The Relationship Between Confidence Intervals for Failure Probabilities and Life Time Quantiles

Yili Hong, William Q. Meeker, and Luis A. Escobar

Abstract

The failure probability of a product $F(t)$ and the life time quantile t_p are commonly used metrics in reliability applications. Confidence intervals are used to quantify the statistical uncertainty of estimators of these two metrics. In practice, a set of pointwise confidence intervals for $F(t)$ or the quantiles t_p are often computed and plotted on one graph, which we refer to as pointwise "confidence bands." These confidence bands for $F(t)$ or t_p can be obtained through normal approximation, likelihood, or other procedures. In this paper, we compare normal approximation and likelihood methods and introduce a new procedure to get the confidence intervals for $F(t)$ by inverting the pointwise confidence bands of the quantile t_p function. We show that it is valid to interpret the set of pointwise confidence intervals for the quantile function as a set of pointwise confidence intervals for $F(t)$ and vice-versa. Our results also indicate that the likelihood based pointwise confidence bands have desirable statistical properties, beyond those that were known previously.

Index Terms

Asymptotic approximation; Confidence bands; Life data analysis; Likelihood confidence interval; Maximum likelihood; Normal approximation; Reliability.

ACRONYMS

cdf	cumulative distribution function
ML	maximum likelihood

NOTATION

χ^2_1	chi-square distribution with 1 degree of freedom
NOR$(0,1)$	standard normal distribution

Y. Hong and W. Q. Meeker are with Department of Statistics, Iowa State University, Ames, IA, 50011, USA (e-mail: hong@iastate.edu; wqmeeker@iastate.edu)

L. A. Escobar is with Department of Experimental Statistics, Louisiana State University, Baton Rouge, LA, 70803, USA (e-mail: luis@lsu.edu)

I. Introduction

A. Motivation

The cdf $F(t)$ of a random variable T can be interpreted as the probability that a unit will fail by time t or the proportion of units in the population that will fail by time t. In particular, the cdf can be used to estimate the proportion of a product that will fail before a particular point in time such as the end of the warranty period. The quantile function t_p is the inverse of the cdf and corresponding to the time at which a specified proportion p of the population fails. For example, if it is felt that it is acceptable to repair/replace no more that 5% of a product population during the warranty period, then the warranty period should be at most equal to $t_{.05}$, the .05 quantile.

Confidence intervals are used to quantify statistical uncertainty. For reliability applications, it is standard practice to plot on one graph a set of pointwise confidence intervals for $F(t)$ over a range of t values or a set of pointwise confidence intervals for t_p over a range of p values (e.g., MINTAB [1], WEIBULL++ [2], PROC RELIABILITY in SAS [3], S-PLUS/SPLIDA in [4] provide such graphics). We will refer to these pointwise sets as “confidence bands.” Figure 1 shows pointwise confidence intervals for the failure probability $F(t)$ when $10 \le t \le 100$. Details for the computation Figure 1 are given in Section VI. Plotting the confidence intervals for an entire interval of values of t (or p) relieves the user from having to specify the particular time (or quantile) of interest, making the software easier to use. In general, the plot obtained from the pointwise bands for $F(t)$ is not exactly the same obtained from the pointwise bands for t_p.

The results in this paper show that it is valid to interpret the set of pointwise confidence intervals for the quantile function as a set of pointwise confidence intervals for $F(t)$ and vice-versa. In particular, we show that normal approximation based pointwise confidence bands for the cdf and the quantile function are asymptotically equivalent and that likelihood based pointwise confidence bands for the cdf and the quantile function are equivalent. Our results are presented for the family of log-location-scale distributions, which includes the commonly used Weibull and lognormal distributions as special cases.

B. Literature Review

Statistical methods (including confidence intervals) for log-location-scale distributions, especially with application to lifetime studies are given, for example, in Chapters 6 and 8 of Nelson [5], Chapter 8 of Meeker and Escobar [6], and Chapter 5 of Lawless [7].

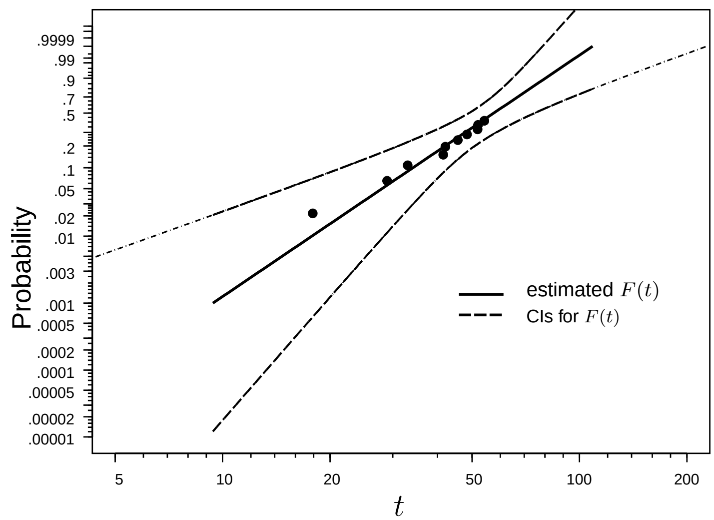

Fig. 1. Weibull Probability Plot of the Censored Ball Bearing Life Test Data with ML Estimate and Pointwise Confidence Bands Based on Likelihood Procedures

