_1}(z)$ are p.d.f's of T^* and Z^* respectively and $U_{\sigma_1}(z) = \int_0^\infty h^*(z,t)\, g_{\sigma_1}(t)\, dt$). Differentiating both sides of (3.9.4) with respect to μ, it follows that $U_{\sigma_1}(\mu) = 0$ a.e for all real μ, that is

$$\int_0^\infty h^*(\mu,t)\, g_{\sigma_1}(t)\, dt = 0 \quad \text{a.e for all fixed } \mu$$

Now, since for each fixed μ, T^* is complete for σ_1, thus,

$$\int_0^\infty h^*(\mu,t)\, g_{\sigma_1}(t)\, dt = 0 \quad \forall\, \sigma_1(\ 0),\ \text{all fixed } \mu$$

$\Longleftrightarrow h^*(\mu,t) = 0$ for all real μ and $t > 0$.

Therefore, (Z^*,T^*) is jointly complete sufficient statistic for (μ,σ_1). Thus, from the above discussion, it follows that, for every $\rho_i = \rho_i^0$

$$\hat{\mu}^*_{MVE} = Z^* - [(\sum_{i=1}^{P} n_i - 1)(n_1 + \sum_{i=2}^{P} n_i (\rho_i^0)^{-1}]^{-1}\, T^* \tag{3.9.5}$$

is the MVUE of common location parameter (μ) when $\rho_i^0 = \sigma_i\sigma_1^{-1}$, $(i = 1,\dots,P)$ are known.

From (3.9.5), it follows that

$$E\hat{\mu}^*_{MVE} = \mu$$

and

$$\mathrm{Var}(\hat{\mu}^*_{MVE}) = (\sum_{i=1}^{P} n_i)[(\sum_{i=1}^{P} n_i - 1)(n_1 + \sum_{i=2}^{P} n_i\rho_i^{-1})^2]^{-1}\sigma_1^2$$

3.10. Modified Maximum Likelihood Estimator for Common Location Parameter (μ) in P-Populations

As we have seen in section 3.2, the maximum likelihood estimator of common location parameter and scale parameters in the case of unknown and unequal scale parameters are given respectively by $\hat{\mu}_{pMLE} = Z^*$ and

$$\hat{\sigma}_i = n_i^{-1}\sum_{j=1}^{n_i}(X_{ij} - Z^*).$$

Generalizing the results of Section 2, a modified MLE for μ is proposed as follows

$$\hat{\mu}_{PMLE^*} = Z^* - (\sum_{i=1}^{P} n_i\hat{\sigma}_i^{-1})^{-1}$$

We next attempt to compare $\hat{\mu}_{PMLE}$ and $\hat{\mu}_{PMLE^*}$ in terms of their distributions as well as the mean squared error criterion.

3.11. Comparison of MLE and Modified MLE in Terms of Their Distribution and MSE

We have already seen that $(\sum_{i=1}^{P} n_i\sigma_i^{-1})(Z^*-\mu)$ has a simple exponential distribution with scale parameter 1. Thus, $E(Z^*-\mu) = (\sum_{i=1}^{P} n_i\sigma_i^{-1})^{-1}$. Therefore, using Chebyshev's inequality $Z^* \to \mu$ in probability as $\min(n_1,\ldots,n_p) \to \infty$. Also, using the weak law of large numbers, $n_i^{-1}\sum_{j=1}^{n_i} X_{ij} \to \mu + \sigma_i$ $(\forall\, i = 1,\ldots,P)$ in probability as $n_i \to \infty$ and it follows that $\hat{\sigma}_i \xrightarrow{P} \sigma_i$ $(\forall\, i = 1,\ldots,P)$ as $n_i \to \infty$. Hence, as $\min(n_1,\ldots,n_p) \to \infty$, $(\Sigma n_i\sigma_i^{-1})/(\Sigma n_i\hat{\sigma}_i^{-1}) \to 1$ in probability. Thus, as $\min(n_1,n_2,\ldots n_p) \to \infty$,

$$(\sum_{i=1}^{P} n_i\sigma_i^{-1})(\hat{\mu}_{PMLE^*}-\mu) = -(\sum_{i=1}^{P} n_i\sigma_i^{-1})/(\sum_{i=1}^{P} n_i\hat{\sigma}_i^{-1}) + (\sum_{i=1}^{P} n_i\sigma_i^{-1})(Z^*-\mu)$$

$\xrightarrow{L} Y - 1$, where Y is a simple exponential with scale parameter 1.

We next compare $\hat{\mu}_{PMLE}$ and $\hat{\mu}_{PMLE^*}$ in terms of their MSE's.

First note that,

$$E(\hat{\mu}_{PMLE}-\mu)^2 = E(Z^*-\mu)^2 = 2(\sum_{i=1}^{P} n_i\sigma_i^{-1})^{-2} \tag{3.11.1}$$

Next, observe that,

$$E(\hat{\mu}_{PMLE*}-\mu)^2 = MSE(\hat{\mu}_{PMLE})-2(\sum_{i=1}^{P} n_i\sigma_i^{-1})E(\sum_{i=1}^{P} n_i\sigma_i^{-1})^{-1}+E(\sum_{i=1}^{P} n_i\hat{\sigma}_i^{-1})^{-2}$$

$$= MSE(\hat{\mu}_{PMLE})-(\sum_{i=1}^{P} n_i\sigma_i^{-1})^{-2}+E\{(\sum_{i=1}^{P} n_i\hat{\sigma}_i^{-1})^{-1}-(\sum_{i=1}^{P} n_i\sigma_i^{-1})^{-1}\}^2 \quad (3.11.2)$$

In order to prove the next theorem, the following assumptions are made. For any pair of $(n_i,n_j,\ i,j = 1,\ldots,p)$

$$0 < d_1 \le d_{1i} < \liminf_{n\to\infty} n_i/n \le \limsup_{n\to\infty} n_i/n \le d_{2i} \le d_2 < \infty$$

$$\text{where } n = \sum_{i=1}^{P} n_i,\quad d_1 = \min_{1\le i\le p} d_{1i} \text{ and } d_2 = \max_{1\le i\le p} d_{2i} \quad (3.11.3)$$

Theorem 3.11.1 Under the assumption (3.11.3),

$$E[(\sum_{i=1}^{P} n_i\hat{\sigma}_i^{-1})^{-1}-(\sum_{i=1}^{P} n_i\sigma_i^{-1})^{-1}]^2 = O(n^{-3}) \quad (3.11.4)$$

Proof Let $g(\sigma_1,\sigma_2,\ldots,\sigma_p) = (\sum_{i=1}^{P} n_i\sigma_i^{-1})^{-1}$. Use a Taylor expansion to get

$$g(\hat{\sigma}_1,\ldots,\hat{\sigma}_p) - g(\sigma_1,\ldots,\sigma_p) = \sum_{i=1}^{P} (\hat{\sigma}_i-\sigma_i)(\partial g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i)$$

$$+ \tfrac{1}{2}[\sum_{i=1}^{P}(\hat{\sigma}_i-\sigma_i)^2(\partial^2 g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i^2)_{\sigma_i=\sigma_i^*}$$

$$+ 2\sum\sum_{1\le i<j\le p} (\hat{\sigma}_i-\sigma_i)(\hat{\sigma}_j-\sigma_j)(\partial^2 g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i\partial\sigma_j)_{\substack{\sigma_i=\sigma_i^*\\ \sigma_j=\sigma_j^*}}] \quad (3.11.5)$$

where $|\sigma_i^*-\sigma_i|<|\hat{\sigma}_i-\sigma_i|,\quad \forall\ i=1,\ldots,P$.

Write

$$A_1 = \sum_{i=1}^{P} (\hat{\sigma}_i - \sigma_i)(\partial g(\sigma_1, \ldots, \sigma_p)/\partial \sigma_i) \tag{3.11.6}$$

and write A_2 for the remaining terms in (3.11.5). Thus,

$$E[(\sum_{i=1}^{P} n_i \hat{\sigma}_i^{-1})^{-1} - (\sum_{i=1}^{P} n_i \sigma_i^{-1})^{-1}]^2 = E[A_1 + A_2]^2 \leq 2(EA_1^2 + EA_2^2) .$$

Next we obtain expressions for $E(\hat{\sigma}_i - \sigma_i)^2$, $(i = 1, \ldots, P)$ and

$E(\hat{\sigma}_i - \sigma_i)(\hat{\sigma}_j - \sigma_j)$, $(i \neq j = 1, \ldots, P)$,

$$\begin{aligned} E(\hat{\sigma}_i - \sigma_i)^2 &= E\{n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - Z^*) - \sigma_i\}^2 \\ &= \sum_{k=1}^{P} E\{[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - Z^*) - \sigma_i]^2 \, I_{[\underset{\sim}{X}_{i(1)} > \underset{\sim}{X}_{k(1)}, \ \forall\, i = 1, \ldots, P(\neq k)]}\} \end{aligned} \tag{3.11.7}$$

Using the independence of $\sum_{j=1}^{n_i} (X_{ij} - \underset{\sim}{X}_{k(1)})$, $(i = 1, \ldots, P)$ and

$(\underset{\sim}{X}_{i(1)}, \ i = 1, \ldots, P)$ one gets from (3.11.7),

$$E(\hat{\sigma}_i - \sigma_i)^2 = \sum_{k=1}^{P} E[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - \underset{\sim}{X}_{k(1)}) - \sigma_i]^2 \, P(\underset{\sim}{X}_{\delta(1)} > \underset{\sim}{X}_{k(1)} \quad \forall\, \delta(\neq k)). \tag{3.11.8}$$

In Section 2.10 of Chapter 2. we proved that for $k = i$,

$$\begin{aligned} E[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - \underset{\sim}{X}_{k(1)}) - \sigma_i]^2 &= E[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - \underset{\sim}{X}_{i(1)}) - \sigma_i]^2 \\ &= n_i^{-1} \sigma_i^2 \end{aligned} \tag{3.11.9}$$

