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STABILITY OF POWER SYSTEMS AND OTHER SYSTEMS OF SECOND ORDER  
DIFFERENTIAL EQUATIONS

Iowa State University

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Stability of power systems and other systems of  
second order differential equations

by

Sherwin James Skar

A Dissertation Submitted to the  
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1980

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## INTRODUCTION

Interest in the stability of the swing equations, which are a model for multi-machine power systems, was generated by the beginning of the electric age. At first, most information about these systems was obtained by simulations on a digital or analog computer. More recently analytical techniques such as Liapunov's direct method have been applied to the problem, but nearly always in the special case of power systems with negligible transfer conductances.

In this dissertation it is shown that stability criteria applied to power systems with negligible transfer conductances do not generalize to systems with non-trivial transfer conductances. Alternative stability criteria will be proved. It is shown by some examples that a commonly accepted, but unproved, stability criterion for the swing equations is not valid.

Most authors investigating the stability of power systems with negligible transfer conductances assume the existence of a stable equilibrium state with rotor angles less than  $90^\circ$  out of phase. In this dissertation it is shown that even though most of its rotor angles are more than  $90^\circ$  out of phase, some even  $180^\circ$  out of phase, an equilibrium solution of the swing equations may be stable.

Although the techniques used in this dissertation are applied to the swing equations, they also apply to a more general class of systems which may be called second order vector differential equations.

Since many models of physical systems involve Newton's laws of motion, they take the form of the second order vector differential equation

$$M\ddot{x} = F(x, \dot{x}) \quad (1)$$

where  $x \in R^k$ ,  $M$  is a  $k \times k$  matrix, and  $F$  maps a subset of  $R^{2k}$  into  $R^k$ . If  $x$  represents the position variables for the system, then  $M$  may represent the masses or moments of inertia of the objects involved and  $F$  may represent a force depending on position and velocity. In many cases of interest (such as the swing equations)  $M$  will be positive definite and diagonal.

With a standard change of variables, for example,  $z = x$ ,  $y = \dot{x}$ ,  $f(y, z) = M^{-1}F(z, y)$ , system (1) may be written in the form

$$\dot{y} = f(y, z) \quad (2a)$$

$$\dot{z} = By \quad (2b)$$

where  $y \in R^k$ ,  $z \in R^\ell$ ,  $m = k + \ell$ ,  $B$  is an  $\ell \times k$  matrix, and  $f$  maps a subset of  $R^m$  into  $R^k$ . In some cases, such as power systems, it is possible to uncouple some of the variables during the transformation from (1) to (2) so that  $k \neq \ell$ . Although the methods of Part I may be applied to system (2), the investigation of this dissertation will study only the case in which  $f(y, z) = -Ay - g(z)$  for some  $k \times k$  matrix  $A$  and some mapping  $g$  from  $R^\ell$  into  $R^k$ . Then system (2) becomes

$$\dot{y} = -Ay - g(z) \quad (3a)$$

$$\dot{z} = By \quad (3b)$$

In Part I system (3) will be studied by a linearization technique. It will be shown that if  $A = 0$ ; then this linearization can never be asymptotically stable, although it may be unstable. Sufficient conditions for the asymptotic stability of an equilibrium solution of system (3) will be proved in the case that the matrix  $A$  is positive definite. A criterion will be found for the uniqueness of an equilibrium solution of system (3) in a convex region. These results will be applied to the swing equations with non-trivial transfer conductances and examples will be given to show that such systems may be unstable under unexpected circumstances.

In Part II the stability of system (3) is studied under conditions that it is damped "Hamiltonian-like" in form. A Hamiltonian-like system shares many of the properties of Hamiltonian systems, although  $y$  and  $z$  may have unlike dimensions (as they do for the swing equations). Criteria for the stability, instability, and uniqueness of equilibrium states of system (3) will be proved. These results will be applied to the swing equations with negligible transfer conductances. A criterion for the existence of a stable equilibrium solution for the swing equations will be proved. Examples will be given of stable equilibrium solutions of the swing equations most of whose rotor angles are more than  $90^\circ$  out of phase with each other; some may even be  $180^\circ$  out of phase with other rotor angles.



In the Discussion section, the results of Parts I and II are discussed and related. A result on the continuous dependence on parameters of stable equilibrium solutions of system (3) is proved and related to power systems. It is shown that the linearization technique for stability provides a necessary and sufficient condition for exponential stability. The need for such a rapid rate of convergence in power system stability analysis is demonstrated. A detailed discussion of synchronous solutions of the power system equations is given. Examples illustrating the need for some of the hypotheses for the results of Parts I and II are given. A result given in Part I is extended, using the notions of observability and controllability, to include the case in which the damping matrix  $A$  is positive semi-definite, but not necessarily positive definite.

One of the stability criteria discussed in Part I is that all the eigenvalues of a certain matrix are real and non-positive. An algorithm for testing this hypothesis is developed in the Appendix.

## EXPLANATION OF THE FORMAT

This dissertation contains two papers, written by the author of this dissertation, which have been accepted for publication in professional mathematics journals. In this dissertation these papers are labeled Part I and Part II. They are self-contained and therefore contain some duplication in their introductory material. The dissertation may be considered as four distinct parts - the introductory material preceding Part I, Part I, Part II, and the summary material following Part II. In each of these parts, equations and highlighted items such as theorems are numbered consecutively but separately from the other parts of the dissertation. In the introductory and summary parts, all reference numbers refer to the list of references at the end of the dissertation. The reference numbers in Part I and Part II refer to the separate reference lists contained in those two parts.

Part I, entitled "Stability of Multi-Machine Power Systems with Non-Trivial Transfer Conductances", will appear in the SIAM Journal on Applied Mathematics. A version of Part II, entitled "Synchronous Solutions of Power Systems: Existence, Uniqueness, and Stability", will appear in the Quarterly of Applied Mathematics. At the request of the referee, the version submitted for publication has been shortened by a few pages. So that the reader may more easily follow the proofs, the longer original version is included in this dissertation.

## BACKGROUND AND NOTATION

## General

Let  $R^n$  be the  $n$  dimensional real vector space with the Euclidean norm  $|| \cdot ||$ . Let  $A^T$  and  $A^{-1}$  be the transpose and (if  $A$  is square and non-singular) the inverse, respectively, of a  $k \times \ell$  matrix  $A = (a_{ij})$  where  $a_{ij}$  is the  $i$ - $j$  th element of  $A$  for  $i = 1, \dots, k$  and  $j = 1, \dots, \ell$ . If  $x \in R^n$ , then  $x$  is a column vector and  $x^T$  is a row vector.

A submatrix of a matrix  $A$  is a matrix formed by deleting from  $A$  some of its rows or columns (and suitably renumbering). A principal submatrix of a square matrix  $A$  is formed by deleting a row from  $A$  if and only if the column with the same index is also deleted. A principal minor determinant is the determinant of a principal submatrix.

In several arguments the theory of M-matrices (also called K-matrices) will be used. By definition, a square matrix is an M- (semi M-) matrix if all off-diagonal elements are non-positive and all principal minor determinants are positive (non-negative). It is known that if  $A$  is an M-matrix, then  $A^{-1}$  exists and has only non-negative elements and only positive diagonal elements. If all off-diagonal elements of a square matrix are non-negative and  $A$  is (row sum or column sum) diagonally dominant, then  $A$  is an M-matrix. (See [4, 17].)

If  $g(z) \in R^k$  for  $z \in R^\ell$  and  $g$  is differentiable, then let

$$\frac{\partial g}{\partial z} = \left( \frac{\partial g_i}{\partial z_j} \right)$$

be the Jacobian matrix of  $g$ . If  $h(y, z) \in \mathbb{R}$  for  $y \in \mathbb{R}^k$  and  $z \in \mathbb{R}^\ell$  and  $h$  is differentiable, then let

$$\nabla_y h = \left( \frac{\partial h}{\partial y_1}, \dots, \frac{\partial h}{\partial y_k} \right)^T$$

$$\nabla_z h = \left( \frac{\partial h}{\partial z_1}, \dots, \frac{\partial h}{\partial z_\ell} \right)^T$$

The pair of constant vectors  $y^*, z^*$  with  $y^* \in \mathbb{R}^k$  and  $z^* \in \mathbb{R}^\ell$  is an equilibrium point of system (3) if  $Ay^* + g(z^*) = 0$  and  $By^* = 0$ . This dissertation will investigate the stability of equilibrium points of system (3).

1. Definition: An equilibrium point  $y^*, z^*$  of system (3) is stable if for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that for any solution  $y(t), z(t)$  of (3) for which  $||y(0) - y^*|| + ||z(0) - z^*|| < \delta$  it is true that  $||y(t) - y^*|| + ||z(t) - z^*|| < \epsilon$  for all  $t \geq 0$ .

2. Definition: An equilibrium point  $y^*, z^*$  of system (3) is asymptotically stable if it is stable and if there exists a  $\delta > 0$  such that for any solution  $y(t), z(t)$  of (3) for which  $||y(0) - y^*|| + ||z(0) - z^*|| < \delta$  it is true that

$$\lim_{t \rightarrow \infty} (||y(t) - y^*|| + ||z(t) - z^*||) = 0.$$

The criteria for stability used in this paper include the linearization theorems and the invariance theorems which are described here as applied to system (3).

Linearization theorems [9]: Assume that  $g(z) = -Cz + h(z)$  where  $C$  is a constant  $k \times \ell$  matrix and

$$\lim_{\|z\| \rightarrow 0} \frac{\|h(z)\|}{\|z\|} = 0$$

for some continuous function  $h(z) \in \mathbb{R}^k$  defined on a neighborhood of the origin in  $\mathbb{R}^\ell$ . Define

$$G = \begin{pmatrix} -A & C \\ B & 0 \end{pmatrix}$$

(a) If all eigenvalues of  $G$  have negative real parts, then the trivial solution  $y(t) \equiv 0$ ,  $z(t) \equiv 0$  of system (3) is asymptotically stable.

(b) If one eigenvalue of  $G$  has positive real part, then the trivial solution of system (3) is not stable.

The linearization theorems will be used in Part I. It will be shown in the Discussion section that the hypothesis of part (a) of the previous theorem is a necessary and sufficient condition for exponential stability.

The invariance theorems use some definitions which are now discussed.

A subset  $N$  of  $\mathbb{R}^{k+\ell}$  is called invariant for system (3) if for every solution  $y(t)$ ,  $z(t)$  of (3) such that  $(y(t_0)^T, z(t_0)^T)^T \in N$ , it is true that  $(y(t)^T, z(t)^T)^T \in N$  for every  $t \geq t_0$ .

Let  $y(t)$ ,  $z(t)$  be a solution of system (3) and let  $L_n(y(\cdot), z(\cdot))$  be the closure of the set  $\{(y(t)^T, z(t)^T)^T \in \mathbb{R}^{k+\ell} : t \geq n\}$ . Then the

limit set for  $y(\cdot), z(\cdot)$  is  $L(y(\cdot), z(\cdot)) = \bigcap_{n=1}^{\infty} L_n(y(\cdot), z(\cdot))$ .

Let  $V(y, z) \in \mathbb{R}$  be a differentiable function for  $y \in \mathbb{R}^k, z \in \mathbb{R}^l$ .  
Define the derivative of V along solutions of (3) to be

$$\frac{dV}{dt} = (\nabla_y V(y, z))^T (Ay + g(z)) + (\nabla_z V(y, z))^T Bz.$$

Invariance theorems [7, 24]: Let  $y^*, z^*$  be an equilibrium point of system (3). Let  $M$  be a bounded open neighborhood of  $(y^{*T}, z^{*T})^T \in \mathbb{R}^{k+l}$ . Let  $V(y, z)$  be a real valued continuously differentiable function on  $M$  such that  $dV/dt \leq 0$  in  $M$ .

(a) Suppose that  $V(y^*, z^*)$  is a minimum for  $V$  on  $M$  and that the only invariant subset of

$$\{(y^T, z^T)^T : \frac{dV}{dt}(y, z) = 0\} \cap M$$

is  $\{(y^{*T}, z^{*T})^T\}$ . Then  $y^*, z^*$  is an asymptotically stable equilibrium point of system (3).

(b) Let  $y(t), z(t)$  be a bounded (for  $t \geq 0$ ) solution of system (3). Then the limit set  $L(y(\cdot), z(\cdot))$  is a non-empty compact invariant set for system (3).

(c) Let  $M_1$  be a compact neighborhood of  $(y^{*T}, z^{*T})^T$  such that  $M_1 \subset M$ . Suppose that  $(y(t)^T, z(t)^T)^T \in M_1$  for all  $t \geq 0$ . Then

$$\lim_{t \rightarrow \infty} \frac{dV}{dt}(y(t), z(t)) = 0.$$

Hence,  $L(y(\cdot), z(\cdot)) \subset \{(y^T, z^T)^T \in \mathbb{R}^{k+l} : \frac{dV}{dt}(y, z) = 0\}$ .

The invariance theorems will be used in Part II.

### Power Systems

The study of power system stability is generally divided into two problems - the steady state stability problem and the transient stability problem.<sup>1</sup> For the steady state stability problem the power system is assumed to be operating at or near a stable equilibrium state; the problem is to determine the effects of small disturbances. The transient stability problem is to determine the effects on the power system due to sudden large disturbances during which the equilibrium states are changed.

In the study of steady state stability and control, the first step is to develop a proper model for the system. The system model, generally a system of many interconnected non-linear differential equations, is then linearized about the operating point. The problem is then studied as a linear control problem [15, 32, 50]. Standard methods such as the Routh-Hurwitz criterion are used for the stability problem. The time periods involved are larger than those involved in the transient stability problem and more variables are used in the analysis. The system matrix for these models often does not have the form analyzed in Part I of this dissertation so that the results may not be applicable to the steady state stability problem in many of the suggested models.

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<sup>1</sup> There is a consensus in the literature that the two problems should be considered separately, but there is no consensus about the names of the two problems. See [15, p. 37], [18, p. 1], and [54, p. 334].

For the transient stability problem it is assumed that the system is originally operating at an equilibrium or synchronous state so that the rotor of each synchronous generator is rotating at the same constant frequency. Then a "fault" or a major disturbance occurs in the system which changes the system parameters and probably makes it unstable. The fault is "cleared", perhaps by temporarily shutting off the power supply to the area causing the disturbance. The problems are to determine whether an asymptotically stable equilibrium state exists in the post-fault system and, if so, whether the system after the fault is cleared is operating within the region of attraction of such an equilibrium state. This dissertation will investigate only the former problem for a model which has not been successfully studied by direct methods. In addition, most of the direct methods used in a simpler model will allow one to conclude the stability, but not the asymptotic stability, of an equilibrium point. (This shortcoming was discussed by Weiss [52].) In this dissertation it is shown that the standard methods of transient stability analysis (namely, Liapunov theory) do imply the asymptotic stability of an equilibrium point if the invariance theorems are applied.

For transient stability analysis a power system consisting of  $n$  interconnected rotary generators is generally modeled by a set of ordinary differential equations called the swing equations. The following arguments are intended to motivate the use of the swing equations as well as to demonstrate their validity and limitations as a model for a power system. (See [10, 15].)



The main ingredients of the energy involved in a synchronous generator are the kinetic energy of the rotor, the mechanical input energy, the electrical output energy, and the damping energy. Since power is the time derivative of energy, the conservation of energy requires that

$$\frac{dK_i}{dt} + P_{Di} + P_{ei} = P_{mi} \quad (4)$$

where for the  $i$ -th generator  $K_i$  is the kinetic energy and  $P_{Di}$ ,  $P_{ei}$ , and  $P_{mi}$  are the damping power, the electrical power output, and the mechanical power input, respectively.

Several approximations are made in order to write system (4) in a form which can be analyzed. Since the voltage output is caused by a rotating magnetic field, the angle of rotation for the generator rotor is identified with the phase angle of the output voltage. This angle is called the rotor angle  $\delta_i$  for the  $i$ -th machine. Without loss of generality (by a change of variables  $\delta_i' = \delta_i + \omega_o t$ ), it is assumed that each rotor angle is measured relative to a reference angle rotating at a constant angular frequency  $\omega_o$ .

The angular momentum  $J_i$  of the  $i$ -th generator is assumed to be large so that in the post-fault system the angular velocity does not change significantly, even for large changes in angle, from the pre-fault constant angular velocity. Hence,  $d\delta_i/dt \approx \omega_o$  for  $i = 1, \dots, n$  and for some constant  $\omega_o > 0$ . Thus, because  $K_i = (1/2) J_i (d\delta_i/dt)^2$  (see, for example, [22, p. 149]) it is true that

$$\frac{dK_i}{dt} = J_i \frac{d\delta_i}{dt} \frac{d^2\delta_i}{dt^2} \approx \omega_o J_i \frac{d^2\delta_i}{dt^2} = M_i \frac{d^2\delta_i}{dt^2} \quad (5)$$

where  $M_i = \omega_o J_i > 0$  for  $i = 1, \dots, n$ .

It is assumed that the damping power loss, due to friction and damper windings added to the system for a stabilizing effect, is proportional to the angular velocity; that is

$$P_{Di} = d_i \frac{d\delta_i}{dt} \quad (6)$$

for some  $d_i > 0$ .

Since the machines are rotating very rapidly with a high angular momentum and since the fault occurs and is cleared in a short time, the mechanical power input,  $P_{mi}$ , does not change much and is assumed to be constant.

The electric power output into the network can be considered as the real part of the standard electric power equation (in this paragraph, let  $k \in \{1, \dots, n\}$  and  $i = \sqrt{-1}$ )

$$P_{ek} + iQ_{ek} = E_k \bar{I}_k = E_k (\bar{I}_{kk} - \sum_{\substack{j=1 \\ j \neq k}}^n \bar{I}_{kj}) \quad (7)$$

for  $k = 1, \dots, n$  where  $Q_{ei}$  is the "reactive" power,  $E_k = |E_k| \exp(i\delta_k)$  is the complex "voltage behind the transient reactance",  $\bar{I}_k$  is the complex conjugate of the complex current, and the current  $I_{kj}$ ,  $j = 1, \dots, n$ , is that which flows due to the voltage output of the  $j$ -th generator. The currents  $I_{kj}$  and the voltages  $E_j$  are related

by the equation  $I_{kj} = E_j / Z_{kj}$  for  $k, j = 1, \dots, n$  where  $Z_{kj} = |Z_{kj}| \exp(i \theta_{kj})$  is the transfer impedance between machines  $k$  and  $j$  for  $k \neq j$  and the driving point impedance for  $k = j$ . It is assumed that the impedances  $Z_{kj} = Z_{jk}$  are constant and that  $|E_j|$  is constant for  $j, k = 1, \dots, n$ . Then equation (7) has the form

$$P_{ek} + iQ_{ek} = |E_k| e^{i\delta_k} \left[ \frac{|E_k|}{|Z_{kk}|} e^{-i(\delta_k - \theta_{kk})} - \sum_{\substack{j=1 \\ j \neq k}}^n \frac{|E_j|}{|Z_{jk}|} e^{-i(\delta_j - \theta_{kj})} \right]$$

$$= \frac{|E_k|^2}{|Z_{kk}|} e^{i\theta_{kk}} - \sum_{\substack{j=1 \\ j \neq k}}^n \frac{|E_k E_j|}{|Z_{kj}|} e^{i(\delta_k - \delta_j + \theta_{kj})}$$

Let  $\alpha_{kj} = \pi/2 - \theta_{kj}$  for  $k, j = 1, \dots, n$ . Then

$$P_{ek} = \frac{|E_k|^2}{|Z_{kk}|} \sin \alpha_{kk} + \sum_{\substack{j=1 \\ j \neq k}}^n \frac{|E_k E_j|}{|Z_{kj}|} \sin (\delta_k - \delta_j - \alpha_{kj}) \quad (8)$$

The parameter  $\alpha_{kj}$  is the transfer conductance parameter. The transfer conductances are the parameters  $G_{kj} = \text{Re}(1/Z_{kj}) = (1/|Z_{kj}|) \sin \alpha_{kj}$  for  $k, j = 1, \dots, n, k \neq j$ . They are due essentially to the resistance of the lines connecting machines  $k$  and  $j$ . If  $G_{kj} = \alpha_{kj} = 0$ , then there is no resistance between machines  $k$  and  $j$ . In most studies of transient stability transfer conductances are neglected. However, in practice the transfer conductances may not be negligible. (See [18, 49].)

Substituting equations (5), (6), and (8) into equation (4), one obtains the swing equations as a model for  $n$  interconnected

synchronous generators.

$$\begin{aligned}
 M_i \frac{d^2 \delta_i}{dt^2} = & -d_i \frac{d\delta_i}{dt} + P_{mi} - \frac{|E_i|^2}{|Z_{ii}|} \sin \alpha_{ii} \\
 & - \sum_{\substack{j=1 \\ j \neq i}}^n \frac{|E_i E_j|}{|Z_{ij}|} \sin (\delta_i - \delta_j - \alpha_{ij})
 \end{aligned} \tag{9}$$

for  $i = 1, \dots, n$ . In this analysis all parameters except the  $\delta_i$  are considered to be constant. Note that the swing equations have the same form as Newton's equations of motion, and they can be derived in that way. Note also that the swing equations have the form of equation (1) and can be put into the form of system (3). This transformation is described in Parts I and II.

## REVIEW OF RELATED LITERATURE

## General

Problems related to stability have interested scientists for thousands of years [36]. However, only in the last two centuries have useful analytic methods been developed to test for the stability of systems modeled by differential equations [41]. These analytic methods may be applied to either linearized or to non-linear systems.

Linear systems

Although linear systems are a special but important class of systems, the methods developed for this special case can often be applied to non-linear systems with differentiable right-hand sides. The system may be "linearized" about an equilibrium point whose local stability properties can, in the generic case, be determined from the stability properties of the related linear system.

Important contributions to the study of the stability of systems of autonomous linear differential equations have been made by many investigators, including Hermite (1854), Maxwell (1868), Vyschnegradskii (1876), Routh (1877), Liapunov (1892), and Hurwitz (1895). For a discussion of these contributions, see the survey article of Parks [41]. At least some of the stability criteria they developed are discussed in almost any book on stability theory or linear systems analysis. These methods are applicable to general linear systems and do not use the special properties of systems of interconnected second order differential equations such as those discussed in Part I. Studies

of systems of second order differential equations have given useful information about the damping matrix, for example, determining the effects of dissipative or of gyroscopic damping. (See [57].)

### Non-linear systems

The first major result in non-linear system stability analysis was the minimum potential energy principle formulated by Lagrange and later proved by Dirichlet (see [33, p. 77]). This direct method is related directly to Newtonian mechanics. It was generalized by Liapunov [35] whose results are generally considered to be the basis of modern stability theory for non-linear systems with many degrees of freedom. A major advantage of the Liapunov direct method (over linearization methods) is that it can be used to determine regions of attraction rather than only local stability properties.

The Liapunov direct method, which requires a negative definite derivative for the Liapunov function, has been extended in the twentieth century by the invariance theorem. (See [6, 31, 34].) Using this result, one may sometimes conclude that an equilibrium point is asymptotically stable even though the Liapunov function does not have a negative definite derivative.

### Power Systems

Transient stability studies for power systems generally use the standard swing equations model (equation (9)). The standard procedure for analyzing the stability of power systems is to use computer simulations [47]. However, there have been many studies, especially in

the last 15 years, using direct methods for transient stability analysis. Reviews by Willems [54], Ribbens-Pavella [44], and Fouad [18] discuss the results of these studies.

The first direct method, the classical "equal area" criterion, is applied only to a one machine system [10, 15, 40]. The method was generalized to more dimensions in the form of an "energy integral" which was used with no damping to study the Lagrange stability of the system [5, 37]. The energy function derived in this way can also be used as a Liapunov function [12, 21].

In order to find the largest stability regions possible, there has been a search for improved Liapunov functions and some controversy about the results. Fallside and Patel [16] and Rao [43] constructed Liapunov functions by generalizing from methods used in other studies, for example, Rao's function is a generalization to the non-linear case of a quadratic form of the type used in linear systems.

A series of authors - for example, Walker and McClamrock (1967) [51], J. L. Willems and J. C. Willems (1970) [56], J. L. Willems (1974) [55], and Henner (1974) [27] - considered the swing equations (system (9)) in the form of a control system and used a generalized Popov criterion [1, 29, 39] to develop a Liapunov function of the Lur'e type. They used the system variables  $\sigma_{ij} = \delta_i - \delta_j - \delta_i^* + \delta_j^*$ ,  $i, j = 1, \dots, n, i \neq j$  where  $(\delta_1^*, \dots, \delta_n^*)$  is an equilibrium point for system (9). By changing the constant term  $P_{mi}$  as required, the non-linear term in

equation (9) can be considered as a sum of terms of the form

$$f_{ij}(\sigma_{ij}) = \frac{|E_i E_j|}{|Z_{ij}|} [\sin(\sigma_{ij} + \delta_i^* - \delta_j^* - \alpha_{ij}) - \sin(\delta_i^* - \delta_j^* - \alpha_{ij})]$$

If  $|\delta_i^* - \delta_j^* - \alpha_{ij}| < \pi/2$ , then  $f_{ij}$  satisfies the sector condition  $0 \leq \sigma f_{ij}(\sigma) \leq \sigma^2$  and  $f'_{ij}(\sigma) > 0$  for small enough values of  $|\sigma|$ .

