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Bounds for Faber coefficients of functions univalent in an ellipse

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Bounds for Faber coefficients of functions univalent in an ellipse

by

Engin Haliloğlu

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DEDICATION

This thesis is dedicated to my father.

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CHAPTER 1. INTRODUCTION

Functions Univalent in the Unit Disc

In this section we introduce the class S of univalent functions and some important related classes. We also give examples of the use of extreme point theory in solving linear extremal problems.

Definition 1.1 *A function $f(z)$ is called univalent in a domain $\Omega \subset \mathbf{C}$ if $f(z)$ is defined in Ω and $f(z_1) \neq f(z_2)$ for each pair of points z_1 and z_2 with $z_1 \neq z_2$ in Ω.*

Definition 1.2 *A function $f(z)$ is said to be in the class S if it is analytic and univalent in the unit disc $\Delta(0,1) = \{z : |z| < 1\}$, and satisfies the conditions $f(0) = 0$ and $f'(0) = 1$.*

The study of the class S seems to have started with Koebe at the beginning of this century. Koebe [12] proved the existence of an absolute constant r such that the disk $\{|z| < r\}$ is contained in $f(\Delta(0,1))$ for every $f(z) \in S$. Bieberbach [2,3] found the best possible value of r to be $\frac{1}{4}$. In 1916, Bieberbach [3] showed that if $f(z) \in S$ is given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

then $|a_2| \leq 2$. He then conjectured that in addition,

$$|a_n| \leq n, \quad (n = 2, 3, 4, \cdots).$$

This conjecture was one of the most challenging open problems of mathematics for 68 years. It was proved by L. deBranges [4] in 1984. deBranges also proved that equality holds in (1.2) if and only if $f(z)$ is given by

$$f(z) = e^{-i\theta}k(e^{i\theta}), \quad \theta \in [0, 2\pi),$$

where

$$k(z) = \frac{z}{(1-z)^2}.$$

The function $k(z)$ is known as the Koebe function.

We now define some subclasses of S that have nice geometric properties and serve as "test cases" for problems about S.

Definition 1.3 *A function $f(z) \in S$ is called convex if $f(\Delta(0,1))$ is a convex set. We denote the class of convex functions by C.*

Definition 1.4 *A function $f(z) \in S$ is called typically real if $f(z)$ is real for real values of z and nonreal for nonreal values of z. We denote the class of typically real functions by T.*

Definition 1.5 *The class of odd functions in S is denoted by $S^{(2)}$.*

The class C is closely related to the following subclass of analytic functions.

Definition 1.6 *A function $p(z)$ analytic in $\Delta(0,1)$ and satisfying the conditions $p(0) = 1$ and $Re\{p(z)\} > 0$ for $z \in \Delta(0,1)$ is said to be of positive real part. We denote the class of such functions by P.*

In this thesis we will study the problem of maximizing certain linear functionals over some compact subsets of functions analytic in $\Delta(0,1)$. It was shown that [5] extreme point theory is useful in solving this type of problems.

Definition 1.7 *Let A be a subset of a vector space X and $f \in A$. We say f is an extreme point of A if when $f_1, f_2 \in A$ and $0 < t < 1$ with*

$$f = tf_1 + (1-t)f_2,$$

then $f_1 = f_2$. We denote extreme points of A by $extA$.

Definition 1.8 *Let X be a topological vector space. Then the closed convex hull of $A \subset X$, denoted by $\overline{co}(A)$, is the smallest closed convex set containing A.*

The following theorem (See [17]) shows that to find the maximum value of a linear functional over a compact set it suffices to find the maximum over the set of extreme points.

Theorem 1.1 *Let X be a locally convex linear topological space and let $K \subset X$ be a compact set. If $l(x)$ is a continuous linear functional on X, then*

$$\max_{x\in K} Re\{l(x)\} = \max_{x \in extK} Re\{l(x)\}.$$

If, in addition, $\overline{co}(K)$ is compact, then

$$\max_{x\in K} Re\{l(x)\} = \max_{x \in ext(\overline{co}(K))} Re\{l(x)\}.$$

The following theorem of Brannan, Clunie, and Kirwan [5] identifies $ext(\overline{co}(P))$.

Theorem 1.2 *The extreme points of the closed convex hull of P consists of functions $p_\theta(z)$ given by*

$$p_\theta(z) = \frac{1+e^{i\theta}z}{1-e^{i\theta}z}, \qquad \theta \in [0, 2\pi). \tag{1.2}$$

The following two theorems due to L. Brickman, T. H. Mac Gregor, and D. R. Wilken [6] determine $\mathrm{ext}(\overline{\mathrm{co}}(C))$ and $\mathrm{ext}(\overline{\mathrm{co}}(T))$, respectively.

Theorem 1.3 *The extreme points of the closed convex hull of C consists of functions $c_\theta(z)$ given by*

$$c_\theta(z) = \frac{z}{1-e^{i\theta}z}, \qquad \theta \in [0, 2\pi). \tag{1.3}$$

Theorem 1.4 *The extreme points of the closed convex hull of T consists of functions $t_\theta(z)$ given by*

$$t_\theta(z) = \frac{z}{1-2z\cos\theta+z^2}, \qquad \theta \in [0, \pi]. \tag{1.4}$$

The problem of determining the collection of all extreme points of $S^{(2)}$ and S are still open problems.

The following three theorems are immediate consequences of applications of the Theorems 1.2, 1.3, and 1.4, respectively.

Theorem 1.5 *If*

$$f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \in P, \tag{1.5}$$

then

$$|b_n| \le 2, \quad (n = 1, 2, 3, \cdots) \tag{1.6}$$

This inequality is sharp.

Theorem 1.6 *If $f(z) \in C$ is given by (1.1), then*

$$|a_n| \leq 1, \quad (n = 2, 3, 4, \cdots). \tag{1.7}$$

Strict inequality holds in (1.7) unless $f(z) = c_\theta(z)$, for some $\theta \in [0, 2\pi)$.

Theorem 1.7 *If $f(z) \in T$ is given by (1.1), then*

$$|a_n| \leq n, \quad (n = 2, 3, 4, \cdots). \tag{1.8}$$

This inequality is sharp for each fixed n.

The coefficient problem for the class $S^{(2)}$ is still an open problem. Littlewood and Paley [14] showed that for functions $f(z) = z + \sum_{n=1}^{\infty} a_{2n+1} z^{2n+1} \in S^{(2)}$ there exists an absolute constant such that $|a_{2n+1}| \leq A$, $(n = 1, 2, 3, \cdots)$. Then they went on to conjecture that $|a_{2n+1}| \leq 1$, $(n = 1, 2, 3, \cdots)$. This conjecture was proved to be false at least for $n = 2$, by Fekete and Szegö [10]. Milin [15] has shown that $|a_{2n+1}| \leq 1.14$, $(n = 1, 2, 3, \cdots)$. This is the best global result known, though sharp results for some individual coefficients have been found.

Faber Polynomials

In this section we give a brief discussion of the Faber polynomials associated with a simply connected and bounded domain $\Omega \subset \mathbf{C}$ with analytic boundary.

Definition 1.9 *Define Σ to be the class of functions $g(z)$ analytic and univalent in $\Delta = \{z : |z| > 1\}$ with*

$$g(z) = z + \sum_{n=0}^{\infty} \frac{c_n}{z^n}, \quad (z \in \Delta). \tag{1.9}$$

Definition 1.10 *Let Ω be a simply connected, bounded domain in* $\mathbf{C}$ *with* $0 \in \Omega$. *Assume Ω has capacity 1 and a function $g(z) \in \Sigma$ is the unique mapping of Δ onto* $\mathbf{C} \setminus \overline{\Omega}$. *Then the Faber polynomials, $\{\Phi_n(z)\}_{n=0}^{\infty}$, associated with Ω (or $g(z)$) are defined by the generating function relation [7, p.218]*

$$\frac{\xi g'(\xi)}{g(\xi) - z} = \sum_{n=0}^{\infty} \Phi_n(z)\xi^{-n} \tag{1.10}$$

Substituting (1.9) into (1.10) and equating like powers of ξ yields

$$\Phi_0(z) = 1, \tag{1.11}$$

$$\Phi_1(z) = z - c_0, \tag{1.12}$$

and the recursion relation,

$$\Phi_{n+1}(z) = (z - c_0)\Phi_n(z) - \sum_{k=1}^{n-1} c_{n-k}\Phi_k(z) - (n+1)c_n, \quad (n = 1, 2, 3, \cdots) \tag{1.13}$$

Hence $\Phi_n(z)$ is a monic polynomial of degree n

The following theorem [18, p.130] provides another proof that $\Phi_n(z)$ is a monic polynomial of degree n.

