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RESISTIVE N-PORT NETWORK SYNTHESIS UTILIZING LAGRANGIAN TREE STRUCTURES.

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RESISTIVE N-PORT NETWORK SYNTHESIS

UTILIZING LAGRANGIAN TREE STRUCTURES

by

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TABLE OF CONTENTS

LIST OF FIGURES

Page

I. INTRODUCTION

The problem of synthesis of a resistive n-port network from its port-admittance matrix, Y, is certainly not a new one. Much investigation has been carried on as cited in the Literature Search section; but, the general problem seems far from being solved. Even though many of the known resistive synthesis procedures and their implications can be applied to networks with R, L, and C elements the author restricts the developments to resistive elements. Since the derivations are carried out utilizing nodal analysis terminology the element values of the realized network are referred to as conductances.

Linear graph theory plays an important role in analysis and synthesis of networks. After defining many of the basic terms involved with linear graph theory and its applications much of the investigation relies upon the augmentation of the original port-admittance matrix with reference to linear transformations and cut-set notation.

Throughout the development and discussion of the realization techniques the n ports of interest corresponding to the (nxn) port-admittance matrix are arranged in a basic Lagrangian tree or in k ($2 \le k \le n$) Lagrangian subtree structures. This form of port structure lends itself nicely to nodal methods. Two forms of (n+2)-node synthesis methods are developed. One form relies entirely upon the element values of the given port-admittance matrix and requires no solving for unknown quantities. The second method depends upon the solving of a set of independent equalities and inequalities. Compared to the first method this method is a much more flexible; but, involved technique.

A third synthesis procedure, which concerns itself with the general case, is developed. Like the second method it requires satisfying a set of independent inequalities. This independent set of inequalities is derived from the augmented port-admittance matrix. The network synthesized by this procedure or by the second procedure is certainly not unique since the solution for the unknown quantities in general will be in the form of a bounded solution. Also, the initial step of each method restricts the topology of the network to basic Lagrangian tree structures.

II. TERMS AND DEFINITIONS

The language associated with linear graph theory and its applications does not always follow a precise standard. Thus, many of the terms possess various meanings depending upon the presentation or publication where they are found. In this section the author wishes to clarify the meaning of the terms or operations that are pertinent for the understanding of the material found in the following sections. The definitions that the author gives are those of the references cited and of personal preference according to the context of the following sections. They are thought to be the most standard representations.

1. (14) A matrix is a rectangular array containing m rows and n columns of elements of a scalar field F. It is called an (mxn) matrix over F.
2. (14) The transpose of matrix, $A = [a_{ij}]$, denoted by A', is defined as $A' = [b_{ij}]$ where $b_{ij} = a_{ji}$.
3. (14) An (nxn) matrix A is non-singular if and only if a matrix B, its inverse, exists such that $AB = I$. Otherwise, A is said to be singular. We denote the inverse as $B = A^{-1}$.
4. (14) A linear transformation, T, from a vector space, V, to a vector space, W, both over the scalar field, F, is a mapping of V into W such that for all $\alpha,\beta \epsilon V$ and for all $a,b \epsilon F$,

   $$(a\alpha + b\beta)T = a(\alpha T) + b(\beta T). \qquad (1)$$

   The linear transformation that is used in Section IVF is described as $Y_2 = SY_1S'$ where S' is the non-singular matrix that gives the set of voltages, V_1, associated with the admittance matrix, Y_1, in terms of the set of voltages, V_2, associated with the admittance matrix, Y_2.

5. A dominant (nxn) matrix, D, satisfies the condition that

$$d_{ii} \geq \sum_{\substack{i \neq j \\ j=1}}^{n} |d_{ij}|, \; i = 1, \cdot \cdot \cdot, n. \tag{2}$$

6. (26) Further restricting the matrix D, a hyperdominant matrix also must satisfy the condition,

$$d_{ij} \leq 0; \; i \neq j; \; i = 1, \cdot \cdot \cdot, n; \; j = 1, \cdot \cdot \cdot, n. \tag{3}$$

7. (25) A real matrix is defined as a paramount matrix if each principal minor of the matrix is not less than the absolute value of any minor built from the same rows.

8. (24) The short-circuit admittance matrix, Y, is defined as the coefficient matrix of the system of equations,

$$\begin{bmatrix} I_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ I_n \end{bmatrix} = \begin{bmatrix} y_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & y_{1n} \\ \cdot & \cdot & & & & & \cdot \\ \cdot & & \cdot & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & \cdot \\ y_{n1} & & \cdot & \cdot & \cdot & \cdot & y_{nn} \end{bmatrix} \begin{bmatrix} V_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ V_n \end{bmatrix} \tag{4}$$

where upon setting the appropriate voltages equal to zero the y_{ij} functions are equated to the current-voltage ratio. The author will sometimes refer to this matrix as the port-admittance matrix.

9. (24) A cut-set is a set of edges of a connected graph G such that the removal of these edges from G reduces the rank of G

by one, provided that no proper subset of this set reduces the rank of G by one when it is removed from G. Here the rank of a graph is defined as (v-p) where there are v vertices and p maximal connected subgraphs.

10. (24) The cut-set matrix of a graph with v vertices and e edges is given by $Q_a = [q_{ij}]$ which has one row for each cut-set of the graph and e columns, such that

$q_{ij} = 1$ if edge j is in cut-set i and the orientations agree,

$q_{ij} = -1$ if edge j is in cut-set i and the orientations are opposite, and

$q_{ij} = 0$ if the edge j is not in cut-set i.

Also, a matrix formed from (v-1) independent rows of Q_a will be labeled the cut-set matrix, Q.

11. (7) A port is an accessible terminal pair regarded as a single entity.

12. (24) A node is an endpoint of an edge. An edge is defined as a line segment together with its distinct endpoints. In a graph an edge may represent a circuit element such as a resistor, capacitor, inductor, etc. Since the author deals only with resistive networks and the derivations and synthesis procedures deal with nodal methods, in many instances an edge will be considered a conductance.

13. (13) An oriented edge is an edge with orientation shown by an arrowhead on the edge pointing away from the first node and toward the second node.

14. (13) With each network element (edge) there are two real valued functions of bounded variation of the real variable, t. These are termed <u>element</u> (<u>edge</u>) <u>voltage</u> and <u>element</u> (<u>edge</u>) <u>current</u>. The orientation that will be used for the network element and edge is shown in Figure 1.

15. (24) A <u>linear graph</u> is a collection of edges, no two of which have a point in common that is not a node.

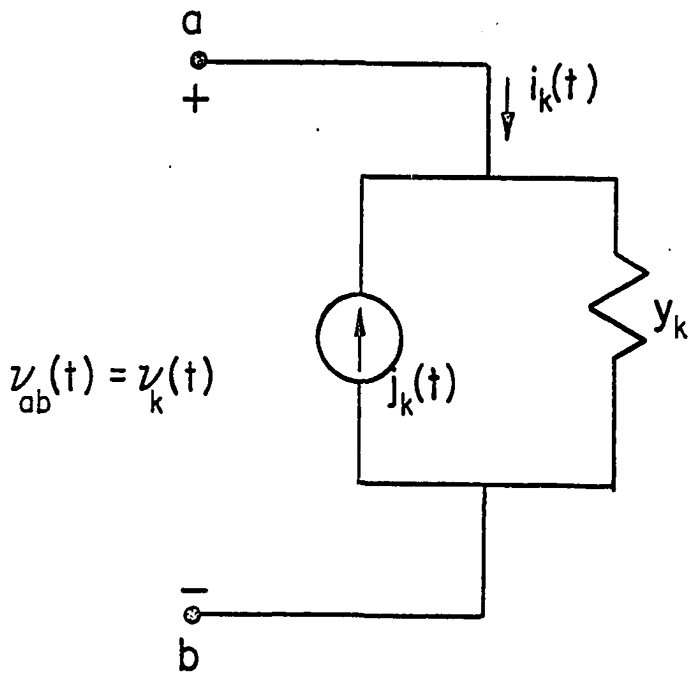
16. (24) A <u>subgraph</u> is a subset of the edges of the graph; therefore, it is a graph itself.

    (6) If a graph $G' = (V',E',\Gamma')$ is so related to a graph $G = (V,E,\Gamma)$ that $V' \subset V$, $E' \subset E$, and $\Gamma'(e) = \Gamma(e)$ for every edge $e \epsilon E'$ then the first graph is said to be a subgraph of the second.

17. (6) A <u>directed graph</u> is a mathematical system consisting of two sets V and E, together with a mapping, Δ, of E into VXV. V refers to vertices and E to edges.

18. (6) A <u>total graph</u> is a directed graph such that, for every two distinct vertices v and w, there is a path from v to w or one from w to v (or both). Here a <u>path</u> is defined as an open curve composed of consistently directed edges.

19. (6) A <u>connected graph</u> is such that every pair of distinct vertices are joined by at least one <u>chain</u>. Here a <u>chain</u> is defined as a set of edges which form an open curve. In considering only the designated set of edges, the degree of each end vertex is 1 with all other vertices in the chain each having degree 2.



(a) Circuit element representation

(b) Graph edge representation

Figure 1. Voltage and current orientation for circuit representation and graph representation

20. (6) A circuit is a set of edges which form a closed curve. In considering only these edges the degree of each vertex in the circuit is 2.

21. (13) A graph G with v vertices is a complete graph if each pair of vertices is connected by an edge (a series or parallel connection of edges is not allowed). The degree of each vertex of a complete graph is (v-1). A complete graph has $\frac{v(v-1)}{2}$ edges.

22. (6) A tree is a connected graph which has no circuits (closed paths). The author will use the terms subgraph and subtree which will refer to a connected set of nodes which are a subset of the total set of nodes.

23. A port tree is a connected set of edges each of which connects a terminal pair from which voltages and currents of interest may be measured.

24. A Lagrangian tree is a connected set of L edges corresponding to L ports of interest, all of which have a common node. Thus, a Lagrangian tree containing L ports has (L+1) nodes. The orientation of the edges, as considered by the author, will be from the + node to the - node with the common node being the negative node. An example of a Lagrangian tree is given in Figure 2. A Lagrangian subtree as used by the author is simply a subset of the n ports of interest given by the port-admittance matrix which can be grouped into a tree structure such as illustrated in Figure 2. The convention of circling the port

representations will be carried throughout the text.

25. A <u>linear tree</u> is a connected set of S edges corresponding to S ports of interest with (S+1) nodes. (S-1) of these nodes have degree two while two nodes have degree one (end nodes). An example of a linear tree is given in Figure 3.

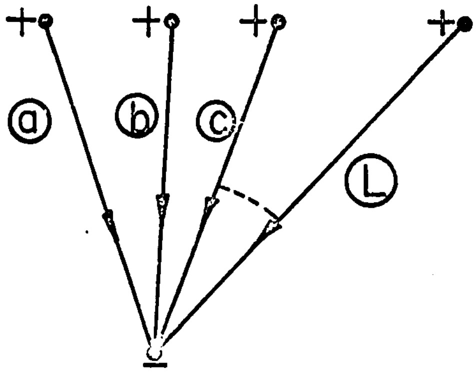


Figure 2. Basic form for the Lagrangian tree (or subtree) with L ports of interest

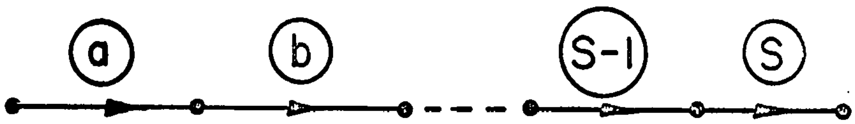


Figure 3. Basic form for the linear tree (or subtree) with S ports of interest all oriented in the same direction

III. LITERATURE SEARCH

A. Concentration on (n+1)-Node Resistive Networks

Even though much effort has been spent concerning the synthesis of n-port resistive networks the general problem seems far from being solved. The results of previous investigations concerning (n+1)-, (n+2)-, and on up to 2n-node (2n nodes are sufficient if the network is at all realizable) synthesis of resistive networks will be presented with the deserving references cited.

The (n+1)-node network is the simplest network (with reference to the minimum number of nodes) that can be realized from an (nxn) port-admittance matrix. At a first glance it seems that it would only be necessary to find a tree of the voltage variables used for defining the matrix. Indeed, this is an important step in most of the synthesis procedures. When restricting an (n+1)-node network to a Lagrangian tree structure containing the ports of interest an inspection of the matrix provides the simple necessary and sufficient conditions for realization (26). These conditions restrict the matrix to one having a hyperdominant or potentially hyperdominant form.

If the port-admittance matrix is based upon a linear tree of voltage variables then the matrix must possess the property of being a uniformly tapered matrix (or capable of being arranged in this form). This result was derived independently by Guillemin (17) and slightly later by Biorci and Civalleri (1) in their use of the sign matrix. If G_{ik} represents the typical element of an $n^{\underline{th}}$ order matrix, G, then the uniformly tapered condition is defined by

$$G_{ik} \geq 0 \tag{5}$$

and

$$G_{ij} + G_{i-1,j+1} \geq G_{i-1,j} + G_{i,j+1}, \ i \leq j, \tag{6}$$

where by definition

$$G_{i,n+1} = G_{0,j} = 0. \tag{7}$$

The preceding results concerning linear and Lagrangian trees were also justified by Brown and Tokad (5) with a slightly different form of derivation. The necessary and sufficient conditions for the realization of a given (nxn) real symmetric matrix, $A = [a_{ij}]_n$, as the short-circuit conductance matrix of a linear tree terminal graph using a complete graph of R elements with (n+1) nodes appear in the following form as found in the cited reference.

1. The sign pattern of A must be such that after a finite number of cross-sign changes, all the entries are non-negative.
2. It must be possible to find a rearrangement of rows and corresponding columns such that when $\Delta_i^n \equiv |a_{in}| - |a_{i-1,n}|$, $i \neq j$; $\Delta_i^{()} \equiv |a_{1()}|$ and $\Delta_n^{n+1} \equiv 0$; then

$$\begin{aligned}
&\Delta_i^n \geq 0 \\
&\Delta_i^{n-1} - \Delta_i^n \geq 0 \\
&\Delta_i^{n-2} - \Delta_i^{n-1} \geq 0 \\
&\quad\vdots \\
&\Delta_i^i - \Delta_i^{i+1} \geq 0 \qquad (i = 1, \cdot \cdot \cdot, n).
\end{aligned} \tag{8}$$

Also, the corresponding requirements for the realization with a Lagrangian tree are given (5).

1. The sign pattern of $A = [a_{ij}]_n$, $a_{ii} > 0$ must be such that after a finite number of cross-sign changes all the off-diagonal entries of A are non-positive.

2. $2a_{ii} \geq \sum_{j=1}^{n} |a_{ij}|$ with $i = 1, \cdot\cdot\cdot, n.$ (9)

Many of the basic features of (n+1)-node synthesis were formulated by Guillemin (17). Much of the initial work was concerned with the sign matrix. Except for row and column interchanges each sign matrix uniquely specifies a geometrical tree pattern and vice versa. It is interesting to note that the linear tree is the only one for which all signs in the port-admittance matrix are positive. Thus, if the signs of the element values of a matrix (port-admittance matrix) are all positive the matrix must be a uniformly tapered matrix or else it has no realization in an (n+1)-node network.

Guillemin's "tree-growing method" (16) recognizes that construction of a tree from a given matrix can be done by inspection once the pattern of growth is established. Therefore, a sorting method is used on a cut-set matrix that weeds out the outermost tips and twigs of the tree. Once these are removed the remainder of the matrix then possesses additional tips. The process is continued until the growth pattern is established. The method is actually regarded as a test for necessary and sufficient conditions for realization since the procedure cannot fail to yield a graph if one does indeed exist for the matrix. A detailed discussion of the tree growing method is given in the reference (16).

Once the tree structure is established then a linear transformation which carries the given matrix over to one corresponding to a linear tree or Lagrangian tree is used. The new matrix must satisfy previously mentioned conditions based upon the linear or Lagrangian tree. Thus, the existence of a tree does not assure a realization. The topological method concerning the tree growing process is certainly not fool-proof and it is possible for a matrix to be realizable even though the (n+1)-node realization fails.

Biorci and Civalleri (2) formulated several definitions and basic properties which are applicable to (n+1)-node synthesis. They proposed a topological solution for finding the tree of the graph. In their work they used the following ideas.

1. The complete tree is the set of n ports.
2. A tree-path is a path of the complete tree.
3. The mutual conductance, G_{ij}, between ports i and j is positive (negative) if the orientation of port j is the same as (opposite to) that of the tree-path [i,j].
4. If G_{ij} is positive (negative) and G_{ik} and G_{jk} have the same (opposite) sign the tree-path [i,j,k] exists.
5. Likewise, if G_{ij} is positive (negative) and G_{ik} and G_{jk} have the opposite (same) sign the tree-path [i,j,k] does not exist.
6. The theorem stating that of the two mutual conductances, G_{ac} and G_{bc}, the larger in absolute value is that between port c and that of the two ports a and b which is closer to c in the tree path (c,a,b) proves to be a very useful theorem in their work. This theorem is used to determine the order of

ports which appear in a "series" portion of the tree. All of the other ports are uniquely determined by the sign of the off-diagonal elements of the given matrix if a tree does exist.

A necessary condition that the conductance matrix, G, be realizable with (n+1) nodes is that G be expressible as

$$G = ADA' \tag{10}$$

where D is a diagonal matrix with non-negative elements, A is an E matrix (that is, a matrix whose elements and subdeterminants take on only the values $\pm$ 1 of 0), and A′ is the transpose of A (8). In other words, each element of G can be expressed as a sum of some conductances of the branches of the network (if it exists), which are non-negative. These sums must be consistent with a possible connection of the branches themselves. This method requires many computations — examining third-order determinants of G, computing subdeterminants of A, and finally determining the structure of the network from A.

The condition of paramountcy as a necessary condition for a matrix, Y or Z, to be the admittance or impedance matrix of a resistive n-port was established by Cederbaum (10). This same property is sufficient for synthesis of a resistive 3-port from its admittance or impedance matrix. A method for reducing the main-diagonal elements of a paramount matrix, leading to its irreducible form is presented by Cederbaum (10).

Topological implications of irreducibility of the admittance or impedance matrix of an n-port network are investigated further by Cederbaum in another reference (11). In special cases where (n-1) rows of an $n^{\underline{th}}$ order matrix contain a diagonal element which is equal to the absolute value of an off-diagonal element in that particular row, Cederbaum (11) proves that the conditions for realizability indicate a network which is realizable by (n+1) nodes or exactly n independent circuits. Also, paramountcy is proved not to be a sufficient condition for realizability.

Essentially, the problem of synthesis of a resistive n-port with exactly (n+1) nodes or n independent circuits may now be viewed as solved (9). However, the class of matrices whose realization in the Y or Z form may be accomplished by a straight-forward procedure is small. This group consists only of matrices decomposable in the unimodular congruence transformation with the corresponding E° matrix yielding nicely to topological methods, dominant Y matrices, and their realization in the Z form if they can be considered planar networks. All paramount matrices of order 3 fall into this class.

The idea of equivalent networks presents another problem in n-port synthesis. As stated by Cederbaum (9), if an n-port realization exists there exist an infinite number of equivalent realizations. At the time of the published reference (9) (and the author knows of no recent contributions on the subject) there was no general theory of equivalence nor a known method of getting from an algebraically feasible solution containing negative conductances to a solution represented by non-negative conductances.

At first, work with (n+1)-node synthesis was concerned only with matrices having all non-zero off-diagonal elements. When a zero off-diagonal element appears it presents a question as to what sign should be associated with the element. Cederbaum, Halkias, and Kim (12) found that k zero elements above the main diagonal presented 2^k different sign patterns thus establishing a much more complicated, but more flexible type problem. The authors of the above cited reference proposed a systematic procedure for obtaining the n-port structure of (n+1)-nodes associated with a Y matrix. Enumerating the particular features of this method the following list is given.

1. A set of necessary conditions which eliminates a large class of matrices is presented.
2. A procedure is derived for the determination of a port structure when a few or no zero elements are present in the matrix.
3. A procedure which may also be applied to a Z matrix when the matrix contains a large number of zero elements is given.
4. A procedure illustrates that more than one network each with (n+1) nodes and containing different port structures may exist for a given Y matrix.

Most of the work of Cederbaum (8, 9, 10, 11), Guillemin (16, 17), and Biorci and Civalleri (2) lead to procedures one would call realization rather than realizability criteria. In a later reference (1) Biorci and Civalleri presented a realizability criterion which is applicable to a matrix without going through the actual realization of the tree. Basic forms (which the author will not repeat here) for the sign matrix are given. If a proper arrangement of the given sign matrix can be achieved to agree with one of the basic forms then this constitutes sufficient proof of realizability.

Olivares (23) gives an algebraic approach for topological analysis and synthesis of the star-tree equilibrium immittances — node conductance matrix or loop resistance matrix. This method utilizes a complete graph on (n+1) nodes. The ideas of tie-sets, cut-sets, and incidence matrices are combined to determine directly the diagonal branch immittance submatrices from the node conductance matrix or the loop resistance matrix. Since, at the time of Olivares's investigations, the necessary and sufficient conditions for realizing the star-tree (Lagrangian tree) and linear tree node conductance matrices were known, he stressed the unknown, but seemingly similar conditions for the loop resistance matrix.

One of the more recent contributions to the area of resistive (n+1)-node synthesis problems is presented by Boesch and Youla (4). This new technique for determining the realizability of an n-port on (n+1)-nodes eliminates Cederbaum's method (10, 11), and avoids the "tree-growing" process of Guillemin (16, 17), and Biorci and Civalleri (2). It is algebraic in nature and has only one restriction — the matrix must have no zero elements. Some of the salient features of this method are listed here.

1. It uses a new equation to relate the short-circuit admittance matrix to the inverse of the connection matrix of port voltages.
2. A unique solution of the equation for the connection matrix is obtained by using a standard form for the short-circuit admittance matrix.
3. A congruence transformation for hyperdominance is checked, completing the realizability of the matrix.

4. Tip ports are recognized by a new method.

5. This latter statement is applied to the realization of a linear tree and eliminates the permutation process of getting a linearly tapered form.