Using these facts, Liapunov functions for the case of negligible transfer conductances ( $\alpha_{ij} = 0$ ) have been developed. For example, Willems (1974) [55] developed a Liapunov function of the form

$$\begin{aligned} V(\delta_1, \dots, \delta_n, \dot{\delta}_1, \dots, \dot{\delta}_n) &= q \sum_{i=1}^n M_i \dot{\delta}_i^2 + r \left( \sum_{i=1}^n M_i \delta_i \right)^2 \\ &+ 2p \sum_{i=1}^n d_i (\delta_i - \delta_i^*)^2 + 2s \left( \sum_{i=1}^n d_i (\delta_i - \delta_i^*) \right) \left( \sum_{j=1}^n M_j \dot{\delta}_j \right) \\ &+ p \sum_{i=1}^n d_i (\delta_i - \delta_i^*)^2 + s \left( \sum_{i=1}^n d_i (\delta_i - \delta_i^*) \right)^2 \\ &+ q \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \int_0^{\sigma_{ij}} f_{ij}(\sigma) d\sigma \end{aligned}$$

where  $p \geq 0$ ,  $q > 0$ ,  $r$ , and  $s$  are real numbers. Willems showed that, by choosing  $p$ ,  $q$ ,  $r$ , and  $s$  properly, he could derive many of the Liapunov functions suggested previously by him and other authors. For example,



by choosing  $q = 1/2$  and  $r = p = s = 0$ , the Liapunov function used in Part II of this dissertation is formed.

In the case of negligible transfer conductances ( $\alpha_{ij} = 0$ ), the fact that  $f_{ij}(\sigma) = -f_{ji}(-\sigma)$  makes this analysis possible (see Part II).

Under these conditions, it is seen that an equilibrium state

$(\delta_1^*, \dots, \delta_n^*)$  is stable if  $|\delta_i^* - \delta_j^*| < \pi/2$  for all  $i, j = 1, \dots, n$  (so that  $f_{ij}$  satisfies a sector condition). One author [23] attempted to extend this work to the case of non-trivial transfer conductances ( $\alpha_{ij} \neq 0$ ). However, errors were found in his work [26, 45, 53].

Nevertheless, it appears to be widely believed that if

$|\delta_i^* - \delta_j^* - \alpha_{ij}| < \pi/2$  for  $i, j = 1, \dots, n, i \neq j$ , then the equilibrium state  $(\delta_1^*, \dots, \delta_n^*)$  is stable. This conjecture is found to be false in Part I of this dissertation.

A shortcoming of the Liapunov functions mentioned previously is that their derivative along solutions is negative semi-definite, but not negative definite. As a result, it is not possible to conclude that an equilibrium state is asymptotically stable by using the classical Liapunov theorems. This shortcoming has been often ignored in the literature. However, Weiss [52] recognized the problem and constructed a Liapunov function with a negative definite derivative along solutions. In Part II of this dissertation it is shown that by applying invariance theory it is possible to conclude asymptotic stability using the simplest of the recommended Liapunov functions (that is, the energy function which is generally a special case of any of the Liapunov functions recommended for transient stability studies).

More recently the techniques of decomposition and aggregation have been applied to the transient stability problem [11, 28]. The large scale power system is divided into smaller subsystems whose stability is studied separately. Conditions on the interconnections between these subsystems are found so that the entire system is stable. Although one of these papers [28] does not assume negligible transfer conductances, the interconnections between machines is assumed to be smaller than is practical. Other investigations, which include non-trivial transfer conductances, have either approximated the effect of the energy of the transfer conductances in a semi-direct method [49] or have considered the transfer conductances as a perturbation [30].

Most studies of transient stability, in order to use the sector condition mentioned previously, consider only those stable equilibrium states  $(\delta_1^*, \dots, \delta_n^*)$  with  $|\delta_i^* - \delta_j^*| < \pi/2$  for  $i, j = 1, \dots, n$ . In Part II it is shown that stable equilibrium states exist with  $|\delta_i^* - \delta_j^*| > \pi/2$  for most rotor angle pairs  $\delta_i^*, \delta_j^*$  or even  $|\delta_i^* - \delta_j^*| = \pi$  for some rotor angle pairs. The problem of whether a post-fault equilibrium state exists is generally solved by numerical calculation [14, 42]. In Part II an analytic criterion for the existence of stable equilibrium points is given.

The most extensive study of equilibrium points for the swing equations was performed by Tavora and Smith [48] under the assumption that transfer conductances are negligible. They studied the uniqueness of equilibrium points. However, this writer was notable to follow their

proof. Conditions for the uniqueness of equilibrium points are proved in Parts I and II of this dissertation.

PART I: STABILITY OF MULTI-MACHINE POWER  
SYSTEMS WITH NON-TRIVIAL TRANSFER CONDUCTANCES

## ABSTRACT

The local stability of second order vector differential equations with linear damping is examined by linearization. It is shown that without damping such systems are stable only if the eigenvalues of a certain matrix are real and non-positive. Sufficient conditions for the asymptotic stability of the damped system are developed. The results are applied to power systems with non-trivial transfer conductances. An important consequence is that unstable equilibrium solutions for the power system swing equations may exist even though the rotor angles are less than  $90^\circ$  out of phase, that is, even though  $|\delta_i - \delta_j - \alpha_{ij}| < \pi/2$  for all rotor angle pairs  $\delta_i, \delta_j$  and all phases  $(\alpha_{ij} + \pi/2)$  in the transfer admittance matrix. It is also shown that there can be at most one equilibrium solution (up to a constant phase added to all rotor angles) of the swing equations with  $|\delta_i - \delta_j - \alpha_{ij}| < \pi/2$  for all  $\delta_i, \delta_j, \alpha_{ij}$ .

## I. INTRODUCTION

Nearly all work in the transient stability of power systems uses a model neglecting transfer conductances. In this model an equilibrium solution for which the rotor angles are less than  $90^\circ$  out of phase is known to be stable. It is usually assumed that an analogous property holds if transfer conductances are included. In this paper we show that this assumption is not valid. We also develop some sufficient conditions for asymptotic stability of power systems with non-trivial transfer conductances.

Recalling Newton's laws of motion, we see that many models of physical systems, including power systems, have the form

$$M\ddot{x} + D\dot{x} + f(x) = 0$$

where  $x \in \mathbb{R}^k$  may be the position variables, the  $k \times k$  matrix  $M$  may represent masses or moments of inertia, the  $k \times k$  matrix  $D$  represents a damping force, and  $f: \mathbb{R}^k \rightarrow \mathbb{R}^k$  may be a position dependent force.

After making a canonical transformation we can write the system in an equivalent form

$$\dot{y} = -Ay - g(z) \tag{1a}$$

$$\dot{z} = By \tag{1b}$$

where  $y \in \mathbb{R}^k$ ,  $z \in \mathbb{R}^\ell$ ,  $k + \ell = m$ ,  $A$  and  $B$  are  $k \times k$  and  $\ell \times k$  real matrices, respectively, and  $g: \mathbb{R}^\ell \rightarrow \mathbb{R}^k$  may be non-linear. In some

cases, such as power systems, it is possible to decouple some variables during the transformation so that  $k \neq \ell$ . In these cases the asymptotic stability of system (1) will correspond to the attractiveness of a family of solutions of the original system.

Useful information about the local stability of system (1) will be developed in Section II by applying some matrix theory to the linearization of system (1). We will see that, without damping, system (1) can be stable only if the eigenvalues of a certain matrix (essentially the Jacobian of  $-f$ ) are all real and non-positive, and we will develop some sufficient conditions for the asymptotic stability of system (1). We apply these results to power systems with non-trivial transfer conductances in Section III.

There is special interest in synchronous solutions to the power system swing equations with the property  $|\delta_i - \delta_j - \alpha_{ij}| < \pi/2$  for all rotor angle pairs  $\delta_i, \delta_j$  and all transfer conductance parameters  $\alpha_{ij}$ . We will find that there is at most one such synchronous solution (up to a constant phase added to all rotor angles) for a given power system. The widely held assumption that such synchronous solutions are necessarily stable will be invalidated by a family of three machine counterexamples developed in Section IV.

Most of the notation used is standard and will be used without explanation. The symbols  $A^T$  and  $A^{-1}$  will denote the transpose and inverse, respectively, of a matrix  $A = (a_{ij})$  whose  $i$ - $j$  th element is  $a_{ij}$ . All vectors  $x$  will be column vectors; hence  $x^T$  is a row vector. The constant vectors  $y^* \in \mathbb{R}^k$ ,  $z^* \in \mathbb{R}^\ell$  are on equilibrium point of

system (1) if  $Ay^* + g(z^*) = 0$  and  $By^* = 0$ . Let  $I_n$  be the  $n \times n$  identity matrix. We let  $\partial g / \partial z$  be the Jacobian matrix of  $g$ .

In several arguments in this paper we will use the theory of M-matrices (also called K-matrices). A square matrix is called an M (semi-M) matrix if all off-diagonal elements are non-positive and all principal minor determinants are positive (non-negative). If  $A$  is an M-matrix, then  $A^{-1}$  exists and has only non-negative elements with positive diagonal elements. If off-diagonal elements of  $A$  are non-positive and  $A$  is diagonally dominant, then  $A$  is an M-matrix. (See [2, 5].)



## II. THE GENERAL SYSTEM

We will study the stability of system (1) by linearizing about an equilibrium point. We assume that system (1) has an equilibrium point  $y^*$ ,  $z^*$  and without loss of generality we let  $y^* = 0 \in \mathbb{R}^k$ ,  $z^* = 0 \in \mathbb{R}^\ell$ . We further assume that

$$g(z) = -Cz + h(z) \text{ and } \lim_{\|z\| \rightarrow 0} \frac{\|h(z)\|}{\|z\|} = 0 \quad (2)$$

for some constant real  $k \times \ell$  matrix  $C$  and some mapping  $h: \mathbb{R}^\ell \rightarrow \mathbb{R}^k$ . For example, if  $g(z)$  is differentiable we let  $C = -(\partial g / \partial z)|_{z=0}$ .

With these assumptions the stability of the equilibrium point at the origin is related in a well-known fashion to the eigenvalues of the matrices

$$G = \begin{pmatrix} -A & C \\ B & 0 \end{pmatrix} \text{ and } G_0 = \begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix} \quad (3)$$

If  $G_0$  has an eigenvalue with positive real part, then, for matrices  $A$  which are small in some matrix norm, the matrix  $G$  will also have an eigenvalue with positive real part. The origin will be an asymptotically stable (an unstable) equilibrium point of (1) if all eigenvalues of  $G$  have negative real parts (if  $G$  has an eigenvalue with positive real part). (See, for example, [10].)

We begin by studying the eigenvalues of  $G_0$ . The following result will be useful.

2.1 Remark: Let  $A$ ,  $B$ ,  $C$ , and  $D$  be constant real  $k \times k$ ,  $\ell \times k$ ,  $k \times \ell$ , and  $\ell \times \ell$  matrices, respectively. Let

$$H = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

(a) If  $\det A \neq 0$ , then  $\det H = \det A \det (D - BA^{-1}C)$ .

(b) If  $\det D \neq 0$ , then  $\det H = \det D \det (A - CD^{-1}B)$ .

For a proof see [7, p. 45].

2.2 Lemma: Let  $B$ ,  $C$ , and  $G_0$  be defined as in equation (3). Then  $\lambda$  is a non-zero eigenvalue of  $G_0$  if and only if  $\lambda^2$  is a non-zero eigenvalue of  $CB$ .

Proof: Let  $\lambda$  be a non-zero complex number. By Remark 2.1

$$\det(G_0 - \lambda I_m) = \det(-\lambda I_\ell) \det(-\lambda I_k - C(-\lambda I_\ell)^{-1}B) = (-1)^\ell \lambda^{\ell-k} \det(CB - \lambda^2 I_k).$$

2.3 Theorem: Let  $B$ ,  $C$ , and  $G_0$  be defined as in Equation 3. The following are equivalent:

- (i) All eigenvalues of  $G_0$  have non-positive real part.
- (ii) All eigenvalues of  $G_0$  are pure imaginary.
- (iii) All eigenvalues of  $CB$  are real and non-positive.

Proof: Clearly (ii) implies (i). Assume (i) and suppose (iii) is false. For some eigenvalue  $\mu = \rho \exp(i\theta)$  of  $CB$  we have that  $\rho > 0$  and  $|\theta| < \pi$ . By Lemma 2.2 we see that  $\lambda = \sqrt{\rho} \exp(i\theta/2)$  is an eigenvalue of  $G_0$ . Then  $\operatorname{Re} \lambda = \sqrt{\rho} \cos(\theta/2) > 0$  which contradicts (i).

Assume (iii). Let  $\lambda$  be an eigenvalue of  $G_0$ . By Lemma 2.2 either  $\lambda^2$  is an eigenvalue of  $CB$  or  $\lambda = 0$ . By (iii)  $\lambda^2 = -\nu \leq 0$  where  $\nu$  is

real. Hence  $\lambda = \pm i \nu$ .

Theorem 2.3 implies that if  $A = 0$  and  $g(z)$  is linear then system (1) has no asymptotically stable equilibrium points. We also have the following corollary.

2.4 Corollary: In system (1) let  $A = 0$  and let  $C$  be defined in equation (2). If the origin is a stable equilibrium point of (1), then all eigenvalues of  $CB$  are real and non-positive.

We may use a continuity argument to find unstable examples of (1) when  $A$  is small in norm and system (1) satisfies the contrapositive of the previous corollary.

To use the two preceding results we may apply Sturm's theorem to the characteristic equation of a matrix  $H$  to determine whether all the eigenvalues of  $H$  are real and non-positive (see [8, p. 175] or [4]). An algorithm analogous to the Routh-Hurwitz criterion can be developed to implement Sturm's theorem.

We now consider criteria for which all non-zero eigenvalues of  $G$  have negative real parts. To determine whether  $G$  is singular, we may use the following remark.

2.5 Remark: Let  $A$ ,  $B$ ,  $C$ , and  $G$  be defined as in equation (3). If  $k < \ell$ , then  $\det G = 0$ . If  $\det A \neq 0$ , then  $\det G = 0$  if and only if  $\det (BA^{-1}C) = 0$ . The last result follows from Remark 2.1.

If  $\det G \neq 0$  and all non-zero eigenvalues of  $G$  have negative real parts, then the origin is an asymptotically stable equilibrium point of (1). Thus, we may use the previous remark and the following theorem to demonstrate the asymptotic stability of an equilibrium point of (1).

2.6 Theorem: Let A, B, C, and G be defined as in equation (3). If one of the following criteria is satisfied, then all non-zero eigenvalues of G have negative real parts:

- (i)  $CB = -KH$  where A, K, and H are symmetric, A and K are positive definite, H is positive semi-definite, and  $AK = KA$ .
- (ii) All non-zero eigenvalues of CB have negative real parts and  $A = aI_k$  where  $a > 0$  is a real constant such that if  $\xi = \mu + i\nu$  is a non-zero eigenvalue of CB ( $\mu, \nu$  are real), then  $\nu^2 + a^2\mu < 0$ .
- (iii) If we let  $H = -CB = (h_{ij})$ , then

$$h_{ii} \geq \sum_{\substack{j=1 \\ j \neq i}}^k |h_{ij}|$$

and  $A = \text{diag } (a_i)$ , where  $a_i > 0$   
and  $2h_{ii} \leq a_i^2$  for  $i = 1, \dots, k$ .

Proof: Let  $\lambda$  be a non-zero complex number. From Remark 2.1 we see that

$$\det (G - \lambda I_m) = (-\lambda)^{\ell-k} \det (-CB + \lambda A + \lambda^2 I_k) . \quad (4)$$

For criterion (i) we see that K has a square root  $\sqrt{K}$  and  $\sqrt{K} A = A \sqrt{K}$  (see [7, p. 275]). Hence, since  $KH = -CB$ ,

$$\sqrt{K}^{-1}(-CB + \lambda A + \lambda^2 I_k) \sqrt{K} = \sqrt{K} H \sqrt{K} + \lambda A + \lambda^2 I_k .$$

Let  $L = \sqrt{K} H \sqrt{K}$ . Then L is symmetric and positive semi-definite because  $\sqrt{K}$  and H are. Then  $\lambda \neq 0$  is an eigenvalue of G if and only if

$$\det (L + \lambda A + \lambda^2 I_k) = 0.$$

To achieve a contradiction, assume that  $\lambda$  is a non-zero eigenvalue of  $G$  with non-negative real part. We consider two cases.

(a) Suppose  $\lambda$  is real and positive. Then  $L + \lambda A + \lambda^2 I_k$  is positive definite and not singular.

(b) Suppose  $\lambda = \mu + i\nu$  with  $\mu \geq 0$  and  $\nu \neq 0$ . Since  $L + \lambda A + \lambda^2 I_k$  is singular there exists a non-zero vector  $x = y + iz$  where  $y, z \in \mathbb{R}^k$  such that

$$\begin{aligned} 0 &= [L + (\mu + i\nu) A + (\mu^2 - \nu^2 + 2i\mu\nu)I_k] (y + iz) \\ &= [(L + \mu A + (\mu^2 - \nu^2)I_k) y - \nu (A + 2\mu I_k)z] \\ &\quad + i[(L + \mu A + (\mu^2 - \nu^2)I_k) z + \nu (A + 2\mu I_k)y] . \end{aligned}$$

The real and imaginary parts of the above equation must vanish separately. Since  $\mu \geq 0$  we may set  $U = (A + 2\mu I_k)^{-1}$  where  $U$  is symmetric and positive definite. Define  $V = L + \mu A + (\mu^2 - \nu^2)I_k$ . Hence

$$Vy - \nu U^{-1}z = 0 \quad \text{and} \quad Vz + \nu U^{-1}y = 0$$

so that by eliminating  $z$  from these equations we see that

$$U V U V y = -\nu^2 y .$$

Thus,  $U V$  must have a pure imaginary eigenvalue which is impossible because  $U V$  is similar to  $\sqrt{U} V \sqrt{U}$  which is symmetric and has only real eigenvalues.

Therefore, the assumption  $\mu \geq 0$  must be false and (i) is proved.

For criterion (ii) let  $\lambda$  be an eigenvalue of  $G$  with  $\lambda \neq 0$ . From equation (4) and the assumption  $A = aI_k$

$$0 = \det (G - \lambda I_m) = (-\lambda)^{\ell-k} \det (-CB + (a\lambda + \lambda^2)I_k) .$$

Then  $a\lambda + \lambda^2 = \xi = \mu + i\nu$  ( $\mu, \nu$  are real) is a non-zero eigenvalue of  $CB$ . Solving for  $\lambda$  we find that

$$\lambda = \frac{1}{2} \left[ -a \pm \frac{1}{\sqrt{2}} (a^2 + 4\mu + ((a^2 + 4\mu)^2 + 16\nu^2)^{\frac{1}{2}})^{\frac{1}{2}} \right. \\ \left. \pm \frac{i}{\sqrt{2}} (-a^2 - 4\mu + ((a^2 + 4\mu)^2 + 16\nu^2)^{\frac{1}{2}})^{\frac{1}{2}} \right]$$

We wish to show that  $\operatorname{Re} \lambda < 0$ . Since  $\nu^2 + a^2\mu < 0$  we know that

$$(a^2 + 4\mu)^2 + 16\nu^2 < (a^2 - 4\mu)^2$$

and  $a^2 - 4\mu > 0$  since  $\mu < 0$ . Hence

$$((a^2 + 4\mu) + ((a^2 + 4\mu)^2 + 16\nu^2)^{\frac{1}{2}})^{\frac{1}{2}} < a\sqrt{2} .$$

It follows that  $\operatorname{Re} \lambda < 0$  so that (ii) is proved.

For criterion (iii) let  $\lambda + \mu + i\nu$  to be eigenvalue of  $G$  with  $\lambda \neq 0$ . From equation (4) we see that

$$\det (H + \lambda A + \lambda^2 I) = 0 .$$

Hence, there exists a complex non-zero  $k$ -vector  $x$  such that

$(H + \lambda A + \lambda^2 I)x = 0$ . Let  $x$  be normalized so that

$$\max \{ |x_i| : i = 1, \dots, k \} = |x_r| = 1$$

for some component  $x_r$  of  $x$  with  $r \in \{1, \dots, k\}$ .

Then from the matrix multiplication

$$\sum_{j=1}^k h_{rj} x_j + (\lambda a_r + \lambda^2) x_r = 0$$

or

$$(h_{rr} + \lambda a_r + \lambda^2) x_r = - \sum_{\substack{j=1 \\ j \neq r}}^k h_{rj} x_j.$$

To achieve a contradiction suppose that  $\mu \geq 0$ . Then

$$\begin{aligned} & |h_{rr} + \mu a_r + i \nu a_r + \mu^2 - \nu^2 + 2i \mu \nu| \\ & \leq \sum_{\substack{j=1 \\ j \neq r}}^k |h_{rj}| |x_j| \leq \sum_{\substack{j=1 \\ j \neq r}}^k |h_{rj}| \equiv p \end{aligned}$$

where  $p$  is defined here. Hence,

$$\begin{aligned} p^2 & \geq [h_{rr}^2 - 2 \nu^2 h_{rr} + 2(\mu a_r + \mu^2) h_{rr} + (\mu a_r + \mu^2 - \nu^2)^2 \\ & \quad + \nu^2 a_r^2 + 4 \nu^2 (a_r \mu + \mu^2)] \end{aligned}$$

so that

$$\begin{aligned} p^2 - h_{rr}^2 + \nu^2 (2h_{rr} - a_r^2) & \geq 2(\mu a_r + \mu^2) h_{rr} \\ & \quad + (\mu a_r + \mu^2 - \nu^2)^2 + 4 \nu^2 (a_r \mu + \mu^2). \end{aligned}$$

By hypothesis  $p - h_{rr} \leq 0$  and  $2 h_{rr} - a_r^2 \leq 0$ . However, it is easy to see that the right-hand side of the above inequality is positive since  $\lambda \neq 0$ . Thus we have a contradiction.

Therefore  $\mu < 0$  and the theorem is proved.

We see from the following result that criterion (ii) of Theorem 2.6 is almost a necessary and sufficient condition for stability. It fails to apply only when  $\det G = 0$  or  $v^2 + a^2\mu = 0$  for some eigenvalue  $\mu + iv$  of CB.

**2.7 Theorem:** Let A, B, C, and G be defined as in equation (3). Assume that  $A = aI_k$  where  $a > 0$  is a real constant. Suppose that  $\xi = \mu + i v$  ( $\mu, v$  are real) is a non-zero eigenvalue of CB such that  $v^2 + a^2\mu > 0$ . Then G has an eigenvalue with positive real part and the origin is not a stable equilibrium point of (1).

**Proof:** See the proof of Theorem 2.6 (ii). If  $\operatorname{Re} \lambda \leq 0$ , then by a straight forward calculation  $a^2 \geq 4\mu$  and  $v^2 + a^2\mu \leq 0$ .

**2.8 Corollary:** Let A, B, C, and G be defined as in equations (1), (2), and (3). If G satisfies one of the criteria of Theorem 2.6 and if  $\det G \neq 0$ , then the origin is an asymptotically stable equilibrium point of system (1).

Referring to Theorem 2.6 we note that in criterion (i) we have that  $AK = KA$  if both A and K are diagonal. Criterion (ii) indicates that if the origin is a stable equilibrium point of system (1) with no damping ( $A = 0$ ), then the uniformly damped ( $A = aI_k$ ,  $a > 0$ ) version of (1) will have an asymptotically stable equilibrium point at the origin (if  $\det G \neq 0$ ). In the case of  $A = \operatorname{diag}(a_i)$ , criterion (iii) may be applied in slightly modified form to a matrix H with the property that  $W^{-1}HW$  is diagonally dominant for some diagonal matrix W (for example, H may be an M-matrix).



In some systems, such as power systems, information about the uniqueness of equilibrium points of system (1) in certain regions is useful. With this goal in mind we develop a result based on the following definition.

2.9 Definition: Let  $J(x)$  be a continuous  $n \times n$  real matrix valued function of  $x \in \mathbb{R}^m$ . We say that  $J(x)$  is uniformly of class U on a set  $S \subset \mathbb{R}^m$  if for every non-zero element  $y \in \mathbb{R}^n$  there exists a vector  $v \in \mathbb{R}^n$  such that if  $x \in S$  and  $z(x) = v^T J(x) y$ , then  $z(x) \neq 0$ .<sup>1</sup>

2.10 Theorem: Let  $S$  be a convex subset of  $\mathbb{R}^n$  and let  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a function with continuous first partial derivatives on  $S$ . Let  $J(x) = \partial f / \partial x$  be the Jacobian matrix of  $f$ . If  $J(x)$  is uniformly of class U on  $S$ , then  $f$  is one-to-one on  $S$ .

Proof: Let  $x$  and  $y$  be distinct points in  $S$ . Define  $w^* = y - x$  and  $w(u) = uy + (1 - u)x$  for  $u \in \mathbb{R}$ . Then  $w(u) \in S$  for  $0 \leq u \leq 1$  by convexity and  $dw/du = w^*$ . Let  $f(u) = f(w(u))$  and  $J(u) = J(w(u))$ . Since  $w^* \neq 0$ , there exists by Definition 2.9 a vector  $v \in \mathbb{R}^n$  such that  $v^T J(u) w^* \neq 0$  for  $0 \leq u \leq 1$ . Then for  $0 \leq u \leq 1$  we have

$$\frac{d}{du} v^T f(u) = v^T \frac{\partial f}{\partial w} \frac{dw}{du} = v^T J(u) w^* \neq 0.$$

Hence  $v^T f(u)$  is either strictly increasing or strictly decreasing.

In either case

$$v^T f(x) = v^T f(0) \neq v^T f(1) = v^T f(y)$$

so that  $f(x) \neq f(y)$ .

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<sup>1</sup>The author would like to thank the referee for his suggestion in improving this definition.