C. Overview

The remainder of this paper is organized as follows. Section II describes the log-location-scale model used in the paper and ML estimation for the model parameters and functions of the parameters. Sections III and IV describe existing procedures to construct confidence intervals for t_p and $F(t)$ and a new procedure for constructing confidence interval for $F(t)$ by inverting the confidence bands for the quantiles. These procedures are based either on a normal approximation or on the likelihood. Section V presents some equivalence results for the confidence bands of $F(t)$ and t_p. Section VI illustrates the methods and results with some application to real data and Section VII contains concluding remarks. Some technical details are given in the appendix.

II. Model and ML Estimation

A. Model and Data

The results of this paper have been developed specifically for the commonly used location-scale and log-location-scale families, although similar results certainly hold for other families of distributions. A random variable Y belongs to the location-scale family, with location μ and scale σ, if its cdf can be written as $F_Y(y;\mu,\sigma) = \Phi[(y-\mu)/\sigma]$, where $-\infty < y < \infty$, $-\infty < \mu < \infty, \sigma > 0$, and $\Phi(z)$ is the parameter free cdf of $(Y-\mu)/\sigma$. The normal distribution, the smallest extreme value distribution, the largest extreme value distribution, and the logistic distribution are commonly used location-scale distributions. A positive random variable T is a member of the log-location-scale family if $Y = \log(T)$ is a member of the location-scale family. Then the distribution of T is $F(t; \mu, \sigma) = \Phi\left\{[\log(t) - \mu)]/\sigma\right\}$. The lognormal, the Weibull, the Fréchet, and the loglogistic are among the important distributions of this family. For example, the cdf of the Weibull random variable T is $F(t; \mu, \sigma) = \Phi_{\text{sev}}\{[\log(t) - \mu]/\sigma\}$ where $\Phi_{\text{sev}}(z) = 1 - \exp[-\exp(z)]$ is the standard (i.e., $\mu = 0, \sigma = 1$) smallest extreme value cdf. For the lognormal distribution, replace Φ_{sev} above with Φ_{nor}, the standard normal cdf.

Suppose that T is a lifetime that has a distribution in the log-location-scale family. Some quantities of interest are the failure probability $F(t_e) = F(t_e; \mu, \sigma)$ at t_e and the p quantile $t_p = \exp(\mu + z_p\sigma)$ of the distribution where $z_p = \Phi^{-1}(p)$ is the p quantile of $\Phi(z)$.

Life tests often result in censored data. Type I (time) censored data result when unfailed units are removed from test at a prespecified time, usually due to limited time for testing. Type II (failure) censored data result when a test is terminated after a specified number of failures, say $2 \leq r \leq n$. If all units fail, the data are called complete or uncensored data.

The results in this paper hold for complete, Type I and Type II censored data, as well as to noninformative randomly censored data that generally arise in field tracking studies and warranty data analysis.

B. ML Estimation

For a censored sample with n independent exact and right censored observations from a log-location-scale random variable T, the likelihood of the data at $\boldsymbol{\theta} = (\mu, \sigma)'$ is

$$L(\boldsymbol{\theta}) = \mathcal{C}\prod_{i=1}^{n}\left\{\frac{1}{\sigma t_i}\phi\left[\frac{\log(t_i) - \mu}{\sigma}\right]\right\}^{\delta_i}\left\{1 - \Phi\left[\frac{\log(t_i) - \mu}{\sigma}\right]\right\}^{1-\delta_i}$$

where $\delta_i = 1$ if t_i is an exact observation and $\delta_i = 0$ if t_i is a right censored observation, Φ is defined in Section II-A and $\phi(z)$ is the density $d\Phi(z)/dz$, and $\mathcal{C}$ is a constant that does not depend on the unknown parameters. Standard computer software (e.g., JMP, MINITAB, SAS, S-PLUS/SPLIDA) provide ML estimates of $\boldsymbol{\theta}$ and functions of $\boldsymbol{\theta}$ such as quantiles and failure probabilities. We denote the ML estimator of $\boldsymbol{\theta}$ by $\widehat{\boldsymbol{\theta}} = (\widehat{\mu}, \widehat{\sigma})$. From the invariance property of ML estimators, the ML estimator of t_p is $\widehat{t_p} = \exp\left[\widehat{\mu} + \Phi^{-1}(p)\,\widehat{\sigma}\right]$. Similarly, the ML estimator of $F(t)$ at t_e is $\widehat{F}(t_e) = \Phi\{[\log(t_e) - \widehat{\mu}]/\widehat{\sigma}\}$. See, for example, Chapter 8 in Meeker and Escobar [6] for more details.

