Misconceptions on Part-Part-Whole Proportional Relationships Using Proportional Division Problems

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Abstract

This study focuses on proportional division, a specific problem type of proportions and proportional relationships, which embeds a part-part-whole relationship. We investigated how students in grades 6 to 9 solved and perceived three proportional division problems. Analysis of the students' performance on the tasks revealed three types of misconceptions emerged in solving proportional division problems: regarding a ratio as a number, additive reasoning, and confusion about the whole. Confusion about the whole was the misconception that appeared most frequently across all grades. This finding suggests teachers should provide various types of proportional problems, especially with part-part-whole ratios. Proportional division is an essential type of problem in working with a part-part-whole relationship while providing a more contextual method for reasoning with ratios.

KEYWORDS: Proportional division, Proportions, Ratios, Mathematical misconception, Part-Part-Whole relationship

Proportional reasoning is “a milestone in student’s cognitive development” (Lobato & Ellis, 2010, p. 48) and plays a critical role in developing algebraic thinking and function sense (National Council of Teachers of Mathematics, 2013; National Mathematics Advisory Panel, 2008). Despite the importance of proportions, “fractions, ratios, and proportions arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging” (Lamon, 2007, p. 629). In fact, more than 50% of eighth-grade students in the U.S. incorrectly answered proportion problems on the National Assessment of Educational Progress (NAEP) (National Center for Education Statistics, 2013). Our larger study (Martinez & I, 2018), which also investigated students’ understanding of proportional reasoning, revealed that they experienced significant difficulty with proportional division problems that involve ratios of part-part-whole relationships. Hence, this paper particularly focuses on analyzing students’ misconceptions on proportional division problems.

Proportional division is defined as “dividing an amount up according to a particular ratio and sometimes called division in a given ratio” (The Centre for Innovation in Mathematics Teaching, n.d.). The Common Core State Standards for Mathematics (CCSSM) (National Governors Association [NGA] Center for Best Practices & Council of Chief State School Officers [CCSSO], 2010) do not directly address proportional division, although this problem type is often included in assessments (e.g., NAEP). It is assumed that if students can reason proportionally, they should be able to solve proportional division problems successfully. For example, we can look at the proportional division problem, “The ratio of boys to girls to adults at a school party was 6:5:2. There were 78 people at the party. How many of them were adults?” (NAEP, Question ID: 2013-8M7 #13 M164801). Since the whole (78 people) is given, it can be divided proportionally using the ratio 6:5:2. Then, students can reason that the number of adults is $\frac{2}{6+5+2}$ of 78.

Proportional division problems involve a part-part-whole relationship in nature, but U.S. mathematics textbooks and/or classroom instruction mostly use a ratio of a part-part relationship when teaching ratios (Kilpatrick, Swafford, & Findell, 2001). Yet, proportional division problems must involve the whole of a ratio, which is not explicit in a ratio of a part-part relationship. This limited view on ratios may cause difficulties and misconceptions when students deal with other relationships such as a part-part-whole relationship. Thus, the research question driving this study is, What misconceptions do secondary students reveal when they solve proportional division problems involving a part-part-whole relationship?

Literature Review

Students in the U.S. formally begin their learning of ratios and proportions in sixth grade and complete it by the end of grade 7 (NGA Center for Best Practices & CCSSO, 2010), although they keep extending the knowledge beyond these grades (Heinz & Sterba-Boatwright, 2008). Traditionally, the most common forms of proportion tasks in U.S. curricula are missing value problems (Kilpatrick et al., 2001). However, other types of proportion tasks, summarized by de la Torre, Tjoe, Rhoads and Lam (2013) are often included: (1) a missing value problem, (2) a comparison problem, and (3) a qualitative problem. In a missing value problem, three of the four values are given in a proportion $\frac{a}{b} = \frac{c}{d}$ and the task is to find the fourth, or missing, value. In a comparison problem, students are given two ratios and asked to determine whether they are equivalent or not. A qualitative problem asks students to consider the effect of an increase or a decrease in one part of a proportion. Similarly, Lamon (1993) proposed four categories of ratio problems: (1) well-chunked measures, (2) part-part-whole, (3) associated sets, and (4) stretchers and shrinkers. Lamon's classification focused on problem contexts while de la Torre et al. (2013) attended to the structure of problems. Well-chunked measures-type problems involve the comparison of two extensive measures, resulting in an intensive measure (e.g., speed). In a part-part-whole problem, a single subset of a whole is given such as a ratio of X boys to Y girls in a classroom. Associated sets are the ratios of two elements that have an ill-defined connection (e.g., people and pizzas). Finally, the stretchers and shrinkers type is similar to the qualitative problem type de la Torre et al. (2013) suggested, but the ratios of stretchers and shrinkers involve scaling up or down in general geometric or measurement situations. Lamon (1993) argued that associated sets were the easiest, followed by part-part-whole, well-chunked measures, and stretchers and shrinkers. However, Alatorre and Figueras (2005) found that mixture problems—a particular type of part-part-whole problems—are more difficult for students than rate problems (well-chunked measures).

In addition to types of ratios, the relationship between ratios and fractions provides a crucial point to conceptualize ratios and proportions. Clark, Berenson, and Cavey (2003) proposed five models and discussed how teachers conceptualized fractions and ratios and what limitations each model has: (1) ratios as a subset of fractions, (2) fractions as a subset of ratios, (3) ratios and fractions as distinct sets, (4) ratios and fractions as overlapping sets, and (5) ratios and fractions as identical sets. This classification involves both part-part and part-whole ratios. Lobato and Ellis (2010) support the fourth model where ratios and fractions share an intersection. The intersection occurs when a ratio represents a part-whole relationship and the term *fraction* is also used for fractional notation. Being able to distinguish when to set up ratios as part-part or part-whole dictates what approach can be used to reason proportionally. This also requires students to identify what each term in a ratio represents in the given context. Hence, it is important for students to understand the contextual relationship prior to solving a proportion problem. In this sense, Adjiage and Pluvinage (2007) argued the importance of using various physical-empirical situations and representations of mathematical objects when teaching ratios and proportions as a result of their French curriculum analysis.