### III. POWER SYSTEMS

The problem of transient stability of systems of synchronous generators modeled by the "swing equations" has been studied since the beginning of the electric age [1, 3]. At first the problem was studied numerically, but, beginning in 1947 [13], direct methods, particularly Liapunov theory, have been applied to the problem [6, 16]. However, most of the direct methods known to this author may be applied only to the swing equations with zero transfer conductances (however, see [12]). In his review of power system stability [6], Fouad regards the assumption of negligible transfer conductances as unsatisfactory.

In this section we apply the results of Section II to the problem of the local stability and the regional uniqueness of equilibrium points of the swing equations with non-zero transfer conductances.

In the classical model for power systems a set of  $n$  interconnected synchronous generators is assumed to be governed by the system of differential equations (see [6, 16])

$$M_i \ddot{\delta}_i + d_i \dot{\delta}_i = [Q_i - E_i^2 G_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^n E_i E_j Y_{ij} \sin(\delta_i - \delta_j - \alpha_{ij})] \quad (5)$$

where  $i = 1, \dots, n$ . For the  $i$ -th generator  $\delta_i$  is the rotor angle,  $M_i > 0$  is the moment of inertia,  $d_i > 0$  is a damping constant,  $Q_i$  is the mechanical power,  $E_i \exp(i \delta_i)$  is the voltage behind the transient

reactance (here  $i \delta_i = \sqrt{-1} \delta_i$ ),  $Y_{ij} \exp(i \theta_{ij}) = G_{ij} + i B_{ij}$  is the  $i - j$ -th element of the short circuit admittance matrix,

$\alpha_{ij} = \alpha_{ji} = \theta_{ij} - (\pi/2)$ , and  $Y_{ij} = Y_{ji} > 0$ . All parameters except  $\delta_i$  are assumed to be time independent.

For convenience we define  $P_{ij} = E_i E_j Y_{ij} > 0$  and  $P_i = Q_i - E_i^2 G_{ii}$  for each  $i, j = 1, \dots, n$  with  $i \neq j$ . Then  $P_{ij} = P_{ji}$ . We make the change of variables

$$\begin{aligned} \omega_i &= \dot{\delta}_i & \text{for } i = 1, \dots, n \\ \sigma_i &= \delta_i - \delta_n & \text{for } i = 1, \dots, n-1 \\ \sigma_n &= 0 \\ \rho &= \delta_n \end{aligned} \tag{6}$$

In these new variables and parameters we can write system (5) in the equivalent form

$$M_i \dot{\omega}_i = -d_i \omega_i + p_i - \sum_{\substack{j=1 \\ j \neq i}}^n P_{ij} \sin(\sigma_i - \sigma_j - \alpha_{ij}) \tag{7a}$$

for  $i = 1, \dots, n$

$$\dot{\sigma}_i = \omega_i - \omega_n \text{ for } i = 1, \dots, n-1 \tag{7b}$$

$$\dot{\rho} = \omega_n \tag{8}$$

By introducing the extra constant parameter  $\sigma_n = 0$  we can write system (7) in a more compact form than otherwise.

For the remainder of this study we will call system (7) the swing equations. The system of equations (7a), (7b), and (8) is equivalent to system (5). However, system (7) is independent of  $\rho$ ; solutions of (5) may be found from solutions of (7) through the integration indicated in equation (8) and the transformation of equation (6). Hence, we will focus our attention on system (7).

3.1 Definition: A solution  $(\delta_1, \dots, \delta_n, \omega_1, \dots, \omega_n)$  of system (5) is called a synchronous solution if  $\omega_i = \omega_0$  is constant for  $i = 1, \dots, n$ . For convenience we define

$$\omega = (\omega_1, \dots, \omega_n)^T \text{ and } \sigma = (\sigma_1, \dots, \sigma_{n-1})^T.$$

3.2 Remark: For any equilibrium point  $\omega^*, \sigma^*$  of system (7) we see that  $\omega_i^* = \omega_0$  for some real constant  $\omega_0$  and for  $i = 1, \dots, n$ . A synchronous solution of system (5) corresponds to an equilibrium point of system (7).

A main goal of a power systems engineer is to assure that a system of synchronous generators approaches a synchronous solution. Hence, by the previous remark we will study attractive synchronous solutions of system (5) by studying the asymptotic stability of equilibrium points of system (7).

The numbers  $\alpha_{ij}$  for  $i, j = 1, \dots, n$  with  $i \neq j$  are the transfer conductance parameters of system (7). We will assume that  $|\alpha_{ij}| < \pi/2$  for all  $i, j = 1, \dots, n$ , but unless specifically stated we will not assume that  $\alpha_{ij} = 0$ .

The open set  $S \subset \mathbb{R}^{n-1}$  defined as

$$S = \{\sigma \in \mathbb{R}^{n-1} : |\sigma_i - \alpha_{in}| < \frac{\pi}{2}, |\sigma_i + \alpha_{in}| < \frac{\pi}{2}, \quad (9)$$

$$|\sigma_i - \sigma_j - \alpha_{ij}| < \frac{\pi}{2} \text{ for } i, j = 1, \dots, n-1; i \neq j\}$$

$$= \{\sigma \in \mathbb{R}^{n-1} : |\sigma_i| < \frac{\pi}{2} - |\alpha_{in}|, |\sigma_i - \sigma_j| < \frac{\pi}{2} - |\alpha_{ij}|$$

$$\text{for } i, j = 1, \dots, n-1; i \neq j\}$$

will be especially significant to our study. We note that  $S$  is a convex set and that if  $\sigma \in S$ , then  $\cos(\sigma_i - \sigma_j - \alpha_{ij}) > 0$  for all  $i, j = 1, \dots, n, i \neq j$  (recall  $\sigma_n = 0$ ).

We will be especially interested in equilibrium points  $\omega^*, \sigma^*$  of (7) with  $\sigma^* \in S$ . In previous studies the assumption has generally been made that the transfer conductances are negligible; that is, that  $\alpha_{ij} = 0$  for  $i, j = 1, \dots, n$ . In this special case of system (7) it has been found that whenever  $\sigma^* \in S$ , then  $\omega^*, \sigma^*$  is a stable equilibrium point of system (7) [6, 16]. One author [9] claimed to develop a Liapunov function to demonstrate that if  $\sigma^* \in S$ , then  $\omega^*, \sigma^*$  is a stable equilibrium point of (7) even if  $\alpha_{ij} \neq 0$ . Although errors have been found in this Liapunov function [11, 14, 16], the conjecture is still widely assumed.

We will see that an equilibrium point  $\omega^*, \sigma^*$  of (7) with  $\sigma^* \in S$  may be either stable or unstable and that (7) can have at most one equilibrium point  $\omega^*, \sigma^*$  with  $\sigma^* \in S$ .

**3.3 Remark:** If the parameters of system (1) are defined in the following way, then system (7) and system (1) are equivalent.

Define  $k = n$ ,  $\ell = n - 1$ ,  $m = 2n - 1$ . Let  $\omega^* \in \mathbb{R}^n$  and  $\sigma^* \in \mathbb{R}^{n-1}$  be constant vectors with  $\omega_i^* = \omega_o$  for some constant  $\omega_o$  and for all  $i = 1, \dots, n$ . By Remark 3.2 all equilibrium points of (7) take this form. Let  $y = \omega - \omega^* \in \mathbb{R}^n$ ,  $z = \sigma - \sigma^* \in \mathbb{R}^{n-1}$ , and  $z_n = \sigma_n = \sigma_n^* = 0$ . Define

$$A = \text{diag} (d_i/M_i) \quad (10)$$

and let  $B = (b_{ij})$  be an  $(n - 1) \times n$  matrix where

$$b_{ij} = \begin{cases} 1 & \text{if } i = j = 1, \dots, n - 1 \\ -1 & \text{if } j = m, i = 1, \dots, n - 1 \\ 0 & \text{if } i \neq j; i, j = 1, \dots, n - 1 \end{cases} \quad (11)$$

For  $i = 1, \dots, n$  (recall  $\sigma_n = \sigma_n^* = z_n = 0$ ) let

$$g_i(z) = \frac{1}{M_i} [d_i \omega_o - P_i + \sum_{\substack{j=1 \\ j \neq i}}^n P_{ij} \sin (z_i - z_j + \sigma_i^* - \sigma_j^* - \alpha_{ij})] \quad (12)$$

and  $g(z) = (g_1(z), \dots, g_n(z))^T$ .

With the above definitions, (7) and (1) are equivalent. Thus we may apply the results of Section II to the swing equations. To do so we must study the matrices  $A$ ,  $CB$ , and  $BA^{-1}C$  where  $A$  and  $B$  are given in (10) and (11) and  $C$  is defined in equation (2).

In the sequel we assume that the vectors  $\omega^*$ ,  $\sigma^*$  of Remark 3.3 are in equilibrium point for system (7). Hence,  $g(0) = 0$  and to be consistent with equation (2), we define the Jacobian matrix

$$C = - \frac{\partial g}{\partial z} \Big|_{z=0} = (c_{ij}) . \quad (13)$$

To help characterize  $C$  and related matrices we define an  $n \times n$  matrix

$U = (u_{ij})$  as

$$u_{ij} = \begin{cases} P_{ij} \cos (\sigma_i^* - \sigma_j^* - \alpha_{ij}) & \text{if } i \neq j; i, j = 1, \dots, n \\ 0 & \text{if } i = j = 1, \dots, n \end{cases} \quad (14)$$

(recall  $\sigma_n^* = 0$ ). From equation (12) we see that

$$c_{ij} = \begin{cases} -\frac{1}{M_i} \sum_{r=1}^n u_{ir} & \text{if } i = j = 1, \dots, n-1 \\ \frac{1}{M_i} u_{ij} & \text{if } i \neq j \text{ and } i, j = 1, \dots, n-1 \\ \frac{1}{M_n} u_{nj} & \text{if } i = n, j = 1, \dots, n-1 \end{cases} \quad (15)$$

Then  $C$  is an  $n \times (n-1)$  matrix as required since  $k = n$  and  $\ell = n-1$ .

Let  $M = \text{diag } (M_i)$ . Using equations (11) and (15) we find that

$$CB = -M^{-1} H \quad (16)$$

where  $H = (h_{ij})$  and

$$h_{ij} = \begin{cases} \sum_{r=1}^n u_{ir} & \text{if } i = j = 1, \dots, n \\ -u_{ij} & \text{if } i \neq j \text{ and } i, j = 1, \dots, n \end{cases} \quad (17)$$

Using equations (10), (11), and (15) we find that

$$BA^{-1}C = -W - Q \quad (18)$$

where  $W = (w_{ij})$ ,  $Q = (q_{ij})$  and

$$w_{ij} = \begin{cases} \frac{1}{d_i} \sum_{r=1}^n u_{ir} & \text{if } i = j = 1, \dots, n-1 \\ -\frac{1}{d_i} u_{ij} & \text{if } i \neq j \text{ and } i, j = 1, \dots, n-1 \end{cases} \quad (19)$$

$$q_{ij} = q_j \equiv \frac{1}{d_n} u_{nj} \quad \text{for } i, j = 1, \dots, n-1. \quad (20)$$

Let  $q = (q_1, \dots, q_{n-1})^T$ . Then  $Q$  is a matrix for which each row is  $q^T$ . In developing the above notation we have partially proved some results of this section.

**3.4 Lemma:** Let matrices  $A$ ,  $B$ , and  $C$  be defined in equations (10), (11), (13), and (15). Suppose that  $\omega^*$ ,  $\sigma^*$  is an equilibrium point of system (7) and that  $\sigma^* \in S$  where  $S$  is defined in equation (9). Then  $BA^{-1}C$  is given by equation (18) and  $\det(BA^{-1}C) \neq 0$ . Let  $G$  be defined in equation (3). Then  $\det G \neq 0$ .

**Proof:** Note that  $\det A \neq 0$  by assumption. By Remark 2.5,  $\det G = 0$  if and only if  $\det(BA^{-1}C) = 0$ . We see from equations (14), (19), and (20) and the comment following equation (9) that  $W$  is an  $M$ -matrix [2, 5] and  $Q$  contains only positive elements.

Suppose  $\det(BA^{-1}C) = \det(-W-Q) = 0$ . Then there exists  $z \in \mathbb{R}^{n-1}$  with  $z \neq 0$  such that

$$(W^T + Q^T)z = - (BA^{-1}C)^T z = 0.$$



Thus

$$z = -(W^T)^{-1} Q^T z = - \left( \sum_{i=1}^{n-1} z_i \right) (W^T)^{-1} q .$$

Since  $W^T$  is also an M-matrix,  $(W^T)^{-1}$  contains only non-negative elements. This leads to a contradiction; for if  $\sum_{i=1}^{n-1} z_i > 0$ , then

$z_j < 0$  for all  $j = 1, \dots, n-1$  and if  $\sum_{i=1}^{n-1} z_i < 0$ , then  $z_j > 0$  for all  $j = 1, \dots, n-1$ .

We now give some stability results for the swing equations. The first result is concerned with the undamped version. Recall that  $\sigma_n = 0$  by convention.

**3.5 Theorem:** In the swing equations (system (7)) suppose that  $\omega^*, \sigma^*$  is an equilibrium point and that  $d_i = 0$  for  $i = 1, \dots, n$ . Let B and C be defined in equations (11), (14), and (15). If  $\omega^*, \sigma^*$  is a stable equilibrium point, then all the eigenvalues of CB are real and non-positive.

Proof: See Corollary 2.4.

We may use the contrapositive of the above theorem to find unstable examples of the swing equations, for if CB has a non-real eigenvalue and  $d_i$  is small enough for  $i = 1, \dots, n$ , then  $\omega^*, \sigma^*$  will be an unstable equilibrium point, even if  $\sigma^* \in S$ . In Section IV we exhibit such unstable examples for  $n = 3$ .

If  $\sigma^* \in S$ , then all eigenvalues of CB have non-positive real parts because  $-CB$  is a semi-M-matrix, but they may not be real (see [2, 5].) To determine whether all eigenvalues of CB are real and non-positive

we may use Sturm's theorem as suggested in the comment following Corollary 2.4.

The following theorem lists some asymptotic stability results for the damped form of the swing equations. (Recall  $\sigma_n = 0$ .)

3.6 Theorem: In the swing equations (system (7)) suppose that  $\omega^*, \sigma^*$  is an equilibrium point and that  $d_i > 0$  for  $i = 1, \dots, n$ . Let  $A, B$ , and  $C$  be defined in equations (10), (11), (14), and (15). Let  $S$  be defined in equation (9). If any of the following criteria is satisfied, then  $\omega^*, \sigma^*$  is an asymptotically stable equilibrium point of the swing equations:

- (i)  $\sigma_i^* = 0$  for  $i = 1, \dots, n - 1$
- (ii)  $\alpha_{ij} = 0$  for  $i, j = 1, \dots, n$  with  $i \neq j$  and  $\sigma^* \in S$
- (iii)  $\det BA^{-1}C \neq 0$ , all non-zero eigenvalues of  $CB$  have negative real part (which occur if  $\sigma^* \in S$ ), and there is a real constant  $a > 0$  such that  $(d_i/M_i) = a$  for  $i = 1, \dots, n$  and  $\nu^2 + a^2\mu < 0$  for any non-zero eigenvalue  $\xi = \mu + i\nu$  of  $CB$  where  $\mu < 0$  and  $\nu$  are real
- (iv)  $\det BA^{-1}C \neq 0$ , all eigenvalues of  $CB$  are real and non-positive, and  $(d_i/M_i) = a > 0$  for  $i = 1, \dots, n$  for some real constant  $a$
- (v)  $\sigma^* \in S$  and

$$d_i^2 \geq 2 M_i \sum_{\substack{j=1 \\ j \neq i}}^n p_{ij} \cos(\sigma_i^* - \sigma_j^* - \alpha_{ij}) \text{ for } i = 1, \dots, n.$$

Proof: See Corollary 2.8. Criteria (i) and (ii) of this theorem follow from criterion (i) of Theorem 2.6. Criteria (iii) and (v) follow from criteria (ii) and (iii), respectively, of Theorem 2.6. Criterion (iv) is a special case of criterion (iii).

The fact that if  $\sigma^* \in S$ , then all non-zero eigenvalues of CB have negative real parts follows from the fact that  $-CB$  is a semi-M-matrix.

We note that Theorem 3.6 is especially effective when applied to swing equations with uniform damping. The assumption of uniform damping is often made in the study of transient stability.

We now consider the uniqueness of equilibrium points in the set  $S$ .

3.7 Lemma: Let  $A$ ,  $B$ , and  $g$  be defined in equations (10), (11), and (12) where  $\omega^* \in \mathbb{R}^n$  and  $\sigma^* \in \mathbb{R}^{n-1}$  are some constant vectors with  $B\omega^* = 0$ . Let  $f(z, y) = Ay + g(z)$  for  $y \in \mathbb{R}^n$  and  $z \in \mathbb{R}^{n-1}$ . Let

$$N_B = \{y \in \mathbb{R}^n : By = 0\} = \{y \in \mathbb{R}^n : y_i = y_j \text{ for } i, j = 1, \dots, n\}$$

and

$$S - \sigma^* = \{z \in \mathbb{R}^{n-1} : z + \sigma^* \in S\}$$

where  $S$  is defined in equation (9). Then  $f$  is one-to-one on  $(S - \sigma^*) \times N_B$ .

Proof: We see that  $(S - \sigma^*) \times N_B$  is convex. We know that  $N_B = \{w u : w \in \mathbb{R}\}$  where  $u = (1, 1, \dots, 1)^T \in \mathbb{R}^n$ . Thus, for  $(z, y) \in (S - \sigma^*) \times N_B$  we let  $f(z, y) = f(z, w) = f(x)$  where  $y = w u$ ,  $z + \sigma^* \in S$  and  $x = (z^T, w)^T \in \mathbb{R}^n$ . Let  $J(x) = \partial f / \partial x$  for  $x \in (S - \sigma^*) \times \mathbb{R}$ . We will show that  $J(x)$  uniformly is of class U on  $(S - \sigma^*) \times \mathbb{R}$ .

Let  $q \in \mathbb{R}^n$  be given with  $q \neq 0$  and let  $r = J(x) q$  for some  $x \in (S - \sigma^*) \times \mathbb{R}$ . Suppose without loss of generality that  $q_n \geq 0$ . Let  $q_k \geq q_j$  for some fixed  $k \in \{1, \dots, n-1\}$  and for all  $j \in \{1, \dots, n-1\}$ . By examining equations (10) and (12) we see that

$$r_k = \frac{1}{M_k} \{d_k q_n + [P_{kn} \cos(x_k + \sigma_k^* - \alpha_{kn})]q_k + \sum_{\substack{j=1 \\ j \neq k}}^{n-1} [P_{kj} \cos(x_k - x_j + \sigma_k^* - \sigma_j^* - \alpha_{kj})](q_k - q_j)\}$$

and

$$r_n = \frac{1}{M_n} \{d_n q_n - \sum_{j=1}^{n-1} [P_{nj} \cos(x_j + \sigma_j^* + \alpha_{nj})]q_j\}.$$

Since  $x \in (S - \sigma^*) \times \mathbb{R}$ , all cosine factors are positive. If  $q_k > 0$ , then  $r_k > 0$ . If  $q_k \leq 0$ , then  $q_j \leq q_k \leq 0$  for all  $j = 1, \dots, n-1$  and either  $q_j < 0$  for some  $j = 1, \dots, n-1$  or  $q_n > 0$ . Thus, if  $q_k \leq 0$ , then  $r_n > 0$ . In Definition 2.9 we choose vector  $v \in \mathbb{R}^n$  such that if  $q_k > 0$ , then  $v_k = 1$ ,  $v_j = 0$  for  $j \neq k$  and if  $q_k \leq 0$ , then  $v_n = 1$ ,  $v_j = 0$  for  $j \neq n$ . Now we apply Theorem 2.10 to complete the proof.

**3.8 Theorem:** System (7) has at most one equilibrium point  $\omega^*, \sigma^*$  with  $\sigma^* \in S$  where  $S$  is defined in equation (9).

Proof: The Theorem is a direct result of Lemma 3.7.

## IV. UNSTABLE EXAMPLES

The swing equations (system (7)) for three machines when written in the form of system (1) according to Remark 3.3 take the form

$$\begin{aligned}
 \dot{y}_1 &= -\frac{d_1}{M_1} y_1 - \frac{1}{M_1} [d_1 \omega_0 - P_1 + P_{12} \sin(z_1 - z_2 + \sigma_1^* - \sigma_2^* - \alpha_{12}) \\
 &\quad + P_{13} \sin(z_1 + \sigma_1^* - \alpha_{13})] \\
 \dot{y}_2 &= -\frac{d_2}{M_2} y_2 - \frac{1}{M_2} [d_2 \omega_0 - P_2 + P_{21} \sin(z_2 - z_1 + \sigma_2^* - \sigma_1^* - \alpha_{21}) \\
 &\quad + P_{23} \sin(z_2 + \sigma_2^* - \alpha_{23})] \\
 \dot{y}_3 &= -\frac{d_3}{M_3} y_3 - \frac{1}{M_3} [d_3 \omega_0 - P_3 - P_{31} \sin(z_1 + \sigma_1^* + \alpha_{31}) \\
 &\quad - P_{32} \sin(z_2 + \sigma_2^* + \alpha_{32})] \\
 \dot{z}_1 &= y_1 - y_3, \quad \dot{z}_2 = y_2 - y_3
 \end{aligned} \tag{21}$$

where we assume that  $y = 0, z = 0$  is an equilibrium point of the above system (so that  $\omega^*, \sigma^*$  with  $\omega_1^* = \omega_2^* = \omega_3^* = \omega_0$  is an equilibrium point of system (7)). We wish to choose the constant parameters of system (21) so that  $\omega_0 = 0, \sigma^* \in S$  (see equation (9)), and  $y = 0, z = 0$  is an unstable equilibrium point of system (21). We will pursue this objective by setting  $d_1 = d_2 = d_3 = 0$  and applying Theorem 3.5.

From equations (16) and (17) we see that

$$CB = \begin{pmatrix} -w_{12} & -w_{13} & w_{12} & w_{13} \\ w_{21} & -w_{21} & -w_{23} & w_{23} \\ w_{31} & w_{32} & -w_{31} & -w_{32} \end{pmatrix} \quad (22)$$

where we use the convention  $\sigma_3 \equiv 0$  and we define

$$w_{ij} = \frac{P_{ij}}{M_i} \cos (\sigma_i^* - \sigma_j^* - \alpha_{ij}) \quad (23)$$

for  $i, j = 1, 2, 3$  with  $i \neq j$ . We wish to choose  $P_{ij}, M_i, \sigma_i^*, \alpha_{ij}$  for  $i, j = 1, 2, 3$  with  $i \neq j$  such that  $\sigma^* \in S$  and  $CB$  has a non-real eigenvalue and then apply Theorem 3.5.

From equation (22) we see that

$$\det (CB - \lambda I_3) = -\lambda(\lambda^2 + b\lambda + c) \quad (24)$$

where

$$b = w_{12} + w_{13} + w_{21} + w_{23} + w_{31} + w_{32} \quad (25)$$

$$c = (w_{12}w_{23} + w_{13}w_{21} + w_{13}w_{23} + w_{12}w_{31} + w_{12}w_{32} \\ + w_{13}w_{32} + w_{21}w_{31} + w_{21}w_{32} + w_{23}w_{31}) . \quad (26)$$

By the quadratic formula and equation (24), matrix  $CB$  has a non-real eigenvalue if and only if  $b^2 - 4c < 0$ .

Define

$$w = (w_{12}, w_{21}, w_{13}, w_{31}, w_{23}, w_{32})^T \in \mathbb{R}^6 \quad (27)$$

We find that

$$b^2 - 4c = w^T F w \quad (28)$$

where

$$F = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \end{pmatrix} \quad (29)$$

The matrix  $F$  has eigenvalues  $\lambda_i$  and corresponding eigenvectors  $v_i$  as follows:

$$\begin{aligned} \lambda_1 &= 4, v_1 = (0, 1, -1, -1, 1, 0)^T \\ \lambda_2 &= 4, v_2 = (2, 1, 1, -1, -1, -2)^T \\ \lambda_3 &= -2, v_3 = (-1, 1, 1, -1, -1, 1)^T \\ \lambda_4 &= 0, v_4 = (1, 0, 0, 0, 0, 1)^T \\ \lambda_5 &= 0, v_5 = (0, 1, 0, 1, 0, 0)^T \\ \lambda_6 &= 0, v_6 = (0, 0, 1, 0, 1, 0)^T \end{aligned} \quad (30)$$

We summarize these results in the following lemma.

4.1 Lemma: Let  $CB$ ,  $b$ ,  $c$ ,  $w$ ,  $F$ , and  $v_i$  for  $i = 1, \dots, 6$  be defined in equations (22), (25), (26), (27), (29), and (30), respectively. Then there exists unique real numbers  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ ,  $\epsilon$ ,  $\zeta$  such that

$$w = \alpha v_3 + \beta v_4 + \gamma v_5 + \delta v_6 + \epsilon v_1 + \zeta v_2. \quad (31)$$

The characteristic polynomial of CB is

$$-\lambda(\lambda^2 + b\lambda + c) .$$

Matrix CB has a non-real eigenvalue if and only if the number

$$b^2 - 4c = w^T Fw = -12 \alpha^2 + 16 \epsilon^2 + 48 \zeta^2 \quad (32)$$

is negative. If  $w_{ij} > 0$  for  $i, j = 1, 2, 3$  with  $i \neq j$ , then all eigenvalues of CB have non-positive real part.

Proof: Much of the lemma is a summary of previous discussion. We verify equations (31) and (32) by noticing that the  $v_i$  are mutually orthogonal eigenvectors of F. If  $w_{ij} > 0$ , then by the quadratic formula all eigenvalues of CB have non-positive real part.