Theorem 1.8 *Let $\{\Phi_n(z)\}_{n=0}^{\infty}$ be the Faber polynomials associated with $g \in \Sigma$. Then $\Phi_n(z)$ is the principal part of $(g^{-1}(z))^n$ at $z = \infty$, i.e.*

$$(g^{-1}(z))^n = \Phi_n(z) + O\left(\frac{1}{z}\right) \quad as \quad z \to \infty. \tag{1.11}$$

Proof: For $r > 1$, let $\Gamma_r = \{g(re^{i\theta}),\ \theta \in [0, 2\pi)\}$ and Ω_r be the inside of Γ_r. (Note $\Omega_1 = \Omega$.) Let $1 < r < \rho < \infty$. By the Cauchy Integral Formula, it follows from (1.10) that

$$\Phi_n(z) = \frac{1}{2\pi i}\int_{|\xi|=\rho} \frac{\xi^n g'(\xi)}{g(\xi) - z}\, d\xi, \qquad z \in \overline{\Omega_r}.$$

Then

$$\Phi_n(z) = \frac{1}{2\pi i}\int_{\Gamma_{r_2}} \frac{(g^{-1}(w))^n}{w-z}\,dw. \tag{1.15}$$

Since $g(\infty)=\infty$ and $g'(\infty)=1$ we may write

$$g^{-1}(z) = z + \sum_{n=0}^{\infty} \frac{d_n}{z^n},$$

for sufficiently large values of z. Thus the function $\dfrac{(g^{-1}(w))^n}{w-z}$ has a pole of order n at $w=\infty$, and we may write (1.15) as

$$\Phi_n(z) = \frac{1}{2\pi i}\int_{|w|=R} \frac{(g^{-1}(w))^n}{w-z}\,dw, \tag{1.16}$$

where R is sufficiently large. Writing

$$(g^{-1}(w))^n = w^n + D_1^{(n)}w^{n-1} + \cdots + D_n^{(n)} + \frac{D_{-1}^{(n)}}{w} + \frac{D_{-2}^{(n)}}{w^2} + \cdots \tag{1.17}$$

and

$$\frac{1}{w-z} = \frac{1}{w} + \frac{z}{w^2} + \cdots + \frac{z^n}{w^{n+1}} + \cdots$$

and then taking the product we obtain from (1.16) that

$$\Phi_n(z) = z^n + D_1^{(n)}z^{n-1} + \cdots + D_n^{(n)}. \tag{1.18}$$

Substituting (1.18) into (1.17) completes the proof of the theorem.

We conclude this section by giving three examples of Faber polynomials:

Example 1.0.1 *Let*

$$g(z) = z.$$

Then

$$g(\Delta) = \Delta = \mathbf{C} \setminus \Delta(0,1).$$

Substituting $g(\xi) = \xi$ into (1.10) yields

$$\frac{\xi}{\xi - z} = \frac{1}{1 - \frac{z}{\xi}} = \sum_{n=0}^{\infty} z^n \xi^{-n} = \sum_{n=0}^{\infty} \Phi_n(z)\xi^{-n}.$$

Hence the Faber polynomials associated with $\Delta(0,1)$ are $\{z^n\}_{n=0}^{\infty}$.

Example 1.0.2 *Let*

$$g(z) = z + \frac{1}{4z}$$

and $r > 1$. Then

$$g(re^{i\theta}) = re^{i\theta} + \frac{1}{4r}e^{-i\theta} = (r + \frac{1}{4r})\cos\theta + i(r - \frac{1}{4r})\sin\theta, \quad \theta \in [0, 2\pi).$$

Hence $g(re^{i\theta})$ is a parametric representation of an ellipse with foci at the points ± 1. We deduce that

$$g(\Delta) = \mathbf{C} \setminus \overline{E} \quad \textit{where}$$

$$E = \{x + iy : \frac{x^2}{(5/4)^2} + \frac{y^2}{(3/4)^2} < 1\}.$$

Substituting $g(\xi) = \xi + \dfrac{1}{4\xi}$ into (1.10) gives

$$\frac{\xi g'(\xi)}{g(\xi) - z} = \frac{4\xi^2 - 1}{4\xi^2 - 4\xi z + 1} = 1 + \frac{z\xi - \frac{1}{2}}{(\xi - \xi_1)(\xi - \xi_2)}$$

where

$$\xi_1 = \frac{z + \sqrt{z^2 - 1}}{2} \quad \textit{and} \quad \xi_2 = \frac{z - \sqrt{z^2 - 1}}{2}.$$

Then we have

$$\frac{\xi g'(\xi)}{g(\xi) - z} = 1 + \frac{1}{\xi_1 - \xi_2}\left(\frac{\xi_1 z - \frac{1}{2}}{\xi - \xi_1} - \frac{\xi_2 z - \frac{1}{2}}{\xi - \xi_2}\right). \tag{1.19}$$

Substituting

$$\xi_1 - \xi_2 = \sqrt{z^2 - 1},$$

$$\xi_1 z - \frac{1}{2} = \frac{z^2 - 1 + z\sqrt{z^2-1}}{2} = \xi_1\sqrt{z^2-1},$$

and

$$\xi_2 z - \frac{1}{2} = \frac{z^2 - 1 - z\sqrt{z^2-1}}{2} = -\xi_2\sqrt{z^2-1}$$

into (1.19) gives

$$\begin{aligned} \frac{\xi g'(\xi)}{g(\xi) - z} &= 1 + \frac{1}{\xi}\left(\frac{\xi_1}{1-\frac{\xi_1}{\xi}} + \frac{\xi_2}{1-\frac{\xi_2}{\xi}}\right) \\ &= 1 + \sum_{n=1}^{\infty} ({\xi_1}^n + {\xi_2}^n)\xi^{-n}. \end{aligned}$$

Hence it follows from (1.10) that

$$\Phi_0(z) = 1,$$

$$\Phi_n(z) = {\xi_1}^n + {\xi_2}^n = 2^{-n}[(z+\sqrt{z^2-1})^n + (z-\sqrt{z^2-1})^n], \qquad (n = 1, 2, 3, \cdots).$$

We see that the Faber polynomials associated with E are the monic Chebyshev polynomials of (-1,1).

Example 1.0.3 *Let*

$$g(z) = z + \frac{1}{2z^2}.$$

Then $g(\Delta)$ is the exterior of the three-cusped hypocycloid (shown in Figure 1.1) whose parametric equation is given by

$$z = e^{i\theta} + \frac{1}{2}e^{-2i\theta}, \qquad 0 \le \theta < 2\pi.$$

Using (1.11), (1.12), and (1.13) we find that the Faber polynomials, $\{\Phi_n(z)\}_{n=0}^{\infty}$, associated with $g(z) = z + \dfrac{1}{2z^2}$ are given by

$$\Phi_0(z) = 1,$$

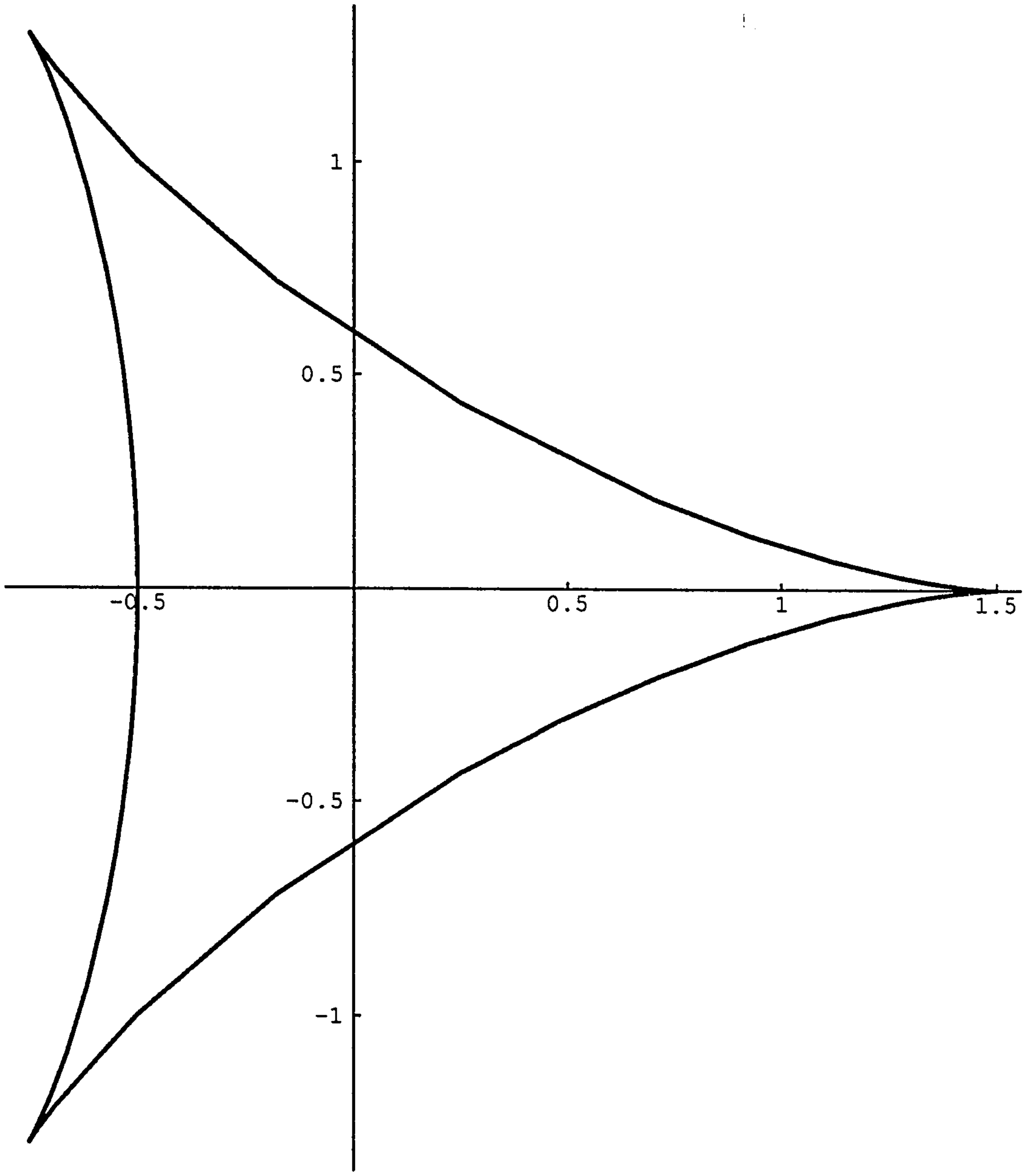

Figure 1.1: Three Cusped Hypocycloid

$$\Phi_1(z) = z,$$

$$\Phi_2(z) = z^2,$$

$$\Phi_3(z) = z^3 - \frac{3}{2},$$

and

$$\Phi_{n+1}(z) = z\Phi_n(z) - \frac{1}{2}\Phi_{n-2}(z), \qquad (n = 3, 4, 5, \cdots).$$

Faber Series

In this section we discuss how Faber polynomials may be used to represent analytic functions in more general domains than disks.