Also, for $k \neq i$,

$$E[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - \underset{\sim}{X}_{k(1)}) - \sigma_i]^2 = E[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - \underset{\sim}{X}_{i(1)}) - \sigma_i + (\underset{\sim}{X}_{i(1)} - \underset{\sim}{X}_{k(1)})]^2$$

$$= n_i^{-1}\sigma_i^2 + 2n_k^{-2}\sigma_k^2 \qquad (3.11.10)$$

In Section 3.3, we have proved

$$P(\underset{\sim}{X}_{\delta(1)} > \underset{\sim}{X}_{k(1)} \quad \forall\, i = 1,\dots,P(\neq k)) = n_k\sigma_k^{-1}(\sum_{\alpha=1}^{P} n_\alpha\sigma_\alpha^{-1})^{-1} \qquad (3.11.11)$$

Thus, from (3.11.8) - (3.11.11), it follows that

$$E(\hat{\sigma}_i - \sigma_i)^2 = (n_i^{-1}\sigma_i^2)(n_i\sigma_i^{-1})(\sum_{\alpha=1}^{P} n_\alpha\sigma_\alpha^{-1})^{-1}$$

$$+ \sum_{\substack{k=1 \\ \neq i}}^{P} (n_i^{-1}\sigma_i^2 + 2n_k^{-2}\sigma_k^2)(n_k\sigma_k^{-1})(\sum_{\alpha=1}^{P} n_\alpha\sigma_\alpha^{-1})^{-1}$$

$$= n_i^{-1}\sigma_i^2 + 2(\sum_{\substack{k=1 \\ (\neq i)}}^{P} n_k^{-1}\sigma_k)(\sum_{\alpha=1}^{P} n_\alpha\sigma_\alpha^{-1})^{-1} \qquad (3.11.12)$$

Now consider

$$E(\hat{\sigma}_i - \sigma_i)(\hat{\sigma}_j - \sigma_j) = E[(n_i^{-1}\sum_{\alpha=1}^{n_i}(X_{i\alpha} - Z^*) - \sigma_i)(n_j^{-1}\sum_{\beta=1}^{n_j}(X_{j\beta} - Z^*) - \sigma_j)]$$

$$= \sum_{k=1}^{P} E[(n_i \sum_{\alpha=1}^{n_i}(X_{i\alpha} - \underset{\sim}{X}_{k(1)}) - \sigma_i)(n_j^{-1}\sum_{\beta=1}^{n_j}(X_{j\beta} - X_{k(1)}) - \sigma_j)]P(\underset{\sim}{X}_{\delta(1)} > \underset{\sim}{X}_{k(1)}$$

$$\forall\, \delta = 1,\dots,P(\neq k))$$

$$(3.11.13)$$

Using once again the fact that $\sum_{j=1}^{n_i}(X_{ij}-X_{\tilde{i}(1)})$ is distributed as the sum of $n_i - 1$ iid $\text{Gamma}(\sigma_i^{-1},1)$ variables say $Y_{i1}, Y_{i2}, \ldots, Y_{i\overline{n_i-1}}$, then we have the following cases.

(I) If $k = i$ and $k \neq j$, then,

$$E[(n_i^{-1}\sum_{\alpha=1}^{n_i}(X_{i\alpha}-X_{\tilde{k}(1)})-\sigma_i)(n_j^{-1}\sum_{\beta=1}^{n_j}(X_{j\beta}-X_{\tilde{k}(1)})-\sigma_j)]$$

$$= E[(n_i^{-1}\sum_{\alpha=1}^{n_i}(X_{i\alpha}-X_{\tilde{i}(1)})-\sigma_i)(n_j^{-1}\sum_{\beta=1}^{n_j}(X_{j\beta}-X_{\tilde{j}(1)})-\sigma_j + (X_{\tilde{j}(1)}-X_{\tilde{i}(1)}))]$$

$$= E[(n_i^{-1}\sum_{\alpha=1}^{n_i-1}(Y_{i\alpha}-\sigma_i)-n_i^{-1}\sigma_i)(n_j^{-1}\sum_{\beta=1}^{n_j-1}(Y_{j\beta}-\sigma_j)-n_j^{-1}\sigma_j)+(X_{\tilde{j}(1)}-X_{\tilde{i}(1)}))]$$

$$= E\{[n_i^{-1}\sum_{\alpha=1}^{n_i-1}(Y_{i\alpha}-\sigma_i)-n_i^{-1}\sigma_i][n_j^{-1}\sum_{\beta=1}^{n_j-1}(Y_{j\beta}-\sigma_j)+(X_{\tilde{j}(1)}-\mu-n_j^{-1}\sigma_j) - (X_{\tilde{i}(1)}-\mu-n_i^{-1}\sigma_i)-n_i^{-1}\sigma_i]\}$$

$$= n_i^{-2}\sigma_i^2 \tag{3.11.14}$$

(II) If $k = j$ and $k \neq i$, then,

$$E[(n_i^{-1}\sum_{\alpha=1}^{n_i}(X_{i\alpha}-X_{\tilde{k}(1)})-\sigma_i)(n_j^{-1}\sum_{\beta=1}^{n_i}(X_{j\beta}-X_{\tilde{k}(1)})-\sigma_j)]$$

$$= E\{[n_i^{-1}\sum_{\alpha=1}^{n_i}(X_{i\alpha}-X_{\tilde{i}(1)})-\sigma_i+(X_{\tilde{i}(1)}-X_{\tilde{j}(1)})][n_j^{-1}\sum_{\beta=1}^{n_j-1}(X_{j\beta}-X_{\tilde{j}(1)})-\sigma_j]\}$$

$$= n_j^{-2}\sigma_j^2 \tag{3.11.15}$$

(III) If $k \neq i$ and $k \neq j$, then,

$$E\{(n_i^{-1}\sum_{\alpha=1}^{n_i}(X_{i\alpha}-\underset{\sim}{X}_{k(1)})-\sigma_i)(n_j^{-1}\sum_{\beta=1}^{n_j}(X_{j\beta}-\underset{\sim}{X}_{k(1)})-\sigma_j)\}$$

$$= E\{[n_i^{-1}\sum_{\alpha=1}^{n_i}(X_{i\alpha}-\underset{\sim}{X}_{i(1)})-\sigma_i+(\underset{\sim}{X}_{i(1)}-\underset{\sim}{X}_{k(1)})][n_j^{-1}\sum_{\beta=1}^{n_j}(X_{j\beta}-\underset{\sim}{X}_{j(1)})-\sigma_j + (\underset{\sim}{X}_{j(1)}-\underset{\sim}{X}_{k(1)})]\}$$

$$= E\{[n_i^{-1}\sum_{\alpha=1}^{n_i-1}(Y_{i\alpha}-\sigma_i)+(\underset{\sim}{X}_{i(1)}-\mu-n_i^{-1}\sigma_i)-(\underset{\sim}{X}_{k(1)}-\mu-n_k^{-1}\sigma_k)-n_k^{-1}\sigma_k]\cdot$$
$$[n_j^{-1}\sum_{\beta=1}^{n_j-1}(Y_{j\beta}-\sigma_j)+(\underset{\sim}{X}_{j(1)}-\mu-n_j^{-1}\sigma_j)-(\underset{\sim}{X}_{k(1)}-\mu-n_k^{-1}\sigma_k)-n_k^{-1}\sigma_k]\}$$

$$= n_k^{-2}\sigma_k^2 + n_k^{-2}\sigma_k^2 = 2n_k^{-2}\sigma_k^2 \tag{3.11.16}$$

Thus, from (3.11.14) - (3.11.16), it follows that

$$E(\hat{\sigma}_i-\sigma_i)(\hat{\sigma}_j-\sigma_j) = n_i^{-2}\sigma_i^2[n_i\sigma_i^{-1}(\sum_{\alpha=1}^{P} n_\alpha\sigma_\alpha^{-1})^{-1}]+n_j^{-2}\sigma_j^2[n_j\sigma_j^{-1}(\sum_{\alpha=1}^{P} n_\alpha\sigma_\alpha^{-1})^{-1}]$$
$$+ 2\sum_{\substack{k\neq i\\ \neq j}}^{P} n_k^{-2}\sigma_k^2[n_k\sigma_k^{-1}(\sum_{\alpha=1}^{P} n_\alpha\sigma_\alpha^{-1})^{-1}]$$

$$= (\sum_{\alpha=1}^{P} n_\alpha\sigma_\alpha^{-1})^{-1}[n_i^{-1}\sigma_i+n_j^{-1}\sigma_j+2\sum_{\substack{k\neq i\\ \neq j}}^{P} n_k^{-1}\sigma_k] \tag{3.11.17}$$

Next observe again that $\partial g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i = n_i\sigma_i^{-2}(\sum_{\alpha=1}^{P} n_\alpha\sigma_\alpha^{-1})^{-2}$ for all $\alpha = 1, \ldots, P$. So by using Minkowski's Inequality,

$$E(A_1^2) \leq 2^{P-1}\, E[\sum_{i=1}^{P}(\hat{\sigma}_i-\sigma_i)^2(\partial g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i)^2]$$

$$= 2^{P-1} \sum_{i=1}^{P}\{[n_i^{-1}\sigma_i^2+2(\sum_{\substack{k=1\\ \neq i}}^{P} n_k^{-1}\sigma_k)(\sum_{\alpha=1}^{P} n_\alpha\sigma_\alpha^{-1})^{-1}][n_i^2\sigma_i^{-4}(\sum_{\alpha=1}^{P} n_\alpha\sigma_\alpha^{-1})^{-4}]$$

$$= O(n^{-3}) \tag{3.11.18}$$

Also under assumption (3.11.3),

$$E(A_2^2) \leq 2^{\frac{1}{2}P(P+1)-2}[\sum_{i=1}^{P} E(\hat{\sigma}_i-\sigma_i)^4(\partial^2 g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i^2)_{\sigma_i=\sigma_i^*}$$

$$+ 2 \sum_{1\leq i<j\leq p}\sum E(\hat{\sigma}_i-\sigma_i)^2(\hat{\sigma}_j-\sigma_j)^2(\partial^2 g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i\partial\sigma_j)^2_{\substack{\sigma_i=\sigma_i^*\\ \sigma_j=\sigma_j^*}}]$$