For decades, a significant body of research reported the difficulty of understanding proportions. Various factors that influence students' struggling with the concepts and skills associated with proportions have been noted, such as additive reasoning (Kaput & West, 1994; Karplus, Pulos, & Stage, 1983); limited knowledge of proportion notations (Silver, 1981); discrete or continuous quantities (Jeong, Levine, & Huttenlocher, 2007); overdependence on textbook representations (Silver, 1981); overuse of proportionality on missing-value problems (Van Dooren, De Bock, Evers, & Verschaffel, 2009); including non-integers in proportions (Lesh, Post, & Behr, 1988; Vergnaud, 1988); number formation of non-integer ratios (Fernandez, Llinares, Modestou, & Gagatsis, 2010; Van Dooren et al., 2009); and identifying proportional relationships and non-proportional relationships (De Bock, Verschaffel, Janssens, Van Dooren, & Claes, 2003; Van Dooren, De Bock, Janssens, & Verschaffel, 2007). However, there has been less attention paid to a misconception related to part-part-whole relationship or proportional division problems. Most studies used only a part-part ratio without considering the whole of the part-part-whole relationship. Although a few researchers (Beckmann & Izsak, 2015; Lamon, 1993) used proportional division problems as example tasks, they did not specify proportional division.

Even though there is a significant body of literature that discusses various students' misconceptions and strategy patterns on proportions, a study focusing on students' understanding of proportional division or proportional reasoning based on a part-part-whole ratio is rare.

Methods

In this section, we provide such details as participants, tasks, and data collection processes as well as our data analysis methods. We begin with describing the theoretical background of the mathematical concepts that provide context for our research design and data analysis.

Mathematical Analysis

It is important to define terms, such as ratios, rates, and proportions, because some of them have multiple meanings or interpretations. As Lamon (2012) stated, "[e]veryday language and usage of rates and ratios is out of control. The media have long employed ratios and rates and the language appropriate to ratios and rates in many different ways, sometimes inconsistently, sometimes interchangeably" (p. 226). The CCSSM progression draft (The Common Core Standards Writing Team, 2012) defines a ratio as a pair of non-negative numbers, A:B, which are not both zero. According to this definition, a ratio is different from a fraction because a ratio represents a relationship between two or more quantities while a fraction generally represents one quantity. Hence, there is no overlap between ratios and fractions with this definition, as described in the third model of Clark et al. (2003). The definition of rates typically goes along with that of ratios. One popular traditional definition set of ratios and rates is that a ratio is a pair of two like quantities with the same units and a rate is a pair of unlike quantities with different units (Vergnaud, 1988). This definition is similar to that of internal ratios and external ratios (Freudenthal, 1983). However, these traditional definitions have limitations. First, ratios and rates are often given without units in textbooks and assessments. Second, the definition set is not aligned with advanced mathematical concepts. For example, slope is a unit rate of change—change in y per change in x—but the x and y might the same unit or might be given without a context of units. To reduce confusion from multiple definitions and provide a common framework, the Common Core Standards Writing Team suggests, "When there are *A* units of one quantity for every *B* units of another quantity, a rate associated with the ratio *A*:*B* is *A/B* units of the first quantity per 1 unit of the second quantity (Note that the two quantities may have different units)" (2012, p. 13).

One of the essential characteristics of ratios is the ability to obtain an equivalent ratio by multiplying or dividing all terms by the same number (Lobato & Ellis, 2010). Related to this, the CCSSM progression draft proposes a way to define an equivalent ratio using *the value of a ratio*: Equivalent ratios are determined when the values of two ratios are equal (the value of a ratio a:b is $\frac{a}{b}$) (the Common Core Standards Writing Team, 2012). Using the value of a ratio is important because we can compare (e.g., greater than or less than) the value of ratios while we cannot compare ratios because of its definition as a relationship. Thus, using this term explicitly helps students avoid the potential confusion of ratios as values. Essentially, the value of a ratio has the equal value to the associated rate (or unit rate) of the ratio. Equivalent ratios are also key to build a proportion, defined by the Common Core Standards Writing Team (2012) as "an equation stating that two ratios are equivalent when equivalent ratios have the same unit rate" (p. 3). This is similar to Lobato and Ellis (2010)'s definition, "a relationship of equality between two ratios" (p. 33). Lamon (2007) also described *proportional reasoning* as "supplying reasons in support of claims made about the structural relationships among four quantities in a context simultaneously involving covariance of quantities and invariance of ratios or products" (pp. 637–638).

Participants

Sixty-four students from grades 6 to 9 at a K–12 private school in the U.S. Midwest participated in this study (see Table 1). The school has two different pathways in the secondary level; (1) an accelerated pathway of pre-algebra A in 7th grade and then algebra in 8th grade; and (2) a regular pathway of pre-algebra B in 8th grade after completing pre-algebra A and then proceeding to algebra in 9th grade. The student population is predominantly from White, middle class families from a suburban area. All the students were taught about the meaning of ratios and simple applications of ratios and rates, such as how to write a ratio from a given situation and how to determine an equivalent ratio before participating in this study.

Table 1

Participant Information from Five Mathematics Classes

Math Class	Number	Grade	Interview participant (pseudonym)
6th grade math	18	All 6th grade	Gina & Denise
Pre-Algebra A (PreA)	14	All 7th grade	Emily
Pre-Algebra B (PreB)	11	7th (3) & 8th (8)	Mike & Haley
Algebra	16	8th (6) & 9th (10)	Solomon & Katlin
Geometry	5	All 9th grade	Amy
Total	64		

Research Design

The data collection consisted of two phases. The first phase was a classroom assessment that included nine open-ended tasks about ratios and proportions (see Appendix A). The nine tasks were developed by the first and third authors to examine students' understanding and misconceptions about ratios and proportions. All 64 students in the five classes participated in this assessment and were asked to solve all nine tasks. However, we decided to include only Tasks 8 and 9 for this study because these two tasks are proportional division problems involving a part-part-whole relationship. The second phase of data collection was a think-aloud interview

(Charters, 2003). From the results of the written assessment, one or two students from each class were selected because their responses implied interesting misconceptions. A total of eight students participated in this individual task-based interview (see Table 1). For this interview, we used the think-aloud method (Charters, 2003) to effectively capture participants' thinking process because the written response on the assessment might not reveal the students' misconceptions of proportional reasoning in depth. The think-aloud protocol includes seven tasks (see Appendix B), and the eight students were given multiple paper cuts of each length A, B, C, and F so they could manipulate those paper cuts to measure and compare lengths. The numbers in the tasks were purposefully chosen so students could not solve the task by using simple multiplication or doubling (building-up strategy), and the context involves various ratio relationships. From the seven tasks, we selected Task 6 to analyze and include in this study due to its use of proportional division.