In large samples, the ML estimator $\widehat{\boldsymbol{\theta}}$ has a distribution that can be approximated by a bivariate normal distribution BVN$(\boldsymbol{\theta}, \Sigma)$, where Σ can be estimated by

$$\widehat{\Sigma}_{\widehat{\boldsymbol{\theta}}} = \left[-\frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right]^{-1}_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = \left[\begin{array}{cc} \widehat{\text{Var}}(\widehat{\mu}) & \widehat{\text{Cov}}(\widehat{\mu}, \widehat{\sigma}) \\ \widehat{\text{Cov}}(\widehat{\mu}, \widehat{\sigma}) & \widehat{\text{Var}}(\widehat{\sigma}) \end{array} \right]$$

and $\mathcal{L}(\boldsymbol{\theta}) = \log[L(\boldsymbol{\theta})]$ is the log likelihood of the data. In the following sections, we also use the scaled estimate of variance-covariance matrix

$$\widehat{\Lambda} = \left(\frac{n}{\widehat{\sigma}^2} \right) \widehat{\Sigma}_{\widehat{\boldsymbol{\theta}}} = \left[\begin{array}{cc} \widehat{\lambda}_{11} & \widehat{\lambda}_{12} \\ \widehat{\lambda}_{12} & \widehat{\lambda}_{22} \end{array} \right]. \tag{1}$$

III. Normal Approximation Based Confidence Interval Procedures

A. Normal Approximation Confidence Procedure for t_p

Under standard regularity conditions met by the log-location-scale distributions used here, properties of ML estimators imply that

$$\frac{\log(\widehat{t}_p) - \log(t_p)}{\widehat{\text{se}}_{\log(\widehat{t}_p)}} \;\; \dot{\sim} \;\; \text{NOR}(0, 1) \tag{2}$$

in large samples, where $\dot{\sim}$ means "approximately distributed" and an estimator of the estimated standard error of $\log(\widehat{t}_p)$ is

$$\widehat{\text{se}}_{\log(\widehat{t}_p)} = \sqrt{\widehat{\text{Var}}(\widehat{\mu}) + 2 z_p \widehat{\text{Cov}}(\widehat{\mu}, \widehat{\sigma}) + z_p^2 \widehat{\text{Var}}(\widehat{\sigma})}\,. \tag{3}$$

Following from (2), a normal approximation confidence interval for $\log(t_p)$ is

$$\left[\log(\underset{\sim}{t}_p), \quad \log(\widetilde{t}_p) \right] = \log(\widehat{t}_p) \mp z_{1-\alpha/2}\, \widehat{\text{se}}_{\log(\widehat{t}_p)}.$$

Thus, the corresponding normal approximation confidence interval for t_p is

$$\left[\underset{\sim}{t}_p, \quad \widetilde{t}_p \right] = \exp \left[\log(\widehat{t}_p) \mp z_{1-\alpha/2}\, \widehat{\text{se}}_{\log(\widehat{t}_p)} \right]. \tag{4}$$

This method is described, for example, in Nelson [5, page 331].

B. $\widehat{z}$ Confidence Interval Procedure for $F(t)$

Similarly, in large samples

$$\frac{\widehat{z} - z}{\widehat{\text{se}}_{\widehat{z}}} \;\; \dot{\sim} \;\; \text{NOR}(0, 1)\,, \tag{5}$$

where $\widehat{z} = \Phi^{-1}[\widehat{F}(t_e)] = [\log(t_e) - \widehat{\mu}]/\widehat{\sigma}$, $z = [\log(t_e) - \mu]/\sigma$, and, by the delta method, an estimator of the standard error of $\widehat{z}$ is

$$\widehat{\text{se}}_{\widehat{z}} = \frac{1}{\widehat{\sigma}} \sqrt{\widehat{\text{Var}}(\widehat{\mu}) + 2 \widehat{z} \widehat{\text{Cov}}(\widehat{\mu}, \widehat{\sigma}) + \widehat{z}^2 \widehat{\text{Var}}(\widehat{\sigma})}\,. \tag{6}$$

Approximation (5) can be used to obtain an approximate confidence interval for $F(t_e)$. In particular, the confidence interval is

$$[\,\underset{\sim}{p}, \quad \widetilde{p}\,] = [\,\Phi(\underset{\sim}{z}), \quad \Phi(\widetilde{z})\,] \tag{7}$$

where $[\,\underset{\sim}{z}, \quad \widetilde{z}\,] = \widehat{z} \mp z_{1-\alpha/2}\,\widehat{\text{se}}_{\widehat{z}}$. This method is described, for example, in Nelson [5, page 332].

