Topics in pricing American type financial contracts

by

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A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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2007

UMI Number: 3274849

UMI Microform 3274849

ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

DEDICATION

I would like to dedicate this thesis to my parents Fanti Meng and Ying Liu without whose support I would not have been able to complete this work.

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CHAPTER 1. Introduction

In financial markets, there exist two types of contracts: European type contracts and American type contracts. European type contracts, such as European options, specify an expiration date. If the option is to be exercised, the exercise must occur on the expiration date. In contrast, an American option contract can be exercised at any time before or on the expiration date. Compared to its European counterpart, the pricing of an American option is much more complicated due to this early exercise feature. If the price of the underlying stock is modeled as a stochastic process, the pricing of an American option on the stock is an optimal stopping problem. In this thesis, we study three pricing problems related to American type financial contracts:

1. We derive a closed form upper bound for American put options. This upper bound can be used in conjunction with traditional Monte Carlo simulation, which usually generates a lower bound, to obtain a better estimate for the option price;

2. A stock can be considered as an American type contract since the owner can sell it at any time. We model the stock price as a diffusion process with a positive probability of jumping to default and find the optimal strategy for the owner to sell the stock. A similar problem has been solved in Oksendal [36] where the stock is considered to be free of default risk;

3. We prove an ordering result for American options with a piecewise linear payoff under a family of equivalent martingale measures used in stochastic volatility mod-

els. The equivalent martingale measure used in this thesis is proposed by Hobson [24]. A similar result for European options is proved by Henderson in [20].

In the remaining part of this chapter, we will discuss these three topics in more detail.

1.1 Upper bound for American option price

In contrast to its European counterpart, the price of an American option does not have a closed-form formula even when the underlying stock price is assumed to be a geometric Brownian motion (such as in the standard Black-Scholes-Merton framework). For American option pricing, researchers and market practitioners have developed several numerical procedures, such as binomial and trinomial trees (see, for example, Cox et. al. [9]), finite difference method (see, for example, Brennan and Schwartz [4], and Hull and White [27]), and Monte Carlo simulation (see, for example, Boyle [3] and Broadie and Glasserman [5]).

Among these numerical techniques, the advantage of Monte Carlo simulation is that it can be used when the payoff of the option depends on the history of the underlying stock price while the other two methods work only when the payoff is dependent only on the terminal value of the stock. One disadvantage of Monte Carlo method is that before the simulation can be performed, one must determine an exercise policy. Unfortunately, the optimal exercise policy is usually not known. For instance, for the standard American put option with payoff $(K - S_t)^+$ where K is the strike price and S_t is the stock price, it is well known that one optimal strategy for the owner is to exercise the option at the first time when the stock price reaches an exercise boundary. Some properties of the exercise boundary are known, see Karatzas and Shreve ([31], section 2.7), or Peskir [37] for recent developments. However, no explicit formula has been derived so far.

Many different methods for determining the exercise boundary in Monte Carlo simulation has been developed, such as the least squares approach and parametrization approach. For an introduction to these methods, we refer to Hull [25], section 20.9. A drawback of Monte Carlo simulation methods is that they always produce lower bounds of the true price since the exercise boundary used in simulation is not necessarily optimal.

On the other hand, it is much harder to obtain an upper bound for American option price using simulation methods. One of the most notable results in this direction is presented in Rogers [39]. Rogers' research is based on the theoretical results of Davis and Karatzas [10]. Rogers' upper bound can be obtained using a Monte Carlo simulation. In this thesis, we derive an upper bound in closed form based on Rogers' result. Since our upper bound is in closed form and no simulation is needed, it can be used as a quick estimate for Rogers' upper bound. We also conduct a comparison between our upper bound and the one proposed by Rogers.

1.2 Jump-to-default models

Default risk is the theme in the pricing and hedging of credit risk. There are two types of models developed by researchers to evaluate default risk. A structural model considers the equity of a firm as a call option of its total asset, and calculates the default probability based on the classic Black-Scholes-Merton model. While a reduced form model defines a default intensity function and a random variable, usually exponentially distributed, and assumes that default occurs at the first time when the accumulated default intensity exceeds the exponentially distributed random variable. The default intensity function in a reduced form model can be chosen as a constant, a deterministic function, or a stochastic process. A review of different intensity functions can be found in Duffie and Singleton [13], section 3.4. A comparison of structural model and reduced form model can be found in Arora, Bohn, and Zhu [1], and Jarrow and Protter [28].

The jump-to-default model used in our study is a reduced form model. The intensity function in our model is a decreasing function of the stock price. Our research is based on the theoretical results by Elliot, Jeanblanc, and Yor [16] for European options. We prove similar results for American options. On the basis of these results, we solve an optimal stopping problem for the owner of a stock. When the stock is free of default risk, Oksendal [36] showed that the owner should hold the stock till the first time when the stock price is greater than or equal to an exercise point. We solve the problem and derive an exercise point under the assumption of a positive default risk. Numerical examples in this thesis show that the owner should actually hold the stock longer under the jump-to-default model.

1.3 Stochastic volatility models

If the log-normal assumption for stock price in the Black-Scholes-Merton model is correct, then the implied volatility, which is obtained by observing option prices from the real market and inverting the Balck-Scholes formula, should not depend on the strike price of the option used in the calculation. However, it is well known that the implied volatility is a (non-constant) function of the strike price of options. This function is called the volatility smile. For reference, see MacBeth and Merville [34], or Lauterbach and Schiltz [32].

Numerous models have been proposed to explain the volatility smile. Merton [35] introduced jumps in the stock distribution; Cox [8] proposed Constant Elasticity Variance

(CEV) model; Dupire [14], as well as Derman and Kani [11] developed a concept called "local volatility". For a review of these models, we refer to Javaheri [29]. In comparison to the models mentioned above, other researchers model the volatility as a stochastic process, such as the Hull-White model [26], the Heston model [22], and the Stein and Stein model [40].

Stochastic volatility models are promising both in theory and in practice. On one hand, it provides a second source of risk affecting the level of instantaneous volatility; On the other hand, a stochastic volatility is usually equivalent to a Generalized Autoregressive Conditional Heteroscedasticity, or GARCH model, which is very popular in empirical economics and finance research. For example, the Heston model corresponds to a special case of GARCH model (see Heston and Nandi [23]).

However, since stochastic volatility models involve two sources of randomness, they are in general very complicated. In particular, since the volatility is not equal to the price of any traded security, its drift in the risk neutral world is not necessarily equal to the risk-free interest rate. Accordingly, there are many different martingale measures, depending on how one determines the stochastic equation for volatility in the risk neutral world. Heston [22] chooses a martingale measure which enables him to derive a closed form pricing formula for European options. However, no theoretical or empirical evidence is provided to justify the selection of Heston's martingale measure. Other popular choices of martingale measures include variance-optimal measure (see Duffie and Richardson [12]) and the minimal entropy measure (see Frittelli [18]). Yet very little empirical research has been conducted to show which martingale measure is more consistent with price of options in the real market.

Recent research by Hobson [24] reveals that the variance-optimal measure and the minimal entropy measure can be integrated into a large family of martingale measures which are known as the q-optimal measures, corresponding to the values $q = 2$ and $q = 1$, respectively. The q-optimal measure, in some sense, is the martingale measure closest to the physical measure. Hobson also derived the form of the q-optimal measure. Similar results and more examples can also be found in Henderson et. al. [21]. Henderson [20] proved that the price of European options with convex payoff is monotonic in the parameter q under the q-optimal measure.

The ordering result for option prices is remarkable since, in conjunction with other properties of the q-optimal measure, it significantly facilitates the selection of martingale measures. For instance, if a stochastic volatility model tends to underprice an option under the q-optimal measure with $q = q_1$, and tends to overprice the same option when $q = q_2$, using the ordering result, one can immediately conclude that the parameter q implied by the option price is between q_1 and q_2.

Our research is based on the proof of the ordering result in Henderson [20]. We point out that one equality in Henderson's proof is not correct, and we fix the problem by choosing an appropriate filtration. We then extend the monotonicity result to American option prices. To the extent of our knowledge, this is the first research on pricing American options under the q-optimal measure.

CHAPTER 2. A closed-form upper bound for American put option

In this chapter, we propose a closed-form upper bound for American put option. Our research is based on a Monte Carlo valuation method developed by L.C.G. Rogers in [39]. Rogers proved that American put value $V \leq \mathbb{E}\left[\sup_{0\leq t\leq T}(Z_t - M_t)\right]$ where Z_t is the discounted intrinsic value of the option and M_t is a martingale satisfying certain conditions. In this chapter, we choose the discounted European put price process as M_t and derive a closed-form upper bound for American put.

2.1 Rogers' Monte Carlo method

Let S_t denote the stock price process and $(\mathcal{F}_t)$ be the filtration generated by S_t. Define the discounted exercise value of the American put

$$Z_t \equiv e^{-rt}(K - S_t)^+ \tag{2.1}$$

The time t value of an American put option that expires at time T is

$$V_A(S_t, T-t) = \sup_{\tau\in\mathcal{S}_{t,T}} \mathbb{E}\left[Z_\tau | S_t\right].$$

where $\mathcal{S}$ is the collection of all $(\mathcal{F}_t)$- stopping times between time t and T. It is well known that it is optimal for the owner of American put to exercise the option at the first time when S_t falls below a threshold called the exercise boundary. Explicit formula for this exercise boundary is unknown. One major difficulty in Monte Carlo valuation of

American options is how to determine the exercise boundary. No matter what boundary is chosen, the traditional simulation method always tends to underprice the option because the boundary is not actually optimal. Therefore, such simulation methods always yield a lower bound for American put. Rogers [39] proposed a new Monte Carlo method based on the work of Davis and Karatzas [10], and proved that this method always gives an upper bound for the value of the American put.

The price of the put option at time 0 is given by $V_A(S_0, T) = Y_0^* \equiv \sup_{\tau \in \mathcal{S}_{[0,T]}} \mathbb{E} Z_\tau$ where Z_t is defined by (2.1). Under the assumptions that $Y_0^* < \infty$, that $\sup_{0 \le t \le T} |Z_t| \in L^p$ for some $p > 1$, and that Z is right continuous, the Snell envelope $Y_t^* \equiv \sup_{\tau \in \mathcal{S}_{t,T}} \mathbb{E}\left[Z_\tau | \mathcal{F}_t\right]$ is a supermartingale and the family $\{Y_\tau^*\}_{\tau \in \mathcal{S}}$ is uniformly integrable. Therefore Y_t^* has a Doob-Meyer decomposition $Y_t^* = Y_0^* + M_t^* - A_t^*$ where M_t^* is a martingale with $M_0^* = 0$, and A_t^* is a previsible integrable increasing process, also vanishing at 0. Rogers showed that $\mathbb{E}\left[\sup_{0 \le t \le T}(Z_t - M_t)\right]$ is an upper bound of Y_0^* for any $M \in H_0^1$ where H_0^1 is the set of all martingales M with $\sup_{0 \le t \le T} |M_t| \in L^1$ and $M_0 = 0$. More precisely, he proved the following theorem.

Theorem 2.1.1. (see also Rogers [39], theorem 2.1)

$$Y_0^* = \inf_{M \in H_0^1} \mathbb{E}\left[\sup_{0 \le t \le T}(Z_t - M_t)\right]. \tag{2.2}$$

The infimum is attained by taking $M = M^*$.

Proof. It follows from the definition of Y_0^* that

$$Y_0^* = \sup_{\tau \in \mathcal{S}_{[0,T]}} \mathbb{E} Z_\tau = \sup_{\tau \in \mathcal{S}_{[0,T]}} \mathbb{E}[Z_\tau - M_\tau] \le \mathbb{E}\left[\sup_{0 \le t \le T}(Z_t - M_t)\right]$$

The last inequality holds, because $\mathbb{E}[Z_\tau - M_\tau] \le \mathbb{E}\left[\sup_{0 \le t \le T}(Z_t - M_t)\right]$ for any stopping time $\tau \in \mathcal{S}_{[0,T]}$.

Now take $M = M^*$,

$$\mathbb{E}\left[\sup_{0\leq t\leq T}(Z_t - M_t^*)\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq T}(Y_t^* - M_t^*)\right] = \mathbb{E}\left[\sup_{0\leq t\leq T}(Y_0^* - A_t^*)\right] = Y_0^*$$

since A_t^* is increasing. From the assumption that $\sup_{0\leq t\leq T}|Z_t| \in L^p$ for some $p > 1$ it follows that M_t^* is actually in H_0^1. This completes the proof.

□

2.2 A closed-form upper bound

Based on (2.2), an upper bound can be obtained using Monte Carlo simulation. Compared to traditional Monte Carlo techniques, Rogers' method not only provides an upper bound rather than a lower bound, it also converges faster because no pre-determined exercise boundary is needed. In this section, we derive a closed-form upper bound. Our upper bound can be used as a quick estimate of the early exercise premium, it may also serve as an error bound for Rogers' method.

2.2.1 Theoretical results

As in the previous section, let $V_A(S_t, T-t)$ be the time t value of an American put option where S_t denotes the stock price and T the expiration date. Furthermore, let $V_U(S_t, T-t)$ be the value of an European put option with the same expiration date, underlying stock and strike price. We will derive a closed-form upper bound for the early exercise premium (EEP) which is defined by $EEP \equiv V_A(S_0, T) - V_U(S_0, T)$.

Throughout this chapter, we assume that the stock price is a Geometric Brownian Motion. According to the risk neutral pricing theory, the stock price under the risk neutral measure is governed by the following stochastic differential equation:

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{2.3}$$

where r represents the risk free rate of return and W_t is a standard Brownian Motion under the risk neutral measure. Under these assumptions, the European put value $V_U(S_0, T)$ is given by the Black-Scholes formula, and therefore a closed-form upper bound for $V_A(S_0, T)$ can be calculated immediately once a closed-form upper bound for EPP is derived.

We first prove the following lemma:

Lemma 2.2.1. Let Z_t be the discounted intrinsic value process defined in (2.1), and $\mathcal{F}_t$ be the filtration generated by the stock price process S_t, then the early exercise premium

$$EEP \leq \mathbb{E} \sup_{t\in[0,T]} \mathbb{E}\left[Z_t - Z_T | \mathcal{F}_t\right]$$

Proof. It is well known that the discounted European put price $e^{-rt}V_U(S_t, T-t)$ is a martingale. Hence $e^{-rt}V_U(S_t, T-t) - V_U(S_0, T)$ is a martingale with starting value 0. It follows from theorem 2.1.1 that

$$\begin{aligned}
V_A(S_0, T) &\leq \mathbb{E} \sup_{t\in[0,T]} \left[Y_t - e^{-rt}V_U(S_t, T-t) + V_U(S_0, T)\right] \\
&= \mathbb{E} \sup_{t\in[0,T]} \left[Y_t - e^{-rt}V_U(S_t, T-t)\right] + V_U(S_0, T)
\end{aligned}$$

Hence

$$EPP = V_A(S_0, T) - V_U(S_0, T) \leq \mathbb{E} \sup_{t\in[0,T]} \left[Y_t - e^{-rt}V_U(S_t, T-t)\right].$$

Moreover the martingale property of $e^{-rt}V_U(S_t, T-t)$ implies that

$$e^{-rT}\mathbb{E}\left[V_U(S_T, 0)|\mathcal{F}_t\right] = e^{-rt}V_U(S_t, T-t)$$

Notice that $V_U(S_T, 0)$ is the payoff of the European put at expiration date and thus $V_U(S_T, 0) = (K - S_T)^+$. It follows that

$$\mathbb{E}\left[Z_T|\mathcal{F}_t\right] = e^{-rt}V_U(S_t, T-t) = e^{-rT}\mathbb{E}\left[(K - S_T)^+\right].$$

Therefore

$$EPP \leq \mathbb{E} \sup_{t\in[0,T]} \left[Z_t - e^{-rt} V_U(S_t, T-t)\right] = \mathbb{E} \sup_{t\in[0,T]} \left[Z_t - \mathbb{E}[Y_T|\mathcal{F}_t]\right]$$

$$= \mathbb{E} \sup_{t\in[0,T]} \mathbb{E}\left[Z_t - Z_T|\mathcal{F}_t\right].$$

This completes the proof.

□

Notice that Z_t is not a C^2 function of S_t. In order to apply Itô's formula, we approximate the function $(K-x)^+$ using smooth functions (see also Chung and Williams [7], page 142).

Define

$$\phi_\epsilon(x) = \begin{cases} K - x & x \leq K - \epsilon \\ (K+\epsilon-x)^2/4\epsilon & K-\epsilon \leq x \leq K+\epsilon \\ 0 & x \geq K+\epsilon \end{cases} \tag{2.4}$$

Then $\lim_{\epsilon\to 0} \phi_\epsilon(x) = (K-x)^+$, and

$$\phi'_\epsilon(x) = \begin{cases} -1 & x \leq K - \epsilon \\ -(K+\epsilon-x)/2\epsilon & K-\epsilon \leq x \leq K+\epsilon \\ 0 & x \geq K+\epsilon \end{cases}$$

$$\phi''_\epsilon(x) = \begin{cases} 0 & x \leq K - \epsilon \\ 1/2\epsilon & K-\epsilon \leq x \leq K+\epsilon \\ 0 & x \geq K+\epsilon \end{cases}$$

Furthermore, define $Z_t^\epsilon = e^{-rt}\phi_\epsilon(S_t)$. Clearly, $\lim_{\epsilon\to 0} Z_t^\epsilon = Z_t$. Next we will prove the main result of this section.