Theorem 1.9 *([14, p.42]) Let $g \in \Sigma$ and $\{\Phi_n(z)\}_{n=0}^{\infty}$ be the Faber polynomials associated with g. Suppose $F(z)$ is a function analytic in Ω_r = inside of $g(|z| = r)$, for some $r > 1$.*

(i) Then $F(z)$ has the representation

$$F(z) = \sum_{n=0}^{\infty} A_n \Phi_n(z), \qquad z \in \Omega_r \tag{1.20}$$

where

$$A_n = \frac{1}{2\pi i}\int_{|z|=\rho} F(g(z)) z^{-n-1}\, dz \quad \text{for any } \rho \text{ with } \quad 1 < \rho < r. \tag{1.21}$$

In addition,

$$\limsup_{n\to\infty} |A_n|^{1/n} \leq \frac{1}{r}.$$

(ii) The series given in (1.20) converges uniformly on compact subsets of Ω_r.

(iii) The expansion of $F(z)$ as Faber series given by (1.20) is unique.

(iv) Conversly, if $\{A_n\}_{n=0}^{\infty}$ is a sequence with $\limsup_{n\to\infty} |A_n|^{1/n} \le \frac{1}{r}$ *then the series* $\sum_{n=0}^{\infty} A_n\Phi_n(z)$ *represents an analytic function in* Ω_r.

Proof: (i) Let $1 < \rho < r$ and let $\Gamma_\rho = g(|z| = \rho)$. For $z \in \overline{\Omega_\rho}$, the Cauchy Integral Formula gives

$$\begin{aligned} F(z) &= \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{F(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{|\xi|=\rho} \frac{F(g(\xi))g'(\xi)}{g(\xi)-z}\,d\xi \\ &= \frac{1}{2\pi i}\int_{|\xi|=\rho} F(g(\xi))\Big(\sum_{n=0}^{\infty}\Phi_n(z)\xi^{-n-1}\Big)\,d\xi \\ &= \sum_{n=0}^{\infty} A_n\Phi_n(z) \end{aligned}$$

where A_n is given by (1.21). Interchange of integration and summation is justified by uniform convergence of the series $\sum_{n=0}^{\infty}\Phi_n(z)\xi^{-n-1}$ to $\dfrac{g'(\xi)}{g(\xi)-z}$ for $|\xi| = \rho$ and z in a compact subset of Ω_ρ. Since

$$M_\rho = \max_{|z|=\rho} |F(g(z))| < \infty$$

it follows from (1.21) that

$$\limsup_{n\to\infty} |A_n|^{1/n} \le \frac{1}{\rho}\limsup_{n\to\infty}\,(\max_{|z|=\rho}|F(g(z))|)^{1/n} = \frac{1}{\rho}.$$

Letting $\rho \to r$ completes the proof of (i).

(ii) Let $K \subset \Omega_r$ be a compact set. Choose ρ so that $K \subset \Omega_\rho$. Then

$$|\Phi_n(z)| = \left|\frac{1}{2\pi i}\int_{|\xi|=\rho} \frac{\xi^n g'(\xi)}{g(\xi)-z}\,d\xi\right| \le M\rho^{n+1}$$

where

$$M = \max_{|\xi|=\rho,\ z\in K}\left|\frac{g'(\xi)}{g(\xi)-z}\right|.$$

Now choose ρ' so that $\rho < \rho' < r$. Then as in the proof of part (i)

$$|A_n| \le \frac{M_{\rho'}}{\rho'^n}.$$

Therefore if $z \in K$, then

$$|A_n \Phi_n(z)| \leq \frac{M_{\rho'}}{\rho'^n} M \rho^{n+1}.$$

Hence, by the Weierstrass M-test, the series $\sum_{n=0}^{\infty} A_n \Phi_n(z)$ converges uniformly on compact subsets of Int Γ_ρ. Letting $\rho \to r$ completes the proof.

(iii) Let

$$F(z) = \sum_{n=0}^{\infty} A_n \Phi_n(z) = \sum_{n=0}^{\infty} B_n \Phi_n(z).$$

Then for $1 < \rho < r$

$$\frac{1}{2\pi i} \int_{|z|=\rho} \left[\sum_{n=0}^{\infty} (A_n - B_n) \Phi_n(g(z)) \right] z^{-m-1} \, dz = 0. \tag{1.22}$$

From (1.14) we have

$$\Phi_n(g(z)) = z^n + O\left(\frac{1}{z}\right) \quad \text{as } z \to \infty. \tag{1.23}$$

Substituting (1.23) into (1.22) yields

$$A_m - B_m = 0, \quad \forall m \geq 0.$$

This proves assertion (iii).

(iv) Let

$$A(z) = \sum_{n=0}^{\infty} A_n z^n.$$

Since $\limsup_{n \to \infty} |A_n|^{1/n} \leq \frac{1}{r}$ the series $\sum_{n=0}^{\infty} A_n z^n$ converges uniformly for $|z| < r$ and $A(z)$ is analytic in $\Delta(0, r)$. For each ρ with $1 < \rho < r$, the function

$$\frac{1}{2\pi i} \int_{|\xi|=\rho} \frac{A(\xi) g'(\xi)}{g(\xi) - z} \, d\xi$$

is analytic for $z \in \Omega_\rho$. Since the series for $A(z)$ converges uniformly for $|z| = \rho$, it follows that

$$\begin{aligned} \frac{1}{2\pi i}\int_{|\xi|=\rho} \frac{A(\xi)g'(\xi)}{g(\xi)-z}\,d\eta &= \sum_{n=0}^{\infty} A_n \frac{1}{2\pi i}\int_{|\xi|=\rho} \xi^n [\sum_{k=1}^{\infty} \Phi_k(z)\xi^{-k-1}]\,d\xi \\ &= \sum_{n=0}^{\infty} A_n \Phi_n(z). \end{aligned}$$

Hence the series $\sum_{n=0}^{\infty} A_n\Phi_n(z)$ converges uniformly to an analytic function in Ω_ρ. Letting $\rho \to r$ completes the proof.

Functions Univalent in a Simply Connected and Bounded Domain

In this section we introduce analogues of the class S and some of its related classes for a general simply connected and bounded domain Ω. Afterwards we state the main results of this thesis.

Let Ω be a simply connected and bounded domain in $\mathbf{C}$ with $\partial\Omega$ analytic. Assume that $0 \in \Omega$. Let $\varphi(z)$ be the unique, one-to-one, and analytic mapping of Ω onto $\Delta(0,1)$ with $\varphi(0) = 0$ and $\varphi'(0) > 0$. We define classes analogues to the classes S, P, C, T, and $S^{(2)}$ for Ω as follows:

Definition 1.11 *Let $S(\Omega)$ denote the class of functions $F(z)$ analytic and univalent in Ω and satisfying $F(0) = 0$ and $F'(0) = 1$.*

Definition 1.12 *A function $P(z)$ analytic in Ω is said to be of positive real part if $P(0) = \dfrac{1}{\varphi'(0)}$ and $Re\{P(z)\} > 0$ for $z \in \Omega$. We use $P(\Omega)$ to denote the class of such functions. (The condition $P(0) = \dfrac{1}{\varphi'(0)}$, instead of $P(0) = 1$, is imposed for convenience.)*

Definition 1.13 *A function $F(z) \in S(\Omega)$ is called convex if $F(\Omega)$ is a convex domain in* $\mathbf{C}$. *We denote the class of convex functions in Ω by $C(\Omega)$.*

Definition 1.14 *Assume that Ω is symmetric about the real axis. Then a function $F(z) \in S(\Omega)$ is called typically real if $F(z)$ is real for real values of $z \in \Omega$ and nonreal for nonreal values of $z \in \Omega$. We denote the class of typically real functions in Ω by $T(\Omega)$.*

Definition 1.15 *If Ω is symmetric about origin then we use $S^{(2)}(\Omega)$ to denote the class of odd functions $F(z) \in S(\Omega)$.*

Assume that Ω has capacity 1, $\partial\Omega$ is analytic, and $g(\Delta) = \mathbf{C} \setminus \overline{\Omega}$ for some $g(z) \in \Sigma$. Then by Theorem 1.9 the functions $F(z)$ in the classes defined above have Faber expansions $F(z) = \sum_{n=0}^{\infty} A_n \Phi_n(z)$ which converge uniformly on compact subsets of Ω.