Next we see that we can choose  $w$  so that CB has non-real eigenvalues.

4.2 Lemma: Let  $b$ ,  $c$ , and  $w$  be defined in equations (25), (26), and (27), respectively.

(a) If  $w_{ij} > 0$  for  $i, j = 1, 2, 3$  with  $i \neq j$ , then  $b^2 > 3c$ .  
 (b) If  $4c > b^2 > 3c > 0$  and  $b > 0$ , then there exist  $\alpha, \beta, \gamma, \delta$  such that the following relations hold:

$$w = (-\alpha + \beta, \alpha + \gamma, \alpha + \delta, -\alpha + \gamma, -\alpha + \delta, \alpha + \beta)^T \quad (33)$$

$$b^2 - 4c = -12 \alpha^2 < 0 \quad (34)$$

$$w_{ij} > 0 \text{ for } i, j = 1, 2, 3 \text{ with } i \neq j . \quad (35)$$



Proof: From equations (27), (30), and (31) we see that

$$w_{12} = 2\zeta - \alpha + \beta \equiv \mu + \beta, \quad w_{32} = -2\zeta + \alpha + \beta \equiv -\mu + \beta \quad (36)$$

$$w_{21} = \epsilon + \zeta + \alpha + \gamma \equiv \nu + \gamma, \quad w_{31} = -\epsilon - \zeta - \alpha + \gamma \equiv -\nu + \gamma$$

$$w_{13} = -\epsilon + \zeta + \alpha + \delta \equiv \rho + \delta, \quad w_{23} = \epsilon - \zeta - \alpha + \delta \equiv -\rho + \delta$$

where  $\mu$ ,  $\nu$ , and  $\rho$  are defined here. Solving for  $\alpha$ ,  $\epsilon$ ,  $\zeta$  in terms of  $\mu$ ,  $\nu$ , and  $\rho$  we see that

$$\alpha = \frac{1}{3}(-\mu + \nu + \rho), \quad \epsilon = \frac{1}{2}(\nu - \rho), \quad \zeta = \frac{1}{6}(2\mu + \nu + \rho). \quad (37)$$

From equation (36) we see that  $w_{ij} > 0$  for  $i, j = 1, 2, 3$  with  $i \neq j$  if and only if

$$\beta > |\mu|, \quad \gamma > |\nu|, \quad \text{and} \quad \delta > |\rho|. \quad (38)$$

For part (a) we see from equation (32) that

$$c = \frac{1}{4}b^2 + 3\alpha^2 - 4\epsilon^2 - 12\zeta^2. \quad (39)$$

From equations (25) and (36) we see that

$$b = 2(\beta + \gamma + \delta). \quad (40)$$

Applying equations (37) - (40), we see that

$$\begin{aligned} b^2 - 3c &= \frac{1}{4}b^2 - 9\alpha^2 + 12\epsilon^2 + 36\zeta^2 \\ &= (\beta + \gamma + \delta)^2 - (-\mu + \nu + \rho)^2 + 12\epsilon^2 + 36\zeta^2 \\ &> (|\mu| + |\nu| + |\rho|)^2 - (-\mu + \nu + \rho)^2 \geq 0. \end{aligned}$$

For part (b) suppose that  $b > 0$  and  $4c > b^2 > 3c > 0$ . In equation (36) let

$$\alpha = \left| \frac{4c - b^2}{12} \right|^{\frac{1}{2}}, \quad \beta = \gamma = \delta = \frac{b}{6}, \quad \epsilon = \zeta = 0. \quad (41)$$

Then equation (40) is satisfied and, by equation (32), equation (34) is satisfied. Hence, the definitions (41) are consistent with (33).

From equations (34) and (41)

$$36 \beta^2 + 12 \alpha^2 = b^2 + 12 \alpha^2 = 4c < \frac{4}{3} b^2 = 48 \beta^2.$$

Hence  $\alpha^2 < \beta^2 = \gamma^2 = \delta^2$  so that relations (38) and thus (35) are true.

If we choose  $P_{ij}$ ,  $M_i$ ,  $\alpha_{ij}$ , and  $\sigma_i^*$  properly, then the previous lemma will allow us to construct unstable swing equations. The following lemma shows that these parameters can be so chosen.

**4.3 Lemma:** For any vector  $w \in \mathbb{R}^6$  of the form (27) which has positive elements and for any positive numbers  $M_1, M_2, M_3$  there exist real constants  $P_{ij} = P_{ji} > 0$ ,  $\sigma_i^*$ , and  $\alpha_{ij} = \alpha_{ji}$  for  $i = 1, 2$  and  $j = 1, 2, 3$  with  $i \neq j$  such that (recall that  $\sigma_3^* = 0$ )

$$w_{ij} = \frac{P_{ij}}{M_i} \cos(\sigma_i^* - \sigma_j^* - \alpha_{ij}) \quad (23)$$

and

$$|\sigma_i^* - \sigma_j^* - \alpha_{ij}| < \frac{\pi}{2} \text{ for } i, j = 1, 2, 3 \text{ with } i \neq j. \quad (42)$$

**Proof:** For  $i, j = 1, 2, 3$  with  $i \neq j$  define  $\theta_{ij} = \sigma_i^* - \sigma_j^* - \alpha_{ij}$  for any numbers  $\sigma_i^*, \sigma_j^*$ , and  $\alpha_{ij}$  with  $\sigma_3^* = 0$  and  $\alpha_{ij} = \alpha_{ji}$ .

Then

$$\theta_{21} - \theta_{12} + \theta_{13} - \theta_{31} + \theta_{32} - \theta_{23} = 0 \quad (43)$$

and

$$\begin{aligned} \sigma_1^* &= \frac{1}{2} (\theta_{13} - \theta_{31}) \\ \sigma_2^* &= \frac{1}{2} (\theta_{23} - \theta_{32}) \\ \alpha_{12} &= -\frac{1}{2} (\theta_{12} + \theta_{21}) \\ \alpha_{13} &= -\frac{1}{2} (\theta_{13} + \theta_{31}) \\ \alpha_{23} &= -\frac{1}{2} (\theta_{23} + \theta_{32}) \end{aligned} \quad (44)$$

We require that  $|\theta_{ij}| < \pi/2$  for  $i, j = 1, 2, 3$  with  $i \neq j$  and equation (43) is satisfied. Then the  $\sigma_i^*$  and  $\alpha_{ij}$  are given by equation (44).

We must show that for any  $w_{ij} > 0$  and  $M_i > 0$  with  $i, j = 1, 2, 3$  and  $i \neq j$  we may choose the  $\theta_{ij}$  so that

$$\frac{M_i w_{ij}}{M_j w_{ji}} = \frac{P_{ij} \cos \theta_{ij}}{P_{ji} \cos \theta_{ji}} = \frac{\cos \theta_{ij}}{\cos \theta_{ji}} \quad (45)$$

and (43) is satisfied.

We first choose appropriate values for  $\theta_{13}, \theta_{31}, \theta_{23}, \theta_{32}$ . For example, assume that  $M_i w_{ij} \leq M_j w_{ji}$ . Let  $\theta_{ji} = 0$ ,  $\cos \theta_{ij} = M_i w_{ij} / M_j w_{ji}$ . Since  $\cos \theta_{ij} > 0$  we may choose  $\theta_{ij}$  so that  $|\theta_{ij}| < \pi/2$ . Since  $\cos \phi = \cos (-\phi)$ , the sign of  $\theta_{ij}$  is arbitrary. Using this freedom we may choose  $\theta_{ij}$  so that if  $\eta = \theta_{31} - \theta_{13} + \theta_{23} - \theta_{32}$ , then  $|\eta| < \pi/2$ .

For example, let  $\{|\theta_{13}|, |\theta_{31}|, |\theta_{23}|, |\theta_{32}|\} = \{\mu, \nu, \xi, \rho\}$  for some numbers such that  $0 \leq \mu \leq \nu \leq \xi \leq \rho < \pi/2$ . Then

$-\pi/2 < \mu - \nu + \rho - \xi < \pi/2$ . Choose the signs of  $\theta_{ij}$  so that

$\eta = \mu - \nu + \rho - \xi$ . If  $\eta = 0$ , then change  $\theta_{13}$  and  $\theta_{31}$  slightly so that equation (45) is satisfied, but  $\eta \neq 0$ . Now we must choose  $\theta_{12}$  and  $\theta_{21}$  so that  $\theta_{21} = \theta_{12} + \eta$  which implies that (43) is satisfied. Assume without loss of generality that  $0 < \eta < \pi/2$ .

Define

$$h(\theta_{12}) = \frac{\cos \theta_{12}}{\cos \theta_{21}} = \frac{\cos \theta_{12}}{\cos (\theta_{12} + \eta)} .$$

Then  $h(-\pi/2) = 0$  and

$$\lim_{\theta \rightarrow (\frac{\pi}{2} - \eta)^-} h(\theta) = \lim_{\theta \rightarrow \frac{\pi}{2}^-} \frac{\sin \eta}{\cos \theta} = +\infty .$$

Hence,  $h(\theta)$  takes on every positive value for  $-\pi/2 < \theta < (\pi/2) - \eta$ .

Hence, we may choose  $\theta_{12}$  so that

$$\frac{\cos \theta_{12}}{\cos \theta_{21}} = h(\theta_{12}) = \frac{M_1 w_{12}}{M_2 w_{21}}$$

and

$$-\frac{\pi}{2} + \eta < \theta_{21} = \theta_{12} + \eta < \frac{\pi}{2} .$$

Thus, we may choose the  $\theta_{ij}$  so that (42), (43), (44), and (45) are satisfied. Let

$$P_{ji} = P_{ij} = \frac{M_i w_{ij}}{\cos \theta_{ij}} > 0 \text{ for } i, j = 1, 2, 3 \text{ with } i < j .$$

Then by equation (45) we see that equation (23) is satisfied. This completes the proof.

Combining the above results we may prove the existence of a large class of unstable three machine swing equations with unstable equilibrium points satisfying (42).

4.4 Theorem: For any polynomial  $\lambda^2 + b\lambda + c$  with  $b > 0$  and  $4c > b^2 > 3c > 0$  there exists a system of the form (21) which has an unstable equilibrium point at the origin such that the relations (24) and (42) are satisfied (i.e.,  $\sigma^* \in S$ ).

Proof: According to Lemmas 4.2 and 4.3 we may choose  $w_{ij} > 0$ ,  $M_i > 0$ ,  $P_{ij} = P_{ji} > 0$ ,  $\alpha_{ij} = \alpha_{ji}$ , and  $\sigma_i^*$  for  $i, j = 1, 2, 3$  with  $i \neq j$  and  $\sigma_3^* = 0$  such that  $b$  and  $c$  are given in equations (25) and (26) and equations (23) and (42) are satisfied. Then CB as defined in equation (22) will have characteristic polynomial (24) so that CB has a non-real eigenvalue.

We use these parameters to define

$$P_i = \sum_{\substack{j=1 \\ j \neq i}}^3 P_{ij} \sin(\sigma_i^* - \sigma_j^* - \alpha_{ij})$$

for  $i = 1, 2, 3$ , and we let  $\omega_0 = 0$ . With any choice of  $d_1, d_2, d_3$  the origin is an equilibrium point of system (21). By continuity of eigenvalues we may choose positive  $d_1, d_2, d_3$  so that the origin is an unstable equilibrium point of (21).

We give the following specific example:

4.5 Example: Define the parameters of system (21) as follows:

$$\omega_0 = 0, M_1 = M_2 = M_3 = 1$$

$$P_{12} = P_{21} = 3.001149$$

$$P_{13} = P_{31} = 1.353902$$

$$P_{23} = P_{32} = 1.353902$$

$$P_1 = -3.217683$$

$$P_2 = -1.376475$$

$$P_3 = 1.547330$$

$$\sigma_1^* = 32.87356^\circ$$

$$\sigma_2^* = 42.87356^\circ$$

$$\alpha_{12} = \alpha_{21} = 73.62308^\circ$$

$$\alpha_{13} = \alpha_{31} = 42.87356^\circ$$

$$\alpha_{23} = \alpha_{32} = -32.87356^\circ$$

$$d = d_1 = d_2 = d_3 \leq 0.5, d > 0 .$$

With these choices of parameters the point  $y = 0, z = 0$  is an unstable equilibrium point of system (21) (within round-off error) even though (42) is satisfied (i.e.,  $\sigma^* \in S$ ).

For this example

$$CB = \frac{1}{3} \begin{pmatrix} -5 & 1 & 4 \\ 4 & -5 & 1 \\ 1 & 4 & -5 \end{pmatrix}$$

which has eigenvalues  $\lambda = 0$  and  $\lambda = (-5 \pm i\sqrt{3})/2$ . By Theorem 3.5 and the comments following it, system (21) has an unstable equilibrium

point at the origin for small  $d_i$ .

According to Theorem 2.7 we set  $\mu = -5/2$ ,  $\nu = \sqrt{3}/2$ ,  $a = d$ . Then  $\nu^2 + a^2\mu \geq .75 - (.25)(2.5) > 0$ . Hence, the associated matrix  $G$  (see equation (3) and Remark (3.3)) has an eigenvalue with positive real part.

We see that the example involves large transfer conductances resulting from the lack of freedom in choosing the parameters in the three machine case. Presumably unstable examples with smaller transfer conductances will occur in higher dimensions where there would be more freedom in choosing parameters. However, for higher dimensions, criteria for non-real eigenvalues are not as simple.

## V. CONCLUSION

It was found that undamped second order vector differential equations can be stable only under very special conditions. Sufficient conditions for asymptotic stability of damped second order vector equations were established. These results were applied to the problem of transient stability for power systems by treating the classical model with non-trivial transfer conductances, a system for which few analytic results have been established.

There is special interest in equilibrium solutions of the swing equations with  $|\delta_i - \delta_j - \alpha_{ij}| < \pi/2$  for all rotor angle pairs  $\delta_i, \delta_j$  and all transfer conductance parameters  $\alpha_{ij}$ . It was found that, contrary to a widely held opinion, such solutions may be unstable. It was also shown that there can be at most one equilibrium solution with this property (except for the same constant added to all rotor angles).

In the model with transfer conductances neglected, an equilibrium solution with  $|\delta_i - \delta_j| < \pi/2$  is stable. It would be useful to find upper bounds on  $|\alpha_{ij}|$  so that the property  $|\delta_i - \delta_j - \alpha_{ij}| < \pi/2$  will guarantee stability.



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PART II: SYNCHRONOUS SOLUTIONS OF POWER  
SYSTEMS: EXISTENCE, UNIQUENESS AND STABILITY

## ABSTRACT

This paper contains a study of the asymptotic stability and uniqueness of equilibrium solutions of multi-dimensional Hamiltonian-like systems. The results are applied to the swing equations, the classical model for power systems. By developing some results in matrix theory, it is shown that asymptotically stable equilibrium solutions may exist even though most rotor angle pairs are more than  $90^\circ$ , some even  $180^\circ$ , out of phase. In contrast to the numerical criteria usually used, an analytic criterion for the existence of asymptotically stable equilibrium solutions of the swing equations is given.

## I. INTRODUCTION

Most authors studying the stability of the swing equations for power systems have investigated only those equilibrium solutions for which all the rotor angles are less than  $90^\circ$  out of phase with each other. In this paper we develop examples of swing equations for which asymptotically stable equilibrium solutions exist even though most of the rotor angle differences are greater than  $90^\circ$  in magnitude; some such differences may be  $180^\circ$ .

It is the usual procedure to determine the existence of a post-fault equilibrium solution by numerical calculations. In this paper we give an analytic criterion for the existence of stable equilibrium solutions.

In general we study the stability, instability, uniqueness and existence of equilibrium solutions of the swing equations. The first three properties are studied by investigating the following system which is a model for many physical systems, including the swing equations:

$$\dot{y} = -Ay - g(z) \quad (1a)$$

$$\dot{z} = By \quad (1b)$$

where  $y \in \mathbb{R}^k$ ,  $z \in \mathbb{R}^\ell$ ,  $k + \ell = m$ ,  $x = (y^T, z^T)^T \in \mathbb{R}^m$ ,  $A$  and  $B$  are constant real  $k \times k$  and  $\ell \times k$  matrices respectively, and  $g: \mathbb{R}^\ell \rightarrow \mathbb{R}^k$  is a continuous map. For the swing equations,  $k \neq \ell$ .

When conservative forces are involved, system (1) will be a damped Hamiltonian-like system. In section II we discuss the stability and uniqueness of equilibrium points of such systems. In section III we

apply these results to the swing equations and prove the existence result. In section IV we develop some matrix theory which will allow us to find swing equations of arbitrarily large dimension for which stable equilibrium solutions exist, even though most of the rotor angle differences are greater than  $90^\circ$  in magnitude.

Most of the notation used is standard and will be used without explanation. The symbols  $A^T$  and  $A^{-1}$  will denote the transpose and inverse, respectively, of a matrix  $A = (a_{ij})$  whose  $i - j$ th element is  $a_{ij}$ . All vectors  $x$  are column vectors; hence,  $x^T$  is a row vector. The constant vectors  $y^* \in R^k$ ,  $z^* \in R^\ell$  are an equilibrium point of system (1) if  $Ay^* + g(z^*) = 0$  and  $By^* = 0$ . Let  $I_n$  be the  $n \times n$  identity matrix. Let  $\partial g / \partial z$  be the Jacobian matrix of  $g$ .

A submatrix of  $A$  is a matrix formed by eliminating from  $A$  certain rows and columns. A principal submatrix of  $A$  is a matrix formed by eliminating a row from  $A$  if and only if the corresponding column is eliminated. We use several characterizations for the positive definiteness of a symmetric matrix (see [8, p. 306]).

For a real valued function  $h(y, z)$  with  $y \in R^k$ ,  $z \in R^\ell$  we let

$$\nabla_y h = (\partial h / \partial y_1, \dots, \partial h / \partial y_k)^T$$

$$\nabla_z h = (\partial h / \partial z_1, \dots, \partial h / \partial z_\ell)^T.$$

## II. HAMILTONIAN-LIKE SYSTEMS

In this section we study the stability of damped Hamiltonian-like systems of the form (1).

2.1 Definition: We say that a system of ordinary differential equations is Hamiltonian-like if it has the form

$$\dot{y} = -L \nabla_z h(y, z) \quad (2a)$$

$$\dot{z} = L^T \nabla_y h(y, z) \quad (2b)$$

where  $y \in \mathbb{R}^k$ ,  $z \in \mathbb{R}^\ell$ ,  $h$  is a differentiable real valued function of  $y$  and  $z$ , and  $L$  is a  $k \times \ell$  real matrix.

The function  $h$  is an integral of system (2) and system (2) is Hamiltonian in form if  $k = \ell$  and  $L$  is the identity matrix. System (2) is called damped if a linear term  $-Ay$  is added to the right-hand side of equation (2a).

In the following theorem if we let  $L = K^{-1}B^T$  and  $h(y, z) = V(x, z^*)$  (see equation (5) below), then system (1) is a damped Hamiltonian-like system. One example of such a system with  $k \neq \ell$  is the swing equations with zero transfer conductances (see section III).

The following theorem lists the results for Hamiltonian-like forms of system (1).

2.2 Theorem: Let  $A$ ,  $B$ , and  $g$  be defined as in equation (1). Let  $K$  be a symmetric positive definite  $k \times k$  matrix. Let  $f(z)$  be a real valued function of  $\ell$  variables with continuous second partial derivatives. Suppose that  $KA + A^TK$  is positive definite, that  $B$  has rank  $\ell$  with  $\ell \leq k$ ,

and that

$$g(z) = K^{-1} B^T \nabla_z f(z) . \quad (3)$$

Define the Hessian matrix

$$J(z) = \left( \frac{\partial^2 f}{\partial z_i \partial z_j} \right) . \quad (4)$$

(a) If  $x^* = (y^{*T}, z^{*T})^T$  is an equilibrium point of system (1), then  $y^* = 0$  and  $g(z^*) = 0$ .

(b) Suppose that  $g(z^*) = 0$  for some  $z^* \in R^\ell$  and that  $J(z^*)$  is positive definite. Then  $x^* = (0^T, z^{*T})^T$  is an asymptotically stable equilibrium point of system (1).

(c) Suppose that  $g(z^*) = 0$  for some  $z^* \in R^\ell$  and that  $J(z^*)$  is not positive semi-definite. Then  $x^* = (0^T, z^{*T})^T$  is not an asymptotically stable equilibrium point of system (1). If in addition  $x^*$  is an isolated equilibrium point of (1), then  $x^*$  is not stable.

(d) Let  $g(z^*) = 0$  for some  $z^* \in R^\ell$  and let  $S$  be a convex subset of  $R^\ell$  with  $z^* \in S$ . Suppose that  $J(z)$  is positive definite and that equation (3) holds for all  $z \in S$ . Then  $x^* = (0^T, z^{*T})^T$  is the only equilibrium point of system (1) in  $R^k \times S$ .

Proof: (a) Suppose that  $x^* = (y^{*T}, z^{*T})^T$  is an equilibrium point of system (1). Then  $Ay^* + g(z^*) = 0$  and  $By^* = 0$ . Thus

$$\frac{1}{2} y^{*T} (A^T K + KA) y^* = y^{*T} K (Ay^*) = -y^{*T} K g(z^*) = -y^{*T} B^T \nabla_z f(z) = 0 .$$

Since  $A^T K + KA$  is positive definite,  $y^* = 0$ .



For parts (b) and (c) define

$$V(x; z^*) = \frac{1}{2} y^T K y + f(z) - f(z^*) \quad (5)$$

where  $z^* \in R^\ell$  is fixed,  $g(z^*) = 0$ , and  $x = (y^T, z^T)^T \in R^m$ . The derivative of  $V$  along solutions of system (1) is

$$\frac{dV}{dt} = y^T K (-Ay - g(z)) + (\nabla_z f(z))^T B y = -\frac{1}{2} y^T (KA + A^T K) y \leq 0.$$

Define

$$G = \{x \in R^m : dV/dt = 0\} = \{x = (y^T, z^T)^T : z \in R^\ell, y \in R^k, y = 0\}.$$

By inspection of system (1) the only solutions of (1) which remain in  $G$  for all  $t$  are the constant solutions or equilibrium points. Define the Hessian matrix

$$H = \left( \frac{\partial^2 V}{\partial x_i \partial x_j} \right) \bigg|_{x=x^*} = \begin{pmatrix} K & 0 \\ 0 & J(z^*) \end{pmatrix}.$$

Since  $K$  is positive definite,  $H$  is positive definite if and only if  $J(z^*)$  is.

Because  $B$  has full rank and  $K^{-1} B^T \nabla_z f(z^*) = g(z^*) = 0$ , we know that  $\nabla_z f(z^*) = 0$ . Hence  $V(x^*; z^*) = 0$  and

$$\nabla_x V \bigg|_{x=x^*} = \begin{pmatrix} Ky^* \\ \nabla_z f(z^*) \end{pmatrix} = 0$$

where  $x^* = (0^T, z^{*T})$ . Thus,  $x^*$  is a zero and a critical point of  $V$ .

(b) Suppose that  $J(z^*)$  and hence  $H$  is positive definite. Then  $x^*$  is a relative minimum point for  $V(x; z^*)$  (see [2, p. 376]).

Since  $\det J(z^*) \neq 0$ ,  $\nabla_z f$  is one-to-one in a neighborhood of  $z^*$  by the inverse function theorem (see [2, p. 372]). Hence,  $g(z)$  is one-to-one in that neighborhood of  $z^*$  because  $K^{-1}$  and  $B^T$  are one-to-one. Thus, there is a neighborhood  $N$  of  $x^*$  such that for  $x \in N$  we have  $V(x; z^*) > 0$  if  $x \neq x^*$  and  $g(z)$  is one-to-one if  $(y^T, z^T)^T = x \in N$ .

Therefore  $x^*$  can be the only equilibrium point and hence, the only invariant subset of  $G \cap N$ . By the invariance theorem [4, 9]  $x^*$  is an asymptotically stable equilibrium point of system (1).

(c) Suppose that  $J(z^*)$  and hence  $H$  is not positive semi-definite. Then  $x^*$  is a saddle point for  $V(x; z^*)$  (see [2]). Hence, there is an open region  $M \subset \mathbb{R}^m$  containing  $x^*$  on its boundary such that  $V(x; z^*) < 0$  for  $x \in M$ . Since  $dV/dt \leq 0$ , a solution starting in  $M$  cannot approach  $x^*$  as  $t$  increases. Thus  $x^*$  is not an asymptotically stable equilibrium point of system (1).

Suppose that, in addition,  $x^*$  is an isolated equilibrium point. Let  $N$  be a bounded neighborhood of  $x^*$  containing no other equilibrium points of (1). By invariance theory (see [4, p. 203]) a solution starting in  $M \cap N$  must either leave the set  $N$  or approach an invariant subset of  $G$ . Since  $\{x^*\}$  is the only invariant subset of  $G$  in  $N$ , the point  $x^*$  cannot be a stable equilibrium point.

(d) By part (a) we need only to show that  $g(z)$  is one-to-one on  $S$ .

Let  $z^1$  and  $z^2$  be two distinct points in  $S$ . Define  $z(u) = uz^2 + (1-u)z^1$  for  $u \in \mathbb{R}$  and  $w = z^2 - z^1$ . Then  $z(u) \in S$  for  $0 \leq u \leq 1$  by convexity of  $S$  and  $dz/du = w$ . Let  $J(u) = J(z(u))$  and  $q(u) = (\nabla_z f(z(u)) - \nabla_z f(z^1))^T w$ . Then

$$\frac{dq}{du} = \frac{d}{du} \sum_{i=1}^{\ell} \left( \frac{\partial f}{\partial z_i} (z(u)) - \frac{\partial f}{\partial z_i} (z^1) \right) w_i = \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} \frac{\partial^2 f}{\partial z_i \partial z_j} w_i w_j = w^T J(u) w .$$

Since  $J(u)$  is positive definite for  $0 \leq u \leq 1$ ,  $dq/du > 0$  for  $0 < u < 1$ .