Proportional Division Problems

The commonalities and differences among Tasks 8 and 9 in the assessment and Task 6 in the interview are described in Table 2. All three tasks involve a part-part-whole relationship, and their contexts require comparing two ratios: one is an abstract ratio simplified to the smallest whole number ratio (e.g., 3:2), and the other is the contextualized ratio consisting of the actual amounts (e.g., 12 cups of lemonade). Commonly, the abstract part-part ratio and the actual amount of the whole (lemonade in the three tasks) are given, and the actual amount of each part (or something similar) is asked to be found. Task 8 includes an additional condition: adding 1 cup of each ingredient to the initial lemonade. Moreover, the numbers in Tasks 8 and 6 form non-integer ratios while Task 9 has integer ratios. Literature reveals non-integer ratios bring much more challenges than integer ratios (Van Dooren et al., 2007). Task 8 involves a more complex situation, adding an equal amount to each part, which aims to examine students' additive reasoning misconception. Although Task 6 requires one more step—finding the actual amount of the whole using subtraction because there was existing lemonade—it is not as complicated as Task 8's added condition.

Table 2

Analysis of Three Tasks of Proportional Division Problems

		Task 8	Task 9	Task 6
Task statement		Eric has 12 cups of lemonade that tastes exactly the same as Brody's (3 cups of water for 2 cups of lemon juice). He needs a larger amount of lemonade. He pours one more cup of water and one more cup of lemon juice. Does his lemonade still taste the same? Why or why not?	Monica uses the same recipe of Brody's lemonade (3 cups of water for 2 cups of lemon juice). She needs to make 45 cups of lemonade. How many cups of water and lemon juice does she need to use to make total 45 cups of lemonade?	To make Aunt Janet's lemonade mix 5 cups of water for 3 cups of lemon juice. Brad has 10 cups of lemonade that tastes exactly the same as Aunt Janet's lemonade. He needs a larger amount of lemonade. How can he make 20 cups of lemonade with the same taste?
Common	Relationship of a ratio	Part-part-whole		
	Structure	Compare two ratios: one ratio of abstract/simplified numbers and the other ratio of an actual amount in context		
	Given information	Two parts of the abstract ratio and the whole of the actual ratio		
Difference	What is the student asked to find	Determine if the final actual ratio is equivalent to the original abstract ratio	Find two parts of the actual ratio	Find two parts of the actual ratio
	Other conditions	Added amount of each part	None	Need to find the actual whole
	Integer/non-integer ratio	Non-integer ratios	Integer ratios	Non-integer ratios

According to a typical way to find the actual amount of each part in Task 8, we first identify the whole term, 5, of the given ratio 3:2. Because the actual amount of the whole is 12 cups of lemonade, we can find the actual amount of water and lemon juice mixed in the lemonade by multiplying 12 by $\frac{3}{5}$ and $\frac{2}{5}$ respectively (see Figure 1). Then, we can add 1 cup of water to $\frac{36}{5}$ cups of water and 1 cup of lemon juice to $\frac{24}{5}$ cups of lemon juice and get $\frac{41}{5}$ and $\frac{29}{5}$. The ratio of these two numbers is 41:29, which is different from 3:2 or its associated rate 1.5:1, so the taste would be different.

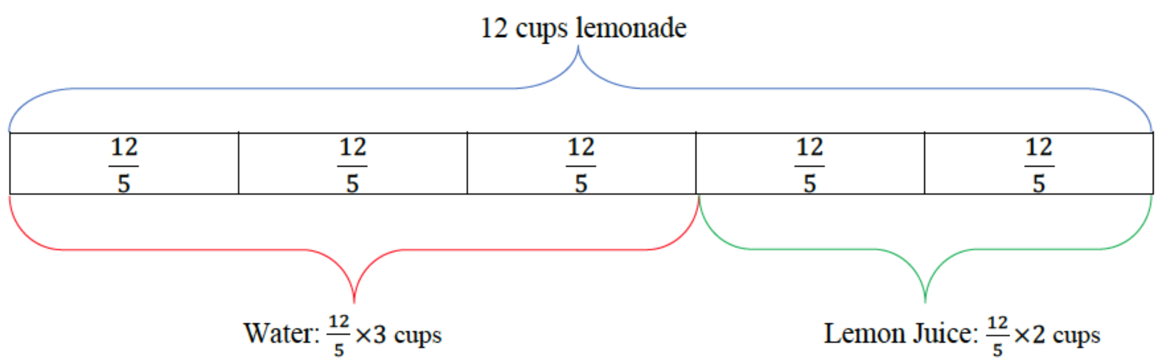


Figure 1. A strip diagram that describes proportional division of 12 cups lemonade by 3:2

If we apply the three types of proportion described by de la Torre et al. (2013), these tasks (Tasks 8, 9, and 6) are similar to qualitative problems but extended in a way that the whole is given instead of one part of the proportion in the given contexts. Using the four problem types described by Lamon (2003), the tasks are part-part-whole problems but involve the stretchers and shrinkers type because they integrate scaling up in general measurement situations.

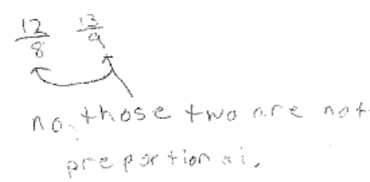
Data Analysis and Sample Codes

We first open-coded students’ responses on Tasks 8 and 9 with respect to students’ misconceptions. The code manual draft was revised several times and finalized to have three codes of misconceptions (see Table 3). Once the coding manual was established for Tasks 8 and

9 it was used to code Task 6 data. To ensure the reliability of our coding, we calculated interrater reliability among two coders which resulted in more than 90% agreement. Then, analytic memos of our interpretations on strong examples were written (Creswell, 2007), leading to emerging themes. After we completed the analysis within tasks, we compared the results across Task 8 and Task 9 to see if the same student had a similar pattern of misconceptions. Task 6 results were not included for the cross-task analysis due to its different setting, a think-aloud protocol.