Note that

$$\partial^2 g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i^2 = [-2\sum_{\substack{k=1\\ \neq i}}^{P} n_i n_k\sigma_i^{-3}\sigma_k^{-1}][\sum_{j=1}^{P} n_j\sigma_j^{-1}]^{-3} \tag{3.11.19}$$

Hence,

$$E[(\hat{\sigma}_i-\sigma_i)^4(\partial^2 g/\partial\sigma_i^2)_{\sigma_i=\sigma_i^*}, \quad i = 1,\ldots,P]$$

$$= 4\sum_{\substack{k=1\\ \neq i}}^{P} n_i^2 n_k^2\, E[(\hat{\sigma}_i-\sigma_i)^4\ \sigma_i^{*-6}\sigma_k^{*-2}(\sum_{j=1}^{P} n_j\sigma_j^{*-1})^{-6}]$$

$$= 4\sum_{\substack{k=1\\ \neq i}}^{P} n_i^2 n_k^2\, E[(\hat{\sigma}_i-\sigma_i)^4\sigma_k^{*-2}(n_i+\sum_{\substack{j=1\\ \neq j}}^{P} n_j\sigma_i^*\sigma_j^{*-1})^{-6}]$$

$$\leq 4\sum_{\substack{k=1\\ \neq i}}^{P} n_i^{-4} n_k^2\, E[(\hat{\sigma}_i-\sigma_i)^4\sigma_k^{*-2}]$$

$$\leq 4 \sum_{\substack{k=1 \\ \neq i}}^{P} n_i^{-4} n_k^2 \{E(\hat{\sigma}_i - \sigma_i)^8 \cdot E(\sigma_k^*)^{-4}\}^{\frac{1}{2}} \tag{3.11.20}$$

Note that

$$E(\hat{\sigma}_i - \sigma_i)^8 = E[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - Z^*) - \sigma_i]^8$$

$$= E\{\sum_{k=1}^{P} [n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - \underset{\sim}{X}_{k(1)}) - \sigma_i] \, I_{[\underset{\sim}{X}_{\delta(1)} > \underset{\sim}{X}_{k(1)} \ \forall \delta = 1, \ldots, P(\neq k)]}\}^8$$

$$\leq 2^{7(P-1)} \sum_{k=1}^{P} E\{[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - \underset{\sim}{X}_{k(1)}) - \sigma_i]^8 \, I_{[\underset{\sim}{X}_{\delta(1)} > \underset{\sim}{X}_{k(1)} \ \forall \delta = 1, \ldots, P(\neq k)]}\}$$

$$\leq 2^{7(P-1)} \sum_{k=1}^{P} \{E[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - \underset{\sim}{X}_{k(1)}) - \sigma_i]^8$$

$$= 2^{7(P-1)} \{E[(n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - \underset{\sim}{X}_{i(1)}) - \sigma_i]^8 + \sum_{\substack{k=1 \\ \neq i}}^{P} E[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - \underset{\sim}{X}_{i(1)}) + (\underset{\sim}{X}_{i(1)} - \underset{\sim}{X}_{k(1)}) - \sigma_i]^8 \tag{3.11.21}$$

Now, using the results of Section 2.10 of Chapter II and (3.11.21),

$$E(\hat{\sigma}_i - \sigma_i)^8 = O(n^{-4}) \tag{3.11.22}$$

Using once again section 2.10 of Chapter II, it follows that

$$E(\sigma_k^*)^{-4} = O(1).$$

Thus, combining (3.11.20) - (3.11.22), it follows that

$$E[(\hat{\sigma}_i-\sigma_i)^4(\partial^2 g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i^2)_{\sigma_i=\sigma_i^*},\quad i=1,\ldots,p] = O(n^{-4}\cdot n^2\cdot n^{-2})$$

$$= O(n^{-4}) \quad \text{for all} \quad i=1,2,\ldots,P$$

(3.11.23)

Finally, since

$$\partial^2 g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i\partial\sigma_j = 2n_i n_j \sigma_i^{-2}\sigma_j^{-2}\left(\sum_{\delta=1}^{P} n_\delta \sigma_\delta^{-1}\right)^{-3}$$

it follows that

$$E\left[(\hat{\sigma}_i-\sigma_i)^2(\hat{\sigma}_j-\sigma_j)^2 \cdot 4n_i^2 n_j^2 \sigma_i^{-4}\sigma_j^{-4}\left(\sum_{\delta=1}^{P} n_\delta\sigma_\delta^{-1}\right)^{-6}\right]_{\substack{\sigma_i=\sigma_i^*\\ \sigma_j=\sigma_j^*}}$$

$$\leq 4n_i^{-4}n_j^2\, E[(\hat{\sigma}_i-\sigma_i)^2(\hat{\sigma}_j-\sigma_j)^2\sigma_i^{*2}\sigma_j^{*-4}] \quad \text{for all} \quad i \neq j = 1,\ldots,P$$

(3.11.24)

From (2.25) and (2.26), it follows after repeated application of Schwarz's inequality that

$$E[(\hat{\sigma}_i-\sigma_i)^2(\hat{\sigma}_j-\sigma_j)^2 \cdot \partial^2 g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i\partial\sigma_j]_{\substack{\sigma_i=\sigma_i^*\\ \sigma_j=\sigma_j^*}} = O(n^{-4})$$

for all paired $\{(i,j),\ i \neq j = 1,\ldots,p\}$ (3.11.25)

Thus, from (3.11.19) and (3.11.25), it follows that

$$E(A_2^2) = O(n^{-4}) \qquad (3.11.26)$$

Combining (3.11.18) and (3.11.26), the result follows.

<u>Remark 3.11.1</u> In view of (3.11.1) and (3.11.2), it follows that

$[\mathrm{MSE}(\hat{\mu}_{PMLE}) - \mathrm{MSE}(\hat{\mu}_{PMLE*})]/[\mathrm{MSE}(\hat{\mu}_{PMLE})] =$

$$\tfrac{1}{2} - \tfrac{1}{2} \left(\sum_{i=1}^{P} n_i \sigma_i^{-1} \right)^2 E\left[\left(\sum_{i=1}^{P} n_i \hat{\sigma}_i^{-1} \right)^{-1} - \left(\sum_{i=1}^{P} n_i \sigma_i^{-1} \right)^{-1} \right]^2$$

Now, in view of Theorem 3.11.1, it appears that for large n_i, use of modified maximum likelihood estimator for common location parameter in P-populations, can result in approximately 50% relative efficiency in terms of the mean squared error criterion.

We next compare $\hat{\mu}_{PMLE}$ and $\hat{\mu}_{MLE*}$ in terms of their biases.

First note that,

$$\mathrm{Bias}(\hat{\mu}_{PMLE}) = E(Z^* - \mu) = \mu + \left(\sum_{i=1}^{P} n_i \sigma_i^{-1} \right)^{-1} - \mu$$

$$= \left(\sum_{i=1}^{P} n_i \sigma_i^{-1} \right)^{-1} \tag{3.11.27}$$

And,

$$\mathrm{Bias}(\hat{\mu}_{MLE*}) = E\{ [Z^* - \left(\sum_{i=1}^{P} n_i \sigma_i^{-1} \right)^{-1}] - \mu \}$$

$$= \mu + (\Sigma n_i \sigma_i^{-1})^{-1} - E\left(\sum_{i=1}^{P} n_i \hat{\sigma}_i^{-1} \right)^{-1} - \mu$$

$$= \left(\sum_{i=1}^{P} n_i \sigma_i^{-1} \right)^{-1} - E\left(\sum_{i=1}^{P} n_i \hat{\sigma}_i^{-1} \right)^{-1} \tag{3.11.28}$$

<u>Lemma 3.11.1</u> Under the assumption (3.11.3)

$$E\left[\left(\sum_{i=1}^{P} n_i \hat{\sigma}_i^{-1} \right)^{-1} - \left(\sum_{i=1}^{P} n_i \sigma_i^{-1} \right)^{-1} \right] = o(n^{-2}) \tag{3.11.29}$$

<u>Proof</u> Once again considering $g(\sigma_1, \ldots, \sigma_p) = \left(\sum_{i=1}^{P} n_i \sigma_i^{-1} \right)^{-1}$ and using a Taylor expansion by (3.12.5), it follows that

$$E(A_1) = \sum_{i=1}^{P} (\partial g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i) \; \{E(\hat{\sigma}_i-\sigma_i)\} \tag{3.11.30}$$

and,

$$E(A_2) = \tfrac{1}{2}\{ \sum_{i=1}^{P} E[(\partial^2 g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i^2)_{\sigma_i=\sigma_i^*} \; (\hat{\sigma}_i-\sigma_i)^2]\}$$

$$+ 2 \sum_{1\le i\le j\le P}\sum E[(\partial^2 g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i\partial\sigma_j)_{\substack{\sigma_i=\sigma_i^* \\ \sigma_j=\sigma_j^*}} \; (\hat{\sigma}_i-\sigma_i)(\hat{\sigma}_j-\sigma_j)]\} \tag{3.11.31}$$

Now,

$$E(\hat{\sigma}_i-\sigma_i) = E\{n_i^{-1} \sum_{j=1}^{n_i} (X_{ij}-Z^*) -\sigma_i\}$$

$$= \sum_{k=1}^{P} E\{[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij}-Z^*)-\sigma_i] \; I_{[\underset{\sim}{X}_{\delta(1)}>\underset{\sim}{X}_{k(1)},\ \forall\ \delta = 1,\ldots,p(\neq k)]}\} \tag{3.11.32}$$