Theorem 2.2.2. Let $p(t,u,x,y)$ be the transition density function of Geometric Brownian Motion, i.e.,

$$p(t,u,x,y) = \frac{1}{y\sigma\sqrt{2\pi(u-t)}} e^{-[\log\frac{y}{x} - (r - \frac{1}{2}\sigma^2)(u-t)]^2/2\sigma^2(u-t)}, \tag{2.5}$$

then

$$\mathbb{E}[Z_t - Z_T|\mathcal{F}_t] = \int_t^T e^{-ru}\left[rk\int_0^K p(t,u,S_t,y)dy - \frac{1}{2}\sigma^2K^2p(t,u,S_t,K)\right]du. \tag{2.6}$$

Proof. Apply Itô's formula to $Z_t^\epsilon - Z_T^\epsilon$. Direct calculation yields

$$\begin{aligned} Z_t^\epsilon - Z_T^\epsilon = & \underbrace{\int_t^T re^{-ru}(\phi_\epsilon(S_u) - S_u\phi'_\epsilon(S_u))du}_{(1)} \\ & - \underbrace{\int_t^T \frac{1}{2}e^{-ru}\phi''_\epsilon(S_u)\sigma^2S_u^2du}_{(2)} \\ & - \underbrace{\int_t^T e^{-ru}\phi'_\epsilon(S_u)\sigma S_u dW_u}_{(3)} \end{aligned} \tag{2.7}$$

Since $|\phi'_\epsilon(x)| \leq 1$, it follows from the properties of stochastic integral that the last term (3) is a martingale and $\mathbb{E}[(3)|\mathcal{F}_t] = 0$ for any ϵ, thus

$$\mathbb{E}[\lim_{\epsilon\to 0}(3)|\mathcal{F}_t] = \lim_{\epsilon\to 0}\mathbb{E}[(3)|\mathcal{F}_t] = 0 \tag{2.8}$$

Now consider the first term. Let $\psi_\epsilon(x) = \phi_\epsilon(x) - x\phi'_\epsilon(x)$, then $|\psi_\epsilon(x)| \geq K + S_u$, and $\mathbb{E}\int_t^T re^{-ru}(K+S_u)du$ is finite. By the dominated convergence theorem,

$$
\begin{aligned}
\mathbb{E}\left[\lim_{\epsilon\to 0}(1)|\mathcal{F}_t\right] &= \mathbb{E}\left[\int_t^T re^{-ru}\lim_{\epsilon\to 0}\phi_\epsilon(S_u) - S_u\phi'_\epsilon(S_u)du|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_t^T re^{-ru}((K-S_u)^+ + S_u\mathbb{I}_{[0,K]}(S_u))du|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_t^T re^{-ru}(K-S_u+S_u)\mathbb{I}_{[0,K]}(S_u)du|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_t^T re^{-ru}K\mathbb{I}_{[0,K]}(S_u)du|\mathcal{F}_t\right] \qquad (2.9)\\
&= \int_t^T re^{-ru}K\mathbb{E}\left[\mathbb{I}_{[0,K]}(S_u)|\mathcal{F}_t\right]du \\
&= \int_t^T re^{-ru}K\mathbb{P}\left[S_u\le K|\mathcal{F}_t\right]du \\
&= \int_t^T re^{-ru}K\int_0^K p(t,u,S_t,y)dydu
\end{aligned}
$$

The last step follows from the Markov property of S_t.

Finally consider the second term. Notice that for $u > t$,

$$
\mathbb{E}[\phi''_\epsilon(S_u)|\mathcal{F}_t] = \mathbb{E}\left[\frac{1}{2\epsilon}\mathbb{I}_{[K-\epsilon,K+\epsilon]}(S_u)\middle|\mathcal{F}_t\right] = \mathbb{P}\left[K-\epsilon\le S_u\le K+\epsilon \mid S_t\right]
$$

Furthermore $\frac{1}{2\epsilon}\mathbb{I}_{[K-\epsilon,K+\epsilon]}(S_u)$ converges to the local time process of S_u at K as ϵ tends to zero and this limit is finite. By the dominated convergence theorem

$$
\begin{aligned}
\mathbb{E}\left[\lim_{\epsilon\to 0}(2)|\mathcal{F}_t\right] &= \int_t^T \frac{1}{2}e^{-ru}\sigma^2\mathbb{E}\left[\lim_{\epsilon\to 0}\frac{S_u^2}{2\epsilon}\mathbb{I}_{[K-\epsilon,K+\epsilon]}(S_u)\middle|\mathcal{F}_t\right] \\
&= \int_t^T\left[\frac{1}{2}e^{-ru}\sigma^2K^2p(t,u,S_t,K)\right]du
\end{aligned}
\qquad (2.10)
$$

Combining (2.7) $\sim$ (2.10) we obtain equality (2.6).

□

Remark: An upper bound for EEP follows immediately from lemma 2.2.1 and theorem 2.2.2:

$$
EEP \le \mathbb{E}\sup_{t\in[0,T]}\int_t^T e^{-ru}\left[rk\int_0^K p(t,u,S_t,y)dy - \frac{1}{2}\sigma^2K^2p(t,u,S_t,K)\right]du, \qquad (2.11)
$$

where $p(t,u,x,y)$ is the transition density function defined by (2.5).

2.2.2 Numerical examples

Based on inequality (2.11), we can derive a closed-form upper bound. In this subsection we will numerically compare our upper bound with Rogers' upper bound. For this purpose we assume that the parameters take the same value as in Rogers [39] (page 277, table 4.1):

$$K = 100, r = 0.06, T = 0.5, \sigma = 0.4 \tag{2.12}$$

For convenience, we define a function $F(t,x)$ by

$$F(t,x) = \int_t^T e^{-ru} \left[f(u-t,x) - g(u-t,x) \right] du$$

where

$$f(u-t,x) \equiv rk \int_0^K p(t,u,x,y) dy \quad \text{and} \quad g(u-t,x) \equiv \frac{1}{2}\sigma^2 K^2 p(t,u,x,K).$$

Our goal is to estimate the quantity $\mathbb{E} \sup_{t\in[0,T]} F(t,S_t)$. To this end, we choose a fixed positive number x^* and consider the following 2 cases:

1. If $S_t < x^*$ for some t, then $f(u-t,x) - g(u-t,x) \leq f(u-t,x)$ since $g(u-t,x)$ is nonnegative. Notice that $f(u-t,x)$ is also nonnegative and $\int_0^K p(t,u,x,y)dy = \mathbb{P}[0 \leq S_u \leq K | S_t = x] \leq 1$ for any x, and hence
$$\sup_{t\in[0,T]} F(t,S_t) \leq \int_0^T e^{-ru} f(u-t,x) du \leq \int_0^T e^{-ru} rK du = K(1-e^{-rT}) \tag{2.13}$$

2. If $S_t \geq x^*$ for all $t \in [0,T]$, we notice that $f(u-t,x)$ is decreasing in x, and $g(u-t,x)$ has a local maximum at $x = \exp(\log(K) - (r - \frac{1}{2}\sigma^2)(u-t) \equiv x^{**}$. Moreover, $g(u-t,x)$ is increasing in x on $(0,x^{**})$ and decreasing on (x^{**},∞). Since both f and g are nonnegative, it follows that
$$f(u-t,S_t) - g(u-t,S_t) \leq \max\{f(u-t,x^*) - g(u-t,x^*),\ \ f(u-t,x^{**})\}$$

whenever $0 \leq u - t \leq T$. Consequently,

$$\sup_{t\in[0,T]} F(t, S_t) \leq \int_0^T e^{-ru} \max\{f(u-t, x^*) - g(u-t, x^*),\ \ f(u-t, x^{**})\} du. \quad (2.14)$$

Combine (2.13) and (2.14) to obtain

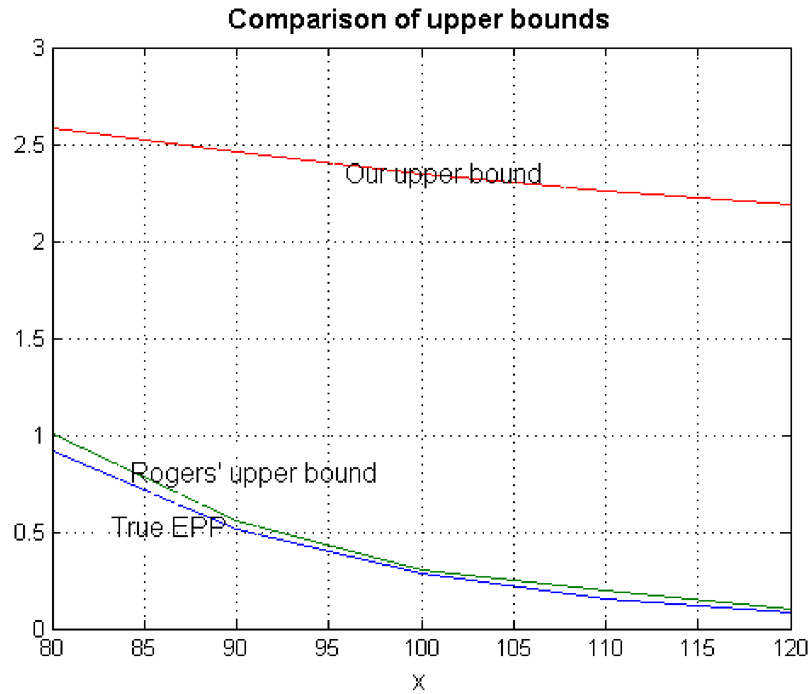
$$\begin{aligned}\mathbb{E} \sup_{t\in[0,T]} F(t, S_t) \leq P \cdot \int_0^T & e^{-ru} \max\{f(u-t, x^*) - g(u-t, x^*),\ \ f(u-t, x^{**})\} du \\ & + (1-P) \cdot K(1 - e^{-rT})\end{aligned} \quad (2.15)$$

where $P = \mathbb{P}[\min_{0\leq t\leq T} S_t \geq x^* | S_0 = x]$.

Next we compare our result with the first example in Rogers [39] (page 277, table 4.1). We calculate EPP when the initial stock price x is equal to $80, 90, 100, 110$, and 120. For each fixed x, we set x^* to be $50, 60, 70, 80$ and choose the minimum upper bound as our final result. The results are shown in the following table and graph.

Table 2.1

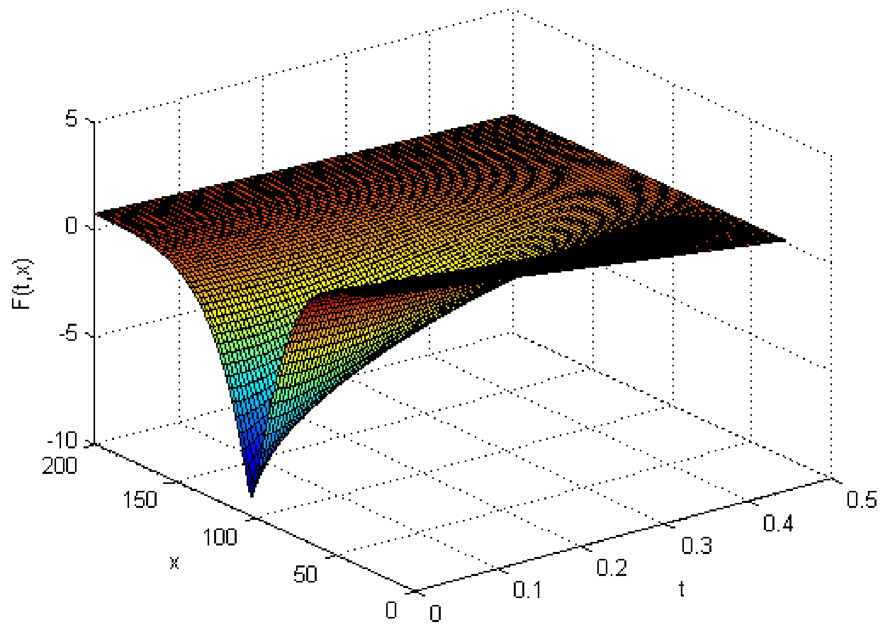
x	EEP (true)	EEP (Rogers)	EEP (ours)
80	0.9166	1.0060	2.5858
90	0.5102	0.5607	2.4607
100	0.2816	0.3061	2.3498
110	0.1555	0.1960	2.2590
120	0.0852	0.1002	2.1846

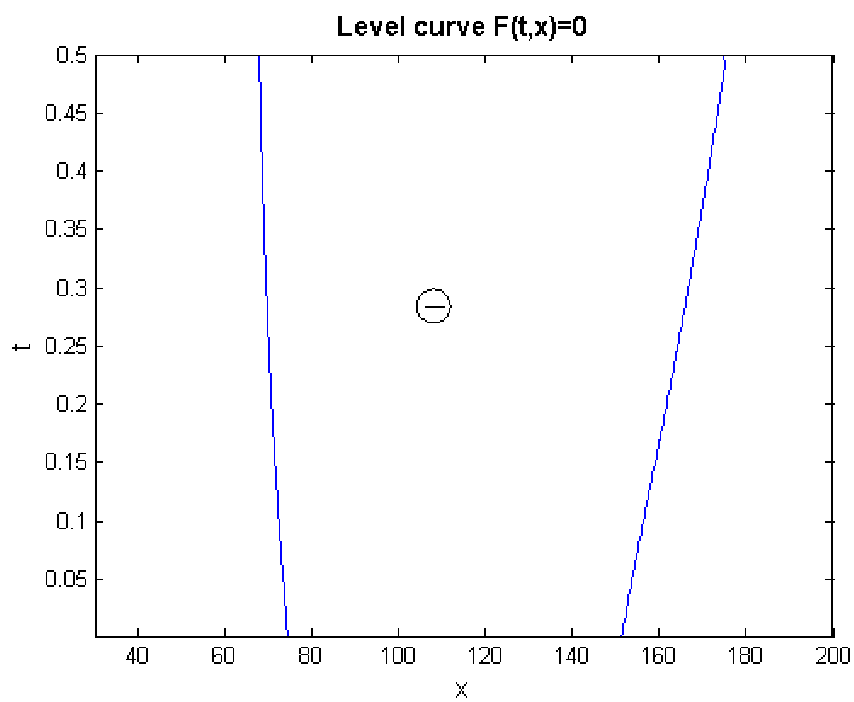


It can be seen that our upper bound works better for in-the-money put options but does not converge to 0 as S_0 increases. We have also tried a different method for at-the-money (i.e., $S_0 = K$) options. The idea is, for a given sample path of S_t, if it stays in a region where $F(t, S_t) \leq 0$ (we call it the $\ominus$ region), then clearly $\sup_{t\in[0,T]} F(t, S_t) = 0$. Otherwise we use the upper bound $K(1 - e^{-rT})$ (see (2.13)). Therefore

$$\mathbb{E} \sup_{t\in[0,T]} F(t, S_t) \leq P_1 \cdot K(1 - e^{-rT}), \tag{2.16}$$

where $P_1 = \mathbb{P}\left[S_t \text{ is not in the } \ominus \text{ region for some } t \in [0, T]\right]$. Now it remains to estimate the probability P_1. For this purpose we plot the graph of the function $F(t, x)$ and the level curve for $F(t, x) = 0$, as shown below.





From the second graph it can be seen that when $t = 0$, $F(t,x)$ has two zeros, call them x_1 and x_2 and suppose $x_1 < x_2$. Furthermore, the rectangle $R \equiv \{0 \leq t \leq T, x_1 \leq x \leq x_2\}$ is in the $\ominus$ region. Since $\log(S_t)$ is a Brownian motion with drift parameter

$(r - \frac{1}{2}\sigma^2) = -0.02$, an upper bound for P_1 can be obtained as follows:

$$P_1 \leq \mathbb{P}\left[\max_{0\leq t\leq T} \log(S_t/S_0) \geq \frac{x_2 - S_0}{\sigma}\right] + \mathbb{P}\left[\min_{0\leq t\leq T} \log(S_t/S_0) \leq \frac{S_0 - x_1}{\sigma}\right] \tag{2.17}$$

When $S_0 = 100$, formulas (2.17) and (2.16) yield an upper bound 2.2484 for EEP, which is smaller than the upper bound 2.3498 from table 4.1.

Remark:

1. This second method works better when S_0 is close to K since otherwise the upper bound (2.17) for P_1 will be close to 1;

2. It can be seen from the graph that the level curve $F(t, x) = 0$ is monotone. However, this has not been proved, and there is no guarantee that this remains true when the parameters change.