Assume that Ω is symmetric with respect to both the real axis and origin. Then we note that if $F(z)$ is in one of the classes defined above then $F(z)$ may be written as

$$F(z) = \frac{f(\varphi(z))}{\varphi'(0)} \tag{1.24}$$

for some function $f(z)$ in the corresponding class of $\Delta(0, 1)$. Therefore for the Faber series of $F(z)$ we use the notation

$$F(z) = \sum_{n=0}^{\infty} A_n(f)\Phi_n(z) \tag{1.25}$$

where $f(z)$ is the function characterized by (1.24). Now we pose the following problem:

Problem: If a function $F(z)$, is in one of the classes defined above, has the Faber expansion given by (1.25), then what can be said about the Faber coefficients $\{A_n(f)\}_{n=0}^{\infty}$? (Note that the case $\Omega = \Delta(0,1)$ reduces to the problems discussed in 1.1, in particular, to the Bieberbach Conjecture and related problems.)

To investigate this problem we must deal with both exterior and interior mappings for Ω. This is very difficult for general domains. Therefore we focus on the elliptical domain E, given in Example 1.0.2, for which exterier and interior mappings are known.

The following theorem [16, p.296] gives the required interior mapping.

Theorem 1.10 *The function*

$$\varphi(z) = \sqrt{k_0}\, sn\left(\frac{2K}{\pi}\sin^{-1} z; \frac{1}{16}\right)$$

is the one-to-one, analytic mapping of E onto $\Delta(0,1)$ with $\varphi(0) = 0$ and $\varphi'(0) = \dfrac{2K\sqrt{k_0}}{\pi} > 0$.

In the next two chapters we investigate bounds for the Faber coefficients of functions $F(z)$ in the classes $S(E)$, $P(E)$, $C(E)$, $T(E)$, and $S^{(2)}(E)$.

For information about Jacobi elliptic functions see the Appendix.

CHAPTER 2. BOUNDS FOR FABER COEFFICIENTS OF FUNCTIONS UNIVALENT IN AN ELLIPSE

We define $S(E)$ as the class of functions $F(z)$ which are analytic and univalent on E and normalized by the conditions $F(0) = 0$ and $F'(0) = 1$. We define two subclasses of $S(E)$ as

$$C(E) = \{F(z) \in S(E) : F(E) \text{ is convex}\},$$

and

$$S^{(2)}(E) = \{F(z) \in S(E) : F(z) \text{ is odd}\}.$$

In addition, we let $P(E)$ denote the class of functions analytic in E and satisfying the conditions $F(0) = \dfrac{1}{\varphi'(0)} = \dfrac{2K\sqrt{k_0}}{\pi}$ and $Re\{F(z)\} > 0$. (The condition $F(0) = \dfrac{1}{\varphi'(0)}$ is imposed for convenience.)

Note that if $F(z)$ is in one of the classes defined above then $F(z)$ may be written as in (1.24) for some $f(z)$ in the corresponding class of $\Delta(0,1)$.

Let the Faber expansions of functions $F(z)$ in the classes indicated above be given by (1.25), where the relation between $F(z)$ and $f(z)$ is given by (1.24). In this chapter, we obtain sharp bounds for the Faber coefficients $A_0(f)$, $A_1(f)$, and $A_2(f)$ for functions in the classes $S(E)$, $C(E)$, and $P(E)$. In each case, equality is attained for functions given by

$$k(z) = k_0(z) = \frac{z}{(1-z)^2}, \tag{2.1}$$

$$c(z) = c_0(z) = \frac{z}{1-z}, \tag{2.2}$$

or

$$p(z) = p_0(z) = \frac{1+z}{1-z}. \tag{2.3}$$

For functions in $S^{(2)}(E)$ a sharp bound for $A_1(f)$ is obtained and the corresponding extremal function in $S^{(2)}$ is shown to be

$$o(z) = \frac{z}{1-z^2}. \tag{2.4}$$

In addition, sharp bounds for certain linear combinations of the Faber coefficients of functions in the classes given above are obtained in Theorems 2.4, 2.5, and 2.6. Extremal functions in the unit disc are given by (2.1), (2.2), (2.3), and (2.4), as before. These theorems lead us to make conjectures for the Faber coefficients $A_n(f)$ of the functions $F(z)$ in the classes $S(E)$, $C(E)$, $P(E)$, and $S^{(2)}(E)$.

Main Results

We will work with the following expansion for the Faber coefficients, $\{A_n\}_{n=0}^{\infty}$, of functions analytic in E.

Lemma 2.1 *If $F(z)$ is analytic in E and has the Faber series given by (1.20) then the Faber coefficients $\{A_n\}_{n=0}^{\infty}$ are given by*

$$A_n = \frac{2^n}{\pi} \int_0^{\pi} F(\cos\theta) \cos n\theta \, d\theta, \quad (n = 0, 1, 2, \cdots)$$

Proof: We have from (1.20)

$$F(\cos\theta) = \sum_{n=0}^{\infty} A_n \Phi_n(\cos\theta). \tag{2.5}$$

where $\Phi_n(x)$ is the monic Chebyshev polynomial of degree n. Substituting

$$\Phi_n(\cos\theta) = 2^{1-n}\cos n\theta$$

into (2.5) gives

$$F(\cos\theta) = \sum_{n=0}^{\infty} A_n 2^{1-n}\cos n\theta. \tag{2.6}$$

Multiplying (2.6) by $\cos m\theta$ and then integrating from 0 to π completes the proof of the lemma.

Corollary 2.1 *If the functions in the classes $S(E)$, $C(E)$, $S^{(2)}(E)$, and $P(E)$ have the Faber expansions, given by (1.25), then the Faber coefficients, $\{A_n(f)\}_{n=0}^{\infty}$, are given by*

$$A_n(f) = \frac{2^n}{2K\sqrt{k_0}}\int_0^{\pi} f(\varphi(\cos\theta))\cos n\theta\, d\theta, \quad (n = 0, 1, 2, \cdots) \tag{2.7}$$

Proof: Corollary 2.1 follows from the Lemma 2.1 and the relation (1.24), since $\varphi'(0) = \dfrac{2K\sqrt{k_0}}{\pi}$.

In the next corollary we show that for functions in $S^{(2)}(E)$, the even Faber coefficients are 0.

Corollary 2.2 *If $F(z) \in S^{(2)}(E)$ then $A_{2n}(f) = 0$, $(n = 0, 1, 2, \cdots)$.*

Proof: From (2.7) we have

$$A_{2n}(f) = \frac{2^{2n}}{2K\sqrt{k_0}}\int_0^{\pi/2} [f(\varphi(\cos\theta)) + f(\varphi(-\cos\theta))]\cos 2n\theta\, d\theta, \quad (n = 0, 1, 2, \cdots).$$

Thus $A_{2n}(f) = 0$, $(n = 0, 1, 2, \cdots)$, because both $f(z)$ and $\varphi(z)$ are odd functions.

Another representation formula for the Faber coefficients, $\{A_n(f)\}_{n=0}^{\infty}$, is given in the following corollary.

Corollary 2.3 *With the same conditions as in the Corollary 2.1.*

$$A_n(f) = \frac{2^{2n} n!}{2K\sqrt{k_0}(2n)!} \int_0^{\pi} \frac{d^n}{dx^n}(f(\varphi(x)))|_{x=\cos\theta} \sin^{2n}\theta \, d\theta, \quad (n = 0, 1, 2, \cdots). \qquad (2.8)$$

Proof: Multiplying the identity

$$\frac{\Phi_n(x)}{\sqrt{1-x^2}} = \frac{(-1)^n 2^{1-n}}{1 \cdot 3 \cdots (2n-1)} \frac{d^n}{dx^n}[(1-x^2)^{n-\frac{1}{2}}]$$

[1, p.785] by $f(\varphi(x))$ and then integrating from -1 to 1 we obtain

$$\int_{-1}^{1} \frac{\Phi_n(x)}{\sqrt{1-x^2}} f(\varphi(x))\, dx = \frac{(-1)^n 2^{1-n}}{1 \cdot 3 \cdots (2n-1)} \int_{-1}^{1} \frac{d^n}{dx^n}[(1-x^2)^{n-\frac{1}{2}}] f(\varphi(x))\, dx. \qquad (2.9)$$

Integrating the right hand side of (2.9) by parts results in

$$\int_{-1}^{1} \frac{\Phi_n(x)}{\sqrt{1-x^2}} f(\varphi(x))\, dx = \frac{(-1)^{n-1} 2^{1-n}}{1 \cdot 3 \cdots (2n-1)} \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}[(1-x^2)^{n-\frac{1}{2}}](f(\varphi(x)))'\, dx.$$

Continuing this process n-times yields

$$\int_{-1}^{1} \frac{\Phi_n(x)}{\sqrt{1-x^2}} f(\varphi(x))\, dx = \frac{2^{1-n}}{1 \cdot 3 \cdots (2n-1)} \int_{-1}^{1} \frac{d^n}{dx^n}(f(\varphi(x)))(1-x^2)^{n-\frac{1}{2}}\, dx. \qquad (2.10)$$

The result follows from Corollary 2.1 by letting $x = \cos\theta$ in (2.10).