Although the method was confined originally to networks with complete graphs, a later correspondence (26) from Boesch indicated that the case of the incomplete graph was solved.

In the following comments the author will try to condense the basic ideas of Boesch's method (4, 26). Noting that

$$I = YV \tag{11}$$

where Y is given, then V may be represented by

$$V = A\phi \tag{12}$$

where ϕ is a column vector representing a set of node-to-datum voltages. A is the transpose of a non-singular incidence matrix. The node-to-datum conductance matrix, Y_d, is then given by the congruence transformation,

$$Y_d = A'YA. \tag{13}$$

Also,

$$\phi = A^{-1}V \tag{14}$$

or

$$\phi = BV. \tag{15}$$

By defining the orientations of the tree branches, matrix B is found as a lower triangular matrix by use of the matrix equation

$$B + B' = \frac{\text{sgn }(Y) + U}{2} + 1_n \tag{16}$$

where U is an $n^{\underline{th}}$ order matrix, each of whose elements is +1, 1_n is the unit matrix of order n, sgn (Y) $\equiv [s_{ij}]$ is the sign matrix of

Y (s_{ij} = +1 if $Y_{ij}>0$, s_{ij} = -1 if $Y_{ij}<0$).

The elements of the lower triangular matrix, B, can be found from $B + B'$ by taking one-half of the diagonal terms and reducing all the elements above the diagonal to zero. From the topological interpretation of B, the tree of port voltages can be drawn if it exists. From this, A is obtained by inspection. Then from Equation 13, Y_d will b· hyperdominant if and only if Y is realizable. For a more complete explanation and an example the reader is referred to the cited reference (3).

B. Considerations on Augmented Admittance Matrices

Later work by Guillemin (18) involved expansion of the short-circuit admittance matrix with up to (n-1) additional rows and columns of zero elements yielding a matrix, G_{exp}. Then the augmented matrix, G_{aug}, which was to be realized, was defined as

$$G_{aug} = G_{exp} + B. \tag{17}$$

Matrix B possessed the following characteristics.

1. B was a matrix having the same order as G_{exp}.
2. The rank of B was directly dependent upon the number of rows (columns) of zero elements in G_{exp}, being exactly equal to this number.

The method gave much freedom in the construction of matrix B, but necessitated a trial and error procedure.

A geometrical consideration in conjunction with a paramount admittance matrix, Y, has been presented by Cederbaum (9). In this cited reference augmentation of the set of n ports to a linear (2n-1) port tree on a complete graph of 2n nodes leads to a system of $\frac{1}{2}n(n+1)$ equations in $q = n(2n-1)$ unknowns. All of the real solutions of this system of equations lead to equivalent n-ports. They are considered points of a manifold L in q-space, E^q. In this cited paper (9) the theory of equivalence is studied and properties of L are presented.

The (n+2)-node realization of an (nxn) port-admittance matrix necessitates the construction of a port connecting two sub-trees which in some arrangement contain the n ports of interest. Halkias and Lupo (19) have presented a realization procedure on an (n+2)-node network where (n-1) ports constitute a linear sub-tree and the remaining port forms the other sub-tree. Their method involves the realization of the $n^{\underline{th}}$ row and column with the $n^{\underline{th}}$ port removed from the other (n-1) ports. The necessary and sufficient conditions are presented for this realization. In Section IVA the author further investigates this realization technique based upon the Lagrangian tree formulation.

The nonlinear nature of the problem involved with (n+2)-node synthesis prompted the investigation by Jambotkar and Tokad (21). An arrangement of (n+1) resistors connected to a particular node in the network represented the unknown parameters. The conclusions from this reference are based on linear subtrees with the above mentioned connecting resistors providing a means to control the number of resistors in the network.

Frisch and Swaminathan (15) have formulated a new set of necessary conditions for the (nxn) port-admittance matrix to be realized on (n+2) nodes. All of the procedures are based upon the fact that the two trees possess linear structures. The derived "supremacy" conditions represent a set of inequalities involving products of pairs of elements in a matrix, S, which is obtained from a linear combination of entries from Y. The matrix Y is augmented by the addition of one port to form a connected port structure (one linear tree consisting of the connecting port and the two linear trees). The uniformly tapered conditions are then applied. A statement concerning the extension of the method to the case of $(n+p)$ nodes $(2<p\leq n)$ is also given.

A new and enlightening approach to resistive $(n+2)$-node synthesis is presented by Halkias and Lupo (20). The method relies heavily upon the principles of $(n+1)$-node synthesis and is based upon the concept of paralleling three networks. One network containing both positive and negative conductances establishes the intersubtree transmission properties. The second network, sometimes called a null network, provides zero intersubtree transmission conductances, but realizes parasitic conductances between the ports of a given subtree. The third network completes the requirement of the subtrees and realizes the remaining portion of the given port-admittance matrix. Since the properties of the linear tree are well established, the port voltages associated with the Y matrix are transformed over to the voltages associated with the linear tree. The synthesis method is quite complicated and definitely needs more extensive development if applied to $(n+p)$ nodes, where p satisfies $2<p\leq n$. As cited by Halkias and Lupo (20) other investigators unknown to the author, are investigating the generalization of this technique and are

1. determining the necessary and sufficient conditions on admittance parameter matrices for their realizations on

multitree-port structures;

2. deriving a systematized procedure for the determination of multitree-port structures for (nxn) short-circuit admittance matrices; and
3. obtaining a more systematized procedure for synthesizing intersubtree null networks.

IV. REALIZATIONS OF RESISTIVE NETWORKS FROM PORT-ADMITTANCE MATRICES

A. Two-Tree Synthesis Process with One Tree Containing Only One Port

The realization of an $n^{\underline{th}}$ order short-circuit admittance matrix, Y, corresponding to n ports of interest with a circuit configuration consisting of (n+2) nodes requires the addition of one port. This port, which, in effect, is the addition of a cut-set, provides the connecting link between the two trees which completely contain the n ports of interest. The general port structure illustrating the connecting port, R_1, is shown in Figure 4.

The $\overline{Y}$ admittance matrix defined by this (n+2)-node structure of Figure 4 is represented in matrix form as

$$\overline{Y} = \begin{bmatrix} \overline{y}_{11} & \overline{y}_{12} & \cdots & \overline{y}_{1n} & \overline{y}_{1r} \\ \overline{y}_{21} & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ \overline{y}_{n1} & \cdots & & \overline{y}_{nn} & \vdots \\ \overline{y}_{r1} & \cdots & \cdots & \cdots & \overline{y}_{rr} \end{bmatrix} \tag{18}$$

where the column matrix $\overline{y}_{ir}$, i = 1, ⋯, n, r, and its transpose $\overline{y}_{rj}$, j = 1, ⋯, n, r, represent the matrix elements corresponding to the additional port, R_1. Matrix $\overline{Y}$ has dimensions (n+1) x (n+1). Also, both $\overline{Y}$ and Y are symmetric matrices. The partitioning of $\overline{Y}$ as in Equation 19

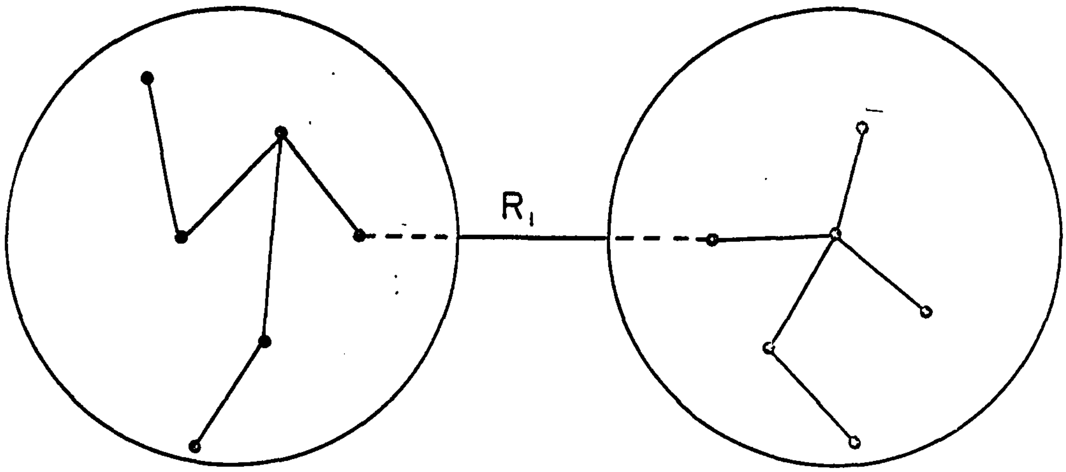


Figure 4. A general port tree structure composed of two subtrees connected by an additional port, R_1, which may connect any node of one subtree to any node of the other subtree

corresponds to the partitions represented by Equation 20. In this particular instance $\overline{Y}_{AB}$ and $\overline{Y}_{BA}$ are column and row submatrices with $\overline{Y}_{BB}$ being a (1x1) submatrix.

$$\overline{Y} = \left[\begin{array}{c|c} \overline{Y}_{AA} & \overline{Y}_{AB} \\ \hline \overline{Y}_{BA} & \overline{Y}_{BB} \end{array}\right] \tag{19}$$

$$\overline{Y} = \left[\begin{array}{ccccc|c} \overline{y}_{11} & \overline{y}_{12} & \cdots & \cdots & \overline{y}_{1n} & \overline{y}_{1r} \\ \overline{y}_{21} & \overline{y}_{22} & & & \vdots & \vdots \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ \overline{y}_{n1} & \cdots & \cdots & \cdots & \overline{y}_{nn} & \vdots \\ \hline \overline{y}_{r1} & \cdots & \cdots & \cdots & \cdots & \overline{y}_{rr} \end{array}\right] \tag{20}$$

To determine the relationships of the elements in the $\overline{Y}$ matrix to the elements in the original Y matrix and thus observe the effects of adding the connecting port, R_1, we may perform a pivotal condensation on $\overline{y}_{rr}$ or $\overline{Y}_{BB}$ and find that

$$\overline{Y}_{AA} - \overline{Y}_{AB}\,\overline{Y}_{BB}^{-1}\,\overline{Y}_{BA} = Y \tag{21}$$

or that

$$\overline{y}_{ij} - \frac{(\overline{y}_{ir})^2}{\overline{y}_{rr}} = y_{ij};\ i=j;\ i=1,\ \cdots,\ n$$

and (22)

$$\overline{y}_{ij} - \frac{(\overline{y}_{ir})(\overline{y}_{rj})}{\overline{y}_{rr}} = y_{ij};\ i\neq j;\ j>i;\ i=1,\ \cdots,\ n;\ j=1,\ \cdots,\ n$$

where

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & \cdots & y_{1n} \\ y_{21} & y_{22} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ y_{n1} & \cdots & \cdots & \cdots & y_{nn} \end{bmatrix} \tag{23}$$

$\overline{Y}_{AB} = \overline{Y}'_{BA}$ need not be restricted to column matrices as indicated by Equation 20. In network analysis the pivotal condensation manipulation as defined by Equations 21 and 22 permits the deletion of one or more cut-sets (ports). This, in turn, reduces the order of the admittance matrix by the number of cut-sets deleted. The new admittance matrix is then related to a new network configuration with the appropriate nodes of the original network which correspond to a deleted port being coalesced by that port deletion. One may reduce an (n+k)x(n+k) admittance matrix to an (nxn) admittance matrix with the removal of the k cut-sets and still retain the voltage-current characteristics of the n ports of interest.

The pivotal condensation operation also becomes a very valuable tool in the synthesis problem. It permits the investigator to view the effects of adding cut-sets (ports) to an original network configuration. However, as shown in Equations 20, 21, and 22, the addition of one port to an (nxn) port-admittance matrix requires the determination of (n+1) unknown quantities which satisfy the relationships of Equation 22. This necessary condition assures that the input-output features of the n ports are maintained.

For (nxn) short-circuit admittance matrices which can be partitioned into a special form, such as Equation 24, having dominant (n-1)x(n-1) and (1x1) submatrices on the diagonal with a dominant $n^{\underline{th}}$ row and column, the Slepian-Weinberg procedure (19) provides for a possible network realization. This procedure excludes having to solve for the (n-1) unknowns. It provides for the addition of a "fictitious" port which connects the one port corresponding to the (1x1) diagonal submatrix to the remaining ports which are contained in a predetermined tree structure.

$$Y = \left[\begin{array}{ccccc|c} y_{11} & y_{12} & \cdots & \cdots & y_{1,n-1} & y_{1n} \\ y_{21} & \cdot & & & \vdots & \vdots \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ y_{n-1,1} & \cdots & \cdots & \cdots & y_{n-1,n-1} & \vdots \\ \hline y_{n1} & \cdots & \cdots & \cdots & \cdots & y_{nn} \end{array}\right] = \left[\begin{array}{c|c} Y_{AA} & Y_{AB} \\ \hline Y_{BA} & Y_{BB} \end{array}\right] \tag{24}$$

In this section the following initial restrictions will be applied to the matrix, Y, providing for the utilization of the Slepian-Weinberg method.

1. Matrix Y must be partitionable into a form with two matrices lying on the diagonal which must correspond to two Lagrangian subtree graphs containing (n-1) ports and one port respectively. According to the definitions of Section II these two submatrices must have hyperdominant forms.

2. $$\sum_{i=1}^{n-1} |y_{in}| \le y_{nn}. \tag{25}$$

3. $y_{in} \leq 0$ for $i = 1, \cdots, (n - 1)$. (26)

The proper form for the Lagrangian subtree structures and the connecting port together with the orientation of ports is shown in Figure 5. Before proceeding with the synthesis process we must be able to justify two important facts pertaining to the selected type of tree structures.

1. The elements of the $[(n-1)x1]$ submatrix, Y_{AB}, and those of its $[1x(n-1)]$ transpose, Y_{BA}, are unaffected by the Lagrangian subtree structure corresponding to submatrix Y_{AA}.
2. The realization of elements in Y_{AB} and Y_{BA} do not cause parasitic realizations on the (n-1) Lagrangian subtree such that negative conductances are required to satisfy the total realization of Y.

Taking into consideration that the off-diagonal elements in the n^{th} column and row are negative, the Slepian-Weinberg procedure allows us to form the 2n-node network of Figure 6 for the realization of the elements in row and column n. Later it will be shown that the possibility of a positive element may arise in row and column n if the orientation of a port in the (n-1)-port Lagrangian subtree allows it to occur.

To show that $Y_{AB} = Y_{BA}'$ is unaffected by a port structure associated with a Lagrangian tree configuration, a node from each of the (n-1) ports can be coalesced into one common node with the tree structure of Figure 7 evolving. The fictitious port, R_1, and connecting branches are shown. Entering of the appropriate cut-sets on this graph allows the formation of the port-admittance matrix associated with the realization of the n^{th} row and column of matrix Y.

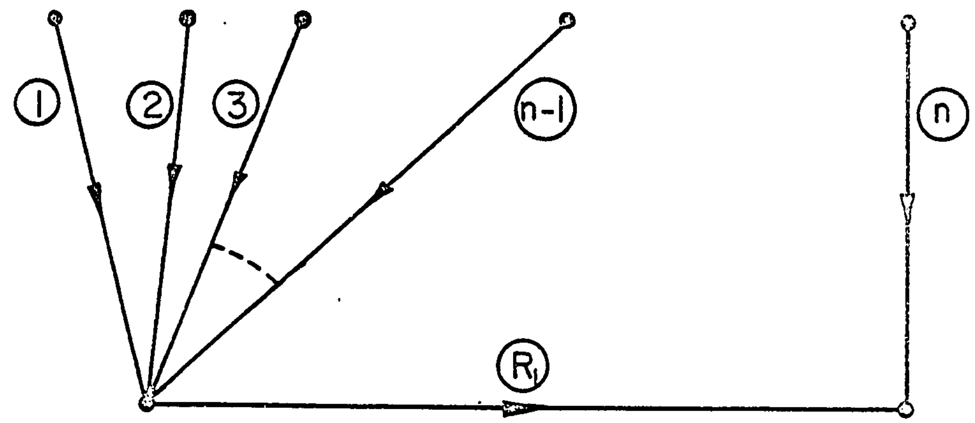


Figure 5. Basic form for port structure with one isolated port

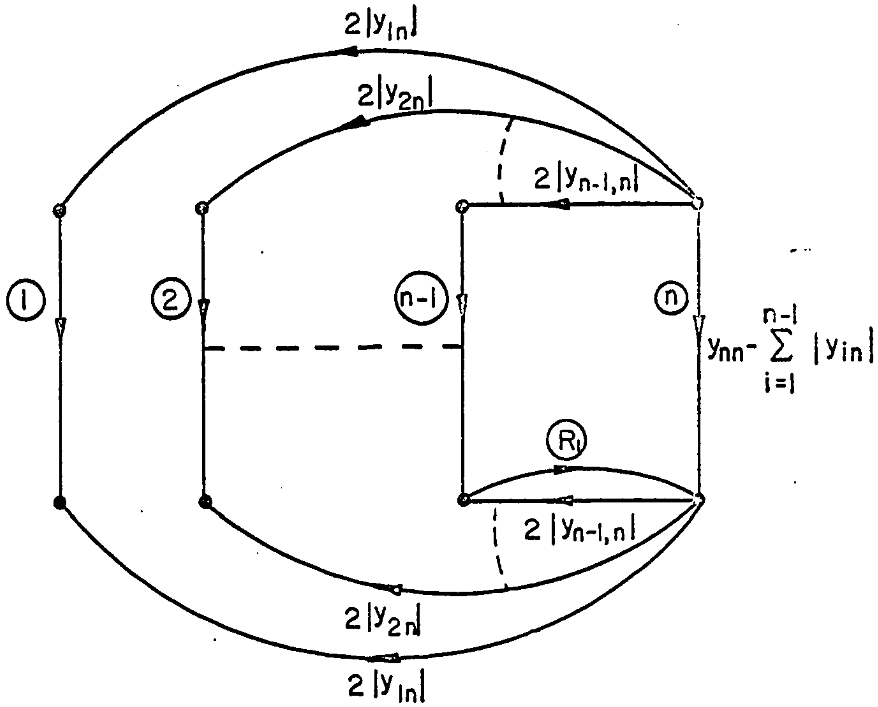


Figure 6. 2n-node realization of the $n^{\underline{th}}$ row and column of matrix Y

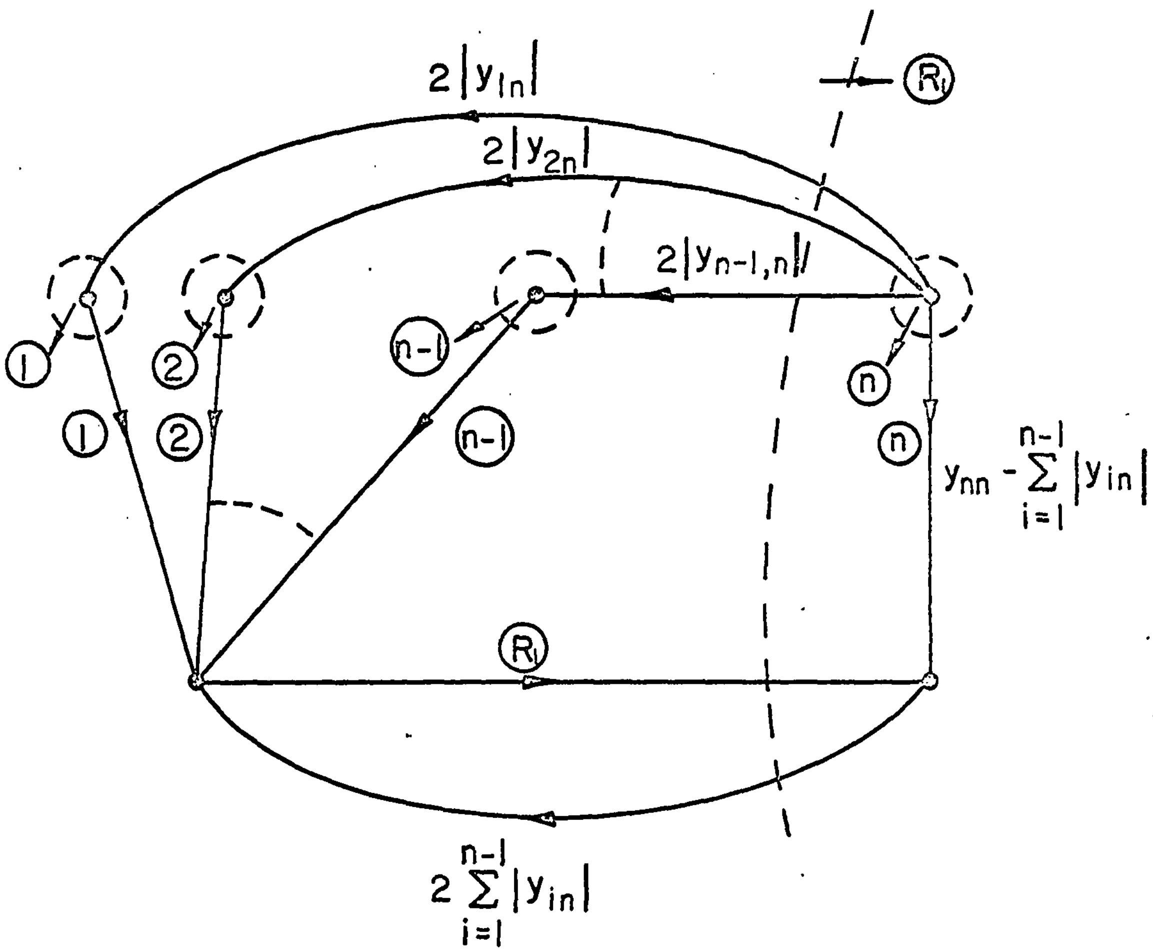


Figure 7. Realization of the $n^{\underline{th}}$ row and column of matrix Y with (n-1) nodes coalesced into one common node

The new port-admittance matrix, $\overline{Y}_{R_1}$, which includes the additional port, R_1, appears as

$$\overline{Y}_{R_1} = \begin{bmatrix} +2|y_{1n}| & 0 & \cdots & 0 & -2|y_{1n}| & +2|y_{1n}| \\ 0 & +2|y_{2n}| & 0 \cdots & 0 & -2|y_{2n}| & +2|y_{2n}| \\ \vdots & 0 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & +2|y_{n-1,n}| & -2|y_{n-1,n}| & +2|y_{n-1,n}| \\ -2|y_{1n}| & -2|y_{2n}| & \cdots & -2|y_{n-1,n}| & (y_{nn}+\sum_{i=1}^{n-1}|y_{in}|) & (-2\sum_{i=1}^{n-1}|y_{in}|) \\ +2|y_{1n}| & +2|y_{2n}| & \cdots & +2|y_{n-1,n}| & (-2\sum_{i=1}^{n-1}|y_{in}|) & (4\sum_{i=1}^{n-1}|y_{in}|) \end{bmatrix}. \tag{27}$$

By performing a pivotal condensation (Equation 22) on the (n+1)x(n+1) element of matrix $\overline{Y}_{R_1}$, the involvement of parasitic terms with the (n-1)x(n-1) submatrix corresponding to the (n-1)-port Lagrangian subtree becomes apparent. Also, the unperturbed values of the elements in the $n^{\underline{th}}$ row and column are observed. Thus, we may say that the $n^{\underline{th}}$ row and column are realized by the branch conductances of Figure 7. The manipulations explained in this paragraph are now illustrated.