Table 3

Final Coding Manual

Code	Description	Example
Regarding a ratio as a number	Treating a ratio as a single quantifiable, non-fraction, value. This shows their lack of understanding that a ratio is a relationship of two or more quantities, not quantities themselves.	In Task 8, some students added 1 to each term of the given ratio, which is not the actual amount of each ingredient.
Additive reasoning	Reasoning additively, not multiplicatively, to determine equivalent ratios.	In Task 9, some students determined 23:22 is equivalent to 3:2 because the two terms of both ratios are one apart.
Confusion about the whole	Confusing which number represents the whole quantity in the problem context. This may show they do not fully understand what a ratio is; they only understand a part-part ratio, not part-whole or part-part-whole ratios.	 “No, those two are not proportional”

Findings

We describe the findings within each task, focusing on the students’ misconceptions. The patterns of the students’ responses within each task are first illustrated and the patterns across the tasks and grades follow.

Analysis within Task 8

Seven out of 64 students provided correct answers with valid strategies on Task 8, which was the lowest among all tasks (10.9%, *n* = 7). Although the most frequent misconception, appearing for 20.3% of all the students, was additive reasoning, confusion about the whole was more pervasive since this misconception was seen in all grade levels (see Table 4). This misconception was found in each class except the algebra class. For example, a sixth-grade student's response, "Yes, because he added the same amount of both ingredients," reveals that this student thought an equivalent ratio can be made by adding the same number to each term of a ratio.

Table 4

Results of Coding on Task 8

Misconceptions	6	PreA	PreB	Algebra	Geometry	Average
Regarding a ratio as a number	22.22%	0%	9.1%	12.5%	60%	15.6%
Additive reasoning	33.33%	35.7%	9.1%	0%	20%	20.3%
Confusion about the whole	5.6%	7.1%	27.3%	25%	20%	15.6%

The other two misconceptions were found in 15.6% of the students' responses. The example in Figure 2 shows that this student added one to each term of the given ratio, 3:2, which involves the misconception of regarding the terms in a ratio as fixed values rather than a relationship. 4:3 is not the correct ratio of the amount of water and lemon juice after adding a cup of each ingredient, but the student used the terms in the abstract ratio as if they are the actual amounts of water and lemon juice and wrote the answer is no because 4/3 is not equivalent to 3/2.

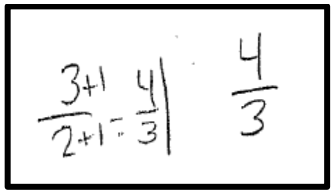
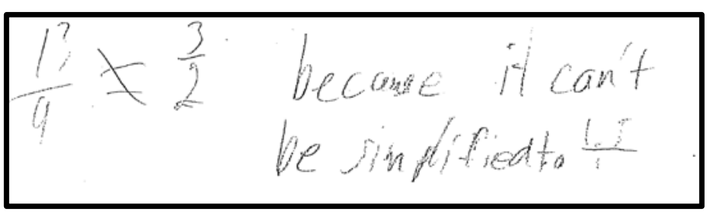
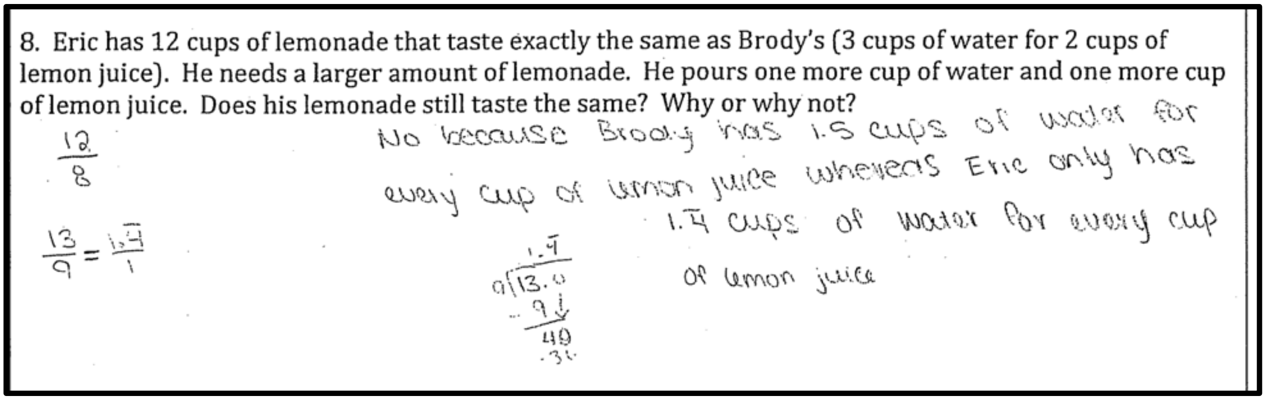


Figure 2. This student in algebra viewed a ratio as a number.

The other misconception was the confusion on finding the whole and parts in the given proportional situation. In Figure 3, although the context of Task 8 indicates the number 12 in “12 cups of lemonade” represents the whole, many students used it as the number of cups of water and deduced 8 proportionally to 12 as the number of cups of lemon juice using the given ratio, 3:2. Two students, shown in Figure 3, added one to each of 12 and 8 and made a ratio, 13:9. Because 13:9 is not equivalent to the given ratio 3:2 or its rate 1.5:1, the students answered no. Although the student (bottom of Figure 3) found the correct rate (“1.5 cups of water for every cups of lemon juice”) of the given ratio, the (incorrectly found) whole-part ratio 13:9 cannot be compared to the part-part ratio 3:2. The students had confusion between a part-part relationship and a part-whole relationship as well as with identifying parts and the whole, although they could use a valid solution strategy (e.g., using a concept of rate). The misconception about the whole appeared in all grades.




8. Eric has 12 cups of lemonade that taste exactly the same as Brody's (3 cups of water for 2 cups of lemon juice). He needs a larger amount of lemonade. He pours one more cup of water and one more cup of lemon juice. Does his lemonade still taste the same? Why or why not?

$\frac{12}{8}$

$\frac{13}{9} = \frac{1.\overline{4}}{1}$

No because Brody has 1.5 cups of water for every cup of lemon juice whereas Eric only has 1.$\overline{4}$ cups of water for every cup of lemon juice

Figure 3. Seventh-grade students' responses that include confusion about the whole.