Theorem 2.1 *If $k(z)$, $c(z)$, and $p(z)$ are given by (2.1), (2.2), and (2.3), respectively, then we have*

$$|A_0(f)| \le A_0(k), \quad f \in S \qquad (2.11)$$

$$|A_0(f)| \le A_0(c), \quad f \in C \qquad (2.12)$$

$$|A_0(f)| \le A_0(p), \quad f \in P. \qquad (2.13)$$

Proof: From (2.7) we have

$$A_0(f) = \frac{1}{2K\sqrt{k_0}} \int_0^{\pi} f(\varphi(\cos\theta))\, d\theta.$$

Since $\varphi(z)$ is an odd function we may write

$$A_0(f) = \frac{1}{2K\sqrt{k_0}} \int_0^{\pi/2} [f(\varphi(\cos\theta)) + f(-\varphi(\cos\theta))]\, d\theta. \tag{2.14}$$

Substituting (1.1) into (2.14) yields

$$A_0(f) = \frac{1}{K\sqrt{k_0}} \int_0^{\pi/2} (\sum_{n=1}^{\infty} a_{2n}\varphi^{2n}(\cos\theta))\, d\theta.$$

Thus

$$|A_0(f)| \le \frac{1}{K\sqrt{k_0}} \int_0^{\pi/2} (\sum_{n=1}^{\infty} |a_{2n}|\varphi^{2n}(\cos\theta))\, d\theta$$

since $\varphi(x) \ge 0$ for $x \in [0,1]$. Hence (2.11) follows from the Bieberbach conjecture as

$$|A_0(f)| \le \frac{1}{K\sqrt{k_0}} \int_0^{\pi/2} (\sum_{n=1}^{\infty} 2n\varphi^{2n}(\cos\theta))\, d\theta = A_0(k)$$

In a similar way, the proof of (2.12) follows from the coefficient estimate (1.7).

Substituting (1.5) into (2.14) gives

$$A_0(f) = \frac{1}{K\sqrt{k_0}} \int_0^{\pi/2} [1 + \sum_{n=1}^{\infty} b_{2n}\varphi^{2n}(\cos\theta)]\, d\theta$$

Thus

$$|A_0(f)| \le \frac{1}{K\sqrt{k_0}} \int_0^{\pi/2} [1 + \sum_{n=1}^{\infty} |b_{2n}|\varphi^{2n}(\cos\theta)]\, d\theta \tag{2.15}$$

since $\varphi(x) \ge 0$ for $x \in [0,1]$. Using the coefficient estimate (1.6) in (2.15) yields (2.13) as

$$|A_0(f)| \le \frac{1}{K\sqrt{k_0}} \int_0^{\pi/2} [1 + 2\sum_{n=1}^{\infty} \varphi^{2n}(\cos\theta)]\, d\theta = A_0(p).$$

Theorem 2.2 *If $k(z)$, $c(z)$, $p(z)$, and $o(z)$ are given by (2.1), (2.2), (2.3) and (2.4), respectively, then*

$$|A_1(f)| \leq A_1(k), \quad f \in S \tag{2.16}$$

$$|A_1(f)| \leq A_1(c), \quad f \in C \tag{2.17}$$

$$|A_1(f)| \leq A_1(p), \quad f \in P \tag{2.18}$$

$$|A_1(f)| \leq A_1(o), \quad f \in S^{(2)}. \tag{2.19}$$

Proof: From (2.7) we have

$$A_1(f) = \frac{1}{K\sqrt{k_0}} \int_0^{\pi/2} [f(\varphi(\cos\theta)) - f(-\varphi(\cos\theta))] \cos\theta \, d\theta, \tag{2.20}$$

since $\varphi(x)$ is an odd function. Substituting (1.1) into (2.20) gives

$$A_1(f) = \frac{2}{K\sqrt{k_0}} \int_0^{\pi/2} [\varphi(\cos\theta) + \sum_{n=1}^{\infty} a_{2n+1}\varphi^{2n+1}(\cos\theta)] \cos\theta \, d\theta.$$

Hence

$$|A_1(f)| \leq \frac{2}{K\sqrt{k_0}} \int_0^{\pi/2} [\varphi(\cos\theta) + \sum_{n=1}^{\infty} |a_{2n+1}|\varphi^{2n+1}(\cos\theta)] \cos\theta \, d\theta,$$

since $\varphi(x) \geq 0$ for $x \in [0, 1]$. As in the proof of Theorem 2.1, inequalities (2.16) and (2.17) result from applying coefficient estimates (1.8) and (1.7), respectively. Similarly, if $f(z) \in P$ is given by (1.5) then

$$|A_1(f)| \leq \frac{2}{K\sqrt{k_0}} \int_0^{\pi/2} [\varphi(\cos\theta) + \sum_{n=1}^{\infty} |b_{2n+1}|\varphi^{2n+1}(\cos\theta)] \cos\theta \, d\theta. \tag{2.21}$$

since $\varphi(x) \geq 0$ for $x \in [0, 1]$. Hence using (1.6) in (2.21) results in (2.18).

For $f \in S^{(2)}$, (2.20) gives

$$A_1(f) = \frac{2}{K\sqrt{k_0}} \int_0^{\pi/2} f(\varphi(\cos\theta)) \cos\theta \, d\theta.$$

Thus

$$A_1(f) \leq \frac{2}{K\sqrt{k_0}} \int_0^{\pi/2} |f(\varphi(\cos\theta))| \cos\theta \, d\theta. \tag{2.22}$$

By the distortion theorem

$$|f(z)| \leq \frac{|z|}{1-|z|^2}, \quad f \in S^{(2)}, \tag{2.23}$$

it follows from (2.22) that

$$|A_1(f)| \leq \frac{1}{K\sqrt{k_0}} \int_0^{\pi/2} \frac{\varphi(\cos\theta)}{1-\varphi^2(\cos\theta)} \cos\theta \, d\theta = A_1(o)$$

because $0 \leq \varphi(x) < 1$ for $x \in [0,1]$.

Remark: We can also obtain (2.16), (2.17), and (2.18) by applying (2.8) instead of (2.7), and noting that $\varphi'(x)|_{x=\cos\theta} \geq 0$, since $\varphi(x)$ is increasing for $x \in [-1,1]$. Using $\varphi'(\cos\theta) \geq 0$ for $\theta \in [0,\pi]$ and the distortion theorem,

$$|f'(z)| \leq \frac{1+|z|^2}{1-|z|^2}, \quad f \in S^{(2)}$$

in (2.8) leads to (2.19).

Theorem 2.3 *If $k(z)$, $c(z)$, and $p(z)$ are given by (2.1), (2.2), and (2.3), respectively, then*

$$|A_2(f)| \leq A_2(k), \quad f \in S \tag{2.24}$$

$$|A_2(f)| \leq A_2(c), \quad f \in C \tag{2.25}$$

$$|A_2(f)| \leq A_2(p), \quad f \in P. \tag{2.26}$$

Proof: From (2.7) we have

$$A_2(f) = \frac{2}{K\sqrt{k_0}} \int_0^{\pi} f(\varphi(\cos\theta)) \cos 2\theta \, d\theta.$$

Then

$$A_2(f) = \frac{2}{K\sqrt{k_0}} \int_0^{\pi/2} [f(\varphi(\cos\theta)) + f(-\varphi(\cos\theta))] \cos 2\theta \, d\theta.$$

Hence

$$A_2(f) = \frac{2}{K\sqrt{k_0}} \int_0^{\pi/4} \{[f(\varphi(\cos\theta))+f(-\varphi(\cos\theta))]-[f(\varphi(\sin\theta))+f(-\varphi(\sin\theta))]\} \cos 2\theta \, d\theta. \tag{2.27}$$

Substituting (1.1) into (2.27) gives

$$A_2(f) = \frac{4}{K\sqrt{k_0}} \int_0^{\pi/4} [\sum_{n=1}^{\infty} a_{2n}(\varphi^{2n}(\cos\theta) - \varphi^{2n}(\sin\theta))] \cos 2\theta \, d\theta.$$

Since $\varphi(x) \geq 0$ and $\varphi(x)$ is increasing for $x \in [0,1]$, we have

$$\varphi^{2n}(\cos\theta) - \varphi^{2n}(\sin\theta) \geq 0, \quad (n = 1,2,3,\cdots) \text{ for } 0 \leq \theta \leq \frac{\pi}{4}.$$

Thus

$$|A_2(f)| \leq \frac{4}{K\sqrt{k_0}} \int_0^{\pi/4} [\sum_{n=1}^{\infty} |a_{2n}|(\varphi^{2n}(\cos\theta) - \varphi^{2n}(\sin\theta))] \cos 2\theta \, d\theta. \tag{2.28}$$

Using (1.8) in (2.28) gives (2.24). In a similar way, (2.25) is obtained from the coefficient estimate (1.7). If $f(z) \in P$ is given by (1.5) then (2.26) follows from

$$|A_2(f)| \leq \frac{4}{K\sqrt{k_0}} \int_0^{\pi/4} [1 + \sum_{n=1}^{\infty} |b_{2n}|(\varphi^{2n}(\cos\theta) - \varphi^{2n}(\sin\theta))] \cos 2\theta \, d\theta$$

by using the coefficient estimate (1.6).