Hence,  $0 = q(0) < q(1)$ .

If  $\nabla_z f(z^1) = \nabla_z f(z^2)$ , then  $q(1) = 0$ , which is a contradiction.

Since  $B^T$  and  $K^{-1}$  are one-to-one,  $g(z^1) \neq g(z^2)$ .

If system (1) is damped Hamiltonian in form, then we can apply

Theorem 2.2 with  $k = \ell$  and  $B = K$ .

### III. POWER SYSTEMS

The problem of transient stability of systems of synchronous generators modeled by the "swing equations" has been studied since the beginning of the electric age [1, 5]. In recent years, Liapunov theory has been applied to the problem [7, 11].

In this section we show that the swing equations take the form of system (1) so that we can apply the results of section II. The Liapunov function used in this section is equivalent to a classical Liapunov function used in the study of power systems. Derivatives of this Liapunov function along solutions are not negative definite so that asymptotic stability does not follow directly from the classical Liapunov theorems. By applying the invariance theorem we can conclude the asymptotic of the equilibrium solutions with a negative semi-definite derivative.<sup>1</sup> We also give results on existence and uniqueness of stable equilibrium solutions for the swing equations.

In the classical model for power systems with negligible transfer conductances, a set of  $n$  interconnected synchronous generators is assumed to be governed by the system of differential equations (see [7, 11])

$$M_i \ddot{\delta}_i + d_i \dot{\delta}_i = Q_i - E_i^2 G_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^n E_i E_j Y_{ij} \sin (\delta_i - \delta_j) \quad (6)$$

---

<sup>1</sup>Professor R. K. Miller of Iowa State University suggested the application of invariance theory to the swing equation.

for  $i = 1, \dots, n$ . For the  $i$ -th generator  $\delta_i$  is the rotor angle,  $M_i > 0$  is the moment of inertia,  $d_i > 0$  is a damping constant,  $Q_i$  is the mechanical power, and the remaining parameters have technical meanings as discussed in the literature (see [7, 11]). All parameters except  $\delta_i$  are assumed to be constant during the transient.

3.1 Definition: A solution  $(\delta_1, \dots, \delta_n)$  is called a synchronous solution of system (6) if  $\dot{\delta}_i = \omega_o$  is constant for  $i = 1, \dots, n$ .

A principal goal of a power systems engineer is to ensure that a system of synchronous generators approaches a synchronous solution.

If we assume that  $(\delta_1, \dots, \delta_n)$  is a synchronous solution of system (6) and we add together all  $n$  equations, we find that

$$\sum_{i=1}^n d_i \dot{\delta}_i = \sum_{i=1}^n (Q_i - E_i^2 G_{ii}) .$$

Hence, we define

$$\omega_o = \left[ \sum_{i=1}^n (Q_i - E_i^2 G_{ii}) \right] / \left( \sum_{i=1}^n d_i \right)$$

to be consistent with Definition 3.1.

For convenience we define

$$P_{ij} = P_{ji} = E_i E_j Y_{ij} \text{ and } P_i = Q_i - E_i^2 G_{ii} - d_i \omega_o . \quad (7a)$$

Then

$$\sum_{i=1}^n P_i = 0 .$$

We make the change of variables

$$\begin{aligned} \omega_i &= \dot{\delta}_i - \omega_0 \quad \text{for } i = 1, \dots, n \\ \sigma_i &= \delta_i - \delta_n \quad \text{for } i = 1, \dots, n-1 \\ \sigma_n &= 0 \\ \rho &= \delta_n \end{aligned} \tag{7b}$$

In the new variables and parameters system (6) takes the form

$$M_i \dot{\omega}_i = -d_i \omega_i + P_i - \sum_{j=1}^n P_{ij} \sin(\sigma_i - \sigma_j) \tag{8a}$$

$$\text{for } i = 1, \dots, n$$

$$\dot{\sigma}_i = \omega_i - \omega_n \quad \text{for } i = 1, \dots, n-1 \tag{8b}$$

$$\dot{\rho} = \omega_n + \omega_0 \tag{9}$$

where

$$\sum_{i=1}^n P_i = 0 \quad \text{and} \quad \sigma_n = 0 .$$

By introducing the extra parameter,  $\sigma_n = 0$ , we can write system (8) in a more compact form than otherwise.

For the remainder of the paper, we will call system (8) the swing equations. The system of equations (8a), (8b), and (9) is equivalent to system (6). However, system (8) is independent of  $\rho$ ; solutions of system (6) may be found from solutions of system (8) by integration indicated by equation (9). Hence, we will focus our attention on system (8).

For convenience, we define

$$\sigma = (\sigma_1, \dots, \sigma_{n-1})^T \quad \text{and} \quad \omega = (\omega_1, \dots, \omega_n)^T.$$

3.2 Remark: A synchronous solution of system (6) corresponds to an equilibrium point  $\omega^*, \sigma^*$  of system (8) with the property that  $\omega^* = 0 \in \mathbb{R}^n$ . We see this by examining the two systems and considering the discussion after Definition 3.1.

Many authors study only stable equilibrium points which occur in the set

$$S = \{\sigma \in \mathbb{R}^{n-1} : |\sigma_i| < \frac{\pi}{2}, |\sigma_i - \sigma_j| < \frac{\pi}{2} \quad (10)$$

$$\text{for } i, j = 1, \dots, n-1, i \neq j\}.$$

We see that  $S$  is convex and that if  $\sigma \in S$ , then  $\cos(\sigma_i - \sigma_j) > 0$  for  $i, j = 1, \dots, n$  (recall  $\sigma_n = 0$ ). We will see that any equilibrium point  $\omega^*, \sigma^*$  with  $\sigma^* \in S$  will be asymptotically stable, a well-known result. However, in section IV we will see that  $\sigma^*$  may be far outside  $S$  and still be an asymptotically stable equilibrium point.

3.3 Remark: If the parameters of system (1) are defined in the following way, then system (1) and system (8) are equivalent.

Let  $k = n$ ,  $\ell = n - 1$ . Let  $y = \omega$ ,  $z = \sigma$  and  $\sigma_n = z_n = 0$ . Define

$$A = \text{diag} (d_i/M_i) \quad (11)$$

and let  $B = (b_{ij})$  be an  $(n-1) \times n$  matrix where

$$b_{ij} = \begin{cases} 1 & \text{if } i = j = 1, \dots, n-1 \\ -1 & \text{if } j = n, i = 1, \dots, n-1 \\ 0 & \text{if } i \neq j; i, j = 1, \dots, n-1 \end{cases} \quad (12)$$

For  $i = 1, \dots, n$  let

$$g_i(z) = \frac{1}{M_i} \left[ -P_i + \sum_{\substack{j=1 \\ j \neq i}}^n P_{ij} \sin(z_i - z_j) \right]$$

and  $g(z) = (g_1(z), \dots, g_n(z))^T$ .

Since systems (1) and (8) are equivalent through the definitions of Remark 3.3, we may apply the results of section II to system (8). To do so, we define

$$\begin{aligned} p_i(\sigma) &= M_i g_i(\sigma) \\ &= -P_i + P_{in} \sin \sigma_i + \sum_{\substack{j=1 \\ j \neq i}}^{n-1} P_{ij} \sin(\sigma_i - \sigma_j) \end{aligned} \quad (13a)$$

for  $i = 1, \dots, n$ . We let

$$p(\sigma) = (p_1(\sigma), \dots, p_{n-1}(\sigma))^T \in \mathbb{R}^{n-1}. \quad (13b)$$



The following result is a direct application of Theorem 2.2. It provides sufficient conditions for the asymptotic stability and the instability of the swing equations (system (8)). It will be used in section IV to find stable equilibrium points of system (8) for which most of the rotor angles are more than  $90^\circ$  out of phase. Unlike many previous authors, we are able to conclude asymptotic stability rather than only stability. The result is almost a necessary and sufficient condition for stability of an equilibrium point. The critical case for which  $\partial p / \partial \sigma$  is positive semi-definite and singular is the only case not treated in the following theorem. The theorem also contains a uniqueness result.

3.4 Theorem: Suppose that  $\omega^*, \sigma^*$  is an equilibrium point of system (8). Then  $\omega^* = 0$  and system (8) is equivalent to a Hamiltonian-like form of system (1). Let  $p$  be defined as in equation (13).

(a) If the Jacobian matrix  $(\partial p / \partial \sigma)|_{\sigma=\sigma^*}$  is positive definite, then  $\omega^*, \sigma^*$  is an asymptotically stable equilibrium point of system (8). If  $\sigma^* \in S$  (see equation (10)), then  $(\partial p / \partial \sigma)|_{\sigma=\sigma^*}$  is positive definite.

(b) If the Jacobian matrix  $(\partial p / \partial \sigma)|_{\sigma=\sigma^*}$  is not positive semi-definite, then  $\omega^*, \sigma^*$  is not asymptotically stable equilibrium point of system (8). If, in addition,  $\sigma^*$  is an isolated solution of  $p(\sigma) = 0$ , then  $\omega^*, \sigma^*$  is not a stable equilibrium point of system (8).

(c) Suppose that  $T$  is a convex subset of  $R^{n-1}$  such that if  $\sigma \in T$ , then  $\partial p / \partial \sigma$  is positive definite. In particular, we may set  $T = S$  where  $S$  is defined in equation (10). Then  $p(\sigma)$  is one-to-one on  $T$ , and  $\omega^*, \sigma^*$  is the only equilibrium point of system (8) with  $\sigma^* \in T$ .

Proof: Let  $A$ ,  $B$ , and  $g$  be defined in Remark 3.3. We apply Theorem 2.2 to system (8) with

$$f(z) = -\sum_{i=1}^{n-1} (P_i z_i + P_{in} \cos z_i) - \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} P_{ij} \cos (z_i - z_j)$$

and with  $K = M = \text{diag } (M_i)$ . We see that  $p(\sigma) = p(z) = \nabla_z f(z)$ . In a straightforward way, we see that the hypotheses of Theorem 2.2 are satisfied and  $\partial p / \partial \sigma|_{\sigma^*} = J(\sigma^*) = J(z^*)$ . If  $\sigma \in S$ , then we see that  $J(\sigma)$  is symmetric and diagonally dominant so that  $J(\sigma)$  is positive definite.

Although  $\partial p / \partial \sigma$  is positive definite in  $S$ , we know by continuity that  $S$  is not the largest region containing the origin in which  $\partial p / \partial \sigma$  is positive definite. Tavora and Smith [10] make a conjecture based on their numerical studies that each connected region in which  $\partial p / \partial \sigma$  is positive definite will be a convex region. If we assume this conjecture, then system (8) can have at most one equilibrium point in any connected region in which  $\partial p / \partial \sigma$  is positive definite. Tavora and Smith also offer a proof that there is at most one equilibrium point in any connected region in which  $\partial p / \partial \sigma$  is positive definite. However, this author was unable to follow their proof.

The existence of an equilibrium point for the swing equations is usually discovered in practice by numerical methods. The next result gives an analytic criterion for the existence of a stable equilibrium point for the swing equations.

3.5 Theorem: Referring to system (8) suppose that

$$\sum_{i=1}^{n-1} \frac{P_i^2}{P_{in}^2} < 1 .$$

Then there exists a unique vector  $\sigma^* \in S$  (see equation (10)) such that  $\omega^* = 0$ ,  $\sigma^*$  is an asymptotically stable equilibrium point of system (8).

Proof: Let  $p$  be the mapping defined in equation (13). By Theorem 3.4,  $p$  is one-to-one on  $S$ . Define  $P = (P_1, \dots, P_{n-1})^T \in R^{n-1}$  and  $Q(\sigma) = p(\sigma) + P$ . Then  $Q$  is one-to-one on  $S$ . Let  $T = Q(S)$  be the range of  $Q$  restricted to  $S$ . Define

$$U = \{q \in R^{n-1} : \sum_{i=1}^{n-1} (q_i^2 / P_{in}^2) < 1\} .$$

Suppose that, as demonstrated later,  $U \subsetneq T$ . Then  $Q^{-1}(U) \subset Q^{-1}(T) = S$ . Since  $P \in U$ , we define  $\sigma^* = Q^{-1}(P) \in S$ . Then  $p(\sigma^*) = Q(\sigma^*) - P = 0$  so that  $\omega^* = 0$ ,  $\sigma^*$  is an asymptotically stable equilibrium point of system (8) by Theorem 3.4.

Thus, we need only to show that  $U \subsetneq T$ . To do so we will show that whenever  $q$  is a boundary point of  $T$ , then

$$\sum_{i=1}^{n-1} (q_i^2 / P_{in}^2) \geq 1$$

so that  $q \notin U$ . However,  $Q(0) = 0 \in U \cap T$  so that  $U$  intersects  $T$  but not the boundary of  $T$ . If  $r \in U$  but  $r \notin T$ , then the straight line from 0 to  $r$  lies in  $U$  and intersects the boundary of  $T$  which is a contradiction.

Hence,  $U \subset T$ .

It remains to show that

$$\sum_{i=1}^{n-1} (q_i^2 / p_{in}^2) \geq 1$$

for all  $q \in R^{n-1}$  on the boundary of  $T$ . Since  $\partial Q / \partial \sigma = \partial p / \partial \sigma$  is positive definite (and hence non-singular) on  $S$  and since  $Q$  is one-to-one on  $S$ , it is easy to see that  $q$  is a boundary point of  $T$  if and only if  $q = Q(\sigma)$  for some  $\sigma$  on the boundary of  $S$ . The point  $\sigma$  is on the boundary of  $S$  if and only if  $\sigma_k = \pi/2$ ,  $\sigma_k = -\pi/2$ , or  $\sigma_k - \sigma_\ell = \pi/2$  for some  $k, \ell \in \{1, \dots, n-1\}$ . We see that if  $\sigma$  is on the boundary of  $S$ , then  $\cos(\sigma_i - \sigma_j) \geq 0$  and  $\cos \sigma_i \geq 0$  for all  $i, j = 1, \dots, n-1$  with  $i \neq j$ . We consider three cases.

(i) Suppose that  $\sigma_k = \pi/2$  where  $k \in \{1, \dots, n-1\}$ . Let  $q = Q(\sigma)$ . Then since  $\sin(\pi/2 - \sigma_i) = \cos \sigma_i \geq 0$  for  $i = 1, \dots, n-1$  we have that

$$\begin{aligned} \frac{q_k^2}{p_{kn}^2} &= \frac{1}{p_{kn}^2} \left[ \sum_{\substack{i=1 \\ i \neq k}}^{n-1} \sum_{\substack{j=1 \\ j \neq k}}^{n-1} p_{ki} p_{kj} \sin(\sigma_k - \sigma_i) \sin(\sigma_k - \sigma_j) \right. \\ &\quad \left. + 2 \sum_{\substack{i=1 \\ i \neq k}}^{n-1} p_{ki} p_{kn} \sin(\sigma_k - \sigma_i) \sin \sigma_k + p_{kn}^2 \sin^2 \sigma_k \right] \end{aligned}$$

$$= \left[ \sum_{\substack{i=1 \\ i \neq k}}^{n-1} \sum_{\substack{j=1 \\ j \neq k}}^{n-1} \frac{P_{ki} P_{kj}}{P_{kn}^2} \cos \sigma_i \cos \sigma_j \right. \\ \left. + 2 \sum_{\substack{i=1 \\ i \neq k}}^{n-1} \frac{P_{ki}}{P_{kn}} \cos \sigma_i + 1 \right] \geq 1 .$$

Hence,

$$\sum_{i=1}^{n-1} (q_i^2 / P_{in}^2) \geq 1 . \quad (14)$$

(ii) Suppose that  $\sigma_k = -\pi/2$  where  $k \in \{1, \dots, n-1\}$ . We repeat the argument of (i) noticing that  $\sin(-\pi/2 - \sigma_i) = -\cos \sigma_i \leq 0$  for  $i = 1, \dots, n-1$ . The negative signs cancel to give the same result.

(iii) Suppose that  $\sigma_k = \sigma_\ell + \pi/2$  for some  $k, \ell \in \{1, \dots, n-1\}$  with  $k \neq \ell$ . Let  $q = Q(\sigma)$ . For  $i = 1, \dots, n-1$ , we make the substitutions  $\sin(\sigma_k - \sigma_i) = \sin(\sigma_\ell + \pi/2 - \sigma_i) = \cos(\sigma_\ell - \sigma_i) \geq 0$ ;  $\sin(\sigma_\ell - \sigma_i) = -\cos(\sigma_k - \sigma_i) \leq 0$ ;  $\sin \sigma_k = \cos \sigma_\ell \geq 0$ ; and  $\sin \sigma_\ell = -\cos \sigma_k \leq 0$  in order to find that

$$\begin{aligned}
\frac{q_k^2}{P_{kn}^2} + \frac{q_\ell^2}{P_{\ell n}^2} &= \left\{ \frac{1}{P_{kn}^2} \left[ \sum_{\substack{i=1 \\ i \neq k}}^{n-1} \sum_{\substack{j=1 \\ j \neq k}}^{n-1} P_{ki} P_{kj} \sin(\sigma_k - \sigma_i) \sin(\sigma_k - \sigma_j) \right. \right. \\
&\quad \left. \left. + 2 \sum_{\substack{i=1 \\ i \neq k}}^{n-1} P_{ki} P_{kn} \sin(\sigma_k - \sigma_i) \sin \sigma_k + P_{kn}^2 \sin^2 \sigma_k \right] \right\} \\
&\quad + \frac{1}{P_{\ell n}^2} \left[ \sum_{\substack{i=1 \\ i \neq \ell}}^{n-1} \sum_{\substack{j=1 \\ j \neq \ell}}^{n-1} P_{\ell i} P_{\ell j} \sin(\sigma_\ell - \sigma_i) \sin(\sigma_\ell - \sigma_j) \right. \\
&\quad \left. + 2 \sum_{\substack{i=1 \\ i \neq \ell}}^{n-1} P_{\ell i} P_{\ell n} \sin(\sigma_\ell - \sigma_i) \sin \sigma_\ell + P_{\ell n}^2 \sin^2 \sigma_\ell \right] \\
&= \sum_{\substack{i=1 \\ i \neq k}}^{n-1} \sum_{\substack{j=1 \\ j \neq k}}^{n-1} \frac{P_{ki} P_{kj}}{P_{kn}^2} \cos(\sigma_\ell - \sigma_i) \cos(\sigma_\ell - \sigma_j) \\
&\quad + \sum_{\substack{i=1 \\ i \neq \ell}}^{n-1} \sum_{\substack{j=1 \\ j \neq \ell}}^{n-1} \frac{P_{\ell i} P_{\ell j}}{P_{\ell n}^2} (-\cos(\sigma_k - \sigma_i)) (-\cos(\sigma_k - \sigma_j)) \\
&\quad + 2 \sum_{\substack{i=1 \\ i \neq k}}^{n-1} \frac{P_{ki}}{P_{kn}} \cos(\sigma_\ell - \sigma_i) \cos \sigma_\ell \\
&\quad + 2 \sum_{\substack{i=1 \\ i \neq \ell}}^{n-1} \frac{P_{\ell i}}{P_{\ell n}} (-\cos(\sigma_k - \sigma_i)) (-\cos \sigma_k) + \sin^2 \sigma_k + \cos^2 \sigma_k \\
&\geq 1.
\end{aligned}$$

Hence,

$$\sum_{i=1}^{n-1} \frac{q_i^2}{P_{in}} \geq 1$$

for all  $q$  on the boundary of  $T$  and the theorem is proved.

Although the previous existence result is conservative since it involves only a few of the parameters in system (8), it does give the best possible result of its kind for two reasons. First, it can be shown that the region  $S$  is the only connected region in  $R^{n-1}$  containing the origin that contains only asymptotically stable equilibrium points of system (8) for any choice of the parameters (consistent with the definition of system (8)). (Consider the limiting cases in which all but  $n - 1$  of the parameters  $P_{ij}$  with  $i \neq j$  are zero and determine where  $\partial p / \partial \sigma$  is positive definite in these cases.) Hence, a general criterion for the existence of asymptotically stable equilibrium points of system (8) will likely involve  $S$ . Second, we see that at some points the inequality (14) becomes an equality for some choices of  $P_{ij}$ . For example, let  $\sigma_i = 0$  if  $i \neq k$  and let  $P_{ik} \rightarrow 0$  for  $i = 1, \dots, n-1$  with  $i \neq k$ . Then the boundary of  $U$  in this limiting case coincides at some points with the boundary of  $T$ . Since the criterion is independent of the limit  $P_{ik} \rightarrow 0$ , we see that the proof will not work for a region larger than  $U$ . Hence, the theorem gives the best criterion of its kind.

3.6 Corollary: Referring to system (6), suppose that there exists  $k \in \{1, \dots, n\}$  such that

$$\sum_{\substack{i=1 \\ i \neq k}}^n \frac{P_i^2}{P_{ik}} < 1$$

where  $P_i$  and  $P_{ij}$  for  $i, j = 1, \dots, n$  with  $i \neq j$  are defined in equations (7a). Then there exist functions  $\delta_1(t), \dots, \delta_n(t)$  which form a synchronous solution of system (6) such that

$$|\delta_i(t) - \delta_j(t)| < \pi/2$$

for all  $i, j = 1, \dots, n$ .

Proof: The Corollary follows from Remark 3.2, Theorem 3.5, and the fact that the numbering of the machines in system (6) is arbitrary.



## IV. STABLE EXAMPLES

Most studies of the Lyapunov stability of power systems discuss only equilibrium points for which the rotor angles are less than  $90^\circ$  out of phase with each other; that is, equilibrium points  $\omega^*$ ,  $\sigma^*$  for which  $\sigma^* \in S$  (see equation (10)). It is known (see [10]) that some equilibrium points outside  $S$  may be stable; a fact which may be established by a continuity argument. However, it appears to be often assumed that stable equilibrium points occur in or near the region  $S$ .

In this section we will see that stable equilibrium points for swing equations may occur far outside the region  $S$ . In fact, there are examples of swing equations which have a stable equilibrium point for which a majority of the rotor angles are more than  $90^\circ$  out of phase; some may be  $180^\circ$  out of phase.

To find such examples we will develop some conditions for positive definiteness and apply Theorem 3.4. We see from equation (13) that

$$\frac{\partial p_i}{\partial \sigma_j} = \begin{cases} P_{in} \cos \sigma_i + \sum_{\substack{r=1 \\ r \neq i}}^{n-1} P_{ir} \cos (\sigma_i - \sigma_r) & \text{if } i = j \\ -P_{ij} \cos (\sigma_i - \sigma_j) & \text{if } i \neq j \end{cases} \quad (15)$$

We will begin the section with a somewhat lengthy, but necessary, search for a criterion based on the signs of the numbers  $P_{ij} \cos (\sigma_i - \sigma_j)$  to determine whether  $\partial p / \partial \sigma$  can be positive definite if the  $P_{ij} > 0$  are chosen properly. Using this criterion we will choose the vector  $\sigma^*$  so that the signs of  $\cos (\sigma_i^* - \sigma_j^*)$  allow  $(\partial p / \partial \sigma) |_{\sigma=\sigma^*}$  to be positive

definite for some choices of  $P_{ij}$ . We will then choose the  $P_i$  so that  $\omega^* = 0$ ,  $\sigma^*$  is an equilibrium point (and by Theorem 3.4 asymptotically stable) of system (8).

The following definitions will be useful. We assume that  $H = (h_{ij})$  is a real  $n \times n$  symmetric matrix (such as  $\partial p / \partial \sigma$ ).

4.1 Definition: Let

$$\begin{aligned} d_{ij} &= -h_{ij} \quad \text{for } i, j = 1, \dots, n \text{ with } i \neq j \\ d_{ii} &= 0 \quad \text{for } i = 1, \dots, n+1 \\ d_{0i} &= d_{i0} = \sum_{j=1}^n h_{ij} \quad \text{for } i = 1, \dots, n. \end{aligned} \tag{16}$$

The matrix  $D(H) = (d_{ij})$  is an  $(n+1) \times (n+1)$  matrix called the configuration matrix for  $H$ .

Note that  $D(H)$  is symmetric and that

$$h_{ii} = \sum_{j=0}^n d_{ij} \quad \text{for } i = 1, \dots, n. \tag{17}$$

There is clearly a one-to-one correspondence between  $H$  and  $D(H)$ . Also, as a map of  $n \times n$  symmetric matrices to  $(n+1) \times (n+1)$  matrices,  $D(H)$  is linear.

4.2 Definition: Let  $D(H) = (d_{ij})$  be the configuration matrix for  $H$ . Define a matrix  $Q(H) = (q_{ij})$  such that

$$q_{ij} = \begin{cases} -1 & \text{if } d_{ij} < 0 \\ 0 & \text{if } d_{ij} = 0 \\ 1 & \text{if } d_{ij} > 0 \end{cases}$$

The matrix  $Q(H)$  is called the sign configuration matrix. Two symmetric matrices  $H$  and  $J$  have the same sign configuration if  $Q(J) = Q(H)$ .

As we will see, the next two definitions are criteria for positive definiteness (or lack thereof). One is the negation of the other.

4.3 Definition: Let  $D(H) = (d_{ij})$ . We say that  $H$  has property R if there exists an ordered  $(n+1)$ -tuple  $k = (k_0, \dots, k_n)$  such that  $k$  is a permutation of elements of  $\{0, \dots, n\}$  and such that for all  $i = 1, \dots, n$  it is true that  $d_{j\ell} > 0$  with  $\ell = k_i$  for some  $j \in \{k_0, \dots, k_{i-1}\}$ .