The notation, $\overline{Y}^{P}_{R_1}$, indicates that a pivotal condensation has been performed on $\overline{Y}_{R_1}$ with respect to the row and column corresponding to port R_1. Thus, with $\overline{Y}_{R_1}$ being symmetrical

$$\bar{Y}^P_{R_1} = \begin{bmatrix} (+2|y_{1n}| - \dfrac{|y_{1n}|^2}{\sum\limits_{i=1}^{n-1} |y_{in}|}) & (- \dfrac{|y_{1n}||y_{2n}|}{\sum\limits_{i=1}^{n-1} |y_{in}|}) & \cdots\cdots & (-|y_{1n}|) \\ (- \dfrac{|y_{1n}||y_{2n}|}{\sum\limits_{i=1}^{n-1} |y_{in}|}) & (+2|y_{2n}| - \dfrac{|y_{2n}|^2}{\sum\limits_{i=1}^{n-1} |y_{in}|}) & \cdots\cdots & (-|y_{2n}|) \\ \vdots & \vdots & \ddots & \vdots \\ (-|y_{1n}|) & (-|y_{2n}|) & \cdots\cdots & (y_{nn}) \end{bmatrix}. \tag{28}$$

Here it is obvious that the $n^{\underline{th}}$ row and column of $\bar{Y}^P_{R_1}$ are identical to the $n^{\underline{th}}$ row and column of Y. At this step of the synthesis process we can remove the portion of the network that is now realized (all branch conductances involved in cut-sets n and R_1 on Figure 7). This portion of the realization will be added in parallel later to yield the total realization of matrix Y. Using a simplified symbol notation for $\bar{Y}^P_{R_1}$ we have in Equation 29 that

$$\bar{Y}^P_{R_1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}. \tag{29}$$

At this point it is appropriate to note that

$$Y - \bar{Y}_{R_1}^{P} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{30}$$

where

$$A_{11} = Y_{AA} - P_{11}$$
$$A_{12} = Y_{AB} - P_{12} = 0$$
$$A_{21} = Y_{BA} - P_{21} = 0$$
$$A_{22} = Y_{BB} - P_{22} = 0. \tag{31}$$

The parasitic effects that the realization of the $n^{\underline{th}}$ row and column of Y have upon the conductance values associated with the (n-1)-port Lagrangian subtree are obtained from the $\bar{Y}_{R_1}^{P}$ matrix, Equation 28, and shown in Figure 8. The magnitude of the $ij^{\underline{th}}$ element, $i \neq j$, of $\bar{Y}_{R_1}^{P}$ corresponds to the value of conductance connecting the noncommon nodes of the $i^{\underline{th}}$ and $j^{\underline{th}}$ ports. The parasitic driving-point conductance, L_a, is derived from matrix P_{11} as

$$p_{ii} - \sum_{\substack{i \neq j \\ j=1}}^{(n-1)} |p_{ij}| = L_a,\ a = i,\ i = 1, \cdot \cdot \cdot, (n-1)$$

or

$$\frac{1}{\sum_{i=1}^{n-1} |y_{in}|} [2|y_{an}|(\sum_{i=1}^{n-1} |y_{in}|) - |y_{an}|^2$$

$$- |y_{an}|(\sum_{i=1}^{a-1} |y_{in}| + \sum_{i=a+1}^{n-1} |y_{in}|)] = L_a,\ a = 1, \cdot \cdot \cdot, (n-1). \tag{32}$$

Factoring the term $|y_{an}|$, we have

Legend

a. $\dfrac{|y_{1n}||y_{n-1,n}|}{\sum_{i=1}^{n-1} |y_{in}|}$

b. $\dfrac{|y_{1n}||y_{3n}|}{\sum_{i=1}^{n-1} |y_{in}|}$

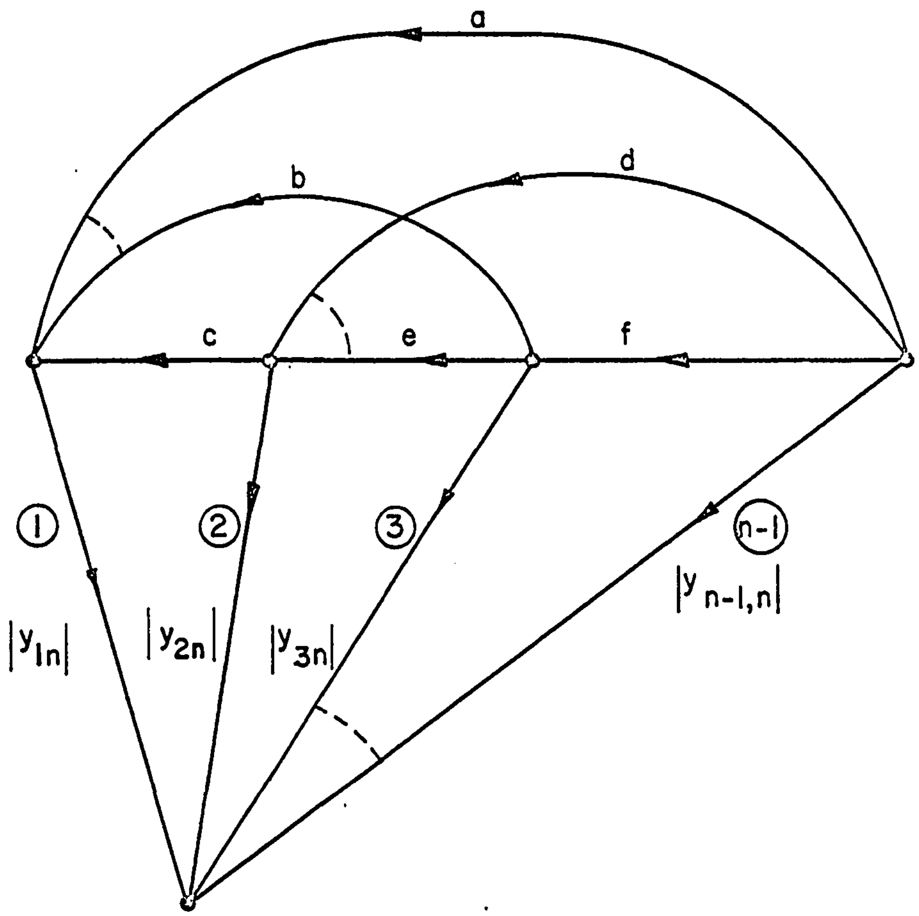
c. $\dfrac{|y_{1n}||y_{2n}|}{\sum_{i=1}^{n-1} |y_{in}|}$

d. $\dfrac{|y_{2n}||y_{n-1,n}|}{\sum_{i=1}^{n-1} |y_{in}|}$

e. $\dfrac{|y_{2n}||y_{3n}|}{\sum_{i=1}^{n-1} |y_{in}|}$

f. $\dfrac{|y_{3n}||y_{n-1,n}|}{\sum_{i=1}^{n-1} |y_{in}|}$

Figure 8. Parasitic realization of n^{th} row and column of matrix Y



a
b
d
c
e
f
1
2
3
n-1
|y1n|
|y2n|
|y3n|
|yn-1,n|

$$\frac{|y_{an}|}{\sum_{i=1}^{n-1}|y_{in}|}[2\sum_{i=1}^{n-1}|y_{in}|-|y_{an}|-\sum_{i=1}^{a-1}|y_{in}|-\sum_{i=a+1}^{n-1}|y_{in}|]$$

$$= L_a,\ a = 1, \cdot \cdot \cdot, (n-1). \tag{33}$$

The terms within the bracket of Equation 33 are easily reducible to $\sum_{i=1}^{n-1}|y_{in}|$, thus giving

$$\frac{|y_{an}|}{\sum_{i=1}^{n-1}|y_{in}|}[\sum_{i=1}^{n-1}|y_{in}|] = |y_{an}| = L_a,\ a = 1, \cdot \cdot \cdot, (n-1). \tag{34}$$

All of the preceding steps in the development of the synthesis procedure assures the important aspect of positive conductances. The final step must now be completed. From Equation 30, the remaining portion of the (nxn) port-admittance matrix to be realized corresponds to the A_{11} submatrix. Once again, to insure having positive conductances for this portion of the network, the matrix A_{11} must satisfy the necessary and sufficient conditions for realization upon the (n-1)-port Lagrangian subtree — A_{11} must be hyperdominant. In terms of the elements of A_{11}, this means that

1. $a_{ii} \geq \sum_{\substack{i \neq j \\ j=1}}^{n-1}|a_{ij}|;\ i = 1, \cdot \cdot \cdot, (n-1)$ and (35)

2. $a_{ij} \leq 0;\ i \neq j;\ i = 1, \cdot \cdot \cdot, (n-1);\ j = 1, \cdot \cdot \cdot, (n-1).$ (36)

The matrix A_{11} appears as

$$A_{11} = \begin{bmatrix} (y_{11}-2|y_{1n}| + \frac{|y_{1n}|^2}{\sum_{i=1}^{n-1}|y_{in}|}) & (y_{12} + \frac{|y_{1n}||y_{2n}|}{\sum_{i=1}^{n-1}|y_{in}|}) & \cdots\cdots & (y_{1,n-1} + \frac{|y_{1n}||y_{n-1,n}|}{\sum_{i=1}^{n-1}|y_{in}|}) \\ (y_{21} + \frac{|y_{1n}||y_{2n}|}{\sum_{i=1}^{n-1}|y_{in}|}) & (y_{22}-2|y_{2n}| + \frac{|y_{2n}|^2}{\sum_{i=1}^{n-1}|y_{in}|}) & \cdots\cdots & (y_{2,n-1} + \frac{|y_{2n}||y_{n-1,n}|}{\sum_{i=1}^{n-1}|y_{in}|}) \\ \vdots & \vdots & \ddots & \vdots \\ (y_{n-1,1} + \frac{|y_{1n}||y_{n-1,n}|}{\sum_{i=1}^{n-1}|y_{in}|}) & (y_{n-1,2} + \frac{|y_{2n}||y_{n-1,n}|}{\sum_{i=1}^{n-1}|y_{in}|}) & \cdots & (y_{n-1,n-1}-2|y_{n-1,n}| + \frac{|y_{n-1,n}|^2}{\sum_{i=1}^{n-1}|y_{in}|}) \end{bmatrix}. \qquad (37)$$

Also, from the basic definition of Y it is shown that $y_{ij} \leq 0$, $i \neq j$. Thus, the off-diagonal elements of the A_{11} matrix must satisfy the following inequality.

$$y_{ab} + \frac{|y_{an}||y_{bn}|}{\sum_{i=1}^{n-1}|y_{in}|} \leq 0;\ a \neq b;\ a=1, \cdots, (n-1);\ b=1, \cdots, (n-1). \qquad (38)$$

Another necessary condition for realizability is that the diagonal elements of A_{11} satisfy Inequality 39,

$$y_{aa} + \frac{|y_{an}|^2}{\sum_{i=1}^{n-1} |y_{in}|} \geq 2|y_{an}|, \ a = 1, \cdot \cdot \cdot, (n-1). \tag{39}$$

Utilizing the conditions derived in Inequalities 38 and 39 in satisfying Inequality 35 it is shown that

$$y_{aa} - 2|y_{an}| + \frac{|y_{an}|^2}{\sum_{i=1}^{n-1} |y_{in}|} - \sum_{\substack{b=1 \\ b\neq a}}^{n-1} \left| y_{ab} + \frac{|y_{an}||y_{bn}|}{\sum_{i=1}^{n-1} |y_{in}|} \right| \geq 0, \ a = 1, \cdot \cdot \cdot, (n-1). \tag{40}$$

If Inequality 38 is satisfied then Inequality 40 may be written in a slightly different form as shown by

$$y_{aa} - 2|y_{an}| + \frac{|y_{an}|^2}{\sum_{i=1}^{n-1} |y_{in}|} + \sum_{\substack{b=1 \\ b\neq a}}^{n-1} \left(y_{ab} + \frac{|y_{an}||y_{bn}|}{\sum_{i=1}^{n-1} |y_{in}|} \right) \geq 0, \ a = 1, \cdot \cdot \cdot, (n-1). \tag{41}$$

Rearranging Inequality 41 gives

$$y_{aa} + \sum_{\substack{b=1\\b\neq a}}^{n-1} y_{ab}$$

$$- \frac{2|y_{an}|\sum_{i=1}^{n-1}|y_{in}| + |y_{an}|^2 + \sum_{\substack{b=1\\b\neq a}}^{n-1}|y_{an}||y_{bn}|}{\sum_{i=1}^{n-1}|y_{in}|} \geq 0$$

$$y_{aa} + \sum_{\substack{b=1\\b\neq a}}^{n-1} y_{ab} - \frac{|y_{an}|\sum_{i=1}^{n-1}|y_{in}|}{\sum_{i=1}^{n-1}|y_{in}|} \geq 0$$

or

$$y_{aa} + \sum_{\substack{b=1\\b\neq a}}^{n-1} y_{ab} - |y_{an}| \geq 0, \; a = 1, \cdot\cdot\cdot, (n-1). \tag{42}$$

Noting that $y_{ab} \leq 0$, $a \neq b$, it is observed that if a port-admittance matrix, Y, is realizable upon (n+2) nodes utilizing the Slepian-Weinberg procedure with Lagrangian subtree formulation, then the original Y matrix must be of a hyperdominant form. Just as before, when the P_{11} matrix was realized on the (n-1)-port Lagrangian subtree graph to illustrate the parasitic effect that the $n^{\underline{th}}$ row and column realization gave, the A_{11} matrix can now be realized. The transfer conductance between the noncommon nodes of the $i^{\underline{th}}$ and $j^{\underline{th}}$ ports are designated by the magnitude of the $ij^{\underline{th}}$ element of A_{11} given by Equation 43.

$$a_{ij} = \left| y_{ij} + \frac{|y_{in}||y_{jn}|}{\sum_{i=1}^{n-1}|y_{in}|} \right|; \; i=1, \cdots, (n-2); \; j=2, \cdots, (n-1);$$

$$i \neq j; \; j > i. \tag{43}$$

Likewise, the driving-point conductances are given by Inequality 40 when the left side is set equal to S_a, $a = 1, \cdots, (n-1)$. The network corresponding to the realization of matrix A_{11} is shown in Figure 9.

Superimposing the two networks that have been realized upon the same set of nodes one obtains the network of Figure 10. This is the full realization of the port-admittance matrix, Y, on (n+2) nodes.

As mentioned previously in this section, a possible sign pattern might develop on the off-diagonal elements of the port-admittance matrix, Y, such that a star arrangement of the ports could still be realizable. Such an arrangement might be viewed as follows. The construction of a tree containing n ports with all ports coalescing at a common node and with orientation toward this common node satisfies the sign matrix of Equation 44.

$$S = \begin{bmatrix} + & - & - & \cdot & \cdot & \cdot & \cdot & \cdot & - \\ - & + & - & \cdot & \cdot & \cdot & \cdot & \cdot & - \\ - & - & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & & & \cdot & \cdot \\ - & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + \end{bmatrix} \qquad (44)$$

If the direction is reversed on a port of this star tree the signs in the row and column corresponding to this port are changed also. Thus, with a given Y matrix possessing both positive and negative off-diagonal elements, if there is a possible port (or ports) direction change such that the sign pattern of Y (Equation 44) is satisfied, the matrix has possibilities of being realizable on this port structure. Of course, the dominant condition must be satisfied also. The above mentioned remarks

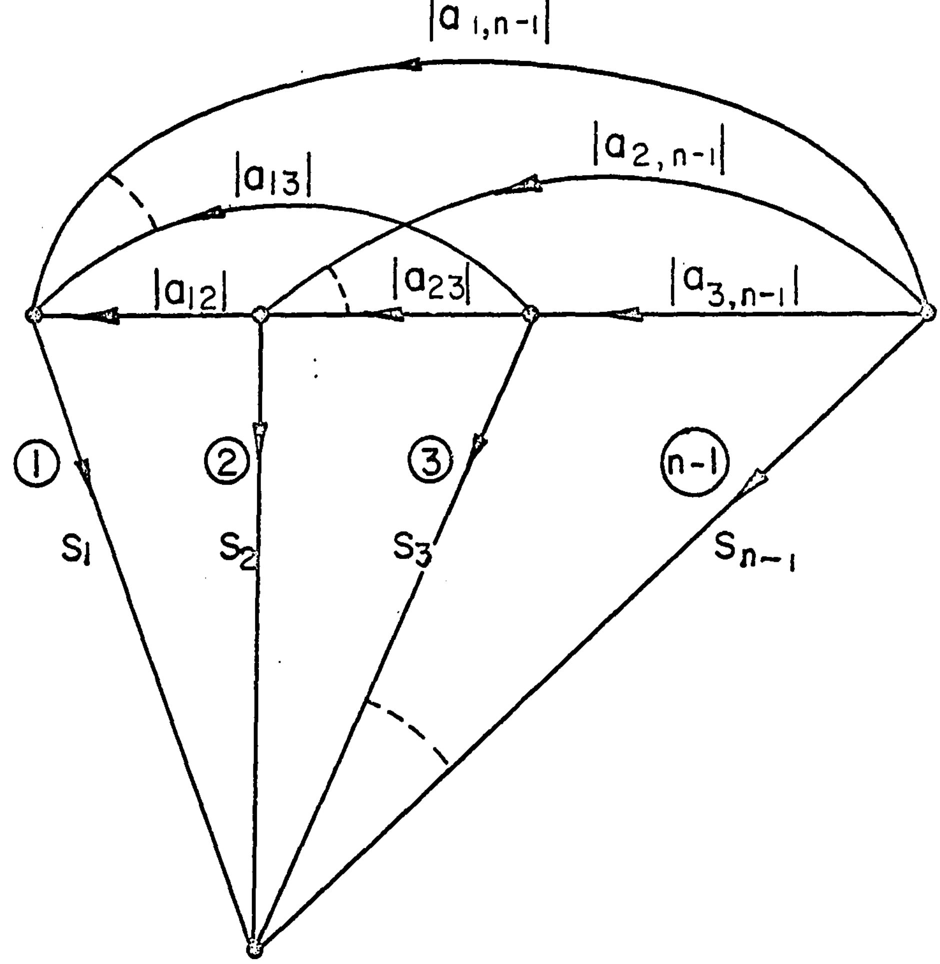


Figure 9. Realization of matrix A_{11}

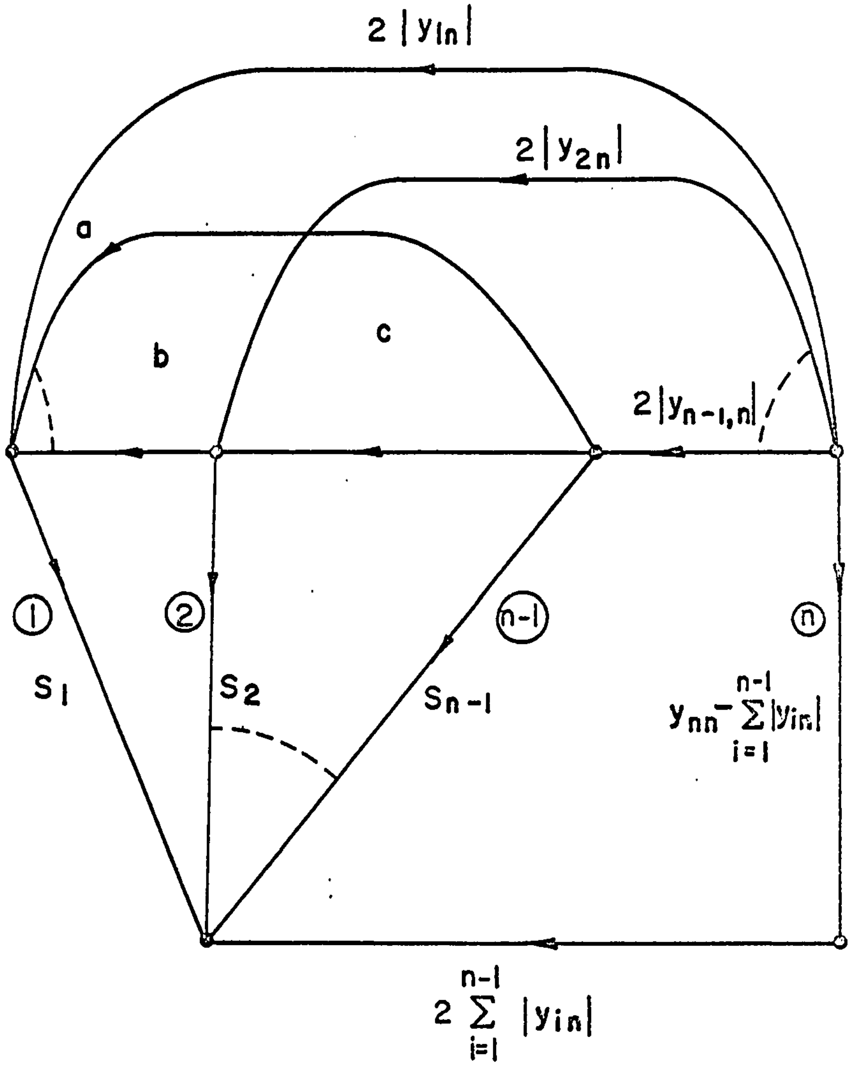
Legend

a. $\left| y_{1,n-1} + \frac{|y_{1n}||y_{n-1,n}|}{\sum_{i=1}^{n-1} |y_{in}|} \right|$

b. $\left| y_{12} + \frac{|y_{1n}||y_{2n}|}{\sum_{i=1}^{n-1} |y_{in}|} \right|$

c. $\left| y_{2,n-1} + \frac{|y_{2n}||y_{n-1,n}|}{\sum_{i=1}^{n-1} |y_{in}|} \right|$

Figure 10. A realization of the port-admittance matrix, Y, on (n+2) nodes



2 |y1n|
2 |y2n|
a
b
c
2 |yn-1,n|
1
2
n-1
n
S1
S2
Sn-1
ynn − Σ i=1 n-1 |yin|
2 Σ i=1 n-1 |yin|

are referred to as a series of cross-sign changes by Brown and Tokad (5). This idea was discussed previously in the Literature Search Section.