Using the independence of $\sum_{j=1}^{n_i} (X_{ij}-\underset{\sim}{X}_{k(1)})$, $(i = 1,\ldots,P)$ and $(\underset{\sim}{X}_{i(1)},\ i = 1,\ldots,p)$, then,

$$E(\hat{\sigma}_i-\sigma_i) = \sum_{k=1}^{P} E[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij}-\underset{\sim}{X}_{k(1)})-\sigma_i] \; P(\underset{\sim}{X}_{\delta(1)}>\underset{\sim}{X}_{k(1)} \ \forall\ i(\neq k)) \tag{3.11.33}$$

Now, it is easy to prove that

$$E[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij}-\underset{\sim}{X}_{k(1)})-\sigma_i] = -n_i^{-1}\sigma_i \qquad \text{if} \quad k = i$$

$$= -n_k^{-1}\sigma_k \qquad \text{if} \quad k \neq i \tag{3.11.34}$$

Thus, from (3.11.32) - (3.11.34) and (3.11.11), it follows that

$$E(\hat{\sigma}_i-\sigma_i) = (-n_i^{-1}\sigma_i)(n_i\sigma_i^{-1})(\sum_{\alpha=1}^{P} n_\alpha\sigma_\alpha^{-1})^{-1}$$

$$- \sum_{k=1(\neq i)}^{P} (n_k^{-1}\sigma_k)(n_k\sigma_k^{-1})(\sum_{\alpha=1}^{P} n_\alpha\sigma_\alpha^{-1})^{-1}$$

$$= -P(\sum_{\alpha=1}^{P} n_\alpha\sigma_\alpha^{-1})^{-1} \qquad (3.11.35)$$

From (3.11.35), it follows that

$$E(A_1) = \sum_{i=1}^{P} [n_i\sigma_i^{-2}(\sum_{i=1}^{P} n_i\sigma_i^{-1})^{-2}][-P(\sum_{\alpha=1}^{P} n_\alpha\sigma_\alpha^{-1})^{-1}]$$

$$= -P(\sum_{i=1}^{P} n_i\sigma_i^{-2})(\sum_{i=1}^{P} n_i\sigma_i^{-1})^{-3}$$

$$= O(n^{-2}) \qquad (3.11.36)$$

And,

$$E\{|(\hat{\sigma}_i-\sigma_i)^2(\partial^2 g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i^2)_{\sigma_i=\sigma_i^*}|\}$$

$$= E\{|(\hat{\sigma}_i-\sigma_i)^2[-2(\sum_{\substack{k=1\\ \neq i}}^{P} n_i n_k \sigma_i^{*-3}\sigma_k^{*-1})(\sum_{j=1}^{P} n_j\sigma_j^{*-1})^{-3}]|\}$$

$$= 2\sum_{\substack{k=1\\ \neq i}}^{P} E\{(\hat{\sigma}_i-\sigma_i)^2[n_i n_k\sigma_k^{*-1}(n_i+\sum_{\substack{j=1\\ \neq i}}^{P}(n_j\sigma_i^*\sigma_j^{*-1})^{-3}]\}$$

$$\leq 2\sum_{\substack{k=1\\ \neq i}}^{P} E\{(\hat{\sigma}_i-\sigma_i)^2(n_i^{-2}n_k\sigma_k^{*-1})\} .$$

$$\leq 2 \sum_{\substack{k=1\\ \neq i}}^{P} n_i^{-2} n_k \{E(\hat{\sigma}_i - \sigma_i)^4 \; E(\sigma_k^*)^{-2}\}^{\frac{1}{2}} \tag{3.11.37}$$

Now,

$$E(\hat{\sigma}_i - \sigma_i)^4 = E[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - Z^*) - \sigma_i]^4$$

$$= E\{\sum_{k=1}^{P} [n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - \underset{\sim}{X}_{k(1)}) - \sigma_i] \; I_{[\underset{\sim}{X}_{\delta(1)} > \underset{\sim}{X}_{k(1)}} \quad \forall \; \delta = 1, \ldots, P(\neq k)\}$$

$$\leq 2^{3(P-1)} \sum_{k=1}^{P} E[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - \underset{\sim}{X}_{k(1)}) - \sigma_i]^4$$

$$= 2^{3(P-1)} \{E[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - \underset{\sim}{X}_{i(1)}) - \sigma_i]^4 + \sum_{\substack{k=1\\ \neq i}}^{P} E[n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - \underset{\sim}{X}_{i(1)}) + (\underset{\sim}{X}_{i(1)} - \underset{\sim}{X}_{k(1)}) - \sigma_i]^4\} \tag{3.11.38}$$

Now, using the results of Section 2.10 of Chapter II and (3.11.37) - (3.11.38), it follows that

$$E(\hat{\sigma}_i - \sigma_i)^4 = O(n^{-2}) \tag{3.11.39}$$

And,

$$E(\sigma_k^*)^{-2} = O(1) \tag{3.11.40}$$

Now, combining (3.11.39) and (3.11.40), it follows that

$$E\{(\hat{\sigma}_i - \sigma_i)^2 (\partial^2 g(\sigma_1, \ldots, \sigma_p)/\partial\sigma_i^2)_{\sigma_i = \sigma_i^*}\} = O(n^{-2}), \quad i = 1, \ldots, P \tag{3.11.41}$$

Finally,

$$E[(\hat{\sigma}_i-\sigma_i)(\hat{\sigma}_j-\sigma_j)(\partial^2 g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i\partial\sigma_j)_{\substack{\sigma_i=\sigma_i^* \\ \sigma_j=\sigma_j^*}}]$$

$$= E\{(\hat{\sigma}_i-\sigma_i)(\hat{\sigma}_j-\sigma_j)[2n_i n_j \sigma_i^{*-2}\sigma_j^{*-2}(\sum_{\delta=1}^{P} n_\delta \sigma_\delta^{-1})^{-3}]\}$$

$$\leq 2n_i^{-2}n_j\ E[(\hat{\sigma}_i-\sigma_i)(\hat{\sigma}_j-\sigma_j)\sigma_i^*\sigma_j^{*-2}]$$

$$\leq 2n_i^{-2}n_j\ \{E(\hat{\sigma}_i-\sigma_i)^4\}^{1/4}\{E(\hat{\sigma}_j-\sigma_j)^4\}^{1/4}\{E(\sigma_i^*)^4\}^{1/4}\cdot\{E(\sigma_j^*)^{-8}\}^{1/4} \qquad (3.11.42)$$

From (3.11.39), (2.10.37) and (2.10.38), it follows that

$$E\{(\hat{\sigma}_i-\sigma_i)(\hat{\sigma}_j-\sigma_j)(\partial^2 g(\sigma_1,\ldots,\sigma_p)/\partial\sigma_i\partial\sigma_j)_{\substack{\sigma_i=\sigma_i^* \\ \sigma_j=\sigma_j^*}}\},$$

$$= O(n^{-2}), \quad (i \neq j = 1,2,\ldots,P) \qquad (3.11.43)$$

Thus, (3.11.41) and (3.11.43) implies that

$$E(A_2) = O(n^{-2}) \qquad (3.11.44)$$

Also, from (3.11.36) and (3.11.44), it follows that

$$E[\sum_{i=1}^{P} n_i\hat{\sigma}_i^{-1})^{-1} - (\sum_{i=1}^{P} n_i\sigma_i^{-1})^{-1}] = E(A_1+A_2) = O(n^{-2}) \qquad (3.11.45)$$

Thus, it follows that for large n_i, $(\forall\ i = 1,\ldots,P)$, use of the modified MLE can result approximately 100% relative efficiency in terms of bias criterion.

4. ESTIMATING THE LOCATION PARAMETER OF AN EXPONENTIAL DISTRIBUTION WITH KNOWN COEFFICIENT OF VARIATION

4.1. The Best Linear Estimator of θ In Class of Unbiased Estimators

Let $X_1, \ldots, X_n$ of fixed size $n \geq 2$ be iid with common pdf

$$f_\theta(x) = (a\theta)^{-1} \exp\{-(x-\theta)/(a\theta)\}, \qquad X \geq \theta$$

where $\theta(>0)$ is unknown, but $a(>0)$ is known. Note that in this case $E_\theta(X_1) = \theta + a\theta = \theta(a+1)$ and $V_\theta(X_1) = a^2\theta^2$. Accordingly, the coefficient of variation turns out to be $[V_\theta(X_1)]^{\frac{1}{2}}/E_\theta(X_1) = a/(a+1)$ which is the same for all $\theta(>0)$.

The joint pdf of $X_1, \ldots, X_n$ can be written as

$$p_\theta(x_1,\ldots,x_n) = (a\theta)^{-n} \exp\{-\Sigma_{i=1}^n (x_i-\theta)/(a\theta)\}\, I_{[\min_{1\leq i\leq n} x_i \geq \theta]}, \qquad (4.1.1)$$

where I denotes the usual indicator function. Writing (4.1.1) in the following form

$$p_\theta(x_1,\ldots,x_n) = (a\theta)^{-n} \exp\{-\sum_{i=1}^{n}(x_i-x_{(1)})/(a\theta)-n(x_{(1)}-\theta)/(a\theta)\}$$
$$\cdot\, I_{[x_{(1)}\geq\theta]} \qquad (4.1.2)$$

where $x_{(1)} = \min_{1\leq i\leq n} x_i$. Then, it is easy to see that (T_1, T_2) is minimal sufficient for θ. However, we will prove that the family of distributions induced by the minimal sufficient statistic is not complete.