C. $\widehat{t}_p$ Confidence Interval Procedure for $F(t)$

This section illustrates a new confidence interval procedure for $F(t_e)$, based on the relationship between estimates of $F(t)$ and t_p. The procedure, which we call the $\widehat{t}_p$ procedure, is defined by inverting the confidence bands for the quantile function. The general idea is illustrated in Figure 2. In particular,

- Compute the confidence intervals $[\underset{\sim}{t}_p, \widetilde{t}_p]$ for the quantiles t_p. In Figure 2 the lower endpoints, $\underset{\sim}{t}_p$, and the upper endpoints, $\widetilde{t}_p$, of these confidence intervals are indicated by $\leftarrow$ and $\rightarrow$, respectively.
- The confidence bands for the cdf $F(t)$, $0 < t < \infty$ are defined as follows. The upper boundary of the confidence band for $F(t)$ is obtained by joining the lower endpoints, $\underset{\sim}{t}_p$, of the quantile confidence intervals. The lower boundary of the confidence bands is obtained by joining the upper endpoints, $\widetilde{t}_p$.
- A pointwise confidence interval for $F(t_e)$ is obtained from the intersections of a vertical line through t_e with the boundaries of the confidence bands for $F(t)$. In Figure 2, this is illustrated for $t_e = 2.0$.

Using (1), one can re-express $\widehat{\text{se}}_{\log(\widehat{t}_p)}$ as $\widehat{\sigma}\sqrt{(\widehat{\lambda}_{11} + 2\widehat{\lambda}_{12} z_p + \widehat{\lambda}_{22} z_p^2)/n}$. Hence

$$\left[\,\log(\underset{\sim}{t}_p), \quad \log(\widetilde{t}_p)\,\right] \;= \log(\widehat{t}_p) \mp \widehat{\sigma}\sqrt{\gamma_{\alpha,n}(\widehat{\lambda}_{11} + 2\widehat{\lambda}_{12} z_p + \widehat{\lambda}_{22} z_p^2)}$$

where $\gamma_{\alpha,n} = z^2_{1-\alpha/2}/n$. In Figure 2, this confidence interval for $F(t_e)$ is indicated with the $\updownarrow$ symbol. Specifically, a $\widehat{t}_p$ procedure confidence interval for $F(t_e)$ based on a normal approximation is given by the solutions $\underset{\sim}{p}$ and $\widetilde{p}$ for the equations

$$\begin{aligned}
\log(t_e) &= \log(\widehat{t}_{\underset{\sim}{p}}) + \widehat{\sigma}\sqrt{\gamma_{\alpha,n}(\widehat{\lambda}_{11} + 2\widehat{\lambda}_{12} z_{\underset{\sim}{p}} + \widehat{\lambda}_{22} z_{\underset{\sim}{p}}^2)} \\
\log(t_e) &= \log(\widehat{t}_{\widetilde{p}}) - \widehat{\sigma}\sqrt{\gamma_{\alpha,n}(\widehat{\lambda}_{11} + 2\widehat{\lambda}_{12} z_{\widetilde{p}} + \widehat{\lambda}_{22} z_{\widetilde{p}}^2)}\ .
\end{aligned}$$

The solutions for $\underset{\sim}{p}$ and $\widetilde{p}$ are

$$[\,\underset{\sim}{p}, \quad \widetilde{p}\,] = [\,\Phi(\underset{\sim}{z}), \quad \Phi(\widetilde{z})\,], \tag{8}$$