All of the synthesis procedures outlined in this section will still apply to a Y matrix with (+) and (-) off-diagonal elements if a finite number of cross-sign changes yield the sign matrix of Equation 44. In retaining the original sign matrix for Y with the corresponding orientation of ports the realization of the $n^{\underline{th}}$ row and column of Y is the same except for Inequality 26, $y_{in} \leq 0$, $i = 1, \cdot\cdot\cdot, (n-1)$. This condition is not necessary if the direction associated with the $2|y_{in}|$ element agrees or disagrees with the orientation of both cut-set i and cut-set n.

The sign pattern for the elements of $\overline{Y}^{P}_{R_1}$ will be the same as the sign pattern for Y. Thus, the parasitic effects will not be altered by a sign change in the Y matrix if all other conditions are agreeable for realization. Subtraction of $\overline{Y}^{P}_{R_1}$ from Y to determine the remaining portion of the port-admittance matrix, Y, to be realized will be performed in exactly the same manner as in Equation 30. If port i and port j form a linear subtree with port orientations in the same direction, then $y_{ij} \geq 0$ and the $ij^{\underline{th}}$ element of A_{11} will appear as

$$y_{ij} - \frac{|y_{in}||y_{jn}|}{\sum_{i=1}^{n-1} |y_{in}|}; \; i \neq j; \; i=1, \cdots, (n-1); \; j=1, \cdots, (n-1). \qquad (45)$$

Thus, Inequality 38 must be changed so as to read

$$y_{ab} - \frac{|y_{an}||y_{bn}|}{\sum_{i=1}^{n-1} |y_{in}|} \geq 0; \; a \neq b; \; a=1, \cdots, (n-1); \; b=1, \cdots, (n-1) \qquad (46)$$

when the sign pattern demands it. If Inequality 46 is satisfied then Inequality 40 will be consistent regardless of the sign pattern of Y. This is true because the value of the term

$$- \sum_{\substack{b=1 \\ b\neq a}}^{n-1} |y_{ab} + \frac{|y_{an}||y_{bn}|}{\sum_{i=1}^{n-1} |y_{in}|} |$$

doesn't change whether the directions of ports a and b are opposite or the same with respect to their common node.

B. Example One Illustrating Qualified Port Removals from Lagrangian Tree Structures

An example will illustrate the fact that not just any one of the n ports can be removed from an (n+1)-node realization involving an n-port Lagrangian tree by the Slepian-Weinberg procedure and still yield a realization with all positive conductances. From this example and other port-admittance matrices studied by the author a possible conjecture might be that a likely possibility for a port removal is a port associated with the off-diagonal element of smallest magnitude.

For the example the port-admittance matrix is given by Equation 47.

$$Y = \begin{bmatrix} 8 & -1 & -4 & -1 \\ -1 & 7 & -1/2 & -3 \\ -4 & -1/2 & 8 & -1 \\ -1 & -3 & -1 & 10 \end{bmatrix} \tag{47}$$

This matrix satisfies the sufficient and necessary conditions for realiza- by an (n+1)-node network as shown in Figure 11.

The (n+2)-node realization with the removal of port 4 progresses as follows. Column 4 and row 4 are realized by the network of Figure 12.

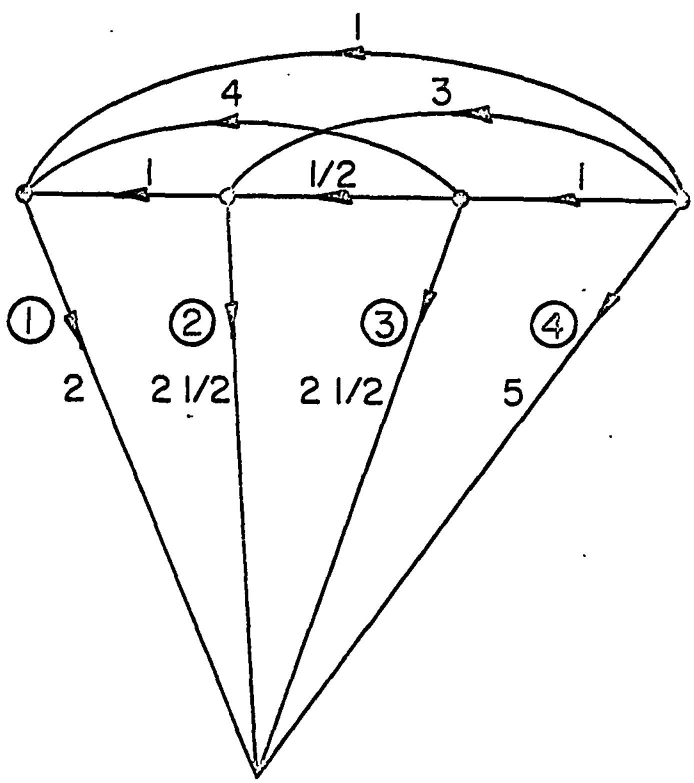


Figure 11. A Lagrnagian tree showing conductance values for an (n+1)-node realization of Equation 47

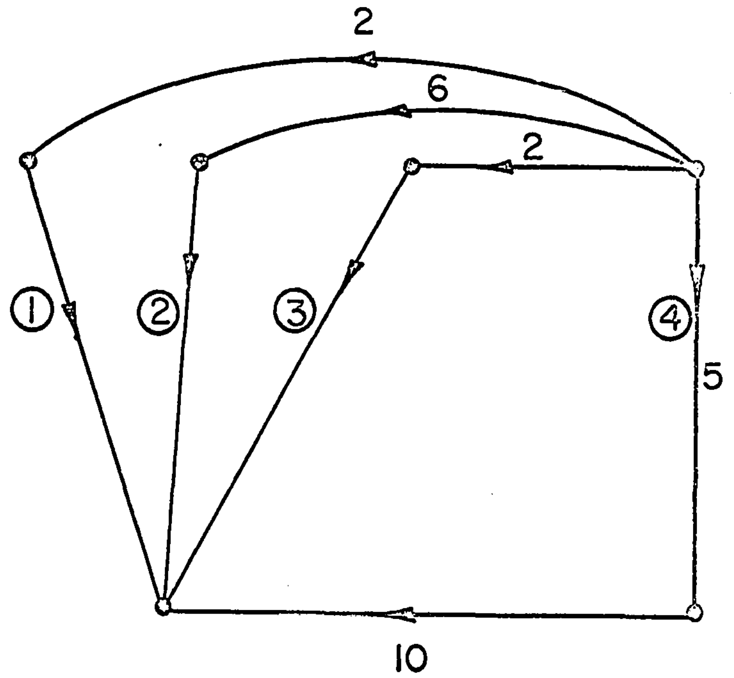


Figure 12. Network with conductance values illustrating the $4^{\underline{th}}$ row and column realization of Equation 47 on (n+2) nodes

Equation 37 then gives the remaining portion of the Y matrix to be realized. Using the appropriate element values the A_{11} matrix is given by Equation 48.

$$A_{11} = \begin{bmatrix} (8-2+\frac{1}{5}) & (-1+\frac{3}{5}) & (-4+\frac{1}{5}) \\ (-1+\frac{3}{5}) & (7-6+\frac{9}{5}) & (-\frac{1}{2}+\frac{3}{5}) \\ (-4+\frac{1}{5}) & (-\frac{1}{2}+\frac{3}{5}) & (8-2+\frac{1}{5}) \end{bmatrix}$$

or

$$A_{11} = \begin{bmatrix} \frac{31}{5} & -\frac{2}{5} & -\frac{19}{5} \\ -\frac{2}{5} & \frac{14}{5} & +\frac{1}{10} \\ -\frac{19}{5} & +\frac{1}{10} & \frac{31}{5} \end{bmatrix} \tag{48}$$

Thus, in trying to synthesize A_{11} upon the (n-1)-port Lagrangian subtree structure, it is found that the + 1/10 element in the A_{11} matrix requires a negative conductance of magnitude 1/10 to be placed between the noncommon nodes of ports two and three. This fact clearly illustrates the idea of not being able to remove any given port from an (n+1)-node Lagrangian tree realization to form an (n+2)-node network using the Slepian-Weinberg procedure.

The (n+2)-node realization with port three removed is now illustrated. Interchanging columns 3 and 4 and rows 3 and 4 the Y matrix appears as Equation 49.

$$Y = \begin{bmatrix} 8 & -1 & -1 & -4 \\ -1 & 7 & -3 & -1/2 \\ -1 & -3 & 10 & -1 \\ -4 & -1/2 & -1 & 8 \end{bmatrix} \tag{49}$$

The portion of the network realized first (row and column 4 of Equation 49) is shown in Figure 13. Forming the A_{11} matrix,

$$A_{11} = \begin{bmatrix} (8-8+\frac{32}{11}) & (-1+\frac{4}{11}) & (-1+\frac{8}{11}) \\ (-1+\frac{4}{11}) & (7-1+\frac{1}{22}) & (-3+\frac{1}{11}) \\ (-1+\frac{8}{11}) & (-3+\frac{1}{11}) & (10-2+\frac{2}{11}) \end{bmatrix}$$

or

$$A_{11} = \begin{bmatrix} \frac{32}{11} & -\frac{7}{11} & -\frac{3}{11} \\ -\frac{7}{11} & \frac{133}{22} & -\frac{32}{11} \\ -\frac{3}{11} & -\frac{32}{11} & \frac{90}{11} \end{bmatrix} . \tag{50}$$

It is readily observed that A_{11} is realizable on the Lagrangian subtree structure with ports 1, 2, and 4. Thus, the parasitic effects do not contribute to the extent of requiring negative conductances for the realization of the port-admittance matrix, Y.

At this point it is instructive, as well as providing a check, to form the Y_{5x5} port-admittance matrix utilizing the cut-sets as shown in Figure 14. By performing a pivotal condensation on the diagonal element corresponding to the $5^{\underline{th}}$ port, the original Y_{4x4} port-admittance matrix is formed. These two matrices are given by Equations 51 and 52 respectively as

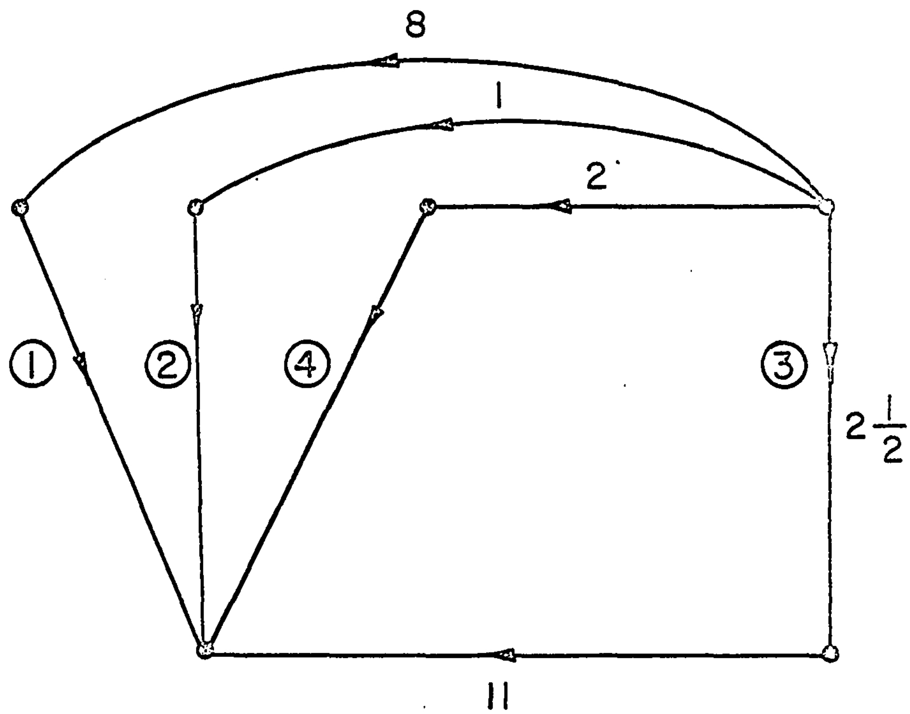


Figure 13. Network with conductance values illustrating realization of the 4^{th} row and column of Equation 49

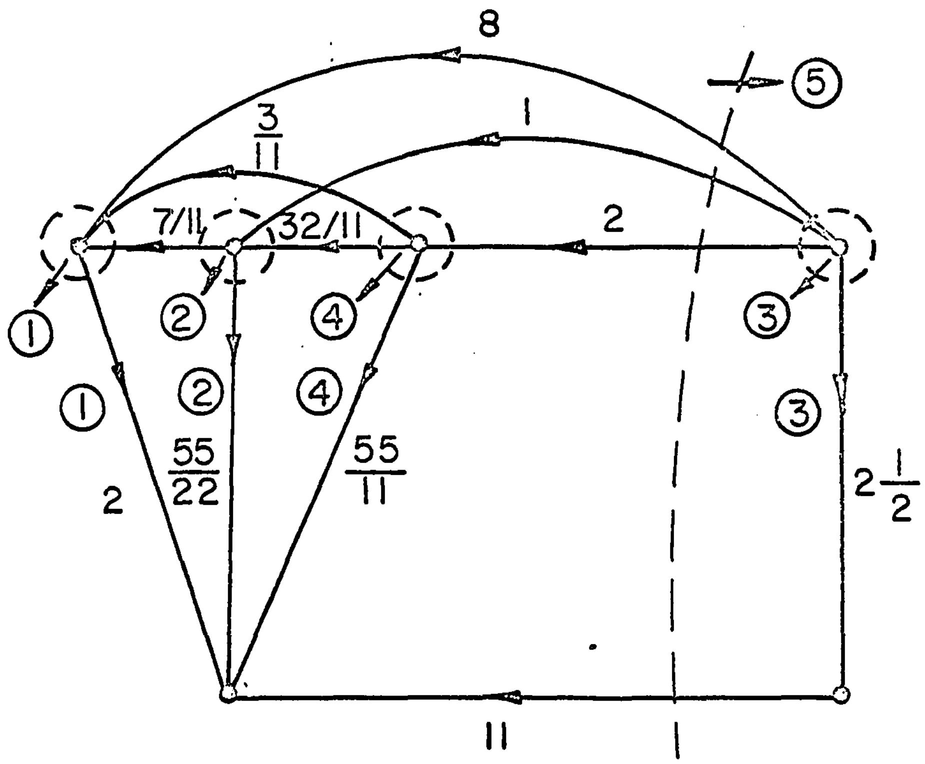


Figure 14. Total realization with (n+2) nodes of Equation 47 or Equation 49

$$Y_{5x5} = \begin{bmatrix} \frac{120}{11} & -\frac{7}{11} & -\frac{3}{11} & -8 & 8 \\ -\frac{7}{11} & \frac{155}{22} & -\frac{32}{11} & -1 & 1 \\ -\frac{3}{11} & -\frac{32}{11} & \frac{112}{11} & -2 & 2 \\ -8 & -1 & -2 & \frac{27}{2} & -11 \\ 8 & 1 & 2 & -11 & 22 \end{bmatrix} \tag{51}$$

and

$$Y_{4x4} = \begin{bmatrix} 8 & -1 & -1 & -4 \\ -1 & 7 & -3 & -1/2 \\ -1 & -3 & 10 & -1 \\ -4 & -1/2 & -1 & 8 \end{bmatrix} . \tag{52}$$

C. Two-Tree Synthesis Process Utilizing Complete Connecting Graph

In Section IVA the port-admittance matrix, Y, was subdivided such that one submatrix corresponded to a 1-port Lagrangian subtree structure. When the matrix Y is partitioned with two submatrices on the diagonal which have dimensions (axa) with a≥2 and (bxb) with b≥2 respectively, the Slepian-Weinberg formulation doesn't apply as readily. In this particular section and in Section IVF the author will provide synthesis procedures which preserve the Lagrangian subtree structure corresponding to the designated submatrices. Also, the necessary equations and inequalities which are required to solve for the unknown quantities that appear with the addition of a connecting cut-set will be given.

The method of synthesis discussed next is also applicable to a matrix with a (1x1) submatrix on the diagonal. For purposes of simplification the examples used in Section IVD and Section IVE correspond to this type of partitioning of the port-admittance matrix.

Here an (nxn) port-admittance matrix, Y, is considered which can be subdivided into two hyperdominant submatrices appearing on the diagonal. Thus, as stated before, these submatrices are realizable on the Lagrangian subtrees of the complete network graph. The matrix and the corresponding Lagrangian subtree network is shown by Equation 53 and Figure 15.

$$Y = \left[\begin{array}{ccc|ccc} y_{aa} & \cdots & y_{af} & y_{ag} & \cdots & y_{an} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y_{fa} & \cdots & y_{ff} & y_{fg} & \cdots & y_{fn} \\ \hline y_{ga} & \cdots & y_{gf} & y_{gg} & \cdots & y_{gn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y_{na} & \cdots & y_{nf} & y_{ng} & \cdots & y_{nn} \end{array}\right] = \left[\begin{array}{c|c} Y_{aa} & Y_{ab} \\ \hline Y_{ba} & Y_{bb} \end{array}\right] \tag{53}$$

Further specifying the structure of the complete network graph, the common nodes of the two Lagrangian subtrees are connected by a port, R_1, as illustrated in Figure 16. Associated with Figure 16 is a port-admittance matrix with dimensions (n+1)x(n+1). The connecting cut-set contains all of the edges which pass from the nodes in one Lagrangian subtree to the nodes in the other Lagrangian subtree. For maximum flexibility a complete connecting graph is assumed. Thus, from each noncommon node in one Lagrangian subtree there emanates two branches of the network corresponding to a port of the other Lagrangian subtree. Considering f ports in one subtree and (n-f) ports in the other subtree

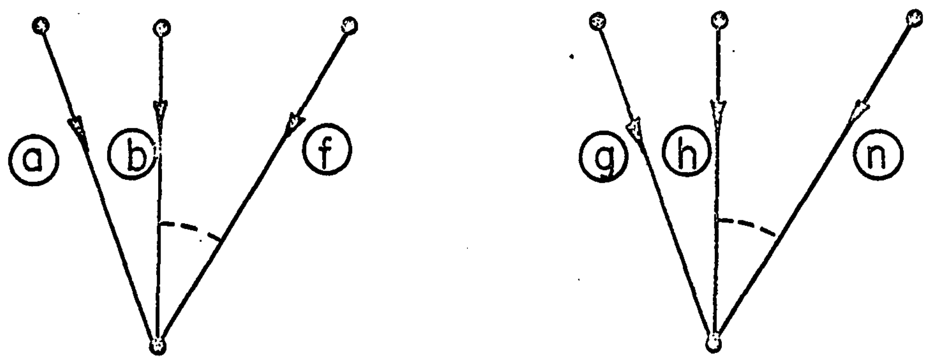


Figure 15. Basic structure for Lagrangian subtrees corresponding to Equation 53

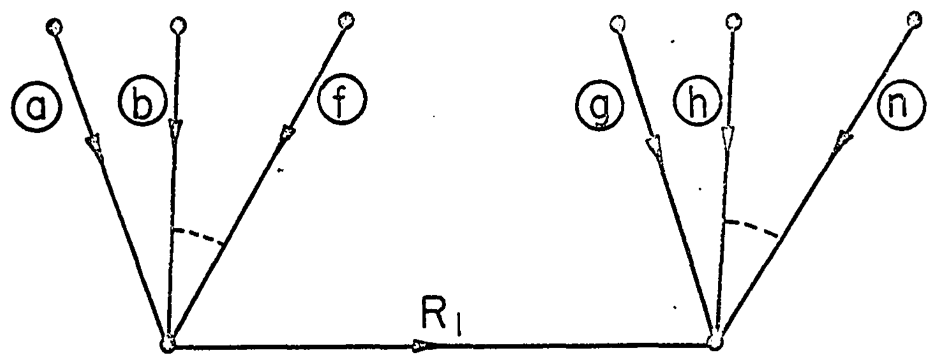


Figure 16. Basic structure for Lagrangian subtrees including connecting port, R_1

we have u = (f+1)(n-f+1) unknowns for which to solve. This number of unknowns is obtained as follows. A term, (n-f)f, equals the number of branches connecting every noncommon node of one subtree to each non-common node of the other subtree. Additional terms, (n-f) and f, represent the number of branches connecting the noncommon nodes of one subtree to the common node of the other subtree. Then another conductance is placed between the common nodes of the two subtrees. Adding these terms we obtain

$$u = (n-f)(f) + (n-f) + f + 1 = (n-f)(f+1) + (f+1)$$

or

$$u = (n-f+1)(f+1) \tag{54}$$

The method of solving for the unknown conductances will now be formulated. First, Figure 16 is expanded to illustrate the unknown conductances and the cut-sets with their proper orientation as in Figure 17. The port-admittance matrix, $\overline{Y}_{R_1}$, for Figure 17 is shown by Equation 55. The partitioning is directly related to the port structure of the Lagrangian subtrees and the connecting cut-set. Submatrix Y_{11} corresponds to the subtree with ports a through f. Submatrix Y_{22} corresponds to the subtree with ports g through n while submatrix Y_{33} is associated with cut-set R_1. Note that cut-set R_1 crosses each connecting edge or all of the unknown conductances. Therefore,

$$\overline{Y}_{R_1} = \left[\begin{array}{c|c|c} Y_{11} & Y_{12} & Y_{13} \\ \hline Y_{21} & Y_{22} & Y_{23} \\ \hline Y_{31} & Y_{32} & Y_{33} \end{array}\right] . \tag{55}$$