It is well-known that $T_1 = X_{(1)}$ and $T_2 = \Sigma_{i=1}^{n}(X_i - X_{(1)})$ are independently distributed, and T_1 and T_2 have respective pdf's

$$f_\theta(t_1) = (n/(a\theta)) \exp\{-n(t_1-\theta)/(a\theta)\} I_{[t_1 \geq \theta]}, \tag{4.1.3}$$

and

$$f_\theta(t_2) = \exp\{-t_2/(a\theta)\} t_2^{n-2}/\{(a\theta)^{n-1}\Gamma(n-1)\} . \tag{4.1.4}$$

Thus we find that T_2 has a Gamma $((a\theta)^{-1}, n-1)$ distribution. Simple computations yield

$$E_\theta(T_1) = \theta + a\theta n^{-1} = (1+an^{-1})\theta \ ; \quad V_\theta(T_1) = (a\theta)^2/n^2 \tag{4.1.5}$$

and

$$E_\theta(T_2) = (n-1)a\theta \ ; \quad V_\theta(T_2) = (n-1)(a\theta)^2 . \tag{4.1.6}$$

Since, $E_\theta(T_1/(1+an^{-1}) - T_2/(a(n-1)) = 0$ for all $\theta > 0$, it follows that the family of distributions induced by (T_1, T_2) is not complete. Also, $T_1^* = T_1/(1+an^{-1})$ and $T_2^* = T_2/\{a(n-1)\}$ are both unbiased estimators of θ with respective variances $(1+an^{-1})^{-2}\{(a\theta)^2/n^2\} = a^2\theta^2(n+a)^{-2}$ and $\theta^2/(n-1)$. Hence, the MVUE of θ in class of all unbiased estimators of the form $CT_1^* + (1-C)T_2^*$ is given by

$$T = \frac{V_\theta(T_2^*)}{V_\theta(T_1^*)+V_\theta(T_2^*)} T_1^* + \frac{V_\theta(T_1^*)}{V_\theta(T_1^*)+V_\theta(T_2^*)} T_2^*$$

$$= (a^{-2}(n+a)^2(1+an^{-1})^{-1} + a^{-1}T_2)/\{a^{-2}(n+a)^2 + n-1\}$$

$$= \{n(n+a)T_1 + aT_2\}/n(n+2a+a^2) \tag{4.1.7}$$

with

$$V_\theta(T) = \{(n+a)^2(a\theta)^2 + a^2(n-1)(a\theta)^2\}/\{n^2(n+2a+a^2)^2\}$$

$$= a^2\theta^2/\{n(n+2a+a^2)\} \qquad (4.1.8)$$

4.2. Linear Minimum Mean Square Estimation

Next we find the smallest MSE estimator of θ in the class of all (not necessarily unbiased) estimators of the form $d_1T_1 + d_2T_2$. The following lemma (see Gleser and Healy, JASA (1976) for a proof) is needed.

<u>Lemma 1</u> Suppose T_1' and T_2' are two uncorrelated unbiased estimators of θ with $V_\theta(T_i')/\theta^2 = v_i (i = 1,2)$ free from θ. Then the smallest MSE estimator of θ of the form $d_1T_1' + d_2T_2'$ is given by

$$T' = (v_2T_1'+v_1T_2')/(v_1+v_2+v_1v_2) = (v_1^{-1}T_1'+v_2^{-1}T_2')/(v_1^{-1}+v_2^{-1}+1). \qquad (4.2.1)$$

It is also easy to show that the MSE of T' in estimating θ is given by

$$E_\theta(T'-\theta)^2 = \theta^2 v_1v_2/(v_1+v_2+v_1v_2) = \theta^2/(v_1^{-1}+v_2^{-1}+1) . \qquad (4.2.2)$$

In the present situation, $T_1' = T_1/(1+an^{-1})$, $T_2' = T_2/\{a(n-1)\}$.

Also, $v_1 = V_\theta(T_1')/\theta^2 = a^2(n+a)^{-2}$, $v_2 = V_\theta(T_2')/\theta^2 = (n-1)^{-1}$. Accordingly, using (4.2.1) and (4.2.2), under squared error loss, the smallest MSE estimator of θ of the form $d_1T_1' + d_2T_2'$ is given by

$$T' = \{a^{-2}(n+a)^2 T_1' + (n-1)T_2'\}/\{a^{-2}(n+a)^2+n-1+1\}$$

$$= \{n(n+a)T_1+aT_2\}/\{(n+a)^2+na^2\} , \quad (4.2.3)$$

with corresponding MSE

$$E_\theta(T'-\theta)^2 = \theta^2/\{a^{-2}(n+a)^2+n-1+1\}$$

$$= a^2\theta^2/\{(n+a)^2+na^2\} . \quad (4.2.4)$$

Once again, using (4.2.1), it follows that T' is a biased estimator of θ with

$$E_\theta(T'-\theta) = -v_1v_2\ \theta/(v_1+v_2+v_1v_2) = -\theta/(v_1^{-1}+v_2^{-1}+1) \quad (4.2.5)$$

Applying this formula, in the present case

$$E_\theta(T'-\theta) = -a^2\theta/\{(n+a)^2+na^2\} \quad (4.2.6)$$

Thus, although the percentage reduction in MSE by the use of T' in place of T is $100\{(MSE(T)-MSE(T'))/MSE(T)\}\% = 100\{a^2/[(n+a)^2+na^2]\}\%$, the magnitude of bias by the use of T' instead of the unbiased estimator T is also $100[a^2/\{(n+a)^2+na^2\}]\%$.

4.3. The Maximum Likelihood Estimator of θ

Next we find the MLE of θ with this end, first write the likelihood function as

$$L(\theta) = (a\theta)^{-n} \exp\{-\Sigma_{i=1}^n(x_i-\theta)/(a\theta)\}\ I_{[\min_{1\le i\le n} x_i \ge \theta]}$$

$$= (a\theta)^{-n} \exp\{-(a\theta)^{-1}\ \Sigma_{i=1}^n x_i+na^{-1}\}\ I_{[\min_{1\le i\le n} x_i\ge\theta]} \quad (4.3.1)$$

Write

$$g(\theta) = -n \log(a\theta) - (a\theta)^{-1}\Sigma_{i=1}^{n}x_i \ , \qquad (4.3.2)$$

so that $g'(\theta) = (\Sigma_{i=1}^{n}x_i - na\theta)/(a\theta^2) \gtrless 0$ according to $\theta \lessgtr a^{-1}n^{-1}\Sigma_{i=1}^{n}x_i$.

From (4.3.1) and (4.3.2), it follows that the MLE of θ is

$$T^* = \min(T_1, a^{-1}(T_1+n^{-1}T_2)) \ . \qquad (4.3.3)$$

In particular, if $0 < a \leq 1$, $T^* = T_1$. From previous calculations, one finds that

$$E_\theta(T_1-\theta) = a\theta/n \ ; \quad E_\theta(T_1-\theta)^2 = 2a^2\theta^2/n^2 \qquad (4.3.4)$$

Since, $an^{-1} - 4a^2\{(n+a)^2+na^2\}^{-1} = a[(n-2a)^2+na^2]\{n[(n+a)^2+na^2]\}^{-1} > 0$ for all $a > 0$, it follows that

$$\frac{|E_\theta(T'-\theta)|}{|E_\theta(T^*-\theta)|} < \frac{1}{4} \ . \qquad (4.3.5)$$

Thus, there is at least 75% reduction in the magnitude of bias by the use of T' instead of T_1. Interestingly, though

$$T_1 - T' = T_1 - \{[n(n+a)T_1+aT_2]/[(n+a)^2+na^2]\}$$

$$= \frac{a(n+a+na)}{(n+a)^2+na^2}T_1 - \frac{a}{(n+a)^2+na^2}T_2 \qquad (4.3.6)$$

It follows from (4.3.4) that $T_1 \xrightarrow{P} \theta$ as $n \to \infty$. Also, since T_2 can be expressed as the sum of $n-1$ iid Gamma $((a\theta)^{-1}, 1)$ variables (see e.g. David (1981, page 20)), it follows that $\{T_2/(n-1)\} \xrightarrow{a.s} a\theta$ as $n \to \infty$. Hence $T_1 - T' \xrightarrow{a.s} 0$ as $n \to \infty$. Also, we will show in below that $E(T_1-T')^2 \to 0$ as $n \to \infty$. From (4.3.6), it follows that

$$E(T_1-T')^2 = \frac{\{a^2(n+a+na)^2\}\{a^2(n+a)^2+n^4\}\theta^2}{[(n+a)^2+na^2]^2[n^2(n+a)^2]} + \frac{a^4n\cdot(n-1)\theta^2}{[(n+a)^2+na^2]^2}$$

$$- \frac{a^3n(n+a+na)(n-1)\theta^2}{(n+a)[(n+a)^2+na^2]^2} \qquad (4.3.7)$$

Hence, $E(T_1-T')^2 \to 0$ as $n \to \infty$. Thus, although, T_1 and T' do not differ asymptotically, T' is in general preferred to T_1 in terms of its superior bias and MSE performance. Also, from (4.3.6),

$$n(T_1-T') = \frac{an(n+a+na)}{(n+a)^2+na^2} T_1 - \frac{an}{(n+a)^2+na^2} T_2 \xrightarrow{a.s.} (a+1)a\theta - a^2\theta = a\theta .$$

Hence, $n(T'-\theta) \xrightarrow{L} Z - a\theta$, where Z is distributed as $\text{Gamma}((a\theta)^{-1},1)$. When $a > 1$, the distribution of T^*, though obtainable in a closed form, is quite involved. We compare below the performance of T^* and T' in terms of the bias and the MSE. First note that

$$E_\theta(T^*-\theta) = E_\theta\{(T_1-\theta)\ I_{[T_1\le a^{-1}(T_1+n^{-1}T_2)]}$$

$$+ [a^{-1}(T_1+n^{-1}T_2)-\theta]\ I_{[T_1>a^{-1}(T_1+n^{-1}T_2)]}\}$$

$$= E_\theta\{(T_1-\theta)\ I_{[T_1\le(a-1)^{-1}n^{-1}T_2]}$$

$$+ a^{-1}(T_1+n^{-1}T_2-a\theta)\ I_{[T_1>(a-1)^{-1}n^{-1}T_2]}\}$$

$$= \theta E_1\{(T_1-1)\, I_{[T_1 \le (a-1)^{-1} n^{-1} T_2]}$$

$$+ a^{-1}(T_1-1+n^{-1}T_2-(a-1))\, I_{[T_1 > (a-1)^{-1} n^{-1} T_2]}\} \qquad (4.3.8)$$