Analysis within Task 9

Thirteen out of 64 students (20.3%) found correct answers on Task 9, though five students who provided correct answers did not write any process. The results of misconception analysis on Task 9 are provided in Table 5. The patterns in misconceptions on Task 9 were more vivid than those of Task 8. Confusion about the whole was the major misconception, with a high percentage of 46.9%. Interestingly, none of them showed the misconception of regarding a ratio as a number. Additive reasoning also appeared less often here than in Task 8, probably because Task 9 did not have an additive situation, as Task 8 did. An additive reasoning misconception was shown by 10.9% of students in Task 9, and this misconception appeared in lower grades only. The most frequent misconception was confusion about the whole (46.9%). The most common case of this misconception was that the sum of two parts was not 45 (see Figure 4).

Table 5

Results of Coding on Task 9

Misconceptions	6	PreA	PreB	Algebra	Geometry	Total
Regarding a ratio as a number	0 %	0 %	0 %	0 %	0 %	0 %
Additive reasoning	27.8 %	7.1 %	9.1 %	0 %	0 %	10.9 %

Confusion about the whole	61.1 %	64.3 %	45.5 %	18.8 %	40 %	46.9 %

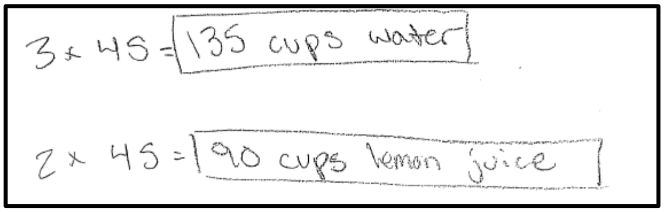


Figure 4. A pre-algebra student's response that shows misconception of the whole

This misconception was pervasively found in all grades. Figure 5 also shows the misconception about the whole where 45 was treated as a part. The student in Figure 5 considered 45 and 3 as amounts of the same kind and applied the multiplicative reasoning with this proportion, 3:2=45:?. Although the calculation was correct, 45 represents the whole (lemonade) and 3 represents a part (water). This implies the student did not find the whole of the abstract ratio, 3:2 and did not check whether the sum of the final two numbers was the given whole, 45.

Figure 5. A pre-algebra student's response that misidentified the whole as a part

Analysis within Task 6

With Task 6, three students showed additive reasoning and one showed confusion about the whole (see Table 6). However, none of the students found a correct answer, although many of them applied valid methods. Among eight students, four students—all in lower grades (6 and 7)—demonstrated misconceptions during their interviews.

Table 6

Results of Coding on Task 6 in Number

Code	6 (2)	PreA (1)	PreB (2)	Algebra (2)	Geometry (1)	Total
Regarding a ratio as a number	-	-	-	-	-	0
Additive reasoning	1	1	1	-	-	3
Confusion about the whole	-	-	1	-	-	1

The most common misconception was additive reasoning. The following excerpt illustrates sixth-grade student Emily's misconception involving additive reasoning.

Emily: If you added those [pause] changes one with respect to... If added them I mean,

Interviewer: You have to mix all of the ingredients.

Emily: yeah so, 5 cups of water plus 3 cups of lemon juice equals 8 cups in general and that is not 10, so you would have to have 6 cups of water and 4 cups of lemon juice to equal 10 cups. Now uh, you would need, 10 [cups] you would have to have like. There is a lot of ways you can do that though?

Emily's final answer was 6 cups of water and 4 cups of lemon juice so the sum would be the desirable amount, 10 cups of lemonade. During this reasoning process, she claimed 6:4 is equivalent to 5:3 because the difference between two terms of the ratios are the same.

Hayley, a seventh-grade student, was the only one who showed the misconception about the whole out of eight students participating in the think-aloud interviews. She first wrote "5/3 10/[blank]," which is similar to "5/3 = 10/*x*," a typical form of a missing-value proportion. Although she applied a valid algorithm, she misinterpreted 10 corresponding to 5, the amount of water (part) while 10 represents the amount of lemonade (whole). The following excerpt demonstrates her confusion on the whole value.

Hayley: I'd say that, to make the 20 cups of lemonade with the same taste he'd have to make 10 more cups of water and 6 more of lemon juice.

Interviewer: Can you explain how you got the answer?

Hayley: Since for the cups she obviously had to make 10 more to make 20 if he had 10 and then I use like if he had like five cups there and went up to 10 so that's times 2, ten time two equals 20 so I did the same with the cups of lemon juice and did 3 times 2 which equals six and then have to take six times and that equaled 12 and then in my mind I kind of subtracted like 12 minus six so I thought about it he need like… Oh, wait. I wrote the wrong thing there he needs 12 more or so I think, wait. He just needs six more cups because if you add that and just think of it as 10 over 6 plus 10 over 6, 20 over 12.

Interviewer: So, if add ten more cups of water and six more cups of lemon juice, then can you make ten cups of lemonade?

Hayley: Um, yes.

In this short excerpt, Hayley explained she could make 10 cups of lemonade from 10 cups of water and 6 cups of lemon juice and the base of this reasoning was the written algorithm that includes 5 and 3 (5/3), 10 and 6 (10/6), and it is easy to make 10 into 20 by doubling. She did not consider what each number represented and did not use her life experience that if she adds 10 cups of water and 6 cups of lemon juice, she would have 16 cups of lemonade, not 10 or 20 cups. The number 20 in her final answer represents the cups of water, not lemonade, which implies her confusion between the whole and a part. This result reveals that knowing and applying an algorithm may lead students to a wrong pathway because they trust the algorithm more than their quantitative reasoning or common sense based on the context.

Analysis Across Tasks and Grades

Since Task 6 was given to a much smaller number of students and conducted through think-aloud interviews instead of written assessments, we compared the results of only Tasks 8 and 9, which had the same participants within the same setting. Among three misconceptions identified from our coding, confusion of the whole appeared most pervasively across all grade levels. However, there was no clear pattern across tasks in terms of misconceptions when we follow individual students' results. Only 11% of the students showed similar misconceptions in Tasks 8 and 9. There was little to no consistency of misconceptions across tasks for most students. The patterns within tasks were stronger than those across tasks. The changes across grades in Task 8 is complex, but in Task 9, when combining two pre-algebra classes and removing the geometry class for its small size (n = 6), a relatively clear pattern was found (see Figure 6).

