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EFFECTS OF THE INDUCED MAGNETIC FIELD ON THE INVISCID MAGNETOHYDRODYNAMIC CHANNEL FLOW

by

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I. INTRODUCTION

Magnetohydrodynamics can be considered as a special part of a wider field, namely, the interaction of electromagnetic fields and flow fields, both gases and liquids.

In astronomy and geophysical studies the electromagnetic-fluid interaction were and still are of great importance in stellar and planetary processes. Then in the last ten years the engineering applications of these interactions began receiving an increasing concentrated efforts as a major area of broad work that is called magnetohydrodynamics (MHD), and sometimes as magnetofluidmechanics (MFD).

Engineering applications (3, 15) of electromagnetic and fluid interactions have been directed to such topics as, conversion of heat energy to electrical energy, ion propulsion studies, radio wave propagation in the ionosphere and controlled nuclear fusion. Another engineering application is connected with the influence of the magnetic field in providing protection of the internal surfaces of channels and nozzles from high temperature, high speed fluids.

A full understanding of MHD is a very necessary step toward adequate dealing with modern plasma physics and applications. A plasma was defined by Langmuir in 1929 as an ionized fluid with an approximately equal densities of ions and electrons.

The designation MHD or MFD is used for a special branche of electromagnetic-fluid interactions in which magnetic forces and energy dominate the corresponding electrical quantities.

Magnetohydrodynamics in the literature covers three different fields. The first is classical MHD in which experiments are conducted on fluids such as Mercury where theoretical results have been developed for the experimental observations noticed, such as Hartmann flow for incompressible, viscous flow of electrically conducting fluids. In other cases it is assumed that the fluid has infinite electrical conductivity and zero viscosity, that is approaching an ideal case. The second path is directed toward a study of conduction in plasma and the electrical discharges which gave a substantial knowledge about radiation properties of conducting fluids. The third path is toward astronomical and geophysical studies based on the kinetic theory initiated by Boltzman and Maxwell and continued later by Spitzer (33), Burgers (2) and others.

Presently concentrated efforts have been conducted toward the use of MHD effects in an inviscid flow (4, 7) fields with infinite (17) electrical conductivity and strong applied magnetic fields. The magnetohydrodynamical effects were found capable of contracting (27, 29, 31) the flow stream, due to the fact that the magnetic field cannot penetrate a perfect conductor, and hence the fluid flow is pushed off the channel walls until a balanced condition between the fluid pressure and the magnetic pressure is reached. Therefore a vacuous frozen layer in the vicinity of the applied magnetic field source is established, with an outer flow field stream free from the magnetic field. The separation (29) of fluid flow from channel walls is due to the emergence of an adverse pressure gradient along the channel wall, where in this region the magnetic body force is larger than the fluid inertia force.

Studies had been conducted on an inviscid, incompressible flow using various types of magnetic field sources, where separation of fluid flow from the internal surface of the channel was obtained and with neglecting the effects of the induced magnetic field.

Before proceeding further, it is very appropriate to present the important MHD parameters which are of importance or related to the problem of this dissertation:

1. The magnetic Reynolds number R_m, (31)

   $R_m = \sigma_o \mu_o V_o L_o$ where

   σ_o = the reference electrical conductivity

   μ_o = the reference magnetic permeability

   V_o = the reference flow velocity

   L_o = the reference length

2. The ordinary fluid Reynolds number R_e, (19)

   $R_e = \dfrac{\rho_o V_o L_o}{M_o}$ where

   ρ_o = the reference fluid density

   M_o = the reference fluid viscosity

   R_e is a measure of the ratio of inertial force to the viscous force.

3. The magnetic Prandtl number, (31)

   $$P_m = \frac{R_m}{R_e}$$

   $$= \frac{\sigma_o \mu_o M_o}{\rho_o}$$

   which is analogous to the gas dynamic number.

4. The magnetic pressure number R_H,

$$R_H = \frac{B_o^{\,2}}{\rho_o \mu_o V_o^{\,2}} \quad \text{where}$$

B_o = the reference magnetic field

R_H is a measure of the ratio of the magnetic pressure $B_o^{\,2}/2\mu_o$ over the dynamic pressure $\rho_o V_o^{\,2}/2$.

5. The magnetic interaction parameter I, (31)

$$I = \frac{\sigma_o L_o B_o^{\,2}}{\rho_o V_o}$$

which is a measure of the magnetic body force to the inertia force.

6. The ordinary Mach number M_o, (19, 24)

$$M_o = \frac{V_o}{A_o} \quad \text{where}$$

A_o = the speed of sound

M_o is a measure of the compressibility of the fluid due to high velocity and defined as the ratio of the fluid flow velocity to the speed of sound A_o.

7. The magnetohydrodynamic Mach number M_m, (31)

$$M_m = \frac{1}{\sqrt{R_H}}$$

which has a great significance in flow problems where wave motion is important.

8. The Hartmann number R_h, (24)

$$R_h = \sqrt{R_e R_m R_H}$$

In the case of incompressible and inviscid conducting fluids through channels, two (31) of the preceeding parameters are of more importance than others. One is the magnetic Reynolds number R_m which determines the change in the applied magnetic field due to electric currents induced in the flow field. The intensity of the magnetic field can be represented by the magnetic pressure $B^2/2\mu$ when the fluid has high electrical conductivity. The induced magnetic field is small compared to the applied magnetic field when the electrical conductivity is small, hence the magnetic force which is the change in the magnetic pressure due to the induced currents is much smaller than the total magnetic pressure. Therefore the second suitable element is the magnetic interaction parameter I. The effective strength of the magnetic field can be measured by I.

Turning again to the magnetic Reynolds number R_m, it can be defined also as

$$R_m = \frac{L_o}{L_e} \quad \text{where}$$

$$L_e = \frac{1}{\sigma_o \mu_o V_o} \quad \text{known as the characteristic length of the flow}$$

L_o = dimension of the flow field.

Also R_m can be defined as:

$$R_m = \frac{V_o}{V_e} \quad \text{where}$$

$$V_e = \frac{1}{\sigma_o \mu_o L_o} \quad \text{known as the characteristic flow velocity, and}$$

V_o = flow field velocity.

Now if $L_o \gg L_e$ we will have what is called the frozen-in fields where the magnetic force lines stay with the conducting fluid. L_e and V_e are respectively characteristic length and velocity by which the magnetic field is moving through the conducting fluid.

When R_m is not zero, the applied magnetic field will face difficulty in penetrating the conducting plasma and the flow field will approach the situation of being aligned with the magnetic field and will be harmonic inside the conducting fluid. This will give rise to the formation of a current sheet as well as a vortex sheet at the surface of the channel. These current and vorticity sheets are in fact two layers. The physical interpretation of those boundary layers is that at the (28, 31) surface there will be a viscous sub-layer of thickness of the order of $1/\sqrt{R_e}$. Since R_e is very large, the thickness of the viscous sub-layer is extremely small. In this sub-layer the vertical and horizontal velocity components are zero at the surface. Then there follows the inviscid-magnetic layer whose thickness is of the order of $1/\sqrt{R_m}$, which will be of finite value. At the inner boundary of this inviscid layer the perpendicular velocity component is zero, and the parallel component is not.

The flow above those viscous and current sheets will approach an irrotational flow to some degree, especially when R_m is less than unity. Figure 1 shows an illustration of the sub-layers.

The significance of assuming the existence of the viscous sub-layer, which is of extremely small thickness, is that of surrounding the flow on each channel. Thus, the flow will be similar to an ideal Poiseuille flow, where the Hartmann number R_h is very large or approaching infinity, because the viscosity coefficient in the viscous sub-layer is small.

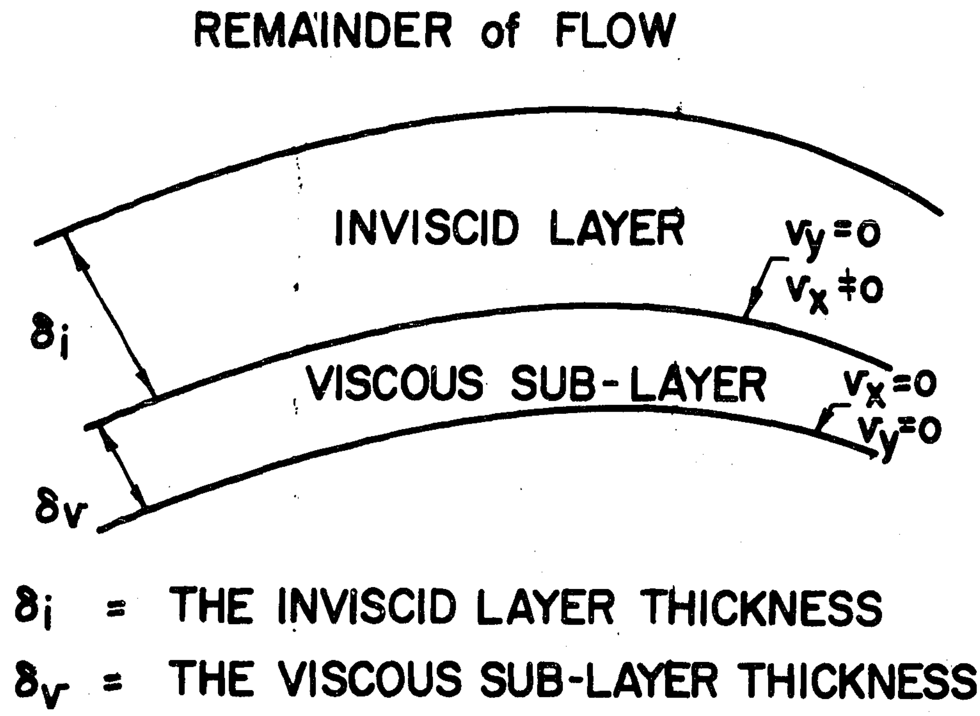


Figure 1. Details of layers formation in the channel flow

The problem of (27, 32) protecting internal surfaces of channels and nozzles from high temperature, high speed flows led to investigations on inviscid, incompressible flow in MHD channels with R_m equal to zero, to the extent that the induced magnetic field is neglected and the applied field is due to various sources such as infinite conductors and linear dipoles. An adverse pressure gradient was obtained at the channel wall and in the region of the applied magnetic field source. This led to the conclusion that the fluid will have a boundary layer separation in that region. This effect will result in preventing the direct contact between the fluid and the wall material and consequently will lead to a reduction in the transfer of heat from the hot fluid to the wall which, in effect, gives protection to the channel walls.

The physical phenomena occurring (31) in the flow field is that the magnetic field acts on the inviscid flow layer in such a manner as to cause a rapid decrease in the free stream velocity, especially in the region near the source of the magnetic field. This will result in a reduction of the wall skin friction. Also the Lorentz force in the boundary layer due to the effect of the magnetic field tends to retard the flow, and so the two effects will result in separating the fluid layer from coming in contact with the channel wall, and this will result in heat transfer reduction from the fluid to the wall. This is considered as an element of great significance in giving more protection to channel walls.

Previous investigators have neglected the induced magnetic field, corresponding to the magnetic Reynolds number being zero.

This dissertation reports an investigation of magnetohydrodynamic flow problems where the fluid is incompressible, inviscid, electrically conducting, thermally non-conducting, of scalar constant conductivity, and with stable ionization of partial degree such that the magnetic Reynolds number is substantially smaller than unity but cannot be neglected. The flow is between infinite parallel plates of arbitrary separation and the applied magnetic field is nonuniform, due to an infinite current carrying conductor located at a small distance below the lower channel wall. The treatment involves the use of perturbation theory in the expansion of the magnetic field and flow field components in a double power series in R_m and I.

The ultimate goal is the development of a certain criterion underwhich the magnetic Reynolds number R_m must be considered or can be neglected. Therefore the main objective is to observe the behavior of the pressure gradient along the channel wall in order to determine the effects of the induced magnetic field on the fluid separation from the channel wall and hence a reduction in heat transfer to the channel wall.

II. REVIEW OF LITERATURE

The application of a magnetic field to control the motion of an electrically conducting fluid has been investigated under different conditions: with respect to the nature of the conducting fluid, the geometry of the channel in which the fluid is flowing, the source of the magnetic field applied, the degree of ionization of the fluid and finally the strength of the magnetic field.

V. J. Rossow (26) conducted an analysis for the flow of an incompessible boundary layer over a flat plate when the applied magnetic field is transverse with respect to the direction of flow and confined the magnetic field effect to the boundary layer only. The fluid flow was viscous and he obtained numerical solutions which showed the behavior of pressure, temperature, and skin friction and illustrated the effect of the magnetic field in reducing heat transfer to the channel wall and the significance of using infinite current carrying conductors in forming the boundary layer (19) separation which prevents hot conducting fluid from coming in contact with the channel walls. Kemp and Petschek (16) analyzed a two dimensional flow of an incompressible constant conductivity fluid through an elliptically shaped solenoid with the magnetic field perpendicular to the flow plane. They considered the effects of ion-slip and Hall current and applied the generalized Ohm's Law in the analysis with the use of the perturbation theory in obtaining solutions for the force and moment on the solenoid. They carried the perturbation approach to the first order in the magnetic Reynolds number R_m and to the first order in the magnetic interaction parameter I. This means that the magnetic field is modified but the flow field is not and the flow field is distorted but the magnetic field is not.

Also Petschek and his colleagues (9) investigated a two dimensional supersonic magnetohydrodynamic flow using the perturbation theory in a double power series of the magnetic Reynolds number R_m and the magnetic interaction parameter I, with consideration of the ion-slip and Hall current. They used a loosely wound circular solenoid as a source of the magnetic field where the flow was perpendicular to the axis of the solenoid and had a weak interaction with it.

The validity of using the perturbation procedure was based on the assumption that the two coefficients in the power series expansion R_m and I must be substantially smaller than unity. This group (9) also extended their study to an experimental analysis for the drag and lift on the solenoid in order to match the theoretical results.

Much work in the field of magnetohydrodynamics has been directed to the effect of the fluid conductivity, whether it is a scalar constant, a tensor constant, a function of position or a function of the current density, on the characteristics of the magnetohydrodynamic flow for various types of magnetic field sources. Kogain (17) conducted his study for a fluid of infinite electrical conductivity and the magnetic field not parallel to the flow field. A. Sherman (29) extended the analysis for the interaction between an inviscid magnetohydrodynamic flow and a non-uniform magnetic field of an infinite current carrying conductor. The flow was between two parallel plates of arbitrary separation and the current carrying conductor was located at unit distance below the lower channel wall. In this analysis, Sherman considered the flow field to be slightly distorted by taking the interaction parameter I small and the total magnetic field undistorted and equal to the original applied magnetic field, and hence the magnetic Reynolds number R_m zero, and no induced magnetic field. He obtained numerical solutions for the velocities, temperature, pressure and pressure gradient along the channel wall, and showed that the high temperature fluid flow will probably have a boundary layer separation in the region near the source of the applied magnetic field due to the adverse pressure gradient. This led to the conclusion that heat transfer from the fluid to the channel wall will be reduced. this conclusion raised hopes for reliance on the non-uniform magnetic field for protection of the channel walls from the high temperature, high speed gaseous plasma.

R. H. Levy (20, 21) continued along the same direction to investigate the effect of the interaction between magnetic field sources of various two dimensional types and fluid plasma at low conductivity. He neglected the effect of the induced magnetic field, such that the flow field is modified but the magnetic field is not. In his analysis, Levy considered magnetic field sources of one current-carrying conductor of infinite extent below the channel wall, two conductors separated by a certain distance below the channel wall and extending infinitely, and also a magnetic dipole below the lower wall. He showed clearly the effect of the magnetic field and flow field interaction in forming an adverse pressure gradient near the magnetic field source and its influence on the protection of the magnetohydrodynamic channel wall from the high temperature gas. Furthermore, Levy analyzed supersonic and subsonic flows with the electrical conductivity as a scalar function of position and used the linearized perturbation theory to the first order in the magnetic interaction parameter I, but R_m was zero.

F. D. Hains and Y. A. Yoler (12) conducted their analysis for magnetohydrodynamic flow through a circular cross-sectional channel with a compressible fluid and where the magnetic field was slightly distorted such that R_m was very small. They checked the calculated results of boundary layer thickness, skin friction, heat transfer and pressure gradient experimentally. Later, C. Chu (4) conducted an investigation on the flow of an inviscid fluid over an insulated flat plate using the linearized perturbation procedure and a magnetic field which was transverse with respect to the direction of fluid flow.

The problem of heat transfer to the channel wall has been under continuous investigation. Siegel and Perlmutter (32) reported an extensive work in the area of heat transfer in the fluid in a transverse magnetic field. Horlock (14) conducted his studies on inviscid magnetohydrodynamic flow at low electrical conductivity with cross and parallel magnetic fields. He used a very small R_m, to the extent that the applied magnetic field remained un-distorted, but he also considered the flow field to be distorted substantially such that the magnetic interaction parameter I was equal to unity.

Timofeev (35) investigated the convection of a weakly ionized plasma with a non-uniform magnetic field due to direct current. He obtained results concerning the value of the critical magnetic field for the condition of instability where it was known that convection of completely ionized plasma is unstable in a non-uniform magnetic field and a similar effect also in a weakly ionized plasma in a strong magnetic field. F. E. Ehler (8) conducted a study on the use of linearized methods in magnetohydrodynamic flow in a circular channel flow with the applied magnetic field having axial symmetry due to a circular solenoid.

The effect of magnetohydrodynamic interaction when the plasma is of infinite electrical conductivity also received great emphasis. Sears (28) conducted a study on the class of steady plane and axisymmetric magnetohydrodynamic flows of inviscid character and analyzed the properties of the boundary layers formed on solid bodies. Sakurai (27) investigated the two dimensional hypersonic channel flow of a perfect gas with infinite electrical conductivity in the non-uniform magnetic field produced by two anti-parallel line currents. He concluded that the magnetic field is capable of contracting the fluid flow away from the magnetic field source and that the wind tunnel wall can be protected from the high temperature, high speed gas flow by the magnetohydrodynamical effect. R. H. Levy (22) investigated the flow of a plasma having infinite conductivity past a two-dimensional dipole to find a bounding line at which the magnetic pressure balances the dynamic pressure. Thommen and Yoshihara (34) investigated the case of a weak magnetic dipole moving in a plasma composed of protons and electrons of infinite conductivity.

The strength of the magnetic field has a big effect on whether the electrical conductivity is a scalar or a tensor. For a weak magnetic field and in a low degree of ionization the induced current is parallel to the electric field as seen by the gas and is proportional to it with the magnetic Reynolds number as the constant of proportionality. At a higher intensity of magnetic field and with a higher degree of ionization, the electrons can make several cyclotron orbits between collisions. They drift in a direction perpendicular to the electric and magnetic fields and thus a Hall current will be produced normal to the electric field direction. At a still higher intensity of the magnetic field, the electrons and ions are held so strongly by the magnetic field that there is a relative motion between the ionized and unionized portions of the fluid and the effective conductivity will be reduced since the ion slip will reduce the induced emf. A. Sherman (29) investigated magnetohydrodynamic flow with a fluid plasma having a tensor conductivity and in a non-equilibrium state by assuming the electrical conductivity to be a linear function of the current density. In his analysis Sherman used the perturbation method and carried the expansion to the zeroth order in the magnetic Reynolds number and to the first order in the expansion parameter. His conclusion was that non-equilibrium ionization tends to reduce the Hall potential even when there is no current leakage between electrode pairs.

The preceding discussion presented the various stages of investigation conducted on the interaction between a fluid plasma at different degrees of ionization (with the electrical conductivity as either scalar constant function, a special function of position, a special function of current density or a tensor constant value), and the applied magnetic field (either transverse, cross-parallel or non-uniform in the plane of the flow). Viscous and inviscid, compressible and incompressible fluids have also been treated. The problem of protection of the internal walls of magnetohydrodynamic channels from high temperature, high speed gaseous plasmas have been receiving continuous attention for the last ten years. In most of those investigations, solutions were obtained by neglecting the effects of the induced magnetic field with R_m thus zero or R_m was taken to be small and in the perturbation procedure the expansion was carried only to the first order in R_m and to the first order in I. For a channel of two infinite parallel plates with an infinite current carrying conductor as a magnetic field source located unit distance below the lower wall, only solutions to the first order in the magnetic interaction parameter I have been considered while the magnetic Reynolds number was taken as zero since the applied magnetic field remained without distortion. The fluid treated was incompressible, inviscid and thermally non-conducting. Solutions have been obtained by converting the partial differential equations to the finite difference form and solving with a digital computer. It has been concluded that a separation of the fluid boundary layer is indicated by the appearance of an adverse pressure gradient at the lower channel wall near the source of the applied magnetic field. This fluid displacement results in a reduction of the heat transfer from the high temperature gaseous plasma to the channel wall and hence this will add a new protective element to the channel. Also theoretical investigations of the case of infinite electrical conductivity fluid have been carried out. Again the magnetohydrodynamical displacement (30) effect in separating the fluid from the channel walls has been demonstrated. The magnetic field cannot penetrate a perfect electrical conductor and hence the fluid flow is frozen and contracted. This case corresponds to the condition of infinite value of the magnetic Reynolds number and results in a substantial reduction in the heat transfer from the fluid to the channel walls.

III. STATEMENT OF THE PROBLEM

The effect of the non-uniform magnetic field due to a current carrying conductor of infinite extent laying at a small distance below the lower plate of a magnetohydrodynamic channel of arbitrary height will be considered. The fluid plasma is incompressible, inviscid, thermally non-conducting, electrically conducting and has constant scalar conductivity. The induced magnetic field is small compared to the applied field, to the extent that the magnetic Reynolds number R_m is small, i.e. R_m is less than unity but not zero.

Solution of the problem will be carried out by the perturbation approach in a double power series expansion in the magnetic Reynolds number and the magnetic interaction parameter, both smaller than unity. The appearance of an adverse pressure gradient at the lower channel wall and in the vicinity of the magnetic field source, implying the probability of boundary layer (19, 31) separation and consequently the reduction of heat transfer from the high temperature plasma to the channel wall material, will be investigated.

In this dissertation the perturbation method will be carried to the first order in the magnetic Reynolds number R_m and then to the first order in the product of R_m and I.

Fourier transforms will be used to solve the partial differential equations arising from the perturbation method using certain justified approximations.

Expressions for the hydrostatic and magnetic pressure gradient along the channel wall will be obtained from which the effects of the induced magnetic field will be determined. A criterion under which the induced magnetic field can be considered or neglected will be developed.

The applied magnetic field is non-uniform and due to an infinite conductor carrying direct current and is represented by

$$B_{oy} = \left(\frac{I\mu_o}{2\pi}\right) \frac{x}{x^2 + (y+y_o)^2} \tag{1}$$

$$B_{ox} = -\left(\frac{I\mu_o}{2\pi}\right) \frac{(y+y_o)}{x^2 + (y+y_o)^2} \tag{2}$$

The conductor is imbedded below the channel wall at a distance y_o. The reference magnetic field is the absolute value of the applied field at $x = 0$, $y = 0$ and the reference distance is y_o.

The fluid is electrically conducting and assumed to have a constant initial velocity V_o parallel to the channel wall at the entrance of the channel and with initial pressure P_o. The initial velocity V_o is considered equal to the reference velocity.

The significance of the induced magnetic field will be explored to the extent of showing that the area under the adverse hydrostatic pressure gradient curve at the wall is increasing more than the case when the induced magnetic field is assumed to be zero. Also the effects of the arbitrary height of the channel flow on pressure distribution, both static and magnetic along the wall, and its influence in producing more boundary layer separation will be explored.

IV. THEORY

Consider a control volume of fluid in the presence of an electromagnetic field. The fluid is assumed to have the normal properties of fluids. The mathematical equations describing the fluid flow are not independent from the electromagnetic field equations. The two sets of equations are coupled. These equations are given in the following sections.