CHAPTER 3. Jump-to-default models

3.1 Jump-to-default models

The risk of default of a security is not taken into consideration in the standard Black-Scholes model. On the contrary, in recent years, default risk is in the center of study on corporate bonds. In this chapter, we consider a jump-to-default model in which the default is assumed to be the first jump time of a doubly stochastic Poisson process (Cox process). Our work is motivated by Elliott, Jeanblanc, and Yor [16], and Linetsky [33]. Formally, we assume that the pre-bankruptcy stock price under the equivalent martingale measure is governed by the SDE

$$dS_t = (r + h(S_t))S_t dt + \sigma S_t dW_t \tag{3.1}$$

where the function $h(\cdot)$ is called the default intensity. As in Linetsky [33], we assume that $h(\cdot)$ is a C^1 function, which is strictly decreasing and has the following limits:

$$\lim_{x\to 0} h(x) = +\infty, \qquad \lim_{x\to\infty} h(x) = 0$$

We model the default time τ_0 as

$$\tau_0 = \inf\left\{ t \geq 0 : \int_0^t h(S_u)du \geq e \right\}$$

where e is an exponentially distributed random variable with mean 1. It is assumed that the random variable e is independent of $(W_t)_{t\geq 0}$.

We denote by $S_t^{\triangle}$ the stock price process subject to default. Throughout this chapter we assume zero recovery in the case of default. Define the bankruptcy indicator process (D_t) by $D_t = \mathbb{I}_{[t \geq \tau_0]}$, then $S_t^{\triangle}$ follows a process in the form

$$dS_t^{\triangle} = S_{t-}^{\triangle}(rdt + \sigma dW_t - dM_t), \tag{3.2}$$

where

$$M_t = D_t - \int_0^{t \wedge \tau_0} h(S_u)du$$

is a martingale. For more details, we refer to Elliott, Jeanblanc, and Yor [16], and Linetsky [33]. Linetsky proved that for any deterministic time T,

$$\mathbb{E}[\mathbb{I}_{[\tau_0 > T]} | \mathcal{F}_T] = e^{-\int_0^T h(S_t)dt}. \tag{3.3}$$

Notice that from the stochastic equation (3.2), it readily follows that the discounted price process $e^{-rt}S_t^{\triangle}$ is a martingale. In other words, the additional term $h(S_t)$ in the drift of the pre-bankruptcy stock price process S_t compensates for the bankruptcy jump so that under the equivalent martingale measure, the total rate of return of $S_t^{\triangle}$ remains the same as the risk free interest rate. See also Linetsky [33].

In the next section we will prove that Rogers' method for obtaining upper bound can still be used for American options when the stock is subject to bankruptcy. We also prove an equality similar to (3.3) but the deterministic time T is replaced by any stopping time τ with respect to the filtration $(\mathcal{F}_t)$ generated by S_t. The problem tackled in the last section is an optimal stopping problem solved by Oksendal in [36] where the stock price follows a geometric Brownian motion. We derive the optimal strategy when the stock has a positive possibility of default.

3.2 Upper bound for American option

When the stock is subject to default, the discounted exercise value of American option becomes

$$Z_t = e^{-rt}(K - S_t^{\triangle})^+. \tag{3.4}$$

In this section we will prove that Rogers' upper bound (2.2) still applies when the stock price process is subject to default. In particular we will prove that $\sup_{\tau \in \mathcal{S}_{[0,T]}} \mathbb{E} Z_\tau < \infty$, $\sup_{0 \le t \le T} |Z_t| \in L^p$ for some $p > 1$, and Z_t is right continuous.

Lemma 3.2.1. Let $(\mathcal{F}_t)$ be a given filtration and τ be a discrete valued $(\mathcal{F}_t)$-stopping time with range $\{t_1, t_2, \cdots\}$. If Y is a nonnegative random variable with $\mathbb{E}[Y] < \infty$, then

$$\mathbb{E}[Y|\mathcal{F}_\tau] = \sum_{k=1}^{\infty} \mathbb{I}_{[\tau = t_k]} \mathbb{E}[Y|\mathcal{F}_{t_k}]$$

where the stopped σ-algebra $\mathcal{F}_\tau$ is defined by

$$\mathcal{F}_\tau \equiv \{A : A \cap [\tau \le t] \in \mathcal{F}_t \text{ for each } t \ge 0\}$$

Proof. We claim that, for any k,

1. $\mathbb{E}[Y \mathbb{I}_{[\tau = t_k]} | \mathcal{F}_\tau]$ is $\mathcal{F}_{t_k}$ measurable.

2. $\mathbb{E}[Y \mathbb{I}_{[\tau = t_k]} | \mathcal{F}_{t_k}]$ is $\mathcal{F}_\tau$ measurable.

3. $\mathbb{E}[Y \mathbb{I}_{[\tau = t_k]} | \mathcal{F}_\tau] = \mathbb{I}_{[\tau = t_k]} \mathbb{E}[Y|\mathcal{F}_{t_k}]$.

To prove claim 1, let $U = \mathbb{E}[Y \mathbb{I}_{[\tau = t_k]} | \mathcal{F}_\tau]$, then for any nonnegative number l, the set

$$[U \mathbb{I}_{[\tau = t_k]} \le l] = ([U \le l] \cap [\tau = t_k]) \cup ([\tau \ne t_k])$$

Since U is $\mathcal{F}_\tau$ measurable, it follows from the definition of $\mathcal{F}_\tau$ that the set $([U \le l] \cap [\tau = t_k])$ is in $\mathcal{F}_{t_k}$. Hence $U \mathbb{I}_{[\tau = t_k]}$ is $\mathcal{F}_{t_k}$ measurable since the set $([\tau \ne t_k])$ is also in $\mathcal{F}_{t_k}$.

On the other hand, we notice that

$$U\mathbb{I}_{[\tau=t_k]} = \mathbb{E}[Y\mathbb{I}_{[\tau=t_k]}|\mathcal{F}_\tau]\cdot\mathbb{I}_{[\tau=t_k]} = \mathbb{E}[Y\mathbb{I}_{[\tau=t_k]}\mathbb{I}_{[\tau=t_k]}|\mathcal{F}_\tau] = \mathbb{E}[Y\mathbb{I}_{[\tau=t_k]}|\mathcal{F}_\tau] = U$$

Therefore U itself is $\mathcal{F}_{t_k}$ measurable. This proves the first claim.

To prove claim 2, let $Z = \mathbb{E}[Y\mathbb{I}_{[\tau=t_k]}|\mathcal{F}_{t_k}]$. Notice that $Z = \mathbb{E}[Y|\mathcal{F}_{t_k}]\mathbb{I}_{[\tau=t_k]}$. Then for any nonnegative number l, the set $[Z \le l] \cap [\tau \le t] = A_1 \cup A_2$, where

$A_1 = [\mathbb{E}[Y|\mathcal{F}_{t_k}] \le l] \cap [\tau = t_k] \cap [\tau \le t]$, and

$A_2 = [\tau \ne t_k] \cap [\tau \le t]$.

To show that Z is $\mathcal{F}_\tau$ measurable, it's enough to show that both A_1 and A_2 are in $\mathcal{F}_t$. To this end we consider two cases. If $t < t_k$, $A_1 = \Phi \in \mathcal{F}_t$ and $A_2 = [\tau \le t] \in \mathcal{F}_t$. If $t \ge t_k$, $A_1 = [\mathbb{E}[Y|\mathcal{F}_{t_k}] \le l] \cap [\tau = t_k] \in \mathcal{F}_{t_k} \subseteq \mathcal{F}_t$, and

$$A_2 = \bigcup_{t_j \le t} [\tau = t_j] \in \mathcal{F}_t.$$

This completes the proof of claim 2.

Now suppose V is a nonnegative $\mathcal{F}_{t_k}$ measurable random variable, then by claim 2, $V\mathbb{I}_{[\tau=t_k]}$ is $\mathcal{F}_\tau$ measurable. Therefore

$$\begin{aligned}\mathbb{E}[V\cdot U] &= \mathbb{E}\left[V\cdot\mathbb{E}[Y\mathbb{I}_{[\tau=t_k]}|\mathcal{F}_\tau]\right] = \mathbb{E}\left[V\mathbb{I}_{[\tau=t_k]}\mathbb{E}[Y|\mathcal{F}_\tau]\right] \\ &= \mathbb{E}\left[\mathbb{E}[YV\mathbb{I}_{[\tau=t_k]}|\mathcal{F}_\tau]\right] = \mathbb{E}[YV\mathbb{I}_{[\tau=t_k]}] = \mathbb{E}\left[V\mathbb{I}_{[\tau=t_k]}\mathbb{E}[Y|\mathcal{F}_{t_k}]\right]\end{aligned}$$

This implies that $U = \mathbb{I}_{[\tau=t_k]}\mathbb{E}[Y|\mathcal{F}_{t_k}]$ almost surely. This completes the proof of the last claim. Since this is true for any k, it follows that

$$\mathbb{E}[Y|\mathcal{F}_\tau] = \mathbb{E}\left[\sum_k (Y\mathbb{I}_{\tau=t_k})|\mathcal{F}_\tau\right] = \sum_k \mathbb{E}\left[Y\mathbb{I}_{\tau=t_k}|\mathcal{F}_\tau\right] = \sum_k \mathbb{I}_{\tau=t_k}\mathbb{E}[Y|\mathcal{F}_{t_k}]$$

The proof of the lemma is complete.

$\square$

Let τ_0 be the default time defined in section 3.1. Note that τ_0 is not an $(\mathcal{F}_t)$-stopping time. Take the nonnegative random variable Y to be the indicator function $\mathbb{I}_{[\tau<\tau_0]}$ for any $(\mathcal{F}_t)$-stopping time τ, then we have the following lemma:

Lemma 3.2.2. For any $(\mathcal{F}_t)$-stopping time τ,

$$\mathbb{E}\left[\mathbb{I}_{[\tau<\tau_0]}|\mathcal{F}_\tau\right] = e^{-\int_0^\tau h(S_u)du}.$$

Proof. First we assume that the stopping time τ take values $t_1, t_2, t_3, \ldots$. In this case, by lemma 3.2.1,

$$\mathbb{E}\left[\mathbb{I}_{[\tau<\tau_0]}|\mathcal{F}_\tau\right] = \sum_k \mathbb{I}_{[\tau=t_k]}\mathbb{E}\left[\mathbb{I}_{[\tau<\tau_0]}|\mathcal{F}_{t_k}\right]$$

Notice that for each k, $\mathbb{I}_{[\tau=t_k]} = \mathbb{I}_{[\tau=t_k]} \cdot \mathbb{I}_{[\tau=t_k]}$ and that $\mathbb{I}_{[\tau=t_k]}$ is $\mathcal{F}_{t_k}$-measurable, therefore

$$\begin{aligned}
& \mathbb{I}_{[\tau=t_k]} \cdot \mathbb{E}\left[\mathbb{I}_{[\tau<\tau_0]}|\mathcal{F}_{t_k}\right] \\
= \;& \mathbb{I}_{[\tau=t_k]} \cdot \mathbb{I}_{[\tau=t_k]} \cdot \mathbb{E}\left[\mathbb{I}_{[\tau<\tau_0]}|\mathcal{F}_{t_k}\right] \\
= \;& \mathbb{I}_{[\tau=t_k]} \cdot \mathbb{E}\left[\mathbb{I}_{[\tau<\tau_0]} \cdot \mathbb{I}_{[\tau=t_k]}|\mathcal{F}_{t_k}\right] \\
= \;& \mathbb{I}_{[\tau=t_k]} \cdot \mathbb{E}\left[\mathbb{I}_{[t_k<\tau_0]}|\mathcal{F}_{t_k}\right] \\
= \;& \mathbb{I}_{[\tau=t_k]} \cdot e^{-\int_0^{t_k} h(S_u)du} \quad \text{by equality (3.3).}
\end{aligned}$$

It follows that

$$\mathbb{E}\left[\mathbb{I}_{[\tau<\tau_0]}|\mathcal{F}_\tau\right] = \sum_k \mathbb{I}_{[\tau=t_k]} \cdot e^{-\int_0^{t_k} h(S_u)du} = e^{-\int_0^\tau h(S_u)du}$$

Now suppose τ is any $\mathcal{F}_t$-stopping time. Define a sequence of discrete valued stopping times by

$$\tau_n = \frac{[2^n\tau]+1}{2^n} \tag{3.5}$$

where $[a]$ denotes the largest integer that is less than or equal to a. Then τ_n is decreasing and $\tau_n \to \tau$ as $n \to \infty$. Moreover, $\mathcal{F}_\tau = \bigcap_n \mathcal{F}_{\tau_n}$ (see Durrett [15], page 348, theorem 6).

For each n, by lemma 3.2.2, $\mathbb{E}\left[\mathbb{I}_{[\tau_n<\tau_0]}|\mathcal{F}_{\tau_n}\right] = e^{-\int_0^{\tau_n} h(S_u)du} \to e^{-\int_0^{\tau} h(S_u)du}$ almost surly. By the dominated convergence theorem for conditional expectations (see, for example, Durrett [15], page 227, (5.7)), $\mathbb{E}[\mathbb{I}_{[\tau_n<\tau_0]}|\mathcal{F}_{\tau_n}] \to \mathbb{E}[\mathbb{I}_{[\tau<\tau_0]}|\mathcal{F}_{\tau}]$ almost surly. So we conclude that $\mathbb{E}\left[\mathbb{I}_{[\tau<\tau_0]}|\mathcal{F}_{\tau}\right] = e^{-\int_0^{\tau} h(S_u)du}$.

This completes the proof.

□

Using the above lemma we can write $\mathbb{E}Z_t$ in terms of the pre-bankruptcy stock price process S_t instead of $S_t^{\triangle}$. We will show that $\mathbb{E}Z_t$ can be written as an expectation of a bounded function.

Theorem 3.2.3. Let Z_t be the discounted exercise value of American option defined by (3.4), and τ be any stopping time with respect to the filtration generated by $S_t^{\triangle}$, then

$$\mathbb{E}Z_\tau = \mathbb{E}\left[Ke^{-r\tau} - (s_\tau \wedge K)e^{-\int_0^{\tau}(r+h(S_u))du}\right] \tag{3.6}$$

where the operator $\wedge$ describes the minimum of two numbers or variables.

Proof. Write $\mathbb{E}Z_\tau$ as the sum of two terms:

$$\mathbb{E}Z_\tau = \mathbb{E}\left[e^{-r\tau}(K-S_\tau^{\triangle})^+\right] = \mathbb{E}\left[e^{-r\tau}(K-S_\tau^{\triangle})^+\mathbb{I}_{[\tau<\tau_0]}\right] + \mathbb{E}\left[e^{-r\tau}(K-S_\tau^{\triangle})^+\mathbb{I}_{[\tau\geq\tau_0]}\right]$$

It follows from lemma 3.2.2 that the first term

$$\begin{aligned}
& \mathbb{E}\left[e^{-r\tau}(K-S_\tau^{\triangle})^+\mathbb{I}_{[\tau<\tau_0]}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{-r\tau}(K-S_\tau)^+\mathbb{I}_{[\tau<\tau_0]}\middle|\,\mathcal{F}_\tau\right]\right] \\
= \;& \mathbb{E}\left[e^{-r\tau}(K-S_\tau)^+\mathbb{E}\left[\mathbb{I}_{[\tau<\tau_0]}\middle|\,\mathcal{F}_\tau\right]\right] = \mathbb{E}\left[e^{-r\tau}(K-S_\tau)^+e^{-\int_0^{\tau} h(S_u)du}\right]
\end{aligned}$$

since $e^{-r\tau}(K-S_\tau)^+$ is $\mathcal{F}_\tau$-measurable. Similarly, the second term

$$\begin{aligned}
& \mathbb{E}\left[e^{-r\tau}(K-S_\tau^{\triangle})^+\mathbb{I}_{[\tau\geq\tau_0]}\right] = \mathbb{E}\left[e^{-r\tau}(K-S_\tau^{\triangle})^+\left(1-e^{-\int_0^{\tau} h(S_u)du}\right)\right] \\
= \;& \mathbb{E}\left[e^{-r\tau}K\left(1-e^{-\int_0^{\tau} h(S_u)du}\right)\right].
\end{aligned}$$

The second equality follows from the assumption that $s_\tau^\triangle = 0$ when $\tau \geq \tau_0$. Put the two terms together we obtain

$$\begin{aligned}\mathbb{E}Z_\tau &= \mathbb{E}\left[Ke^{-r\tau} + e^{-\int_0^\tau (r+h(S_u))du}[(K - S_\tau)^+ - K]\right] \\ &= \mathbb{E}\left[Ke^{-r\tau} - (S_\tau \wedge K)e^{-\int_0^\tau (r+h(S_u))du}\right],\end{aligned}$$

as needed.