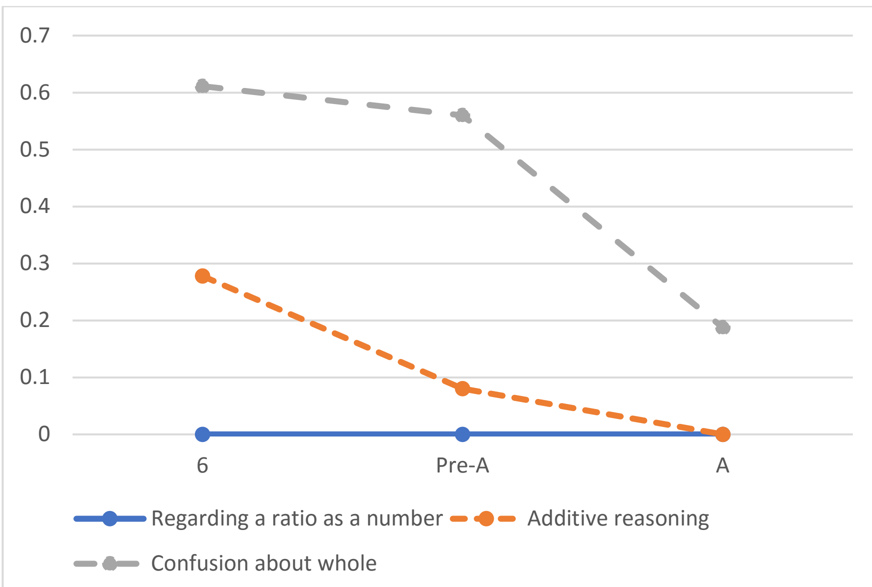


Figure 6. Analysis from 6th grade to algebra classes of Task 9

In Figure 6, the misconception about the whole has the highest rate overall and is consistently decreasing. The misconception of additive reasoning also decreases throughout the three grade levels.

Discussion and Implication

Our results indicate that missing the basic understanding of the definition of ratios (as a relationship) is the base of all misconceptions. Lesh et al. (1988) used the term “pre-PR” to indicate misused proportional reasoning. Examples of this include the inability to recognize the structural similarity of two equivalent relationships of a given proportion, additive reasoning, and the blind application of cross-multiplication. Our study also found some examples of pre-PR (e.g., additive reasoning), but the confusion about the whole of a ratio and treating a ratio as one quantity seem to be stronger misconceptions than pre-PR types of misconceptions. Furthermore, confusion related to the whole or to additive reasoning appeared less frequently as students’ math class level advances (see Figure 6). The reason for this tendency could be that higher-grade students have had more exposure to various proportional problems and have more skills to support their solving processes. The different results from Tasks 8 and 9 suggest that students need to explore various task types, such as integer and non-integer, in order to express their misconceptions and have an opportunity to learn from their misunderstanding.

Previous research has found the types of tasks affect students’ performance. For instance, students tend to use additive reasoning on non-integer ratios and proportional approaches on integer ratios (Van Dooren, De Bock, Evers, & Verschaffel, 2009). This is aligned with our results, because additive reasoning was the most frequent misconception in the non-integer ratio problems (Tasks 8 and 6). Task 8 was designed to capture the additive misconception by using the context of adding 1 cup to each ingredient. Thus, it was not surprising to see more additive reasoning in Task 8 than in other tasks. However, additive reasoning was still strong in Task 6, which uses non-integer ratios, although this task was not intentionally designed to see this misconception. Moreover, the factor of integer or non-integer ratios seemed influential to the difference of the results between Tasks 8 and 9 (Fernandez et al., 2010) because the misconception involving additive reasoning did not appear often in Task 9, which uses integer ratios. Based on this result, we found that when other factors (e.g., non-integer ratios) were not included, confusion about the whole became the most prevalent misconception when solving proportional division problems, as we saw in the result of Task 9. Identifying the whole of a part-part ratio may not be familiar to students because U.S. textbooks do not explicitly address this method (Kilpatrick et al., 2001) and do not provide sufficient proportional division problems.

Moreover, Adjiage and Pluvinage (2007) proposed a framework that describes how students solve a contextual ratio problem, which consists of two levels: physical-empirical situations and semiotic registers for expressing rational numbers. The students who showed confusion of the whole seemed to struggle in the level of physical-empirical situations because they did not fully comprehend the given situation. The misinterpretation on the problem situation misled students to find incorrect semiotic registers or mathematical representations.

Therefore, the results of this study suggest that teachers and curriculum developers provide students with various proportional division problems, because this type of problem can expand students' conceptual understanding of ratios and proportions involving a part-part-whole relationship. With proportional division problems, students can develop their ability to identify relationships within and between ratios and may apply their knowledge of rational numbers. Furthermore, rational or real numbers with ratios and proportions are crucial in advanced mathematics. For example, students will use ratios such as $1:2:\sqrt{3}$ or $1:1:\sqrt{2}$ with special triangles. If students cannot think beyond whole numbers, their mathematical reasoning will be limited. Our results also indicate that students reveal different misconceptions depending on what tasks are given. Hence, one type of task may not be enough to fully understand proportional division. If only Task 9 was given, it would have been difficult to see multiple misconceptions and thus adjust instruction to alleviate them.

Some limitations of this study may include the small number of participants and the absence of intervention. It would be important to investigate how the inclusion of proportional division instruction influenced students' proportional reasoning for future research. This study can be a precursor to more involved studies that conduct pre- and post-assessments of well-designed intervention of proportional division. Moreover, developing proportional division problems aligned with the CCSSM and curriculum analysis of proportion tasks would provide important contributions to the field. Developing the concept of proportional division and understanding students' misconceptions will allow for a robust curriculum analysis. Another implication could be developing an assessment of proportional reasoning. Based on our findings about the common misconceptions around proportional division, there will be a higher chance to develop a more appropriate assessment tool to measure students' understanding of ratios and proportions.

References

Adjiage, R., & Pluvinage, F. (2007). An experiment in teaching ratio and proportion. *Educational Studies in Mathematics*, *65*(2), 149–175.

Alatorre, S., & Figueras, O. (2005). A developmental model for proportional reasoning in ratio comparison tasks. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 25–32). Melbourne, Australia.