For any  $T \subset \{0, \dots, n\}$  we let  $\bar{T} = \{i \in \{0, \dots, n\} : i \notin T\}$ .

4.4 Definition: Let  $D(H) = (d_{ij})$ . We say that  $H$  has property N if there exists a proper subset  $T$  of  $\{0, \dots, n\}$  such that  $d_{ij} \leq 0$  whenever  $i \in T$  and  $j \in \bar{T}$ .

4.5 Lemma: Let  $H$  be a real  $n \times n$  symmetric matrix. Then  $H$  has property R if and only if  $H$  does not have property N.

Proof: First, suppose that  $H$  has property N and property R. Let  $k$  and  $T$  be the  $(n+1)$ -tuple and the subset of  $\{0, \dots, n\}$  of Definitions 4.3 and 4.4, respectively, where  $D(H) = (d_{ij})$ . By property R  $d_{j\ell} > 0$  where  $j = k_0$  and  $\ell = k_1$ . Thus, either  $\{k_0, k_1\} \subset T$  or  $\{k_0, k_1\} \subset \bar{T}$ . Assume without loss of generality that  $\{k_0, k_1\} \subset T$ . Proceed by induction. Suppose that  $\{k_0, \dots, k_{i-1}\} \subset T$ . Then  $d_{j\ell} > 0$

where  $j = k_i$  and  $\ell = k_r$  where  $r \in \{0, \dots, i-1\}$ . Hence,  $k_i \in T$ . By induction  $\{k_0, \dots, k_n\} = \{0, \dots, n\} \subset T$  so that  $T$  is not a proper subset of  $\{0, \dots, n\}$ . This is a contradiction.

Second, suppose that  $H$  does not have property  $N$ . Let  $k_0 = 0$ . There exists  $k_1 \in \{1, \dots, n\}$  such that  $d_{j\ell} > 0$  if  $j = k_0$ ,  $\ell = k_1$  (otherwise we let  $T = \{0\}$  in Definition 4.4). Define the  $(n+1)$ -tuple  $k$  in Definition 4.3 by induction. Assume that  $k_0, \dots, k_{i-1}$  have been defined. Then there exists  $k_i \in \{k_0, \dots, k_{i-1}\}$  such that  $d_{j\ell} > 0$  with  $j = k_i$  and  $\ell = k_r$  for some  $r \in \{0, \dots, i-1\}$  (otherwise let  $T = \{k_0, \dots, k_{i-1}\}$  in Definition 4.4). By induction we may define the  $(n+1)$ -tuple  $k$  of Definition 4.3.

In the proofs of the following results we will use several characterizations for a positive definite matrix (see [8, p. 306]).

**4.6 Lemma:** Let  $H$  be a real  $n \times n$  symmetric matrix with  $D(H) = (d_{ij})$ . If there exists  $k \in \{0, \dots, n\}$  such that  $d_{kj} \leq 0$  for all  $j \in \{0, \dots, n\}$ , then  $H$  is not positive definite.

**Proof:** If  $1 \leq k \leq n$ , then  $h_{kk} = \sum_{j=0}^n d_{kj} \leq 0$  so that  $H$  is not positive definite because  $H$  has a non-positive diagonal element.

Assume  $k = 0$ . Define  $F = (f_{ij})$  as follows:

$$f_{ij} = \begin{cases} 1 & \text{if } i = j = 1, \dots, n \\ 1 & \text{if } j = 1, i = 1, \dots, n \\ 0 & \text{if } j \neq 1 \text{ and } i \neq j; i, j = 1, \dots, n \end{cases}$$

Then  $\det F = 1$ . Let  $H' = F^T H F$ . For any  $x \in R^n$ ,

$$x^T H' x = (Fx)^T H (Fx) .$$

Hence  $H$  is positive definite if and only if  $H'$  is positive definite.

The element  $h'_{11}$  of  $H'$  is

$$h'_{11} = \sum_{i=1}^n \sum_{j=1}^n f_{i1} h_{ij} f_{j1} = \sum_{i=1}^n \sum_{j=1}^n h_{ij} = \sum_{i=1}^n d_{i0} \leq 0 .$$

Hence,  $H'$  is not positive definite.

4.7 Lemma: Let  $H$  be a real  $n \times n$  symmetric matrix with  $D(H) = (d_{ij})$  such that  $d_{ij} \geq 0$  for  $i, j = 0, \dots, n$ . If some principal submatrix  $F$  of  $H$  has property  $N$ , then  $H$  has property  $N$ .

Proof: Let  $F$  be any  $r \times r$  matrix. We may define  $F$  in an arbitrary way as follows. Let  $P$  and  $\bar{P}$  be non-empty disjoint sets such that  $P \cup \bar{P} = \{0, \dots, n\}$  and such that  $0 \in \bar{P}$ . Let  $F$  be formed from  $H$  by eliminating from  $H$  all rows and columns indexed by elements of  $\bar{P}$  (excluding 0). Then for any  $\{i, j\} \subset \{1, \dots, r\}$  there exist unique  $\{k, \ell\} \subset P$  such that  $f_{ij} = h_{k\ell}$ .

Let  $D(F) = (g_{ij})$ . For  $i \neq j$  we have by equation (16)

$$g_{ij} = -f_{ij} = -h_{k\ell} = d_{k\ell}$$

and

$$f_{ii} = h_{kk} = \sum_{j \in P} d_{kj} + \sum_{j \in \bar{P}} d_{kj} = \sum_{j=1}^r g_{ij} + \sum_{j \in \bar{P}} d_{kj} .$$

Hence,

$$g_{i0} = \sum_{j \in \bar{P}} d_{kj} .$$

By the hypothesis we may choose a proper subset  $T$  of  $\{0, \dots, r\}$  such that  $g_{ij} \leq 0$  if  $i \in T$  and  $j \in T$  with  $j \in \{0, \dots, r\}$ . Without loss of generality assume  $0 \in T$ . Define  $T'$  to be the set of all integers  $\ell \in P$  such that row (and column)  $\ell$  of  $H$  corresponds to row (and column)  $j$  of  $F$  for some  $j \in T$ . Then  $T'$  will be the set required by property N as shown below.

Let  $k \in T'$ ,  $\ell \in \bar{T}'$ . If  $\ell \in P$ , then  $d_{k\ell} = g_{ij} \leq 0$  for some  $i, j \in \{1, \dots, r\}$ . Suppose  $\ell \in P$ . Let  $i$  be the row of  $F$  corresponding to row  $k$  of  $H$ . Then  $i \in T$  and (since  $d_{kj} \geq 0$  for  $j = 0, \dots, n$ )

$$0 \leq \sum_{j \in \bar{P}} d_{kj} = g_{i0} \leq 0 .$$

In particular,  $d_{k\ell} = 0$ . Thus,

$$d_{k\ell} \leq 0 \text{ for all } k \in T' \text{ and } \ell \in \bar{T}' .$$

The following result is the main matrix result of the section.

4.8 Lemma: Let  $H$  be a real  $n \times n$  symmetric matrix with  $D(H) = (d_{ij})$ .

(a) If  $H$  has property N, then  $H$  is not positive definite.

(b) If  $d_{ij} > 0$  for  $i, j = 0, \dots, n$  with  $i \neq j$ , then  $H$  is positive definite.

(c) Suppose that  $d_{ij} \geq 0$  for  $i, j = 0, \dots, n$ . Then  $H$  is positive definite if and only if  $H$  has property R.

(d) If  $H$  has property  $R$ , then there exists a real  $n \times n$  symmetric matrix  $H'$  with the same sign configuration as  $H$  such that  $H'$  is positive definite.

(e) If  $d_{ij} < 0$  for some  $i, j \in \{0, \dots, n\}$  then there exists a real  $n \times n$  symmetric matrix  $H'$  with the same sign configuration as  $H$  such that  $H'$  is not positive definite.

Proof: (a) Let  $T$  be a proper subset of  $\{0, \dots, n\}$  such that  $d_{ij} \leq 0$  whenever  $i \in T$  and  $j \in \bar{T}$ . Assume  $0 \in T$ . Let  $T$  have  $r$  elements. Let  $F$  be the principal submatrix of  $H$  formed by eliminating the rows and columns with indices in  $\bar{T}$ . Let  $D(F) = (g_{ij})$ . By the arguments in the proof of the previous lemma, for each  $\{i, j\} \subset \{0, \dots, r\}$  there exists unique  $\{k, \ell\} \subset T$  such that

$$g_{ij} = d_{k\ell} \text{ if } i \neq j \text{ (and } k \neq \ell)$$

$$g_{i0} = \sum_{m \in \bar{T}} d_{km} \leq 0$$

where the inequality holds because  $d_{km} \leq 0$  if  $k \in T$  and  $m \in \bar{T}$ . By Lemma 4.6, matrix  $F$  and hence,  $H$  are not positive definite.

Part (b) follows from part (c).

(c) We see that one direction follows from part (a).

We prove the other direction by induction. The result for  $2 \times 2$  matrices follows by an easy calculation.

Assume that (c) is true for all  $(n-1) \times (n-1)$  matrices. Let  $H$  be the  $n \times n$  matrix of the hypothesis. Since  $H$  does not have property  $N$ , there exists an integer  $\ell$  with  $1 \leq \ell \leq n$  such that  $d_{0\ell} > 0$ . Without

loss of generality, let  $\ell = n$ . Let

$$H = \begin{pmatrix} A & b \\ b^T & c \end{pmatrix}$$

where

$$b^T = (h_{in}, \dots, h_{n-1 n}) = (-d_{in}, \dots, -d_{n-1 n})$$

$$c = h_{nn} = \sum_{i=0}^n d_{in}$$

and  $A$  is the principal submatrix of  $H$  formed by eliminating the last row and column of  $H$ . By Lemmas 4.5 and 4.7, the submatrix  $A$  has property  $R$  so that  $A$  is positive definite by the induction hypothesis. Let

$A = (a_{ij})$  and  $A^{-1} = (\tilde{a}_{ij})$ . Since  $A$  is an  $M$ -matrix (see [3, 6]), we know that  $\tilde{a}_{ij} \geq 0$  for all  $i, j = 1, \dots, n$ .

Let  $e = (1, 1, \dots, 1)^T \in R^{n-1}$ . Then the  $i$ -th component of  $Ae$  is

$$(Ae)_i = \sum_{j=1}^{n-1} h_{ij} = d_{i0} + d_{in} \geq d_{in}.$$

Thus, for  $j = 1, \dots, n-1$

$$1 = e_j = (A^{-1}Ae)_j = \sum_{i=1}^{n-1} \tilde{a}_{ji} (Ae)_i \geq \sum_{i=1}^{n-1} \tilde{a}_{ji} d_{in}.$$

Since  $A$  is positive definite,  $H$  is positive definite if and only if  $\det H > 0$ . By a result from matrix theory (see [8, p. 45])

$$\det H = (c - b^T A^{-1} b) \det A.$$



But

$$\begin{aligned}
 c - b^T A^{-1} b &= \sum_{i=0}^n d_{in} - \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (-d_{in}) \tilde{a}_{ij} (-d_{jn}) \\
 &= d_{0n} + \sum_{i=1}^{n-1} d_{in} \left(1 - \sum_{j=1}^{n-1} \tilde{a}_{ij} d_{jn}\right) \\
 &\geq d_{0n} > 0 .
 \end{aligned}$$

Hence,  $\det H > 0$  so that  $H$  is positive definite.

(d) Assume that  $H$  has property  $R$  with  $D(H) = (d_{ij})$ . Let  $f_{ij} = d_{ij}$  if  $d_{ij} \geq 0$  and  $f_{ij} = \epsilon d_{ij}$  for  $\epsilon > 0$  if  $d_{ij} \leq 0$ . Define  $\bar{H}(\epsilon)$  as  $D(\bar{H}(\epsilon)) = (f_{ij}(\epsilon))$ . Then  $\bar{H}(0)$  is positive definite by part (c). By continuity of eigenvalues, there exists  $\epsilon > 0$  such that  $\bar{H}(\epsilon)$  is positive definite, but clearly  $\bar{H}(\epsilon)$  has the same sign configuration as  $H$  for any  $\epsilon > 0$ .

(e) For this part we choose  $d_{ij} < 0$  so small that

$$h_{ii} = \sum_{\ell=0}^n d_{i\ell} < 0 .$$

We may use Lemma 4.7 to show that many examples of the swing equations exist with asymptotically stable equilibrium points far outside the region usually studied.

**4.9 Theorem:** Let  $\sigma^* \in R^{n-1}$ . Suppose that there is an  $n$ -tuple  $k = (k_1, \dots, k_n)$  such that  $k$  is a permutation of elements of  $\{1, \dots, n\}$  and such that for all  $i = 2, \dots, n$  it is true that

$\cos(\sigma_j^* - \sigma_\ell^*) > 0$  if  $\ell = k_i$  for some  $j \in \{k_1, \dots, k_{i-1}\}$ . (Recall  $\sigma_n^* = 0$ .) Then we may choose real numbers  $P_i$  and  $P_{ij} = P_{ji} > 0$  for  $i, j = 1, \dots, n$  with  $i \neq j$  such that  $\omega^* = 0$ ,  $\sigma^*$  is an asymptotically stable equilibrium point of system (8) for any  $M_i > 0$  and  $d_i > 0$ ,  $i = 1, \dots, n$ .

Proof: Let  $p(\sigma)$  be defined in equation (13). Let  $H = (\partial p / \partial \sigma)|_{\sigma=\sigma^*}$  and  $D(H) = (d_{ij})$  (see Definition 4.1). Then,  $d_{ij} = P_{ij} \cos(\sigma_i^* - \sigma_j^*)$  for all  $i, j = 1, \dots, n-1$  with  $i \neq j$  and  $d_{i0} = P_{in} \cos \sigma_i^*$  for  $i = 1, \dots, n-1$ . The hypothesis determines that  $H$  has property R for any choices of the  $P_{ij}$ . By Lemma 4.8 (d) we may choose the  $P_{ij}$  so that  $H$  is positive definite. With these choices of  $P_{ij}$ , we choose the  $P_i$  so that  $\omega^* = 0$ ,  $\sigma^*$  is an equilibrium point of system (8).

In applying the previous theorem we are restricted in the possible sign configurations of  $\partial p / \partial \sigma$  by the fact that the right-hand side of equation (8a) is periodic in each component of  $\sigma$ . Hence, if  $\omega^* = 0$ ,  $\sigma^*$  is an equilibrium point of system (8), then we may assume without loss of generality that  $|\sigma_i^*| \leq \pi$  for  $i = 1, \dots, n-1$ . This assumption restricts the possible choices of  $\sigma^*$  for which  $\cos(\sigma_i^* - \sigma_j^*) \leq 0$ . Except in low dimensions, for all  $\sigma^*$  many of the numbers  $\cos(\sigma_i^* - \sigma_j^*)$  are positive. Nevertheless, examples of stable equilibrium points exist for which most of the numbers  $\cos(\sigma_i^* - \sigma_j^*)$  are negative.

4.10 Example: Let  $n = 3\ell + 3$  for some positive integer  $\ell$ . Let

$$\sigma_1^* = \pi/3, \quad \sigma_2^* = -\pi/3$$

$$\sigma_{2+i}^* = 0, \quad \sigma_{2+\ell+i}^* = 2\pi/3, \quad \sigma_{2+2\ell+i}^* = -2\pi/3$$

for  $i = 1, \dots, \ell$ . With  $k = (1, \dots, n)$  we apply Theorem 4.8 to see that  $\omega^* = 0$ ,  $\sigma^*$  is an asymptotically stable equilibrium point of system (8) for some choices of parameters. Of the numbers  $\cos \sigma_i^*$ , a total of  $\ell + 2$  are positive and  $2\ell$  are negative. Of the numbers  $\cos (\sigma_i^* - \sigma_j^*)$ , a total of  $(3\ell^2 + 5\ell)/2$  are positive and  $(6\ell^2 + 4\ell + 2)/2$  are negative. For large  $\ell$  there are about twice as many negative as positive numbers of this kind; that is, the associated synchronous solution is attractive even though most of the rotor angles are more than  $90^\circ$  out of phase with each other.

4.11 Example: As a specific example, we define the parameters of system (8) as follows:  $n = 4$ ,  $d_i > 0$ ,  $M_i > 0$  for  $i = 1, 2, 3, 4$ , and

$$P_1 = 11\sqrt{3}/2 = -P_4, P_2 = \sqrt{3}/2 = -P_3$$

$$P_{12} = 10, P_{13} = 1, P_{14} = 1, P_{23} = 10, P_{24} = 1, P_{34} = 10.$$

This system has an asymptotically stable equilibrium point  $\omega^* = 0$ ,  $\sigma^*$  with

$$\sigma_1^* = \pi, \sigma_2^* = 2\pi/3, \sigma_3^* = \pi/3$$

$$\sigma_1^* - \sigma_2^* = \pi/3, \sigma_1^* - \sigma_3^* = 2\pi/3, \sigma_2^* - \sigma_3^* = \pi/3.$$

In this example,

$$\left. \frac{\partial p}{\partial \sigma} \right|_{\sigma=\sigma^*} = \frac{1}{2} \begin{pmatrix} 7 & -10 & 1 \\ -10 & 19 & -10 \\ 1 & -10 & 19 \end{pmatrix}$$

which is positive definite so that  $\omega^*, \sigma^*$  is asymptotically stable by Theorem 3.4 (a). Thus, a synchronous solution may be attractive even though some of the rotor angles are  $180^\circ$  out of phase.

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## DISCUSSION

The stability results of Part I and Part II were obtained by totally different methods. The linearization technique used in Part I can be used only for local stability analysis. However, the results of Part I can be used to study a wide variety of systems including those studied in Part II and power systems with non-trivial transfer conductances which cannot be studied by the methods of Part II. In addition, the linearization technique will assure a rapid (exponential) rate of convergence to the equilibrium point, a consequence which does not follow from the methods of Part II even though it has an important consequence that is discussed in this section.

The method of Part II, using Liapunov functions and the invariance theorem, has been successfully applied to power systems only when transfer conductances are neglected. Although the invariance theorem is not usually stated explicitly, this procedure has been used to find regions of asymptotic stability. Using the concept of continuous dependence on parameters, which is discussed in this section, one can use the analysis in Part II to study power systems with small transfer conductances.

This section contains a discussion of the continuous dependence of the swing equations on parameters, a discussion of exponential stability and its consequences for power systems, a detailed discussion of the relationships between the reduced systems (with one variable omitted) and the swing equations from which they were derived, a series of examples

illustrating the necessity of some of the hypotheses for the results of Parts I and II, and an extension of one of the results of Part I to allow for the possibility of a positive semi-definite damping matrix.

### Continuous Dependence

In Parts I and II there was occasionally a reference to the continuous dependence of solutions on parameters (see [9]), for example, in the paragraph following Corollary 2.4 of Part I. Information of this kind is useful in other ways as well; for example, if the post-fault system is not much different from the pre-fault system, it would be useful to know that the post-fault system has a stable equilibrium point. It would also be useful to know that if the transfer conductances are small, then an analysis such as that in Part II which neglects them will give some information about the original problem. With these ideas in mind, consider the following system (see system (1) of Part I)

$$\dot{y} = -D(\phi)y - h(z; \phi) \quad (1a)$$

$$z = F(\phi)y \quad (1b)$$

where  $y \in R^k$ ,  $z \in R^\ell$ ,  $D(\phi)$  and  $F(\phi)$  are  $k \times k$  and  $\ell \times k$  matrix valued functions respectively of  $\phi \in R^n$ , and  $h(z, \phi) \in R^k$  for  $z \in R^\ell$  and  $\phi \in R^n$ .

1. Theorem: In system (1) (of this section) let  $D(\phi)$ ,  $F(\phi)$ , and  $h(z; \phi)$  have continuous first partial derivatives in all variables. Let  $A$ ,  $B$ , and  $g$  be defined as in system (1) of Part I. Suppose that  $A = D(0)$ ,  $B = F(0)$ , and  $g(z) = h(z; 0)$  for all  $z \in R^\ell$ . Suppose that  $x^* = (y^{*T}, z^{*T})^T$  is an equilibrium point of system (1) of Part I.

Let

$$C = - \left. \frac{\partial g}{\partial z} \right|_{z=z^*} = - \left. \frac{\partial h}{\partial z} \right|_{\substack{z=z^* \\ \phi=0}}$$

Suppose that  $\det G \neq 0$  where  $G$  is defined in equation (3) of Part I.

Then the following statements are true:

(a) For every  $\epsilon > 0$  there is a  $\gamma > 0$  such that if  $\|\phi\| < \gamma$ , then system (1) (of this section) has an equilibrium point  $\bar{x}^* = (\bar{y}^{*T}, \bar{z}^{*T})^T$  with  $\|\bar{x}^* - x^*\| < \epsilon$ .

(b) If all eigenvalues of  $G$  have negative real parts, then the  $\gamma$  of part (a) can be chosen so that  $\bar{x}^*$  is an asymptotically stable equilibrium point of system (1) (of this section).

Proof: Let  $f: \mathbb{R}^{k+\ell} \times \mathbb{R}^n \rightarrow \mathbb{R}^{k+\ell}$  be defined

$$f(x; \phi) = \begin{pmatrix} -D(\phi)y - h(z; \phi) \\ F(\phi)y \end{pmatrix}$$

where  $x = (y^T, z^T)^T$ ,  $y \in \mathbb{R}^k$ ,  $z \in \mathbb{R}^\ell$ ,  $\phi \in \mathbb{R}^n$ . Since  $x^*$  is an equilibrium point of system (1) of Part I, it is true that  $f(x^*, 0) = 0$ . Let

$$J(z; \phi) = \begin{pmatrix} -D(\phi) & -\frac{\partial h}{\partial z}(z; \phi) \\ F(\phi) & 0 \end{pmatrix} = \frac{\partial f}{\partial x}(x; \phi)$$

Then  $J(z^*; 0) = G$  and  $\det G \neq 0$ . By the implicit function theorem (see, for example, [3, p. 374]) there is a neighborhood  $N$  of  $0 \in \mathbb{R}^n$  and a unique continuous differentiable function  $x(\phi)$  defined on  $N$  such that  $x(0) = x^*$  and  $f(x(\phi), \phi) = 0$  for  $\phi \in N$ . Let  $\epsilon > 0$  be given. Then there



exists  $\gamma > 0$  such that if  $||\phi|| < \gamma$ , then  $\phi \in N$  and  $||x^* - x(\phi)|| < \epsilon$ .

This proves part (a).

Since the eigenvalues of  $J(z(0); 0) = G$  have negative real parts and since they depend continuously on  $\phi$ , all eigenvalues of  $J$  will have negative real parts if  $||\phi||$  is chosen to be small enough. This proves part (b).

This result may easily be applied to the swing equations. Consider the system (see system (7) of Part I)

$$N_i \dot{\omega}_i = -e_i \omega_i + Q_i - \sum_{\substack{j=1 \\ j \neq i}}^n Q_{ij} \sin(\sigma_i - \sigma_j - \beta_{ij}) \quad (2a)$$

for  $i = 1, \dots, n$

$$\dot{\sigma}_i = \omega_i - \omega_n \text{ for } i = 1, \dots, n-1 \quad (2b)$$

where  $N_i, e_i, Q_i, Q_{ij}, \beta_{ij}$  are constants for  $i, j = 1, \dots, n, i \neq j$  and  $\sigma_n \equiv 0$ .

2. Corollary: Let  $A, B, C$ , and  $G$  be defined as in equations (3), (10), (11), (14), and (15) of Part I. Suppose that  $\omega^*, \sigma^*$  is an asymptotically stable equilibrium point of system (7) of Part I in the sense that all eigenvalues of  $G$  have negative real parts. Then for any  $\epsilon > 0$  there is a  $\gamma > 0$  such that if  $|d_i - e_i| < \gamma, |P_{ij} - Q_{ij}| < \gamma, |\alpha_{ij} - \beta_{ij}| < \gamma$ , and  $|M_i - N_i| < \gamma$  for  $i, j = 1, \dots, n, i \neq j$ , then system (2) (of this section) has an asymptotically stable equilibrium point  $\bar{\omega}^*, \bar{\sigma}^*$  with  $|\bar{\omega}_i^* - \omega_i^*| < \epsilon$  for  $i = 1, \dots, n$  and  $|\bar{\sigma}_i^* - \sigma_i^*| < \epsilon$  for  $i = 1, \dots, n-1$ .

A result of the previous corollary is that if a power system, after clearing a fault, is not much different from the stable pre-fault system, then the post-fault system will (according to the swing equations model) have an asymptotically stable equilibrium point near the pre-fault stable equilibrium point. If the transfer conductance parameters  $\alpha_{ij}$  are small in magnitude, then the stability of the swing equations can be studied by setting  $\alpha_{ij} = 0$  for  $i, j = 1, \dots, n, i \neq j$ .

### Exponential Stability

Using the linearization results of Part I, one can predict an exponential rate of convergence to an equilibrium point. In fact, it will be shown that a non-linear system has an exponentially stable equilibrium point if and only if the matrix of the associated linear system has only eigenvalues with negative real parts. It will be shown that this result has an important consequence in the study of the swing equations by uncoupling one of the system components.

Consider the differential equation

$$\dot{x} = Ax + f(x, t) \quad (3)$$

where  $x \in \mathbb{R}^n$ ,  $A$  is an  $n \times n$  matrix, and  $f: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  is a continuous function on a neighborhood of  $0 \in \mathbb{R}^n$  such that

$$\lim_{||x|| \rightarrow 0} \frac{||f(x, t)||}{||x||} = 0$$

uniformly in  $t$  for all  $t \geq 0$ .