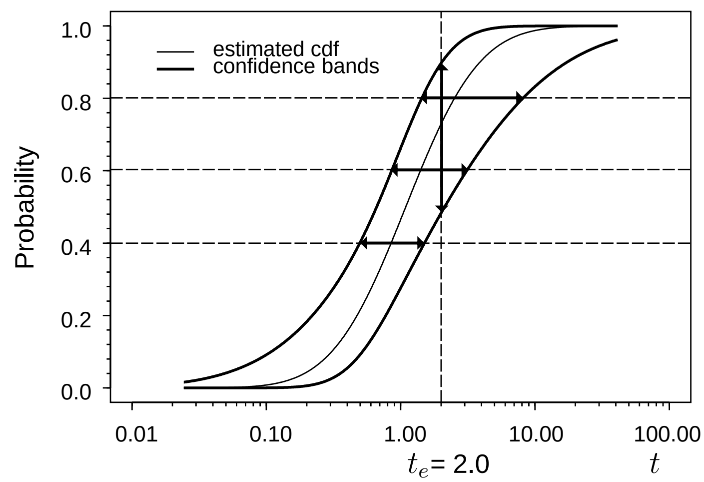

Fig. 2. Illustration of the $\widehat{t_p}$ Procedure

where

$$[\underset{\sim}{z}, \quad \widetilde{z}] = \widehat{z} + \frac{\gamma_{\alpha,n}(\widehat{\lambda}_{12} + \widehat{z}\widehat{\lambda}_{22})}{1 - \gamma_{\alpha,n}\widehat{\lambda}_{22}} \mp \frac{\sqrt{\gamma_{\alpha,n}\left(\widehat{\lambda}_{11} + 2\widehat{z}\widehat{\lambda}_{12} + \widehat{z}^2\widehat{\lambda}_{22}\right) - \gamma_{\alpha,n}^2\left(\widehat{\lambda}_{11}\widehat{\lambda}_{22} - \widehat{\lambda}_{12}^2\right)}}{1 - \gamma_{\alpha,n}\widehat{\lambda}_{22}}.$$

The procedure leading to the interval (8) requires that $\gamma_{\alpha,n}\widehat{\lambda}_{22} < 1$ for the solution to be unique. If $\gamma_{\alpha,n}\widehat{\lambda}_{22} \geq 1$, one or both roots might be complex, infinite, or non-unique. One can show that this anomaly occurs when a joint confidence region for (μ, σ) includes non-positive values of σ, in which case the confidence bands provided by procedure (4) and (7) are not monotonically increasing.

The $\widehat{t_p}$ procedure links together the procedures for constructing confidence intervals for t_p and for $F(t)$. This link allows us to show analytically the relationships between the two procedures. An alternative procedure for defining confidence intervals of t_p can be similarly obtained based on an inversion of the confidence bands for $F(t)$. We do not give the details here.

IV. Likelihood Based Confidence Interval Procedures

This section introduces likelihood based procedures for computing confidence intervals for t_p and $F(t)$. We also introduce a likelihood based $\widehat{t_p}$ procedure for $F(t)$ in a manner similar to the $\widehat{t_p}$ procedure in Section III-C. Generally, there are no closed forms for these likelihood procedures and numerical methods are needed.

A. Confidence Intervals for t_p

Standard large sample theory also provides the result that

$$2\left\{\mathcal{L}\left(\widehat{\mu}\,,\widehat{\sigma}\right)-\max_{\sigma}\mathcal{L}\left[\log(t)-z_p\,\sigma\,,\sigma\right]\right\}\quad\dot{\sim}\quad\chi^2_1 \tag{9}$$

for fixed $0<p<1$ and $t=t_p$ (e.g. Meeker and Escobar [6, page 182]). This result would be the basis for a likelihood ratio test for a quantile t_p. Likelihood confidence intervals for t_p can be obtained by inverting likelihood ratio tests. In particular, a $100(1-\alpha)\%$ likelihood based confidence interval for the t_p using (9) is

$$\left[\undertilde{t}_p\,,\quad \widetilde{t}_p\right] \tag{10}$$

where

$$\begin{aligned}
\undertilde{t}_p &= \min\left\{t: t \text{ satisfying } \max_{\sigma}\mathcal{L}\left[\log(t)-z_p\,\sigma\,,\sigma\right]\geq \mathcal{L}(\widehat{\mu}\,,\widehat{\sigma})-\frac{1}{2}\chi^2_{1;1-\alpha}\right\}\\
\widetilde{t}_p &= \max\left\{t: t \text{ satisfying } \max_{\sigma}\mathcal{L}\left[\log(t)-z_p\,\sigma\,,\sigma\right]\geq \mathcal{L}(\widehat{\mu}\,,\widehat{\sigma})-\frac{1}{2}\chi^2_{1;1-\alpha}\right\}.
\end{aligned}$$

B. Confidence Intervals for $F(t)$

Similarly, we have the fact that

$$2\left\{\mathcal{L}\left(\widehat{\mu}\,,\widehat{\sigma}\right)-\max_{\sigma}\mathcal{L}\left[\log(t_e)-z_p\,\sigma\,,\sigma\right]\right\}\quad\dot{\sim}\quad\chi^2_1 \tag{11}$$

for fixed $t_e>0$ and $p=F(t_e)$. Using (11), the likelihood based confidence interval for $F(t_e)$ is

$$[\,\undertilde{p}\,,\quad \widetilde{p}\,] \tag{12}$$

where

$$\begin{aligned}
\undertilde{p} &= \min\left\{p: p \text{ satisfying } \max_{\sigma}\mathcal{L}\left[\log(t_e)-z_p\,\sigma\,,\sigma\right]\geq \mathcal{L}(\widehat{\mu},\widehat{\sigma})-\frac{1}{2}\chi^2_{1;1-\alpha}\right\}\\
\widetilde{p} &= \max\left\{p: p \text{ satisfying } \max_{\sigma}\mathcal{L}\left[\log(t_e)-z_p\,\sigma\,,\sigma\right]\geq \mathcal{L}(\widehat{\mu},\widehat{\sigma})-\frac{1}{2}\chi^2_{1;1-\alpha}\right\}.
\end{aligned}$$

Note that both procedure (10) and (12) also give confidence bands for the quantile function and the cdf, respectively. One can show that these bands are always monotonically increasing, which is in contrast to the normal approximation confidence bands in Section III that can, especially in small samples, be non-monotone.