A. Flow Field Equations

1. Continuity equation (3, 24)

The mathematical statement of the conservation of mas is

$$\frac{\partial \rho}{\partial t} + \bar{\nabla} \cdot (\rho \bar{V}) = 0 \tag{3}$$

where

$(\bar{\ })$ indicates vector quantity

ρ = fluid mass density

Equation 3 can be rewritten as

$$\frac{\partial \rho}{\partial t} + \rho\, \bar{\nabla} \cdot \bar{V} + \bar{V} \cdot \bar{\nabla} \rho = 0 \tag{4}$$

Then,

$(\frac{\partial \rho}{\partial t} + \bar{V} \cdot \bar{\nabla} \rho)$ can be replaced with the convective derivative term

$$\frac{D\rho}{Dt}$$

and Equation 4 becomes

$$\frac{D\rho}{Dt} + \rho\, \bar{\nabla} \cdot \bar{V} = 0 \tag{5}$$

The continuity equation is unaffected by the electromagnetic effects since forces never appear in it.

The definition for an incompressible fluid is that the convective derivative is equal to zero.

Therefore, Equation 5 becomes

$$\overline{\nabla} \cdot \overline{V} = 0 \tag{6}$$

2. Momentum equation (24, 31)

The contribution to momentum flux comes from

a. surface forces due to pressures acting on the control volume surface.

b. body forces such as magnetic, electric and gravitational forces.

Surface forces are represented by the pressure tensor

$$\overline{\overline{P}} = -P\overline{\overline{I}} + \tau_{ij} \tag{7}$$

where

P = hydrostatic pressure

$\overline{\overline{I}}$ = unit tensor

$\overline{\overline{\tau}}_{ij}$ = viscosity tensor

The electromagnetic forces are given by the Lorentz equation

$$\overline{\overline{F}}_{em} = \overline{j} \times \overline{B} + \rho_e \overline{E} \tag{8}$$

where

E = electric field intensity

ρ_e = excess electric charge

The surface and body forces can be combined in a momentum equation

$$\frac{D}{Dt} \int_V (\rho V) dv = \int_S F_s \cdot n \, da + \int_V F_b \, dv \tag{9}$$

where

F_s = surface force

and

F_b = body force

Then, for an arbitrary volume of incompressible and inviscid fluid, with no applied electric field and neglecting gravitational force, the momentum equation becomes

$$\rho(\overline{V} \cdot \overline{\nabla})\overline{V} = - \overline{\nabla} P + \overline{j} \times \overline{B} \tag{10}$$

where

B = magnetic flux density

upon application of the Divergence Theorem.

3. Energy equation (24, 31)

Conservation of energy in magnetohydrodynamic flow is represented by the following:

a. The rate of increase of the fluid energy is the sum of the rate of increase of kinetic energy and the rate of increase of internal energy.

b. The rate of energy input comes from the sum of

   (1) The rate at which electromagnetic energy enters.

   (2) The rate at which energy due to heat conduction and diffusion enters.

   (3) The rate of energy input resulting from surface forces.

Then for incompressible, inviscid and steady flow the energy equation is given by

$$\rho(\bar{V} \cdot \bar{\nabla}) \left[\frac{1}{2} V^2 + \frac{P}{\rho} + C_v T\right] = \bar{E} \cdot \bar{j} \tag{11}$$

where

E = total electric field intensity

C_v = specific heat at constant volume

T = temperature

j = current density

B. Electromagnetic Field Equations (15, 31)

1. The charge continuity equation

$$\bar{\nabla} \cdot \bar{j} + \frac{\partial \rho_e}{\partial t} = 0 \tag{12}$$

2. Ampere's Law

$$\bar{\nabla} \times \bar{B} = \mu_o \bar{j}_t \tag{13}$$

where

j_t = total current density

$$\bar{\nabla} \cdot \bar{B} = 0 \tag{14}$$

3. Faraday's equation

$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \tag{15}$$

(The remaining Maxwell equation which relates the divergence of the electric field to the net charge density can be replaced by the condition that electron and ion densities are equal in magnetohydrodynamic flows.)

4. Ohm's Law

$$\bar{j}_c = \sigma(\bar{E}_a + \bar{V} \times \bar{B}) \tag{16}$$

where

j_c = conduction current density neglecting Hall effect and ion slip

σ = electrical conductivity

E_a = applied electric field intensity

5. The energy equation (18, 31)

$$W_{em} = \bar{E}_a \cdot \bar{j}_t - \frac{j_c^2}{\sigma} \tag{17}$$

where W_{em} is the rate at which electromagnetic energy enters the flow field.

If $E_a = 0$

$$W_{em} = -\frac{j_c^2}{\sigma} \tag{18}$$

For MHD flow having the properties of being incompressible, inviscid and steady and with no applied electric field the magnetohydrodynamic equations are

$$\begin{aligned}
&\bar{\nabla} \cdot \bar{V} = 0 \\
&\rho(\bar{V} \cdot \bar{\nabla})\bar{V} = -\bar{\nabla} P + \sigma(\bar{V}\times\bar{B}) \times \bar{B} \\
&\bar{\nabla} \cdot \bar{B} = 0 \\
&\bar{\nabla} \times \bar{B} = \mu_o \bar{j} \\
&\rho(\bar{V} \cdot \bar{\nabla})(\tfrac{1}{2}\bar{V}^2 + \tfrac{P}{\rho} + C_v T) = -\frac{j_c^2}{\sigma}
\end{aligned} \tag{19}$$

if the Hall effect and ion slip are neglected.

Next, Equation 19 can be made non-dimensional as follows:

$$B^* = \frac{B}{B_{ref}}$$

$$V^* = \frac{V}{V_{ref}}$$

$$P^* = \frac{P}{\rho V_{ref}^2} \tag{20}$$

$$l^* = \frac{l}{l_{ref}}$$

$$\rho^* = \frac{\rho}{\rho_{ref}}$$

where B_{ref}, V_{ref}, l_{ref} and ρ_{ref} are reference values of magnetic field density, velocity, length and fluid density respectively and ( )* represents a non-dimensional, we get

$$\bar{\nabla} \cdot \bar{V} = 0$$

$$\bar{\nabla} \cdot \bar{B} = 0 \tag{21}$$

$$\bar{\nabla} \times \bar{B} = R_m(\bar{V} \times \bar{B})$$

$$(\bar{V} \cdot \bar{\nabla})\bar{V} = -\bar{\nabla} P + I(\bar{V} \times \bar{B}) \times \bar{B}$$

where

$$R_m = \mu_o \sigma_o V_{ref} y_o \ ,$$

$$I = \frac{\sigma_o y_o \ B_{ref}^2}{\rho_o \ V_{ref}}$$

and

$$IR_m = \frac{\sigma y_o \ B_{ref}^2}{\rho \ V_{ref}} \sigma y_o \mu \ V_{ref}$$

$$= \mu(\sigma^2 y_o^2 \ B_{ref}^2 \ V_{ref}^2/\rho V_{ref}^2)$$

If $V_{ref} = V_o$ (the initial flow velocity) and $B_{ref} = B_o$ (the applied magnetic field at (0, 0))

$$l_{ref} = y_o$$

$$R_m = \mu_o \sigma_o V_o y_o \tag{22}$$

$$I = \frac{\sigma_o y_o B_o^2}{\rho_o V_o} \tag{23}$$

Therefore,

$$IR_m = \mu_o (\sigma_o{}^2 y_o{}^2 B_o{}^2 V_o{}^2)/\rho_o V_o{}^2$$

V. DERIVATION OF THE MAGNETIC AND FLOW FIELD PERTURBATION EQUATIONS

A. Application of the Perturbation Method (8, 31)

Let

$\overline{V}$ represent the velocity of the conducting fluid

P represent the pressure of the conducting fluid

$\overline{B}$ represent the total magnetic flux density

Then, expressing $\overline{V}$, P and $\overline{B}$ as a double power series in I and R_m

$$\overline{V} = \overline{V}_o + \sum_{n+k=1} I^n R_m^k \, \overline{V}^{(n,k)} \qquad n,\ k = 0,\ 1,\ 2\ ---$$

$$\overline{B} = \overline{B}_o + \sum_{n+k=1} I^n R_m^k \, \overline{B}^{(n,k)} \tag{24}$$

$$P = P_o + \sum_{n+k=1} I^n R_m^k \, P^{(n,k)}$$

$$\overline{V} = \overline{V}_o + R_m \overline{V}_1 + I\overline{V}_2 + IR_m \overline{V}_3 + \cdots$$

$$\overline{B} = \overline{B}_o + R_m \overline{B}_1 + I\overline{B}_2 + IR_m \overline{B}_3 + \cdots \tag{25}$$

$$P = P_o + R_m P_1 + IP_2 + IR_m P_3 + \cdots$$

Rewriting Equations 21

$$\overline{\nabla} \times \overline{B} = R_m(\overline{V} \times \overline{B})$$

$$\overline{\nabla} \cdot \overline{B} = 0$$

$$(\overline{V} \cdot \overline{\nabla})\overline{V} = - \overline{\nabla} P + I(\overline{V} \times \overline{B}) \times \overline{B} \tag{21}$$

$$\overline{\nabla} \cdot \overline{V} = 0$$

Substituting Equations 25 in Equations 21 and equating the sum of coefficients of like powers in I and R_m to zero, the following system of equations is obtained.

1. Zeroth order

$$
\left.
\begin{aligned}
&\bar{\nabla} \cdot \bar{B}_o = 0 \\
&\bar{\nabla} \times \bar{B}_o = 0 \\
&\bar{\nabla} \cdot \bar{V}_o = 0 \\
&(\bar{V}_o \cdot \bar{\nabla})\bar{V}_o + \overline{}\, P_o = 0
\end{aligned}
\right\}
\qquad (26)
$$

2. First order in I

$$
\left.
\begin{aligned}
&\bar{\nabla} \cdot \bar{B}_2 = 0 \\
&\bar{\nabla} \times \bar{B}_2 = 0 \\
&\bar{\nabla} \cdot \bar{V}_2 = 0 \\
&(\bar{V}_o \cdot \bar{\nabla})\bar{V}_2 + (\bar{V}_2 \cdot \bar{\nabla})\bar{V}_o + \bar{\nabla} P_2 = (\bar{V}_o \times \bar{B}_o) \times \bar{B}_o
\end{aligned}
\right\}
\qquad (27)
$$

3. First order in R_m

$$
\left.
\begin{aligned}
&\bar{\nabla} \cdot \bar{B}_1 = 0 \\
&\bar{\nabla} \times \bar{B}_1 = \bar{V}_o \times \bar{B}_o \\
&\bar{\nabla} \cdot \bar{V}_1 = 0 \\
&(\bar{V}_o \cdot \bar{\nabla})\bar{V}_1 + (\bar{V}_1 \cdot \bar{\nabla})\bar{V}_o + \bar{\nabla} P_1 = 0
\end{aligned}
\right\}
\qquad (28)
$$

4. First order in $\mathbb{R}_m$

$$
\begin{aligned}
&\bar{\nabla} \cdot \bar{B}_3 = 0 \\
&\bar{\nabla} \times \bar{B}_3 = \bar{V}_o \times \bar{B}_2 + \bar{V}_2 \times \bar{B}_o \\
&\bar{\nabla} \cdot \bar{V}_3 = 0 \\
&(\bar{V}_1 \cdot \bar{\nabla})\bar{V}_2 + (\bar{V}_2 \cdot \bar{\nabla})\bar{V}_1 + (\bar{V}_3 \cdot \bar{\nabla})\bar{V}_o + (\bar{V}_o \cdot \bar{\nabla})\bar{V}_3 + \bar{\nabla} P_3 \\
&\qquad = (\bar{V}_o \times \bar{B}_1) \times \bar{B}_o + (\bar{V}_o \times \bar{B}_o) \times \bar{B}_1 + (\bar{V}_1 \times \bar{B}_o) \\
&\qquad \times \bar{B}_o
\end{aligned}
\tag{29}
$$

To solve this system of equations we proceed as follows:

1. To the zeroth order, the magnetic flux density is that of the applied magnetic field with fluid velocity and pressure equal to the initial value V_o and P_o, respectively.
2. To the first order in I, the magnetic interaction parameter, the fluid flow will be slightly distorted but the induced magnetic field is neglected and so the applied magnetic field will remain undistorted. This implies $B_2 = 0$ and V_2, P_2 must be solved for, based on the zeroth order solution. Sherman and Levy (20, 31) have obtained solutions for V_2 and P_2 for this particular case.
3. To the first order in R_m, the magnetic Reynolds number, the applied magnetic field will be distorted due to the effect of the induced field. This distortion will be slight because R_m is substantially less than unity. The flow field will remain undistorted, so that V_1, $P_1 = 0$ and the induced magnetic field B_1 can be solved for, based on the zeroth-order solutions.

4. To the first order in the product of IR_m the distorted flow field V_3, P_3 can be solved for, knowing V_o, V_1, V_2, B_o and B_1. B_3 can also be determined, knowing V_2 and B_o. In this part both the magnetic field and the flow field are distorted.

B. Derivation of the Induced Magnetic Field Equations

Rewriting Equation 28

$$\bar{\nabla} \cdot \bar{B}_1 = 0$$

$$\bar{\nabla} \times \bar{B}_1 = \bar{V}_o \times \bar{B}_o$$

$$\bar{\nabla} \cdot \bar{V}_1 = 0$$

$$(\bar{V}_o \cdot \bar{\nabla})\bar{V}_1 + (\bar{V}_1 \cdot \bar{\nabla})\bar{V}_o \times \bar{\nabla} P_1 = 0$$

This set of equations represents the case where the applied magnetic field of flux density B_o will be distorted (32) to some degree. This results in the induced magnetic flux density B_1. The flow field will remain undistorted, i.e. V_1, $P_1 = 0$.

V_o, the velocity of flow at the entrance of the channel, is considered the reference velocity and hence

$$V_o = 1 \text{ (non-dimensional)}$$

Since the initial flow is in the x direction,

$$\bar{V}_o = \bar{a}_x$$

where $\bar{a}_x$ is a unit vector in the x direction.

Equation 28 can be written as

$$\bar{\nabla} \cdot \bar{B}_1 = 0$$

$$\bar{\nabla} \times \bar{B}_1 = \bar{a}_x \times \bar{B}_o \qquad (30)$$

$$\bar{\nabla} \cdot \bar{V}_1 = 0$$

$$(\bar{a}_x \cdot \bar{\nabla})\bar{V}_1 + (\bar{V}_1 \cdot \bar{\nabla})\bar{a}_x + \bar{\nabla}P_1 = 0$$

The flow will remain un-distorted, i.e. V_1, $P_1 = 0$.

Equation 30 becomes

$$\bar{\nabla} \cdot \bar{B}_1 = 0$$

$$\bar{\nabla} \times \bar{B}_1 = \bar{a}_x \times (\bar{a}_x B_{ox} + \bar{a}_y B_{oy}) = \bar{a}_z B_{oy} \qquad (31)$$

The two equations to be solved to obtain $\bar{B}_1$ are

$$\bar{\nabla} \cdot \bar{B}_1 = 0$$

$$\bar{\nabla} \times \bar{B}_1 = \bar{a}_z B_{oy} \qquad (32)$$

where B_{oy} is the y component of the external field B_o.

From Equation 1

$$B_{oy} = \frac{x}{x^2 + (1+y)^2}$$

Then

$$\bar{\nabla} \times \bar{B}_1 = \frac{x}{x^2 + (1+y)^2} \qquad (33)$$

$$\bar{\nabla} \cdot \bar{B}_1 = 0$$

Since B_1 will be two dimensional,

$$\bar{\nabla} \times \bar{B}_1 = \frac{\partial B_{1y}}{\partial x} - \frac{\partial B_{1x}}{\partial y} \tag{34}$$

$$\frac{\partial B_{1y}}{\partial x} - \frac{\partial B_{1x}}{\partial y} = \frac{x}{x^2 + (1+y)^2} \tag{35}$$

$$\frac{\partial B_{1y}}{\partial y} + \frac{\partial B_{1x}}{\partial x} = 0 \tag{36}$$

Now differentiate Equation 35 with respect to y and Equation 36 with respect to x and then subtract with the result that

$$\frac{\partial^2 B_{1y}}{\partial x \partial y} - \frac{\partial^2 B_{1x}}{\partial y^2} = \frac{-2x(1+y)}{[x^2+(1+y)^2]^2} \tag{37}$$

$$\frac{\partial^2 B_{1y}}{\partial x \partial y} + \frac{\partial^2 B_{1x}}{\partial x^2} = 0 \tag{38}$$

Subtracting Equation 37 from 38 we obtain

$$\frac{\partial^2 B_{1x}}{\partial x^2} + \frac{\partial^2 B_{1x}}{\partial y^2} = \frac{2x(1+y)}{[x^2+(1+y)^2]^2} \tag{39}$$

It is now necessary to establish the boundary conditions on B_{1x}.

It is shown in Figure 1 that in addition to the main flow, there is an inviscid (28,31) layer of finite thickness. Beneath this layer there is the viscous sub-layer, laying on the channel wall, of very small thickness, of the order of $1/\sqrt{R_e}$ with R_e assumed to have a very large value.

In the viscous sub-layer of the fluid the vertical and horizontal velocity components will have zero value, then just after this sub-layer, the flow assumes the undistorted velocity V_o in the x direction. Therefore the flow is very similar to an ideal Hartmann (24) type where the Hartmann number R_h is very large.

The following approach, based on boundary layer approximation, is similar to that presented in the Sears paper (28) and also in the book by Sherman and Sutton (31).

Rewrite the second of Equations 28 as

$$\overline{\nabla} \times \overline{B}_1 = \overline{V}_o \times \overline{B}_o$$

$\overline{V}_o$ is only in the x direction and

$$\overline{B}_o = \overline{a}_x B_{ox} + \overline{a}_y B_{oy}$$

Then, in two dimensions,

$$\frac{\partial \overline{B}_{1x}}{\partial y} - \frac{\partial \overline{B}_{1y}}{\partial x} = \overline{a}_z V_o B_{oy} \qquad (40)$$

In the viscous sub-layer, the flow field has zero velocity. In other words, V_o will be replaced by zero in the viscous sub-layer.

Therefore, Equation 40 in the viscous sub-layer becomes

$$\frac{\partial \overline{B}_{1x}}{\partial y} - \frac{\partial \overline{B}_{1y}}{\partial x} = 0 \qquad (41)$$

and also in this sub-layer

$$\frac{\partial}{\partial y} \text{ is of the order of } 1/\delta_v$$

where

$$\delta_v = \text{thickness of the viscous sub-layer} = 1/\sqrt{R_e}$$

and

$$\frac{\partial}{\partial x} \text{ is of the order of } 1$$

Therefore, Equation 41 becomes

$$\sqrt{R_e}\; B_{1x} = B_{1y} \qquad (42)$$

or

$$B_{1x} = \frac{B_{1y}}{\sqrt{R_e}} \tag{43}$$

Since R_e is a large number in the viscous sub-layer, or at the surface of the channel wall,

$$B_{1x} \approx 0 \tag{44}$$

at $y = 0$ and L

where

L = Arbitrary channel height

Therefore, the partial differential equation of B_{1x}, together with the established boundary conditions, becomes

$$\frac{\partial^2 B_{1x}}{\partial x^2} + \frac{\partial^2 B_{1x}}{\partial y^2} = \frac{2x(1+y)}{[x^2 + (1+y)^2]^2} \tag{45}$$

$$B_{1x} = 0 \text{ at } y = 0, L$$

and (46)

$$B_{1x} = 0 \text{ at } x = \pm \infty$$

In a similar problem, Pai (24) indicated in his textbook that boundary condition on B_x for an insulated wall can be taken to be zero at the boundary.

C. Derivation of Flow Field Equations (31)

Rewriting the fourth of Equations 29

$$(\overline{V}_1 \cdot \overline{\nabla})\overline{V}_2 + (\overline{V}_2 \cdot \overline{\nabla})\overline{V}_1 + (\overline{V}_3 \cdot \overline{\nabla})\overline{V}_o + (\overline{V}_o \cdot \overline{\nabla})\overline{V}_3 + \overline{\nabla} P_3$$
$$= (\overline{V}_o \times \overline{B}_1) \times \overline{B}_o + (\overline{V}_o \times \overline{B}_o) \times \overline{B}_1 + (\overline{V}_1 \times \overline{B}_o) \times \overline{B}_o$$

$\overline{V}_1$, P_1 are the perturbations in the flow field when only the magnetic field is distorted, and hence both are equal to zero.

Thus,

$$\overline{V}_1 = P_1 = 0$$

and

$$\overline{V}_o = \overline{a}_x \text{ (dimensionless)}$$

Now the above equation becomes

$$(\overline{V}_3 \cdot \overline{\nabla})\overline{V}_o + (\overline{V}_o \cdot \overline{\nabla})\overline{V}_3 + \overline{\nabla} P_3$$
$$= (\overline{V}_o \times \overline{B}_1) \times \overline{B}_o + (\overline{V}_o \times \overline{B}_o) \times \overline{B}_1 \tag{47}$$

The terms on the right of Equation 47 are

$$(\overline{V}_o \times \overline{B}_1) \times \overline{B}_o = \overline{a}_z B_{1y} \times [\overline{a}_x B_{ox} + \overline{a}_y B_{oy}]$$
$$= \overline{a}_y B_{1y} B_{ox} - \overline{a}_x B_{1y} B_{oy} \tag{48}$$

and

$$(\overline{V}_o \times \overline{B}_o) \times \overline{B}_1 = \overline{a}_y B_{1x} B_{oy} - \overline{a}_x B_{1y} B_{oy} \tag{49}$$

$$(\overline{V}_3 \cdot \overline{\nabla})\overline{V}_o + (\overline{V}_o \cdot \overline{\nabla})\overline{V}_3 + \overline{\nabla} P_3$$
$$= \overline{a}_y[B_{1y} B_{ox} + B_{1x} B_{oy}] - \overline{a}_x[2\, B_{1y} B_{oy}] \tag{50}$$

Now we look at the left side of Equation 49. The first expression is zero since V_o is a constant value, hence

$$(\overline{a}_x) \circ [\overline{a}_x \frac{\partial}{\partial x} + \overline{a}_y \frac{\partial}{\partial y}] [\overline{a}_x V_{3x} + \overline{a}_y V_{3y}]$$
$$= \overline{a}_x \frac{\partial V_{3x}}{\partial x} + \overline{a}_y \frac{\partial V_{3y}}{\partial x}$$

and

$$\overline{\nabla} P_3 = \overline{a}_x \frac{\partial P}{\partial x} + \overline{a}_y \frac{\partial P}{\partial y}$$

Equation 49 becomes

$$\bar{a}_x \frac{\partial V_{3x}}{\partial x} + \bar{a}_y \frac{\partial V_{3y}}{\partial x} + \bar{a}_x \frac{\partial P}{\partial x} + \bar{a}_y \frac{\partial P}{\partial y}$$

$$= \bar{a}_y[B_{1y}B_{ox} + B_{1x}B_{oy}] - \bar{a}_x[2B_{1y}B_{oy}] \tag{51}$$

and

$$\frac{\partial V_{3x}}{\partial x} + \frac{\partial P}{\partial x} = -2B_{1y}B_{oy} \tag{52}$$

$$\frac{\partial V_{3y}}{\partial x} + \frac{\partial P}{\partial y} = B_{1y}B_{ox} + B_{1x}B_{oy} \tag{53}$$

Equation 51 can also be written as

$$-\frac{\partial P}{\partial x} = \frac{\partial V_{3x}}{\partial x} + 2\,B_{1y}B_{oy} \tag{54}$$

The term $2\,B_{1y}B_{oy}$ represents the magnetic pressure along the channel flow. The term $\frac{\partial V_{3x}}{\partial x}$ represents the accelerating or decelerating pressure along the channel. When $-\frac{dP}{dx}$ is negative, it will correspond to an adverse pressure gradient that gives the indication of boundary layer separation.