Now,

$$E_1[(T_1-1)\, I_{[T_1 \le (a-1)^{-1} n^{-1} T_2]}]$$

$$= \int_{n(a-1)}^{\infty} \{ \int_1^{\frac{t_2}{n(a-1)}} (t_1-1)\, \frac{n}{a} \exp(-\frac{n(t_1-1)}{a})dt_1\} \exp(-\frac{t_2}{a}) t_2^{n-2} \,/$$

$$\{a^{n-1}\, (n-1)\}dt_2 \,. \qquad (4.3.9)$$

The inner integral of (4.3.9) is

$$= \frac{a}{n} \int_0^{\frac{t_2}{a(a-1)} - \frac{n}{a}} z \exp(-z)\, dz$$

$$= \frac{a}{n} [1-\{1+(\frac{t_2}{a(a-1)} - \frac{n}{a})\} \exp(-\frac{t_2}{a(a-1)} + \frac{n}{a})] \qquad (4.3.10)$$

From (4.3.9) and (4.3.10) one gets

$$E_1[(T_1-1)\, I_{[T_1 \le (a-1)^{-1} n^{-1} T_2]}]$$

$$= \frac{a}{n} [\int_{n(a-1)}^{\infty} \exp(-t_{2/a}) t_2^{n-2}\, a^{-(n-1)}/\Gamma(n-1)\, dt_2$$

$$- \int_{n(a-1)}^{\infty} \exp(-\frac{t_2}{a-1} + \frac{n}{a})\, \{1+ \frac{1}{a}\, (\frac{t_2}{a-1} - n)\}\, \frac{t_2^{n-2}}{a^{n-1}\Gamma(n-1)}\, dt_2]$$

$$= \frac{a}{n} [\int_{n(a-1)/a}^{\infty} \exp(-t_2) t_2^{n-2}/\Gamma(n-1) dt_2$$

$$- \exp(\frac{n}{a})(\frac{a-1}{a})^{n-1} \int_n^{\infty} [1+ \frac{z-n}{a}] \exp(-z)\, z^{n-2}/\Gamma(n-1) dz] \qquad (4.3.11)$$

Now,

$$\int_n^\infty \exp(-z)\ z^{n-1}/\Gamma(n-1)dz$$

$$= \exp(-n)\ n^{n-1}/\Gamma(n-1) + (n-1)\int_n^\infty \exp(-z)\ z^{n-2}/\Gamma(n-1)dz \qquad (4.3.12)$$

Combining (4.3.11) and (4.3.12), it follows that

$$E_1[(T_1-1)\ I_{[T_1 \le a^{-1}(n-1)^{-1}T_2]}]$$

$$= \frac{a}{n}\ \{\int_{n(a-1)/a}^\infty \exp(-t_2)t_2^{n-2}/\Gamma(n-1)dt_2 - \exp(\frac{n}{a})(\frac{a-1}{a})^{n-1}(1-\frac{n}{a})\ \cdot$$

$$\int_n^\infty \exp(z)\ z^{n-2}/\Gamma(n-1)dz - \exp(\frac{n}{a})(\frac{a-1}{a})^{n-1}$$

$$[\frac{1}{a}\exp(-n)\ n^{n-1}/\Gamma(n-1) + \frac{1}{a}(n-1)\int_n^\infty \exp(z)\ z^{n-2}/\Gamma(n-1)dz]\}$$

$$= \frac{a}{n}\ \{\int_{n(a-1)/a}^\infty \exp(-t_2)t_2^{n-2}/\Gamma(n-1)dt_2 - \exp(\frac{n}{a})(\frac{a-1}{a})^{n}\ \cdot$$

$$\int_n^\infty \exp(z)\ z^{n-2}/\Gamma(n-1)dz - \exp(\frac{n}{a})(\frac{a-1}{a})^{n-1}\frac{1}{a}\exp(-n)n^{n-1}/\Gamma(n-1)\}$$

$$\geq \frac{a}{n}\ [P(U_n' > \frac{n(a-1)}{a}) - \exp(\frac{n}{a})(\frac{a-1}{a})^{n}(\frac{a}{a-1})\ P(U_n'>n)$$

$$- \exp(\frac{n}{a})(\frac{a-1}{a})^{n}(\frac{a}{a-1})\ \exp(-n)\ n^{n-1}/\Gamma(n-1)]$$

$$\geq \frac{a}{n}\ [P(U_n' > \frac{n(a-1)}{a}) - \exp(-n/2a^2)(\frac{a}{a-1})\ P(U_n'>n)$$

$$- \exp(-n/2a^2)(\frac{a}{a-1})\ \exp(-n)\ n^{n-1}/\exp(-(n-2))\ \sqrt{2\pi}\ (n-2)^{n-3/2}$$

$$\exp(\frac{1}{12(n-2)})] \qquad (4.3.13)$$

where in the last step, we have used the inequality $(a-1)^n/a^n \leq \exp(-\frac{n}{a} - \frac{n}{2a^2})$ and Stirling's formula. Again,

$$E_1[T_1-1)\ I_{[T_1>n^{-1}(a-1)^{-1}T_2]}]$$

$$= \int_0^{n(a-1)} [\ \int_1^{\infty}(t_1-1)\ \frac{n}{a}\ \exp(-n(t_1-1)/a)dt_1]\ \exp(-t_{2/a})t_2^{n-2}a^{-(n-1)}/\Gamma(n-1)dt_2$$

$$+ \int_{n(a-1)}^{\infty} [\ \int_{\frac{t_2}{n(a-1)}}^{\infty} (t_1-1)\ \frac{n}{a}\ \exp(-n(t_1-1)/a)dt_1]\ \exp(-t_{2/a})t_2^{n-2}a^{-(n-1)}/\Gamma(n-1)dt_2$$

$$= \frac{a}{n}\ [P(U_n' \le \frac{n(a-1)}{a}) + \int_{n(a-1)}^{\infty} \{[1+ (\frac{t_2}{a(a-1)}) - \frac{n}{a})]\ \exp(\frac{t_2}{a(a-1)} + \frac{n}{a})\ \frac{\exp(-\frac{t_2}{a})t_2^{n-2}a^{-(n-1)}}{\Gamma(n-1)}\}dt_2$$

$$= \frac{a}{n}\ \{P(U_n' \le \frac{n(a-1)}{a}) + \int_{n(a-1)}^{\infty} \exp(-\frac{t_2}{a-1} + \frac{n}{a})[1+ \frac{1}{a}(\frac{t_2}{a-1} - n)]\ \frac{t_2^{n-2}}{a^{n-1}\Gamma(n-1)}\ dt_2\}$$

$$= \frac{a}{n}\ \{P(U_n' \le \frac{n(a-1)}{a}) + \exp(\frac{n}{a})(\frac{a-1}{a})^{n-1} \int_n^{\infty}[1+ \frac{z-n}{a}]\ \exp(-z)z^{n-2}/\Gamma(n-1)dz\}$$

$$\ge \frac{a}{n}\ \{P(U_n' \le \frac{n(a-1)}{a}) + (\frac{a-1}{a})^{n-1}\ \exp(\frac{n}{a}) \cdot \frac{1}{a} \int_n^{\infty}(z-n)\ \exp(-z)\ \frac{z^{n-2}}{\Gamma(n-1)}\ dz\}$$

$$\ge \frac{a}{n}\ \{P(U_n' \le \frac{n(a-1)}{a}) + \exp(\frac{n}{a})(\frac{a-1}{a})^{n-1}(1- \frac{n}{a}) \cdot \int_n^{\infty} \exp(-a)z^{n-2}/\Gamma(n-1)dz + \exp(\frac{n}{a})(\frac{a-1}{a})^{n-1}.$$

$$[\ \frac{1}{a}\exp(-n)\ n^{n-1}/\Gamma(n-1)+\frac{1}{a}\ (n-1)\ \int_n^\infty \exp(-z)\ z^{n-2}/\Gamma(n-1)dz]\}$$

$$\geq \frac{a}{n}\ P(U'_n \leq \frac{n(a-1)}{a}\ ) - \exp(\ \frac{n}{a}\ )(\ \frac{a-1}{a}\ )^{n-1}\ P(U'_n>n)\}$$

$$\geq \frac{a}{n}\ P(U'_n \leq \frac{n(a-1)}{a}\ ) - \frac{a}{a-1}\ \exp(-\ \frac{n}{2a^2}\ )\ P(U'_n>n)\} \qquad (4.3.14)$$

Finally,

$P_1[T_1>(a-1)^{-1}n^{-1}T_2]$

$$= \int_0^{n(a-1)}[\ \int_1^\infty \frac{n}{a}\ \exp\{-\ \frac{n}{a}\ (t_1-1)dt_1]\ \exp(-\ \frac{t_2}{a}\ )t_2^{n-2}\ a^{-(n-1)}/\Gamma(n-1)dt_2$$

$$+ \int_{n(a-1)}^\infty [\ \int_{n^{-1}(a-1)^{-1}t_2}^\infty \frac{n}{a}\ \exp\{-\ \frac{n}{a}(t_1-1)dt_1]\exp(-\ \frac{t_2}{a}\ )t_2^{n-2}a^{-(n-1)}/$$

$$\Gamma(n-1)dt_2$$

$$= \int_0^{n(a-1)/a} e^{-z}z^{n-2}/\Gamma(n-1)dz + \int_{n(a-1)}^\infty \exp(\ \frac{n}{a}\ )\cdot\ \exp(-\frac{t_2}{a-1}\ )t_2^{n-2}a^{-(n-1)}/$$

$$\Gamma(n-1)dt_2$$

$$= P(U'_n \leq n(a-1)/a) + \exp(\ \frac{n}{a}\ )\ \cdot\ (\ \frac{a-1}{a}\ )^{n-1}\ P(U'_n>n) \qquad (4.3.15)$$

Now, using Bernstein's inequality,

$$P(U'_n \leq n(a-1)/a) \leq \inf_{t\geq 0}\ E[e^{t(\frac{n(a-1)}{a})-U'_n}]$$

$$= \inf_{t\geq 0} e^{\frac{nt(a-1)}{a}}\ (1+t)^{-(n-1)} \qquad (4.3.16)$$

Write

$g(t) = nt(a-1)/a - (n-1)\log(1+t)$

Hence,

$$g'(t) = \frac{n(a-1)}{a} - \frac{n-1}{1+t}, \qquad g''(t) = \frac{n-1}{(1+t)^2} \ (>0) \tag{4.3.17}$$

Hence, $g(t)$ is minimized at $t = \frac{a(n-1)}{n(a-1)} - 1 = \frac{n-a}{n(a-1)}$ (>0) for $n > a$.