Theorem 2.4 *If $k(z)$, $c(z)$, and $p(z)$ are given by (2.1), (2.2), and (2.3), respectively, then for $n = 0,1,2,\cdots$, we have*

$$|A_0(f) \pm 2^{-2n} A_{2n}(f)| \leq A_0(k) \pm 2^{-2n} A_{2n}(k), \quad f \in S, \tag{2.29}$$

$$|A_0(f) \pm 2^{-2n} A_{2n}(f)| \leq A_0(c) \pm 2^{-2n} A_{2n}(c), \quad f \in C, \tag{2.30}$$

and

$$|A_0(f) \pm 2^{-2n} A_{2n}(f)| \leq A_0(p) \pm 2^{-2n} A_{2n}(p), \quad f \in P. \tag{2.31}$$

Proof: To prove (2.28) let $f \in S$ be given by (1.1) and consider

$$I_n = \int_0^\pi f(\varphi(\cos\theta))(1 \pm \cos 2n\theta)\, d\theta, \quad (n = 0, 1, 2, \cdots).$$

Then

$$\begin{aligned} I_n &= \int_0^{\pi/2} [f(\varphi(\cos\theta)) + f(-\varphi(\cos\theta))](1 \pm \cos 2n\theta)\, d\theta \\ &= 2\int_0^{\pi/2} [\sum_{m=1}^{\infty} a_{2m} \varphi^{2m}(\cos\theta)](1 \pm \cos 2n\theta)\, d\theta. \end{aligned}$$

Thus

$$|I_n| \leq 2\int_0^{\pi/2} [\sum_{m=1}^{\infty} |a_{2m}| \varphi^{2m}(\cos\theta)](1 \pm \cos 2n\theta)\, d\theta. \tag{2.32}$$

Hence (2.29) follows from (2.32) by using the coefficient estimate (1.8).

Inequalities (2.30) and (2.31) may be proved in the same way by applying inequalities (1.7) and (1.6), respectively. Note that the case $n = 0$ (with + sign) in Theorem 2.4 yields Theorem 2.1.

Theorem 2.5 *If $k(z)$, $c(z)$, $p(z)$, and $o(z)$ are given by (2.1), (2.2), (2.3), and (2.4), respectively, then for $n = 0, 1, 2, \cdots$, we have*

$$\begin{aligned} |A_1(f) \pm 2^{-2n-2} A_{2n+1}(f) \pm 2^{-2n} A_{2n-1}(f)| &\leq \\ A_1(k) \pm 2^{-2n-2} A_{2n+1}(k) \pm 2^{-2n} A_{2n-1}(k), &\qquad f \in S, \end{aligned} \tag{2.33}$$

$$\begin{aligned} |A_1(f) \pm 2^{-2n-2} A_{2n+1}(f) \pm 2^{-2n} A_{2n-1}(f)| &\leq \\ A_1(c) \pm 2^{-2n-2} A_{2n+1}(c) \pm 2^{-2n} A_{2n-1}(c), &\qquad f \in C, \end{aligned} \tag{2.34}$$

$$|A_1(f) \pm 2^{-2n-2}A_{2n+1}(f) \pm 2^{-2n}A_{2n-1}(f)| \leq$$

$$A_1(p) \pm 2^{-2n-2}A_{2n+1}(p) \pm 2^{-2n}A_{2n-1}(p), \qquad f \in P, \tag{2.35}$$

$$|A_1(f) \pm 2^{-2n-2}A_{2n+1}(f) \pm 2^{-2n}A_{2n-1}(f)| \leq$$

$$A_1(o) \pm 2^{-2n-2}A_{2n+1}(o) \pm 2^{-2n}A_{2n-1}(o), \qquad f \in S^{(2)}. \tag{2.36}$$

where $A_{-1}(f) = 0$.

Proof: Let $f \in S$ be given by (1.1) and consider

$$L_n = \int_0^{\pi} f(\varphi(\cos\theta))\cos\theta(1 \pm \cos 2n\theta)\,d\theta.$$

Then

$$\begin{aligned} L_n &= \int_0^{\pi/2}[f(\varphi(\cos\theta)) - f(-\varphi(\cos\theta))]\cos\theta(1 \pm \cos 2n\theta)\,d\theta \\ &= 2\int_0^{\pi/2}[\varphi(\cos\theta) + \sum_{m=1}^{\infty} a_{2m+1}\varphi^{2m+1}(\cos\theta)]\cos\theta(1 \pm \cos 2n\theta)\,d\theta. \end{aligned}$$

Hence

$$|L_n| \leq 2\int_0^{\pi/2}[\varphi(\cos\theta) + \sum_{m=1}^{\infty} |a_{2m+1}|\varphi^{2m+1}(\cos\theta)]\cos\theta(1 \pm \cos 2n\theta)\,d\theta. \tag{2.37}$$

Using (1.8) in (2.37) yields (2.33).

In a similar way, (2.34) and (2.35) result from the inequalities (1.7) and (1.6), respectively.

If $f \in S^{(2)}$ then

$$L_n = 2\int_0^{\pi/2} f(\varphi(\cos\theta))\cos\theta(1 \pm \cos 2n\theta)\,d\theta.$$

Thus

$$|L_n| \leq 2\int_0^{\pi/2} |f(\varphi(\cos\theta))|\cos\theta(1 \pm \cos 2n\theta)\,d\theta. \tag{2.38}$$

So (2.36) follows from (2.38), by using the distortion theorem (2.23).

Note that the case $n = 0$ (with + sign) in Theorem 2.5 yields Theorem 2.2.

Theorem 2.6 *If $k(z)$, $c(z)$, $p(z)$, and $o(z)$ are given by (2.1), (2.2), (2.3), and (2.4), respectively, then for $n = 0, 1, 2, \cdots$, we have*

$$|(2n+1)^2 A_1(f) \pm 2^{-2n-1} A_{2n+1}(f)| \le (2n+1)^2 A_1(k) \pm 2^{-2n-1} A_{2n+1}(k), \quad f \in S,$$

$$|(2n+1)^2 A_1(f) \pm 2^{-2n-1} A_{2n+1}(f)| \le (2n+1)^2 A_1(c) \pm 2^{-2n-1} A_{2n+1}(c), \quad f \in C,$$

$$|(2n+1)^2 A_1(f) \pm 2^{-2n-1} A_{2n+1}(f)| \le (2n+1)^2 A_1(p) \pm 2^{-2n-1} A_{2n+1}(p), \quad f \in P,$$

$$|(2n+1)^2 A_1(f) \pm 2^{-2n-1} A_{2n+1}(f)| \le (2n+1)^2 A_1(o) \pm 2^{-2n-2} A_{2n+1}(o), \quad f \in S^{(2)}.$$

Proof: Let

$$M_n = \int_0^{\pi} f(\varphi(\cos\theta))[(2n+1)^2 \cos\theta \pm \cos(2n+1)\theta]\, d\theta.$$

Then Theorem 2.6 is proved by using the argument of Theorem 2.3 and noting that

$$(2n+1)^2 \cos\theta \pm \cos(2n+1)\theta \ge 0, \quad \theta \in [0, 2\pi).$$

We note that the case $n = 0$ (with + sign) in Theorem 2.6 gives Theorem 2.2.

Theorems 2.1, 2.2, 2.3, 2.4, 2.5, and 2.6 lead us to make the following conjectures:

Conjecture 2.1 $|A_n(f)| \le A_n(k)$, $(n = 0, 1, 2, \cdots)$, $f \in S$.

Conjecture 2.2 $|A_n(f)| \le A_n(c)$, $(n = 0, 1, 2, \cdots)$, $f \in C$.

Conjecture 2.3 $|A_n(f)| \le A_n(p)$, $(n = 0, 1, 2, \cdots)$, $f \in P$.

Conjecture 2.4 $|A_n(f)| \le A_n(o)$, $(n = 0, 1, 2, \cdots)$, $f \in S^{(2)}$.

CHAPTER 3. ON THE FABER COEFFICIENTS OF FUNCTIONS UNIVALENT IN AN ELLIPSE

In this chapter, we obtain sharp bounds for the Faber coefficients $\{A_n(f)\}_{n=0}^{\infty}$ of functions $F(z)$ in the classes $C(E)$, $T(E)$, and $P(E)$. We also show that equality holds if and only if $f(z) = c(z)$, $f(z) = k(z)$, and $f(z) = p(z)$, where $c(z)$, $k(z)$, and $p(z)$ are given by (2.2), (2.1), and (2.3), respectively. Hence, in all three cases, extremal functions are unique unlike the analogues results for the $\Delta(0,1)$. This shows that the Conjectures 2 and 3 made in Chapter 2 are true. In addition, we restate Conjectures 1 and 4 made in Chapter 2 explicitly by evaluating $A_n(k)$ and $A_n(o)$ where $k(z)$ is the Koebe function given by (2.1) and $o(z)$ is given by (2.4).

Main Results

Let $\mathcal{F}$ denote one of the sets C, T, and P. Then $\mathcal{F}$ is a compact set. Hence the closed convex hull of $\mathcal{F}$, $\overline{\mathrm{co}}(\mathcal{F})$, is also compact and since $A_n(f)$ is a continuous linear functional

$$M = \max_{f \in \overline{\mathrm{co}}(\mathcal{F})} |A_n(f)|$$

exists. In addition, we have

$$\max_{f \in \mathcal{F}} |A_n(f)| = \max_{\mathrm{ext}(\overline{\mathrm{co}}(\mathcal{F}))} |A_n(f)|. \tag{3.1}$$

Using (3.1) with Theorems 1.3, 1.4, and 1.2, we see that the problem of maximizing $|A_n(f)|$ over the classes C, T, and P reduces to the problem of maximizing the values of $|A_n(c_\theta)|$ ($\theta \in [0, 2\pi)$), $|A_n(t_\theta)|$ ($\theta \in [0, \pi]$), and $|A_n(p_\theta)|$ ($\theta \in [0, 2\pi)$) over θ, respectively.