Differentiating Equation 52 with respect to y and Equation 53 with respect to x, we get

$$\frac{\partial^2 V_{3x}}{\partial x \partial y} + \frac{\partial^2 P}{\partial x \partial y} = -2\frac{\partial}{\partial y}(B_{1y}B_{oy}) \tag{55}$$

$$\frac{\partial^2 V_{3y}}{\partial x^2} + \frac{\partial^2 P}{\partial y \partial x} = \frac{\partial}{\partial x}[B_{1y}B_{ox} + B_{1x}B_{oy}] \tag{56}$$

Subtracting Equation 54 from Equation 55, we get

$$\frac{\partial^2 V_{3y}}{\partial x^2} - \frac{\partial^2 V_{3x}}{\partial x \partial y} = \frac{\partial}{\partial x}[B_{1y}B_{ox} + B_{1x}B_{oy}] + 2\frac{\partial}{\partial y}[B_{1y}B_{oy}] \tag{57}$$

For incompressible fluids

$$\bar{\nabla} \cdot \bar{V} = 0$$

or

$$\frac{\partial V_{3x}}{\partial x} + \frac{\partial V_{3y}}{\partial y} = 0$$

or

$$\frac{\partial V_{3x}}{\partial x} = - \frac{\partial V_{3y}}{\partial y}$$

Therefore, Equation 57 becomes

$$\nabla^2 V_{3y} = \frac{\partial^2 V_{3y}}{\partial x^2} + \frac{\partial^2 V_{3y}}{\partial y^2} = \frac{\partial}{\partial x} [B_{1y} B_{ox} + B_{1x} B_{oy}] + 2 \frac{\partial}{\partial y} [B_{1y} B_{oy}] \tag{58}$$

or

$$\frac{\partial^2 V_{3y}}{\partial x^2} + \frac{\partial^2 V_{3y}}{\partial y^2} = B_{1y} \frac{\partial B_{ox}}{\partial x} + B_{ox} \frac{\partial B_{1y}}{\partial x} + B_{1x} \frac{\partial B_{oy}}{\partial x} + B_{oy} \frac{\partial B_{1x}}{\partial x} + 2 B_{1y} \frac{\partial B_{oy}}{\partial y} + 2 B_{oy} \frac{\partial B_{1y}}{\partial y}$$

Since $\frac{\partial B_{ox}}{\partial x} = - \frac{\partial B_{oy}}{\partial y}$ and $\frac{\partial B_{1x}}{\partial x} = - \frac{\partial B_{1y}}{\partial y}$

the result is

$$\frac{\partial^2 V_{3y}}{\partial x^2} + \frac{\partial^2 V_{3y}}{\partial y^2} = B_{1y} \frac{\partial B_{oy}}{\partial y} + B_{1x} \frac{\partial B_{oy}}{\partial x} - B_{oy} \frac{\partial B_{1x}}{\partial x} + B_{ox} \frac{\partial B_{1y}}{\partial y} \tag{59}$$

with the boundary conditions

$$\begin{aligned} V_{3y} &= 0 \text{ at } x = \pm \infty \\ &= 0 \text{ at } y = 0, L \end{aligned} \tag{60}$$

VI. THE SOLUTION OF THE INDUCED MAGNETIC FIELD EQUATIONS

Equation 45 and boundary conditions of the first of Equations 46 are reproduced here

$$\frac{\partial^2 B_{1x}}{\partial x^2} + \frac{\partial^2 B_{1x}}{\partial y^2} = \frac{2x(1+y)}{[x^2+(1+y)^2]^2}$$

$$B_{1x} = 0 \text{ at } y = 0,\ L$$

Also there is the boundary condition that B_{1x} must be bounded at

$x = \pm\infty$, i.e. $B_{1x} = 0$ at $x = \pm\infty$

Now define the Fourier transform of a function $u(x, y)$ as

$$\overline{u}(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, y)\, e^{i\alpha x}\, dx \tag{61}$$

where

α = Fourier transform variable

We note that in the remaining part of this dissertation the symbol $\overline{u}$ will denote the transform of u rather than vector u.

Multiplying Equation 45 by $\frac{1}{\sqrt{2\pi}} e^{i\alpha x}$ and integrating with respect to x, and using the definition given in Equation 61, we get (see Appendix A)

$$\frac{\partial^2 \overline{B}_{1x}}{\partial y^2} - \alpha^2 \overline{B}_{1x} = -i\, \frac{\sqrt{\pi}}{\sqrt{2}} \cdot \alpha \cdot e^{-|\alpha|(1+y)} \tag{62}$$

The transformed boundary conditions are

$$\begin{aligned} \overline{B}_{1x}(\alpha, 0) &= 0 \\ \overline{B}_{1x}(\alpha, L) &= 0 \end{aligned} \tag{63}$$

Then the general solution of $\overline{B}_{1x}$ is obtained as

$$\bar{B}_{1x} = \frac{i\sqrt{\pi}}{2\sqrt{2}} \left[\frac{\alpha}{|\alpha|} y e^{-|\alpha|(1+y)} - \frac{\alpha}{|\alpha|} L \frac{\sinh \alpha y}{\sinh \alpha L} \cdot e^{-|\alpha|(1+L)}\right] \tag{64}$$

Now the inverse Fourier transform of a function $\bar{u}(\alpha, y)$ is given by

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{u}(\alpha, y)\, e^{-i\alpha x}\, d\alpha \tag{65}$$

Taking the inverse transform of $\bar{B}_{1x}$ from Equation 64 gives the following result (see Appendix A).

$$B_{1x} = \frac{1}{2} \left[\frac{xy}{x^2 + (1+y)^2} + \sum_{n=0}^{\infty} \frac{xL}{x^2 + (1+L+2nL+2y)^2} - \sum_{n=0}^{\infty} \frac{xL}{x^2 + (1+L+2nL)^2}\right] \tag{66}$$

B_{1y} can be found from the divergence relation given in Equation 36, repeated below.

$$\frac{\partial B_{1x}}{\partial x} + \frac{\partial B_{1y}}{\partial y} = 0$$

Therefore

$$B_{1y} = - \int \frac{\partial B_{1x}}{\partial x}\, dy + \phi(x) \tag{67}$$

where $\phi(x)$ is a function of x only. In order to validate the differentiation of B_{1x} which has an infinite series solution, the uniform convergence (13) of the infinite series in Equation 66 must be proven.

The Weirstrass M test of uniform convergence is used for this purpose. Take the 2nd series expression in Equation 66,

$$\sum_{n=0}^{\infty} \frac{xL}{x^2 + [1+L+2nL]^2}$$

In this series $|U_n| = \dfrac{|xL|}{x^2 + [1+L+2nL]^2}$

then,

$$\frac{|x|}{x^2 + [1+L+2nL]^2} \leq \frac{1}{[1+L+2nL]^2}$$

For all x and n and

$$M_n = \frac{1}{[1+L+2nL]^2}$$

$$\frac{1}{[1+L+2nL]^2} \leq \frac{1}{n^2}$$

Since the series $\Sigma \dfrac{1}{n^2}$ converges uniformly, the original series,

$$\sum_{n=0}^{\infty} \frac{xL}{x^2 + [1+L+2nL]^2}$$

converges uniformly and absolutely for all values of x and n. Hence, it can be differentiated term by term. Similarly the first series expression in Equation 66,

$$\sum_{n=0}^{\infty} \frac{xL}{x^2 + [1+L+2nL+2y]^2}$$

can be shown to converge uniformly and absolutely for all values of x and n, and hence it can be differentiated term by term.

Now in order to calculate B_{1y} using Equation 67, $\dfrac{\partial B_{1x}}{\partial x}$ is found as

$$\frac{\partial B_{1x}}{\partial x} = \frac{1}{2}\,\frac{y(1+y)^2}{[x^2+(1+y)^2]^2} - \frac{1}{2}\,\frac{yx^2}{[x^2+(1+y)^2]^2}$$

$$+ \frac{L}{2}\sum_{n=0}^{\infty}\frac{[1+L+2nL+2y]^2}{[x^2 + \{1+L+2nL+2y\}^2]^2}$$

$$- \frac{L}{2}\sum_{n=0}^{\infty}\frac{x^2}{[x^2 + \{1+L+2nL+2y\}^2]^2}$$

$$- \frac{L}{2}\sum_{n=0}^{\infty}\frac{\{1+L+2nL\}^2 - x^2}{[x^2 + \{1+L+2nL\}^2]^2} \tag{68}$$

To find B_{1y}, Equation 68 must be integrated with respect to y.

In order that the above infinite series expression in $\frac{\partial B_{1x}}{\partial x}$ can be integrated term by term, we must have ordinary convergence. This was checked by using the Weirstrass M test. They converge uniformly and absolutely for all x and n.

Integrating Equation 68 with respect to y, the following result is obtained.

$$B_{1y} = -\frac{1}{2}\,\frac{x^2 + 1 + y}{x^2 + (1+y)^2} + \frac{1}{4}\ln\,[x^2+(1+y)^2]$$

$$+ \frac{L}{4}\sum_{n=0}^{\infty}\frac{[1+L+2nL+2y]}{[x^2 + \{1+L+2nL+2y\}^2]}$$

$$+ \frac{L}{2}\sum_{n=0}^{\infty}\frac{[\{1+L+2nL\}^2 - x^2]}{[\{1+L+2nL\}^2 + x^2]^2}\,y$$

$$+ \phi\,(x) \tag{69}$$

In order to find $\phi\,(x)$, the boundary condition that B_{1y} must go to zero at x equal $\pm\infty$ is used, and

$$\phi(x) = \cdot \frac{1}{4} \text{Ln}\,[x^2+(1+\delta)^2] + \frac{1}{2}$$

where δ is chosen as representing the arbitrary thickness of inviscid layer of the conducting fluid from the lower channel wall.

Equation 69 thus becomes

$$B_{1y} = \frac{-1}{2} \frac{x^2 + 1 + y}{x^2 + (1+y)^2} \div \frac{1}{4} \text{Ln} \frac{x^2 + (1+y)^2}{x^2 + (1+\delta)^2}$$

$$+ \frac{L}{4} \sum_{n=0}^{\infty} \frac{[1+L+2nL+2y]}{[x^2 + \{1+L+2nL+2y\}^2]}$$

$$+ \frac{L}{2} \sum_{n=0}^{\infty} \frac{y[\{1+L\ 2nL\}^2 - x^2]}{[\{1+L+2nL\}^2 + x^2]^2} + \frac{1}{2} \tag{70}$$

Since the main interest is in the behavior of the inviscid boundary layer of thickness δ in the gaseous flow, only the solution for small values of y will be examined. The following approximation is used.

$$y \ll (1+L+2nL)\ ,\ n = 0, 1, 2 \cdots$$

Therefore,

$$B_{1x} \approx \frac{1}{2} \frac{xy}{x^2 + (1+y)^2}$$

and (71)

$$B_{1y} \approx \frac{-1}{2} \frac{x^2 + 1 + y}{x^2 + (1+y)^2} + \frac{L}{4} \sum_{n=0}^{\infty} \frac{[1+L+2nL]}{[x^2 + \{1+L+2nL\}^2]} + \frac{1}{2}$$

Therefore the total magnetic field density with the effect of the induced field included, is given by

$$B_x = B_{ox} + R_m B_{1x}$$

and

$$B_y = B_{oy} + R_m B_{1y}$$

or

$$B_x = -\frac{1+y}{x^2+(1+y)^2} + R_m\left[\frac{xy}{x^2+(1+y)^2}\right.$$

$$\left.+\frac{1}{2}\sum_{n=0}^{\infty}\frac{xL}{x^2+[1+L+2nL+2y]^2} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{xL}{x^2+[1+L+2nL]^2}\right] \tag{72}$$

and

$$B_y = \frac{x}{x^2+(1+y)^2} + R_m\left[-\frac{1}{2}\,\frac{x^2+1+y}{x^2+(1+y)^2}\right.$$

$$-\frac{1}{4}\,\mathrm{Ln}\,\frac{x^2+(1+y)^2}{x^2+(1+\delta)^2} + \frac{L}{4}\sum_{n=0}^{\infty}\frac{[1+L+2nL+2y]}{[x^2+\{1+L+2nL+2y\}^2]}$$

$$\left.+\frac{L}{2}\sum_{n=0}^{\infty}\frac{y[\{1+L+2nL\}^2-x^2]}{[\{1+L+2nL\}^2+x^2]^2} + \frac{1}{2}\right] \tag{73}$$

VII. THE SOLUTION OF FLOW FIELD PERTURBATION EQUATIONS

A. The Solution for V_{3y}

Equation 59 is repeated here for convenient reference.

$$\frac{\partial^2 V_{3y}}{\partial x^2} + \frac{\partial^2 V_{3y}}{\partial y^2} = B_{1y}\frac{\partial B_{oy}}{\partial y} + B_{1x}\frac{\partial B_{oy}}{\partial x} - B_{oy}\frac{\partial B_{1x}}{\partial x} + B_{ox}\frac{\partial B_{1y}}{\partial y} \tag{59}$$

with the boundary conditions

$$\begin{aligned} V_{3y} &= 0 \text{ at } x = \pm\infty \\ &= 0 \text{ at } y = 0,\ L \end{aligned} \tag{60}$$

In Equation 59, the right hand side is known since B_{oy}, B_{ox}, B_{1x} and B_{1y} are determined from Equations 1, 2, 66 and 70 respectively. We will obtain the solution for the region of interest, namely for small values of y.

With the approximation $y \ll (1+L+2nL)$, Equation 59 reduces to the following:

$$\frac{\partial^2 V_{3y}}{\partial x^2} + \frac{\partial^2 V_{3y}}{\partial y^2} = \frac{L}{2}\frac{x(1+y)}{x^2 + (1+y)^2}\sum_{n=0}^{\infty}\frac{1+L+2nL}{[x^2 + \{1+L+2nL\}^2]^2} - \frac{L}{2}\frac{x(1+y)}{[x^2 + (1+y)^2]^2}\sum_{n=0}^{\infty}\frac{1+L+2nL}{x^2 + \{1+L+2n\}^2} \tag{74}$$

Then letting

$$L_n = 1 + L + 2nL$$

and

$$Y = 1 + y$$

Equation 74 becomes

$$\frac{\partial^2 V_{3y}}{\partial x^2} + \frac{\partial^2 V_{3y}}{\partial y^2} = \frac{L}{2} \frac{xY}{x^2 + Y^2} \sum_{n=0}^{\infty} \frac{L_n}{[x^2+L_n^2]^2}$$

$$- \frac{L}{2} \frac{xY}{[x^2 + Y^2]^2} \sum_{n=0}^{\infty} \frac{L_n}{x^2 + L_n^2} \tag{75}$$

The right hand side of Equation 75 is expanded by partial fractions in order to obtain Fourier transforms easily.

Equation 75 becomes

$$\frac{\partial^2 V_{3y}}{\partial x^2} + \frac{\partial^2 V_{3y}}{\partial y^2} = \sum_{n=0}^{\infty} \frac{LY\, L_n}{2(L_n^2 - Y^2)^2} \frac{1}{(x+iY)}$$

$$+ \sum_{n=0}^{\infty} \frac{YL\, L_n}{2(L_n^2 - Y^2)^2} \frac{1}{(x-iY)}$$

$$- \sum_{n=0}^{\infty} \frac{YL\, L_n}{2(L_n^2 - Y^2)^2} \frac{1}{(x+iL_n)}$$

$$- \sum_{n=0}^{\infty} \frac{YL\, L_n}{2(L_n^2 - Y^2)^2} \frac{1}{(x-iL_n)}$$

$$- \sum_{n=0}^{\infty} \frac{YL\, L_n}{2(L_n^2 - Y^2)} \frac{x}{(x^2+Y^2)^2}$$

$$- \sum_{n=0}^{\infty} \frac{YL\, L_n}{2(L_n^2 - Y^2)} \frac{x}{(x^2+L_n^2)^2} \tag{76}$$

Then, for $y \ll (1+L+2nL)$, Equation 76 becomes

$$\frac{\partial^2 V_{3y}}{\partial x^2} + \frac{\partial^2 V_{3y}}{\partial y^2} = \sum_{n=0}^{\infty} \frac{YL\ L_n}{2(L_n^2-1)^2} \frac{1}{(x+iY)}$$

$$+ \sum_{n=0}^{\infty} \frac{YL\ L_n}{2(L_n^2-1)^2} \frac{1}{(x-iY)}$$

$$- \sum_{n=0}^{\infty} \frac{YL\ L_n}{2(L_n^2-1)^2} \frac{1}{(x+iL_n)}$$

$$- \sum_{n=0}^{\infty} \frac{YL\ L_n}{2(L_n^2-1)^2} \frac{1}{(x-iL_n)}$$

$$- \sum_{n=0}^{\infty} \frac{YL\ L_n}{2(L_n^2-1)} \frac{x}{(x^2+Y^2)^2}$$

$$- \sum_{n=0}^{\infty} \frac{YL\ L_n}{2(L_n^2-1)} \frac{x}{(x^2+L_n^2)^2} \qquad (77)$$

with the boundary conditions on V_{3y} being

$$V_{3y} = 0 \text{ at } x = \pm\infty$$

$$V_{3y} = 0 \text{ at } y = 0$$

$$y = L$$

The Fourier transforms with respect to x are used with Equation 77.

The partial differential equation is converted to an ordinary differential equation in y as shown on the following page (see Appendix B).

$$\frac{\partial^2 \overline{V}_{3y}}{\partial y^2} - \alpha^2 \overline{V}_{3y}$$

$$= -\sum_{n=0}^{\infty} \frac{i\sqrt{\pi}/2 \; L \; L_n}{(L_n^2-1)^2} \left[(1+y) \; e^{\alpha(1+y)}\right] \quad \text{for } \alpha < 0$$

$$+ \sum_{n=0}^{\infty} \frac{i\sqrt{\pi}/2 \; L \; L_n}{(L_n^2-1)^2} \left[(1+y) \; e^{-\alpha(1+y)}\right] \quad \text{for } \alpha > 0$$

$$+ \sum_{n=0}^{\infty} \frac{i\sqrt{\pi}/2 \; L \; L_n}{(L_n^2-1)^2} \cdot e^{\alpha L_n} \left[1+y\right] \quad \text{for } \alpha < 0$$

$$- \sum_{n=0}^{\infty} \frac{i\sqrt{\pi}/2 \; L \; L_n}{(L_n^2-1)^2} \cdot e^{-\alpha L_n} \left[1+y\right] \quad \text{for } \alpha > 0$$

$$+ \sum_{n=0}^{\infty} \frac{i\sqrt{\pi}/2 \; L \; L_n \; \alpha}{(L_n^2-1)} \left[e^{-|\alpha|(1+y)}\right] \quad \text{for all } \alpha$$

$$+ \sum_{n=0}^{\infty} \frac{i\sqrt{\pi}/2 \; L\alpha}{(L_n^2-1)} \cdot e^{-|\alpha| L_n} \left[1+y\right] \quad \text{for all } \alpha \tag{78}$$

where α is the Fourier transform variable.

Equation 78 is an ordinary differential equation with respect to y with the following boundary conditions

$$\overline{V}_{3y} = 0 \text{ at } y = 0, L \tag{79}$$

The solution of Equation 78 with boundary conditions as in Equation 79 is obtained as follows:

$$\overline{V}_{3y} = \sum_{n=0}^{\infty} \frac{i\sqrt{\pi}/2\ L\ L_m}{(L_m^2-1)^2} \Big[\Big(\frac{L^2\ e^{-\alpha(1+L)}\ \sinh\alpha y - y^2\ e^{-\alpha(1+y)}\ \sinh\alpha L}{4\alpha\ \sinh\alpha L}$$

$$+ \frac{L\ e^{-\alpha(1+L)}\ \sinh\alpha y - y\ e^{-\alpha(1+y)}\ \sinh\alpha L}{4\alpha^2\ \sinh\alpha L}\Big) \quad \text{for } \alpha > 0$$

$$+ \Big(\frac{L^2\ e^{\alpha(1+L)}\ \sinh\alpha y - y^2\ e^{\alpha(1+y)}\ \sinh\alpha L}{4\alpha\ \sinh\alpha L}$$

$$+ \frac{y\ e^{\alpha(1+y)}\ \sinh\alpha L - L\ e^{\alpha(1+L)}\ \sinh\alpha y}{4\alpha^2\ \sinh\alpha L}\Big) \quad \text{for } \alpha < 0$$

$$+ \Big(\frac{L\ e^{-\alpha(1+L)}\ \sinh\alpha y - y\ e^{-\alpha(1+y)}\ \sinh\alpha L}{4\alpha\ \sinh\alpha L}\Big) \quad \text{for } \alpha > 0$$

$$+ \Big(\frac{L\ e^{\alpha(1+L)}\ \sinh\alpha y - y\ e^{\alpha(1+y)}\ \sinh\alpha L}{4\alpha\ \sinh\alpha L}\Big) \quad \text{for } \alpha < 0$$

$$+ \Big(\frac{y\ e^{-\alpha L_n}\ \sinh\alpha L - L\ e^{-\alpha L_n}\ \sinh\alpha y}{\alpha^2\ \sinh\alpha L}\Big) \quad \text{for } \alpha > 0$$

$$+ \Big(\frac{L\ e^{\alpha L_n}\ \sinh\alpha y - y\ e^{\alpha L_n}\ \sinh\alpha L}{\alpha^2\ \sinh\alpha L}\Big) \quad \text{for } \alpha < 0$$

$$+ \frac{(e^{\alpha L_n}\ \sinh\alpha y + e^{\alpha(L_n+y)}\ \sinh\alpha L)}{4\alpha^2\ \sinh\alpha L} \quad \text{for } \alpha < 0$$

$$- \frac{(e^{\alpha L_n}\ \sinh\alpha L + e^{\alpha(L+L_n)}\ \sinh\alpha y)}{4\alpha^2\ \sinh\alpha L} \quad \text{for } \alpha < 0$$

$$+ \frac{(e^{-\alpha L_n}\ \sinh\alpha L + e^{-\alpha(L+L_n)}\ \sinh\alpha y)}{4\alpha^2\ \sinh\alpha L} \quad \text{for } \alpha > 0$$

$$- \frac{(e^{-\alpha(L_n+y)}\ \sinh\alpha L + e^{-\alpha(L_n+L)}\ \sinh\alpha y)}{4\alpha^2\ \sinh\alpha L}\Big] \quad \text{for } \alpha > 0$$

$$+\sum_{n=0}^{\infty} \frac{i\sqrt{\pi}/2\ L\ L_m}{(L_m^2-1)} \left[\frac{L}{2}\frac{\alpha}{|\alpha|} e^{-|\alpha|(1+L)} \frac{\sinh \alpha y}{\sinh \alpha L} - \frac{y}{2}\frac{\alpha}{|\alpha|} e^{-|\alpha|(1+y)}\right] \quad \text{for all } \alpha$$

$$+\sum_{n=0}^{\infty} \frac{i\sqrt{\pi}/2\ L}{(L_m^2-1)} \left[\left(\frac{L\ e^{-\alpha L_n} \sinh \alpha y - y\ e^{-\alpha L_n} \sinh \alpha L}{\alpha \sinh \alpha L}\right) \quad \text{for } \alpha > 0\right.$$

$$+\left(\frac{L\ e^{\alpha L_n} \sinh \alpha y - y\ e^{\alpha L_n} \sinh \alpha L}{\alpha \sinh \alpha L}\right) \quad \text{for } \alpha < 0$$

$$+\frac{(e^{\alpha L_n} \sinh \alpha y + e^{\alpha(y+L_n)} \sinh \alpha L)}{4\alpha \sinh \alpha L} \quad \text{for } \alpha < 0$$

$$-\frac{(e^{\alpha L_n} \sinh \alpha L + e^{\alpha(L+L_n)} \sinh \alpha y)}{4\alpha \sinh \alpha L} \quad \text{for } \alpha < 0$$

$$+\frac{(e^{-\alpha L_n} \sinh \alpha y + e^{-\alpha(y+L_n)} \sinh \alpha L)}{4\alpha \sinh \alpha L} \quad \text{for } \alpha > 0$$

$$\left.-\frac{(e^{-\alpha L_n} \sinh \alpha L + e^{-\alpha(L_n+L)} \sinh \alpha y)}{4\alpha \sinh \alpha L}\right] \quad \text{for } \alpha > 0 \qquad (80)$$