□

Notice that $\left|Ke^{-r\tau} - (S_\tau \wedge K)e^{-\int_0^\tau (r+h(S_u))du}\right| \leq 2K$, it follows immediately that the conditions in Rogers' proof are satisfied, i.e., $\sup_{\tau \in \mathcal{S}_{[0,T]}} \mathbb{E}Z_\tau < \infty$, $\sup_{0 \leq t \leq T} |Z_t| \in L^p$ for some $p > 1$, and Z_t is right continuous.

3.3 The optimal time to sell the stock

In this section we consider an investor who needs to choose the optimal time to sell a stock so that the discounted value of the stock is maximized. The stock price is assumed to follow the jump-to-default model introduced in section 3.1. As in Oksendal [36], We also assume that the discount rate $\rho > r$ and that there is a financial charge of a dollars when the investor sell the stock. The investor may deicide to sell the stock at any $(\mathcal{F}_t)$-stopping time τ where $(\mathcal{F}_t)$ represents the filtration generated by $S_t^\triangle$. If default occurs before the investor sell the stock, the payoff to the investor is 0. So this problem can be described as the following optimal stopping problem:

$$\text{Maximize } \mathbb{E}_x\left[e^{-\rho\tau}(S_\tau^\triangle - a)\mathbb{I}_{[\tau < \tau_0]}\right]$$

over all $(\mathcal{F}_t)$-stopping times τ, where τ_0 again denotes the default time.

Notice that $S_\tau^\triangle = S_\tau$ when $\tau < \tau_0$, it follows from theorem 3.2.3 that the value function for this optimal stopping problem can be written as

$$V(x) \equiv \sup_\tau \mathbb{E}_x \left[e^{-\int_0^\tau (\rho + h(S_u))du} (S_\tau - a) \right] \tag{3.7}$$

where the supremum is taken over all $(\mathcal{F}_t)$-stopping times τ. To obtain the solution to this optimal stopping problem, we need to compare solutions to certain second order differential equations. For this, we intend to use the maximum principle for ordinary differential equations (see,for example, Protter and Wenberger [38]).

In this section, we first state and prove an upper bound of the objective function, then we construct a strategy such that this upper bound can be achieved. The construction of an optimal strategy will be provided at the end of the section.

3.3.1 The optimal strategy

We first state and prove the verification lemma, which helps us to sort out an optimal stopping time.

Lemma 3.3.1. (Verification lemma) Let Q be a nonnegative C^1 function which is piecewise C^2. Also assume that the limits $\lim_{x \to c-} Q''(x)$ and $\lim_{x \to c+} Q''(x)$ exist and are finite for each c. Let Q satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\max\{\frac{\sigma^2}{2} x^2 Q''(x) + (r + h(x))xQ'(x) - (\rho + h(x))Q(x), (x - a) - Q(x)\} = 0 \tag{3.8}$$

for almost all x in $[0, \infty)$, then

$$Q(x) \geq V(x)$$

where $V(x)$ is the value function defined by (3.7).

Proof. For any $\mathcal{F}_t$-stopping time τ, using a localization procedure and applying Itô's formula to $e^{-\int_0^\tau (\rho + h(S_u))du} Q(S_\tau)$ and taking expectation yields

$$\mathbb{E}_x \left[e^{-\int_0^\tau (\rho + h(S_u))du} Q(S_\tau) \right] = Q(x) + \mathbb{E}_x \left[\int_0^\tau e^{-\int_0^u (\rho + h(S_r))dr} \mathcal{L}(Q(S_u)) \cdot du \right]$$

where the differential operator $\mathcal{L}$ is defined by

$$\mathcal{L} \equiv \frac{\sigma^2}{2}x^2\frac{d^2}{dx^2} + (r + h(x))x\frac{d}{dx} - (\rho + h(x)).$$

It follows from the HJB equation that both $\mathcal{L}(Q(x))$ and $(x - a) - Q(x)$ are non-positive. Therefore

$$Q(x) \geq \mathbb{E}_x\left[e^{-\int_0^\tau(\rho+h(S_u))du}Q(S_\tau)\right] \geq \mathbb{E}_x\left[e^{-\int_0^\tau(\rho+h(S_u))du}(S_\tau - a)\right]$$

The conclusion in the lemma follows since the above inequality holds for every $(\mathcal{F}_t)$ stopping time.

$\square$

Now we assume that there exists a point $x^* > a$ and a increasing C^2 function $\widetilde{Q}_*(\cdot)$ which is defined on $\mathbb{R}$ such that it satisfies the differential equation $\mathcal{L}(\widetilde{Q}_*(x)) = 0$ everywhere on $\mathbb{R}$. We also assume that $\widetilde{Q}_*(x^*) = x^* - a$, $\widetilde{Q}'_*(x^*) = 1$ and $\widetilde{Q}_*(x) > x - a$ for all $x < x^*$. The existence of the point x^* and such a function $\widetilde{Q}_*(\cdot)$ will be shown in the next subsection.

Consider a function Q_* defined by

$$Q_*(x) = \begin{cases} \widetilde{Q}_*(x) & \text{if } x \leq x^* \\ x - a & \text{if } x \geq x^* \end{cases}$$

Then $Q_*(x)$ satisfies the following conditions:

(i) $\mathcal{L}(Q_*(x)) = 0$ and $Q_*(x) > x - a$ if $x < x^*$;

(ii) $Q_*(x) = x - a$ if $x \geq x^*$; and

(iii) $Q'_*(x)$ is continuous everywhere, and $Q''_*(x)$ is continuous everywhere except at x^*. Furthermore, $Q''_*(x^*-)$ and $Q''_*(x^*+)$ are finite.

Since $Q''_*(x^*-) = \widetilde{Q}''_*(x^*) \geq 0$ from theorem 3.3.6, and the fact that $\mathcal{L}(\widetilde{Q}_*(x^*)) = 0$, it follows that $(\rho - r)x^* - a(\rho + h(x^*)) \geq 0$. Therefore $(\rho - r)x - a(\rho + h(x)) \geq (\rho - r)x^* - a(\rho + h(x^*)) \geq 0$ for all $x \geq x^*$, since $\rho > r$ and $h(\cdot)$ is strictly decreasing. Then it is straightforward to check that $Q_*(x)$ satisfies the HJB equation and it is C^1 on $\mathbb{R}$ and a piecewise C^2 function. Hence, by the verification lemma, $Q_*(x)$ is an upper bound for the value function $V(x)$. We will now construct a strategy such that this upper bound is achieved.

Define the $(\mathcal{F}_t)$-stopping time

$$\tau^* \equiv \inf\{t \geq 0 : S_t \geq x^*\},$$

and

$$V_*(x) \equiv \mathbb{E}_x \left[e^{-\int_0^{\tau^*} (\rho + h(S_u)) du} (S_{\tau^*} - a) \right].$$

We intend to prove that

$$V_*(x) = Q_*(x) \tag{3.9}$$

and τ_{x^*} is an optimal stopping time.

When $x \geq x^*$, obviously $Q_*(x) = V_*(x) = x - a$. If $x < x^*$, we need to show that $Q_*(x) = V_*(x)$. To this end, we apply Itô's formula to $Q_*(S_{\tau_*}) e^{-\int_0^{\tau^*} (\rho + h(S_r)) dr}$ and take expectation to get

$$\begin{aligned} & \mathbb{E}_x \left[Q_*(S_{\tau^*}) e^{-\int_0^{\tau^*} (\rho + h(S_r)) dr} \right] \\ = \; & Q_*(x) + \mathbb{E}_x \left[\int_0^{\tau^*} \mathcal{L}(Q(S_u)) \cdot e^{-\int_0^u (\rho + h(S_r)) dr} du \right] \\ = \; & Q_*(x) \end{aligned}$$

The second equality holds since $\mathcal{L}(Q_*(x)) = 0$ for any x. Moreover, by the definition of τ^*, we notice that $Q_*(S_{\tau^*}) = Q_*(x^*) = S_{\tau^*} - a$. Therefore

$$Q_*(x) = \mathbb{E}_x \left[(S_{\tau^*} - a) e^{-\int_0^{\tau^*} (\rho + h(S_r)) dr} \right] = V_*(x)$$

We have proved that the strategy that the investor sell the stock at the first time when the stock price reaches x^* achieves the upper bound and hence this strategy is optimal. What's left is to show the existence of the point x^* and the function $\widetilde{Q}_*(x)$, which will be done in the next subsection.

3.3.2 Existence of the function $\widetilde{Q}_*(x)$

In this subsection we will prove the existence of the point x^* and the function $\widetilde{Q}_*(x)$. We first consider the following transformation: let $y = \ln(x)$, and $Y_t = \ln(S_t)$, then the differential equation $\mathcal{L}(Q) = 0$ becomes $\mathcal{T}(Q) = 0$ with the operator $\mathcal{T}$ defined as

$$\mathcal{T} = \frac{\sigma^2}{2}\frac{d^2}{dy^2} + (r - \frac{\sigma^2}{2} + \psi(y))\frac{d}{dy} - (\rho + \psi(y))$$

where $\psi(y) = h(e^y)$ is the transformed default density function that satisfies $\lim_{y\to-\infty}\psi(y) = \infty$ and $\lim_{y\to\infty}\psi(y) = 0$. Furthermore, the process Y_t is governed by the stochastic differential equation

$$dY_t = (r - \frac{\sigma^2}{2} + \psi(Y_t))dt + \sigma dW_t \tag{3.10}$$

To prove the existence of x^* and $\widetilde{Q}_*(x)$, it will be sufficient to find a number y^* and a function $Q(y)$ such that the following problem (3.11) has a non-negative solution, then choose the point $x^* = e^{y^*}$ and the function $\widetilde{Q}_*(x) = Q(\ln(x))$.

$$\begin{cases} \mathcal{T}(Q(y)) = 0 \\ Q(y^*) = e^{y^*} - a \text{ and } Q'(y^*) = e^{y^*} \\ Q(y) > e^y - a \text{ for all } y < y^* \end{cases} \tag{3.11}$$

The proof will be divided into 3 steps:

Step 1: prove that for any large number b, the following boundary value problem has a solution:

$$\begin{cases} \mathcal{T}(Q(y)) = 0 \\ Q(b) = e^b - a \\ \lim_{y\to-\infty} Q(y) = l_b \text{ for some } l_b \geq 0 \end{cases} \tag{3.12}$$

Step 2: show that for large b, the solution Q to (3.12) satisfies $Q'(b) > e^b$ hence $Q(\cdot)$ will cross the curve $e^y - a$ at least twice.

Step 3: prove the existence of a solution to problem (3.11).

We start the proof by introducing a function $Q_b(y)$ defined on the interval $(-\infty, b]$:

$$Q_b(y) \equiv \mathbb{E}_y \left[e^{-\int_0^{\tau_b}(\rho+\psi(Y_s))ds} \right] (e^b - a), \tag{3.13}$$

where b is a positive number such that $e^b > a$ and $\tau_b \equiv \inf\{t \geq 0 : Y_t = b\}$. Our aim is to show that $Q_b(y)$ is a bounded solution to problem (3.12) on the interval $(-\infty, b]$. The major difficulty is how to analyze the behavior of $Q_b(y)$ as y goes to $-\infty$. To avoid this difficulty we first consider functions defined on finite intervals.

Lemma 3.3.2. Let b be a positive number such that $e^b > a$, Let Y_t be the process satisfying (3.10). For each positive integer n, define the stopping time τ_n and the function Q_n by

$$\tau_n \equiv \inf\{t \geq 0 : Y_t = b \text{ or } Y_t = -n\}, \text{ and}$$

$$Q_n(y) \equiv \mathbb{E}_y \left[e^{-\int_0^{\tau_n}(\rho+\psi(Y_s))ds} \mathbb{I}_{[Y_{\tau_n}=b]} \right] (e^b - a)$$

then

(i) $Q_n(y)$ satisfies $\mathcal{T}(Q_n(y)) = 0$, $Q_n(-n) = 0$, and $Q_n(b) = e^b - a$;

(ii) $Q_n(y)$ has no local extrema in $(-n, b)$;

(iii) $Q_n'(y) > 0$ for $y \in (-n, b)$; and

(iv) For any fixed $y < b$, the sequence $\{Q_n(y)\}_{n>-y}$ is strictly increasing;

(v) $\lim_{n\to\infty} Q_n(y) = Q_b(y)$.

Proof. The proof of (i) is essentially the same as the proof of (3.9).

To prove (ii), we notice that $Q_n(y)$ is nonnegative by its definition. Suppose $Q_n(d) = 0$ for some d in $(-n, b)$, then $Q_n'(d) = 0$ since $Q_n \geq 0$ on $[-n, b]$. Now by the uniqueness of the solution to the initial value problem

$$\mathcal{T}Q_n = 0, \quad Q_n(d) = Q_n'(d) = 0,$$

$Q_n(y) = 0$ for all y in $[-n, b]$. This contradicts with $Q_n(b) = e^b - a > 0$. Therefore $Q_n(y)$ is strictly positive on $(-n, b]$. Furthermore, $Q_n(y)$ satisfies the differential equation $\mathcal{T}(Q_n) = 0$, so we have

$$\frac{1}{2}\sigma^2 Q_n''(y) = (\rho + \psi(y))Q_n(y) - (r - \frac{\sigma^2}{2} + \psi(y))Q_n'(y) \tag{3.14}$$

Suppose $c \in (-n, b)$ is a local maximum of Q_n, then $Q_n'(c) = 0$ so the right hand side of (3.14) is strictly positive hence $Q_n''(c) > 0$, contradicting the assumption that $y = c$ is a local maximum. Therefore Q_n has no local maximum. Now suppose c is a local minimum, since $Q_n(c) > 0$ and $Q_n(-n) = 0$, there must be a local maximum between $-n$ and c, which is impossible. This proves (ii).

It follows from (ii) that Q_n is monotone. Moreover, equation (3.14) implies that $Q_n''(y) > 0$ whenever $Q_n'(y) = 0$ hence Q_n has no saddle points and Q_n must be strictly monotonic. Combined with the fact that $Q_n(-n) < Q_n(b)$, we conclude that Q_n is strictly increasing.

To prove (iv), it's enough to show that for any n, $Q_{n+1}(y) > Q_n(y)$ on the interval $(-n, b)$. By (iii), $Q_{n+1}(y)$ is strictly increasing hence $Q_{n+1}(-n) > Q_{n+1}(-n-1) = 0$.

If there exists a point $c \in (-n, b)$ such that $Q_{n+1}(c) \le Q_n(c)$, there must be a point between $-n$ and c at which $Q_{n+1}(c)$ and $Q_n(c)$ are equal. But $Q_{n+1}(b) = Q_n(b) = e^b - a$, by the maximum principle, Q_{n+1} and Q_n must be the same function, contradicting $Q_{n+1}(-n) > 0 = Q_n(-n)$. This completes the proof of (iv).

To prove the convergence of $Q_n(y)$ to $Q_b(y)$, we first need to show that $\mathbb{I}_{[Y_{\tau_n}=b]} \to 1$ as $n \to \infty$. For this purpose we consider the scale function $S(y)$ of the process Y_t.

$$S(y) \equiv \int_c^y \exp\left[-\int_c^z \frac{2(r - \sigma^2/2 + \psi(u))}{\sigma^2} du\right] dz$$

where c is a fixed number. Then the probability that Y_t reaches b before $-n$, or equivalently, $Y_{\tau_n} = b$, can be represented as (see Bhattacharya and Waymire [2], page 419)

$$P_y[Y_{\tau_n} = b] = \frac{S(y) - S(-n)}{S(b) - S(-n)} = \frac{\int_{-n}^y \exp\left[-\int_c^z \frac{2(r-\sigma^2/2+\psi(u))}{\sigma^2} du\right] dz}{\int_{-n}^b \exp\left[-\int_c^z \frac{2(r-\sigma^2/2+\psi(u))}{\sigma^2} du\right] dz}$$

Choose a number $d < c$,

$$\int_{-n}^d \exp\left[-\int_c^z \frac{2(r - \sigma^2/2 + \psi(u))}{\sigma^2} du\right] dz = \int_{-n}^d \exp\left[\int_z^c \frac{2(r - \sigma^2/2 + \psi(u))}{\sigma^2} du\right] dz \to \infty$$

since $\psi(u) \to \infty$ as $u \to -\infty$. Therefore

$$\begin{aligned} P_y[Y_{\tau_n} = b] &= \frac{\int_{-n}^d \exp\left[-\int_c^z \frac{2(r-\sigma^2/2+\psi(u))}{\sigma^2} du\right] dz + \int_d^y \exp\left[-\int_c^z \frac{2(r-\sigma^2/2+\psi(u))}{\sigma^2} du\right] dz}{\int_{-n}^d \exp\left[-\int_c^z \frac{2(r-\sigma^2/2+\psi(u))}{\sigma^2} du\right] dz + \int_d^b \exp\left[-\int_c^z \frac{2(r-\sigma^2/2+\psi(u))}{\sigma^2} du\right] dz} \\ &\to 1 \text{ as } n \to \infty \end{aligned}$$

Consequently $\mathbb{I}_{[Y_{\tau_n}=b]} \to 1$ almost surely and $Q_n(y) \to Q_b(y)$ for any $y \le b$.