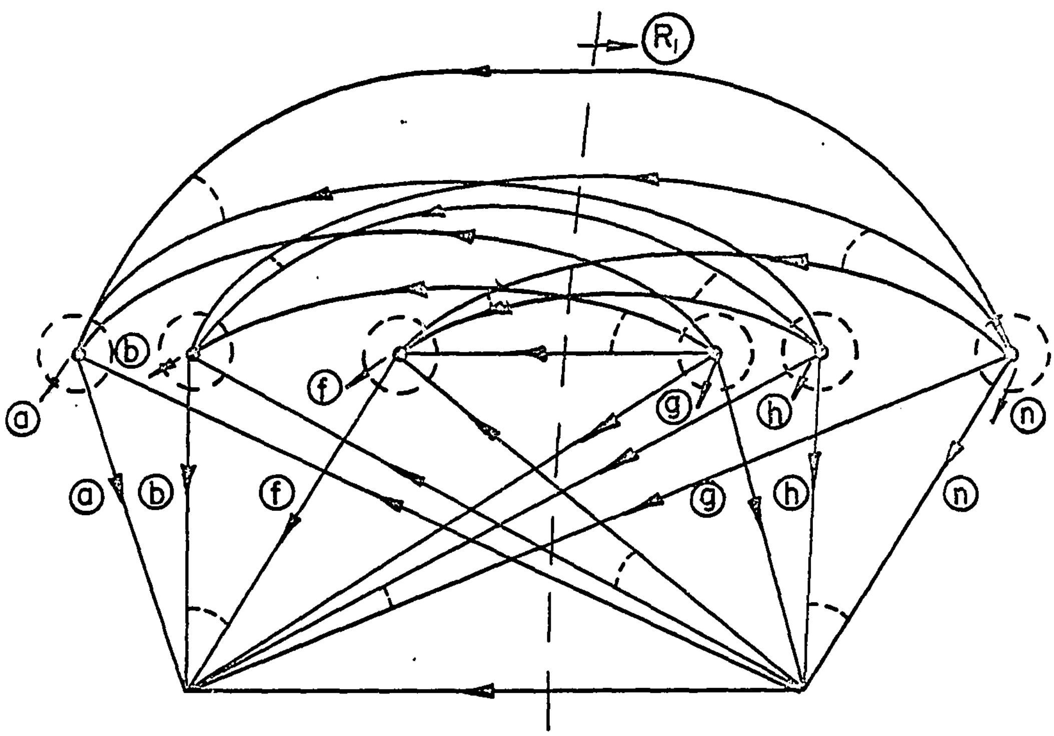


Figure 17. Graph illustrating the two Lagrangian subtrees and their interconnecting set of unknown conductances

Forming Y_{11} from Figure 17 we have that

$$Y_{11} = \begin{bmatrix} \left(\begin{array}{l}\Sigma\text{ conductances} \\ \text{crossing cut-} \\ \text{set a}\end{array}\right) & 0 & \cdots & 0 \\ 0 & \left(\begin{array}{l}\Sigma\text{ conductances} \\ \text{crossing cut-} \\ \text{set b}\end{array}\right) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \left(\begin{array}{l}\Sigma\text{ conductances} \\ \text{crossing cut-} \\ \text{set f}\end{array}\right) \end{bmatrix} . \qquad (56)$$

Zero elements arise in matrix Y_{11} since there are no conductances common to the cut-sets in the same Lagrangian subtree. Likewise,

$$Y_{22} = \begin{bmatrix} \left(\begin{array}{l}\Sigma\text{ conductances} \\ \text{crossing cut-} \\ \text{set g}\end{array}\right) & 0 & \cdots & 0 \\ 0 & \left(\begin{array}{l}\Sigma\text{ conductances} \\ \text{crossing cut-} \\ \text{set h}\end{array}\right) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \left(\begin{array}{l}\Sigma\text{ conductances} \\ \text{crossing cut-} \\ \text{set n}\end{array}\right) \end{bmatrix} . \qquad (57)$$

Continuing to form the submatrices of $\overline{Y}_{R1}$ we have the following forms for $Y_{12} = Y_{21}'$, $Y_{13} = Y_{31}'$, $Y_{23} = Y_{32}'$, and Y_{33}.

$$Y_{12} = Y_{21}' = \begin{bmatrix} \left(\begin{array}{l}-\text{ conductance} \\ \text{common to cut-} \\ \text{sets a and g}\end{array}\right) & \left(\begin{array}{l}-\text{ conductance} \\ \text{common to cut-} \\ \text{sets a and h}\end{array}\right) & \cdots & \left(\begin{array}{l}-\text{ conductance} \\ \text{common to cut-} \\ \text{sets a and n}\end{array}\right) \\ \left(\begin{array}{l}-\text{ conductance} \\ \text{common to cut-} \\ \text{sets b and g}\end{array}\right) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \left(\begin{array}{l}-\text{ conductance} \\ \text{common to cut-} \\ \text{sets f and g}\end{array}\right) & \cdots & \cdots & \left(\begin{array}{l}-\text{ conductance} \\ \text{common to cut-} \\ \text{sets f and n}\end{array}\right) \end{bmatrix} \qquad (58)$$

$$Y_{13} = Y'_{31} = \begin{bmatrix} \Sigma \text{ conductances common to} \\ \text{cut-sets a and } R_1 \\ \Sigma \text{ conductances common to} \\ \text{cut-sets b and } R_1 \\ \vdots \\ \Sigma \text{ conductances common to} \\ \text{cut-sets f and } R_1 \end{bmatrix} \tag{59}$$

$$Y_{23} = Y'_{32} \quad \begin{bmatrix} - \Sigma \text{ conductances common to} \\ \text{cut-sets g and } R_1 \\ - \Sigma \text{ conductances common to} \\ \text{cut-sets h and } R_1 \\ \vdots \\ - \Sigma \text{ conductances common to} \\ \text{cut-sets n and } R_1 \end{bmatrix} \tag{60}$$

$$Y_{33} = [\Sigma \text{ conductances common to cut-sets } R_1] \tag{61}$$

Upon performing a pivotal condensation on Y_{33} in Equation 55 we have that

$$\bar{Y}^P_{R_1} = \begin{bmatrix} (Y_{11} - Y_{13}\, Y_{33}^{-1}\, Y_{31}) & (Y_{12} - Y_{13}\, Y_{33}^{-1}\, Y_{32}) \\ (Y_{21} - Y_{23}\, Y_{33}^{-1}\, Y_{31}) & (Y_{22} - Y_{23}\, Y_{33}^{-1}\, Y_{32}) \end{bmatrix}. \tag{62}$$

Now, $Y_{ab} = Y'_{ba}$ of Equation 53 must be realized exactly by the conductances connecting the two Lagrangian subtrees. With this fact in mind, the obvious conclusion to draw is that

$$Y_{ab} = Y'_{ba} = Y_{12} - Y_{13}\, Y_{33}^{-1}\, Y_{32}\,. \tag{63}$$

Just as with the synthesis process of Section IVA, the realization of submatrices $Y_{13} = Y'_{31}$, $Y_{23} = Y'_{32}$, and Y_{33} will cause parasitic realizations upon the two Lagrangian subtrees. These parasitic realizations must not cause negative conductances to be used so as to fully realize Y. Therefore, the remainder of the Y matrix to be realized, Y_1, is found as

$$Y_1 = Y - \overline{Y}^P_{R_1} = \begin{bmatrix} (Y_{aa}-Y_{11}+Y_{13}Y_{33}^{-1}Y_{31}) & 0 \\ & \\ 0 & (Y_{bb}-Y_{22}+Y_{23}Y_{33}^{-1}Y_{32}) \end{bmatrix}. \qquad (64)$$

Utilizing Equations 56-61, 63 and 64, relationships involving the various unknown conductances can be formulated in conjunction with the elements of the port-admittance matrix, Y. Substituting Equations 58-61 into Equation 63 yields

$$Y_{ab} = \begin{bmatrix} (-ag+\frac{(\Sigma aR_1)(\Sigma gR_1)}{\Sigma R_1}) & (-ah+\frac{(\Sigma aR_1)(\Sigma hR_1)}{\Sigma R_1}) & \cdots & (-an+\frac{(\Sigma aR_1)(\Sigma nR_1)}{\Sigma R_1}) \\ (-bg+\frac{(\Sigma bR_1)(\Sigma gR_1)}{\Sigma R_1}) & (-bh+\frac{(\Sigma bR_1)(\Sigma hR_1)}{\Sigma R_1}) & \cdots & (-bn+\frac{(\Sigma bR_1)(\Sigma nR_1)}{\Sigma R_1}) \\ \vdots & \vdots & \ddots & \vdots \\ (-fg+\frac{(\Sigma fR_1)(\Sigma gR_1)}{\Sigma R_1}) & (-fh+\frac{(\Sigma fR_1)(\Sigma hR_1)}{\Sigma R_1}) & \cdots & (-fn+\frac{(\Sigma fR_1)(\Sigma nR_1)}{\Sigma R_1}) \end{bmatrix}$$

(65)

Here the author uses a simplified notation. A term such as (ag) refers to the conductance common to cut-set a and cut-set g. The term ΣaR_1 refers to the sum of conductances common to cut-set a and cut-set R_1.

Also, ΣR_1 refers to the sum of all conductances connecting the two Lagrangian subtrees. Now, the general expression for the ij$\underline{\text{th}}$ element of Equation 65 where cut-set i and cut-set j are in separate Lagrangian subtrees can be written as

$$y_{ij} = \frac{-\begin{pmatrix}\text{conductance}\\ \text{common to}\\ \text{cut-sets}\\ \text{i and j}\end{pmatrix}\begin{pmatrix}\Sigma\text{ conduct-}\\ \text{ances not}\\ \text{in cut-set}\\ \text{i or cut-}\\ \text{set j}\end{pmatrix}+\begin{pmatrix}\Sigma\text{ conduct-}\\ \text{ances in}\\ \text{cut-set i}\\ \text{and not in}\\ \text{cut-set j}\end{pmatrix}\begin{pmatrix}\Sigma\text{ conduct-}\\ \text{ances in}\\ \text{cut-set j}\\ \text{and not in}\\ \text{cut-set i}\end{pmatrix}}{\begin{matrix}\Sigma\text{ conductances}\\ \text{common to cut-}\\ \text{set } R_1\end{matrix}} . \quad (66)$$

As expected, the matrix of Equation 65 is not symmetrical.

Working with the off-diagonal elements of Y_{aa} and Y_{bb} (conductances common to cut-sets in the same Lagrangian subtree) and applying the necessary and sufficient conditions for realization, the following inequalities are determined.

$$Y_{aa} - Y_{11} + Y_{13}Y_{33}^{-1}Y_{31} \leq 0 \quad (67)$$

$$Y_{bb} - Y_{22} + Y_{23}Y_{33}^{-1}Y_{32} \leq 0 \quad . \quad (68)$$

However, the off-diagonal elements of matrices Y_{11} and Y_{22} are equal to zero. Thus, Inequalities 67 and 68 reduce to

$$Y_{aa} + Y_{13}Y_{33}^{-1}Y_{31} \leq 0 \quad (69)$$

and

$$Y_{bb} + Y_{23}Y_{33}^{-1}Y_{32} \leq 0. \quad (70)$$

After making the proper substitutions and performing the indicated matrix operations, Inequalities 69 and 70 produce submatrices containing the general ij$\underline{\text{th}}$ element,

$$y_{ij} + \frac{\begin{pmatrix}\Sigma \text{ conductances common} \\ \text{to cut-sets } i \text{ and} \\ R_1\end{pmatrix}\begin{pmatrix}\Sigma \text{ conductances common} \\ \text{to cut-sets } j \text{ and} \\ R_1\end{pmatrix}}{\Sigma \text{ conductances common to cut-set } R_1} \le 0, \; i \ne j, \tag{71}$$

with cut-sets i and j located in the same Lagrangian subtree. At this point the fact should be reviewed that the orientation of the ports of the Lagrangian subtree in conjunction with the signs of the elements of matrix Y dictate that y_{ij} is negative. The second term of Inequality 71 will always be positive. Therefore, the parasitic coupling effect — second term of Inequality 71 — between ports located in the same Lagrangian subtree cannot be greater than $|y_{ij}|$.

The parasitic driving-point effect should be considered at this stage of the synthesis process. This is obtained by the comparison of the diagonal elements of Y and the contribution of the pivotal condensation calculation upon the diagonal elements of $\bar{Y}^P_{R_1}$. Using Equations 56-61 it is easily shown that the pivotal condensation operation produces terms on the diagonal of $\bar{Y}^P_{R_1}$ having the following form.

$$\begin{pmatrix}\Sigma \text{ conductances} \\ \text{common to cut-} \\ \text{sets } i \text{ and } R_1\end{pmatrix} - \frac{\begin{pmatrix}\Sigma \text{ conductances common to} \\ \text{cut-set } i \text{ and } R_1\end{pmatrix}^2}{\begin{pmatrix}\Sigma \text{ conductances common to} \\ \text{cut-set } R_1\end{pmatrix}} \tag{72}$$

or

$$\frac{\begin{pmatrix}\Sigma \text{ conductances common} \\ \text{to cut-sets } i \text{ and } R_1\end{pmatrix}\begin{pmatrix}\Sigma \text{ conductances not common} \\ \text{to cut-sets } i \text{ and } R_1\end{pmatrix}}{\begin{pmatrix}\Sigma \text{ conductances common} \\ \text{to cut-set } R_1\end{pmatrix}} \tag{73}$$

Now, considering Equation 64, the diagonal elements of Y_1 must all be positive. Thus, Inequality 74 must be satisfied. The set of n inequalities obtained from

$$y_{ii} - \frac{(\Sigma \text{ conductances common to cut-sets } i \text{ and } R_1)\ (\Sigma \text{ conductances not common to cut-sets } i \text{ and } R_1)}{(\Sigma \text{ conductances common to cut-set } R_1)} \geq 0 \tag{74}$$

is not used to solve for the unknowns since it becomes a part of another set of n inequalities involved with the magnitude conditions associated with the Lagrangian subtree networks. The latter set of n inequalities evolves from matrix Y_1. Using Inequalities 71 and 74 and remembering that $y_{ij} \leq 0$ with cut-sets i and j in the same Lagrangian subtree, we have the set of n inequalities from

$$y_{ii} - \frac{(\Sigma \text{ conductances common to cut-sets } i \text{ and } R_1)(\Sigma \text{ conductances not common to cut-sets } i \text{ and } R_1)}{(\Sigma \text{ conductances common to cut-set } R_1)}$$

$$- (- y_{ij} - \frac{(\Sigma \text{ conductances common to cut-sets } i \text{ and } R_1)\ (\Sigma \text{ conductances common to cut-sets } j \text{ and } R_1)}{(\Sigma \text{ conductances common to cut-set } R_1)} \geq 0. \tag{75}$$

Simplifying Inequality 75 we have that

$$y_{ii}+y_{ij} - \frac{(\Sigma \text{ conductances common to cut-sets } i \text{ and } R_1)}{(\Sigma \text{ conductances common to cut-set } R_1)} \left[(\Sigma \text{ conductances not common to cut-sets } i \text{ and } R_1) - (\Sigma \text{ conductances common to cut-sets } j \text{ and } R_1) \right] \geq 0. \tag{76}$$

To complete the synthesis process the conductance values which are placed between the noncommon nodes located in the same Lagrangian subtree and the conductance values placed across the port terminals are required.

When Inequality 71 is set equal to L_{ij}, it represents the conductance value that is common to cut-sets i and j of a Lagrangian subtree. In the same manner, when Inequality 76 is set equal to L_i, it represents the conductance value that connects the $i^{\underline{th}}$ node to the common node of a Lagrangian subtree — the shunt conductance of port i.

The number of equations and inequalities that can be obtained from an (nxn) port-admittance matrix, Y, with one Lagrangian subtree containing f ports and the other Lagrangian subtree containing (n-f) ports is $\frac{n(n+1)}{2}$. Referring to Equation 53, matrix Y_{ab} contributes f(n-f) equations which are required to be satisfied exactly. Equation 66 yields this set of equations. Inequality 71 provides

$$\frac{f^2-f}{2} + \frac{(n-f)^2-(n-f)}{2} = \frac{2f^2-2nf+n^2-n}{2}$$

inequalities. In addition, Inequality 76 gives n inequalities which are used to solve for the unknown conductance values.

In comparing Equation 54 $(u = (n-f+1)(f+1))$ with $\frac{n(n+1)}{2}$ it is interesting to note that $\frac{n(n+1)}{2}$ is greater than or equal to the number of unknown conductances for all values of n except n = 2. For finding the maximum number of unknowns for a particular n one can use

$$u(f) = (n - f + 1)(f + 1). \tag{77}$$

Taking the derivative of u(f) with respect to f and setting the result equal to zero yields

$$\frac{d(u(f))}{df} = n - 2f = 0.$$

Solving for f in terms of n, $f = \frac{n}{2}$. By performing the second derivative one finds that $f = \frac{n}{2}$ represents the value of f which gives the maximum value for Equation 77. Substituting $f = \frac{n}{2}$ into Equation 77 yields

$$u_{max}(f) = \frac{n^2 + 4n + 4}{4} \tag{78}$$

which is greater than $\frac{n(n+1)}{2}$ only for n = 2 corresponding to allowable values for f. The network realized by the synthesis procedure just explained is shown in Figure 18.

D. Example Two Illustrating (n+2)-Node Synthesis of a Port-Admittance Matrix Which is Nonrealizable by an (n+1)-Node Lagrangian Tree

The following example illustrates the method of synthesis as described in Section IVC. A (3x3) port-admittance matrix is given by Equation 79.

$$Y = \begin{bmatrix} \frac{200}{24} & -\frac{145}{24} & -\frac{108}{24} \\ -\frac{145}{24} & \frac{167}{24} & \frac{108}{24} \\ -\frac{108}{24} & \frac{108}{24} & \frac{192}{24} \end{bmatrix} \tag{79}$$

It is readily seen that a network with (n+1) nodes based upon a 3-port Lagrangian tree cannot be synthesized from Y because of the nondominant condition. But the Y matrix can be partitioned such that there is a (2x2) submatrix on the diagonal and a (1x1) submatrix on the diagonal which correspond to a Lagrangian subtree with two ports and a Lagrangian subtree with one port respectively. In Figure 19 ports 1 and 2 are shown in one Lagrangian subtree and port 3 is shown in the other Lagrangian subtree. The connecting port corresponding to cut-set 4 along with the complete set of six connecting conductances are also shown. The six conductances represent the unknown quantities. Equation 66 and

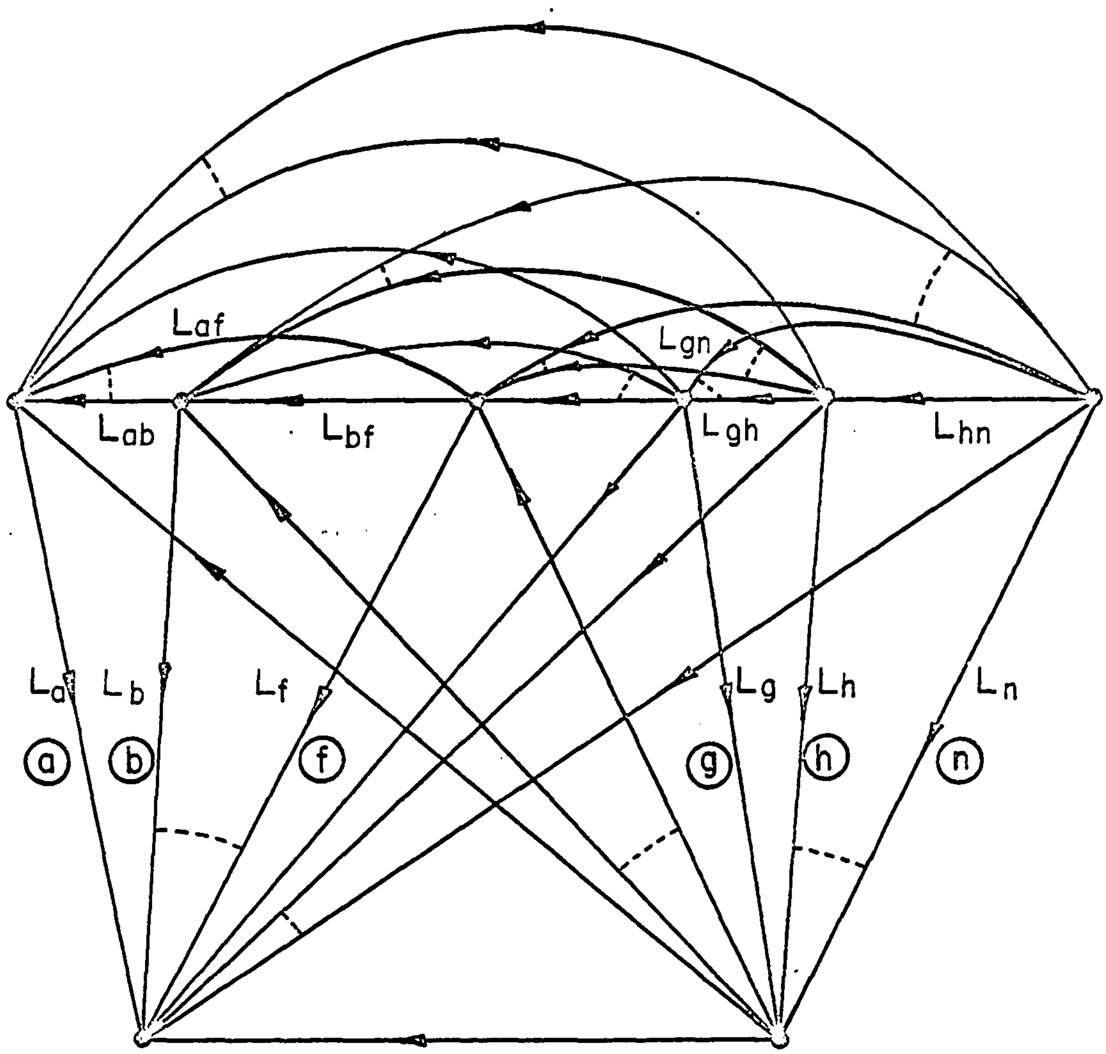


Figure 18. (n+2)-node network with unidentified edges as unknowns based on two Lagrangian subtree structures for a realizable port-admittance matrix