The inverse Fourier transform of Equation 80 is taken (see Appendix B) and we obtain for V_{3y}

$$V_{3y} = \sum_{m=0}^{\infty} \frac{-L\ L_m}{(L_m^2-1)^2} \left[\frac{y^2}{4} \tan^{-1} \frac{x}{1+y} + \sum_{n=0}^{\infty} \left\{\frac{xy}{8} L_n \frac{x^2 + (L_n+L+y)^2}{x^2 + (L_n-L+y)^2}\right.\right.$$

$$+\frac{y}{4}(L_n+L+y) \tan^{-1} \frac{x}{L_n + L + y}$$

$$-\frac{y}{4}(L_n-L+y) \tan^{-1} \frac{x}{L_n - L + y}$$

$$-\frac{L^2}{4} \tan^{-1} \frac{x}{L_n + L + y} + \frac{L^2}{4} \tan^{-1} \frac{x}{L_n + L + y}$$

$$-\frac{Lx}{8} L_n \frac{x^2 + (L_n+L+y)^2}{x^2 + (L_n+L-y)^2}$$

$$- \frac{L}{4} (L_n+L+y) \tan^{-1} \frac{x}{L_n + L + y}$$

$$+ \frac{L}{4} (L_n+L-y) \tan^{-1} \frac{x}{L_n + L - y}$$

$$- \frac{L}{4} \tan^{-1} \frac{x}{L_n + L + y} + \frac{L}{4} \tan^{-1} \frac{x}{L_n + L - y}\}$$

$$+ \frac{y}{4} \tan^{-1} \frac{x}{1 + y}]$$

$$- \sum_{m=0}^{\infty} \frac{L L_m}{(L_m^2-1)^2} \sum_{n=0}^{\infty} [- \frac{xy}{8} L_n \frac{x^2 + (L_n+2L+2nL)^2}{x^2 + (L_n+2nL)^2}$$

$$- \frac{y}{4} (L_n+2L+2nL) \tan^{-1} \frac{x}{L_n + 2L + 2nL}$$

$$+ \frac{y}{4} (L_n + 2nL) \tan^{-1} \frac{x}{L_n + 2nL}$$

$$+ \frac{L}{4} x L_n \frac{x^2 + (L+L_n+2nL+y)^2}{x^2 + (L+L_n+2nL-y)^2}$$

$$+ \frac{L}{4} (L+L_n+2nL+y) \tan^{-1} \frac{x}{L + L_n + 2nL + y}$$

$$- \frac{L}{4} (L+L_n+2nL-y) \tan^{-1} \frac{x}{L + L_n + 2nL - y}$$

$$+ \frac{x}{4} L_n \frac{x^2 + (L_n+2nL)^2}{x^2 + (2L+2nL+L_n)^2}$$

$$- \frac{1}{4} (2L+2nL+L_n) \tan^{-1} \frac{x}{2L + 2nL + L_n} .$$

$$+ \frac{1}{4} (2nL+L_n) \tan^{-1} \frac{x}{2nL + L_n}$$

$$+ \frac{x}{4} L_n \frac{x^2 + (2L+L_n+2nL+y)^2}{x^2 + (2nL+L_n+y)^2}$$

$$+ \frac{1}{4} (2L+2nL+L_n+y) \tan^{-1} \frac{x}{2L + 2nL + L_n + y}$$

$$- \frac{1}{4} (2nL+L_n+y) \tan^{-1} \frac{x}{2nL + L_n + y}$$

$$+ \frac{x}{4} L_n \frac{x^2 + (L_n+L+2nL+y)^2}{x^2 + (L_n+L+2nL-y)^2}$$

$$+ \frac{1}{4} (L_n+L+2nL+y) \tan^{-1} \frac{x}{L_n + L + 2nL + y}$$

$$- \frac{1}{4} (L_n+L+2nL-y) \tan^{-1} \frac{x}{L_n + L + 2nL - y}$$

$$- \frac{x}{4} L_n \frac{x^2 + (2L+2nL+L_n+y)^2}{x^2 + (2L+2nL+L_n-y)^2}$$

$$+ \frac{1}{4} (2L+2nL+L_n-y) \tan^{-1} \frac{x}{2L + 2nL + L_n - y}$$

$$- \frac{1}{4} (2L+2nL+L_n+y) \tan^{-1} \frac{x}{2L + 2nL + L_n + y}]$$

$$- \sum_{m=0}^{\infty} \frac{L L_m}{(L_m^2-1)} [\frac{1}{8} \frac{xy}{x^2 + (1+y)^2}$$

$$- \frac{L}{4} \sum_{n=0}^{\infty} \{ \frac{x}{x^2 + (1+2L+2nL-y)^2} + \frac{x}{x^2 + (1+2L+2nL+y)^2} \}]$$

$$- \sum_{m=0}^{\infty} \frac{L}{(L_m^2-1)} \sum_{n=0}^{\infty} [\frac{y}{4} \tan^{-1} \frac{x}{L_n}$$

$$- \frac{L}{4} \tan^{-1} \frac{x}{L + L_n + 2nL - y}$$

$$+ \frac{L}{4} \tan^{-1} \frac{x}{L + L_n + 2nL + y}$$

$$+ \frac{L}{4} \tan^{-1} \frac{x}{L_n + 2nL} - \frac{1}{4} \tan^{-1} \frac{x}{2L + 2nL + L_n}$$

$$- \frac{L}{4} \tan^{-1} \frac{x}{L_n + 2nL + y} + \frac{1}{4} \tan^{-1} \frac{x}{L_n + 2nL + L_n + y}$$

$$- \frac{L}{4} \tan^{-1} \frac{x}{L + L_n + 2nL - y} + \frac{1}{4} \tan^{-1} \frac{x}{L + L_n + 2nL + y}$$

$$- \frac{L}{4} \tan^{-1} \frac{x}{L_n + 2nL + 2L + y}$$

$$+ \frac{L}{4} \tan^{-1} \frac{x}{L_n + 2nL + 2L - y}] \tag{81}$$

To simplify Equation 81 we let

$$y << (1 + L + 2nL)$$

with the result that

$$V_{3y} = \sum_{m=0}^{\infty} \frac{-L\ L_m}{(L_m^2-1)^2} [\frac{y^2}{4} \tan^{-1} \frac{x}{1 + y}$$

$$+ \frac{y}{4} \tan^{-1} \frac{x}{1 + y} + \sum_{n=0}^{\infty} [\frac{xy}{8} L_n \frac{x^2 + (L_n+L)^2}{x^2 + (L_n-L)^2}$$

$$- \frac{xy}{8} L_n \frac{x^2 + (L_n+2L+2nL)^2}{x^2 + (L_n+2nL)^2}$$

$$+ \frac{y}{4} (L+L_n) \tan^{-1} \frac{x}{L_n + L}$$

$$- \frac{y}{4} (L_n-L) \tan^{-1} \frac{x}{L_n - L}$$

$$+ \frac{y}{4} (L_n+2nL) \tan^{-1} \frac{x}{L_n + 2nL}$$

$$- \frac{y}{4} (L_n+2L+2nL) \tan^{-1} \frac{x}{L_n + 2L + 2nL}]$$

$$- \sum_{m=0}^{\infty} \frac{L\ L_m}{(L_m^2-1)} [\frac{1}{8} \frac{xy}{x^2 + (1+y)^2}]$$

$$- \sum_{m=0}^{\infty} \frac{L}{(L_m^2-1)} \sum_{n=0}^{\infty} \frac{y}{4} \tan^{-1} \frac{x}{L_n} \qquad (82)$$

B. The Solution for Static and Magnetic Pressure Gradients

Now, we return to Equation 54, rewritten below as

$$\frac{\partial V_{3x}}{\partial x} + \frac{\partial P}{\partial x} = - 2\ B_{1y} B_{oy} \qquad (54)$$

Since

$$\frac{\partial V_{3x}}{\partial x} = - \frac{\partial V_{3y}}{\partial y}$$

we obtain

$$- \frac{\partial V_{3y}}{\partial y} + \frac{\partial P}{\partial x} = - 2\ B_{1y} B_{oy}$$

or

$$- \frac{\partial P}{\partial x} = 2\ B_{1y} B_{oy} - \frac{\partial V_{3y}}{\partial y} \qquad (83)$$

where $- \frac{\partial P}{\partial x}$ is the negative of the pressure gradient.

The term $2\ B_{1y}B_{oy}$ from Equation 82 represents the magnetic pressure in the flow field.

$$- \frac{\partial V_{3y}}{\partial y} = \sum_{m=0}^{\infty} \frac{L\ L_m}{4(L_m^2-1)^2} [(1+2y) \tan^{-1} \frac{x}{1+y}$$

$$- \frac{xy\ (1+y)}{x^2 + (1+y)^2} + \sum_{n=0}^{\infty} [\frac{x}{2} L_n \frac{x^2 + (L_n+L)^2}{x^2 + (L_n-L)^2}$$

$$- \frac{x}{2} L_n \frac{x^2 + (L_n+2L+2nL)^2}{x^2 + (L_n+2nL)^2}$$

$$+ (L_n+L) \tan^{-1} \frac{x}{L_n + L}$$

$$- (L_n-L) \tan^{-1} \frac{x}{L_n - L}$$

$$+ (L_n+2nL) \tan^{-1} \frac{x}{L_n + 2nL}$$

$$- (L_n + 2L + 2nL) \tan^{-1} \frac{x}{L_n + 2L + 2nL}]$$

$$+ \sum_{m=0}^{\infty} \frac{L\ L_m}{8(L_m^2-1)} \frac{x(x^2-y^2+1)}{[x^2+(1+y)^2]^2}$$

$$+ \sum_{m=0}^{\infty} \frac{L}{(L_m^2-1)} \sum_{n=0}^{\infty} \frac{1}{4} \tan^{-1} \frac{x}{L_n} \tag{84}$$

The term $2\ B_{1y}B_{oy}$ with $y \ll (1 + L + 2nL)$ becomes

$$2\, B_{1y} B_{oy} = \frac{-xy\ (1+y)}{[x^2+(1+y)^2]^2}$$

$$+ \frac{xL}{x^2 + (1+y)^2} \sum_{n=0}^{\infty} \frac{1 + L + 2nL}{x^2 + (1+L+2nL)^2} \tag{85}$$

Combining Equations 84 and 85 and setting $y = 0$, we obtain the negative of the pressure gradient along the lower channel wall as

$$- \frac{dP}{dx} = \sum_{m=0}^{\infty} \frac{L\, L_m}{4(L_m^2-1)^2} [\tan^{-1} x + \sum_{n=0}^{\infty}$$

$$[\frac{x}{2} L_n \frac{x^2 + (L_n+L)^2}{x^2 + (L_n-L)^2}$$

$$- \frac{x}{2} L_n \frac{x^2 + (L_n+2nL+2L)^2}{x^2 + (L_n+2nL)^2}$$

$$+ (L_n+L) \tan^{-1} \frac{x}{L_n + L}$$

$$- (L_n-L) \tan^{-1} \frac{x}{L_n - L}$$

$$+ (L_n+2nL) \tan^{-1} \frac{x}{L_n + 2nL}$$

$$- (L_n+2nL+2L) \tan^{-1} \frac{x}{L_n + 2nL + 2L}]$$

$$+ \sum_{m=0}^{\infty} \frac{L\, L_m}{8(L_m^2-1)} \frac{x}{1 + x^2}$$

$$+ \sum_{m=0}^{\infty} \frac{L}{4(L_m^2-1)} \sum_{n=0}^{\infty} \tan^{-1} \frac{x}{L_n}$$

$$+ \sum_{m=0}^{\infty} \frac{L\ L_m}{(x^2+L_m^2)} \frac{x}{(1+x^2)} \tag{86}$$

and the magnetic pressure along the lower channel wall is given by

$$- \frac{dP_m}{dx} = - \sum_{m=0}^{\infty} \frac{L\ L_m}{(x^2+L_m^2)} \frac{x}{1 + x^2} \tag{87}$$

VIII. RESULTS

In the previous sections algebraic expressions for the magnetic field and the velocity component V_{3y} were developed. From these, expressions for the static and magnetic pressure gradients $-\frac{dP}{dx}$ and $-\frac{dP_m}{dx}$ along the channel wall were obtained.

The pressure gradients as given by Equation 86 and 87 were numerically calculated for different value of R_m, I, and L using an IBM 360 digital computer. The results are tabulated in Appendix C, in Tables 1 to 8. The data is displayed graphically in Figures 2 to 16, described as follows:

1. In Figures 2 to 5 the total adverse static pressure gradient along the channel wall is plotted against x for a given value of R_m, with I and L held constant at 0.10 and 6.4 respectively. The case of $R_m = 0$ as given in reference (27) is also shown for easy comparison. Figure 2 shows the total adverse static pressure gradient along the channel wall when $R_m = 0.05$ and $I = 0.10$. It is recognized that the increase in the net negative area with respect to that when R_m = zero is insignificant. Figure 3 shows the total adverse static pressure gradient with $R_m = 0.075$, $I = 0.10$. It indicates a slight increase in the net negative area from that when $R_m = 0$. Figure 4 shows the total adverse static pressure gradient with $R_m = 0.1$, $I = 0.1$. It indicates a significant increase in the net negative area from that when $R_m = 0$. Figure 5 confirms the increasing influence of the induced field where, with $R_m = 0.15$, $I = 0.1$, the adverse pressure gradient is significantly greater than that with $R_m = 0$.

2. Figures 6 through 12 show the behavior of the static and magnetic pressure gradient due to the effects of the induced magnetic field for channel heights of L = 6.4, 9.6, 16, 32, 64, 96 and, for a very large height, L = 9999. Those curves show the increasing influence of the induced magnetic field as the channel height increases.
3. Figures 13 and 14 show the adverse static pressure gradient due to the induced magnetic field for various channel heights. They indicate the increasing effects of the induced magnetic field as channel height increases.
4. Figures 15 and 16 show the adverse magnetic pressure gradients due to the induced magnetic field. They indicate also the increasing effects of the induced magnetic field as channel height increases.
5. Figures 17 and 18 show the magnitude of the maximum value of static and magnetic pressure gradient for various channel heights.

Figure 2. Total static pressure gradient at lower wall

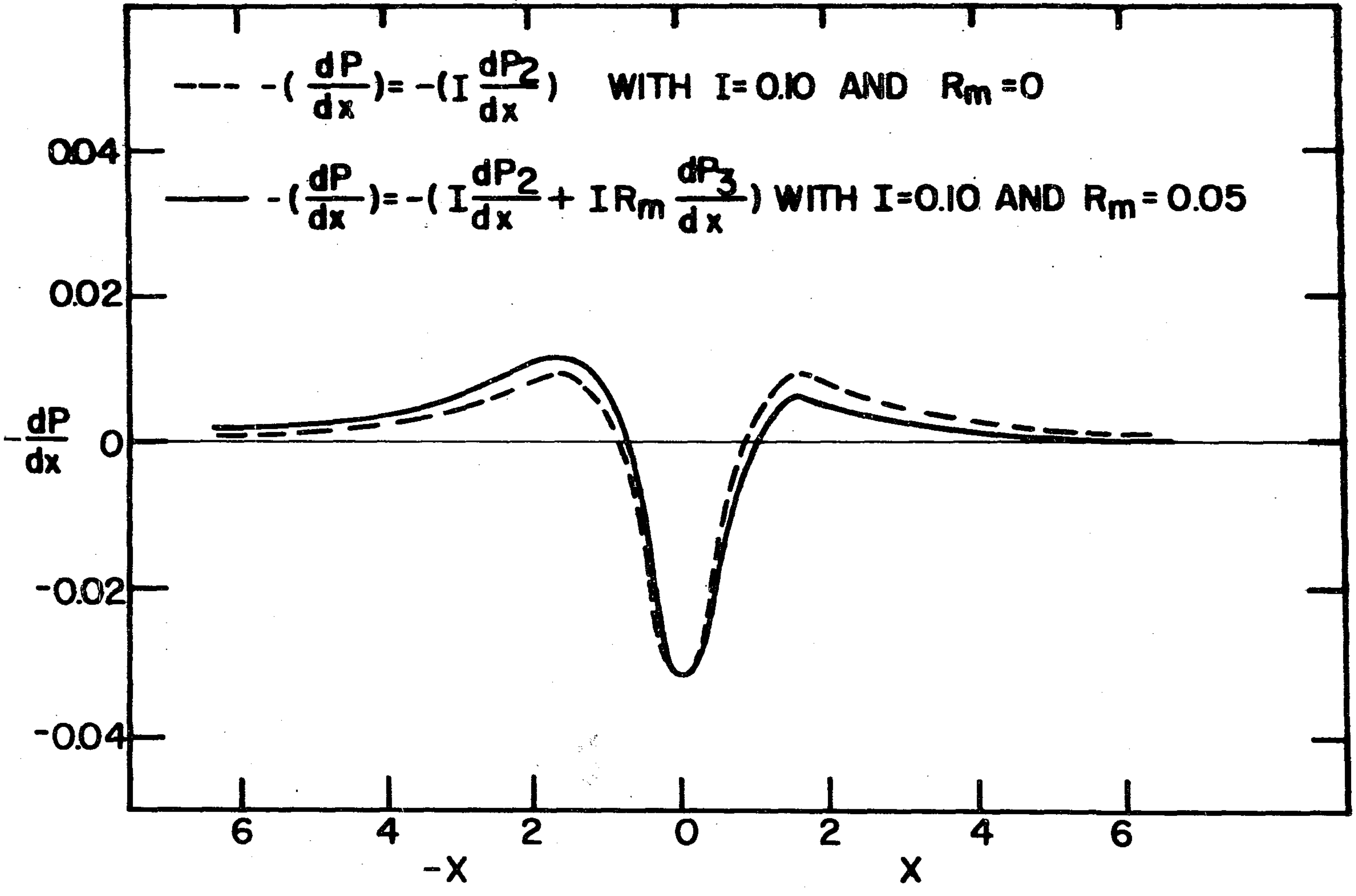
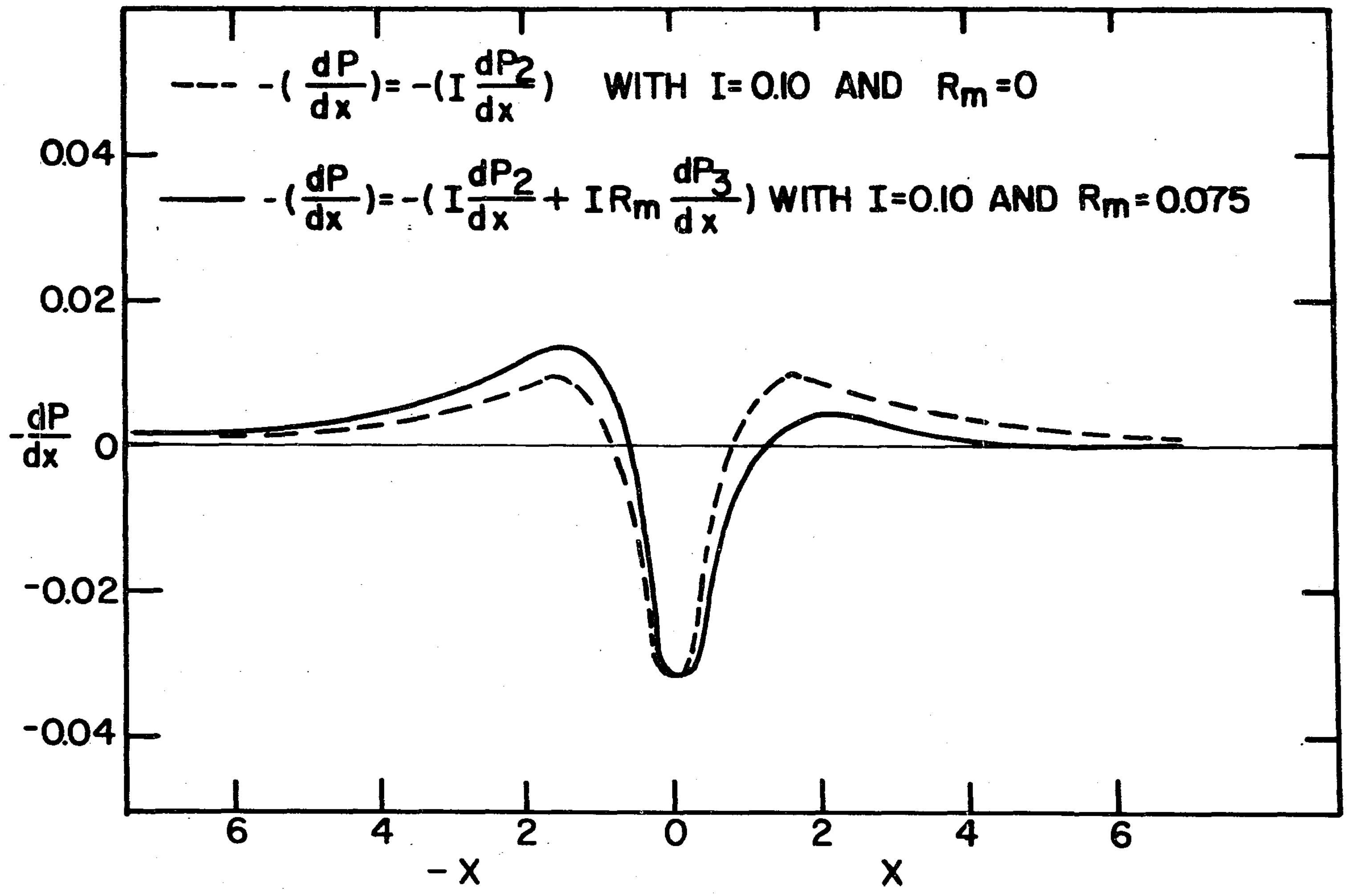
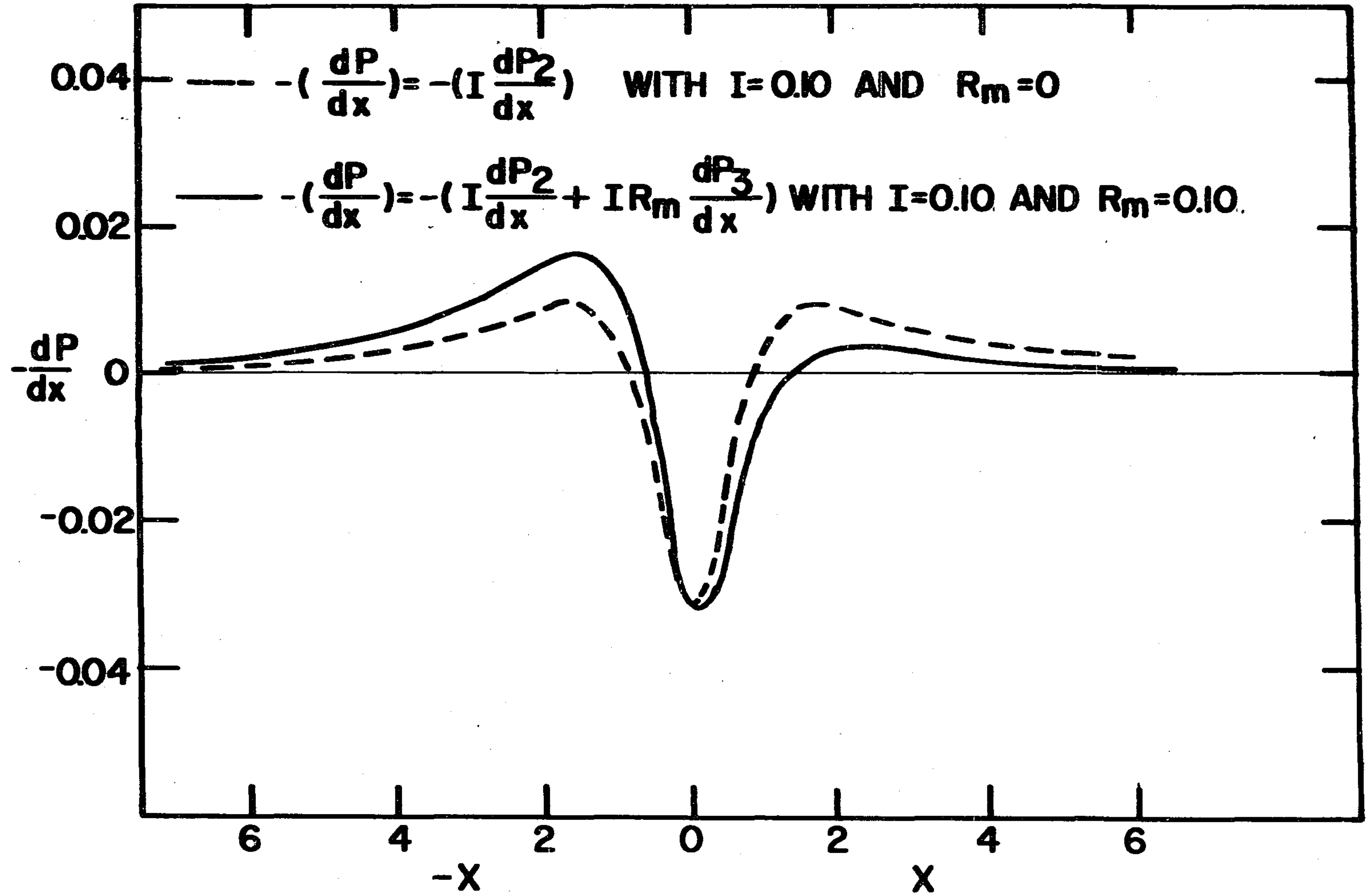