The proof of lemma is complete.

□

Since the function Q_b is the limit of Q_n, it is not surprising that they share many properties, which will be described and proved in the next proposition.

Proposition 3.3.3. The function Q_b defined by (3.13) is a bounded and nonnegative solution to $\mathcal{T}(Q_b) = 0$ and satisfies the boundary condition $Q_b(b) = e^b - a$. Furthermore,

(i) Q_b has no local extrema in $(-\infty, b)$;

(ii) $Q_b'(y) > 0$;

(iii) $Q_b(y)$ is bounded on $(-\infty, b]$;

(iv) $\lim_{y\to-\infty} Q_b(y) = l_b$ where $l_b \geq 0$ is a finite number.

Proof. In order to show that $\mathcal{T}(Q_b) = 0$, we notice that Q_n satisfies this differential equation. Integrate the equation $\mathcal{T}(Q_n) = 0$ to yield:

$$\frac{\sigma^2}{2}Q_n'(b) = \frac{\sigma^2}{2}Q_n'(y) - \int_y^b (r - \frac{\sigma^2}{2} + \psi(u))Q_n'(u)du + \int_y^b (\rho + \psi(r))dr$$

Applying integration by parts to the second term on the right hand side gives

$$\begin{aligned}\frac{\sigma^2}{2}Q_n'(b) = \frac{\sigma^2}{2}Q_n'(x) \quad &- \quad (r - \frac{\sigma^2}{2} + \psi(b))Q_n(b) + (r - \frac{\sigma^2}{2} + \psi(y))Q_n(y) \\ &+ \quad \int_y^b Q_n(r)\psi'(r)dr + \int_y^b (\rho + \psi(r))Q_n(r)dr\end{aligned}$$

Integrate once again to obtain:

$$\begin{aligned}\frac{\sigma^2}{2}Q_n'(b)(b-y) =& \frac{\sigma^2}{2}[Q_n(b) - Q_n(y)] - (r - \frac{\sigma^2}{2} + \psi(b))Q_n(b)(b-y) \\ &+ \int_y^b (r - \frac{\sigma^2}{2} + \psi(u))Q_n(u)du + \int_y^b \int_u^b Q_n(r)\psi'(r)drdu \\ &+ \int_y^b \int_u^b (\rho + \psi(r))Q_n(r)drdu\end{aligned} \tag{3.15}$$

By lemma 3.3.2, $Q_n(y) \to Q_b(y)$ as $n \to \infty$. Notice that $Q_n(y)$ is bounded by $e^b - a$ on $(-\infty, b)$. By the bounded convergence theorem, the right hand side of (3.15) is convergent hence the left hand side must also converge. In other words, $Q_n'(b)$ converges to a finite number λ. The limit of equation (3.15) is

$$\begin{aligned}\frac{\sigma^2}{2}\lambda(b-y) =& \frac{\sigma^2}{2}[Q_b(b)-Q_b(y)] - (r-\frac{\sigma^2}{2}+\psi(b))Q_b(b)(b-y) \\ &+ \int_y^b (r-\frac{\sigma^2}{2}+\psi(u))Q_b(u)du + \int_y^b \int_u^b Q_b(r)\psi'(r)drdu \\ &+ \int_y^b \int_u^b (\rho+\psi(r))Q_b(r)drdu\end{aligned} \tag{3.16}$$

Notice that, using (3.16), $Q_b(y)$ can be written as a linear combination of several differentiable functions, and hence it is also differentiable. Differentiate (3.16) at b we get

$$\begin{aligned}-\frac{\sigma^2}{2}\lambda =& -\frac{\sigma^2}{2}Q_b'(b) + (r-\frac{\sigma^2}{2}+\psi(b))Q_b(b)b - (r-\frac{\sigma^2}{2}+\psi(b)) \\ =& -\frac{\sigma^2}{2}Q_b'(b)\end{aligned}$$

So $Q_b'(b) = \lambda$. Using this result in (3.16), we find that $Q_b(y)$ and $Q_n(y)$ satisfies the same integral equation hence they must satisfy the same differential equation $\mathcal{T}(Q) = 0$. It is obvious that $Q_b(y)$ satisfies the boundary condition $Q_b(b) = e^b - a$.

The proof of claim (i) and (ii) in the proposition is exactly the same as that of (ii) and (iii) in lemma 3.3.2. For claim (iii), $Q_b(y)$ is bounded on $(-\infty, b]$ since each Q_n is bounded by $e^b - a$. Claim (iv) follows since $Q_b(y)$ is decreasing and bounded below by 0 as $y \to -\infty$.

This completes the proof of the proposition.

$\square$

Remark: Since $\tau_b < \infty$ with probability one, strong Markov property yields the following relationship for the family $\{Q_b : b > \ln a\}$: if $b_1 > b_2$, then $Q_{b_1}(y) = Q_{b_2}(y) \cdot Q_{b_1}(b_2)$. Since Q_{b_1} and Q_{b_2} satisfy the same ODE, it follows that they are constant multiples of each other.

The function $Q_b(y)$ can be easily extended on the whole real line by setting $Q_b(y)$ to be the solution of the following initial value problem:

$$\mathcal{T}(Q) = 0, \quad Q(b) = e^b - a, \quad \text{and} \quad Q'(b+) = Q'(b-).$$

Obviously this extended function inherits properties (i), (ii), and (iv) in proposition 3.3.3 from the original function, i.e., it has no local extrema, it is strictly increasing, and it converges to some nonnegative number l_b as y goes to $-\infty$. In the remaining part of this section, we use $Q_b(y)$ to represent the extended function.

Notice that for any real number $c > \ln(a)$, the function $Q_c(y)$ defined by

$$Q_c(y) \equiv \frac{e^c - a}{Q_b(c)} Q_b(y)$$

solves the boundary value problem (3.12) with b replaced by c and l_b replaced by $l_c \equiv \frac{(e^c-a)}{Q_b(c)} l_b$. Consider the family of functions $\{Q_b\}_{b>a}$, we intend to show that there exists a b^* such that Q_{b^*} not only solves (3.12), but also satisfies the additional conditions that $Q'_{b^*} = e^{b^*}$, and $Q_{b^*}(y) \geq e^y - a$ on $(-\infty, b^*]$. Loosely speaking, we want to find a $Q_b(y)$ which meets $e^b - a$ tangentially at the point b. Notice that $Q_b(y)$ is a continuous function of b, so it's enough to find a $Q_{b_1}(y)$ which crosses $e^y - a$ at least twice and a $Q_{b_2}(y)$ which never intersects with $e^b - a$. It turns out that $Q_{b_2}(y)$ can be obtained by modifying $Q_{b_1}(y)$, and for such a $Q_{b_1}(y)$ to exist, a sufficient condition is that $Q_{b_1}(b_1) = e^{b_1} - a$ and $Q'_{b_1}(b_1) > e^{b_1}$. In fact, we will show that $Q'_b(b) - Q_b(b)$ tends to infinity as b goes to infinity. First we find a lower bound for $Q'_b(y) - Q_b(y)$:

Lemma 3.3.4. For any y and any $b > \ln(a)$,

$$Q'_b(y) - Q_b(y) \geq \frac{2}{\sigma^2}(\rho - r) \int_{-\infty}^{y} Q_b(u) e^{-\frac{2}{\sigma^2}\int_u^y (r+\psi(s))ds} du > 0 \tag{3.17}$$

Proof. Introduce $H_b(y) \equiv Q'_b(y) - Q_b(y)$, then H_b satisfies

$$\frac{\sigma^2}{2} H'_b(y) + (r + \psi(y))H_b(y) = (\rho - r)Q_b(y)$$

Multiply the equation by $\exp(\int_c^y \frac{2}{\sigma^2}(r+\psi(u))du)$ and integrate the equation to obtain

$$H_b(y) = H_b(c)e^{-\frac{2}{\sigma^2}\int_c^y(r+\psi(u))du} + \frac{2(\rho-r)}{\sigma^2}\int_c^y Q_b(u)e^{-\frac{2}{\sigma^2}\int_u^y(r+\psi(s))ds}du$$

where c is any real number. Since $Q_b'(y) > 0$, $Q_b \to l_b$, and $-\frac{2}{\sigma^2}\int_c^y(r+\psi(u))du \to -\infty$ as $c \to -\infty$, we have $H_b(c) \geq -Q_b(c)$ and

$$\lim_{c\to-\infty} H_b(c)e^{-\frac{2}{\sigma^2}\int_c^y(r+\psi(u))du} \geq -\lim_{c\to-\infty} Q_b(c)e^{-\frac{2}{\sigma^2}\int_c^y(r+\psi(u))du} = 0$$

Therefore

$$H_b(y) \geq \frac{2(\rho-r)}{\sigma^2}\int_{-\infty}^y Q_b(u)e^{-\frac{2}{\sigma^2}\int_u^y(r+\psi(s))ds}du > 0 \tag{3.18}$$

□

As discussed earlier, each function in the family $\{Q_b\}_{b>a}$ is a constant multiple of any other function in the family. Now we choose an arbitrary (but fixed) b_0 and for convenience, denote the corresponding function Q_{b_0} by Q_0, then the inequality (3.18) at $y = b$ can be written as

$$H_b(b) \geq \frac{2(\rho-r)}{\sigma^2}\frac{(e^b-a)}{Q_0(b)}\int_{-\infty}^b Q_0(u)e^{-\frac{2}{\sigma^2}\int_u^y(r+\psi(r))dr}du > 0 \tag{3.19}$$

Formula (3.19) gives a lower bound for $Q_b'(b) - Q(b)$, we intend to show that this lower bound tends to infinity as b goes to infinity. For this purpose, we notice that $\psi(b) \to 0$ as $b \to \infty$. Hence for large y, the solution to $\mathcal{T}Q = 0$ behaves similar to the solution to the following constant coefficient ODE

$$Q''(y) + \frac{2}{\sigma^2}(r - \frac{\sigma^2}{2})Q'(y) - \frac{2}{\sigma^2}\rho Q(y) = 0. \tag{3.20}$$

A fundamental set of ODE (3.20) is $\{e^{\lambda(r)y}, e^{-\gamma(r)y}\}$ where

$$\lambda(r) = \frac{2\rho}{\sqrt{(r-\frac{\sigma^2}{2})^2 + 2\rho\sigma^2} + (r-\frac{\sigma^2}{2})}, \text{ and}$$

$$\gamma(r) = \frac{2\rho}{\sqrt{(r-\frac{\sigma^2}{2})^2 + 2\rho\sigma^2} - (r - \frac{\sigma^2}{2})}.$$

We use the notation $\lambda(r), \gamma(r)$ to denote that they depend on the parameter r.

We will now prove the following lemma:

Lemma 3.3.5.

$$\lim_{b\to\infty} \frac{(e^b - a)}{Q_0(b)} \int_{-\infty}^{b} Q_0(u) e^{-\frac{2}{\sigma^2}\int_u^b (r+\psi(z))dz} du = \infty$$

Proof. Since the function Q_0 appears on both the numerator and the denominator, we need to find upper and lower estimates for the function Q_0. Choose positive numbers $\theta > r$ and ϵ such that $0 < \lambda(r) - \lambda(\theta) < 0.5$ and $0 < \epsilon < \theta - r$. Since $\psi(y)$ is a decreasing function that goes to 0 as $y \to \infty$, we can choose a number y_ϵ so that $\psi(y) < \epsilon$ for all $y > y_\epsilon$. Consider two functions $Q_1(y)$ and $Q_2(y)$ such that $Q_1(y)$ satisfies equation (3.20) and $Q_2(y)$ satisfies (3.20) with r replaced by θ. Moreover, they satisfy the initial conditions $Q_1(y_\epsilon) = Q_2(y_\epsilon) = Q_0(y_\epsilon)$, $Q_1'(y_\epsilon) = m_1$ and $Q_2'(y_\epsilon) = m_2$ with $m_1 > Q_0'(y_\epsilon) > m_2$. We want to show that for any $y > y_\epsilon$, we have $Q_1(y) > Q_0(y) > Q_2(y)$.

By the assumptions $Q_1(y_\epsilon) = Q_0(y_\epsilon)$ and $Q_1'(y_\epsilon) > Q_0'(y_\epsilon)$ it follows that $Q_1(y) > Q_0(y)$ on some interval $(y_\epsilon, y_\epsilon + \delta)$. Now assume that $Q_1(y) \le Q_0(y)$ for some $y > y_\epsilon$, then $d \equiv \inf\{y > y_\epsilon : Q_1(y) \le Q_0(y)\}$ exists and is finite. By continuity of Q_1 and Q_0 we have $Q_1(d) = Q_0(d)$. Also, $d \ge y_\epsilon + \delta$. So Q_1 and Q_0 coincide at $y = y_\epsilon$ and $y = d$, and $Q_1 > Q_0$ on the interval (c, d).

Define the operator $\mathcal{A}$ by

$$\mathcal{A} \equiv \frac{\sigma^2}{2}\frac{d^2}{dx^2} + (r - \frac{\sigma^2}{2})\frac{d}{dx} - \rho,$$

then $\mathcal{A}(Q_1) = 0$. Recall that $\psi(y) > 0$ and $Q_0'(y) > Q_0(y)$ from lemma 3.3.4. Therefore

$$\mathcal{A}(Q_0) = \mathcal{T}(Q_0) - \psi(y)Q_0'(y) + \psi(y)Q_0(y) = \psi(y)[Q_0(y) - Q_0'(y)] < 0$$

Consequently $\mathcal{A}(Q_0 - Q_1) < 0$, $Q_0(y_\epsilon)$, and $Q_0(d) = Q_1(d)$. Hence we can apply the maximum principle of ordinary differential equations to conclude that $Q_0(y) > Q_1(y)$ on (y_ϵ, d). This is a contradiction since $Q_1(y) > Q_0(y)$ on the interval $(y_\epsilon, y_\epsilon + \delta)$. This proves that $Q_1(y)$ is an upper bound for $Q_0(y)$ on (y_ϵ, ∞). By a similar argument it can be proved that $Q_2(y)$ is a lower bound for $Q_0(y)$ on the same interval.

In fact, $Q_1(y)$ can be explicitly expressed as

$$Q_1(y) = \frac{(\gamma(r)Q_0(y_\epsilon) + m_1)}{\lambda(r) + \gamma(r)} e^{\lambda(r)\cdot(y-y_\epsilon)} + \frac{(\lambda(r) - m_1)}{\lambda(r) + \gamma(r)} e^{-\gamma(r)\cdot(y-y_\epsilon)}$$

So there exists a constants K such that $Q_0(y) < Q_1(y) < Ke^{\lambda(r)\cdot y}$ if $y > y_\epsilon$. Similarly it can be shown that there is a constant C such that $Q_0(y) > Q_2(y) > Ce^{-\lambda(\theta)\cdot y}$ for $y > y_\epsilon$. Since $\lambda(r)$ is continuous in r, and $0 < \lambda(r) - \lambda(\theta) < 0.5$ as described at the beginning of the proof, we have $1 + \lambda(\theta) - \lambda(r) > 0$. We use this estimate below.

Now for $b > y_\epsilon$,

$$\begin{aligned}
&\frac{(e^b - a)}{Q_0(b)} \int_{-\infty}^{b} Q_0(u) e^{-\frac{2}{\sigma^2}\int_u^b (r+\psi(z))dz} du \\
\geq &\frac{(e^b - a)}{Ke^{\lambda(r)\cdot b}} \int_{y_\epsilon}^{b} Ce^{-\lambda(\theta)\cdot u} e^{-\frac{2}{\sigma^2}(r+\epsilon)(b-u)} du \\
\geq &\frac{(e^b - a)}{Ke^{\lambda(r)\cdot b}} \cdot C \cdot \left[e^{-\lambda(\theta)\cdot b} - e^{-\frac{2}{\sigma^2}(r+\epsilon)(b-y_\epsilon)+\lambda(\theta)\cdot y_\epsilon} \right] \\
= &\frac{C}{K} \cdot e^{(1+\lambda(\theta)-\lambda(r))b} + o(b) \quad \text{ as } b \to \infty \\
\to &\infty \quad \text{ as } b \to \infty \quad \text{ since } 1 + \lambda(\theta) - \lambda(r) > 0.
\end{aligned}$$

The proof is complete.

□

Lemma 3.3.5 and inequality (3.19) together implies that $Q_b'(b) - Q_b(b) \to \infty$ as $b \to \infty$. Therefore, we can choose a b such that $Q_b'(b) > Q_b(b) + a = e^b$, then the graph of $Q_b(y)$ crosses the curve $e^y - a$ with $Q_b'(y) > (e^y - a)'$ at $y = b$. This implies that $Q_b(y) < e^b - a$ on some interval $(b - \delta, b)$. But $Q_b(y) \geq 0 > e^y - a$ when $y < \ln a$, so there must be another point $c < b$ which satisfies the following condition:

$$\begin{gathered} Q_b(c) = e^c - a, \text{ and there exists a } \delta > 0 \text{ such that } Q_b(y) > e^y - a \\ \text{on } (c - \delta, c) \text{ and } Q_b(y) < e^y - a \text{ on } (c, c + \delta). \end{gathered} \tag{3.21}$$

Notice that $Q_b'(c) < e^c$ implies that c satisfies condition (3.21), which in turn implies that $Q_b'(c) \leq e^c$.