C. $\widehat{t}_p$ Confidence Interval Procedure for $F(t)$

To show the relationship between the likelihood confidence bands obtained from (10) and (12), we define a likelihood based $\widehat{t}_p$ procedure confidence interval for $F(t)$ in a manner similar to the normal approximation $\widehat{t}_p$ procedure in Section III-C. A confidence interval from this procedure, as illustrated in Figure 2, is

$$[\ \underset{\sim}{p}\,,\quad \widetilde{p}\ ] \tag{13}$$

where $\underset{\sim}{p}$ and $\widetilde{p}$ are obtained by solving from equations $t_e = \widetilde{t}_{\underset{\sim}{p}}$ and $t_e = \underset{\sim}{t}_{\widetilde{p}}$. That is, $\underset{\sim}{p}$ is chosen such that the upper endpoint of the confidence interval for the $\underset{\sim}{p}$ quantile is t_e. Similarly, $\widetilde{p}$ is chosen such that the lower endpoint of the confidence interval for the $\widetilde{p}$ quantile is t_e.

V. Equivalence Results

This section outlines some equivalence results among the confidence interval procedures given in Section III and IV.

Result 1: The normal approximation based confidence interval procedures for $F(t)$ defined by (7) and (8) are asymptotically equivalent.
This result implies that there is a difference in these procedures, but that the difference becomes smaller in large samples. This result also suggests that it is valid to interpret the set of approximate pointwise confidence intervals for the quantile function as a set of approximate pointwise confidence intervals for $F(t)$ and vice-versa. See Appendix A for a proof.

Result 2: The likelihood based confidence interval procedures (12) and (13) for $F(t)$ are equivalent.
This result shows that if one uses the likelihood based procedures, it makes no difference whether one computes pointwise confidence bands for $F(t)$ or t_p; the bands will be the same. See Appendix B for a proof. The property in **Result** 2 is in addition to the property that likelihood based confidence intervals procedures tend to have better coverage properties (e.g., as described in Jeng and Meeker [8]).

VI. Application To The Ball Bearing Data

To illustrate the procedures, we consider a well-known subset of the Lieblein and Zelen [9] ball bearing life test data. As described in Lawless [7, page 98], this data set has 23 observations on millions of cycles to failure for each ball bearing. To introduce censoring to the data, we assume the life test ended after the first 10 bearing failures.

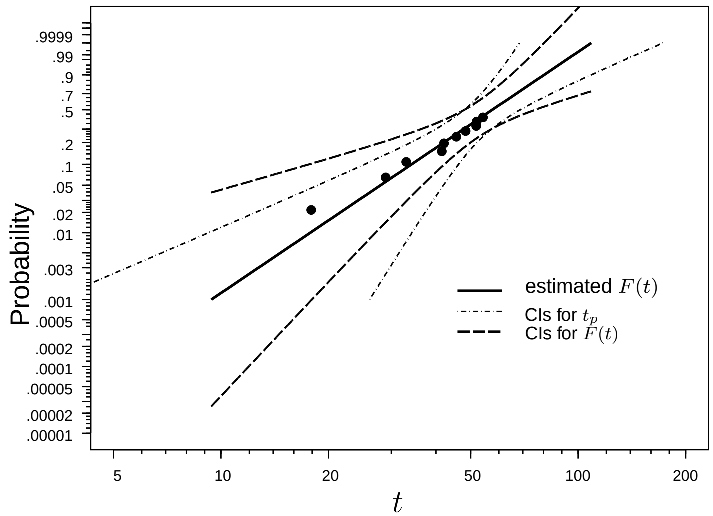

Fig. 3. Weibull Probability Plot of the Censored Ball Bearing Life Test Data with ML Estimate and Pointwise Confidence Bands Based on Normal Approximation Procedures

Figures 1 and 3 are Weibull probability plots of the Weibull ML estimate and pointwise confidence bands for $F(t)$ using the censored ball bearing life test data. Figure 3 shows the pointwise confidence bands for $F(t)$ based on procedures (4) and (7). As **Result 1** shows, these two sets of confidence bands are not exactly the same. But if the sample size were to get larger, we would expect these two sets of confidence bands to get closer. Because of **Result 2**, only the pointwise confidence bands for $F(t)$ based on procedure (10) are shown in Figure 1, the bands obtained from procedure (12) are exactly the same.