$-(\frac{dP}{dx}) = -(I\frac{dP_2}{dx})$ WITH I=0.10 AND R_m=0
$-(\frac{dP}{dx}) = -(I\frac{dP_2}{dx} + IR_m\frac{dP_3}{dx})$ WITH I=0.10 AND R_m=0.05
$-\frac{dP}{dx}$
0.04
0.02
0
-0.02
-0.04
6
4
2
0
2
4
6
-X
X

Figure 3. Total static pressure gradient at lower wall



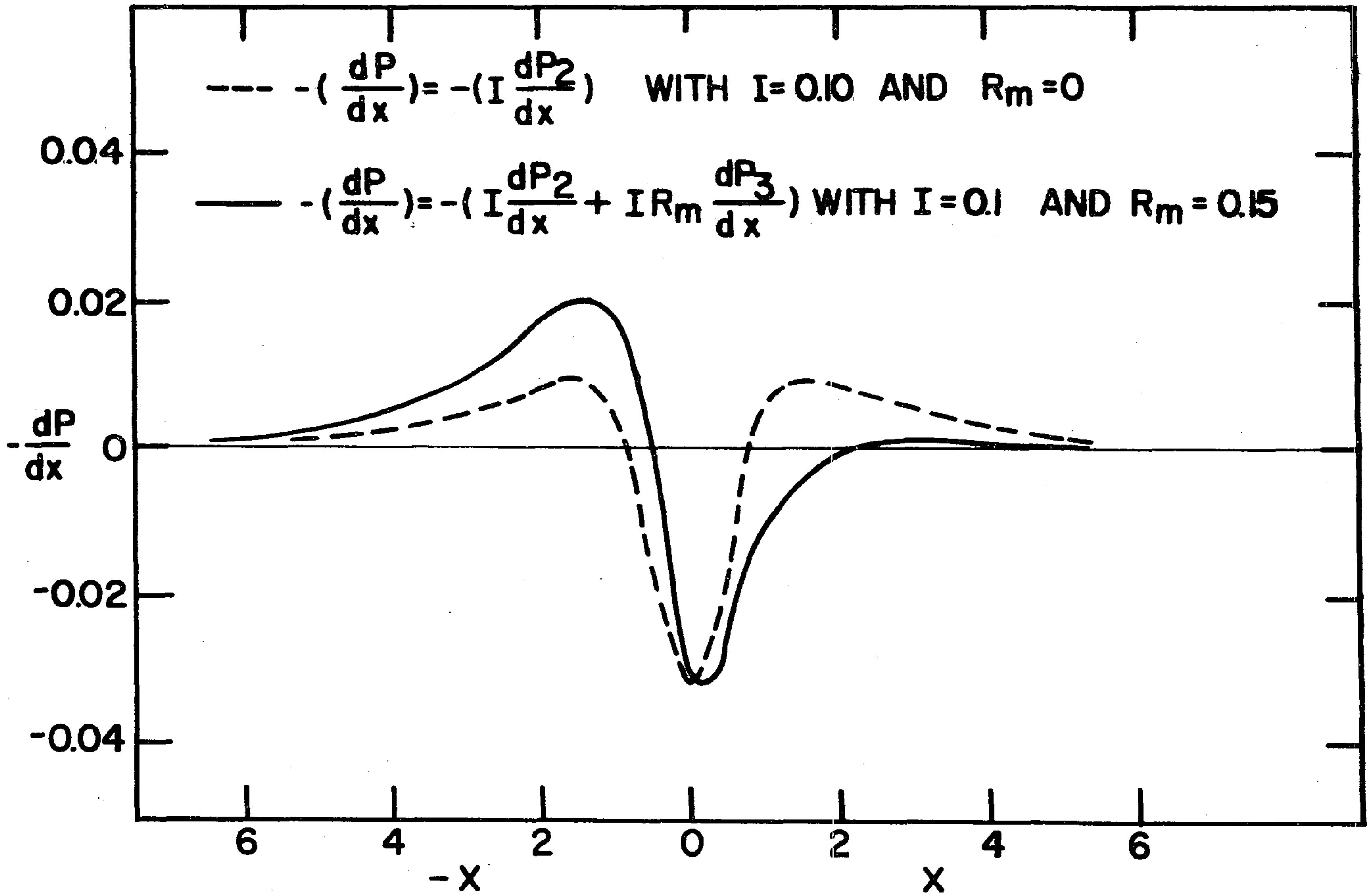
$-\left(\frac{dP}{dx}\right) = -\left(I\frac{dP_2}{dx}\right)$ WITH $I = 0.10$ AND $R_m = 0$
$-\left(\frac{dP}{dx}\right) = -\left(I\frac{dP_2}{dx} + IR_m\frac{dP_3}{dx}\right)$ WITH $I = 0.10$ AND $R_m = 0.075$
0.04
0.02
$-\frac{dP}{dx}$ 0
-0.02
-0.04
6
4
2
0
2
4
6
- X
X

Figure 4. Total static pressure gradient at lower wall



$-\left(\frac{dP}{dx}\right) = -\left(I\frac{dP_2}{dx}\right)$ WITH I=0.10 AND R_m=0
$-\left(\frac{dP}{dx}\right) = -\left(I\frac{dP_2}{dx} + IR_m\frac{dP_3}{dx}\right)$ WITH I=0.10 AND R_m=0.10
0.04
0.02
$-\frac{dP}{dx}$ 0
-0.02
-0.04
6
4
2
0
2
4
6
-X
X

Figure 5. Total static pressure gradient at lower wall



$-\left(\frac{dP}{dx}\right) = -\left(I\frac{dP_2}{dx}\right)$ WITH $I = 0.10$ AND $R_m = 0$
$-\left(\frac{dP}{dx}\right) = -\left(I\frac{dP_2}{dx} + IR_m\frac{dP_3}{dx}\right)$ WITH $I = 0.1$ AND $R_m = 0.15$
0.04
0.02
0
-0.02
-0.04
$-\frac{dP}{dx}$
6
4
2
0
2
4
6
-X
X

Figure 6. Static and magnetic pressure gradient at lower wall due to induced magnetic field

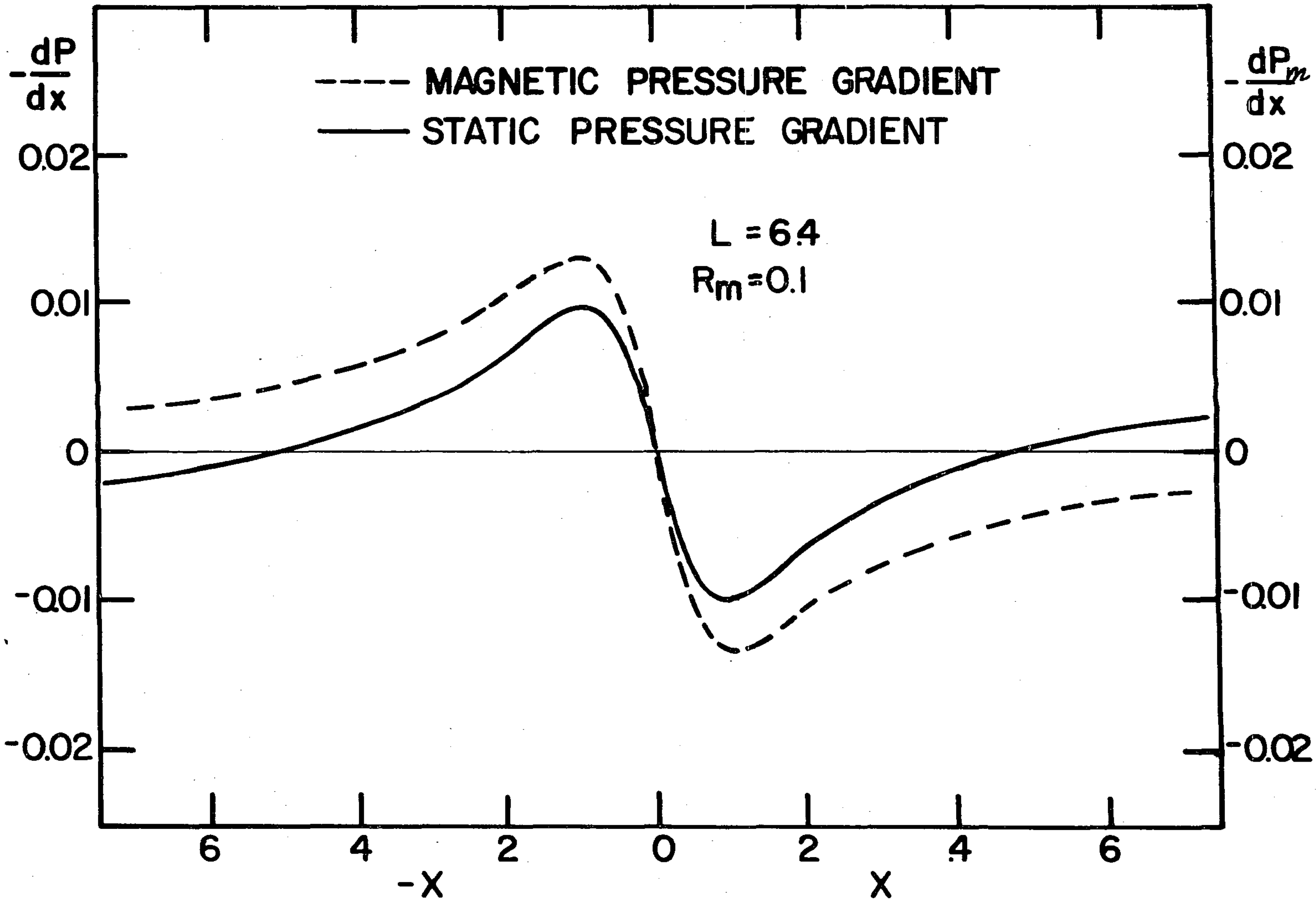

MAGNETIC PRESSURE GRADIENT
STATIC PRESSURE GRADIENT
- dP/dx
- dPm/dx
L = 6.4
Rm = 0.1
0.02
0.01
0
-0.01
-0.02
6
4
2
0
2
4
6
-X
X

Figure 7. Static and magnetic pressure gradient at lower wall due to induced magnetic field

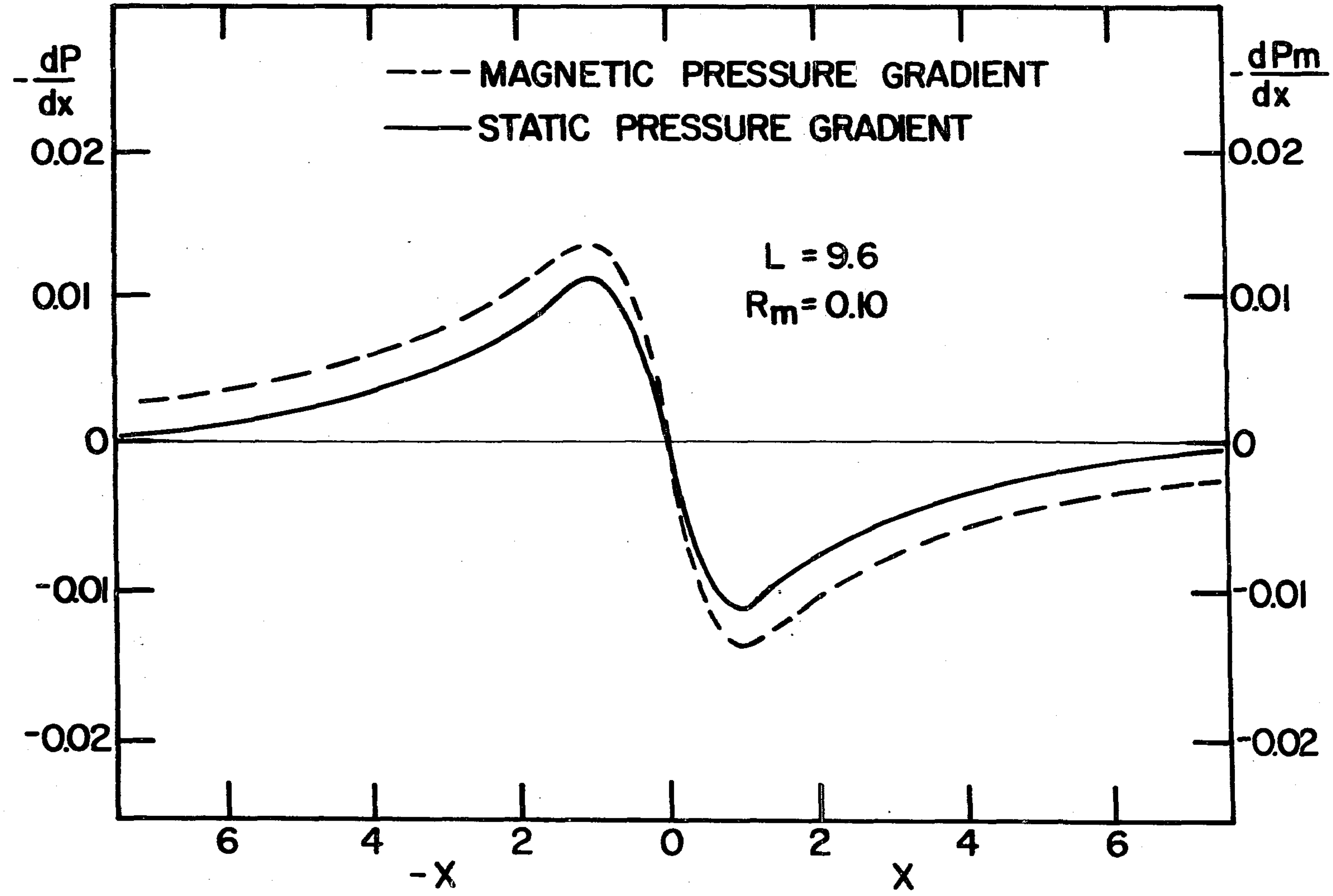

MAGNETIC PRESSURE GRADIENT
STATIC PRESSURE GRADIENT
$-\frac{dP}{dx}$
$-\frac{dPm}{dx}$
L = 9.6
R_m = 0.10
0.02
0.01
0
-0.01
-0.02
6
4
2
0
2
4
6
-X
X

Figure 8. Static and magnetic pressure gradient at lower wall due to induced magnetic field

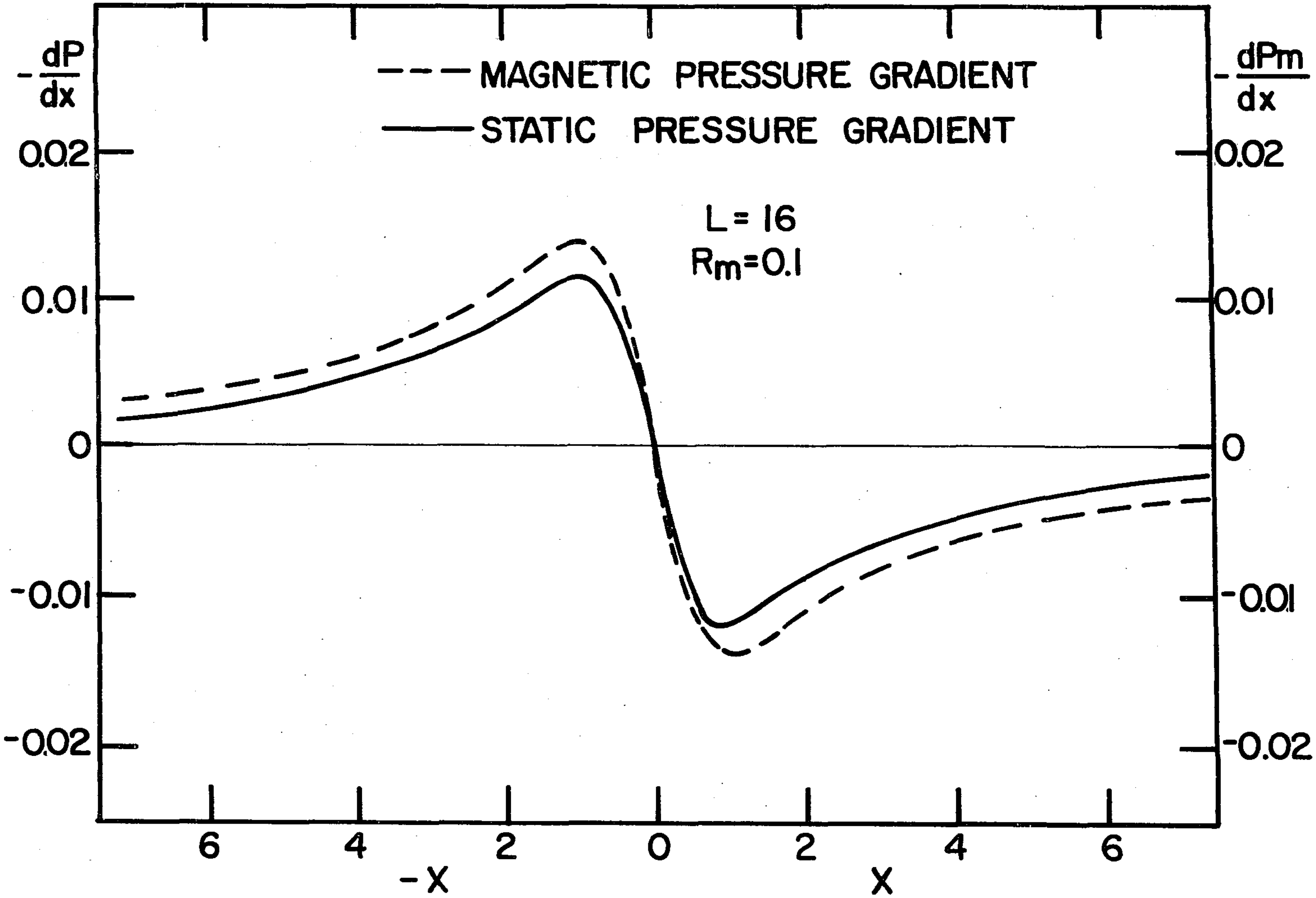
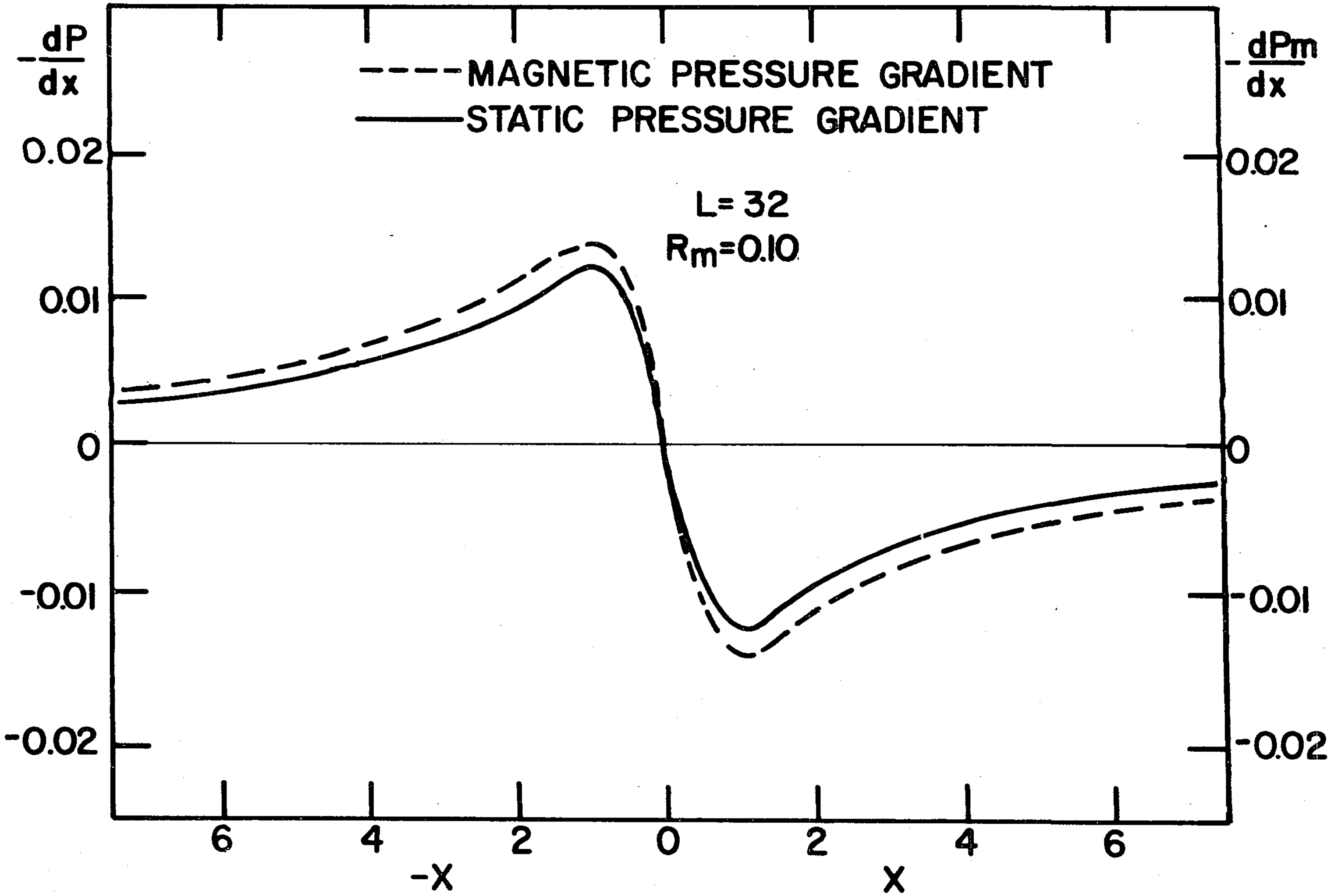

- dP/dx
- dPm/dx
MAGNETIC PRESSURE GRADIENT
STATIC PRESSURE GRADIENT
L = 16
Rm=0.1
0.02
0.01
0
-0.01
-0.02
6
4
2
0
2
4
6
-X
X

Figure 9. Static and magnetic pressure gradient at lower wall due to induced magnetic field