3. Definition: The trivial solution  $x \equiv 0$  of system (3) is exponentially stable if there exists a  $\eta > 0$  such that for every  $\epsilon > 0$  there is a  $\gamma > 0$  such that

$$||x(t)|| \leq \epsilon e^{-\eta(t-t_0)}$$

for all  $t \geq t_0$  whenever  $x(t)$  is a solution of system (3) for which  $||x(t_0)|| < \gamma$ . (See [24].)

4. Theorem: The trivial solution  $x \equiv 0$  of equation (3) is exponentially stable if and only if all eigenvalues of the matrix  $A$  have negative real parts.

Proof: If all eigenvalues of  $A$  have negative real parts, then a standard proof of asymptotic stability (see [9, p. 314]) shows that the trivial solution is exponentially stable. If an eigenvalue of  $A$  has positive real part, then it is known that the trivial solution is not stable [9].

Assume that all eigenvalues of  $A$  have non-positive real part but that some eigenvalues of  $A$  have zero real part. Let

$$PAP^{-1} = B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

where  $B$  is in Jordan canonical form and all eigenvalues of the  $k \times k$  ( $k > 0$ ) matrix  $B_1$  have zero real part. Let  $y = Px$  and  $g(y, t) = Pf(P^{-1}y, t)$ . Then the system

$$\dot{y} = By + g(y, t) \tag{4}$$

is equivalent to system (3) and

$$\lim_{||y|| \rightarrow 0} \frac{||g(y, t)||}{||y||} = 0$$

uniformly in  $t$  for  $t \geq 0$ .

Let  $PR^n = \{y: y = Px \text{ for some } x \in R^n\}$ . Let  $PR_1^n = \{y_1: (y_1^T, y_2^T)^T \in PR^n \text{ and } y_1 \text{ is a } k\text{-vector}\}$ . Let  $PR_2^n = \{y_2: (y_1^T, y_2^T)^T \in PR^n \text{ and } y_2 \text{ is an } n-k \text{ vector}\}$ . Let  $g = (g_1^T, g_2^T)^T$  where  $g_1(y, t) \in PR_1^n$  and  $g_2(y, t) \in PR_2^n$  for all  $t \geq 0$  and for all  $y \in PR^n$ .

Assume that the trivial solution of system (3) and, therefore, of system (4) is exponentially stable. This assumption will lead to a contradiction.

By the nature of  $B_1$  and  $B_2$  there exist an integer  $\ell \geq 0$ , real constants  $\sigma > 0$ ,  $K_1 > 0$ ,  $K_2 > 0$ ,  $K_3 > 0$ , and a positive definite function  $K_0(y_1)$  for  $y_1 \in PR_1^n$  such that if  $y_1 \in PR_1^n$ ,  $y_2 \in PR_2^n$ , and  $t \in R$ , then

$$K_0(y_1) \leq ||e^{B_1 t} y_1|| \leq (K_1 + K_2 |t|^\ell) ||y_1||$$

and

$$||e^{B_2 t} y_2|| \leq K_3 e^{-\sigma t} ||y_2||.$$

Let  $\eta$  be given according to Definition 3. Let  $\nu < \min\{\sigma, \eta\}$ . Choose  $\epsilon > 0$  such that

$$\epsilon < \frac{1}{2} \left( \frac{K_1}{\nu} + \frac{K_2 \ell!}{(\nu)^{\ell+1}} \right)^{-1}$$

and

$$\epsilon < \frac{\sigma - \nu}{2 K_3}$$

There exists  $\gamma_1 > 0$  such that if  $\|y\| < \gamma_1$  for  $y \in \mathbb{R}^n$ , then  $\|g(y, t)\| < \epsilon \|y\|$  for all  $t \geq 0$ . There exists  $\gamma_2 > 0$  such that if  $\|y(0)\| < \gamma_2$  for some solution  $y(t)$  of equation (4), then  $\|y(t)\| < \gamma_1 e^{-\eta t} \leq \gamma_1 e^{-\nu t} \leq \gamma_1$  for all  $t \geq 0$ .

Let  $y(t) = (y_1^T(t), y_2^T(t))^T$ , where  $y_1(t) \in \mathbb{R}_1^n$  and  $y_2(t) \in \mathbb{R}_2^n$ , be a solution of equation (4) which satisfies the initial conditions  $y_2(0) = 0$  and  $y_1(0) = a$  with  $0 < \|a\| < \gamma_2$ . Then  $\|y(0)\| = \|y_1(0)\| < \gamma_2$ . Hence  $\|y(t)\| < \gamma_1 e^{-\nu t} \leq \gamma_1$  for all  $t \geq 0$ . Let

$$d(y) = \sup_{t \geq 0} e^{\nu t} \|y(t)\|.$$

By the variation of constants for formula (see [9]), there exists  $b \in \mathbb{R}_1^n$  such that

$$y_1(t) = e^{tB_1} b - \int_t^\infty e^{B_1(t-s)} g_1(y(s), s) ds \quad (5)$$

$$y_2(t) = \int_0^t e^{B_2(t-s)} g_2(y(s), s) ds \quad (6)$$

for all  $t \geq 0$ .

From equation (5) it follows that for  $t \geq 0$

$$\begin{aligned}
 ||y_1(t) - e^{tB_1} b|| &\leq \int_t^\infty ||e^{B_1(t-s)} g_1(y(s), s)|| ds \\
 &\leq \int_t^\infty (K_1 + K_2(s-t)^\ell) ||g_1(y(s), s)|| ds \\
 &\leq \int_t^\infty (K_1 + K_2(s-t)^\ell) \varepsilon ||y(s)|| ds \\
 &\leq \int_t^\infty (K_1 + K_2(s-t)^\ell) \varepsilon d(y) e^{-\nu s} ds \\
 &\leq \varepsilon d(y) \left( \frac{K_1}{\nu} e^{-\nu t} + \frac{K_2 \ell!}{\nu^{\ell+1}} e^{-\nu t} \right) .
 \end{aligned}$$

Allowing  $t \rightarrow \infty$  in the above inequality, one finds that  $0 \leq K_0(b) \leq 0$ .

Hence  $b = 0$ . Therefore for  $t \geq 0$

$$e^{\nu t} ||y_1(t)|| \leq \varepsilon \left( \frac{K_1}{\nu} + \frac{K_2 \ell!}{\nu^{\ell+1}} \right) d(y) < \frac{1}{2} d(y) .$$

From equation (6) it follows that for  $t \geq 0$

$$\begin{aligned}
 ||y_2(t)|| &\leq \int_0^t ||e^{B_2(t-s)} g_2(y(s), s)|| ds \\
 &\leq \int_0^t K_3 e^{-\sigma(t-s)} ||g_2(y(s), s)|| ds \\
 &\leq \int_0^t K_3 e^{-\sigma t} e^{\sigma s} \varepsilon ||y(s)|| ds \\
 &\leq \int_0^t \varepsilon K_3 e^{-\sigma t} e^{\sigma s} d(y) e^{-\nu s} ds
 \end{aligned}$$

$$\begin{aligned} &\leq \epsilon K_3 d(y) e^{-\sigma t} \frac{e^{(\sigma-\nu)t} - 1}{\sigma - \nu} \\ &\leq \epsilon K_3 d(y) \frac{e^{-\nu t} - e^{-\sigma t}}{\sigma - \nu} \end{aligned}$$

Hence, for  $t \geq 0$

$$e^{\nu t} ||y_2(t)|| \leq \epsilon K_3 d(y) \frac{1 - e^{-(\sigma-\nu)t}}{\sigma - \nu} \leq \frac{1}{2} d(y) .$$

Therefore

$$\begin{aligned} d(y) &= \sup_{t \geq 0} e^{\nu t} ||y(t)|| \\ &= \sup_{t \geq 0} e^{\nu t} (||y_1(t)||^2 + ||y_2(t)||^2)^{\frac{1}{2}} \\ &\leq \sup_{t \geq 0} (\frac{1}{2} d^2(y) + \frac{1}{2} d^2(y))^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{2}} d(y) . \end{aligned}$$

Hence  $d(y) = 0$  so that  $y(t) = 0$  for  $t \geq 0$ . This contradicts the assumption that  $||y_1(0)|| = ||a|| > 0$ .

### Synchronous States

In Parts I and II the original swing equations (equation (5) of Part I or equation (6) of Part II) were considered to be almost synonymous with the "reduced system" (system (7) of Part I or system (8) of Part II) in which one of the system variables is uncoupled from the rest of the system. In this section the relationship between these two systems is discussed in more detail, particularly with respect to stability. Equation (9) of the Background and Notation section or

equation (5) of Part I will be referred to as the swing equations.

System (7) of Part I will be called the reduced system.

Let  $\delta(t) = (\delta_1(t), \dots, \delta_n(t))$  represent a solution of the swing equations. A solution  $\delta^*(t) = (\delta_1^*(t), \dots, \delta_n^*(t))$  is called a synchronous solution if  $\dot{\delta}_i^* = \omega_0$  is constant for  $i = 1, \dots, n$ . The components of a synchronous solution may be written in the form

$$\delta_i^*(t) = \omega_0 t + \sigma_i^*$$

where  $\sigma_i^*$  is constant for  $i = 1, \dots, n$ . Note that  $\delta_i^* - \delta_j^* = \sigma_i^* - \sigma_j^*$  is constant for  $i, j = 1, \dots, n$ . Two synchronous solutions  $\delta^*$  and  $\delta^{**}$  are called equivalent if there is a constant  $\delta_0$  such that  $\delta_i^{**}(t) = \delta_i^*(t) + \delta_0$  for  $i = 1, \dots, n$ . That is, two synchronous solutions differing by a constant phase are equivalent.

It is easy to see that a synchronous solution of the swing equations is never unique. If  $\delta^*$  is a synchronous solution, then for any real number  $\delta_0$  and any set of integers  $\{k_1, \dots, k_n\}$  the vector

$$\delta^{**} = (\delta_1^* + 2k_1\pi + \delta_0, \dots, \delta_n^* + 2k_n\pi + \delta_0)$$

is also a synchronous solution. Theorem 3.8 of Part I, interpreted in the language of the present discussion, may be rewritten as follows.

5. Theorem: Suppose that  $\delta^*$  and  $\delta^{**}$  are synchronous solutions of the swing equations with  $|\delta_i^* - \delta_j^* - \alpha_{ij}| < \pi/2$  and  $|\delta_i^{**} - \delta_j^{**} - \alpha_{ij}| < \pi/2$  for  $i, j = 1, \dots, n, i \neq j$ . Then there is a constant  $\delta_0$  such that  $\delta_i^{**} = \delta_i^* + \delta_0$  for  $i = 1, \dots, n$ ; that is,  $\delta^*$  and  $\delta^{**}$  are equivalent.



Theorem 3.4(c) of Part II may be reinterpreted in a similar fashion.

The system composed of equations (7a), (7b), and (8) of Part I is equivalent to the swing equations ((5) of Part I). The relationship between the two systems is given by the linear transformation (6) of Part I. Each constant solution or equilibrium point of the reduced system ((7) of Part I) corresponds to a one parameter family of equivalent synchronous solutions of the swing equations; the parameter is the integration constant used to solve equation (8) of Part I. The asymptotic stability of an equilibrium point of the reduced system imposes the following notion of stability on the corresponding family of synchronous solutions of the swing equations (see [46]).

6. Definition: A synchronous solution  $\delta^*$  of the swing equations is synchronously stable if it is stable (see Definition 1 of the Background and Notation section) and there is a  $\gamma > 0$  such that if  $\delta(t)$  is a solution of the swing equations for which  $|\delta_1(0) - \delta_1^*(0)| < \gamma$  and  $|\dot{\delta}_1(0) - \dot{\delta}_1^*(0)| < \gamma$  for  $i = 1, \dots, n$ , then there is a constant  $\delta_0$  such that

$$\lim_{t \rightarrow \infty} |\delta_i(t) - \delta_i^*(t) - \delta_0| = 0$$

and

$$\lim_{t \rightarrow \infty} |\dot{\delta}_i(t) - \dot{\delta}_i^*(t)| = 0$$

for  $i = 1, \dots, n$ .

Thus, a synchronous solution  $\delta^*$  is synchronously stable if it is stable and solutions of the swing equations initially close to  $\delta^*$  asymptotically approach a synchronous state equivalent to  $\delta^*$ . Because the system is autonomous, the synchronous stability of  $\delta^*$  implies the synchronous stability of all synchronous solutions equivalent to  $\delta^*$ . It is a straightforward procedure to show that if  $\delta^*$  is synchronously stable, then the related equilibrium point  $\omega^*, \sigma^*$  of the reduced system (see transformation (6) of Part I) is asymptotically stable. The converse will be investigated here.

Note that the results of Part I imply that an equilibrium point of the reduced system ((7) of Part I) is exponentially stable. This fact will imply that the related synchronous solution is synchronously stable. If the equilibrium point were asymptotically stable but not exponentially stable, then this result might not hold.

7. Example: (See equations (5) through (8) of Part I), let  $\omega^*, \sigma^*$  be an equilibrium point of the reduced system (7) of Part I. Suppose that for some solution  $\omega(t), \sigma(t)$  initially close to  $\omega^*, \sigma^*$  it is true that

$$\lim_{t \rightarrow \infty} \omega(t) = \omega^*$$

but that there exist positive constants  $T$  and  $K$  such that

$$\omega_n(t) - \omega_n^* \geq \frac{K}{t}$$

for  $t \geq T$ . A related solution of the swing equations has the form

$$\delta_i(t) = \sigma_i(t) + \delta_n(t)$$

for  $i = 1, \dots, n - 1$  and

$$\delta_n(t) = \int_0^t \omega_n(s) ds + b$$

for some constant  $b$ . A synchronous solution related to the equilibrium point  $\omega^*, \sigma^*$  has the form

$$\delta_i^*(t) = \omega_0 t + \sigma_i^* + a$$

where  $\omega_0 = \omega_i^*$  for  $i = 1, \dots, n - 1$  for some constant  $a$  and

$$\delta_n^*(t) = \omega_0 t + a$$

where  $\omega_0 = \omega_n^*$ . For any constant  $\delta_0$  it is true that

$$\begin{aligned} |\delta_n(t) - \delta_n^*(t) - \delta_0| &= \left| \int_0^t (\omega_n(s) - \omega_0) ds + b - a - \delta_0 \right| \\ &\geq \left| \int_T^t (\omega_n(s) - \omega_0) ds \right| - \left| \int_0^T (\omega_n(s) - \omega_0) ds + b - a - \delta_0 \right| \\ &\geq \int_T^t \frac{K}{s} ds - \left| \int_0^T (\omega_n(s) - \omega_0) ds + b - a - \delta_0 \right| \\ &\geq K \log \frac{t}{T} - \left| \int_0^T (\omega_n(s) - \omega_0) ds + b - a - \delta_0 \right|. \end{aligned}$$

Hence,  $\delta^*$  is not synchronously stable or even stable.

In order to ensure that a stability result for the reduced system implies the stability of a synchronous solution, it is necessary that

the rate of asymptotic approach to the equilibrium is fairly rapid, for example, an exponential approach is sufficient.

Let  $\omega^*, \sigma^*$  be an equilibrium point of the reduced system. A synchronous solution  $\delta^*$  which corresponds to  $\omega^*, \sigma^*$  has the form (see transformation (6) of Part I)

$$\delta^*(t) = (\omega_0 t + \sigma_1^* + \delta_0, \dots, \omega_0 t + \sigma_{n-1}^* + \delta_0, \omega_0 t + \delta_0)$$

for some constant  $\delta_0$ .

8. Theorem: Let  $\omega^*, \sigma^*$  be an exponentially stable equilibrium point of system (7) of Part I.

(a) A synchronous solution  $\delta^*$  of the swing equations ((5) of Part I) which corresponds to  $\omega^*, \sigma^*$  is synchronously stable.

(b) There is an  $\eta > 0$  such that for every  $\epsilon > 0$  there is a  $\gamma > 0$  such that if  $\delta(t)$  is a solution of the swing equations for which

$$|\delta_i(0) - \delta_i^*(0)| < \gamma \text{ and } |\dot{\delta}_i(0) - \omega_0| < \gamma$$

for  $i = 1, \dots, n$ , then

$$|\dot{\delta}_i(t) - \omega_0| < \epsilon e^{-\eta t}$$

where  $\omega_0 = \dot{\delta}_i^*(t)$  for  $i = 1, \dots, n$  and  $\delta^*$  is the synchronous solution of (a).

Proof: Let  $\eta$  be given according to Definition 3 for exponential stability. Let  $\epsilon > 0$  be given. Choose  $\gamma > 0$  so that if  $\omega(t), \sigma(t)$  is a solution of the reduced system and  $|\sigma_i(0) - \sigma_i^*| < \gamma$  for  $i = 1, \dots, n-1$  and  $|\omega_1(0) - \omega_0| < \gamma$  for  $i = 1, \dots, n$ , then for  $t \geq 0$

$$|\sigma_i(t) - \sigma_i^*| < \frac{\epsilon}{2} e^{-\eta t} \quad \text{for } i = 1, \dots, n-1$$

and

$$|\omega_i(t) - \omega_0| < \frac{\epsilon\eta}{2} e^{-\eta t} \quad \text{for } i = 1, \dots, n.$$

Since  $\dot{\delta}_i(t) = \omega_i(t)$ , part (b) of the theorem is proved because  $\epsilon$  is arbitrary.

Let  $\delta(t)$  be a solution of the swing equations with

$$|\delta_i(0) - \delta_i^*(0)| < \mu \quad \text{and} \quad |\dot{\delta}_i(0) - \dot{\delta}_i^*(0)| < \mu \quad \text{for } i = 1, \dots, n$$

where  $\mu < \gamma/2$  and  $\mu < \epsilon/3$ . By transformation (6) of Part I it is true that there is a solution  $\omega(t)$ ,  $\sigma(t)$  of the reduced system such that

$$\delta_i(t) = \sigma_i(t) + \delta_n(t) \quad \text{for } i = 1, \dots, n-1$$

$$\dot{\delta}_i(t) = \omega_i(t) \quad \text{for } i = 1, \dots, n$$

$$\delta_n(t) = \int_0^t \omega_n(s) ds + a$$

for some constant  $a$ . Also

$$\delta_i^*(t) = \sigma_i^* + \delta_n^*(t) \quad \text{for } i = 1, \dots, n-1$$

$$\dot{\delta}_i^*(t) = \omega_0 = \omega_i^* \quad \text{for } i = 1, \dots, n$$

$$\delta_n^*(t) = \omega_0 t + b$$

for some constant  $b$ . Now

$$|\omega_i(0) - \omega_0| = |\dot{\delta}_i(0) - \dot{\delta}_i^*(0)| < \mu < \gamma \text{ for } i = 1, \dots, n$$

$$\begin{aligned} |\sigma_i(0) - \sigma_i^*| &= |\delta_i(0) - \delta_n(0) - \delta_i^*(0) + \delta_n^*(0)| \\ &\leq |\delta_i(0) - \delta_i^*(0)| + |\delta_n(0) - \delta_n^*(0)| < 2\mu < \gamma \\ |a - b| &= |\delta_n(0) - \delta_n^*(0)| < \mu. \end{aligned}$$

Thus, for  $t \geq 0$

$$\begin{aligned} |\delta_n(t) - \delta_n^*(t)| &= \left| \int_0^t (\omega_n(s) - \omega_0) ds + a - b \right| \\ &\leq \int_0^t |\omega_n(s) - \omega_0| ds + |a - b| \\ &\leq \int_0^t \frac{\epsilon\eta}{2} e^{-\eta s} ds + |a - b| \\ &\leq \frac{\epsilon}{2} (1 - e^{-\eta t}) + \mu < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

and

$$\begin{aligned} |\delta_i(t) - \delta_i^*(t)| &= |\sigma_i(t) + \delta_n(t) - \sigma_i^* - \delta_n^*(t)| \\ &\leq |\sigma_i(t) - \sigma_i^*| + |\delta_n(t) - \delta_n^*(t)| \\ &< \frac{\epsilon}{2} e^{-\eta t} + \frac{\epsilon}{2} (1 - e^{-\eta t}) + \mu < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for  $i = 1, \dots, n - 1$ . The stability of the synchronous state  $\delta^*$  follows from the above inequalities and part (b) of this theorem.

Since

$$|\dot{\delta}_n(t) - \dot{\delta}_n^*(t)| = |\omega_n(t) - \omega_o| < \frac{\epsilon\eta}{2} e^{-\eta t}$$

for  $t \geq 0$ , it is true that if  $u \leq v$ , then

$$-\frac{\epsilon\eta}{2} \int_u^v e^{-\eta s} ds \leq (\delta_n(u) - \delta_n^*(u)) - (\delta_n(v) - \delta_n^*(v)) \leq \frac{\epsilon\eta}{2} \int_u^v e^{-\eta s} ds$$

Hence,

$$|(\delta_n(u) - \delta_n^*(u)) - (\delta_n(v) - \delta_n^*(v))| < \frac{\epsilon}{2} (e^{-\eta u} - e^{-\eta v}) \leq \frac{\epsilon}{2} e^{-\eta u}$$

By the Cauchy criterion the limit

$$\lim_{t \rightarrow \infty} (\delta_n(t) - \delta_n^*(t)) \equiv \delta_o$$

exists. Hence, part (a) of the theorem is proved.

### Examples

In this section some examples are given which show that the hypotheses of the theorems of Parts I and II are needed. For convenience the linear version of system (1) of Part I is written here and has the form

$$\dot{y} = -Ay + Cz \tag{7a}$$

$$\dot{z} = By \tag{7b}$$

where  $y \in R^k$ ,  $z \in R^\ell$ ,  $A$ ,  $B$ , and  $C$  are  $k \times k$ ,  $\ell \times k$ , and  $k \times \ell$  matrices, respectively. If the damping matrix  $A$  is omitted, then system (7)

has the form

$$\dot{y} = Cz \quad (8a)$$

$$\dot{z} = By \quad (8b)$$

The first example shows that the converse of Corollary 2.4 of Part I is not true even in the linear case. As one might expect, it may fail when eigenvalues of CB of multiplicity greater than one occur.

9. Example: Let  $k = \ell = 2$  and

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad CB = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$$

Then all eigenvalues of CB are real and negative. However,

$y_1(t) = \cos t$ ,  $y_2(t) = t \sin t$ ,  $z_1(t) = t \cos t$ ,  $z_2(t) = \sin t$  are the components of a solution of system (8) so that the trivial solution is not stable.

In criterion (i) of Theorem 2.6 of Part I it is assumed that H is symmetric. The following two examples show that such a hypothesis is needed. The examples also show that even though the trivial solution is a stable solution of the system  $\ddot{x} + Hx = 0$  or of system (8) and even though the damping matrix A is positive definite, the trivial solution may not be a stable solution of the system  $\ddot{x} + A\dot{x} + Hx = 0$  or of system (7).



10. Example: Let  $k = \ell = 2$  and

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad H = C = \begin{pmatrix} -2 & 0 \\ -14 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The eigenvalues of  $G_0$  (see equation (3) of Part I) are  $\{\pm i, \pm i\sqrt{2}\}$  so that the trivial solution of system (8) is stable. The characteristic polynomial of  $G$  (see equation (3) of Part I) can be computed from equation (4) of Part I. The characteristic polynomial of  $G$  is

$$\det (-CB + A\lambda + \lambda^2 I_2) = \lambda^4 + 3\lambda^3 + 4\lambda^2 - 10\lambda + 2.$$

Hence,  $\lambda = 1$  is an eigenvalue of  $G$  so that the trivial solution of system (7) is not stable.

The following is a similar example with diagonal damping. It also shows that the hypothesis of uniform damping, that is,  $A = a I_k$ , in criterion (ii) of Theorem 2.6 of Part I is needed.

11. Example: Let  $k = \ell = 2$  and

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = C = \begin{pmatrix} -3 & -5/2 \\ 3/2 & 1 \end{pmatrix}, \quad B = I_2$$

The eigenvalues of  $CB = H = C$  are  $\{-1/2, -3/2\}$  and the eigenvalues of  $G_0$  are  $\{\pm i/\sqrt{2}, \pm i\sqrt{3/2}\}$ . Hence, except for uniform damping, the hypotheses of criterion (ii) of Theorem 2.6 of Part I are satisfied. In addition, the trivial solution of system (8) is stable. The characteristic polynomial of  $G$  is

$$\lambda^4 + 5\lambda^3 + 8\lambda^2 - \lambda + 3/4.$$

By using the Routh-Hurwitz criterion (see [8, 24]), one can show that  $G$  has two eigenvalues with positive real part.

In Theorem 2.2 of Part II it is assumed that  $J$  is positive definite. The following two examples show that in the critical case for which  $J$  is singular but positive semi-definite, the trivial solution may be either unstable or asymptotically stable.

12. Example: Let  $k = \ell = 1$ ,  $A = B = K = 1$ ,  $g(z) = z^3$ , and  $f(z) = z^4/4$ . Then, by a straightforward application of the Lyapunov function  $V = y^2/2 + z^4/4$  and the invariance theorems (see the Background and Notation section) it can be shown that the trivial solution of system (1) of Part II is asymptotically stable even though

$$J(o) = \frac{df}{dz}(o) = 0$$

is negative semi-definite.

13. Example: Let  $k = \ell = 1$ ,  $A = B = K = 1$ ,  $g(z) = z^2$ , and  $f(z) = z^3/3$ . Using the Lyapunov function  $V = y^2/2 + z^3/3$  and the invariance theorems, it can be shown that the trivial solution of system (1) of Part I is not stable even though

$$J(o) = \frac{df}{dz}(o) = 0$$

is positive semi-definite.