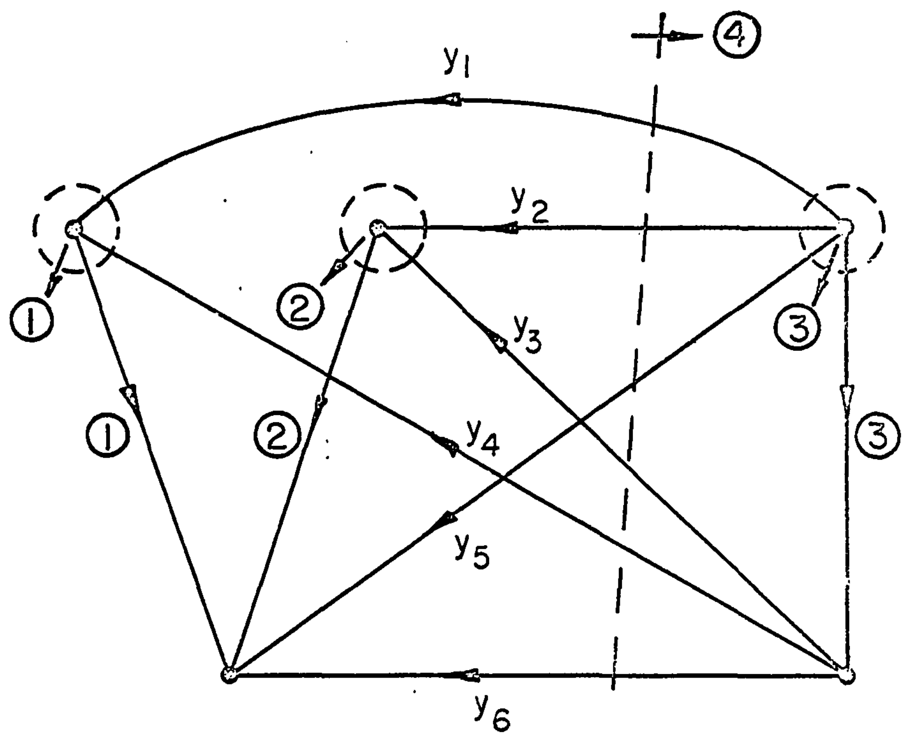


Figure 19. Graph illustrating the two Lagrangian subtrees with their set of six unknown connecting conductances

Inequalities 71 and 76 are used to solve for the six unknowns. From Equation 66 one gets

$$- \frac{108}{24} = \frac{- y_1(y_3+y_6) + y_4(y_2+y_5)}{\sum_{i=1}^{6} y_i} \tag{80}$$

and

$$+ \frac{108}{24} = \frac{- y_2(y_4+y_6) + y_3(y_1+y_5)}{\sum_{i=1}^{6} y_i} . \tag{81}$$

Inequality 71 yields

$$- \frac{145}{24} + \frac{(y_1+y_4)(y_2+y_3)}{\sum_{i=1}^{6} y_i} \leq 0. \tag{82}$$

Inequality 76 yields the following three inequalities.

$$\frac{200}{24} - \frac{145}{24} - \frac{(y_1+y_4)(y_5+y_6)}{\sum_{i=1}^{6} y_i} \geq 0 \tag{83}$$

$$\frac{167}{24} - \frac{145}{24} - \frac{(y_2+y_3)}{\sum_{i=1}^{6} y_i}(y_5+y_6) \geq 0 \tag{84}$$

$$\frac{192}{24} + 0 - \frac{(y_1+y_2+y_5)}{\sum_{i=1}^{6} y_i} (y_3+y_4+y_6) \geq 0 \tag{85}$$

One set of values which constitutes a solution for the preceding equations and inequalities is $y_1=y_3=10$, $y_2=y_4=y_5=y_6=1$.

Continuing with the synthesis method described in Section IVC, the conductance y_a common to cut-sets 1 and 2 is found by taking the magnitude of the left-hand side of Inequality 82 and designating it y_a. Likewise, y_b, y_c, and y_d representing the shunt conductance values of ports 1, 2, and 3 respectively are found by taking the value of the left-hand sides of Inequalities 83, 84, and 85. The complete realization of Y (Equation 79) is shown in Figure 20.

As a check on the realization of Figure 20, the port-admittance matrix can be formed as in Equation 86.

$$\bar{Y}_{4x4} = \begin{bmatrix} \frac{321}{24} & -1 & -10 & 11 \\ -1 & 12 & -1 & 11 \\ -10 & -1 & 14 & -12 \\ 11 & 11 & -12 & 24 \end{bmatrix} \qquad (86)$$

Upon performing a pivotal condensation on the y_{44} element of Equation 86, the original Y matrix (Equation 79) is obtained.

E. Example Three Illustrating (n+2)-Node Synthesis of a Port-Admittance Matrix Which is Nonrealizable by an (n+1)-Node Network

In the previous example, even though the port-admittance matrix was not realizable upon an (n+1)-node Lagrangian tree, it was realizable upon an (n+1)-node linear tree. The author would now like to consider the classic example which is not realizable by any (n+1)-node network. The port-admittance matrix is given by Equation 87.

$$Y = \begin{bmatrix} 9 & -5 & 5 \\ -5 & 9 & 1 \\ 5 & 1 & 9 \end{bmatrix} \qquad (87)$$

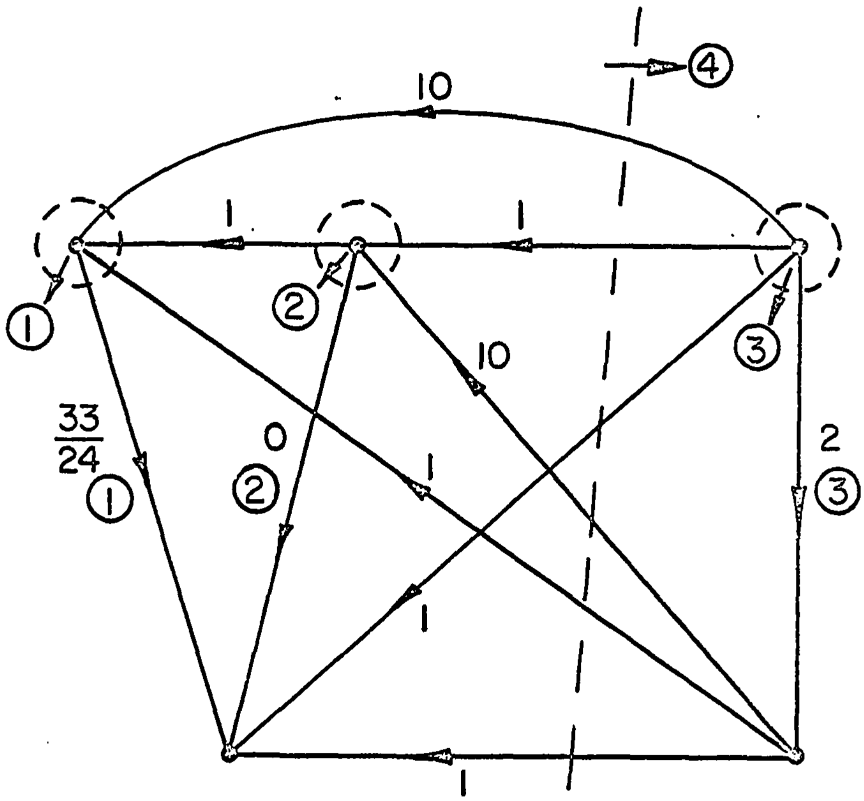


Figure 20. Complete realization on (n+2) nodes for Equation 79 with conductance values given

The network that is finally synthesized is shown in the cited reference (25). This section illustrates that the same network realization may be obtained by the synthesis methods described in Section IVC. The Y matrix is partitioned with ports 1 and 2 in one Lagrangian subtree and port 3 in the other Lagrangian subtree. The basic structure is illustrated in Figure 19 and is not duplicated here.

Upon using Equation 66 and Inequalities 71 and 76, the six unknown conductances are placed in the following relationships. From Equation 66,

$$5 = \frac{-\,y_1(y_3+y_6) + y_4(y_2+y_5)}{\sum_{i=1}^{6} y_i} \tag{88}$$

and

$$1 = \frac{-\,y_2(y_4+y_6) + y_3(y_1+y_5)}{\sum_{i=1}^{6} y_i}. \tag{89}$$

Inequality 71 yields

$$-\,5 + \frac{(y_1+y_4)(y_2+y_3)}{\sum_{i=1}^{6} y_i} \leq 0. \tag{90}$$

Inequality 76 yields the following three inequalities.

$$9 - 5 - \frac{(y_1+y_4)(y_5+y_6)}{\sum_{i=1}^{6} y_i} \geq 0 \tag{91}$$

$$9 - 5 - \frac{(y_2+y_3)(y_5+y_6)}{\sum_{i=1}^{6} y_i} \geq 0 \tag{92}$$

$$9 + 0 - \frac{(y_1+y_2+y_5)(y_3+y_4+y_6)}{\sum_{i=1}^{6} y_i} \geq 0 \tag{93}$$

A set of values that constitutes a solution for the Equations and Inequalities 88-93 is $y_1=y_6=0$, $y_2=\frac{17}{8}$, $y_3=\frac{17}{2}$, $y_4=17$, and $y_5=\frac{17}{2}$.

As with the previous example, the conductance y_a common to cut-sets 1 and 2 is found by taking the magnitude of the left-hand side of Inequality 90 and designating it y_a. Also, y_b, y_c, and y_d representing the shunt conductance values of ports 1, 2, and 3 respectively are found by taking the values of the left-hand side of Inequalities 91, 92, and 93. Figure 21 shows the complete realization.

Forming the port-admittance matrix of Figure 21, Equation 94, and then performing a pivotal condensation on the y_{44} element, the original Y matrix (Equation 87) is obtained.

$$\overline{Y}_{4x4} = \begin{bmatrix} 17 & 0 & 0 & 17 \\ 0 & \frac{97}{8} & -\frac{17}{8} & \frac{85}{8} \\ 0 & -\frac{17}{8} & \frac{97}{8} & -\frac{85}{8} \\ 17 & \frac{85}{8} & -\frac{85}{8} & \frac{289}{8} \end{bmatrix} \tag{94}$$

These steps serve as a check for the realization of Y.

F. K-Tree Synthesis Process Utilizing Lagrangian Subtrees

Proceeding to a much more general problem, the author wishes to consider not only the possibility of an (nxn) port-admittance matrix, Y, being subdivided with two hyperdominant submatrices on the diagonal, but k hyperdominant submatrices on the diagonal. Once again, these

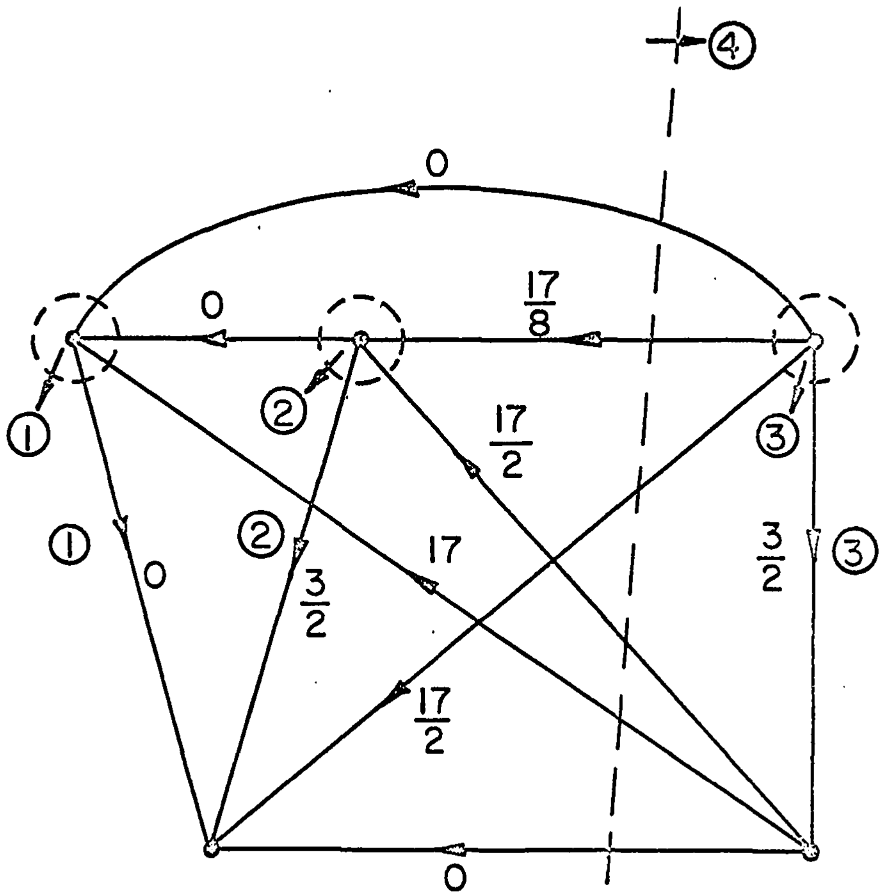


Figure 21. Complete realization on (n+2) nodes for Equation 87 with conductance values given

hyperdominant submatrices correspond to Lagrangian subtrees with the ports directed toward the common nodes. As indicated previously there is the possibility of positive off-diagonal elements in these submatrices if the port directions permit. Under the prescribed conditions, the Y matrix takes on the following form.

$$Y = \left[\begin{array}{ccc|ccc|c|ccc}
y_{11} & \cdots & y_{1a} & y_{1,a+1} & \cdots & y_{1b} & \cdots & y_{1,n-r} & \cdots & y_{1n} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
y_{a1} & \cdots & y_{aa} & y_{a,a+1} & \cdots & y_{ab} & \cdots & y_{a,n-r} & \cdots & y_{an} \\
\hline
y_{a+1,1} & \cdots & y_{a+1,a} & y_{a+1,a+1} & \cdots & y_{a+1,b} & \cdots & y_{a+1,n-r} & \cdots & y_{a+1,n} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
y_{b1} & \cdots & y_{ba} & y_{b,a+1} & \cdots & y_{bb} & \cdots & y_{b,n-r} & \cdots & y_{bn} \\
\hline
 & \vdots & & & \vdots & & \ddots & & \vdots & \\
\hline
y_{n-r,1} & \cdots & y_{n-r,a} & y_{n-r,a+1} & \cdots & y_{n-r,b} & \cdots & y_{n-r,n-r} & \cdots & y_{n-r,n} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
y_{n1} & \cdots & y_{na} & y_{n,a+1} & \cdots & y_{n,b} & \cdots & y_{n,n-r} & \cdots & y_{nn}
\end{array}\right] \tag{95}$$

Corresponding to Equation 95, Figure 22 illustrates the form of the graph of Lagrangian subtrees with the connecting ports, R_{n+1}, R_{n+2}, ..., R_{n+k-1}, shown linking the common nodes in a linear subtree arrangement. The cut-sets are also shown. A new port-admittance matrix, $\overline{Y}_{k-1}$, with dimensions (n+k-1)x(n+k-1) associated with Figure 22 can now be defined as

$$
\bar{Y}_{k-1} = \left[\begin{array}{ccc|ccc|c|ccc|cccc}
x_{11} & \cdots & x_{1a} & x_{1,a+1} & \cdots & x_{1b} & & x_{1,n-r} & \cdots & x_{1n} & r_{1,n+1} & r_{1,n+2} & \cdots & r_{1,n+k-1} \\
\vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
x_{a1} & \cdots & x_{aa} & x_{a,a+1} & \cdots & x_{ab} & & x_{a,n-r} & \cdots & x_{an} & & & & \\
\hline
x_{a+1,1} & \cdots & x_{a+1,a} & x_{a+1,a+1} & \cdots & x_{a+1,b} & & x_{a+1,n-r} & \cdots & x_{a+1,n} & r_{a+1,n+1} & r_{a+1,n+2} & \cdots & r_{a+1,n+k-1} \\
\vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
x_{b1} & \cdots & x_{ba} & x_{b,a+1} & \cdots & x_{bb} & & x_{b,n-r} & \cdots & x_{bn} & & & & \\
\hline
\vdots & & & \vdots & & & \ddots & & \vdots & & & \vdots & & \\
\hline
x_{n-r,1} & \cdots & x_{n-r,a} & x_{n-r,a+1} & \cdots & x_{n-r,b} & & x_{n-r,n-r} & \cdots & x_{n-r,n} & r_{n-r,n+1} & r_{n-r,n+2} & \cdots & r_{n-r,n+k-1} \\
\vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
x_{n1} & \cdots & x_{na} & x_{n,a+1} & \cdots & x_{nb} & & x_{n,n-r} & \cdots & x_{nn} & & & & \\
\hline
r_{n+1,1} & \cdots & \cdots & r_{n+1,a+1} & \cdots & \cdots & & r_{n+1,n-r} & \cdots & \cdots & r_{n+1,n+1} & r_{n+1,n+2} & \cdots & r_{n+1,n+k-1} \\
r_{n+2,1} & \cdots & \cdots & r_{n+2,a+1} & \cdots & \cdots & & r_{n+2,n-r} & \cdots & \cdots & r_{n+2,n+1} & r_{n+2,n+2} & \cdots & \vdots \\
\vdots & & & & & & \cdots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\
r_{n+k-1,1} & \cdots & \cdots & r_{n+k-1,a+1} & \cdots & \cdots & & r_{n+k-1,n-r} & \cdots & \cdots & r_{n+k-1,n+1} & \cdots & \cdots & r_{n+k-1,n+k-1}
\end{array}\right] \quad (96)
$$

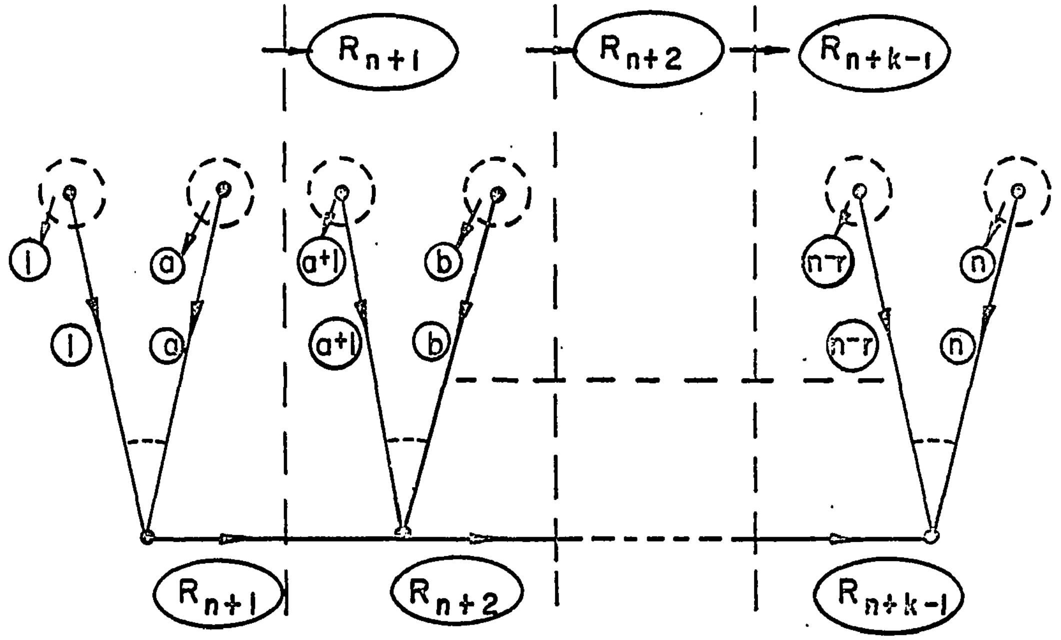


Figure 22. Graph of k Lagrangian subtrees and (k-1) connecting ports

Partitioning $\overline{Y}_{k-1}$ into a more compact form we have

$$\overline{Y}_{k-1} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \tag{97}$$

where the submatrix R_{11} refers to the portion of the $\overline{Y}_{k-1}$ matrix designated with x elements. Since R_{11} can be written in terms of R_{12}, R_{21}, R_{22} and Y as

$$Y = R_{11} - R_{12}R_{22}^{-1}R_{21}$$

or

$$R_{11} = Y + R_{12}R_{22}^{-1}R_{21} \tag{98}$$

we have only the following number of unknowns in the $\overline{Y}_{k-1}$ matrix — Number of unknowns $Z = n(k+1) + \frac{(k-1)^2-(k-1)}{2} + k-1$. This is simplified to Equation 99,

$$Z = n(k-1) + \frac{k^2}{2} - \frac{k}{2}. \tag{99}$$

In contrast to the method presented in Section IVC where only one cut-set was added, the elements of the matrix $\overline{Y}_{k-1}$ are the unknown quantities and not the circuit conductances. Perhaps one could use Equation 98 and satisfy the off-diagonal submatrices exactly and carry out the synthesis procedure utilizing the idea of parasitic effects while realizing the various k Lagrangian subtree networks. More will be mentioned concerning the merits and limitations of this procedure in Section V.

Now the author wishes to pursue a new method of synthesis for the Y matrix with k submatrices on the diagonal. The principles associated with Lagrangian tree synthesis methods will still be utilized. The development of the procedure for the general Y matrix follows.