Thus, from (4.3.16)

$$\begin{aligned} P(U_n' \le n(a-1)/a) &\le \exp\left(\frac{n-a}{a}\right) \cdot \left(\frac{na-a}{n(a-1)}\right)^{-(n-1)} \\ &= \exp\left(\frac{n}{a}-1\right)\left(\frac{n-1}{n}\right)^{-(n-1)}\left(\frac{a}{a-1}\right)^{-(n-1)} \\ &\le \exp\left(\frac{n}{a}-1\right)\left(\frac{a-1}{a}\right)^{n} \cdot e \cdot \frac{a}{a-1} \\ &\le \exp\left(-\frac{n}{2a^2}\right)\frac{a}{a-1} \end{aligned} \tag{4.3.18}$$

Combining (4.3.8), (4.3.14), (4.3.15) and (4.3.18)

$$\begin{aligned} E_\theta[T^*-\theta] &\ge \theta\Big\{\frac{a}{n} P\left(U_n' > \frac{n(a-1)}{a}\right) - \exp\left(-\frac{n}{2a^2}\right) \\ &\quad - \frac{n+a+1}{n} \exp(-n/2a^2)\, P(U_n' > n) \\ &\quad - \frac{a^2}{n(a-1)} \exp\left(-\frac{n}{2a^2}\right) e^{-n} n^{n-1}/\exp(-n-2)\sqrt{2\pi}\,(n-2)^{n-\frac{3}{2}}\Big\} \\ &\ge \frac{a\theta}{n}[1-A_n] \end{aligned} \tag{4.3.19}$$

where $A_n \le Kn^{\frac{1}{2}} \exp\left(-\frac{n}{2a^2}\right)$.

Thus, the use of T' instead of T^* results in about 50% savings in bias even for moderate n.

Next we compare the MSE of T^* with that of T'. First write

$$E_\theta(T^*-\theta)^2 = E_\theta[(T_1-\theta)^2 I_{[T_1\leq(a-1)^{-1}n^{-1}T_2]} + \{a^{-1}(T_1+n^{-1}T_2)-\theta^2\} I_{[T_1>(a-1)^{-1}n^{-1}T_2]}]$$

$$= \theta^2 E_1[(T_1-1)^2 I_{[T_1\leq(a-1)^{-1}n^{-1}T_2]} + \{a^{-1}(T_1+n^{-1}T_2)-1\}^2 I_{[T_1>(a-1)^{-1}n^{-1}T_2]}] \tag{4.3.20}$$

Now,

$$E_1[(T_1-1)^2 I_{[T_1\leq(a-1)^{-1}n^{-1}T_2]}] =$$

$$\int_{n(a-1)}^{\infty}\left[\int_1^{t_2[n(a-1)]}(t_1-1)^2\exp[-\tfrac{n}{a}(t_1-1)]\cdot\tfrac{n}{a}\,dt_1\right] \exp\left(-\frac{t_2}{a}\right)t_2^{n-2}a^{-(n-1)}/\Gamma(n-1)\,dt_2 \tag{4.3.21}$$

The inner integral of (4.3.21) is

$$= \frac{a^2}{n^2}\int_0^{\frac{t_2}{a(a-1)}-\frac{n}{a}} z^2\exp(-z)\,dz$$

$$= \frac{a^2}{n^2}\left[2-\left(2+2\left(\frac{t_2}{a(a-1)}-\frac{n}{a}\right)+\left(\frac{t_2}{a(a-1)}-\frac{n}{a}\right)^2\right)\exp\left(-\frac{t_2}{a(a-1)}+\frac{n}{a}\right)\right]. \tag{4.3.22}$$

Combining (4.3.21) and (4.3.22) it follows that

$$E_1[(T_1-1)^2 I_{[T_1\le(a-1)^{-1}n^{-1}T_2]}]$$

$$= \frac{a^2}{n^2}\{2P(U_n' > \frac{n(a-1)}{a}) - \int_{n(a-1)}^{\infty}[2+\frac{2}{a}(\frac{t_2}{a-1}-n) + \frac{1}{a^2}$$

$$(\frac{t_2}{a-1}-n)^2]\exp(-\frac{t_2}{a-1}+\frac{n}{a})\frac{t_2^{n-2}a^{-(n-1)}}{\Gamma(n-1)}dt_2\}$$

$$= \frac{a^2}{n^2}\{2P(U_n' > \frac{n(a-1)}{a}) - \int_n^{\infty}[2+\frac{2}{a}(z-n) + \frac{1}{a^2}(z-n)^2$$

$$\exp(-z)\exp(n/a)\frac{z^{n-2}(\frac{a-1}{a})^{n-1}}{\Gamma(n-1)}dz\} \tag{4.3.23}$$

But,

$$\int_n^{\infty}(z-n)\exp(-z)z^{n-2}/\Gamma(n-1)dz$$

$$= \exp(-n)n^{n-1}/\Gamma(n-1) - \int_n^{\infty}\exp(-z)\cdot z^{n-2}/\Gamma(n-1)dz \tag{4.3.24}$$

and,

$$\int_n^{\infty}(z-n)^2\exp(-z)z^{n-2}/\Gamma(n-1)dz =$$

$$\int_n^{\infty}\exp(-z)z^n/\Gamma(n-1)dz - 2n\int_n^{\infty}\exp(-z)z^{n-1}/\Gamma(n-1)dz$$

$$+ n^2\int_n^{\infty}\exp(-z)z^{n-2}/\Gamma(n-1)dz \tag{4.3.25}$$

Now write

$$\int_n^{\infty}\exp(-z)\frac{z^n}{\Gamma(n-1)}dz = \exp(-n)\cdot n^n/\Gamma(n-1) + n\int_n^{\infty}\exp(-z)\frac{z^{n-1}}{(n-1)}dz$$

$$= \exp(-n)\cdot n^n/\Gamma(n-1) + n\exp(-n)\cdot n^{n-1}/\Gamma(n-1) + n(n-1)\int_n^{\infty}\exp(-z)\frac{z^{n-2}}{(n-1)}dz$$

$$= 2\exp(-n)\cdot n^{n}/\Gamma(n-1) + n(n-1) \int_n^\infty \exp(-z)\cdot z^{n-2}/\Gamma(n-1)\, dz; \tag{4.3.26}$$

And,

$$\int_n^\infty \exp(-z) \frac{z^{n-1}}{\Gamma(n-1)} dz = \exp(-n)\cdot n^{n-1}/\Gamma(n-1) + (n-1) \int_n^\infty \exp(-z) \frac{z^{n-2}}{\Gamma(n-1)} dz \tag{4.3.27}$$

Now combine (4.3.23) - (4.3.27) to get

$$E_1[(T_1-1)^2 I_{[T_1 \leq (a-1)^{-1} n^{-1} T_2]}]$$

$$= \frac{a^2}{n^2} \{2P(U_n' > \frac{n(a-1)}{a}) - (\frac{a-1}{a})^{n-1} \exp(\frac{n}{a})[(2-\frac{2}{a}+\frac{1}{a^2}(n^2+n^2-n-2n^2+2n))$$

$$\cdot P(U_n'>n) + (\frac{2}{a} n^{n-1} + \frac{1}{a^2} \cdot 2n^n - \frac{1}{a^2} 2n\cdot n^{n-1}) e^{-n}/\Gamma(n-1)]\}$$

$$\geq \frac{a^2}{n^2} \{2P(U_n' > \frac{n(a-1)}{a}) - 2(\frac{a-1}{a})^n \exp(n/a) - \frac{n}{a^2}(\frac{a-1}{a})^{n-1} \exp(n/a)$$

$$- (\frac{a-1}{a})^{n-1} \exp(\frac{n}{a}) \cdot \frac{2n^{n-1}}{a} \cdot \exp(-n)/\Gamma(n-1)\}$$

$$\geq \frac{a^2}{n^2} \{2P(U_n' > \frac{n(a-1)}{a}) - 2\exp(-n/2a^2) - \frac{n}{a^2} \exp(-n/2a^2) (\frac{a}{a-1})$$

$$- \frac{2}{a} (\frac{a}{a-1}) \exp(-n/2a^2)\cdot \exp(-n)\cdot n^{n-1}\}$$

$$\geq \frac{a^2}{n^2} \{2P(U_n' > \frac{n(a-1)}{a}) - 2\exp(-\frac{n}{2a^2})(\frac{a}{a-1}) - \frac{2n}{a^2} \exp(-n/2a^2)$$

$$- \frac{2}{a} [\exp(-n)\, n^{n-1}/\Gamma(n-1)] \exp(-n/2a^2)\} \tag{4.3.28}$$

Using the result

$$P(U_n' > \frac{n(a-1)}{a}) \geq 1 - \frac{a}{a-1}\exp(-\frac{n}{2a^2}) \tag{4.3.29}$$

it follows from (4.3.28) that

$$E_1[(T_1-1)^2 I_{[T_1 \leq (a-1)^{-1}n^{-1}T_2]}] \geq 2a^2n^{-2}(1-A_n) \tag{4.3.30}$$

where $A_n = K \cdot n \cdot \exp(-n/2a^2)$. Since the left hand side of (4.3.30) is a lower bound for $E_\theta(T^*-\theta)^2$, comparing (4.3.30) with (4.3.4) it follows that,

$$MSE(T^*) - 2\cdot MSE(T') = (\frac{2a^2}{n^2} - \frac{2a^2}{(n+a)^2+na^2} - \frac{K\cdot e^{-n/2a^2}}{n})\ \theta$$

$$= \{\frac{(2a^4+4a^3)n+4a^4}{n^2[(n+a)^2+na^2]} - \frac{K\ e^{-n/2a^2}}{n}\}\ \theta \tag{4.3.31}$$

From (4.3.31), it follows that, there is approximately 50% MSE reduction by the use of T' in place of T^* .