Let $d \equiv \inf\{y : Q_b(y) = e^y - a\}$. The set $\{y : Q_b(y) = e^y - a\}$ is closed and bounded below by $\ln a$, so d is finite and d is also in the set, i.e., $Q_b(d) = e^d - a$. Since d is the infimum, $Q_b(y)$ and $e^y - a$ does not intersect at any $y < d$. This implies that $Q_b(y) > e^y - a$ on $(-\infty, d)$. Apparently $Q_b'(d) \leq e^d$ because otherwise, if $Q_b'(d) > e^d$, by the argument in the previous paragraph, there must a $c < d$ at which $Q_b(y)$ and $e^y - a$ intersect, contradiction. If $Q_b'(d) = e^d$, the existence of solution to problem (3.11) follows immediately by taking $Q(y) = Q_b(y)$ and $y^* = d$, so we can assume that $Q_b'(d) < e^d$. This implies that d satisfies condition (3.21).

Next we consider the family $\{tQ_b(y)\}_{t \geq 1}$ parameterized by $t \geq 1$. Clearly $\mathcal{T}(tQ_b) = 0$ for every t. Define $d_t \equiv \inf\{y < b : tQ_b(y) = e^y - a\}$ (d_t is well defined at least for $t = 1$). Notice that d_t is a strictly increasing function of t and by definition $d_t < b$. Let

$$t^* \equiv \sup\{t \geq 1 : tQ_b(y) \text{ and } e^y - a \text{ intersect at least once in } (d, b)\}.$$

The set $\{t \geq 1 : tQ_b(y)$ and $e^y - a$ intersect at least once in $(d, b)\}$ is closed. Therefore $t^*Q_b(y)$ intersects $e^y - a$ at least once in (d, b), and consequently, $d^* \equiv d_{t^*}$ is well defined. Furthermore $t^*Q_b(y) > e^y - a$ on $(-\infty, d^*)$ and $t^*Q_b'(d^*) \leq e^{d^*}$. We claim that $t^*Q_b'(d^*) = e^{d^*}$. Suppose $t^*Q_b'(d^*) < e^{d^*}$, then d^* satisfies condition (3.21) with Q_b replaced by t^*Q_b. This implies there exists $\delta > 0$ such that $t^*Q_b(y) < e^y - a$ on $(d^*, d^* + \delta)$.

We can choose δ such that $d^* + \delta < b$. Now choose $\epsilon > 0$ such that $(t^* + \epsilon)Q_b(y) < e^y - a$ at $y = d^* + \delta$. Since $(t^* + \epsilon)Q_b(y) > t^*Q_b(y) = e^y - a$ at $y = d^*$, there must be a point d^{**} in $(d^*, d^* + \delta)$ where $(t^* + \epsilon)Q_b(y)$ and $e^y - a$ intersect, contradicting t^* is the largest number such that $tQ_b(y)$ and $e^y - a$ intersect in (d, b).

We have proved that the point d^* and function $t^*Q_b(x)$ solves problem (3.11). Formally, we have proved the following existence theorem:

Theorem 3.3.6. There exists a point $y^* > \ln(a)$ and a C^2 function $Q(y)$ such that $\mathcal{T}Q = 0$ on $(-\infty, y^*)$, $Q(y^*) = e^{y^*} - a$, $Q'(y^*) = e^{y^*}$, and $Q''(y^*) - Q'(y^*) > 0$. Furthermore, $Q(y) > y - a$ for $y < y^*$, i.e., y^* and $Q(y)$ solves problem (3.11).

Proof. Choose y^* equal to b^*, and set $Q(y) = t^*Q_b(y)$. It can be seen from the above discussion that y^* and $Q(y)$ satisfies all conditions in the theorem, and hence solves problem (3.11). The condition $Q''(y^*) - Q'(y^*) > 0$ follows from lemma 3.3.4. $\square$

As mentioned in the beginning of this subsection, this is equivalent to the existence of x^* and $\widetilde{Q}_*(x)$ if we set $x^* = e^{y^*}$ and $\widetilde{Q}_*(x) = Q(\ln x)$. Note that the condition $Q''(y^*) - Q'(y^*) > 0$ implies that $\widetilde{Q}''_*(x^*) > 0$.

3.3.3 Numerical examples

When a stock is subject to default, on one hand, the investor tends to sell the stock earlier due to the default risk; on the other hand, the pre-bankruptcy stock price process has a higher drift than the risk free interest rate, so the investor also has an incentive to hold the stock longer. In other words, as can be seen from (3.7), when default risk is considered, the default intensity function is added both to the drift r of stock price and to the discount factor ρ. An interesting question is, how does the default intensity function affect the optimal strategy and the value function? In this subsection, we consider a family $\{cx^{-p}\}$ of default intensity functions parameterized by c and p and investigate the influence of these two parameters on the boundary x^* and the value function $V(x)$. We intend to answer the following questions:

1. What's the relationship between the boundary x^*, the value function $V(x)$ and the default risk?

2. When the default risk is taken into consideration, how will other parameters affect the optimal strategy? In particular, we will study the influence of the financial charge a on the boundary x^*. The reason we choose the parameter a is that it has a simple relationship with x^* in the no-default model. As shown in Oksendal ([36], page 209, formula (10.2.13)), x^* is a linear function of a when default risk is not considered. We want to test if this linear relationship still holds when there is default risk.

Unless otherwise specified, the parameters are set as follows:

$$\rho = 0.07, \quad r = 0.06, \quad \sigma = 0.4, \quad c = 1, \quad p = 1, \quad a = 1 \tag{3.22}$$

In the remaining part of this section, we solve the boundary value problem (3.11) numerically and analyze the results. We will test the effect of parameters p, c, and a on the boundary x^* the the value function $V(x)$.

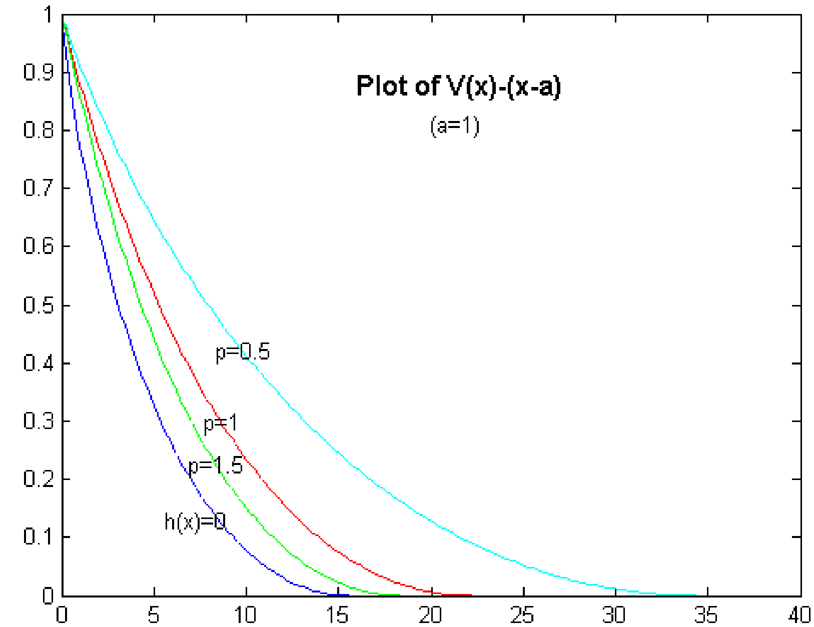
1. Effect of the parameter p

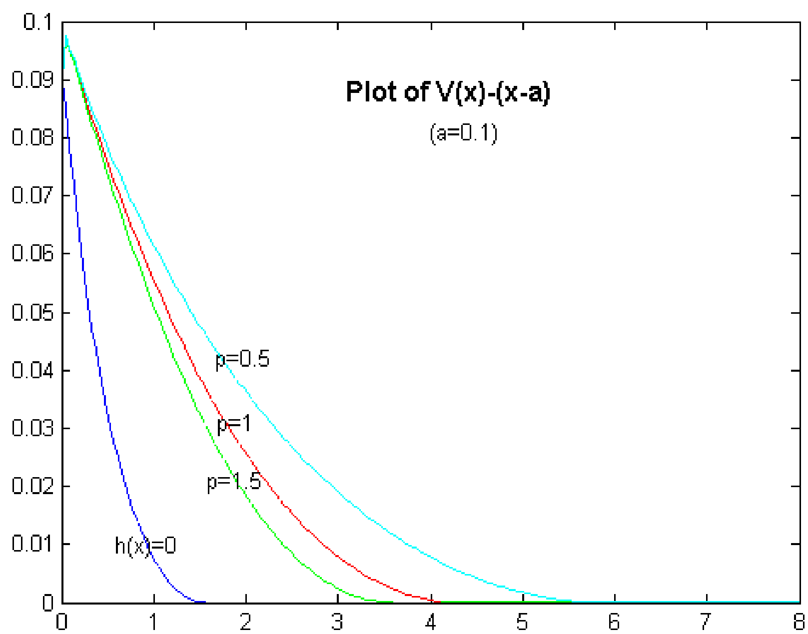
We first test the effect of p when the financial charge $a = 1$ and $a = 0.1$. All other parameters are set according to (3.22). The boundary x^* are listed in the following table:

Default intensity	$x^*(a=1)$	$x^*(a=0.1)$
$h(x)=0$	15.55	1.555
$h(x)=x^{-0.5}$	34.42	5.914
$h(x)=x^{-1}$	22.08	4.310
$h(x)=x^{-1.5}$	18.31	3.582

It can be seen that the x^* obtained from jump-to-default models is always larger than that from the no-default model. Furthermore, x^* is decreasing in p. For $a=1$ case, the x^* obtained from the model with $h(x)=x^{-1.5}$ is close to that from the no-default model. This is not surprising since when the stock price is greater than 1, when p becomes larger, the default probability $(1-e^{-\int_0^t h(S_u)du})$ in the jump to default model goes to zero.

Accordingly, the value function $V(x)$ is also decreasing in p, and the value function obtained from any jump-to-default model is always above that obtained from the no-default model, as can be seen from the following figures. We plot the graph of $V(x)-(x-a)$, rather than $V(x)$, against the initial stock price x.



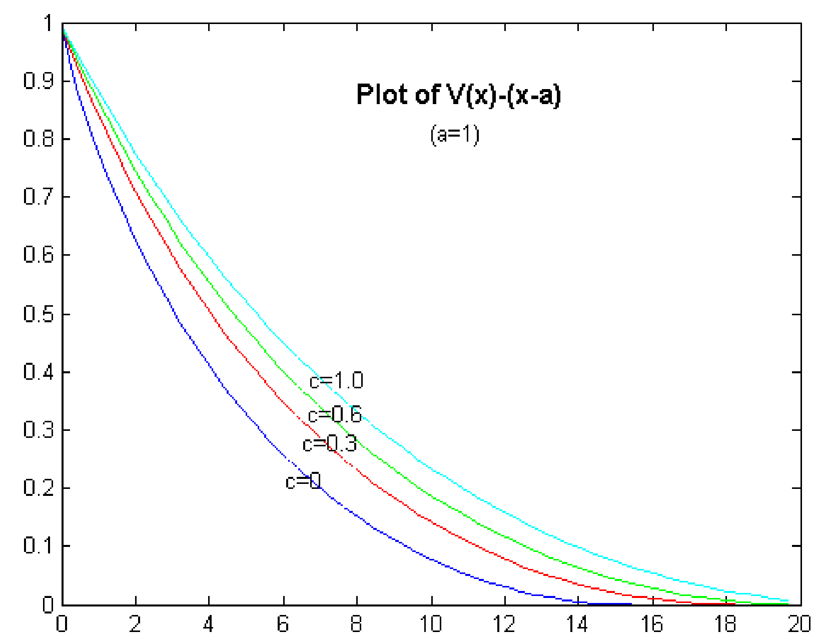


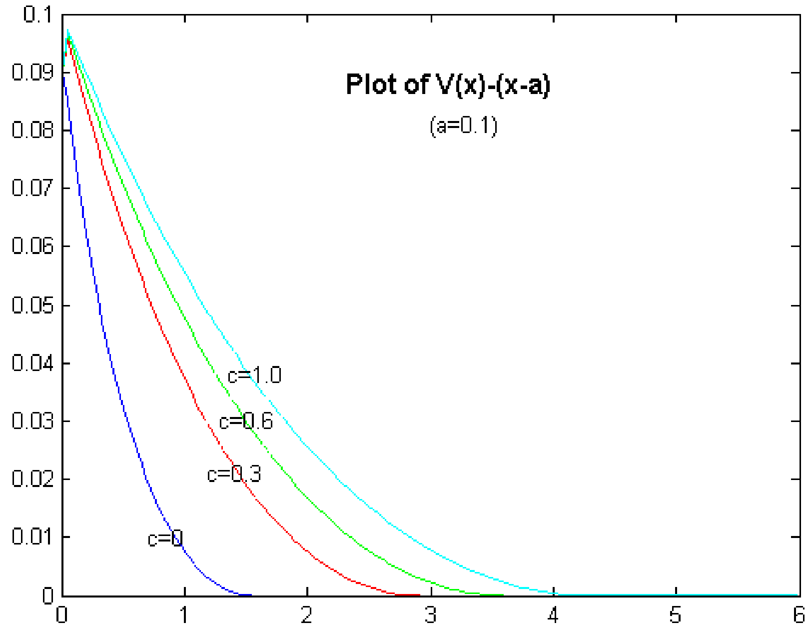
2. Effect of the parameter c

Next we fix $p = 1$ and test the effect of the parameter c. Again we set $a = 1$ and $a = 0.1$ in the two tests. We list the results in the following table and draw the graph of

$V(x) - (x - a)$ against x, as in the previous tests. It can be seen that when the parameter c is larger (corresponding to higher default risk), the boundary x^* is larger and the value function $V(x)$ is higher. Combining this result with the result for the parameter p, we conclude that the increase in the drift of stock price, in some sense, "dominates" the default risk. More precisely, when the default risk is taken into consideration, the investor will hold the stock longer and the value of the stock is higher. Moreover, the higher the default risk is, the larger the boundary x^* and the value function $V(x)$ are. This answers the first question raised at the beginning of this subsection.

	$x^*(a = 1)$	$x^*(a = 0.1)$
$c = 0$	15.55	1.555
$c = 0.3$	18.22	2.901
$c = 0.6$	20.08	3.604
$c = 1.0$	22.08	4.310

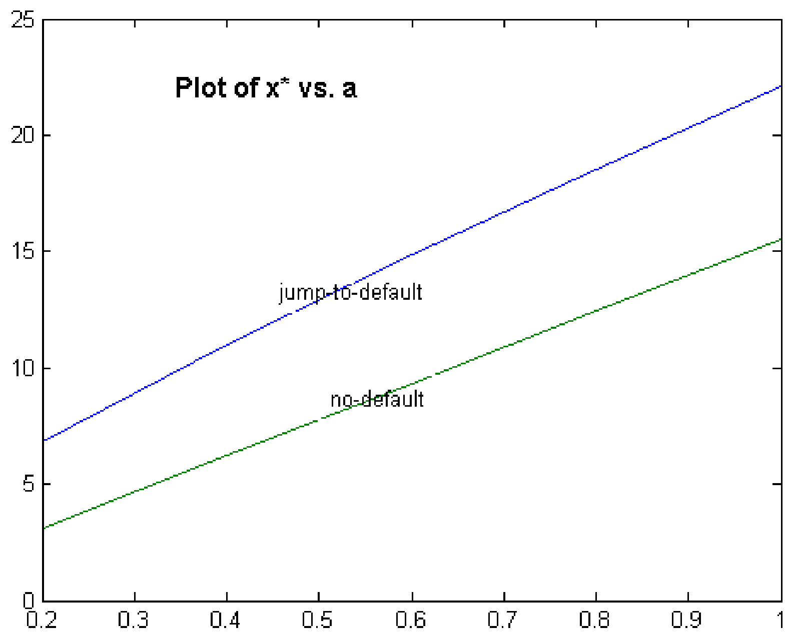




3. Effect of the financial charge a on x^*

In the no-default case, x^* is linear function of the financial charge a, now we set the default intensity $h(x) = 1/x$ and study how a affects x^* in the jump-to-default model. All other parameters are set according to (3.22). The results are shown in the table below. We also plot the graph of x^* against a. The result shows that under the jump-to-default model, the boundary x^* still increase approximately linearly as a function of the financial charge a. It is not unreasonable to assume that other parameters, such as ρ and r, influence the boundary x^* in a similar way in the jump-to-default model as in the no-default model.

a	jump-to-default	no-default
0.2	6.79	3.11
0.4	11.02	6.22
0.6	14.87	9.33
0.8	18.53	12.44
1.0	22.08	15.55



CHAPTER 4. Stochastic volatility models

4.1 Introduction to stochastic volatility models

In the Black-Scholes option pricing model, the volatility of stock price is assumed to be a constant. This constant volatility model fails to explain two phenomena in the real market:

1. The implied volatility is a function of strike price. This is called the "volatility smile";

2. The stock price distribution has a fat tail compared to log-normal distribution.

To address these two problems, researchers have modified the standard Black-Scholes model in several different ways, among them are jump diffusion models, level dependent volatility models, local volatility approach, and stochastic volatility models. A review of these models can be found in Javaheri [29].