$-\frac{dP}{dx}$
$-\frac{dPm}{dx}$
MAGNETIC PRESSURE GRADIENT
STATIC PRESSURE GRADIENT
L= 32
R_m=0.10
0.02
0.01
0
-0.01
-0.02
6
4
2
0
2
4
6
-X
X

Figure 10. Static and magnetic pressure gradient at lower wall due to induced magnetic field

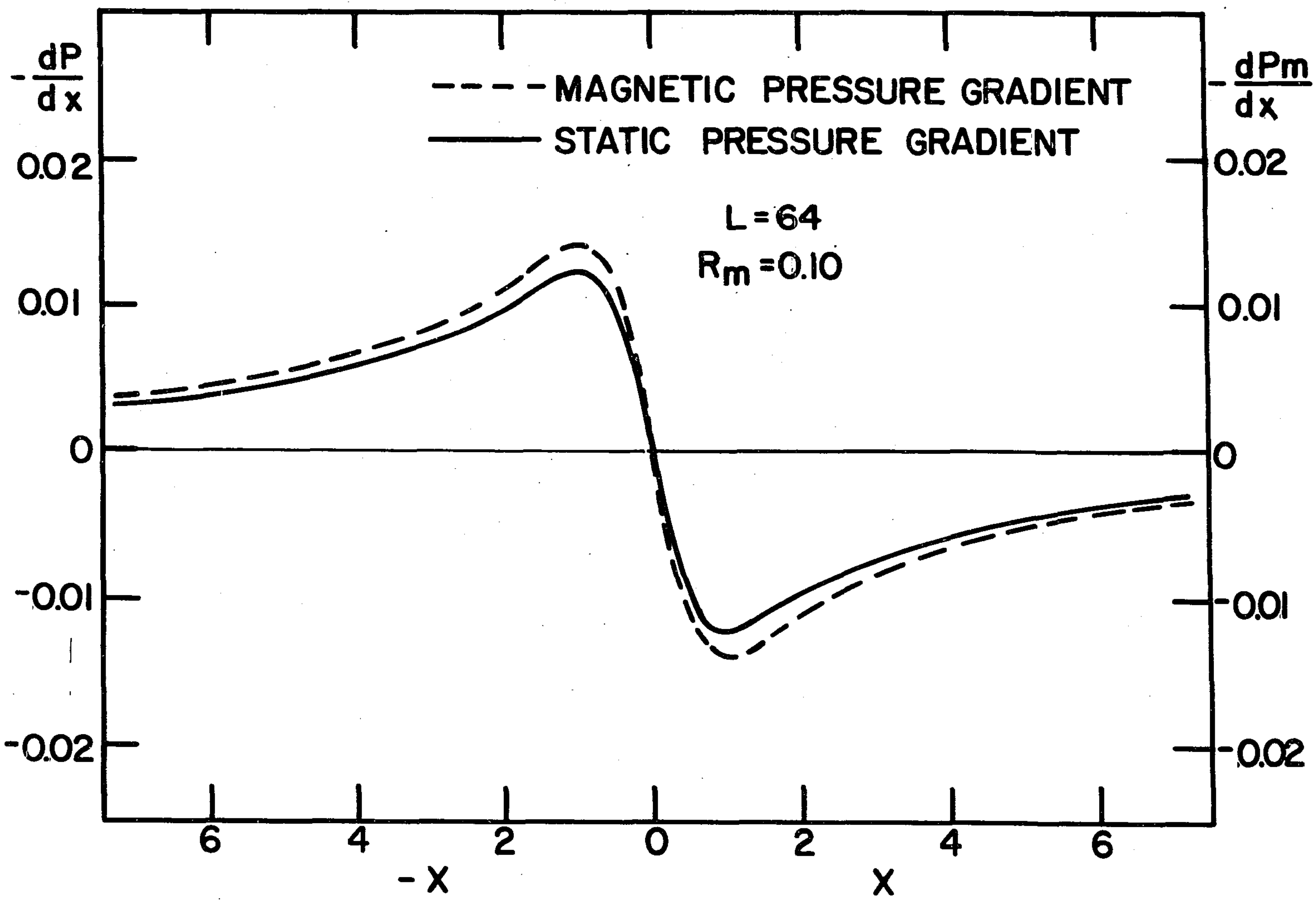

- dP/dx
- dPm/dx
MAGNETIC PRESSURE GRADIENT
STATIC PRESSURE GRADIENT
L = 64
Rm = 0.10
0.02
0.01
0
-0.01
-0.02
6
4
2
0
2
4
6
- X
X

Figure 11. Static and magnetic pressure gradient at lower wall due to induced magnetic field

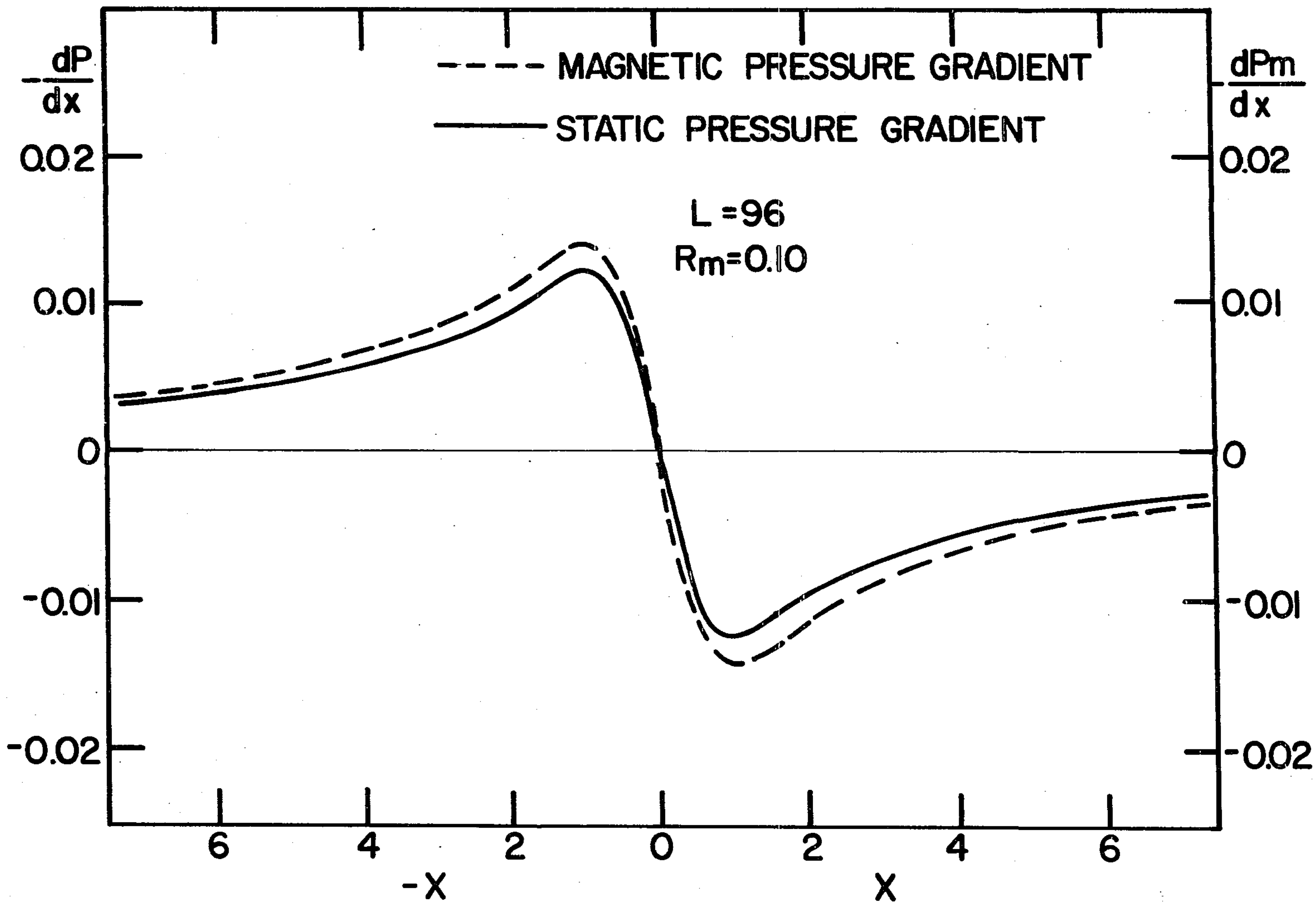
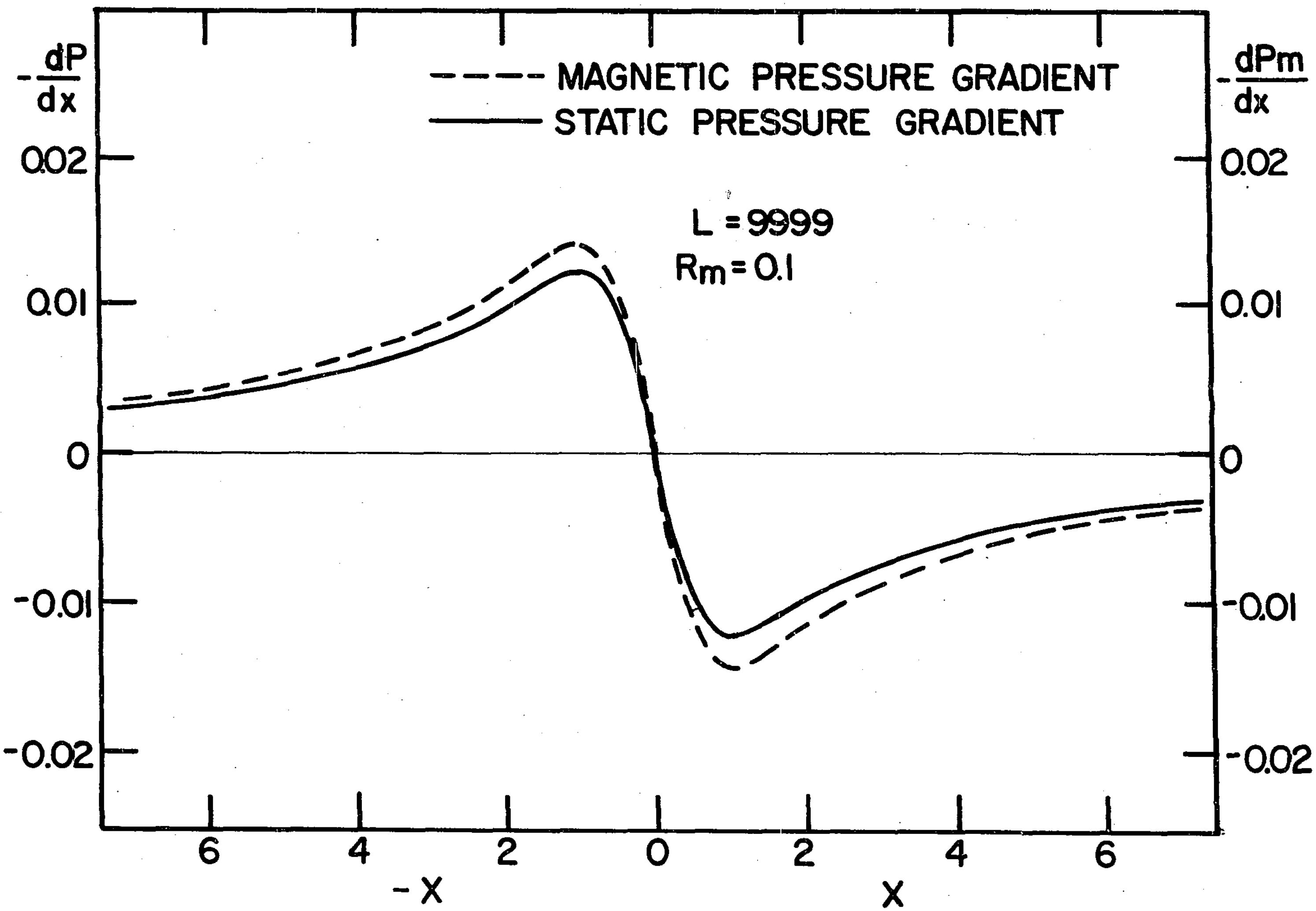
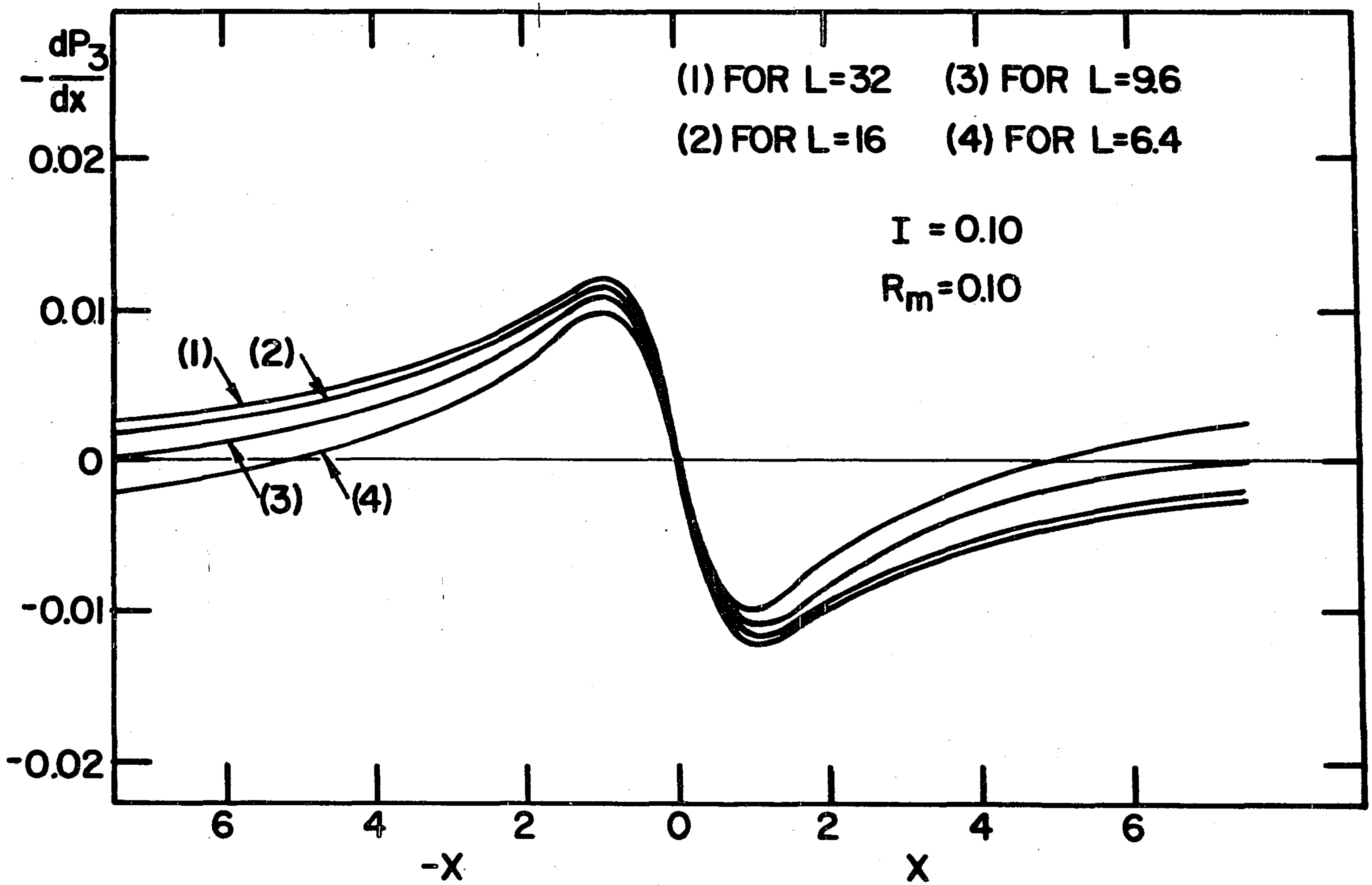

MAGNETIC PRESSURE GRADIENT
STATIC PRESSURE GRADIENT
L = 96
Rm = 0.10
-dP/dx
dPm/dx
0.02
0.01
0
-0.01
-0.02
6
4
2
0
2
4
6
-X
X

Figure 12. Static and magnetic pressure gradient at lower wall due to induced magnetic field



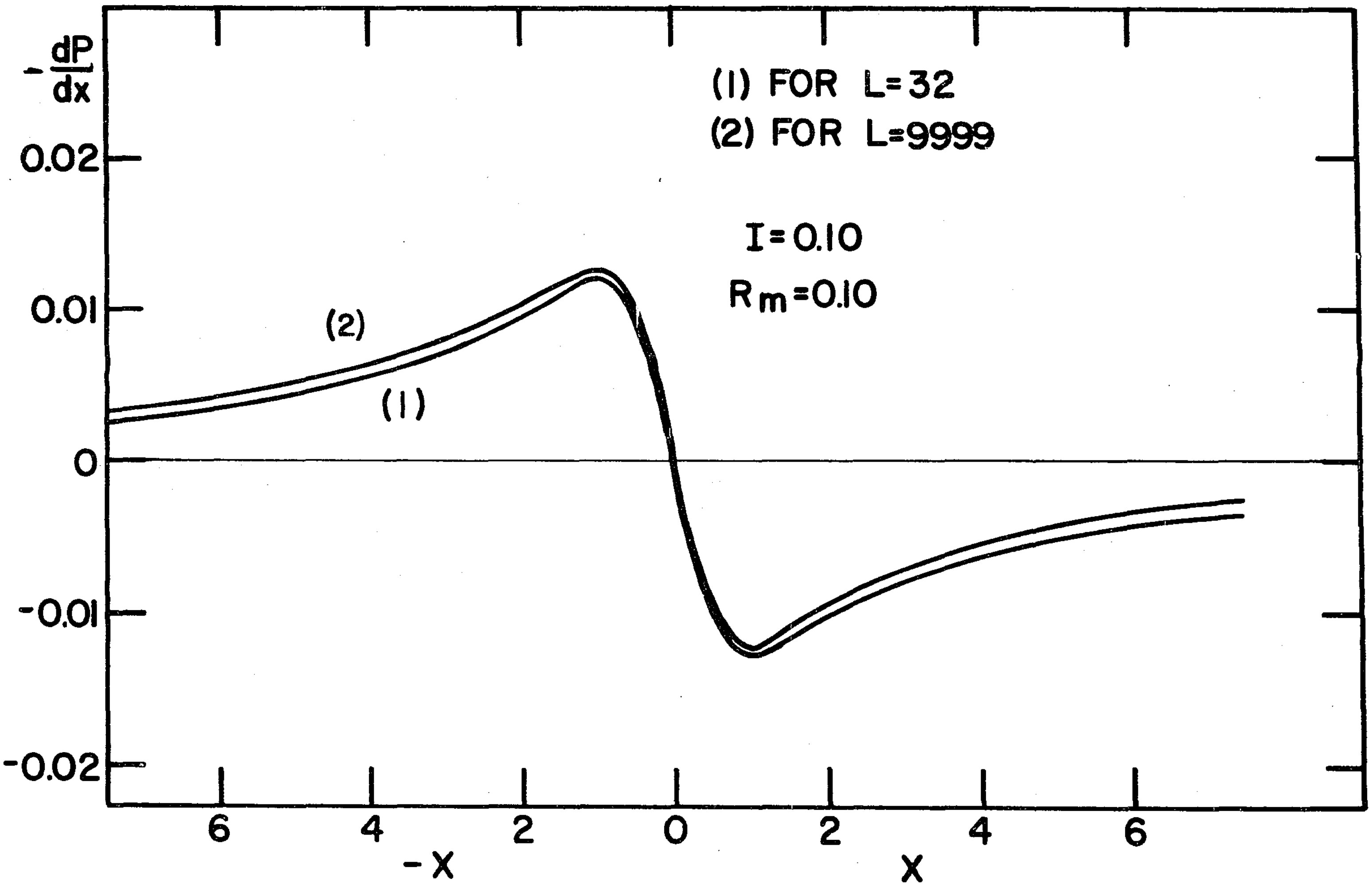
$-\frac{dP}{dx}$
$-\frac{dPm}{dx}$
MAGNETIC PRESSURE GRADIENT
STATIC PRESSURE GRADIENT
L = 9999
$R_m = 0.1$
0.02
0.01
0
-0.01
-0.02
6
4
2
0
2
4
6
- X
X

Figure 13. Static pressure gradient due to induced magnetic field at lower wall at various values of L



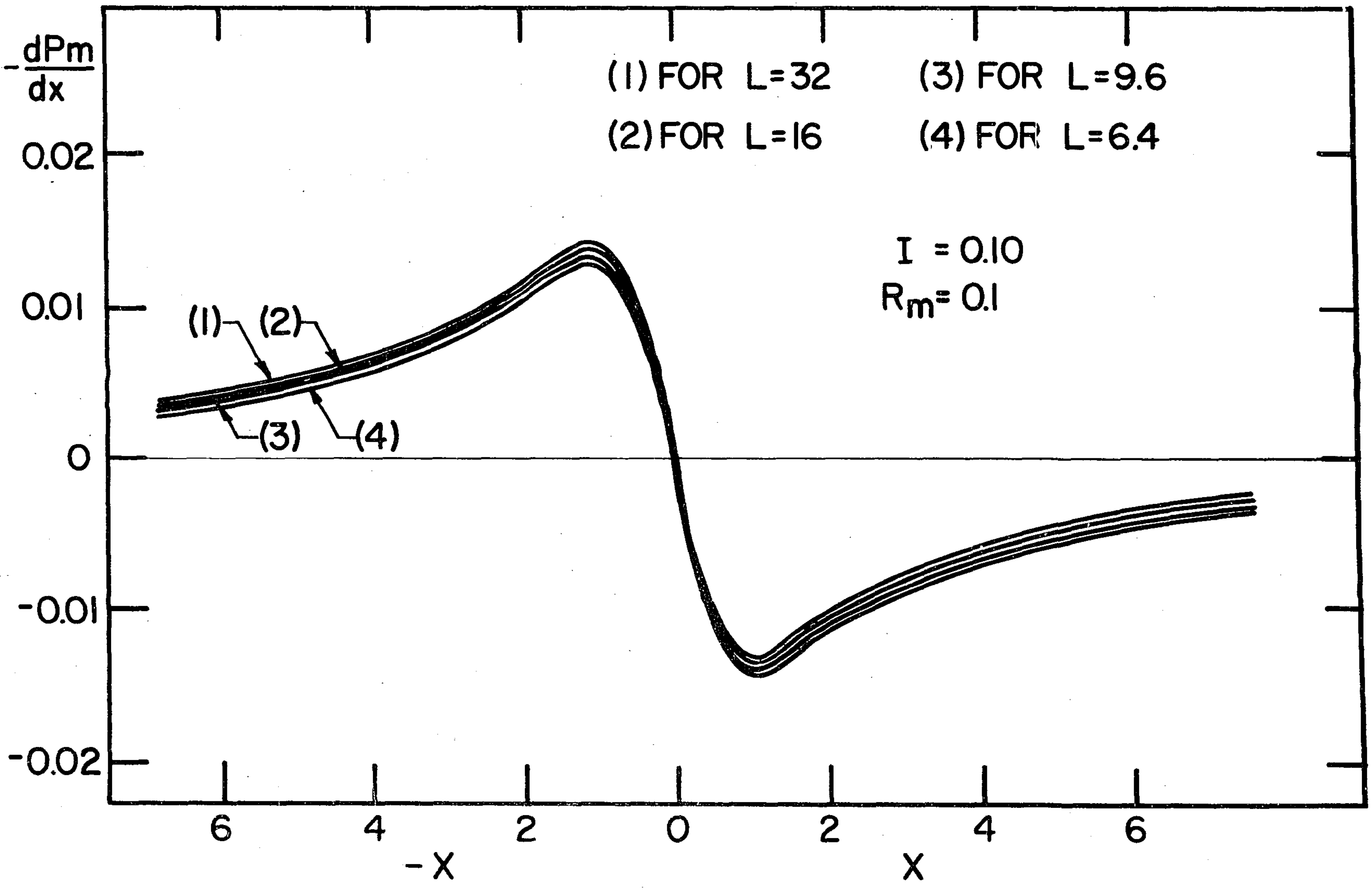
$-\frac{dP_3}{dx}$
0.02
0.01
0
-0.01
-0.02
(1) FOR L=32
(3) FOR L=9.6
(2) FOR L=16
(4) FOR L=6.4
I = 0.10
R_m=0.10
(1)
(2)
(3)
(4)
6
4
2
0
2
4
6
-X
X