4.4. Best Scale Invariant Estimates

We have already noted in Section 4.3 that (T_1, T_2) is minimal sufficient for θ. An estimator $g(T_1, T_2)$ of θ is said to be a scale invariant estimator if for any $c(>0)$,

$$g(cT_1, cT_2) = cg(T_1, T_2)\ . \tag{4.4.1}$$

Note that the estimators T, T' and T^* proposed in above Sections are all scale-invariant estimators.

Since T_1 and T_2 are scale-invariant, all scale-invariant estimators $g(T_1, T_2)$ must have the form

$$g(T_1, T_2) = U_1 d(U_2) \tag{4.4.2}$$

where

$$U_1 = T_1 + n^{-1} T_2 \quad \text{and} \quad U_2 = T_1/(T_1+n^{-1}T_2) \tag{4.4.3}$$

To show this fact, simply note that for $g(T_1, T_2)$ to be scale-invariant, we must have $g(T_1, T_2) = c^{-1}g(cT_1, cT_2)$ for all $c > 0$. Letting

$$C = U_1^{-1}$$

and,

$$d(U_2) = g(U_1^{-1}T_1, U_1^{-1}T_2) = g(U_2, n(1-U_2))$$

one gets,

$$g(T_1, T_2) = U_1 d(U_2) .$$

We want to find the smallest MSE estimator of θ in the class of all scale-invariant estimators.

With this end, first observe that

$$\begin{aligned} E_\theta[U_1 d(U_2) - \theta]^2 &= \theta^2 E_\theta[(U_1/\theta) d(U_2) - 1]^2 \\ &= \theta^2 E_1[U_1 d(U_2) - 1]^2, \end{aligned} \tag{4.4.4}$$

since the joint distribution of (U_1/θ) and U_2 is independent of $\theta(>0)$. Since,

$$E_1[U_1 d(U_2)-1]^2 = E_1[d^2(U_2)\ E(U_1^2 | U_2) - 2d(U_2)\ E(U_1 | U_2)+1] \tag{4.4.5}$$

From (4.4.5), the best scale invariant estimate of θ is given by $U_1 d^*(U_2)$, where

$$d^*(u_2) = E_1(U_1 \mid U_2=u_2)/E_1(U_1^2 \mid U_2=u_2). \tag{4.4.6}$$

In order to find $d^*(u_2)$, first write the joint pdf of T_1 and T_2 (when $\theta = 1$) as

$$f_1(t_1,t_2) = \frac{n}{a^n\Gamma(n-1)} \exp(-n(t_1-1)/a) \exp(-t_{2/a})\, t_2^{n-2}$$

$$= \frac{n}{a^n\Gamma(n-1)} \exp(1/a) \exp\{-n(t_1+n^{-1}t_2)/a\}t_2^{n-2} \tag{4.4.7}$$

Let,

$u_1 = t_1 + n^{-1}t_2.\qquad u_2 = t_1/u_1$

Thus,

$t_1 = u_1u_2.\qquad t_2 = nu_1(1-u_2)$

Then the Jacobian of transformation is nu_1 . Hence, U_1 and U_2 have joint pdf

$$f(u_1,u_2) = \frac{n}{a^n\Gamma(n-1)} \exp(1/a) \exp\{-nu_{1/a}\}(nu_1(1-u_2))^{n-2}\cdot nu_1$$

$$= \frac{(n/a)^n}{\Gamma(n-1)} \exp(1/a)\cdot\exp\{-nu_{1/a}\} \cdot u_1^{n-1}(1-u_2)^{n-2},$$

$$u_1u_2 \geq 1,\quad 0 < u_2 < 1 \tag{4.4.8}$$

Hence,

$$d^*(u_2) = \frac{E_1(U_1|U_2)}{E_1(U_1^2|U_2)} = \frac{\int_{u_2^{-1}}^{\infty} u_1 f(u_1,u_2)du_1}{\int_{u_2^{-1}}^{\infty} u_1^2 f(u_1,u_2)du_1}$$

$$= \frac{\int_{u_2^{-1}}^{\infty} \exp(-nu_{1/a})\, u_1^n du_1}{\int_{u_2^{-1}}^{\infty} \exp(-nu_{1/a}) u_1^{n+1} du_1}$$

$$= \frac{n \cdot n!}{a(n+1)!} \cdot \frac{\int_{n(au_2)^{-1}}^{\infty} \exp(-z)\ z^n/n!\ dz}{\int_{n(au_2)^{-1}}^{\infty} \exp(-z)\ z^{n+1}/(n+1)!\ dz}$$

Thus,

$$d^*(u_2) = \frac{n}{a(n+1)} \cdot \frac{\Sigma_{j=0}^{n} \exp(-n\ /au_2)(n/au_2)^j/j!}{\Sigma_{j=1}^{n+1} \exp(-n/au_2)(n/au_2)^j/j!} \tag{4.4.9}$$

Hence, the best scale-invariant estimator is given by

$$T^{**} = u_1 d^*(u_2) = \frac{nu_1}{a(n+1)} \cdot \frac{\Sigma_{j=0}^{n} \exp(-n/au_2)(n/au_2)^j/j!}{\Sigma_{j=0}^{n+1} \exp(-n/au_2)(n/au_2)^j/j!} \tag{4.4.10}$$

4.5. A Class of Bayes Estimators

Next we construct a class of Bayes estimates for θ and identify the best scale-invariant estimate as a limiting Bayes estimate. With this end, first write the joint pdf of T_1 and T_2 as

$$f_\theta(t_1,t_2) = \frac{n}{a\theta} \exp(-\frac{n}{a\theta}(t_1-1))\cdot\exp(-t_2/(a\theta))t_2^{n-2}(a\theta)^{-(n-1)}/\Gamma(n-1)$$

$$\propto \theta^{-n} \exp(-(nt_1+t_2)/(a\theta)) \tag{4.5.1}$$

Consider a prior density $g(\theta)$ for θ of the form

$$g_{\alpha,r}(\theta) = \frac{r^{\alpha}}{(a\theta)^{\alpha+1}\Gamma(\alpha)} e^{-\frac{r}{a\theta}}, \quad \text{if} \quad \theta > 0$$

$$= 0 \qquad\qquad \text{otherwise} \qquad\qquad (4.5.2)$$

where $r > 0$ is a given positive number, and $\alpha > 0$ is a given integer. Then, the posterior distribution of θ given $T_1 = t_1$ and $T_2 = t_2$ is

$$f(\theta|t_1,t_2) \propto \theta^{-n-\alpha-1}\exp(-(nt_1+t_2+r)/(a\theta)), \quad 0 \le \theta \le t_1 \qquad (4.5.3)$$

Now, assuming the loss $L(\theta,a.) = (\theta-a.)^2/\theta^2$, the Bayes estimate of θ with respect to the prior $g(\theta)$ is given by

$$\hat{\theta}_{\alpha,r} = E(\theta|x_1,\ldots,x_n) = E(\theta|T_1,T_2)$$

$$= \frac{\int_0^{t_1} \theta^{-1} \exp(-(nt_1+t_2+r)/(a\theta))\theta^{-n-\alpha-1}\, d\theta}{\int_0^{t_2} \theta^{-2} \exp(-(nt_1+t_2+r)/(a\theta))\theta^{-n-\alpha-1}\, d\theta}$$

$$= \frac{\int_{t_1^{-1}}^{\infty} \exp(-(n_1t_1+t_2+r)\phi/a)\ \phi^{n+\alpha}\, d\phi}{\int_{t_1^{-1}}^{\infty} \exp(-(n_1t_1+t_2+r)\phi/a)\ \phi^{n+\alpha+1}\, d\phi}$$

$$= \frac{\{(nt_1+t_2+r)/a\} \int_{(nt_1+t_2+r)(at_1)^{-1}}^{\infty} \exp(-z)\ z^{n+\alpha}\, dz}{\int_{(nt_1+t_2+r)(at_1)^{-1}}^{\infty} \exp(-z)\ z^{n+\alpha+1}\, dz}$$

Thus,

$$\hat{\theta}_{\alpha,r} = \frac{a^{-1}(na_1+r) \int_{(nu_1+r)/(u_1u_2)}^{\infty} \exp(-z)\; z^{n+\alpha}\, dz}{\int_{(nu_1+r)/(u_1u_2)}^{\infty} \exp(-z)\; z^{n+\alpha+1}\, dz}\,, \tag{4.5.4}$$

$$\longrightarrow \frac{nu_1}{a(n+1)} \cdot \frac{\int_{nu_2^{-1}}^{\infty} \exp(-z)\; z^{n}/n!\, dz}{\int_{nu_2^{-1}}^{\infty} \exp(-z)\; z^{n+1}/(n+1)!\, dz}$$

as $\alpha \to 0$ and $r \to 0$.

Comparing (4.5.4) with (4.4.10), we see that

$$T^{**} = \hat{\theta}_{0,0}\,, \tag{4.5.6}$$

That is, for squared error loss, the minimum risk scale-invariant estimate T^{**} is the limit $\lim\limits_{\substack{\alpha\to 0\\ r\to 0}} \hat{\theta}_{\alpha,r}$ of the Bayes rules against the priors $g_{\alpha,r}(\theta)$. It is easily seen that the limit as $\alpha \to 0$ exists and can be taken inside the integral in (4.5.4).

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