VII. Concluding Remarks

This paper compares confidence interval procedures for distribution probabilities and quantiles. We also show the relationships between the pointwise confidence bands for $F(t)$ and the pointwise confidence bands for t_p and show that the bands for $F(t)$ and t_p are the same in the case of likelihood based intervals. The results provide additional motivation (e.g., beyond motivation described in Jeng and Meeker [8]) to move from the traditional normal approximation intervals to likelihood based intervals.

Even though the advantages of likelihood-based intervals has been known to many statisticians for the past 15 to 20 years, as far as we know, the only commercial computer packages to have implemented likelihood-based confidence intervals for functions of parameters are SAS PROC RELIABILITY (only for quantiles, not for probabilities) and Weibull++ (for both probabilities and quantiles). The results in this paper show that if the likelihood based intervals are used, only one set of confidence bands, either confidence bands for quantile function or for cdf, are needed because another set of confidence bands are exact the same.

ACKNOWLEDGEMENTS

We would like to thank Marci Connors (formerly from Hewlett Packard) and Fritz Scholz (formerly from Boeing) for independently suggesting that we should study the issues outlined in this paper.

APPENDIX

A. Proof of Result 1

This appendix shows the asymptotic equivalence of confidence bands from the $\widehat{z}$ procedure in (7) and the $\widehat{t_p}$ procedure in (8). In either case, the confidence band for the cdf can be expressed as $[\underset{\sim}{p},\quad \widetilde{p}] = [\Phi(\underset{\sim}{z}),\quad \Phi(\widetilde{z})]$. It suffices to consider the lower band because the proof for the upper band is similar. Using (1), one can re-express $\widehat{\text{se}}_{\widehat{z}}$ as $\sqrt{(\widehat{\lambda}_{11} + 2\widehat{\lambda}_{12}\widehat{z} + \widehat{\lambda}_{22}\widehat{z}^2)/n}$. For confidence bands defined by (7), $\underset{\sim}{z}_1 = \widehat{z} - z_{1-\alpha/2}\,\widehat{\text{se}}_{\widehat{z}} = \widehat{z} - \sqrt{\gamma_{\alpha,n}(\widehat{\lambda}_{11} + 2\widehat{\lambda}_{12}\widehat{z} + \widehat{\lambda}_{22}\widehat{z}^2)}$. For confidence bands defined by (8),

$$\underset{\sim}{z}_2 = \widehat{z} + \frac{\gamma_{\alpha,n}(\widehat{\lambda}_{12} + \widehat{z}\widehat{\lambda}_{22})}{1 - \gamma_{\alpha,n}\widehat{\lambda}_{22}} - \frac{\sqrt{\gamma_{\alpha,n}\left(\widehat{\lambda}_{11} + 2\widehat{z}\widehat{\lambda}_{12} + \widehat{z}^2\widehat{\lambda}_{22}\right) - \gamma_{\alpha,n}^2\left(\widehat{\lambda}_{11}\widehat{\lambda}_{22} - \widehat{\lambda}_{12}^2\right)}}{1 - \gamma_{\alpha,n}\widehat{\lambda}_{22}}.$$

Note that

$$\frac{\underset{\sim}{z}_2 - \widehat{z}}{\underset{\sim}{z}_1 - \widehat{z}} = \frac{\sqrt{\left(\widehat{\lambda}_{11} + 2\widehat{z}\widehat{\lambda}_{12} + \widehat{z}^2\widehat{\lambda}_{22}\right) - \gamma_{\alpha,n}\left(\widehat{\lambda}_{11}\widehat{\lambda}_{22} - \widehat{\lambda}_{12}^2\right)} - \sqrt{\gamma_{\alpha,n}}(\widehat{\lambda}_{12} + \widehat{z}\widehat{\lambda}_{22})}{(1 - \gamma_{\alpha,n}\widehat{\lambda}_{22})\sqrt{(\widehat{\lambda}_{11} + 2\widehat{\lambda}_{12}\widehat{z} + \widehat{\lambda}_{22}\widehat{z}^2)}} \to 1$$

as $n \to \infty$ because $\gamma_{\alpha,n} = z_{1-\alpha/2}^2/n \to 0$ (holding r/n or expectation of r/n constant). Thus the confidence bands defined by (7) and defined by (8) are asymptotically equivalent.

B. Proof of Result 2

This appendix shows the equivalence of the likelihood based confidence interval procedures for $F(t)$ in (12) and in (13). The claims is that the upper band for the quantile function is exactly the same as the lower band of the cdf and that the lower band for the quantile function is exactly the same as the upper band of the cdf. Only the proof of the first case is given, as the proof of the second case is similar.