In stochastic models, popular choice of processes for the volatility includes Ornstein-Uhlenbeck process and similar processes such as Cox-Ingersoll-Ross process, see, for example, the Stein and Stein model in [40] and the Heston model in [22]. Since stochastic volatility models involve two stochastic processes, they are in general more complicated than other models. One of the difficulties is that, since the volatility of a stock can not be the price of any securities traded in the market, stochastic volatility models are incomplete. This implies that the Equivalent Martingale Measure (EMM) is not unique, and therefore one may derive many different values for the price of the same option. One major difficulty in option pricing under stochastic volatility models is how to choose the EMM, or equivalently, how to determine the market price of volatility risk. Later in this section and also in the next section, we will introduce the EMM used by Heston in [22], and also an optimization method to choose EMM proposed by Hobson and Henderson in [24] and [20]. Hobson and Henderson's EMM is known as the q-optimal measure.

In [22], Heston modeled the volatility σ_t as an Ornstein-Uhlenbeck process:

$$\begin{cases} dS_t = \mu S_t dt + \sigma_t S_t dW_t \\ d\sigma_t = -\beta\sigma_t dt + \delta dB_t \end{cases} \tag{4.1}$$

where W_t and B_t are standard Brownian motions with correlation ρ. The variance $v_t = \sigma_t^2$ follows a Cox-Ingersoll-Ross (CIR) process and the model can be written as:

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t \\ dv_t = \kappa(\theta - v_t)dt + \eta\sqrt{v_t} dB_t \end{cases} \tag{4.2}$$

By the risk-neutrality argument, the drift of the stock price under EMM is equal to the risk free rate of return r and the drift of the variance process is $\kappa(\theta - v_t) - \lambda(S_t, v_t, t)\eta\sqrt{v_t}$ where $\lambda(S_t, v_t, t)$ is called the market price of volatility risk. Heston assumes that the market price of volatility risk is proportional to v_t, i.e., $\lambda(S_t, v_t, t) = \lambda_t v_t$. Under this assumption, Heston was able to obtain a closed-form pricing formula for the value of European call option expired at time T in the following form:

$$C(S_t, v_t, t) = S_t P_1 - Ke^{-r(T-t)} P_2 \tag{4.3}$$

where P_1 and P_2 satisfy certain differential equation and terminal conditions. Using a Fourier transform technique, Heston derived an explicit expression for P_1 and P_2. Heston's result is remarkable since it is the first such closed form pricing formula for a stochastic volatility model. In the next section we will discuss the drawbacks of Heston's model and introduce the EMM proposed by Hobson and Henderson.

4.2 The q-optimal measure

In Heston's model, the EMM is determined so that a closed form pricing formula can be obtained. Although the assumption that the market price of volatility risk is proportional to the variance is not completely unreasonable, there is not much empirical evidence or theoretical foundation that supports this assumption. In this section we will introduce a family of EMM's — the q-optimal measure proposed by Hobson in [24]. Compared to the measure in Heston's model, the q-optimal measure is theoretically promising since it is, in some sense, the EMM closest to the physical measure. The q-optimal measure also includes as special cases some of the most popular EMM's, such as the variance optimal measure and the minimal entropy measure. Furthermore, the collection of q-optimal measures is actually a family of measures and as Henderson showed in [20], the European option price under q-optimal measure is monotonic in q thus it is convenient to calibrate the model and choose the correct q based on real market data. In this section, we will first introduce the definition and properties of the q-optimal measure; then we will prove the monotonicity property of American option price under q-optimal measure.

4.2.1 Definition and properties of q-optimal measure

Consider the more general model for stock price S_t and volatility σ_t under the physical measure:

$$\begin{cases} \frac{dS_t}{S_t} = \sigma_t(\alpha(t,\sigma_t)dt + dW_t) \\ d\sigma_t = a(t,\sigma_t)dt + b(t,\sigma_t)dB_t \end{cases} \tag{4.4}$$

where W_t and B_t are two standard Brownian motions with constant correlation coefficient ρ. Write $\overline{\rho} = \sqrt{1-\rho^2}$ then B_t can be written as $B_t = \rho W_t + \overline{\rho} Z_t$, where Z is a Brownian motion independent of W. Notice that a and b are assumed to be independent of S_t so that the volatility process is an autonomous diffusion.

Assume that the drift of stock price under EMM is 0. According to the risk neutral pricing theory (see Henderson et. al. [21] or Frey [17]), the Radon-Nikodym derivative of any EMM $\mathbb{Q}$ with respect to the physical measure $\mathbb{P}$ is

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = M_T$$

where M_T is the terminal value of a martingale M_t given by

$$M_t = \exp\left(\int_0^t \left[-\alpha(u,\sigma_u)dW_u - \frac{1}{2}\alpha(u,\sigma_u)^2 du - \lambda_u dZ_u - \frac{1}{2}\lambda_u^2 du\right]\right) \tag{4.5}$$

The family of EMM's is parameterized by λ_t, which is the change in the drift of Z process. λ_t is also known as the market price of Z risk. By Girsanov's theorem, under the equivalent martingale measure $\mathbb{Q}$, the processes $W_t^{\mathbb{Q}}$ and $Z_t^{\mathbb{Q}}$ defined by

$$dW_t^{\mathbb{Q}} = dW_t + \alpha(t,\sigma_t)dt$$

$$dZ_t^{\mathbb{Q}} = dZ_t + \lambda_t dt$$

are two independent standard Brownian motions. Accordingly, the change of drift on σ_t under $\mathbb{Q}$ is $(\rho\alpha(t,\sigma_t)+\overline{\rho}\lambda_t)b(t,\sigma_t)$. Thus under $\mathbb{Q}$, the stock price and volatility processes are governed by the following stochastic differential equations:

$$\begin{cases} \frac{dS_t}{S_t} = \sigma_t dW_t^{\mathbb{Q}} \\ d\sigma_t = [a(t,\sigma_t) - \rho\alpha(t,\sigma_t)b(t,\sigma_t) + \overline{\rho}\lambda_t b(t,\sigma_t)]dt + b(t,\sigma_t)dB_t^{\mathbb{Q}} \end{cases} \tag{4.6}$$

It can be seen from (4.5) that to determine the EMM, it's equivalent to choose the market price λ_t of Z risk. First of all, we expect λ_t to satisfy the Novikov condition so that M_t is a martingale (see Karatzas and Shreve [30], page 199). The fundamental idea of the q-optimal measure is to choose a martingale measure $\mathbb{Q}$ as close as possible to the physical measure $\mathbb{P}$.

Define the q-distance $H_q(\mathbb{P},\mathbb{Q})$ between the physical measure and the martingale measure as follows:

If $q \notin \{0,1\}$,

$$H_q(\mathbb{P},\mathbb{Q}) = \begin{cases} \mathbb{E}\left[\frac{q}{q-1}(M_T)^q\right] & \text{if } \mathbb{Q} \ll \mathbb{P} \\ \infty & \text{otherwise,} \end{cases}$$

and if $q \in \{0,1\}$,

$$H_q(\mathbb{P},\mathbb{Q}) = \begin{cases} \mathbb{E}\left[(-1)^{1+q} M_T^q \ln(M_T)\right] & \text{if } \mathbb{Q} \ll \mathbb{P} \\ \infty & \text{otherwise,} \end{cases}$$

Hobson [24] shows that, for each q, there exists a measure that minimizes $H_q(\mathbb{P},\mathbb{Q})$. This measure is called the q-optimal measure.

Hobson [24] derives a representation equation associated with the q-optimal measure and finds the form of the corresponding market price $\lambda^q(t,\sigma_t)$ of Z risk. Let $A = 1 - q\rho^2$, then $\lambda^q(t,\sigma_t) = \overline{\rho} b(t,\sigma_t)\frac{\partial f}{\partial \sigma_t}(t,\sigma_t)$, where

$$f(t,x) = \begin{cases} 0 & \text{if } q = 0 \\ -\frac{1}{A}\log \hat{\mathbb{E}}_x\left[\exp\left(-\frac{q}{2}A\int_t^T \alpha(u,\sigma_u)^2 du\right)\right] & \text{if } q \neq 0 \text{ and } A \neq 0 \\ \hat{\mathbb{E}}_x\left[\frac{q}{2}\int_t^T \alpha(u,\sigma_u)^2 du\right] & \text{if } q \neq 0 \text{ and } A = 0 \end{cases} \tag{4.7}$$

The expectation $\hat{\mathbb{E}}$ is taken under the probability measure $\hat{\mathbb{P}}$ under which the volatility has dynamics

$$d\sigma_t = (\alpha(t,\sigma_t) - q\rho\alpha(t,\sigma_t)b(t,\sigma_t))dt + b(t,\sigma_t)d\hat{W}_t$$

with $\hat{\mathbb{P}}$-Brownian motion $\hat{W}_t$.

From the representation (4.7), it can be seen that if $q > 0$ and $\rho^2 < 1/q$, or equivalently $qA < 0$, then f is positive and finite hence the q-optimal measure is well-defined for all time $t \geq 0$. On the other hand, if $qA < 0$, the function f explodes at a finite time hence the q-optimal measure is not defined beyond that time horizon.

By Feynman-Kac formula, f solves the PDE

$$\frac{q}{2}\alpha(t,\sigma)^2 - q\rho b(t,\sigma)\alpha(t,\sigma)f_\sigma - \frac{A}{2}b(t,\sigma)^2(f_\sigma)^2 + a(t,\sigma)f_\sigma + \frac{1}{2}b(t,\sigma)^2 f_{\sigma\sigma} + f_t = 0$$

with the boundary condition $f(T,\sigma) = 0$ where T is the expiration date of the option.

Hobson [24] and Henderson et. al. [21] also gives the form of f in some special cases. For example, Hobson shows that if $\alpha(t,\sigma) = \alpha_1\sigma$ for some constant α_1, $a(t,\sigma) = \kappa(m/\sigma - \sigma)$ for some constants κ and m, and $b(t,\sigma)$ is constant, then f can be represented in the form

$$f(t,\sigma) = \sigma^2 F(T-t)/2 + G(T-t),$$

where F and G satisfies certain ODE's and initial conditions. For details we refer to Hobson [24], section 5, and Henderson et. al. [21], section 5.

4.2.2 Ordering of American option prices under q-optimal measure

Throughout this subsection, we assume that the Brownian motions W and B in model (4.4) are independent. In this case the market price $\lambda^q(t,\sigma)$ of Z risk is the same as market price of volatility risk. Under this assumption Henderson [20] proved the monotonicity of European option price as a function of q under q-optimal measure. In this subsection we are able to extend Henderson's result for American option prices. Earlier work on comparison theorems for the expected values of convex function of diffusion processes can be found in Hajek [19].

Theorem 4.2.1. (see also theorem 4.2 in Henderson [20]) Assume that the correlation coefficient ρ is 0 in the stochastic volatility model (4.4), then the market price $\lambda^q(t,\sigma)$ of Z risk is nondecreasing in q if the market price $\alpha(t,\sigma)$ of W risk is nondecreasing in σ, and $\lambda^q(t,\sigma)$ is nonincreasing in q if $\alpha(t,\sigma)$ is nonincreasing in σ.

Proof. Since this theorem does not involve option prices, the proof in Henderson [20] is still valid.

$\square$

Now that the monotonicity of $\lambda^q(t,\sigma)$ in q has been proved, it remains to show that American option price is monotonic in λ. We assume that the payoff of the option is a convex function of stock price.

Theorem 4.2.2. (see also theorem 3.1 in Henderson [20]) Suppose $\lambda^q(t,\sigma)$ and $\gamma^q(t,\sigma)$ are two market price of volatility risk corresponding to martingale measure $\mathbb{Q}^\lambda$ and $\mathbb{Q}^\gamma$, respectively. The stock price and volatility processes under $\mathbb{Q}^\lambda$ are given by

$$\begin{cases} \frac{dS_t^\lambda}{S_t^\lambda} = \sigma_t^\lambda dW_t^{\mathbb{Q}^\lambda} \\ d\sigma_t^\lambda = [a(t,\sigma_t^\lambda) - \lambda(t,\sigma_t^\lambda)b(t,\sigma_t^\lambda)]dt + b(t,\sigma_t^\lambda)dB_t^{\mathbb{Q}^\lambda} \end{cases} \tag{4.8}$$

where $W^{\mathbb{Q}^\lambda}$ and $B^{\mathbb{Q}^\lambda}$ are independent $\mathbb{Q}^\lambda$-Brownian motions. The stock price and volatility processes under $\mathbb{Q}^\gamma$ are given by

$$\begin{cases} \frac{dS_t^\gamma}{S_t^\gamma} = \sigma_t^\gamma dW_t^{\mathbb{Q}^\gamma} \\ d\sigma_t^\gamma = [a(t,\sigma_t^\gamma) - \gamma(t,\sigma_t^\gamma)b(t,\sigma_t^\gamma)]dt + b(t,\sigma_t^\gamma)dB_t^{\mathbb{Q}^\gamma} \end{cases} \tag{4.9}$$

where $W^{\mathbb{Q}^\gamma}$ and $B^{\mathbb{Q}^\gamma}$ are independent $\mathbb{Q}^\gamma$-Brownian motions. Let $T > 0$ be a fixed constant, then for any convex function $h(\cdot)$ which satisfies $\mathbb{E}^{\mathbb{Q}^\gamma}[|h(S_T^\gamma)|] < \infty$,

$$\mathbb{E}^{\mathbb{Q}^\lambda} h(S_T^\lambda) \leq \mathbb{E}^{\mathbb{Q}^\gamma} h(S_T^\gamma)$$

if $\lambda(t,\sigma) \geq \gamma(t,\sigma)$ for all t and σ.

Proof. Consider a new measure $\hat{\mathbb{Q}}$ under which $\hat{W}$ and $\hat{B}$ are two independent Brownian motions. Let $\hat{\sigma}^\lambda$ and $\hat{\sigma}^\gamma$ be two volatility processes governed by stochastic equations

$$d\hat{\sigma}_t^\lambda = [a(t,\hat{\sigma}_t^\lambda) - \lambda(t,\hat{\sigma}_t^\lambda)b(t,\hat{\sigma}_t^\lambda)]dt + b(t,\hat{\sigma}_t^\lambda)d\hat{B}_t$$

and

$$d\hat{\sigma}_t^\gamma = [a(t,\hat{\sigma}_t^\gamma) - \gamma(t,\hat{\sigma}_t^\gamma)b(t,\hat{\sigma}_t^\gamma)]dt + b(t,\hat{\sigma}_t^\gamma)d\hat{B}_t,$$

respectively. Since $\lambda(t,\sigma) \geq \gamma(t,\sigma)$, a standard stochastic comparison theorem (see Karatzas and Shreve [30], section 5.2, prop. 2.18) yields $\hat{\sigma}_t^\lambda \leq \hat{\sigma}_t^\gamma$ if these two processes have the same initial value. The intuition here is that once the two processes coincide, then their increment have the same random term, but the drift term of $\hat{\sigma}_t^\lambda$ is always less than or equal to the drift of $\hat{\sigma}_t^\gamma$, hence $\hat{\sigma}_t^\lambda$ can never exceed $\hat{\sigma}_t^\gamma$.