A port representation of the resistive network in question is shown in Figure 22 with the port equations having the form

$$I(t) = \overline{Y}_{k-1}V(t)$$

or

$$\begin{bmatrix} I_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ I_n \\ \hline 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = \left[\begin{array}{c|c} R_{11} & R_{12} \\ \hline R_{21} & R_{22} \end{array}\right] \begin{bmatrix} V_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ V_n \\ \hline V_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ V_{n+k-1} \end{bmatrix} . \tag{100}$$

If the port-admittance matrix, $\overline{Y}_{k-1}$, is realizable upon the tree of Figure 22 it is realizable upon a complete graph (each node is connected by a branch of the network to every other node) corresponding to this tree. The complete graph will provide the maximum flexibility for realization since zero-valued conductances will be allowed. Using the principles of a complete graph it is readily apparent that a new Lagrangian tree can be formed upon the entire set of nodes in Figure 22. The next step is to perform a linear transformation from the graph corresponding to the connected set of k Lagrangian subtrees to a graph of only one Lagrangian tree while preserving the original port relationships. Consider this new Lagrangian tree to take the form of Figure 23 with the designated cut-set notation. The common node of this

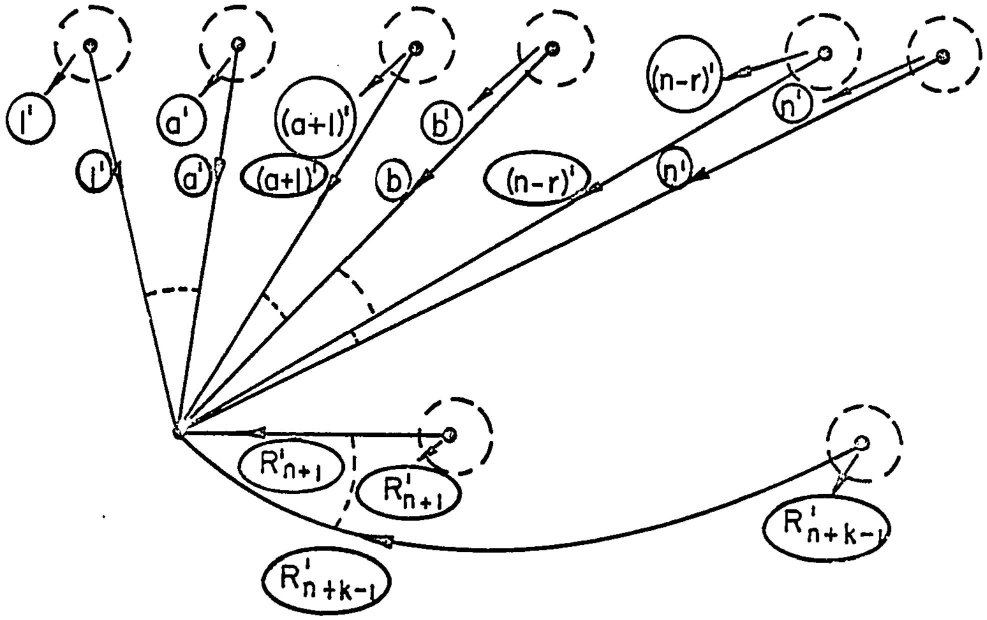


Figure 23. Graph of one Lagrangian tree utilizing all (n+k) nodes of the network

Lagrangian tree is an arbitrary choice and affects only the form of the transformation matrix.

Figure 24 illustrates the superimposing of Figure 23 on Figure 22 to form a new graph. The tree of the new graph, Figure 24, is chosen as the branches of the Lagrangian tree of Figure 23. Then the cut-set matrix, Q, for the new graph can be easily formed. It is given by Equation 101 which maintains the various Lagrangian subtree partitionings associated with Figure 22. Matrix Q appears as Equation 101 with the matrix I representing an (n+k-1)x(n+k-1) unit matrix. (Equation 101 appears on the following page).

Writing matrix Q as

$$Q = [Q_{11} \quad Q_{12}] \tag{102}$$

with $Q_{11} = I$, the new port-admittance matrix, Y^L_{k-1}, is given as

$$Y^L_{k-1} = Q_{12}\bar{Y}_{k-1}Q'_{12}. \tag{103}$$

The Y^L_{k-1} matrix is now the matrix to which the Lagrangian tree principles are applied. Matrix Y^L_{k-1} has dimensions (n+k-1)x(n+k-1). Thus, the hyperdominant restriction provides for a set of $\frac{(n+k-1)(n+k-2)}{2}$ inequalities. In addition, the dominant condition, as applied to the diagonal elements, contributes (n+k-1) inequalities. The total number of inequalities that are available to solve for the unknown terms in Equation 96 are

$$\frac{(n+k-1)(n+k-2)}{2} + (n+k-1) = (n+k-1)[\frac{(n+k)}{2}]. \tag{104}$$

It is easily shown that the number of independent inequalities given by Equation 104 will always be greater than the number of unknown terms indicated by Equation 99.

$$
Q = \begin{array}{c}
 \\ 1' \\ \vdots \\ a' \\ (a+1)' \\ \vdots \\ b' \\ \vdots \\ (n-r)' \\ \vdots \\ n' \\ R'_{n+1} \\ \vdots \\ R'_{n+k-1}
\end{array}
\begin{array}{c}
1' \cdots\cdots R'_{n+k-1} \quad 1 \cdots\cdots a \quad (a+1) \cdots\cdots b \quad \cdots\cdots\cdots \quad (n-r) \cdots\cdots n \quad R_{n+1} \cdots\cdots R_{n+k-1} \\
\left[\begin{array}{c|c|c|c|c|c}
 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & & \cdots\cdots\cdots & & \\ \hline
 & & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \cdots\cdots\cdots & & \\ \hline
I & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline
 & & & \cdots\cdots\cdots & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \\ \hline
 & & \begin{matrix} -1 & -1 & \cdots & -1 \\ & & & \\ & & & \end{matrix} & \cdots\cdots\cdots & \begin{matrix} & & & \\ & & & \\ -1 & -1 & \cdots & -1 \end{matrix} & \begin{matrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & \ddots & \ddots & \\ & & & & & 1 \\ & & & & & -1 \end{matrix}
\end{array}\right]
\end{array}
$$

(101)

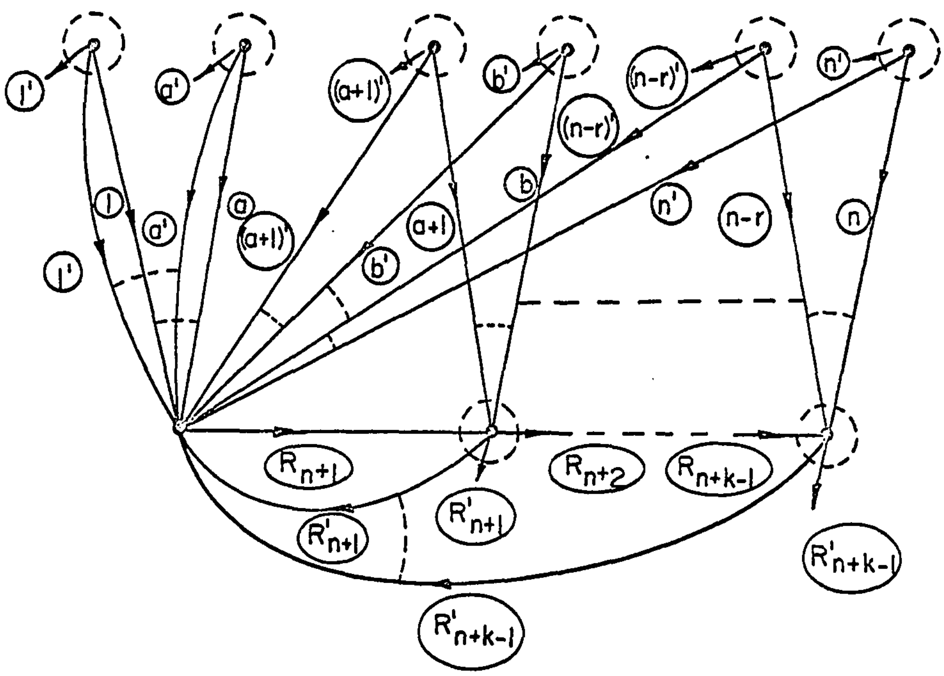


Figure 24. Two different trees whose combination forms a new graph covering the same set of nodes and providing for the formation of the transformation matrix, Q

Since $\overline{Y}_{k-1}$ is symmetrical and since there is a unit submatrix contained in matrix Q_{12}, the submatrix R_{11} of $\overline{Y}_{k-1}$ will be unchanged by the matrix operations of Equation 103. Remembering the established matrix partitions the author wishes to write Y^L_{k-1} as Equation 105 and in a more compact form — Equation 106.

$$Y^L_{k-1} = \left[\begin{array}{ccccc|c} X_{11} & X_{12} & \cdots & \cdots & X_{1k} & X_{1r} \\ X_{21} & \ddots & & & \vdots & X_{2r} \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots & \vdots \\ X_{k1} & \cdots & \cdots & \cdots & X_{kk} & X_{kr} \\ \hline X_{r1} & X_{r2} & \cdots & \cdots & X_{rk} & X_{rr} \end{array}\right] \tag{105}$$

$$Y^L_{k-1} = \left[\begin{array}{c|c} L_{11} & L_{12} \\ \hline L_{21} & L_{22} \end{array}\right] \tag{106}$$

Thus, $L_{11} = R_{11}$ with the submatrices X_{ij}; $i=1, \ldots, k$; $j=1, \ldots, k$ of Equation 105 being exactly the same as the corresponding submatrices of R_{11} in Equation 96. After performing the linear transformation the submatrices $L_{12} = L'_{21}$ and L_{22} have their forms as shown by Equations 107-110.

$$X_{1r} = X'_{r1} = \begin{bmatrix} [(-\sum_{i=a+1}^{b} x_{1i}) - r_{1,n+1} + r_{1,n+2}] & \cdots & [(-\sum_{i=n-r}^{n} x_{1i}) - r_{1,n+k-1}] \\ \vdots & \ddots & \vdots \\ [(-\sum_{i=a+1}^{b} x_{ai}) - r_{a,n+1} + r_{a,n+2}] & \cdots & [(-\sum_{i=n-r}^{n} x_{ai}) - r_{a,n+k-1}] \end{bmatrix} \tag{107}$$

$$X_{2r} = X'_{r2} = \begin{bmatrix} \left[\left(-\sum_{i=a+1}^{b} x_{a+1,i}\right) - r_{a+1,n+1} + r_{a+1,n+2}\right] & \cdots & \left[\left(-\sum_{i=n-r}^{n} x_{a+1,i}\right) - r_{a+1,n+k-1}\right] \\ \vdots & \ddots & \vdots \\ \left[\left(-\sum_{i=a+1}^{b} x_{bi}\right) - r_{b,n+1} + r_{b,n+2}\right] & \cdots & \left[\left(-\sum_{i=n-r}^{n} x_{bi}\right) - r_{b,n+k-1}\right] \end{bmatrix} \tag{108}$$

$$\vdots \qquad\qquad \vdots$$

$$X_{kr} = X'_{rk} = \begin{bmatrix} \left[\left(-\sum_{i=a+1}^{b} x_{n-r,i}\right) - r_{n-r,n+1} + r_{n-r,n+2}\right] & \cdots & \left[\left(-\sum_{i=n-r}^{n} x_{n-r,i}\right) - r_{n-r,n+k-1}\right] \\ \vdots & \ddots & \vdots \\ \left[\left(-\sum_{i=a+1}^{b} x_{ni}\right) - r_{n,n+1} + r_{n,n+2}\right] & \cdots & \left[\left(-\sum_{i=n-r}^{n} x_{ni}\right) - r_{n,n+k-1}\right] \end{bmatrix} \tag{109}$$

The form of the pre- and post-multiplying matrices contribute to a more complicated form for $X_{rr} = L_{22}$ than for the submatrices $L_{12} = L'_{21}$. Matrix X_{rr} appears as in Equation 110.

After performing the linear transformation, we have a matrix, namely Y^L_{k-1}, which is realizable by the Lagrangian tree synthesis procedures if it is realizable at all. Also, at this point there are still $\frac{n(n+1)}{2}$ unknowns in the x_{ij} notation where i and j range from 1 to n. These can be eliminated with the proper substitution and utilization of Equation 98.

$$
X_{rr} = \begin{bmatrix}
\begin{array}{l} [(\sum\limits_{i=a+1}^{b}\sum\limits_{j=a+1}^{b} x_{ij})+2\sum\limits_{i=a+1}^{b} r_{i,n+1} \\ -2\sum\limits_{i=a+1}^{b} r_{i,n+2}+(r_{n+1,n+1} \\ -2r_{n+1,n+2}+r_{n+2,n+2})] \end{array} &
\begin{array}{l} [(\sum\limits_{i=a+1}^{b}\sum\limits_{j=b+1}^{c} x_{ij})+\sum\limits_{i=b+1}^{c} r_{i,n+1} \\ -\sum\limits_{i=b+1}^{c} r_{i,n+2}+\sum\limits_{i=a+1}^{b} r_{i,n+2} \\ -\sum\limits_{i=a+1}^{b} r_{i,n+3}+(r_{n+1,n+2} \\ -r_{n+2,n+2}-r_{n+1,n+3}+r_{n+2,n+3})] \end{array} & \cdots &
\begin{array}{l} [(\sum\limits_{i=a+1}^{b}\sum\limits_{j=n-r}^{n} x_{ij})+\sum\limits_{i=n-r}^{n} r_{i,n+1} \\ -\sum\limits_{i=n-r}^{n} r_{i,n+2}+\sum\limits_{i=a+1}^{b} r_{i,n+k-1} \\ +(r_{n+1,n+k-1}-r_{n+2,n+k-1})] \end{array} \\
\begin{array}{l} [(\sum\limits_{i=a+1}^{b}\sum\limits_{j=b+1}^{c} x_{ij})+\sum\limits_{i=b+1}^{c} r_{i,n+1} \\ -\sum\limits_{i=b+1}^{c} r_{i,n+2}+\sum\limits_{i=a+1}^{b} r_{i,n+2} \\ -\sum\limits_{i=a+1}^{b} r_{i,n+3}+(r_{n+1,n+2} \\ -r_{n+2,n+2}-r_{n+1,n+3}+r_{n+2,n+3})] \end{array} &
\begin{array}{l} [(\sum\limits_{i=b+1}^{c}\sum\limits_{j=b+1}^{c} x_{ij})+2\sum\limits_{i=b+1}^{c} r_{i,n+2} \\ -2\sum\limits_{i=b+1}^{c} r_{i,n+3}+(r_{n+2,n+2} \\ -2r_{n+2,n+3}+r_{n+3,n+3})] \end{array} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\begin{array}{l} [(\sum\limits_{i=a+1}^{b}\sum\limits_{j=n-r}^{n} x_{ij})+\sum\limits_{i=n-r}^{n} r_{i,n+1} \\ -\sum\limits_{i=n-r}^{n} r_{i,n+2}+\sum\limits_{i=a+1}^{b} r_{i,n+k-1} \\ +(r_{n+1,n+k-1}-r_{n+2,n+k-1})] \end{array} & \cdots & \cdots &
\begin{array}{l} [(\sum\limits_{i=n-r}^{n}\sum\limits_{j=n-r}^{n} x_{ij})+2\sum\limits_{i=n-r}^{n} r_{i,n+k-1} \\ +(r_{n+k-1,n+k-1})] \end{array}
\end{bmatrix} \tag{110}
$$

With reference to Equations 96 and 97 we have that

$$x_{ii} = y_{ii} + \sum_{p=n+1}^{n+k-1} r_{ip} \, (R_{22})^{-1} \sum_{p=n+1}^{n+k-1} r_{pi}; \; i=1, \ldots, n \tag{111}$$

and

$$x_{ij} = y_{ij} + \sum_{p=n+1}^{n+k-1} r_{ip} \, (R_{22})^{-1} \sum_{p=n+1}^{n+k-1} r_{pj}; \; i=1, \ldots, n; \; j=1, \ldots, n. \tag{112}$$

After these substitutions have been made, Equation 106 will have all of its elements in terms of y_{ij}; $i=1, \ldots, n$; $j=1, \ldots, n$ and the unknown quantities, r_{st}; $s=1, \ldots, n+k-1$; $t=1, \ldots, n+k-1$. Of course, $y_{ij} = y_{ji}$ and $r_{st} = r_{ts}$. The set of inequalities that must be satisfied for a network realization with reference to matrix Y_{k-1}^{L} are given by

$$l_{i'j'} \leq 0; \; i' \neq j'; \; i'=1', \ldots, (n+k-1)'; \; j'=1', \ldots, (n+k-1)' \tag{113}$$

and

$$l_{i'i'} \geq \sum_{\substack{j' \neq i' \\ j'=1'}}^{(n+k-1)'} |l_{i'j'}|; \; i'=1', \ldots, (n+k-1)'. \tag{114}$$

If these $(n+k-1)\,\frac{(n+k)}{2}$ inequalities can be satisfied, Y_{k-1}^{L} will be realizable with positive conductances; therefore, the Y matrix will also be realized. Just as with the previous synthesis methods $|l_{i'j'}|$ represents the conductance common to cut-set i' and cut-set j' of the Lagrangian tree in Figure 23. The conductance which is common to cut-set i' and connects to the common node of the Lagrangian tree of Figure 23 is given by

$$l_{i'i'} - \sum_{\substack{j' \neq i' \\ j'=1'}}^{(n+k-1)'} |l_{i'j'}|.$$

To further clarify how the realization of Y is obtained by the realization of Y^L_{k-1}, the conductance common to cut-set n' and cut-set R'_{n+k-1} in Figure 23 is the same conductance that appears across the $n^{\underline{th}}$ port of interest associated with the port-admittance matrix, Y. Likewise, the conductance common to cut-sets n' and $(n-r)'$ is the same as the conductance connecting the noncommon nodes of ports n and (n-r) which are located in the $k^{\underline{th}}$ Lagrangian subtree with reference to matrix Y.

G. Example Four Utilizing K-Tree Synthesis of a Port-Admittance Matrix Which is Nonrealizable by an (n+1)-Node Lagrangian Tree

The method of synthesis as described in Section IVF can also be used to realize the port-admittance matrix given by Equation 79 and presented here as Equation 115.

$$Y = \left[\begin{array}{cc|c} \frac{200}{24} & -\frac{145}{24} & -\frac{108}{24} \\ -\frac{145}{24} & \frac{167}{24} & \frac{108}{24} \\ \hline -\frac{108}{24} & \frac{108}{24} & \frac{192}{24} \end{array}\right] \tag{115}$$

The author will use the same notation as in Section IVF. Partitioning Y as in example two we have k=2. With k=2 and n=3, Equation 99 gives the number of unknowns as four. These will be designated as the elements in the $4^{\underline{th}}$ row and column of $\overline{Y}_1$ — Equation 96. Remembering the symmetry of matrix $\overline{Y}_{k-1}$, Equation 116 gives

$$\bar{Y}_{k-1}\Big|_{k=2} = \bar{Y}_1 = \begin{bmatrix} x_{11} & x_{12} & x_{13} & r_{14} \\ x_{12} & x_{22} & x_{23} & r_{24} \\ x_{13} & x_{23} & x_{33} & r_{34} \\ r_{14} & r_{24} & r_{34} & r_{44} \end{bmatrix}. \tag{116}$$

By utilizing Equations 111 and 112, the x_{ij} terms (i=1,2,3; j=1,2,3) can be replaced by equivalent expressions involving the unknown quantities and the elements of the port-admittance matrix, Y. Thus, $\bar{Y}_1$ can be written as

$$\bar{Y}_1 = \begin{bmatrix} (\frac{200}{14} + \frac{r_{14}^2}{r_{44}}) & (-\frac{145}{24} + \frac{r_{14}r_{24}}{r_{44}}) & (-\frac{108}{24} + \frac{r_{14}r_{34}}{r_{44}}) & r_{14} \\ (-\frac{145}{24} + \frac{r_{14}r_{24}}{r_{44}}) & (\frac{167}{24} + \frac{r_{24}^2}{r_{44}}) & (\frac{108}{24} + \frac{r_{24}r_{34}}{r_{44}}) & r_{24} \\ (-\frac{108}{24} + \frac{r_{14}r_{34}}{r_{44}}) & (\frac{108}{24} + \frac{r_{24}r_{34}}{r_{44}}) & (\frac{192}{24} + \frac{r_{34}^2}{r_{44}}) & r_{34} \\ r_{14} & r_{24} & r_{34} & r_{44} \end{bmatrix}. \tag{117}$$

Figures 22, 23 and 24 are now applied to the example. These are shown by Figures 25, 26, and 27. The cut-set matrix, Q, is constructed from Figure 23 as

$$Q = [Q_{11} \quad Q_{12}] = \left[\begin{array}{c|c} 1000 & 10\ 0\ 0 \\ 0100 & 01\ 0\ 0 \\ 0010 & 00\ 1\ 0 \\ 0001 & 00\text{-}1\text{-}1 \end{array}\right] . \tag{118}$$

Performing the matrix operations of Equation 105 on the matrix $\bar{Y}_1$, the following matrix is formed and is designated as Y_1^L.