In the following theorems we evaluate the values of $A_n(c_\theta)$, $A_n(t_\theta)$, and $A_n(p_\theta)$, where $A_n(f)$ is given by (2.7). We need to use different countours for different quadrants of θ. So each theorem includes one quadrant of θ.

Theorem 3.1 *If $c_\theta(z)$ is given by (1.3) then*

$$A_n(c_\theta) = \frac{\pi^2 e^{-i\theta}(e^{in\alpha(\theta)} - 2^{-2n}e^{-in\alpha(\theta)})}{4K^2\sqrt{k_0}\,(1-2^{-4n})(1+k_0^2-2k_0\cos 2\theta)^{1/2}}, \quad 0 \le \theta \le \frac{\pi}{2}, \quad (n = 0, 1, 2, \cdots)$$

where $0 \le \alpha(\theta) \le \frac{\pi}{2}$ is given by

$$\varphi[\cos(\alpha(\theta) + \frac{\pi\tau}{4})] = e^{-i\theta}, \quad 0 \le \theta \le \frac{\pi}{2} \quad \textit{with} \quad \tau = \frac{4i\ln 2}{\pi}.$$

Proof: The function $\cos z$ maps the rectangle R with vertices at the points $-\frac{\pi\tau}{4}$, $\pi - \frac{\pi\tau}{4}$, $\pi + \frac{\pi\tau}{4}$, and $\frac{\pi\tau}{4}$ onto E. Therefore the function $\varphi(\cos z)$ maps R onto $\Delta(0, 1)$ with

$$\varphi[\cos(\alpha(t) + \frac{\pi\tau}{4})] = e^{-it}, \qquad 0 \le t \le \frac{\pi}{2} \tag{3.2}$$

where $\alpha(t)$ increases from 0 to $\frac{\pi}{2}$ as t increases from 0 to $\frac{\pi}{2}$.

Integrate the function $h(z) = c_\theta(\varphi(\cos z))e^{inz}$ over the parallelogram $ABCD$ with vertices at the points $-\pi$, π, $\pi\tau$, and $\pi\tau - 2\pi$, respectively. From (3.2) we see that $\alpha(\theta) + \frac{\pi\tau}{4}$ is a pole of $h(z)$ inside $ABCD$.

Let

$$iK' = K\tau$$

and refer to $\mathrm{sn}\left(z; \frac{1}{16}\right)$ as $\mathrm{sn}\, z$ for convenience. Then

$$\varphi(\cos(\pi\tau - z)) = \sqrt{k_0}\,\mathrm{sn}\left(\frac{2K}{\pi}\left(\frac{\pi}{2} - \pi\tau + z\right)\right) = \sqrt{k_0}\,\mathrm{sn}\left(\frac{2K}{\pi}\left(\frac{\pi}{2} + z\right)\right),$$

since $\mathrm{sn}\, z$ is doubly periodic with periods $2iK'$ and $4K$. Thus

$$\varphi(\cos z) = \varphi(\cos(-z)) = \varphi(\cos(\pi\tau - z)). \tag{3.3}$$

It follows from (3.3) that $-\alpha(\theta) + \frac{3\pi\tau}{4}$ is the other pole of $h(z)$ inside $ABCD$. So by the residue theorem

$$\oint_{ABCD} h(z)\,dz = 2\pi i(\mathrm{Res}_{\alpha(\theta)+\frac{\pi\tau}{4}} + \mathrm{Res}_{-\alpha(\theta)+\frac{3\pi\tau}{4}}) \tag{3.4}$$

where Res_{z_0} denotes the residue of the function $h(z)$ at the point $z = z_0$.

The contribution of the integrals on BC and DA cancel each other because $h(z)$ is a periodic function with period 2π. Now

$$\int_{AB} h(z)\,dz = \int_{-\pi}^{\pi} h(x)\,dx = 2\int_0^{\pi} c_\theta(\varphi(\cos x))\cos nx\,dx \tag{3.5}$$

and

$$\int_{CD} h(z)\,dz = \int_{2\pi}^{0} h(x + \pi\tau - 2\pi)\,dx = -\int_0^{2\pi} h(x + \pi\tau)\,dx$$

From (3.3) we obtain

$$\int_{CD} h(z)\,dz = -\int_0^{2\pi} e^{in(x+\pi\tau)}c_\theta(\varphi(\cos x))\,dx = 2\cdot 2^{-4n}\int_0^{\pi} c_\theta(\varphi(\cos x))\cos nx\,dx. \tag{3.6}$$

Then adding (3.5) and (3.6) results in

$$\oint_{ABCD} h(z)\,dz = 2(1 - 2^{-4n})\int_0^{\pi} c_\theta(\varphi(\cos x))\cos nx\,dx. \tag{3.7}$$

To evaluate $\mathrm{Res}_{\alpha(\theta)+\frac{\pi\tau}{4}}$, expand the function $c_\theta(\sqrt{k_0}\,\mathrm{sn}(u+u_0))$ about $u=0$, where

$$u_0 = \frac{2K}{\pi}\left(\frac{\pi}{2} - \alpha(\theta) - \frac{\pi\tau}{4}\right). \tag{3.8}$$

The addition formula for $\mathrm{sn}\,u$ [8, p.33] yields

$$\sqrt{k_0}\,\mathrm{sn}\,(u+u_0) = \frac{\sqrt{k_0}\,\mathrm{sn}\,u\,\mathrm{cn}\,u_0\,\mathrm{dn}\,u_0 + \sqrt{k_0}\,\mathrm{sn}\,u_0\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{1-k_0^2\,\mathrm{sn}^2 u_0\,\mathrm{sn}^2 u} \tag{3.9}$$

where $\mathrm{cn}\,z$ and $\mathrm{dn}\,z$ refer to $\mathrm{cn}\left(z;\dfrac{1}{16}\right)$ and $\mathrm{dn}\left(z;\dfrac{1}{16}\right)$, respectively. It follows from (3.2) that

$$\sqrt{k_0}\,\mathrm{sn}\,u_0 = e^{-i\theta}, \qquad 0 \le \theta \le \frac{\pi}{2}. \tag{3.10}$$

To evaluate $\mathrm{cn}\,u_0$ and $\mathrm{dn}\,u_0$ employ the identities

$$\mathrm{sn}^2 z + \mathrm{cn}^2 z = 1 \quad [8, \mathrm{p}.25] \tag{3.11}$$

and

$$k_0^2\mathrm{sn}^2 z + \mathrm{dn}^2 z = 1 \quad [8, \mathrm{p}.25]. \tag{3.12}$$

To determine whether to use + or − sign for $\mathrm{cn}\,u_0$ and $\mathrm{dn}\,u_0$ check the signs of $Re\{\mathrm{cn}\,(x-\frac{iK'}{2})\}$ and $Re\{\mathrm{dn}\,(x-\frac{iK'}{2})\}$, respectively. Deduce from the addition formulas for $\mathrm{cn}\,u$ and $\mathrm{dn}\,u$ [8, p.34]

$$\mathrm{cn}\left(x-\frac{iK'}{2}\right) = \sqrt{\frac{1+k_0}{k_0}}\,\frac{\mathrm{cn}\,x + i\,\mathrm{sn}\,x\,\mathrm{dn}\,x}{1+k_0\,\mathrm{sn}^2 x}$$

and

$$\mathrm{dn}\left(x-\frac{iK'}{2}\right) = \frac{\sqrt{1+k_0}\,(\mathrm{dn}\,x + ik_0\,\mathrm{sn}\,x\,\mathrm{cn}\,x)}{1+k_0\,\mathrm{sn}^2 x}.$$

Thus $Re\{\mathrm{cn}\,(x-\frac{iK'}{2})\} \ge 0$ and $Re\{\mathrm{dn}\,(x-\frac{iK'}{2})\} \ge 0$ for $x \in [0, K]$ since $\mathrm{cn}\,x$ decreases from 1 to 0 and $\mathrm{dn}\,x$ decreases from 1 to $\sqrt{1-k_0^2}$ for $x \in [0, K]$. Hence using (3.11) and (3.12) we obtain

$$\operatorname{cn} u_0 = \sqrt{1 - \frac{e^{-2i\theta}}{k_0}}$$

and

$$\operatorname{dn} u_0 = \sqrt{1 - k_0 e^{-2i\theta}}.$$

Choosing the principal branch as $-\pi < \arg z \leq \pi$ we obtain

$$0 \leq \arg(\operatorname{cn} u_0) \leq \frac{\pi}{2}$$

and

$$0 \leq \arg(\operatorname{dn} u_0) \leq \frac{\pi}{4}.$$

Therefore

$$0 \leq \arg(\operatorname{cn} u_0 \operatorname{dn} u_0) \leq \frac{3\pi}{4}$$

which implies

$$\sqrt{k_0}\operatorname{cn} u_0 \operatorname{dn} u_0 = ie^{-i\theta}(1 + k_0^2 - 2k_0\cos 2\theta)^{1/2}. \tag{3.13}$$