Figure 14. Static pressure gradient due to induced magnetic field at lower wall at various values of L



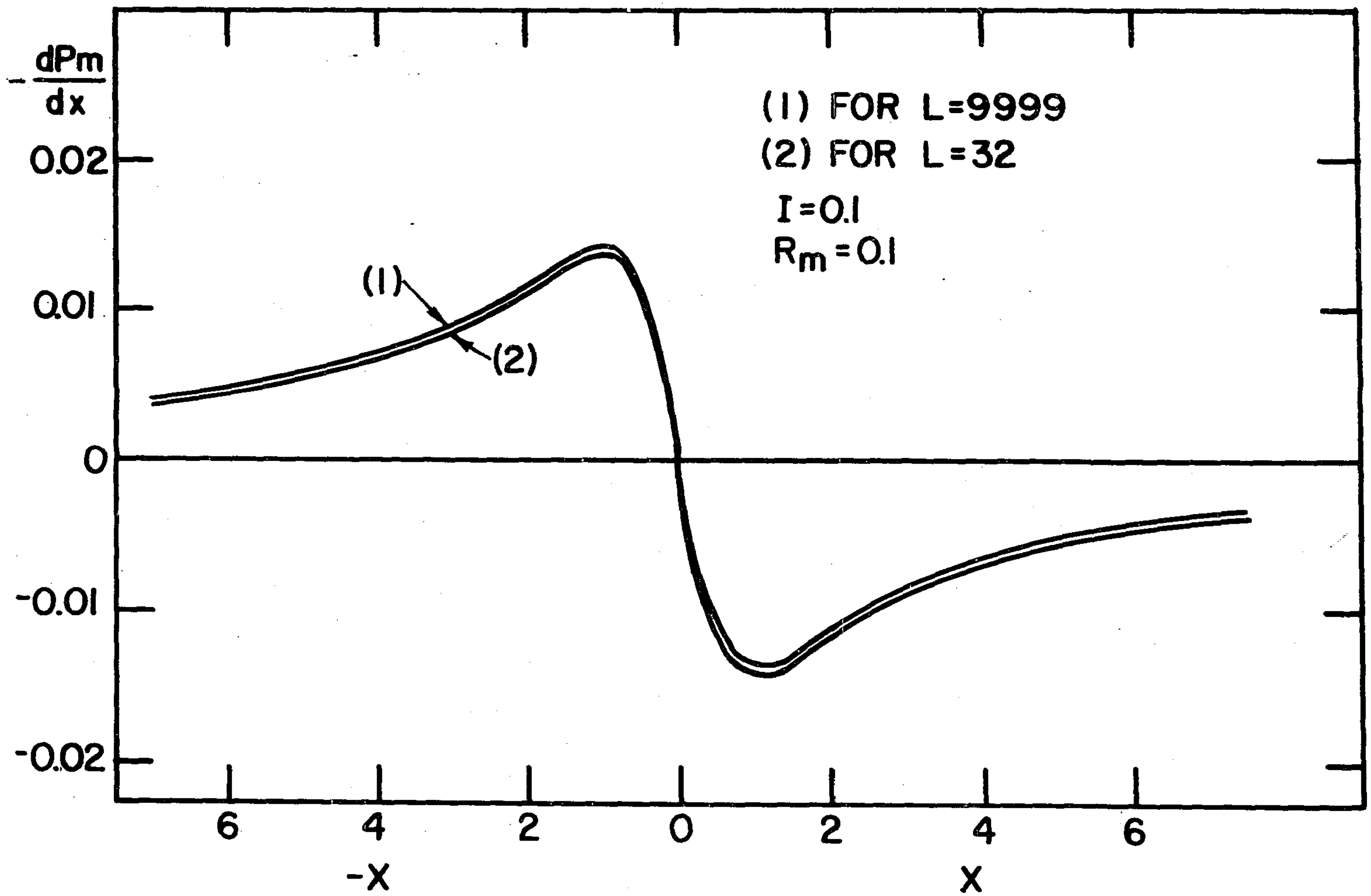
$-\frac{dP}{dx}$
(1) FOR L=32
(2) FOR L=9999
I=0.10
Rm=0.10
0.02
0.01
0
-0.01
-0.02
(1)
(2)
6
4
2
0
2
4
6
-X
X

Figure 15. Magnetic pressure gradient due to induced magnetic field at lower wall for various L



$-\frac{dPm}{dx}$
0.02
0.01
0
-0.01
-0.02
(1) FOR L=32
(2) FOR L=16
(3) FOR L=9.6
(4) FOR L=6.4
I = 0.10
R_m= 0.1
(1)
(2)
(3)
(4)
6
4
2
0
2
4
6
-X
X

Figure 16. Magnetic pressure gradient due to induced magnetic field at lower wall for various L



$-\frac{dPm}{dx}$
0.02
0.01
0
-0.01
-0.02
6
4
2
0
2
4
6
-X
X
(1) FOR L=9999
(2) FOR L=32
I=0.1
R_m=0.1
(1)
(2)

Figure 17. Magnitude of maximum static pressure gradient due to induced magnetic field

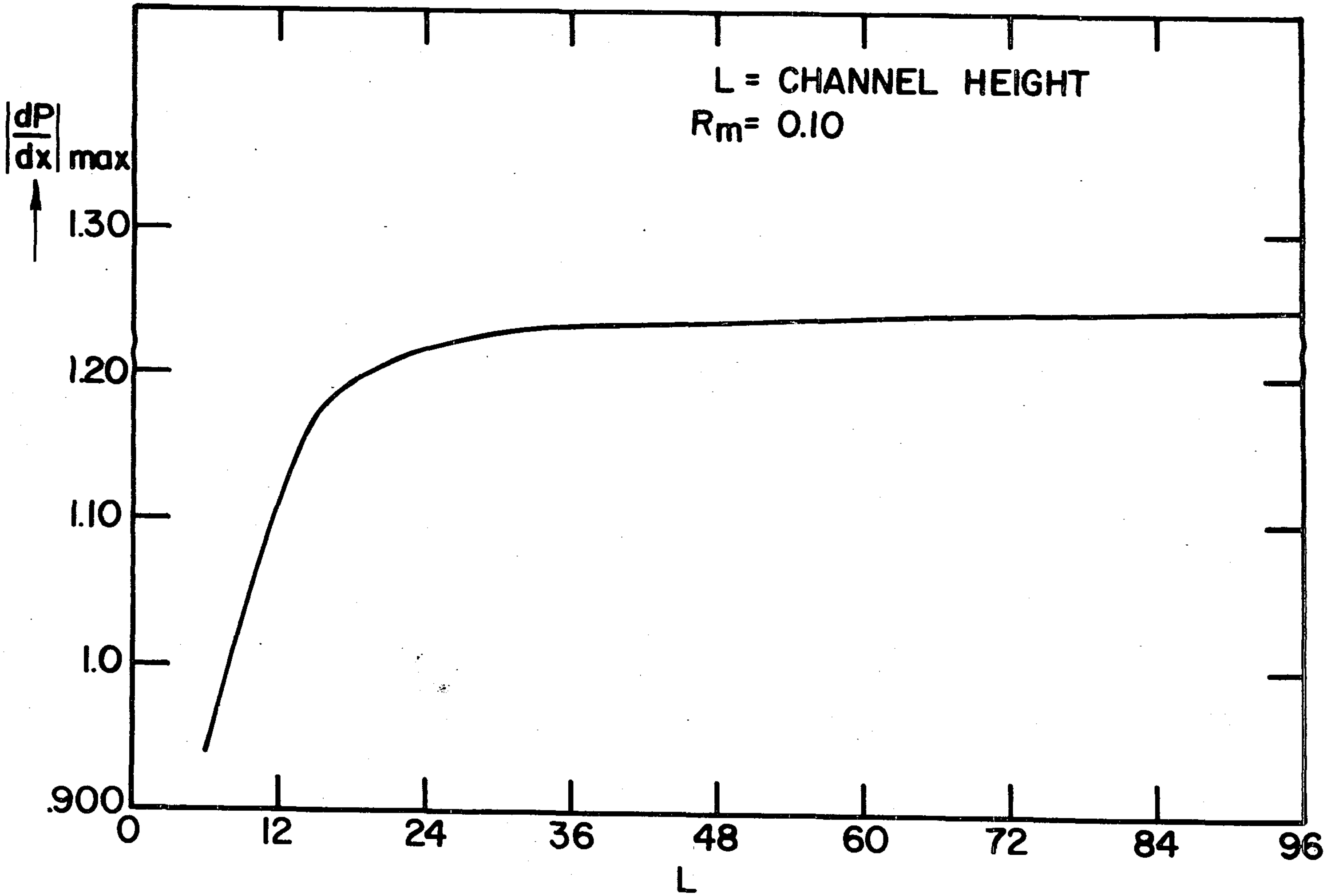
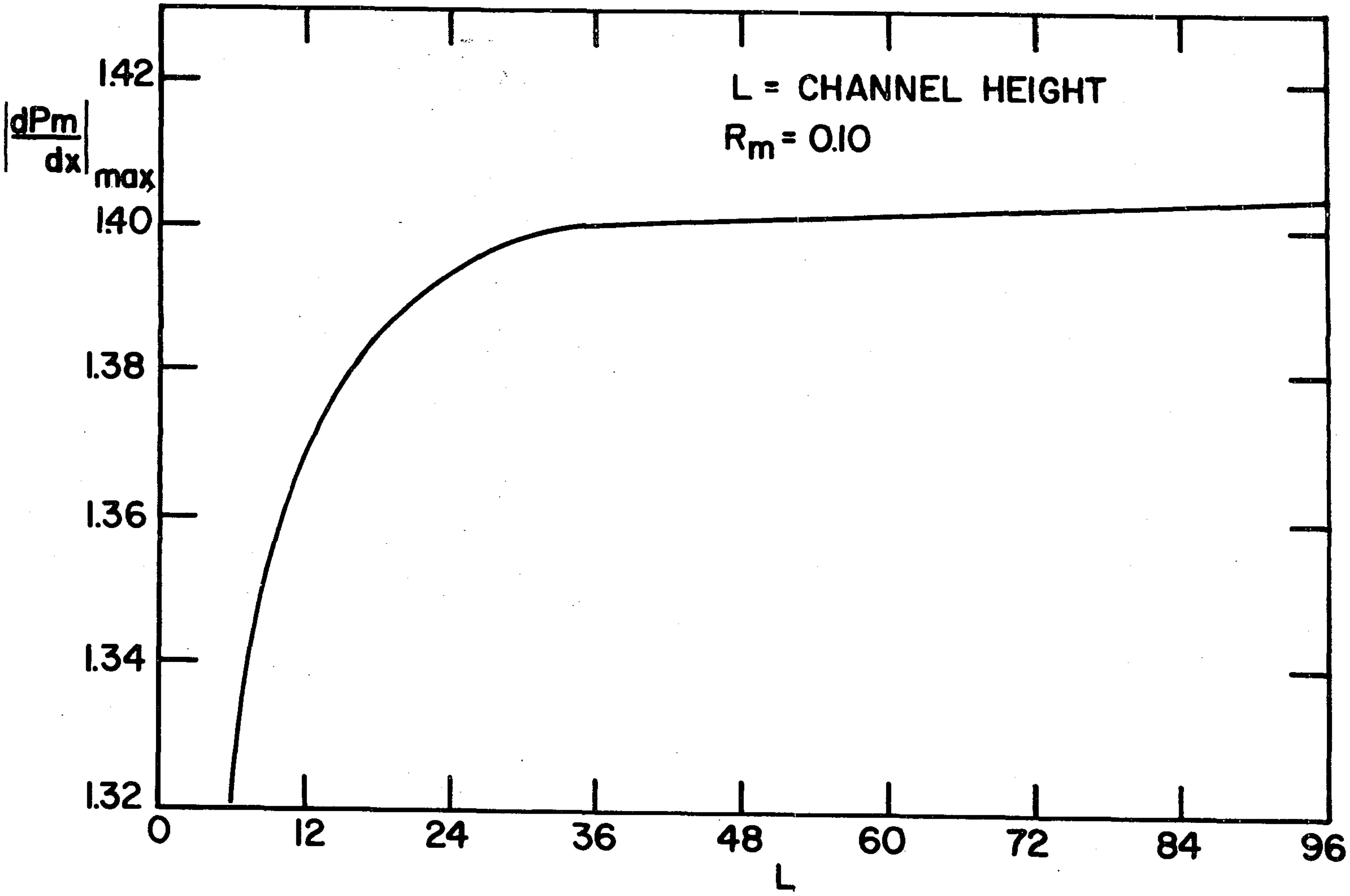

L = CHANNEL HEIGHT
R_m= 0.10
$\left|\frac{dP}{dx}\right|_{max}$
1.30
1.20
1.10
1.0
.900
0
12
24
36
48
60
72
84
96
L

Figure 18. Magnitude of maximum magnetic pressure gradient due to induced magnetic field


L = CHANNEL HEIGHT
$R_m = 0.10$
$\left|\frac{dPm}{dx}\right|_{max}$
1.42
1.40
1.38
1.36
1.34
1.32
0
12
24
36
48
60
72
84
96
L

IX. CONCLUSIONS

The importance of preventing a hot conducting fluid from coming in contact with the internal walls of channels and nozzles, and hence of protecting those walls is the main point of study when the induced magnetic field is to be considered.

From the results obtained in Section VIII, the following conclusions concerning the effects of the induced magnetic field on protecting the channel walls from hot conducting fluid are recognized.

1. Figure 2 indicates that for I = 0.10 a negligible increase in the net area of the adverse pressure gradient is observed with R_m = 0.05. With $R_m = 0.075$, the increase in the net negative area is slight but recognizable as shown in Figure 3. It is shown that with R_m equal to 0.1 and 0.15 the obvious result is a clear net increase in the negative area of $-\frac{dP}{dx}$.

   Those results indicate that with R_m of the order of 0.1 the induced magnetic field cannot be neglected and therefore must be included in any treatment of inviscid magnetohydrodynamic flow. Also, the conclusion is reached that with R_m less than 0.1 it can be neglected. We recall that $R_m = \sigma\mu V_o y_o$.

   Therefore, for fluid flow with electrical conductivity σ and magnetic permeability μ and with $R_m \geq 0.1$

   $$\frac{1}{10\mu\sigma} \geq V_o y_o$$

   or

   $$\frac{1}{V_o y_o} \geq (10\mu\sigma)$$

As a consequence of $R_m \geq 0.1$, an improvement in boundary layer separation will result. This will add more to the protection of the channel walls from the high temperature fluid.

2. Figures 6 to 12 show the hydrostatic and magnetic pressure gradient due to the induced magnetic field along the channel wall for several values of L. The plots indicate that in the up-stream region of the flow field the magnetic pressure gradient is acting to help the inertia forces. As the flow approaches the source of the applied magnetic field, this effect decreases. At the start of the down stream flow, the magnetic pressure gradient is acting in opposition to the inertia forces of the flow field. As a result, separation of fluid particles from the channel wall will occur.
3. Figures 13 to 16 show the hydrostatic and magnetic pressure gradients due to the induced magnetic field along the channel wall for several channel heights. It is indicated that as the two channel walls move apart, the differences among the hydrostatic pressure gradients and also among magnetic pressure gradients are diminishing. This is due to the fact that the magnetic field has more freedom in deflecting the flow and slowing it down as the height of the channel is increasing.
4. Another fact demonstrated in Figures 6 through 12 is that as the channel height increases from 6.4 to 96, the order of magnitude of the hydrostatic pressure gradient is approaching that of the magnetic pressure gradient. This is due to the increased

influence of the magnetic field in slowing down the flow as the channel width increases.

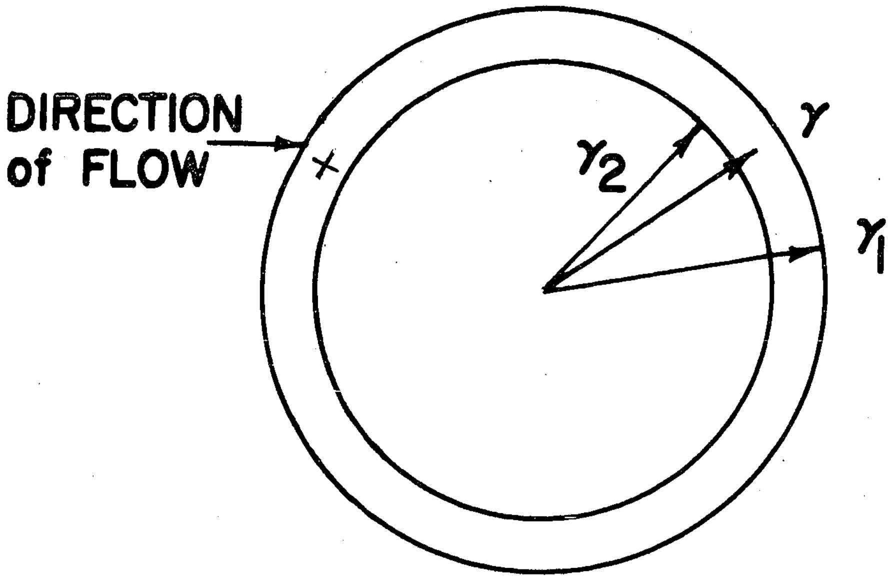
5. In Figures 17 and 18 it is shown that the maximum magnitude of the static and magnetic pressure gradient due to the induced magnetic field along the channel wall is increasing with L until L is about 30. Past this point the increase is small due to the fact that in such a case the magnetic forces have more freedom to slow down the fluid flow.

It should also be mentioned that the problem studied in this dissertation, namely that of MHD flow between two parallel plates of infinite extent, can be considered equivalent to that of an annular duct of radii r_1 and r_2 such that the mean circumference of the duct is much larger than (r_1-r_2), i.e.

if r = mean radius of the duct

$$(2\pi r) \gg (r_1-r_2)$$

In other words the length of the channel is much greater than its height as shown in Figures 19 and 20.



$$2\pi\gamma \gg (\gamma_1 - \gamma_2)$$

Figure 19. Hot conducting fluid flow in an annular duct

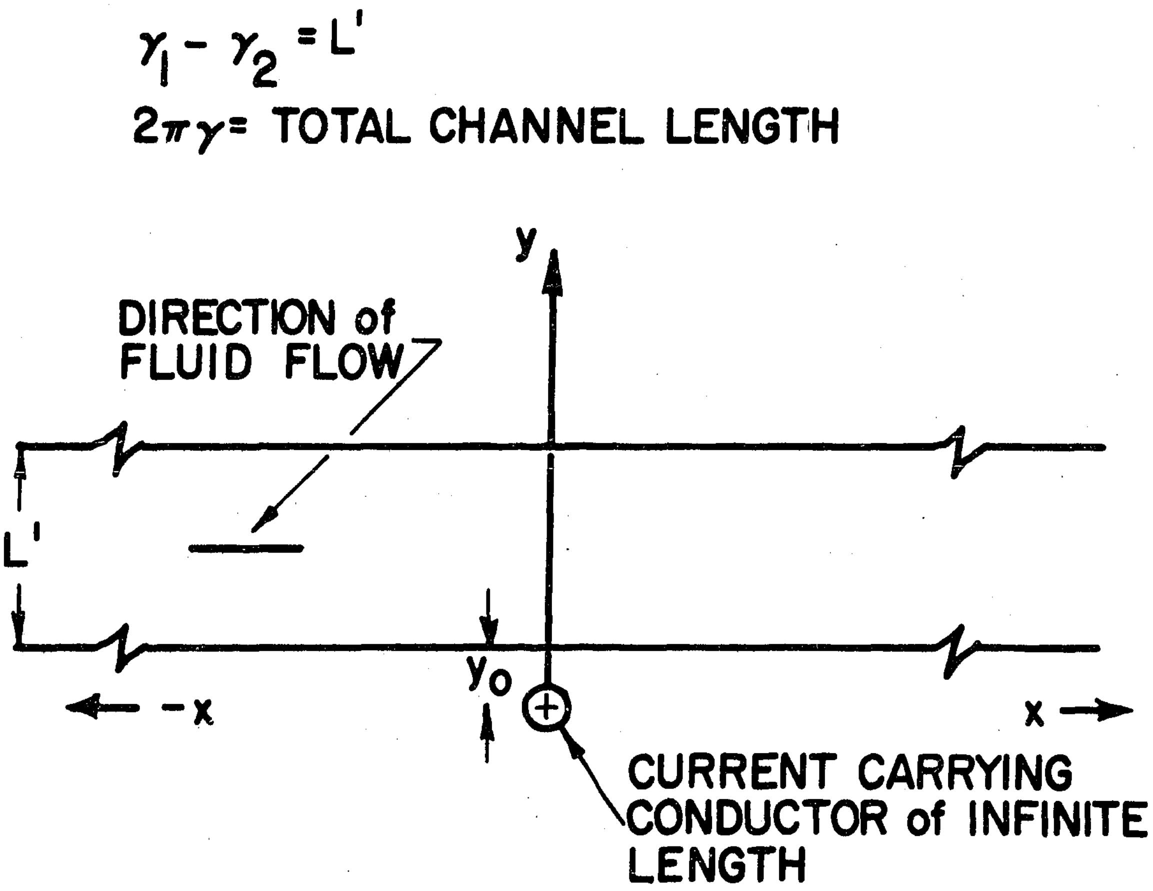


Figure 20. Hot conducting fluid flow in a two parallel plate channel

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XII. APPENDIX A - FOURIER TRANSFORMATION OF EQUATION 45

The equation to be transformed is

$$\frac{\partial^2 B_{1x}}{\partial x^2} + \frac{\partial^2 B_{1x}}{\partial y^2} = \frac{2\ x\ (1+y)}{[x^2 + (1+y)^2]^2} \tag{45}$$

with the boundary conditions given by Equation 46

$$B_{1x} = 0 \text{ at } y = 0,\ L$$

and (46)

$$B_{1x} = 0 \text{ at } x = \pm\infty$$

Now, take the exponential Fourier transformation of the above partial differential equation with respect to x, resulting in

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{\partial^2 B_{1x}}{\partial x^2} + \frac{\partial^2 B_{1x}}{\partial y^2}\right] e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2\ x\ (1+y)\ e^{i\alpha x}}{[x^2 + (1+y)^2]^2}\, dx \tag{88}$$

where α is the Fourier transform variable. This operation reduces the partial differential equation to an ordinary one. Equation 88 becomes

$$\frac{\partial^2 \overline{B}_{1x}}{\partial y^2} - \alpha^2 \overline{B}_{1x} = \frac{2(1+y)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{x\ e^{i\alpha x}\ dx}{[x^2 + (1+y)^2]^2} \tag{89}$$

where $\overline{B}_{1x}$ is the Fourier transform of B_{1x}. To find the Fourier transform of the right hand side of Equation 89, we proceed as follows:

$$\int_{-\infty}^{\infty} \frac{x\ e^{i\alpha x}\ dx}{[x^2 + (1+y)^2]^2} = \int_{-\infty}^{\infty} e^{i\alpha x} \frac{d}{dx} \left[\frac{-1}{2} \frac{1}{x^2 + (1+y)^2}\right] \tag{90}$$

$$= [e^{i\alpha x} \frac{\frac{1}{2}}{x^2 + (1+y)^2} \Big|_{-\infty}^{\infty} - (\frac{-1}{2}) \int_{-\infty}^{\infty} \frac{(i\alpha)\, e^{i\alpha x}}{x^2 + (1+y)^2}\, dx \tag{91}$$