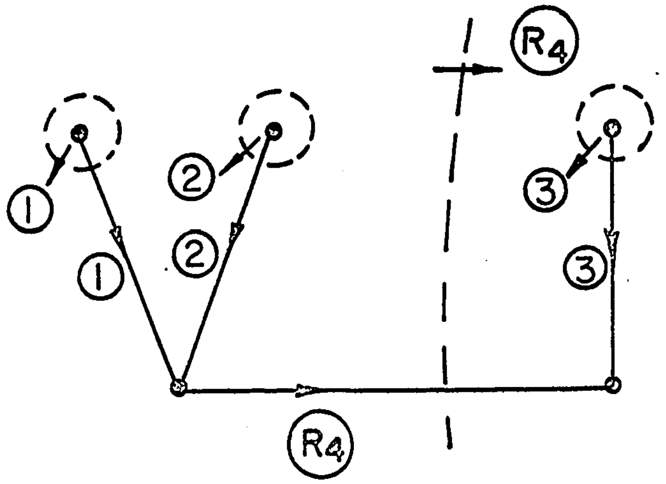


Figure 25. Graph of two Lagrangian subtrees with one connecting port

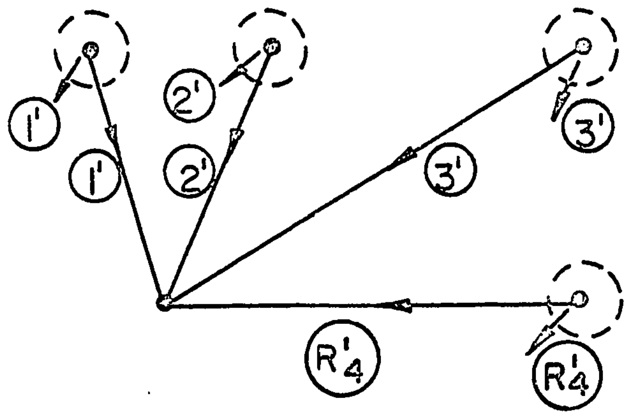


Figure 26. Graph of one Lagrangian tree utilizing all five nodes of the network

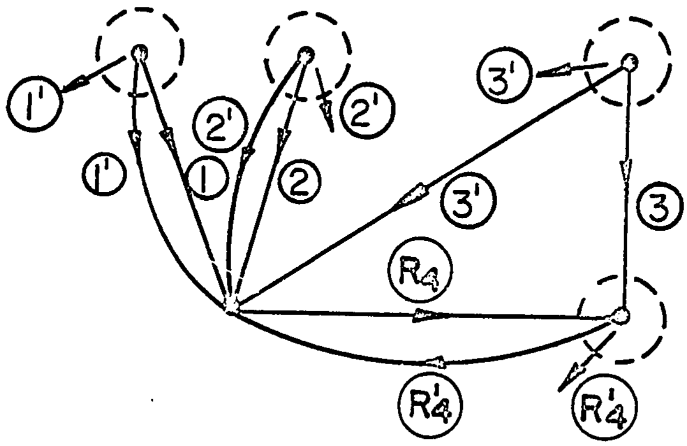


Figure 27. Superposition of Figure 26 on Figure 25 providing for the formation of the transformation matrix, Q

$$Y_1^L = \begin{bmatrix} (\frac{200}{14} + \frac{r_{14}^2}{r_{44}}) & (-\frac{145}{24} + \frac{r_{14}r_{24}}{r_{44}}) & (-\frac{108}{24} + \frac{r_{14}r_{34}}{r_{44}}) & (\frac{108}{24} - \frac{r_{14}r_{34}}{r_{44}} - r_{14}) \\ (-\frac{145}{24} + \frac{r_{14}r_{24}}{r_{44}}) & (\frac{167}{24} + \frac{r_{24}^2}{r_{44}}) & (\frac{108}{24} + \frac{r_{24}r_{34}}{r_{44}}) & (-\frac{108}{24} - \frac{r_{24}r_{34}}{r_{44}} - r_{24}) \\ (-\frac{108}{24} + \frac{r_{14}r_{34}}{r_{44}}) & (\frac{108}{24} + \frac{r_{24}r_{34}}{r_{44}}) & (\frac{192}{24} + \frac{r_{34}^2}{r_{44}}) & (-\frac{192}{24} - \frac{r_{34}^2}{r_{44}} - r_{34}) \\ (\frac{108}{24} - \frac{r_{14}r_{34}}{r_{44}} - r_{14}) & (-\frac{108}{24} - \frac{r_{24}r_{34}}{r_{44}} - r_{24}) & (-\frac{192}{24} - \frac{r_{34}^2}{r_{44}} - r_{34}) & (\frac{192}{24} + \frac{r_{34}^2}{r_{44}} + 2r_{34} + r_{44}) \end{bmatrix} \tag{119}$$

One should note here that Equations 106-110 give Y_1^L directly when the proper substitutions corresponding to Equations 111 and 112 are carried through.

Matrix Y_1^L must satisfy the necessary and sufficient conditions for realization upon the Lagrangian tree of Figure 26. Thus, Inequalities 113 and 114 must be satisfied. A solution that exists for a network realization is $r_{14} = r_{24} = 11$, $r_{34} = -12$, and $r_{44} = 24$. Inequality 113 yields the following set of inequalities when the values for the unknown quantities are substituted. Thus,

$$-\frac{145}{24} + \frac{(11)(11)}{24} = -1 \leq 0 \tag{120}$$

$$-\frac{108}{24} + \frac{(11)(-12)}{24} = -10 \leq 0 \tag{121}$$

$$\frac{108}{24} + \frac{(11)(-12)}{24} = -1 \leq 0 \tag{122}$$

$$+\frac{108}{24} - \frac{(11)(-12)}{24} - 11 = -1 \leq 0 \tag{123}$$

$$-\frac{108}{24} - \frac{(11)(-12)}{24} - 11 = -10 \leq 0 \tag{124}$$

and

$$-\frac{192}{24} - \frac{(-12)^2}{24} - (-12) = -2 \leq 0. \tag{125}$$

Likewise, Inequality 114 yields the following set when the values of the unknown quantities are substituted. Here the magnitudes of the appropriate Inequalities 120-125 are used.

$$\frac{200}{14} + \frac{(11)^2}{24} \geq 1 + 10 + 1 = 12$$

or

$$\frac{321}{24} > 12, \tag{126}$$

$$\frac{167}{24} + \frac{(11)^2}{24} \geq 1 + 1 + 10 = 12$$

or

$$12 = 12, \tag{127}$$

$$\frac{192}{24} + \frac{(-12)^2}{24} \geq 10 + 1 + 2 = 13$$

or

$$14 > 13, \tag{128}$$

$$\frac{192}{24} + \frac{(-12)^2}{24} + 2(-12) + 24 \geq 1 + 10 + 2 = 13$$

or

$$14 > 13. \tag{129}$$

Therefore, Y_1^L can be written as

$$Y_1^L = \begin{bmatrix} \frac{321}{24} & -1 & -10 & -1 \\ -1 & 12 & -1 & -10 \\ -10 & -1 & 14 & -2 \\ -1 & -10 & -2 & 14 \end{bmatrix} \tag{130}$$

Realizing matrix Y_1^L directly upon the Lagrangian tree structure of Figure 26 the network of Figure 28 is obtained. The port representation corresponding to the original port-admittance matrix is maintained, thus the realization of Y_1^L also provides a realization for Y. The author wishes to note that the identical realizations of Figures 20 and 28 are by no means an indication of a unique realization for the Y matrix.

H. Example Five Utilizing K-Tree Synthesis of a Port-Admittance Matrix Which is Nonrealizable by an (n+1)-Node Network

The method of synthesis presented in Section IVF will be applied to the port-admittance matrix of Equation 131

$$Y = \left[\begin{array}{cc|c} 9 & -5 & 5 \\ -5 & 9 & 1 \\ \hline 5 & 1 & 9 \end{array}\right] \tag{131}$$

which is not realizable by any network with (n+1) nodes. With the indicated partitioning of Y we have k=2. Thus, there are four unknown quantities for which to solve. Keeping in mind the symmetry condition, the matrix $\overline{Y}_{k-1}$ is formed as

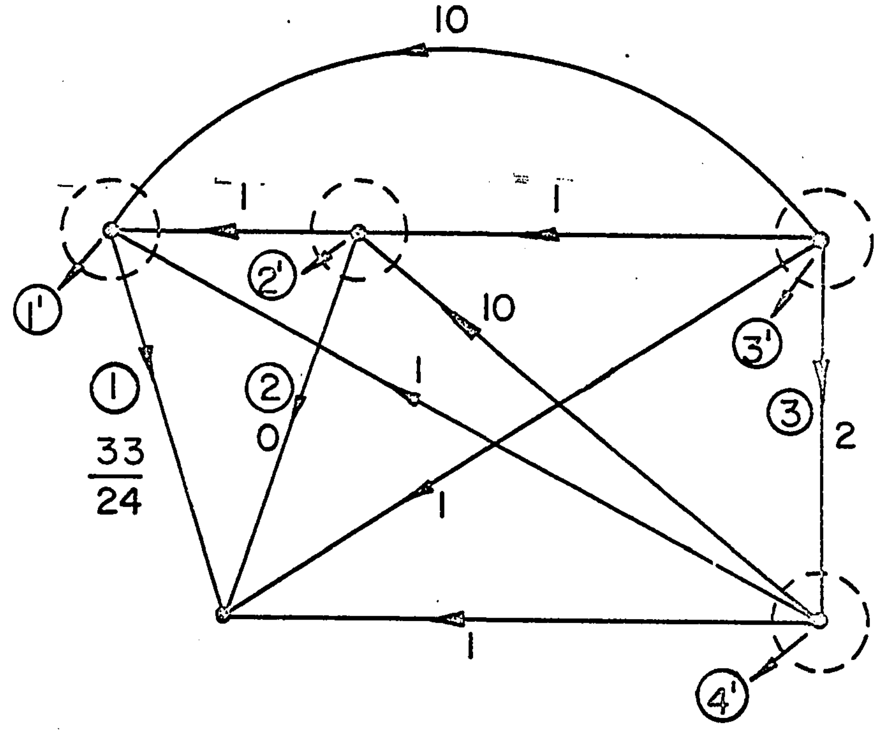


Figure 28. Complete realization for Equation 115 with k = 2 utilizing k-tree synthesis method with conductance values given

$$\bar{Y}_{k-1}\Big|_{k=2} = \bar{Y}_1 = \begin{bmatrix} x_{11} & x_{12} & x_{13} & r_{14} \\ x_{12} & x_{22} & x_{23} & r_{24} \\ x_{13} & x_{23} & x_{33} & r_{34} \\ r_{14} & r_{24} & r_{34} & r_{44} \end{bmatrix} \tag{132}$$

Upon making the proper substitutions as in Equation 117 (example four), $\bar{Y}_1$ can be rewritten as

$$\bar{Y}_1 = \begin{bmatrix} (9 + \frac{r_{14}^2}{r_{44}}) & (-5 + \frac{r_{14}r_{24}}{r_{44}}) & (5 + \frac{r_{14}r_{34}}{r_{44}}) & r_{14} \\ (-5 + \frac{r_{14}r_{24}}{r_{44}}) & (9 + \frac{r_{24}^2}{r_{44}}) & (1 + \frac{r_{24}r_{34}}{r_{44}}) & r_{24} \\ (5 + \frac{r_{14}r_{34}}{r_{44}}) & (1 + \frac{r_{24}r_{34}}{r_{44}}) & (9 + \frac{r_{34}^2}{r_{44}}) & r_{34} \\ r_{14} & r_{24} & r_{34} & r_{44} \end{bmatrix} . \tag{133}$$

Since the port structure of the network corresponds exactly to that of example four the transformation matrix is exactly the same and Figures 25-27 are applicable. Therefore, Equation 118 is applied and the figures are not duplicated here. Performing the appropriate linear transformation, or utilizing the derived general form for matrix Y_1^L, this matrix is given as

$$
Y_1^L = \begin{bmatrix}
(9 + \frac{r_{14}^2}{r_{44}}) & (-5 + \frac{r_{14}r_{24}}{r_{44}}) & (5 + \frac{r_{14}r_{34}}{r_{44}}) & (-5 - \frac{r_{14}r_{34}}{r_{44}} - r_{14}) \\
(-5 + \frac{r_{14}r_{24}}{r_{44}}) & (9 + \frac{r_{24}^2}{r_{44}}) & (1 + \frac{r_{24}r_{34}}{r_{44}}) & (-1 - \frac{r_{24}r_{34}}{r_{44}} - r_{24}) \\
(5 + \frac{r_{14}r_{34}}{r_{44}}) & (1 + \frac{r_{24}r_{34}}{r_{44}}) & (9 + \frac{r_{34}^2}{r_{44}}) & (-9 - \frac{r_{34}^2}{r_{44}} - r_{34}) \\
(-5 - \frac{r_{14}r_{34}}{r_{44}} - r_{14}) & (-1 - \frac{r_{24}r_{34}}{r_{44}} - r_{24}) & (-9 - \frac{r_{34}^2}{r_{44}} - r_{34}) & (9 + \frac{r_{34}^2}{r_{44}} + 2r_{34} + r_{44})
\end{bmatrix}
\tag{134}
$$

If Inequalities 113 and 114 are satisfied, then matrix Y_1^L satisifes the necessary and sufficient conditions for realization upon the Lagrangian tree of Figure 26. One solution that provides for a network realization is $r_{14} = 17$, $r_{24} = \frac{85}{8}$, $r_{34} = -\frac{85}{8}$, $r_{44} = \frac{289}{8}$. When these values for the unknown quantities are substituted into Inequalities 113 and 114 the following results are obtained. From Inequality 113

$$
-5 + \frac{(17)(\frac{85}{8})}{\frac{289}{8}} = 0 \le 0, \tag{135}
$$

$$
5 + \frac{(17)(-\frac{85}{8})}{\frac{289}{8}} = 0 \le 0, \tag{136}
$$

$$1 + \frac{(\frac{85}{8})(-\frac{85}{8})}{\frac{289}{8}} = -\frac{17}{8} \leq 0, \tag{137}$$

$$-5 - \frac{(17)(-\frac{85}{8})}{\frac{289}{8}} = -17 \leq 0, \tag{138}$$

$$-1 - \frac{(\frac{85}{8})(-\frac{85}{8})}{\frac{289}{8}} - \frac{85}{8} = -\frac{17}{2} \leq 0, \tag{139}$$

and

$$-9 - \frac{(-\frac{85}{8})^2}{\frac{289}{8}} - (-\frac{85}{8}) = -\frac{3}{2} \leq 0. \tag{140}$$

Using the proper magnitudes from Inequalities 135-140 in Inequality 114 we have

$$9 + \frac{(17)^2}{\frac{289}{8}} \geq 0 + 0 + 17 = 17$$

or $$17 = 17, \tag{141}$$

$$9 + \frac{(\frac{85}{8})^2}{\frac{289}{8}} \geq 0 + \frac{17}{8} + \frac{17}{2} = \frac{85}{8}$$

or $$\frac{97}{8} > \frac{85}{8}, \tag{142}$$

$$9 + \frac{(-\frac{85}{8})^2}{\frac{289}{8}} \geq 0 + \frac{17}{8} + \frac{3}{2} = \frac{29}{8}$$

or $$\frac{97}{8} > \frac{29}{8}, \tag{143}$$

$$9 + \frac{(-\frac{85}{8})^2}{\frac{289}{8}} + 2(-\frac{85}{8}) + \frac{289}{8} \geq 17 + \frac{17}{2} + \frac{3}{2} = 27$$

or $$27 = 27. \tag{144}$$

Using the foregoing results, Y_1^L becomes

$$Y_1^L = \begin{bmatrix} 17 & 0 & 0 & -17 \\ 0 & \frac{97}{8} & -\frac{17}{8} & -\frac{17}{2} \\ 0 & -\frac{17}{8} & \frac{97}{8} & -\frac{3}{2} \\ -17 & -\frac{17}{2} & -\frac{3}{2} & 27 \end{bmatrix}. \tag{145}$$

Figure 29 shows the realization of the matrix Y_1^L upon the Lagrangian tree of Figure 26. Once again, the port representation corresponding to the original port-admittance matrix is maintained. Therefore, the realization of Y with (n+2) nodes is achieved.

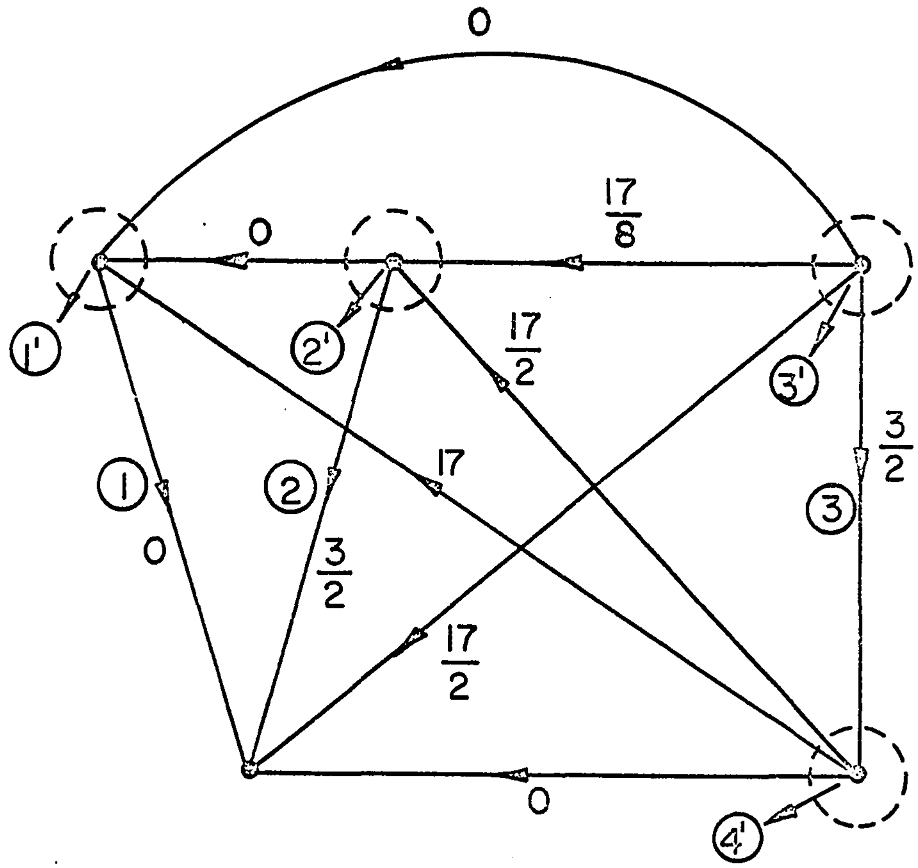


Figure 29. Complete realization for Equation 131 with k = 2 utilizing k-tree synthesis method with conductance values given

V. SUMMARY AND SUGGESTED RESEARCH CONSIDERATIONS

This investigation delves into the synthesis problem of resistive networks containing (n+p) nodes where $2 \leq p \leq n$. The port characteristics of the realized network are defined by a given (nxn) port-admittance matrix, Y. All three of the methods discussed rely upon the basic principles of network topology as applied to the star (Lagrangian) tree.

In the first method which utilizes the Slepian-Weinberg procedure the necessary and sufficient conditions for realization of an (n+2)-node resistive network with a Lagrangian subtree containing (n-1) ports compel the original port-admittance matrix to be hyperdominant. It is also shown that the hyperdominant condition can be relaxed to a dominant condition if a possible sign pattern of Y appears such that Y is potentially hyperdominant (i.e., a matrix that is hyperdominant after a finite number of cross-sign changes). An example clearly illustrates that the magnitudes of the elements in the Y matrix govern the removal of a particular port. One of the valuable features of this synthesis method is the elimination of the necessity for solving for any unknown quantities throughout the entire synthesis process.

The second method of (n+2)-node synthesis applies to any port-admittance matrix which can be partitioned such that there are two submatrices which are hyperdominant or potentially hyperdominant on the diagonal. The two corresponding Lagrangian subtrees are connected by the unknown quantities (conductances) which connect each node in one subtree to every node in the other subtree. The necessary conditions for realization are based upon satisfying the appropriate inequalities and equalities which are derived from the principles of the Lagrangian subtree synthesis and by precisely satisfying the element values of the off-diagonal submatrices of the Y matrix.

A general method of synthesis is presented which allows for the partitioning of the port-admittance matrix into a matrix with k sub-matrices on the diagonal. Once again, these submatrices must be hyper-dominant or potentially hyperdominant. Only the hyperdominant condition is considered in the derivations. To obtain the necessary conditions for realization a linear transformation is used between the structure associated with the k Lagrangian subtrees with their (k-1) connecting ports and the structure of one Lagrangian tree covering the entire set of nodes. The corresponding augmented matrix is realized upon this latter Lagrangian tree thus providing for the realization of the original (nxn) port-admittance matrix. This method also preserves the original port structure and orientation.

The nonlinear nature of the equations and inequalities which contribute to the realization of the port-admittance matrices might possibly be attacked in several ways. The solution will certainly not be unique in the general sense and in most cases it will be a solution having some or all of the unknown quantities bounded. By placing some physical restrictions on the elements of the matrices or conductance values one could eliminate some of the unknown quantities and thus provide for an easier solution. Perhaps arbitrary values could be chosen and all of the inequalities could be changed to equalities giving a set of independent equations to solve. With several judicious choices for the inequalities one might discover a movement pattern for the unknown quantities which might present a clue as to the form of solution.

Since the synthesis procedures are all oriented toward the network which possesses a complete connecting graph it is really immaterial as to the actual positioning of the augmenting port. The cut-sets involved with the augmenting ports include all of the connecting edges so really any one of these edges in a particular cut-set could serve as the augmenting port associated with that cut-set. The end points of this edge would be considered the port terminals. The author has chosen the basic form of the network to appear as in Figure 22.

There are several suggestions which might prove to be worthwhile as further investigations. The Slepian-Weinberg method as used in this investigation applies to only one port removal. Perhaps there could be a form of connecting network defined so that a Lagrangian subtree consisting of more than one port could be removed. This would prove to be a very valuable step since there are no unknowns for which to solve when using the Slepian-Weinberg procedure. The author is relatively sure that similar results are being investigated with linear subtree methods.

Considering the beginning comments of Section IVF, one might wish to pursue the synthesis procedure as based directly on the k Lagrangian subtrees utilizing the idea of parasitic realizations. This would follow in a similar manner as the method of Section IVC except that the unknown quantities would be matrix elements and not conductances. In view of Equation 99 and the available number of independent equations and inequalities there would obviously be a maximum value for k. Also, another limitation which is imposed by the circuit configuration of Figure 22 is that for each $i^{\underline{th}}$ row (column) of $\overline{Y}_{k-1}$, $r_{i,n+x} \geq r_{i,n+y}$ where $x < y$ ($r_{n+x,i} \geq r_{n+y,i}$ where $x < y$) and $x=1, \ldots, (k-1)$, $y=1, \ldots, (k-1)$.

Another suggestion might be to find the optimum time for the elimination of the unknowns which are written in terms of x in Equation 96 with respect to the form of the port-admittance matrix. Perhaps this procedure could be carried further with the goal to be that of finding the necessary and sufficient conditions as applied to the original port admittance matrix for realization with k Lagrangian subtrees with the structure of Figure 22. As previously mentioned in the Literature Search this type of investigation is being conducted with linear subtrees but no known solutions have as yet been presented.

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