Beckmann, S., & Izsak, A. (2015). Two perspectives on proportional relationships: Extending complementary origins of multiplication in terms of quantities. *Journal for Research in Mathematics Education*, *46*(1), 17–38.

The Centre for Innovation in Mathematics Teaching. (n.d.). Unit 7 Section 3: Proportional Division. Retrieved March 15, 2018, from http://www.cimt.org.uk/projects/mepres/book8/bk8i7/bk8_7i3.htm

Charters, E. (2003). The use of think-aloud methods in qualitative research: An introduction to think-aloud methods. *Brock Education*, *12*(2), 68–82.

Clark, M. R., Berenson, S. B., & Cavey, L. O. (2003). A comparison of ratios and fractions and their roles as tools in proportional reasoning. *The Journal of Mathematical Behavior*, *22*(3), 297–317. https://doi.org/10.1016/S0732-3123(03)00023-3

The Common Core Standards Writing Team. (2012). Progressions for the Common Core State Standards in Mathematics: 6-7, ratios and proportional relationships. Retrieved from http://ime.math.arizona.edu/progressions/

Creswell, J. W. (2007). *Qualitative inquiry & research design: Choosing among five approaches* (2nd ed). Thousand Oaks: Sage Publications.

De Bock, D., Verschaffel, L., Janssens, D., Van Dooren, W., & Claes, K. (2003). Do realistic contexts and graphical representations always have a beneficial impact on students' performance? Negative evidence from a study on modelling non-linear geometry problems. *Learning and Instruction*, *13*(4), 441–463. https://doi.org/10.1016/S0959-4752(02)00040-3 de la Torre, J., Tjoe, H., Rhoads, K., & Lam, D. (2013). Conceptual and theoretical issues in proportional reasoning. *International Journal for Studies in Mathematics Education*, *6*(1), 21–38.

Fernandez, C., Llinares, S., Modestou, M., & Gagatsis, A. (2010). Proportional reasoning: How task variables influence the development of students' strategies from primary to secondary school. *Acta Didactica Universitatis Comenianae Mathematics*, (10), 1–18.

Freudenthal, H. (1983). Ratio and proportion. In *Didactical Phenomenology of Mathematical Structures | Hans Freudenthal | Springer* (pp. 178–209). Springer. Retrieved from //www.springer.com/us/book/9789027715357

Heinz, K., & Sterba-Boatwright, B. (2008). The when and why of using proportions. *Mathematics Teachers*, *101*, 528–533.

Jeong, Y., Levine, S. C., & Huttenlocher, J. (2007). The development of proportional reasoning: Effect of continuous versus discrete quantities. *Journal of Cognition and Development*, *8*(2), 237–256.

Kaput, J. J., & West, M. M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The Development of multiplicative reasoning in the learning of mathematics*. Albany: State University of New York Press.

Karplus, R., Pulos, S., & Stage, E. K. (1983). Early adolescents' proportional reasoning on 'rate' problems. *Educational Studies in Mathematics*, *14*(3), 219–233. https://doi.org/10.1007/BF00410539

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington D.C.: National Academies Press. Retrieved from http://www.nap.edu/catalog/9822.html

Lamon, S. J. (1993). Ratio and proportion: Connecting content and children's thinking. *Journal for Research in Mathematics Education, 24*(1), 41–61.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–668). Charlotte, NC: Information Age Publishing.

Lamon, S. J. (2012). *Teaching fractions and ratios for understanding: essential content knowledge and instructional strategies for teachers* (3rd ed.). New York: Routledge.

Lesh, R., Post, T., & Behr, M. J. (1988). Proportional reasoning. In J. Hiebert & M. J. Behr (Eds.), *Number concepts and operations in the middle grades*. [Hillsdale, N.J.]: Reston, Va: Lawrence Erlbaum Associates; National Council of Teachers of Mathematics.

Lobato, J., & Ellis, A. B. (2010). *Developing essential understanding of ratios, proportions, and proportional reasoning for teaching mathematics in grades 6–8*. Reston, VA: National Council of Teachers of Mathematics.

Martinez, R., & I, J. Y. (2018). Four ways to determine equivalent ratios. *Mathematics Teaching in the Middle School*, *24*(1), 48–52.

National Center for Education Statistics. (2013). *The nation's report card mathematics and reading 2013*. Washington, D.C.: Institute of Education Science, U.S. Department of Education.

National Council of Teachers of Mathematics. (2013). *Teaching ratio and proportion in the middle grades* (Research Brief). Reston, VA.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington D.C.: National Governors Association Center for Best Practices, Council of Chief State School Officers. Washington, D.C.

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Washington D.C.: Department of Education.

Silver, E. A. (1981). Young adults' thinking about rational numbers. In T. R. Post & M. P. Roberts (Eds.), *Proceedings of the third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 149–159). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Van Dooren, W., De Bock, D., Evers, M., & Verschaffel, L. (2009). Students' overuse of proportionality on missing-value problems: How numbers may change solutions. *Journal for Research in Mathematics Education*, *40*(2), 187–211.

Van Dooren, W., De Bock, D., Janssens, D., & Verschaffel, L. (2007). Pupils' over-reliance on linearity: A scholastic effect? *British Journal of Educational Psychology*, *77*(2), 307–321. https://doi.org/10.1348/000709906X115967

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. J. Behr (Eds.), *Number concepts and operations in the middle grades*. [Hillsdale, N.J.]: Reston, VA: Lawrence Erlbaum Associates; National Council of Teachers of Mathematics.