The first expression of the right side of Equation 91 is zero and thus we get,

$$\int_{-\infty}^{\infty} \frac{x\, e^{i\alpha x}}{[x^2 + (1+y)^2]^2} = \frac{i\alpha}{2} \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x^2 + (1+y)^2} \tag{92}$$

Then, from a table of Fourier transforms (24), the Fourier transform of

$$[a^2 + (x\pm b)^2]^{-\nu}$$

$$= \frac{1}{\sqrt{2\pi}}\, 2\, e^{\pm ib\alpha} \sqrt{\pi}\, [\quad(\nu)]^{-1} [\frac{|\alpha|}{2a}]^{\nu-\frac{1}{2}} K_{\nu-\frac{1}{2}}(a|\alpha|) \text{ with } R_e\nu > 0$$

For $\nu = 1$, $b = 0$, $a = (1+y)$, the Fourier transform of

$$\frac{1}{a^2 + x^2} = \frac{1}{\sqrt{a}} \sqrt{|\alpha|}\, K_{\frac{1}{2}}(a|\alpha|) \tag{93}$$

Then, using Equation 92, the Fourier transform of

$$\frac{x}{(x^2+a^2)^2} = \frac{i\alpha}{2} \frac{1}{\sqrt{a}} \sqrt{|\alpha|}\, K_{\frac{1}{2}}(a|\alpha|) \tag{94}$$

Also from tables (24),

$$K_\nu(Z) = \frac{\pi}{2}(\sin \pi\nu)^{-1} [I_{-2}(Z) - I_\nu(Z)]$$

From Equation 94,

$\nu = \frac{1}{2}$, $Z = a|\alpha|$, thus

$$K_{\frac{1}{2}}(a|\alpha|) = \frac{\pi}{2}(\sin\frac{\pi}{2})^{-1}\,[I_{-\frac{1}{2}}(a|\alpha|) - I_{\frac{1}{2}}(a|\alpha|)$$

$$= \frac{\pi}{2}[I_{-\frac{1}{2}}(a|\alpha|) - I_{\frac{1}{2}}(a|\alpha|)] \qquad (95)$$

Also from the tables (24)

$$I_\nu(Z) = e^{-\frac{1}{2}i\pi\nu}\,J_\nu[Z\,e^{\frac{i\pi}{2}}]$$

therefore

$$I_{\frac{1}{2}}(a|\alpha|) = e^{-\frac{i\pi}{4}}\,J_{\frac{1}{2}}(ia|\alpha|) \qquad (96)$$

and

$$I_{-\frac{1}{2}}(a|\alpha|) = e^{\frac{i\pi}{4}}\,J_{-\frac{1}{2}}(ia|\alpha|) \qquad (97)$$

Hence

$$K_{\frac{1}{2}}(a|\alpha|) = \frac{\pi}{2}[e^{\frac{i\pi}{4}}\,J_{-\frac{1}{2}}(ia|\alpha|) - e^{-\frac{i\pi}{4}}\,J_{\frac{1}{2}}(ia|\alpha|)] \qquad (98)$$

since

$$\left.\begin{aligned} J_{-\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}}\cos x \\ J_{\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}}\sin x \end{aligned}\right\} \qquad (99)$$

Substituting in Equation 89 we get

$$\frac{\partial^2 \overline{B}_{1x}}{\partial y^2} - \alpha^2\overline{B}_{1x} = -\,i\frac{\sqrt{\pi}}{\sqrt{2}}\cdot\alpha\cdot e^{-|\alpha|(1+y)} \qquad (100)$$

The Fourier transforms of the boundary conditions in Equation 46 are given by,

$$\overline{B}_{1x}\ (\alpha,\ 0) = 0 \tag{101}$$

and

$$\overline{B}_{1x}\ (\alpha,\ L) = 0 \tag{102}$$

The general solution for $\overline{B}_{1x}$ is thus

$$\overline{B}_{1x} = \frac{i\sqrt{\pi}}{2\sqrt{2}}\left[\frac{\alpha}{|\alpha|}\ y\ e^{-|\alpha|(1+y)} - \frac{\alpha}{|\alpha|}\ L\ \frac{\sinh\ \alpha y}{\sinh\ \alpha L}\ e^{-|\alpha|(1+L)}\right] \tag{103}$$

The inverse Fourier transform of $\overline{B}_{1x}$ is

$$B_{1x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{B}_{1x}\ e^{-i\alpha x}\ d\alpha \tag{104}$$

Performing the steps indicated in Equation 104 and using the following relations from tables (10, 11)

$$\int_0^{\infty} \frac{e^{-\mu x} - e^{-\nu x}}{1 - e^{-x}}\ dx = \psi(\nu) - \psi(\mu)\ \ (R_e\ \mu > 0,\ R_e\ \nu > 0) \tag{105}$$

where

$$\psi(\nu) = -\ C - \sum_{n=0}^{\infty} \left(\frac{1}{x+n} - \frac{1}{n+1}\right) \tag{106}$$

and C = Euler's constant, we get

$$B_{1x} = \frac{1}{2}\left[\frac{xy}{x^2 + (1+y)^2} + \sum_{n=0}^{\infty} \frac{xL}{x^2 + (1+L+2nL+2y)^2} - \sum_{n=0}^{\infty} \frac{xL}{x^2 + (1+L+2nL)^2}\right] \tag{107}$$

Further properties of the ψ function are

$$\psi(x+iy) - \psi(x-iy) = \sum_{n=0}^{\infty} \frac{2iy}{y^2 + (x+n)^2} \tag{108}$$

$$\psi(\mu) - \psi(\nu) = \sum_{n=0}^{\infty} \left[\frac{1}{\nu + n} - \frac{1}{\mu + n}\right]$$

$$= \sum_{n=0}^{\infty} \frac{\mu - \nu}{(\mu+n)(\nu+n)} \tag{109}$$

$\mu, \nu \neq 0, -1, -2$

$R_e(\mu), R(\nu) > 0$

and

$$\psi(Z) = \frac{d}{dZ} \log\Gamma(Z) \tag{110}$$

XIII. APPENDIX B - FOURIER TRANSFORMATION OF EQUATION 77

The equation to be transformed is

$$\frac{\partial^2 V_{3y}}{\partial x^2} + \frac{\partial^2 V_{3y}}{\partial y^2} = \sum_{n=0}^{\infty} \frac{YL\ L_n}{2(L_n^2-1)^2} \frac{1}{(x+iY)}$$

$$+ \sum_{n=0}^{\infty} \frac{YL\ L_n}{2(L_n^2-1)^2} \frac{1}{(x-iY)}$$

$$- \sum_{n=0}^{\infty} \frac{YL\ L_n}{2(L_n^2-1)^2} \frac{1}{(x+iL_n)}$$

$$- \sum_{n=0}^{\infty} \frac{YL\ L_n}{2(L_n^2-1)^2} \frac{1}{(x-iL_n)}$$

$$- \sum_{n=0}^{\infty} \frac{YL\ L_n}{2(L_n^2-1)} \frac{x}{(x^2+Y^2)^2}$$

$$- \sum_{n=0}^{\infty} \frac{YL\ L_n}{2(L_n^2-1)} \frac{x}{(x^2+L_n^2)^2} \qquad (77)$$

Use is made of mathematical tables (24) to transform the above partial differential equation.

An example showing the transform of the first term of Equation 77 follows:

Let F.T. = Fourier transform

F.T. of $\frac{1}{x + iY}$ can be obtained from tables (24).

F.T. of $\frac{1}{(a-ix)^{\nu}} = 0$ for $\alpha > 0$

$$= \sqrt{2\pi}\ [\Gamma(\nu)]^{-1}\ (-\alpha)^{\nu-1}\ e^{\alpha a} \text{ for } \alpha < 0 \qquad (111)$$

where

$R_e \nu > 0$

For $\nu = 1$

$$\text{F.T. of } \frac{1}{a - ix} = 0 \quad \text{for } \alpha > 0$$

$$= \sqrt{2\pi}\, e^{\alpha a} \text{ for } \alpha < 0 \qquad (112)$$

or

$$\text{F.T. of } \frac{i}{x + ia} = 0 \quad \text{for } \alpha > 0$$

$$= \sqrt{2\pi}\, e^{\alpha a} \quad \text{for } \alpha < 0 \qquad (113)$$

Therefore

$$\text{F.T. of } \frac{1}{x + iY} = 0 \quad \text{for } \alpha > 0$$

$$= - i\sqrt{2\pi}\, e^{\alpha Y} \quad \text{for } \alpha < 0 \qquad (114)$$

The inverse Fourier transform of Equation 80 was obtained using the relation (1)

$$\int_{-\infty}^{x} f(t)dt = \text{F.T.}^{-1} \left[\frac{F(\alpha)}{- i\alpha}\right] \qquad (115)$$

where

F.T.^{-1} = inverse Fourier transform

$f(t)$ = inverse Fourier transform of $F(\alpha)$

t is a dummy variable

The inverse transform was checked using the convolution (39) theorem. As an example consider the inverse transform of two terms from Equation 80

$$F.T.^{-1}\left[\frac{-y^2}{4\alpha}e^{-\alpha Y}\right]\text{ (where }\alpha>0)$$

$$+\,F.T.^{-1}\left[\frac{-y^2}{4\alpha}e^{\alpha Y}\right]\text{ (where }\alpha<0) \tag{116}$$

$$F.T.^{-1}\left[e^{-\alpha Y}\right]\text{ (where }\alpha>0)$$

$$+\,F.T.^{-1}\left[e^{\alpha Y}\right]\text{ (where }\alpha<0)$$

$$=\frac{1}{\sqrt{2\pi}\,i}\frac{1}{(t-iY)}-\frac{1}{\sqrt{2\pi}\,i}\frac{1}{(t+iY)} \tag{117}$$

Then, using Equation 115

$$F.T.^{-1}\left[\frac{e^{-\alpha Y}}{-i\alpha}\right]\text{ (where }\alpha>0)$$

$$+\,F.T.^{-1}\left[\frac{e^{\alpha Y}}{-i\alpha}\right]\text{ (where }\alpha<0)$$

$$=\frac{1}{\sqrt{2\pi}\,i}\int_{-\infty}^{x}\left[\frac{1}{t-iY}-\frac{1}{t+iY}\right]dt \tag{118}$$

$$=\frac{1}{\sqrt{2\pi}\,i}\left[L_n(t-iY)-L_n(t+iY)\right]_{-\infty}^{x}$$

$$=\frac{1}{\sqrt{2\pi}\,i}\left[L_n\frac{t+iY}{t-iY}\right]_{-\infty}^{x}$$

$$=\frac{-i}{\sqrt{2\pi}}L_n\frac{x+iY}{x-iY}$$

$$=\frac{-i}{\sqrt{2\pi}}\left[L_n\sqrt{x^2+Y^2}+i\tan^{-1}\frac{Y}{x}\right.$$

$$\left.-\,L_n\sqrt{x^2+Y^2}+i\tan^{-1}\frac{Y}{x}\right]$$

$$=\frac{\sqrt{\pi}}{\sqrt{2}}\tan^{-1}\frac{Y}{x} \tag{119}$$

Then, returning to Equation 118,

$$\text{F.T.}^{-1} \left\{ \left[\frac{-y^2 \; e^{-\alpha Y}}{4\alpha}\right] \text{(where } \alpha > 0\text{)} \right.$$

$$\left. + \left[\frac{-y^2 \; e^{\alpha Y}}{4\alpha}\right] \text{(where } \alpha < 0\text{)} \right\}$$

$$= \frac{iy^2 \sqrt{\pi}}{4\sqrt{2\pi}} \tan^{-1} \frac{Y}{x}$$

$$= \frac{iy^2}{4\sqrt{2}} \tan^{-1} \frac{Y}{x} \qquad (120)$$

XIV. APPENDIX C - TABLES

Table 1. Hydrostatic and magnetic pressure gradient vs. distance L = 6.4

x	$-\frac{dP}{dx}$	$-\frac{dP_m}{dx}$
-10	-0.3330	0.1996
- 9	-0.2875	0.2277
- 8	-0.2341	0.2633
- 7	-0.1700	0.3095
- 6	-0.0904	0.3711
- 5	0.0124	0.4560
- 4	0.1517	0.5779
- 3	0.3516	0.7611
- 2	0.6509	1.0423
- 1	0.9871	1.3262
- .9	0.9947	1.3204
- .8	0.9875	1.2968
- .7	0.9613	1.2501
- .6	0.9114	1.1749
- .5	0.8331	1.0659
- .4	0.7231	0.9194
- .3	0.5802	0.7342
- .2	0.4070	0.5132
- .1	0.2100	0.2642

Table 1 (Continued)

x	$-\frac{dP}{dx}$	$-\frac{dP_m}{dx}$
0.0	0.0	0.0
.1	-0.2100	-0.2642
.2	-0.4070	-0.5132
.3	-0.5802	-0.7342
.4	-0.7231	-0.9194
.5	-0.8331	-1.0659
.6	-0.9114	-1.1749
.7	-0.9613	-1.2501
.8	-0.9875	-1.2968
.9	-0.9947	-1.3204
1	-0.9871	-1.3262
2	-0.6509	-1.0423
3	-0.3516	-0.7611
4	-0.1517	-0.5779
5	-0.0124	-0.4560
6	0.0904	-0.3711
7	0.1700	-0.3095
8	0.2341	-0.2033
9	0.2875	-0.2277

Table 1 (Continued)

x	$-\frac{dP}{dx}$	$-\frac{dP_m}{dx}$
10	0.3330	-0.1996

$\frac{dP}{dx}$ = Hydrostatic pressure gradient used in this table and Tables 1-7

$\frac{dP_m}{dx}$ = Magnetic pressure gradient used in this table and Tables 1-7

Table 2. Hydrostatic and magnetic pressure gradient vs. distance L = 9.6

x	$-\frac{dP}{dx}$	$-\frac{dP_m}{dx}$
-10	-0.0680	0.2225
- 9	-0.0300	0.2527
- 8	0.0153	0.2906
- 7	0.0706	0.3391
- 6	0.1402	0.4027
- 5	0.2311	0.4892
- 4	0.3557	0.6118
- 3	0.5366	0.7948
- 2	0.8093	1.0751
- 1	1.0997	1.3563
- .9	1.1002	1.3496
- .8	1.0853	1.3247
- .7	1.0504	1.2764
- .6	0.9907	1.1991
- .5	0.9017	1.0876
- .4	0.7798	0.9378
- .3	0.6239	0.7487
- .2	0.4367	0.5232
- .1	0.2251	0.2694
0.0	0.0	0.0

Table 2 (Continued)

x	$-\frac{dP}{dx}$	$-\frac{dP_m}{dx}$
.1	-0.2251	-0.2694
.2	-0.4367	-0.5232
.3	-0.6239	-0.7487
.4	-0.7798	-0.9378
.5	-0.9017	-1.0876
.6	-0.9907	-1.1991
.7	-1.0504	-1.2764
.8	-1.0853	-1.3247
.9	-1.1002	-1.3496
1	-1.0997	-1.3563
2	-0.8093	-1.0751
3	-0.5366	-0.7948
4	-0.3557	-0.6118
5	-0.2311	-0.4892
6	-0.1402	-0.4027
7	-0.0706	-0.3391
8	-0.0153	-0.2906
9	0.0300	-0.2527
10	0.0680	-0.2225

Table 3. Hydrostatic and magnetic pressure gradient vs. distance L = 16

x	$-\frac{dP}{dx}$	$-\frac{dP_m}{dx}$
-10	0.1075	0.2480
- 9	0.1400	0.2793
- 8	0.1793	0.3179
- 7	0.2278	0.3668
- 6	0.2896	0.4305
- 5	0.3715	0.5165
- 4	0.4856	0.6384
- 3	0.6537	0.8207
- 2	0.9102	1.1008
- 1	1.1735	1.3811
- .9	1.1697	1.3738
- .8	1.1499	1.3480
- .7	1.1095	1.2985
- .6	1.1043	1.2196
- .5	0.9476	1.1059
- .4	0.8179	0.9535
- .3	0.6535	0.7611
- .2	0.4569	0.5318
- .1	0.2353	0.2738
0.0	0.0	0.0

Table 3 (Continued)

x	$-\frac{dP}{dx}$	$-\frac{dP_m}{dx}$
.1	-0.2353	-0.2738
.2	-0.4569	-0.5318
.3	-0.6535	-0.7611
.4	-0.8179	-0.9535
.5	-0.9476	-1.1059
.6	-1.0437	-1.2196
.7	-1.1095	-1.2985
.8	-1.1499	-1.3480
.9	-1.1697	-1.3738
1	-1.1735	-1.3811
2	-0.9102	-1.1008
3	-0.6537	-0.8207
4	-0.4856	-0.6384
5	-0.3715	-0.5165
6	-0.2896	-0.4305
7	-0.2278	-0.3668
8	-0.1793	-0.3179
9	-0.1400	-0.2793
10	-0.1075	-0.2480

Table 4. Hydrostatic and magnetic pressure gradient vs. distance L = 32

x	$-\frac{dP}{dx}$	$-\frac{dP_m}{dx}$
-10	0.2038	0.2688
- 9	0.2324	0.2996
- 8	0.2674	0.3378
- 7	0.3113	0.3860
- 6	0.3683	0.4489
- 5	0.4449	0.5343
- 4	0.5533	0.6556
- 3	0.1752	0.8378
- 2	0.9644	1.1190
- 1	1.2155	1.4001
- .9	1.2095	1.3925
- .8	1.1872	1.3661
- .7	1.1440	1.3158
- .6	1.0747	1.2357
- .5	0.9748	1.1204
- .4	0.8406	0.9659
- .3	0.6711	0.7709
- .2	0.4690	0.5387
- .1	0.2415	0.2773
0.0	0.0	0.0

Table 4 (Continued)

x	$-\frac{dP}{dx}$	$-\frac{dP_m}{dx}$
.1	-0.2415	-0.2773
.2	-0.4690	-0.5387
.3	-0.6711	-0.7709
.4	-0.8406	-0.9659
.5	-0.9748	-1.1204
.6	-1.0747	-1.2357
.7	-1.1440	-1.3158
.8	-1.1872	-1.3661
.9	-1.2095	-1.3925
1	-1.2155	-1.4001
2	-0.9644	-1.1190
3	-0.7152	-0.8378
4	-0.5533	-0.6556
5	-0.4449	-0.5343
6	-0.3683	-0.4489
7	-0.3113	-0.3860
8	-0.2674	-0.3378
9	-0.2324	-0.2996
10	-0.2038	-0.2688

Table 5. Hydrostatic and magnetic pressure gradient vs. distance L = 64

x	$-\frac{dP}{dx}$	$-\frac{dP_m}{dx}$
-10	0.2336	0.2768
- 9	0.2608	0.3073
- 8	0.2943	0.3451
- 7	0.3368	0.3931
- 6	0.3922	0.4558
- 5	0.4673	0.5411
- 4	0.5742	0.6625
- 3	0.7348	0.8452
- 2	0.9826	1.1274
- 1	1.2309	1.4097
- .9	1.2243	1.4019
- .8	1.2013	1.3753
- .7	1.1571	1.3246
- .6	1.0867	1.2439
- .5	0.9854	1.1278
- .4	0.8495	0.9723
- .3	0.6781	0.7760
- .2	0.4738	0.5422
- .1	0.2440	0.2792
0.0	0.0	0.0

Table 5 (Continued)

x	$-\frac{dP}{dx}$	$-\frac{dP_m}{dx}$
.1	-0.2440	-0.2792
.2	-0.4738	-0.5422
.3	-0.6781	-0.7760
.4	-0.8495	-0.9723
.5	-0.9854	-1.1278
.6	-1.0867	-1.2439
.7	-1.1571	-1.3246
.8	-1.2013	-1.3753
.9	-1.2243	-1.4019
1	-1.2309	-1.4097
2	-0.9826	-1.1274
3	-0.7348	-0.8452
4	-0.5742	-0.6625
5	-0.4673	-0.5411
6	-0.3922	-0.4558
7	-0.3368	-0.3931
8	-0.2943	-0.3451
9	-0.2608	-0.3073
10	-0.2336	-0.2768

Table 6. Hydrostatic and magnetic pressure gradient vs. distance L = 96

x	$-\frac{dP}{dx}$	$-\frac{dP_m}{dx}$
-10	0.2399	0.2787
- 9	0.2668	0.3092
- 8	0.3000	0.3469
- 7	0.3422	0.3949
- 6	0.3973	0.4576
- 5	0.4722	0.5429
- 4	0.5789	0.6645
- 3	0.7393	0.8475
- 2	0.9871	1.1302
- 1	1.2351	1.4129
- .9	1.2284	1.4051
- .8	1.2051	1.3784
- .7	1.1607	1.3276
- .6	1.0901	1.2467
- .5	0.9884	1.1303
- .4	0.8521	0.9744
- .3	0.6801	0.7778
- .2	0.4752	0.5434
- .1	0.2447	0.2798
0.0	0.0	0.0

Table 6 (Continued)

x	$-\frac{dP}{dx}$	$-\frac{dP_m}{dx}$
.1	-0.2447	-0.2798
.2	-0.4752	-0.5434
.3	-0.6801	-0.7778
.4	-0.8521	-0.9744
.5	-0.9884	-1.1303
.6	-1.0901	-1.2467
.7	-1.1607	-1.3276
.8	-1.2051	-1.3784
.9	-1.2284	-1.4051
1	-1.2351	-1.4129
2	-0.9871	-1.1302
3	-0.7393	-0.8475
4	-0.5789	-0.6645
5	-0.4722	-0.5429
6	-0.3973	-0.4576
7	-0.3422	-0.3949
8	-0.3000	-0.3469
9	-0.2668	-0.3092
10	-0.2399	-0.2787

Table 7. Hydrostatic and magnetic pressure gradient vs. distance L = 9999

x	$-\frac{dP}{dx}$	$-\frac{dP_m}{dx}$
-10	0.2459	0.2810
- 9	0.2726	0.3115
- 8	0.3057	0.3493
- 7	0.3477	0.3974
- 6	0.4027	0.4603
- 5	0.4776	0.5459
- 4	0.5844	0.6679
- 3	0.7451	0.8515
- 2	0.9934	1.1354
- 1	1.2418	1.4192
- .9	1.2349	1.4114
- .8	1.2115	1.3846
- .7	1.1668	1.3335
- .6	1.0957	1.2522
- .5	0.9934	1.1354
- .4	0.8564	0.9788
- .3	0.6836	0.7812
- .2	0.4776	0.5459
- .1	0.2459	0.2810
0.0	0.0	0.0

Table 7 (Continued)

x	$-\frac{dP}{dx}$	$-\frac{dP_m}{dx}$
.1	-0.2459	-0.2810
.2	-0.4776	-0.5459
.3	-0.6836	-0.7812
.4	-0.8564	-0.9788
.5	-0.9934	-1.1354
.6	-1.0957	-1.2522
.7	-1.1668	-1.3335
.8	-1.2115	-1.3846
.9	-1.2349	-1.4114
1	-1.2418	-1.4192
2	-0.9934	-1.1354
3	-0.7451	-0.8515
4	-0.5844	-0.6679
5	-0.4776	-0.5459
6	-0.4027	-0.4603
7	-0.3477	-0.3974
8	-0.3057	-0.3493
9	-0.2726	-0.3115
10	-0.2459	-0.2810

Table 8. Channel width vs. magnitude of maximum hydrostatic and magnetic pressure gradient

L	$\left\|\frac{dP_m}{dx}\right\|_{max.}$	$\left\|\frac{dP}{dx}\right\|_{max.}$
6.4	1.3262	0.9947
9.6	1.3563	1.1002
16	1.3811	1.1735
32	1.4001	1.2155
64	1.4097	1.2309
96	1.4129	1.2351
9999	1.4192	1.2418

$L = \frac{L'}{y_o}$

L' is actual channel width

y_o is the characteristic length

All points are for $x = 1$ except for the first two which are for $x = 9$