Thus, parts (b) and (c) of Theorem 2.2 of Part II cannot be applied to the critical case in which  $J$  is positive semi-definite and singular.

In criterion (i) of Theorem 2.6 of Part I it is assumed that the damping matrix  $A$  is negative definite. The following example demonstrates

that a negative semi-definite condition for  $A$  is not sufficient for asymptotic stability.

14. Example: Let  $k = \ell = 2$ ,  $-B = C = H = K = I_2$  and

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

Then all hypotheses of criterion (i) of Theorem 2.6 of Part I are satisfied except that  $A$  is negative semi-definite instead of negative definite. The characteristic polynomial for  $G$  is  $(\lambda^2 + 1)(\lambda^2 + \lambda + 1)$  so that not all non-zero eigenvalues of  $G$  have negative real parts. In fact, one solution of system (7) has components  $y_1 \equiv 0$ ,  $y_2 = \epsilon \cos t$ ,  $z_1 \equiv 0$ ,  $z_2 = \epsilon \sin t$  for any  $\epsilon > 0$ . Hence, the trivial solution of system (7) is not asymptotically stable.

In the next section a condition is added to criterion (i) of Theorem 2.6 of Part I to account for the case in which matrix  $A$  is positive semi-definite, but not necessarily positive definite.

#### Partial Damping

In criterion (i) of Theorem 2.6 of Part I it is assumed that the damping matrix  $A$  is positive definite. In this section a condition is developed for replacing "positive definite" with "positive semi-definite". The result is similar to the results of Miller and Michel [38] which relates stability to the concept of observability from control theory. To help prove the result, some concepts from control theory are introduced and a preliminary result is proved.

15. Definition: Let  $M$  and  $N$  be  $m \times m$  and  $m \times n$  matrices, respectively. The pair  $(M, N)$  is called controllable if one of the following criteria is satisfied.

- (i) The rows of  $e^{Mt} N$  are linearly independent on  $[0, \infty)$ .
- (ii) The  $m \times m$  matrix  $(N \quad MN \quad \cdots \quad M^{m-1} N)$  has rank  $m$ .
- (iii)  $x^T N \neq 0$  for every left eigenvector  $x$  of  $M$ . (A non-zero complex  $m$ -vector  $x$  is a left eigenvector of  $M$  if  $x^T M = \lambda x$  for some complex  $\lambda$ .)

The equivalence of characterizations (i) and (ii) of controllability is proved in, for example, [8]. The equivalence of characterization (iii) is proved by Hautus [25].

16. Definition: Let  $M$  and  $P$  be  $m \times m$  and  $n \times m$  matrices, respectively. The pair  $(M, P)$  is called observable if one of the criteria is satisfied.

- (i) The columns of  $P e^{Mt}$  are linearly independent on  $[0, \infty)$
- (ii) The  $m \times m$  matrix

$$\begin{pmatrix} P \\ PM \\ \vdots \\ PM^{m-1} \end{pmatrix}$$

has rank  $m$ .

- (iii)  $Px \neq 0$  for every (right) eigenvector  $x$  of  $M$ .

The equivalence of characterizations (i) and (ii) of observability is proved in [8]. The equivalence of characterization (iii) is proved

by Hautus [25].

The following remark gives a well-known relationship between controllability and observability. (See [8].)

17. Remark: The pair  $(M, N)$  is controllable if and only if the pair  $(M^T, N^T)$  is observable.

18. Theorem: Let  $G$  be defined in equation (3) of Part I. If all eigenvalues of  $G$  have negative real parts, then the pair  $(CB, A)$  is observable and controllable.

Proof: The contrapositive will be proved beginning with the observability condition. Suppose that the pair  $(CB, A)$  is not observable. By criterion (iii) of Definition 16 there is an eigenvector  $x$  of  $CB$  such that  $Ax = 0$ . Let  $CBx = \lambda x$  for some complex  $\lambda$ .

Case 1: Assume that  $\lambda \neq 0$ . Let  $\xi = \sqrt{\lambda}$  be a square root of  $\lambda$  whose real part is non-negative. Then the vector

$$\begin{pmatrix} x \\ \frac{1}{\xi} Bx \end{pmatrix} \neq 0$$

is an eigenvector of  $G$  with eigenvalue  $\xi$  whose real part is non-negative.

Case 2: Assume that  $\lambda = 0$  and  $Bx \neq 0$ . Then the vector  $(x^T, 0^T)^T \neq 0$  is an eigenvector of  $G$  with zero eigenvalue.

Case 3: Assume  $\lambda = 0$  and  $Bx = 0$ . Then the vector

$$\begin{pmatrix} 0 \\ Bx \end{pmatrix} \neq 0$$

is an eigenvector of  $G$  with zero eigenvalue.

For the controllability criterion note that the eigenvalues of

$$G^T = \begin{pmatrix} -A^T & B^T \\ C^T & 0 \end{pmatrix}$$

are the same as the eigenvalues of  $G$ . If all eigenvalues of  $G$  have negative real parts, then the observability result proved previously implies that the pair  $(B^T C^T, A^T)$  is observable. By Remark 17, the pair  $(CB, A)$  is controllable. This completes the proof.

From Definition 16 it is easy to see that the pair  $(CB, A)$  is observable if  $A$  is non-singular. Thus, the above theorem is interesting only when  $A$  is singular. Also, the above theorem cannot be used to determine the instability of a non-linear system by a linearization procedure. However, the following two results can be used to determine instability in this case.

19. Lemma: Suppose that  $Ax = 0$  and  $C Bx = \lambda x$  for some non-zero  $k$ -vector  $x$  where  $\lambda$  is either not real or positive real. Then the matrix  $G$  (see equation (3) of Part I) has an eigenvalue with positive real part.

Proof: Case 1 of the proof of Theorem 18 applies. In this case  $\xi$  has positive real part.

20. Corollary: If  $(CB, A)$  is not observable and if no eigenvalues of  $CB$  are real and non-positive, then the matrix  $G$  has an eigenvalue with positive real part.

Criterion (i) of Theorem 2.6 of Part I may be modified in the following way to include partial damping. This modified theorem uses

the ideas discussed in the first part of this section.

21. Theorem: Let  $A$ ,  $B$ ,  $C$ , and  $G$  be defined according to equation (3) of Part I. Let  $A$ ,  $K$ , and  $H$  be symmetric  $k \times k$  matrices with  $K$  positive definite and  $H$  and  $A$  positive semi-definite. Suppose that  $CB = -KH$ ,  $AK = KA$ , and the pair  $(CB, A)$  is either observable or controllable where  $B$  and  $C$  are  $\ell \times k$  and  $k \times \ell$  matrices, respectively. Then all non-zero eigenvalues of  $G$  have negative real part.

Proof: Refer to the proof of criterion (i) of Theorem 2.6 of Part I. As in that previous proof assume that  $\lambda = \mu + i\nu$  is a non-zero eigenvalue of  $G$  with  $\mu \geq 0$ . This assumption will lead to a contradiction. If  $\mu > 0$ , then the procedure of the previous proof may be used to achieve a contradiction. To complete the proof of the observability part, it will be shown that if  $\mu = 0$ , then  $Ax = 0$  for some eigenvector  $x$  of  $CB$ . Then Definition 16 implies that the pair  $(CB, A)$  is not observable.

Assume that  $\mu = 0$ . Since eigenvalues occur in conjugate pairs, assume without loss of generality that  $\nu > 0$ . As in the previous proof, let  $L = \sqrt{K} H \sqrt{K}$ . By the arguments of the previous proof, there exist real  $k$ -vectors  $y$  and  $z$  such that

$$(L - \nu^2 I_k) y + \nu Az = 0 \quad (9a)$$

and

$$(L - \nu^2 I_k) z - \nu Ay = 0 \quad (9b)$$

Hence

$$z^T (L - v^2 I_k) y = -v z^T A z < 0$$

unless  $Az = 0$  and

$$y^T (L - v^2 I_k) z = v y^T A y > 0$$

unless  $Ay = 0$ .

If  $Ay \neq 0$ , then

$$0 < y^T (L - v^2 I_k) z = z^T (L - v^2 I_k) y \leq 0$$

which is a contradiction. Similarly, the assumption  $Az \neq 0$  leads to a contradiction.

Assume that  $Ay = 0$ . Let  $\bar{z} = \sqrt{K} z$ . Then since (by equation (9b))

$$v^2 z = Lz = \sqrt{K} H \sqrt{K} z = -\sqrt{K}^{-1} CB \bar{z}$$

it is true that

$$CB \bar{z} = -v^2 \bar{z}.$$

Also

$$A\bar{z} = A \sqrt{K} z = \sqrt{K} Az = 0.$$

Hence, by characterization (iii) of Definition 16, the pair  $(CB, A)$  is not observable.

The controllability part of the theorem follows from the fact that the eigenvalues of  $G$  and  $G^T$  are the same. If  $G$  has a non-zero



eigenvalue with non-negative real part, then a proof analogous to the above proof applied to  $G^T$  shows that the pair  $(B^T C^T, A^T)$  is not observable. By Remark 17 the pair  $(CB, A)$  is not controllable.

This completes the proof.

## CONCLUSION

The local stability of equilibrium points of systems of second order differential equations was investigated with the focus on the example of the swing equations for power systems.

The existence of equilibrium points for the swing equations with negligible transfer conductances is usually determined by numerical calculation. In this dissertation an analytic existence criterion was developed. The uniqueness of stable equilibrium points for the swing equations with negligible transfer conductances has been studied previously. In this dissertation the results for that special case were proved in a new way, and a uniqueness result in the more general case of non-trivial transfer conductances was developed.

The traditional Liapunov functions for the swing equations with negligible transfer conductances do not have negative definite derivatives along solutions. In this dissertation it was shown that they can nevertheless be used to prove asymptotic stability if the invariance theorem is also applied. Not much is known about the stability of the swing equations with non-trivial transfer conductances, even with uniform damping. For this case, results of this dissertation essentially (i.e., except for a critical case) classify equilibrium points as unstable or asymptotically stable. These results demonstrate that an equilibrium solution  $\delta^*$  may not be stable even though it satisfies the property  $|\delta_i^* - \delta_j^* - \alpha_{ij}| < \pi/2$  for all rotor angle pairs  $\delta_i^*, \delta_j^*$  where  $\alpha_{ij}$  is the complement of the phase of the transfer

admittance matrix. Hence, the stability criterion for the case of negligible transfer conductances does not generalize to the case of non-trivial transfer conductances.

The stability results for the swing equations with non-trivial transfer conductances were proved by linearization. It was shown that the linearization stability technique is necessary and sufficient for exponential stability. This result implies a rapid rate of convergence in the reduced system for the swing equations. This rapid rate of convergence was shown to have an important consequence in the analysis of the stability of synchronous states in power systems.

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## APPENDIX

In this appendix an algorithm is developed to determine whether all the zeros of a real polynomial  $p(x)$  are real and non-positive. If the characteristic polynomial of a matrix is known, then the algorithm may be used to determine whether the eigenvalues of the matrix are real and non-positive. The algorithm is an application of Sturm's theorem and is similar to the Routh-Hurwitz criterion for stability (see [13] or [20, p. 174]).

It is easy to determine the multiplicity  $N(0)$  of  $x = 0$  as a solution of  $p(x) = 0$ . Hence, Sturm's theorem will be applied to  $p_1(x) = p(x)/x^{N(0)}$  to determine the number (including multiplicity) of zeros of  $p(x)$  in the interval  $(-\infty, 0)$ . To do so, define  $p_2(x) = p_1'(x)$ . Next, find polynomials  $q_i(x)$  and  $p_i(x)$  for  $i = 1, \dots, m$  for some integer  $m$  such that

$$\begin{aligned}
 p_1(x) &= q_1(x) p_2(x) - p_3(x) \\
 p_2(x) &= q_2(x) p_3(x) - p_4(x) \\
 &\dots\dots\dots \\
 p_{j-1}(x) &= q_{j-1}(x) p_j(x) - p_{j+1}(x) \\
 &\dots\dots\dots \\
 p_{m-1}(x) &= q_{m-1}(x) p_m(x)
 \end{aligned} \tag{1}$$

where  $p_m(x)$  is not identically zero and either

(a) the degree of  $p_j$  is less than or equal to the degree of  $p_{j-1}$  and the degree of  $p_{j+1}$  is less than the degree of  $p_{j-1}$  or

(b) the degree of  $p_j$  is greater than the degree of  $p_{j-1}$  and  $p_{j+1} = -p_{j-1}$ .

To apply Sturm's theorem to the problem it is necessary to show that  $p_1, p_2, \dots, p_m$  is a generalized Sturm chain. The sequence of real polynomials

$$f_1, f_2, \dots, f_m$$

is a Sturm chain on  $(-\infty, 0)$  if  $f_m(x) \neq 0$  on  $(-\infty, 0)$  and if whenever  $f_j(x) = 0$  for some  $x \in (-\infty, 0)$  and for some  $j \in \{2, \dots, m-1\}$  then  $f_{j-1}(x) f_{j+1}(x) < 0$ . If  $f_1, f_2, \dots, f_m$  is a Sturm chain and  $g$  is a real polynomial, then

$$gf_1, gf_2, \dots, gf_m$$

is a generalized Sturm chain.

In the sequence  $p_1, \dots, p_m$  it is clear from the equations (1) that  $p_m$  is a divisor of  $p_j$  for  $j = 1, \dots, m$ . The polynomial  $p_m$  will be called the chain divisor. Define the chain  $r_1, \dots, r_m$  by the equations  $p_j = r_j p_m$  for  $j = 1, \dots, m$ . Then  $r_m(x) \equiv 1$  and

$$\begin{aligned} r_{j-1}(x) &= q_{j-1}(x) r_j(x) - r_{j+1}(x) \text{ for } j = 2, \dots, m-1 \\ r_{m-1}(x) &= q_{m-1}(x) r_m(x). \end{aligned} \quad (2)$$

Assume that  $r_j(x) = 0$  for  $2 \leq j \leq m-1$ . Then  $r_{j+1}(x) = 0$  if and only if  $r_{j-1}(x) = 0$ . If  $r_{j+1}(x) = r_j(x) = r_{j-1}(x) = 0$ , then, from equations (2), it follows that  $r_i(x) = 0$  for  $i = 1, \dots, m$ . But

$r_m(x) = 1 \neq 0$ . Hence  $r_{j+1}(x) r_j(x) < 0$  as required. Therefore,  $r_1, \dots, r_m$  is a Sturm chain and  $p_1, \dots, p_m$  is a generalized Sturm chain.

Define the coefficients  $a(j, k)$  for  $j = 1, \dots, m$  and  $k = 0, \dots, n$  so that

$$p_j(x) = \sum_{k=0}^n a(j, k) x^k.$$

The coefficients  $a(j, k)$  form an  $m \times (n-1)$  array  $A$ . Define

$$u_j = \max \{u : a(j, u) \neq 0\}$$

$$v_j = \min \{v : a(j, v) \neq 0\}$$

for  $j = 1, \dots, m$ .

For  $j = 2, \dots, m$ , define

$$q_{j-1}(x) = \begin{cases} \frac{a(j-1, u_{j-1})}{a(j, u_j)} x^{(u_{j-1} - u_j)} & \text{if } u_{j-1} \geq u_j \\ 0 & \text{if } u_{j-1} < u_j \end{cases}$$

Then equations (1) imply that if  $u_{j-1} \geq u_j$  and  $2 \leq j \leq m-1$ , then it is true that

$$\begin{aligned}
& \sum_{k=0}^n a(j+1, k) x^k \\
&= \frac{a(j-1, u_{j-1})}{a(j, u_j)} x^{(u_{j-1}-u_j)} \sum_{k=0}^n a(j, k) x^k \\
&\quad - \sum_{k=0}^n a(j-1, k) x^k \\
&= \sum_{k=0}^{u_{j-1}-u_j-1} a(j-1, k) x^k \\
&\quad + \sum_{k=u_{j-1}-u_j}^{u_{j-1}} \frac{a(j-1, u_{j-1})a(j, k+u_{j-1}-u_j)-a(j, u_j)a(j-1, k)}{a(j, u_j)} x^k
\end{aligned}$$

Hence, under these conditions

$$a(j+1, k) = \begin{cases} -a(j-1, k) & \text{if } 0 \leq k \leq u_{j-1}-u_j-1 \\ \frac{a(j-1, u_{j-1})a(j, k+u_{j-1}-u_j)-a(j, u_j)a(j-1, k)}{a(j, u_j)} & \text{if } u_{j-1}-u_j \leq k \leq u_{j-1}-1 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

Also, equations (1) imply that if  $u_{j-1} < u_j$  and  $j = 2, \dots, m-1$ , then

$$\sum_{k=0}^n a(j+1, k) x^k = - \sum_{k=0}^n a(j-1, k) x^k.$$

Hence, under these conditions

$$a(j+1, k) = -a(j-1, k) \quad \text{for } 0 \leq k \leq n. \quad (4)$$

Thus, the coefficients  $a(j, k)$  form an array similar to the array used in the Routh-Hurwitz criterion. Given polynomials  $p_1$  and  $p_2$  one may compute the coefficients  $a(j, k)$  for  $j \geq 3$  using only equations (3) and (4).

Sturm's theorem relates the number of distinct zeros of  $p(x)$  in the interval  $(-\infty, 0)$  to the difference in the number of changes in sign of the sequence

$$\text{sgn } p_1(0), \dots, \text{sgn } p_m(0)$$

and the sequence

$$\text{sgn } p_1(-\infty), \dots, \text{sgn } p_m(-\infty)$$

where  $\text{sgn } p_j(-\infty) = \lim_{x \rightarrow -\infty} \text{sgn } p_j(x)$  and  $\text{sgn } p_j(0) = \lim_{x \rightarrow 0^+} \text{sgn } p_j(x)$ .

To make this idea more explicit, define

$$\alpha_j = \text{sgn } a(j, v_j) = \text{sgn } p_j(0) \quad (5)$$

$$\beta_j = (-1)^{u_j} a(j, u_j) = \text{sgn } p_j(-\infty) \quad (6)$$

for  $j = 1, \dots, m$ . Let  $N_0(1)$  be the number of ordered pairs in the set

$$\{(j, j+1) : 1 \leq j \leq m-1, \alpha_j \alpha_{j+1} = -1\} \quad (7)$$

and let  $N_{-\infty}(1)$  be the number of ordered pairs in the set

$$\{(j, j+1) : 1 \leq j \leq m-1, \beta_j \beta_{j+1} = -1\} . \quad (8)$$

Sturm's Theorem: Let  $p_1, \dots, p_m$  be a generalized Sturm chain with  $p_2 = p_1'$ . Then

$$N(1) = N_{-\infty}(1) - N_0(1) \quad (9)$$

is the number of distinct zeros of the polynomial  $p_1(x)$  on the interval  $(-\infty, 0)$ . (See [13, 20].)

Since  $p_m(x)$  is the greatest common divisor  $p_1(x)$  and  $p_2(x)$ , the zeros of  $p_m(x)$  are the zeros of  $p_1(x)$  of multiplicity greater than one. To count the zeros of  $p(x)$  on  $(-\infty, 0)$  of multiplicity two, the procedure is repeated with  $p_m$  replacing  $p_1$ . Numbers  $N_0(2)$ ,  $N_{-\infty}(2)$ , and  $N(2) = N_{-\infty}(2) - N_0(2)$  will be determined where  $N(2)$  is the number of distinct zeros of  $p_m(x)$  on  $(-\infty, 0)$ . Equivalently,  $N(2)$  is the number of zeros of  $p(x)$  on  $(-\infty, 0)$  of multiplicity two. The process can be repeated to find  $N_0(i)$ ,  $N_{-\infty}(i)$ , and  $N(i)$  where  $N(i)$  is the number of zeros of  $p(x)$  on  $(-\infty, 0)$  of multiplicity  $i$  for  $i = 1, 2, 3, \dots$ . The procedure ends when the chain divisor is constant. Once the number  $N(i)$  is known for  $i = 1, 2, \dots$ , then

$$N = N(0) + N(1) + N(2) + \dots$$

is the number of zeros of  $p(x)$  on  $(-\infty, 0]$  including multiplicity.

Remark: If  $N = n$  where  $n$  is the degree of  $p(x)$ , then all zeros of  $p(x)$  are real and non-positive. If  $N < n$ , then  $p(x)$  has a zero which

is not real or has a zero which is real and positive.

Note that the number  $N$  could also be calculated by finding

$$N_o = N_o(1) + N_o(2) + \dots$$

and

$$N_{-\infty} = N_{-\infty}(1) + N_{-\infty}(2) + \dots$$

Then

$$N = N(0) + N_{-\infty} - N_o.$$

The numbers  $N_o(j)$  and  $N_{-\infty}(j)$  will not be changed if  $p_j(x)$  is replaced by  $c p_j(x)$  where  $c > 0$ . If  $p_{m+1} = 0$ , then  $p_m$  is the chain divisor.

These facts will be used in the algorithm which is merely a formalization of the procedure discussed previously.

The procedure is illustrated in the following examples.

1. Example: In this example the number of non-positive real zeros of the polynomial

$$\begin{aligned} p(x) &= x(x-1)(x+2)(x+1)(x+1) \\ &= -2x - 3x^2 + x^3 + 3x^4 + x^5 \end{aligned}$$

is found by the methods of this appendix. Thus,  $N(0) = 1$  and

$$p_1(x) = -2 - 3x + x^2 + 3x^3 + x^4. \text{ Hence, } p_2(x) = -3 + 2x + 9x^2 + 4x^3.$$

The array of coefficients  $A = (a(j, k))$  is given in Table 1 along with several comments. Each row is completed by using equations (3)



Table 1. The array of coefficients  $A = (a(j, k))$  for Example 1

$N_0$	$j \backslash k$	0	1	2	3	4	$N_{-\infty}$	Comments
	1	-2	-3	1	3	1		
	2	-3	2	9	4		X	
X	3	8	9	-2	-3		X	$p_3$ is replaced by $4p_3$
X	4	-23	-42	-19			X	$p_4$ is replaced by $3p_4$
	5	-19	-30	-11				$p_5$ is replaced by $19p_5/8$
	6	-1	-1				X	$p_6$ is replaced by $11p_6/108$
X	7	1	1				X	$p_7$ is replaced by $p_7/19$
	8	1	0					$p_7$ is the chain divisor; $p_8=0$ is replaced by $p_7'$
X	9	-1					X	
	10	0						Since the new chain divisor $p_9$ is constant, the test ends

and the previous two rows. For example,

$$a(5, 2) = \frac{(-3)(-42) - (-19)(-2)}{-19} = -\frac{88}{19}$$

$$a(5, 1) = \frac{(-3)(-23) - (-19)(9)}{-19} = -\frac{240}{19}$$

$$a(5, 0) = \frac{(-3)(0) - (-19)(8)}{-19} = -8$$

$$a(6, 1) = \frac{(-19)(-30) - (-11)(-42)}{-11} = -\frac{108}{11}$$

$$a(6, 0) = \frac{(-19)(-19) - (-11)(-23)}{-11} = -\frac{108}{11}$$

In the column marked  $N_0$  an x on the line between rows  $j$  and  $j + 1$  indicates that

$$(\operatorname{sgn} p_j(0)) (\operatorname{sgn} p_{j+1}(0)) = \alpha_j \alpha_{j+1} = -1 .$$

Hence, the number of x's in column  $N_0$  is the number

$N_0 = N_0(1) + N_0(2) + \dots$ . Similarly, the number of x's in the column marked  $N_{-\infty}$  is the number  $N_{-\infty}(1) + N_{-\infty}(2) + \dots$ . Therefore, the number  $N$  of zeros in the interval  $(-\infty, 0]$  is  $N(0) + N_{-\infty} - N_0 = N$ . In this example  $N(0) = 1$ ,  $N_{-\infty} = 7$ ,  $N_0 = 4$ . Hence,  $N = 4$  is the number of zeros of  $p(x)$  which are real and negative, a fact that is clear by the definition of  $p$ .

2. Example: In this example the number of non-positive real zeros of the polynomial

$$\begin{aligned} p(x) &= x^3 (x^2 + x + 1) (x - 1) (x + 1) \\ &= x^3 (x^4 + x^3 - x - 1) \end{aligned}$$

is found by the methods of this appendix. The array  $A = (a(j, k))$  is given in Table 2. Note that  $N(0) = 3$ ,  $N_0 = 3$ ,  $N_{-\infty} = 4$ . Hence,  $p$  has  $N = N(0) + N_{-\infty} - N_0 = 1$  non-positive real zero, a fact which is clear from the definition of  $p$ .

3. Example: In this example the number of non-positive real zeros of the polynomial

$$\begin{aligned} p(x) &= (x^2 + x + 1) (x^2 - x + 1) (x^2 + 1) (x + 1) (x - 1) \\ &= x^8 + x^6 - x^2 - 1 \end{aligned}$$

is found by the methods of this appendix. The array  $A = (a(j, k))$  is given in Table 3. Note that  $N(0) = 0$ ,  $N_0 = 4$ ,  $N_{-\infty} = 5$ . Hence,  $p$  has  $N = N_{-\infty} - N_0 = 5 - 4 = 1$  non-positive real zero, a fact which is clear from the definition of  $p$ .

Table 2. The array of coefficients  $A = (a(j, k))$  for Example 2

$N_0$	$j \backslash k$	0	1	2	3	4	$N_{-\infty}$
	1	-1	-1	0	1	1	
	2	-1	0	3	4		X
X	3	4	3	0	-1		X
X	4	-5	-4	-1			X
	5	-1	-2	-1			
X	6	2	1				
	7	1	0				X
	8	1					
	9	0					

Table 3. The array of coefficients  $A = (a(j, k))$  for Example 3

[illegible]