Appendix A: Classroom Assessment Tasks Given to All 6–9 grade Students

	Task	Rationale/purpose to examine
1	Hannah has 5 erasers and 3 pencils. What is the ratio of pencils to erasers?	Can a student write a ratio?
2	Look at the table. What is the ratio of *x* to *y*? Explain your answer. *x* | *y* 6 | $\frac{2}{3}$ $13\ \frac{1}{2}$ | 1.5 3 | $\frac{1}{3}$	Can a student find an equivalent ratio when rational numbers are involved?
3	If 10 pounds of beans cost $4, how much will 15 pounds of beans cost? Show your work or explain your answer.	Can a student find a missing value using an equivalent ratio/proportion of natural numbers?
4	Is 4:7 equivalent to 5:8? Explain.	Does a student apply additive reasoning when determining an equivalent ratio?
5	Brody uses 3 cups of water for 2 cups of lemon juice to make lemonade. Jason uses 1.5 cups of water for 1 cup of lemon juice to make lemonade. Do Brody's and Jason's lemonades taste the same or different? Explain.	Can a student determine equivalent ratios by comparing two pairs of rational numbers in context? It can be solved by simple doubling or halving.
6	Salina mixes 6 cups of water and 4 cups of lemon juice to make lemonade. Can she make lemonade that has the same taste as Brody's lemonade (3 cups of water for 2 cups of lemon juice)? Note that she mixes all ingredients. If yes, explain why. If no, how many cups of which ingredient (water/lemon juice) does she need?	Can a student determine equivalent ratios by comparing two pairs of quantities in context? It can be solved by simple doubling or halving.
7	Mary mixes 13 cups of water and 8 cups of lemon juice to make lemonade. After she mixes all ingredients, she drinks one cup of lemonade. Now, does her lemonade taste the same as Brody's lemonade (3 cups of water for 2 cups of lemon juice)? Why or why not?	Can a student identify and use correctly *part* and *whole* in context? Can a student understand a proportion is invariant in the mix?
8	Eric has 12 cups of lemonade that taste exactly the same as Brody's (3 cups of water for 2 cups of lemon juice). He needs a larger amount of lemonade. He pours one more cup of water and one more cup of lemon juice. Does his lemonade still taste the same? Why or why not?	Can a student identity and use correctly *part* and *whole* in context? Does a student apply additive reasoning when determining an equivalent ratio? Can a student perform a proportional division problem?
9	Monica uses the same recipe of Brody's lemonade (3 cups of water for 2 cups of lemon juice). She needs to make 45 cups of lemonade. How many cups of water and lemon juice does she need to use each to make total 45 cups of lemonade?	Can a student identity and use correctly *part* and *whole* in context? Can a student perform a proportional division problem?

Appendix B: Think-aloud Interview Tasks

	Task	Rational/Purpose
1	Measure the lengths of A and B. Find all possible ratios of the length of A and the length of B.	Can a student find a ratio using an equal measurement, 2A=3B?
2	Measure the lengths of A and B with unit C. Find a ratio of the length of A to the length of B. Is the ratio equivalent to the ratio you found in the task 1? Why or why not?	Can a student use two layers of ratios?
3	Make a new strip D by adding lengths of A and C. Make a new strip E by adding lengths of B and C. Measure the lengths of D and E using unit C. Find a ratio of the lengths of D to E.	Does a student use additive reasoning related to equivalent ratios?
4	Measure the lengths of A and B with unit F. Find a ratio of the lengths of A to B.	Can a student find a ratio when rational terms less than 1 are given?
5	Brody uses 2A units of water for 3B units of lemon juice to make lemonade. What is the ratio of water to lemon juice he mixes?	Can a student find a ratio with two different units?
6	To make Aunt Janet’s lemonade mix 5 cups of water for 3 cups of lemon juice. Brad has 10 cups of lemonade that taste exactly the same as Aunt Janet’s lemonade. He needs a larger amount of lemonade. How can he make 20 cups of lemonade with the same taste?	Can a student identity and use correctly *part* and *whole* in context? Can a student perform a proportional division problem?
7	Jenifer has a beautiful photo. She wants to make a nice frame for the photo, so she can hang it on the wall of her living room. She enlarged the photo by 300%. After that, she lost the original photo. She has only the 300% enlarged one now, but she happens to need the original size of the photo. So, she copies the enlarged photo by 100%. What can you say about the size of the photo now?	Can a student identity and use correctly *part* and *whole* in context? Can a student calculate a percent using a correct whole?

Note: A is 5/3 longer than B and 5 times longer than C, and F is twice as long as A.

Appendix C: Additional Evidence from the Think-aloud Interviews (Task 6)

Interview with Gina (6th grade): It seemed Gina wanted to make 2 cups of lemonade using the given ratio. Because 2 is a quarter of 8, she needed to divide both terms by 4 using proportional division. However, she made a mistake in this calculation or randomly chose two small numbers summing 2.

> **Gina**: …which would be the five cups and the three cups lemon juice so if he already had the 18 how many cups you would just have to make that so if he already has 10, he would need 10 more cups to make 20 cups
>
> **Interviewer**: Yes, so you could make 18 easily.
>
> **Gina**: Right, right, yeah.
>
> **Interviewer**: But it's 20 cups that you need to make.
>
> **Gina:** So, 2, we need 2 more cups of lemonade and the recipe makes 8 cups, so [deep breath] maybe um, um, I can, if he already has the 10 so you need two and a half cups of water and one half cups of lemon juice extra to make the 20 cups by now from the 18 cups. If he already had 18 cause, I think maybe.
>
> **Interviewer:** Can you write that down just so that we remember what you said?
>
> **Gina:** Um, 18 plus because [begins to write "18 +"] I don't even know I thought that everything okay so [writes "18 + 2 $\frac{1}{2}$ water + $\frac{1}{4}$ juice"] If this makes 8 cups and he already has 10 and he only needs 2 to make 20, you have to divide it by two, and five divided by two is two and a half and three divided by two would be one half, I think, hopefully.

Interview with Solomon (8th grade, Algebra): He tried to write a missing-value proportion, "5/3 = 10/*x*" while he misinterpreted 10 as corresponding to 5, the amount of water (part) while 10 represented the amount of lemonade (whole).

> **Solomon**: Um, I'd say that, to make the 20 cups of lemonade with the same taste he'd have to make 10 more cups of water and 6 more of lemon juice.
>
> **Interviewer**: Can you explain how you got the answer?
>
> **Solomon**: um, since for the cups she obviously had to make 10 more to make 20 if he had 10 and then I use like if he had like five cups there and went up to 10 so that's times 2, ten time two equals 20 so I did the same with the cups of lemon juice and did 3 times 2 which equals six and then have to take six time and that equaled 12 and then in my mind I kind of subtracted like 12 minus six so I thought about it he need like… Oh, wait. I wrote the wrong thing there he

needs 12 more or so I think, wait. He just needs six more cups because if you add that and just think of it as 10 over 6 plus 10 over 6, 20 over 12.

Interviewer: So, if you add ten more cups of water and six more cups of lemon juice, then can you make ten cups of lemonade?

Solomon: um, yes.