Let (t_e, p_L) and (t_e, p_U) (with $p_L < p_U$), as illustrated in Figure 2, be the points at which the vertical line through t_e intersects the confidence bands for the quantile function. Because t_e is the upper ending point of the confidence interval for t_{p_L}, that is, $t_e = \max\{t : t$ satisfying $\max_\sigma \mathcal{L}[\log(t) - \sigma z_{p_L}, \sigma] \geq k\}$ where $k = \mathcal{L}(\widehat{\mu}, \widehat{\sigma}) - \frac{1}{2}\chi^2_{1;1-\alpha}$. Thus,

$$\max_\sigma \mathcal{L}[\log(t_e) - \sigma z_{p_L}, \sigma] \geq k \tag{14}$$

$$\max_\sigma \mathcal{L}[\log(t_e + \delta) - \sigma z_{p_L}, \sigma] < k \text{ for all } \delta > 0. \tag{15}$$

Consider $p < p_L$ and suppose that $\tilde{\sigma}$ maximizes $\mathcal{L}[\log(t_e) - \sigma z_p, \sigma]$. It follows that

$$\begin{aligned}\max_\sigma \mathcal{L}[\log(t_e) - \sigma z_p, \sigma] &= \mathcal{L}[\log(t_e) - \tilde{\sigma} z_p, \tilde{\sigma}] \\ &= \mathcal{L}[\log(t_e) + \tilde{\sigma}(z_{p_L} - z_p) - \tilde{\sigma} z_p, \tilde{\sigma}] \\ &= \mathcal{L}[\log(t_e + \delta) - \tilde{\sigma} z_p, \tilde{\sigma}] < k \end{aligned} \tag{16}$$

where $\delta = t_e\{\exp[\tilde{\sigma}(z_{p_L} - z_p)] - 1\} > 0$ and the inequality in (16) follows from (15). Then from (14) and (16), it follows that

$$p_L = \min\{p : p \text{ satisfying } \max_\sigma \mathcal{L}[\log(t_e) - \sigma z_p, \sigma] \geq k\}$$

which means p_L is the lower ending point of confidence interval for $F(t_e)$ from (12). That is, the upper band for the quantile function is exactly the same as the lower band of the cdf.

References

[1] Minitab, *MINITAB Statistical Software, Release 15 for Windows*, Minitab Inc., State College, Pennsylvania, 2006.

[2] Weibull++, *Life Data Analysis (Weibull Analysis) Software*, ReliaSoft Corporation, http://weibull.reliasoft.com/, 2007.

[3] SAS, *SAS/QC User's Guide, Version 9.1*, SAS Institute Inc, NC, 2003.

[4] W. Q. Meeker and L. A. Escobar, *SPLIDA (S-PLUS Life Data Analysis)*, www.public.iastate.edu/~splida., 2004.

[5] W. Nelson, *Applied Life Data Analysis*. New York: John Wiley and Sons Inc, 1982.

[6] W. Q. Meeker and L. A. Escobar, *Statistical Methods for Reliability Data*. New York: John Wiley and Sons Inc, 1998.

[7] J. F. Lawless, *Statistical Models and Methods for Lifetime Data*, 2nd ed. New York: John Wiley and Sons Inc, 2003.

[8] S.-L. Jeng and W. Q. Meeker, "Comparisons of Weibull distribution approximate confidence intervals procedures for Type I censored data," *Technometrics*, vol. 42, no. 2, pp. 135–148, 2000.

[9] J. Lieblein and M. Zelen, "Statistical investigation of the fatigue life of deep-groove ball bearings," *Journal of Research, National Bureau of Standards*, vol. 57, pp. 273–316, 1956.

Yili Hong received BS in statistics (2004) from University of Science and Technology of China and MS in statistics (2005) from Iowa State University. He is currently a PhD candidate in the Department of Statistics at Iowa State University. His research interests include reliability data analysis and Engineering statistics.

William Q. Meeker is a Professor of Statistics and Distinguished Professor of Liberal Arts and Sciences at Iowa State University. He is a Fellow of the American Statistical Association, an elected member of the International Statistical Institute, and a past Editor of *Technometrics*. He is co-author of the books *Statistical Methods for Reliability Data* with Luis Escobar (1998), and *Statistical Intervals: A Guide for Practitioners* with Gerald Hahn (1991), nine book chapters, and of numerous publications in the engineering and statistical literature. He has won the American Society for Quality (ASQ) Youden prize four times, the ASQ Wilcoxon Prize three times, and the American Statistical Association Best Practical Application Award. He is also an ASQ Shewhart Medalist. He has consulted extensively on problems in reliability data analysis, reliability test planning, accelerated testing, nondestructive evaluation, and statistical computing.

Luis A. Escobar is a Professor in the Department of Experimental Statistics, Louisiana State University. His research and consulting interests include statistical analysis of reliability data, accelerated testing, survival analysis, linear and non-linear models. His publications have appeared in the engineering and statistical literature.