Define $A_t^\lambda \equiv \int_0^t (\hat{\sigma}_u^\lambda)^2 du$, and $A_t^\gamma \equiv \int_0^t (\hat{\sigma}_u^\gamma)^2 du$. Then $A_t^\lambda \le A_t^\gamma$ since $0 \le \hat{\sigma}_t^\lambda \le \hat{\sigma}_t^\gamma$. Consider the process $M_t = \hat{W}_{A_t^\lambda}$ and two filtrations

$$\mathcal{F}_t \equiv \sigma(A_s^\lambda : 0 \le s \le t, M_s : 0 \le s \le t), \text{ and}$$

$$\mathcal{G}_t \equiv \sigma(A_s^\lambda : 0 \le s \le T, M_s : 0 \le s \le t)$$

where $\sigma(X)$ represents the sigma algebra generated by X. We intend to show that M_t is an $(\mathcal{F}_t)$-martingale. The process M_t has continuous sample paths, it remains to show that

$$\hat{\mathbb{E}}\left[M_{t+s} - M_t \middle| \mathcal{F}_t \right] = 0, \quad \text{or equivalently} \quad \hat{\mathbb{E}}\left[\hat{W}_{A_{t+s}^\lambda} - \hat{W}_{A_t^\lambda} \middle| \mathcal{F}_t \right] = 0 \tag{4.10}$$

Notice that A_{t+s}^λ is measurable with respect to $\mathcal{G}_t$ and $\left(\hat{W}_{A_{t+s}^\lambda} - \hat{W}_{A_t^\lambda} \right)$ is a random variable independent of M_t. Therefore

$$\hat{\mathbb{E}}\left[\hat{W}_{A_{t+s}^\lambda} - \hat{W}_{A_t^\lambda} \middle| \mathcal{G}_t \right] = 0$$

but

$$\hat{\mathbb{E}}\left[\hat{W}_{A_{t+s}^\lambda} - \hat{W}_{A_t^\lambda} \middle| \mathcal{F}_t \right] = \hat{\mathbb{E}}\left(\hat{\mathbb{E}}\left[\hat{W}_{A_{t+s}^\lambda} - \hat{W}_{A_t^\lambda} \middle| \mathcal{G}_t \right] \middle| \mathcal{F}_t \right)$$

since $\mathcal{F}_t \subseteq \mathcal{G}_t$. Hence (4.10) follows. Similarly it can be shown that

$$\hat{\mathbb{E}}\left[M_{t+s}^2 \middle| \mathcal{G}_t \right] = M_t^2 + \hat{\mathbb{E}}\left[(M_{t+s} - M_t)^2 \middle| \mathcal{G}_t \right] = M_t^2 + A_{t+s}^\lambda - A_t^\lambda$$

Define $Z_t \equiv M_t^2 - A_t^\lambda$, then Z_t has continuous sample paths. Moreover, $\hat{\mathbb{E}}\left[Z_{t+s} - Z_t \middle| \mathcal{G}_t \right] = 0$. Consequently $\hat{\mathbb{E}}\left[Z_{t+s} - Z_t \middle| \mathcal{F}_t \right] = 0$. Hence Z_t is an $(\mathcal{F}_t)$-martingale. It follows that the quadratic variation process $< M >_t = A_t^\lambda = \int_0^t (\hat{\sigma}_s^\lambda)^2 ds$.

Introduce two new processes $\bar{W}_t^\lambda$ and $\bar{W}_t^\gamma$ by

$$\bar{W}_t^\lambda \equiv \int_0^t \frac{1}{\hat{\sigma}_u^\lambda} d\hat{W}_{A_u^\lambda} \qquad \text{and} \qquad \bar{W}_t^\gamma \equiv \int_0^t \frac{1}{\hat{\sigma}_u^\gamma} d\hat{W}_{A_u^\gamma}$$

The quadratic variation of $\bar{W}_t^\lambda$ is

$$< \bar{W}^\lambda >_t = \int_0^t \frac{1}{(\hat{\sigma}_s^\lambda)^2} d < M >_s = t.$$

We have shown that $\bar{W}_t^\lambda$ is an $(\mathcal{F}_t)$-adapted local martingale with continuous sample paths and quadratic variation t. Hence $\bar{W}_t^\lambda$ is a $\hat{\mathbb{Q}}$-Brownian motion. Similarly $\bar{W}_t^\gamma$ is also a $\hat{\mathbb{Q}}$-Brownian motion.

Notice that if $\hat{S}^\lambda$ solves $d\hat{S}_t^\lambda = \hat{\sigma}_t^\lambda \hat{S}_t^\lambda d\bar{W}_t^\lambda$, then the distribution of $\hat{S}^\lambda$ under $\hat{\mathbb{Q}}$ is the same as the distribution of S_t^λ under $\mathbb{Q}^\lambda$ if they have the same initial value s_0. Hence $\mathbb{E}^{\mathbb{Q}^\lambda} h(S_t^\lambda) = \hat{\mathbb{E}} h(\hat{S}_t^\lambda)$. By a similar argument, $\mathbb{E}^{\mathbb{Q}^\gamma} h(S_t^\gamma) = \hat{\mathbb{E}} h(\hat{S}_t^\gamma)$. Furthermore, for any fixed time t, $\hat{S}_t^\lambda$ and $\hat{S}_t^\gamma$ can be explicitly expressed as

$$\hat{S}_t^\lambda = s_0 \exp\left(\hat{W}_{A_t^\lambda} - \frac{1}{2}A_t^\lambda\right) \quad \text{and} \quad \hat{S}_t^\lambda = s_0 \exp\left(\hat{W}_{A_t^\gamma} - \frac{1}{2}A_t^\gamma\right)$$

For convenience we define $\hat{S}_t = s_0 \exp\left(\hat{W}_t - \frac{1}{2}t\right)$. Then $\hat{S}_t^\lambda = \hat{S}_{A_t^\lambda}$ and $\hat{S}_t^\gamma = \hat{S}_{A_t^\gamma}$

Define the filtrations $(\mathcal{D}_t)$ and $(\mathcal{H}_t)$ by

$$\mathcal{D}_t \equiv \sigma(\hat{W}_s : 0 \leq s \leq t) \quad \text{and } \mathcal{H}_t \equiv \sigma(\hat{W}_s : 0 \leq s \leq t, \hat{B}_s : 0 \leq s \leq T),$$

then both A_t^λ and A_t^γ are measurable with respect to $\mathcal{D}_T$, but $\mathcal{D}_T \subseteq \mathcal{H}_t$ for any $t \geq 0$, hence A_t^λ and A_t^γ are $(\mathcal{H}_t)$-stopping times. Furthermore, A_t^λ and A_t^γ depends only on $\hat{B}_t$ so they are independent of the $\hat{S}_t$ process. We need to show

$$\hat{\mathbb{E}}[h(\hat{S}_T^\lambda)] \leq \hat{\mathbb{E}}[h(\hat{S}_T^\gamma)],$$

or equivalently,

$$\hat{\mathbb{E}}[h(\hat{S}_{A_T^\lambda}) - h(\hat{S}_{A_T^\gamma})] \leq 0 \tag{4.11}$$

For this purpose we use the fact that $A_t^\lambda \leq A_t^\gamma < \infty$ and define the region G in $\mathbb{R}^2$ by

$$G \equiv \{(y_1, y_2) \in \mathbb{R}^2 : y_1 < y_2\}.$$

Then the expectation in (4.11) can be written as

$$\begin{aligned} \hat{\mathbb{E}}[h(\hat{S}_{A_T^\lambda}) - h(\hat{S}_{A_T^\lambda})] &= \hat{\mathbb{E}} \int_G [h(\hat{S}_{y_1}) - h(\hat{S}_{y_2})] dF(y_1, y_2) \\ = \int_G \hat{\mathbb{E}}[h(\hat{S}_{y_1}) - h(\hat{S}_{y_2})] dF(y_1, y_2) & \end{aligned} \tag{4.12}$$

where $F(y_1, y_2)$ is the distribution function of the random vector $(A_T^\lambda, A_T^\gamma)$. The order of integration in (4.12) can be changed since A_t^λ and A_t^γ are independent of the $\hat{S}_t$ process.

Now we notice that $\hat{S}_t$ is a martingale and h is a convex function, and hence $\hat{\mathbb{E}}[h(\hat{S}_{y_1}) - h(\hat{S}_{y_2})] \leq 0$ when $y_1 < y_2$. It follows that

$$\int_G \hat{\mathbb{E}}[h(\hat{S}_{y_1}) - h(\hat{S}_{y_2})] dF(y_1, y_2) \leq 0,$$

and consequently, (4.11) holds, as needed.

This completes the proof.

□

Remark: The proof in Henderson [20] is not complete. In particular, in their proof they claim that $\hat{\mathbb{E}}(\hat{S}_T^\gamma | A_T^\lambda) = \hat{S}_T^\lambda$. But this equality doesn't make sense, since $\hat{S}_T^\lambda$ is not measurable with respect to the sigma-algebra generated by A_T^λ. In our proof, we construct an enlargement of $\sigma(A_T^\lambda)$ with respect to which $\hat{S}_T^\lambda$ is measurable.

The generalization of the above monotonicity result to American type options is not straightforward for the following reason: If τ is a stopping time with respect to the filtration generated by the stock price process, then A_τ^λ and A_τ^γ are not independent of the $\hat{S}_t$ process, and consequently the order of integration in (4.12) can not be changed. We are able to prove the following theorem for American type options when the convex payoff function h satisfies a linear growth condition. In particular, the following theorem remains valid for American call and put options in stochastic volatility models.

Theorem 4.2.3. Let $\lambda^q(t, \sigma)$, $\gamma^q(t, \sigma)$ be the market price of volatility risk defined in theorem 4.2.2, and $\mathbb{Q}^\lambda$, $\mathbb{Q}^\gamma$ be the corresponding martingale measures under which the stock price and volatility processes are governed by (4.8) and (4.9), respectively. In addition, suppose $h(\cdot)$ is a nonnegative convex function which satisfies a linear growth condition $0 \leq h(x) \leq C_0 + C_1 x$ for some positive constants C_0 and C_1, then

$$\sup_{\tau} \mathbb{E}^{\mathbb{Q}^\lambda} h(S_\tau^\lambda) \leq \sup_{\tau} \mathbb{E}^{\mathbb{Q}^\gamma} h(S_\tau^\gamma)$$

if $\lambda(t,\sigma) \geq \gamma(t,\sigma)$ for all t and σ. The supremum is taken over all stopping times $\tau \leq T$ with respect to the filtration generated by the stock price process.

Proof. Let τ be any stopping time adapted to the filtration of the price process $\{\hat{S}_t^\lambda : t \geq 0\}$ and $0 \leq \tau \leq T$. We intend to show that

$$\hat{\mathbb{E}}[h(\hat{S}_\tau^\lambda)] \leq \hat{\mathbb{E}}[h(\hat{S}_\tau^\gamma)].$$

If $\hat{\mathbb{E}}[h(\hat{S}_\tau^\gamma)] = \infty$, this inequality is obvious and hence we assume $\hat{\mathbb{E}}[h(\hat{S}_\tau^\gamma)] < \infty$.

First we notice that the processes $(\hat{S}_t^\lambda, \hat{\sigma}_t^\lambda)$ and $(\hat{S}_t^\gamma, \hat{\sigma}_t^\gamma)$ are adapted to the filtration $(\mathcal{H}_t)$, and it is easy to observe that A_t^λ and A_t^γ are $(\mathcal{H}_t)$-stopping times that satisfy $A_t^\lambda \leq A_t^\gamma < \infty$ for each t in $[0,T]$. Since $\hat{S}_t$ is an $(\mathcal{H}_t)$-martingale for $0 \leq t < \infty$, the process $\hat{S}_t^\lambda$ (or equivalently $\hat{S}_{A_t^\lambda}$) is adapted to the filtration $(\mathcal{H}_{A_t^\lambda})$. Similarly $\hat{S}_t^\gamma$ is $(\mathcal{H}_{A_t^\gamma})$-adapted and $\mathcal{H}_{A_t^\lambda} \subseteq \mathcal{H}_{A_t^\gamma}$ for each $t \geq 0$.

Therefore τ is a $(\mathcal{H}_{A_t^\lambda})$-stopping time as well as a $(\mathcal{H}_{A_t^\gamma})$-stopping time. Now, using a discrete approximation of τ, as in the proof of lemma 3.2.2, it is a straightforward matter to verify that A_τ^λ and A_τ^γ are $(\mathcal{H}_t)$-stopping times which satisfy $A_\tau^\lambda \leq A_\tau^\gamma \leq A_T^\gamma < \infty$. Consider the bounded stopping times $A_\tau^\lambda \wedge m$ and $A_\tau^\gamma \wedge n$ where $0 \leq m \leq n$. Since $\hat{S}_t$ is an $(\mathcal{H}_t)$-martingale and $h(\cdot)$ is convex, it follows that

$$\hat{\mathbb{E}}[h(\hat{S}_{A_\tau^\lambda \wedge m})] \leq \hat{\mathbb{E}}[h(\hat{S}_{A_\tau^\gamma \wedge n})]$$

We first send $n \to \infty$ and show that $\hat{\mathbb{E}}[h(\hat{S}_{A_\tau^\gamma \wedge n})] \to \hat{\mathbb{E}}[h(\hat{S}_{A_\tau^\gamma})]$.

Notice that

$$\hat{\mathbb{E}}[|h(\hat{S}_{A_\tau^\gamma \wedge n}) - h(\hat{S}_{A_\tau^\gamma})|] = \hat{\mathbb{E}}[|h(\hat{S}_n) - h(\hat{S}_{A_\tau^\gamma})|\mathbb{I}_{[A_\tau^\gamma > n]}]$$

$$\leq \hat{\mathbb{E}}[h(\hat{S}_n)\mathbb{I}_{[A_\tau^\gamma > n]}] + \hat{\mathbb{E}}[h(\hat{S}_{A_\tau^\gamma})\mathbb{I}_{[A_\tau^\gamma > n]}]$$

Since $\hat{\mathbb{E}}[h(\hat{S}_{A^\gamma_\tau})] < \infty$, using the dominated convergence theorem to obtain

$$\lim_{n\to\infty} \hat{\mathbb{E}}[h(\hat{S}_{A^\gamma_\tau})\mathbb{I}_{[A^\gamma_\tau > n]}] = \hat{\mathbb{E}}\left[\lim_{n\to\infty} h(\hat{S}_{A^\gamma_\tau})\mathbb{I}_{[A^\gamma_\tau > n]}\right] = 0 \tag{4.13}$$

On the other hand, when $h(\cdot)$ satisfies $0 \leq h(x) \leq C_0 + C_1 x$, note that $\hat{S}_n$ and $\mathbb{I}_{[A^\gamma_\tau > n]}$ are independent, therefore, we obtain

$$\begin{aligned} &\hat{\mathbb{E}}[h(\hat{S}_n)\mathbb{I}_{[A^\gamma_\tau > n]}] \leq \hat{\mathbb{E}}[h(\hat{S}_n)\mathbb{I}_{[A^\gamma_T > n]}] = \hat{\mathbb{E}}[h(\hat{S}_n)] \cdot \hat{\mathbb{P}}[A^\gamma_T > n] \\ \leq &(C_0 + C_1 s_0) \cdot \hat{\mathbb{P}}[A^\gamma_T > n] \to 0 \quad \text{as } n \to \infty \end{aligned} \tag{4.14}$$

Combining (4.13) and (4.14) to conclude that

$$\hat{\mathbb{E}}[|h(\hat{S}_{A^\gamma_\tau \wedge n}) - h(\hat{S}_{A^\gamma_\tau})|] \to 0 \quad \text{as } n \to \infty$$

It follows that $\hat{\mathbb{E}}[h(\hat{S}_{A^\gamma_\tau \wedge n})] \to \hat{\mathbb{E}}[h(\hat{S}_{A^\gamma_\tau})]$ as $n \to \infty$. Hence we obtain $\hat{\mathbb{E}}[h(\hat{S}_{A^\lambda_\tau \wedge m})] \leq \hat{\mathbb{E}}[h(\hat{S}_{A^\gamma_\tau})]$. Now by letting $m \to \infty$ and using Fatou's lemma, we derive

$$\hat{\mathbb{E}}[h(\hat{S}_{A^\lambda_\tau})] \leq \hat{\mathbb{E}}[h(\hat{S}_{A^\gamma_\tau})].$$

The above inequality holds for every stopping time τ with respect to the filtration generated by the stock price process. By taking supremum, it follows immediately that

$$\sup_\tau \mathbb{E}^{\mathbb{Q}^\lambda} h(S^\lambda_\tau) \leq \sup_\tau \mathbb{E}^{\mathbb{Q}^\gamma} h(S^\gamma_\tau)$$

The proof is complete.

$\square$

CHAPTER 5. Future research

- For American option price, while the upper bound derived in this thesis has a closed form and provides a quick estimate, it is still too large for the purpose of option pricing in the real world. More efforts are needed to reduce the estimation error. Furthermore, this closed form upper bound can be extended to cases where the underlying stock is subject to default risk or has stochastic volatility.

- The method used in chapter 3 to solve the when-to-sell-the-stock problem is also applied to other similar optimal stopping problems. For example, it can be used to price a stock loan when the stock has a positive possibility of jumping to default. For an introduction to stock loan, we refer to Xia and Zhou [41].

- In chapter 4 we prove the ordering result for the price of American type options with convex payoff function that satisfies a linear growth condition. Further research needs to be conducted to generalize this result to American options with any convex payoff function.

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ACKNOWLEDGMENTS

I would like to express my deep and sincere gratitude to my Ph.D. supervisor, Dr. Ananda Weerasinghe for his enthusiasm, his inspiration, and his great efforts to explain things clearly and simply. It has truly been a pleasure working with him.

I would also like to thank my committee members, Dr. Arka Ghosh, Dr. Dermot Hayes, Dr. Steven Hou, and Dr. Michael Smiley., for their time and efforts.

I am indebted to my parents and my brother for their love and encouragement. Without their support, I would not have been able to complete this work.