Using

$$\operatorname{sn} u = u - \frac{1}{3!}(1 + k_0^2)u^3 + \cdots \quad [8,\ \text{p.37}], \tag{3.14}$$

$$\operatorname{cn} u = 1 - \frac{1}{2!}u^2 + \cdots \quad [8,\ \text{p.37}], \tag{3.15}$$

$$\operatorname{dn} u = 1 - \frac{1}{2!}k_0^2 u^2 + \cdots \quad [8,\ \text{p.37}], \tag{3.16}$$

and (3.13) in (3.9) and doing necessary calculations result in

$$\sqrt{k_0}\operatorname{sn}(u + u_0) = e^{-i\theta} + ie^{-i\theta}(1 + k_0^2 - 2k_0\cos 2\theta)^{1/2}u + \cdots. \tag{3.17}$$

Thus

$$c_\theta(\sqrt{k_0}\operatorname{sn}(u + u_0)) = \frac{ie^{-i\theta}}{(1 + k_0^2 - 2k_0\cos 2\theta)^{1/2}u} + \cdots$$

or

$$c_\theta\left(\sqrt{k_0}\,\mathrm{sn}\left(\frac{2K}{\pi}\left(\frac{\pi}{2}-z\right)\right)\right) = -\frac{\pi i e^{-i\theta}}{2K(1+k_0^2-2k_0\cos 2\theta)^{1/2}(z-\alpha(\theta)-\frac{\pi\tau}{4})}+\cdots.$$

Hence we obtain

$$\mathrm{Res}_{\alpha(\theta)+\frac{\pi\tau}{4}} = -\frac{\pi i e^{-i\theta}2^{-n}e^{in\alpha(\theta)}}{2K(1+k_0^2-2k_0\cos 2\theta)^{1/2}}. \tag{3.18}$$

In a similar way, residue of $h(z)$ at the point $-\alpha(\theta)+\dfrac{3\pi\tau}{4}$ may be obtained as

$$\mathrm{Res}_{-\alpha(\theta)+\frac{3\pi\tau}{4}} = \frac{\pi i e^{-i\theta}2^{-3n}e^{-in\alpha(\theta)}}{2K(1+k_0^2-2k_0\cos 2\theta)^{1/2}}. \tag{3.19}$$

Substituting (3.18) and (3.19) into (3.4) yields

$$\oint_{ABCD} h(z)\,dz = \frac{\pi^2 e^{-i\theta}2^{-n}}{K(1+k_0^2-2k_0\cos 2\theta)^{1/2}}(e^{in\alpha(\theta)}-2^{-2n}e^{-in\alpha(\theta)}). \tag{3.20}$$

Comparing (3.7) and (3.20) gives the desired result.

For θ in other quadrants, proofs are similar to the proof of Theorem 3.1. Therefore we will state the theorems and then in the proofs indicate only the integration countours and poles of $h(z)$ inside the countours.

Theorem 3.2 *If $c_\theta(z)$ is given by (1.3) then*

$$A_n(c_\theta) = \frac{(-1)^n\pi^2 e^{-i\theta}(e^{-in\alpha(\pi-\theta)}-2^{-2n}e^{in\alpha(\pi-\theta)})}{4K^2\sqrt{k_0}\,(1-2^{-4n})(1+k_0^2-2k_0\cos 2\theta)^{1/2}},\quad \frac{\pi}{2}\le\theta\le\pi,\quad (n=0,1,2,\cdots),$$

where $\alpha(\theta)$ is as in Theorem 3.1.

Proof: Integrate the function $h(z) = c_\theta(\varphi(\cos z))e^{inz}$ over the parallelogram $ABCD$ with vertices at the points 0, 2π, $3\pi+\pi\tau$, $\pi+\pi\tau$. Inside $ABCD$ there are two poles of $h(z)$ at the points $\pi-\alpha(\pi-\theta)+\dfrac{\pi\tau}{4}$ and $\pi+\alpha(\pi-\theta)+\dfrac{3\pi\tau}{4}$.

Theorem 3.3 *If $c_\theta(z)$ is given by (1.3) then*

$$A_n(c_\theta) = \frac{(-1)^n \pi^2 e^{-i\theta}\left(e^{in\alpha(\theta-\pi)} - 2^{-2n} e^{-in\alpha(\theta-\pi)}\right)}{4K^2\sqrt{k_0}\,(1-2^{-4n})(1+k_0^2-2k_0\cos 2\theta)^{1/2}}, \quad \pi \le \theta \le \frac{3\pi}{2}, \quad (n = 0,1,2,\cdots),$$

where $\alpha(\theta)$ is as in Theorem 3.1.

Proof: Integrate the function $h(z) = c_\theta(\varphi(\cos z))e^{inz}$ over the parallelogram $ABCD$ with vertices at the points 0, 2π, $3\pi - \pi\tau$, and $\pi - \pi\tau$. Inside $ABCD$, there are two poles of $h(z)$ at the points $\pi - \alpha(\theta-\pi) - \frac{\pi\tau}{4}$ and $\pi + \alpha(\theta - \pi) - \frac{3\pi\tau}{4}$.

Theorem 3.4 *If $c_\theta(z)$ is given by (1.3) then*

$$A_n(c_\theta) = \frac{\pi^2 e^{-i\theta}\left(e^{-in\alpha(2\pi-\theta)} - 2^{-2n} e^{in\alpha(2\pi-\theta)}\right)}{4K^2\sqrt{k_0}\,(1-2^{-4n})(1+k_0^2-2k_0\cos 2\theta)^{1/2}}, \quad \frac{3\pi}{2} \le \theta < 2\pi, \quad (n = 0,1,2,\cdots),$$

where $\alpha(\theta)$ are is in Theorem 3.1.

Proof: Integrate the function $h(z) = c_\theta(\varphi(\cos z))e^{inz}$ over the parallelogram $ABCD$ with vertices at the points $-\pi$, π, $-\pi\tau$, and $-2\pi - \pi\tau$. Two poles of $h(z)$ occur at the points $\alpha(2\pi - \theta) - \frac{\pi\tau}{4}$ and $-\alpha(2\pi-\theta) - \frac{3\pi\tau}{4}$.

Theorem 3.5 *If $p_\theta(z)$ is given by (1.2) then*

$$A_n(p_\theta) = 2A_n(c_\theta), \qquad 0 \le \theta < 2\pi, \qquad (n = 0,1,2,\cdots).$$

The proof is similar to the proofs of Theorems 3.1-3.4.

Since the function

$$t_\theta(z) = \frac{z}{1 - 2z\cos\theta + z^2}, \qquad 0 \le \theta \le \pi$$

has a double pole at $\theta = 0$ we will treat this case separately. (Note that for $\theta = 0$, $t_\theta(z)$ becomes the Koebe function.)

Theorem 3.6 *If $k(z)$ is given by (2.1) then*

$$A_n(k) = \frac{\pi^3 n}{8K^3\sqrt{k_0}\,(1-k_0)^2(1-2^{-2n})}, \qquad (n = 1, 2, \cdots).$$

Proof: Integrate the function $h(z) = k(\varphi(\cos z))e^{inz}$ over the countour used in the proof of Theorem 3.1. Inside the parallelogram $ABCD$, there are two double poles of $h(z)$ at the points $\frac{\pi\tau}{4}$ and $\frac{3\pi\tau}{4}$. So by the residue theorem

$$\oint_{ABCD} h(z)\,dz = 2\pi i(\mathrm{Res}_{\frac{\pi\tau}{4}} + \mathrm{Res}_{\frac{3\pi\tau}{4}}). \tag{3.21}$$

As in Theorem 3.1,

$$\oint_{ABCD} h(z)\,dz = 2(1-2^{-4n})\int_0^\pi k(\varphi(\cos x))\cos nx\,dx. \tag{3.22}$$

To find $\mathrm{Res}_{\frac{\pi\tau}{4}}$ expand $k\left(\sqrt{k_0}\,\mathrm{sn}\left(u + K - \frac{iK'}{2}\right)\right)$ about $u = 0$. Doing necessary calculations we obtain

$$k\left(\sqrt{k_0}\,\mathrm{sn}\left(\frac{2K}{\pi}\left(\frac{\pi}{2} - z\right)\right)\right) = -\frac{\pi^2}{4K^2(1-k_0)^2(z - \frac{\pi\tau}{4})^2} + \frac{0}{(z - \frac{\pi\tau}{4})} + \cdots. \tag{3.23}$$

Writing

$$e^{inz} = 2^{-n}e^{in(z-\frac{\pi\tau}{4})} = 2^{-n}[1 + in\left(z - \frac{\pi\tau}{4}\right) + \cdots] \tag{3.24}$$

and multiplying (3.23) by (3.24) yields

$$\mathrm{Res}_{\frac{\pi\tau}{4}} = -2^{-n}in\frac{\pi^2}{4K^2(1-k_0)^2}. \tag{3.25}$$

In a similar way , residue of $h(z)$ at $z = \frac{3\pi\tau}{4}$ is obtained as

$$\mathrm{Res}_{\frac{3\pi\tau}{4}} = -2^{-3n}in\frac{\pi^2}{4K^2(1-k_0)^2}. \tag{3.26}$$

Substituting (3.25) and (3.26) into (3.21) yields

$$\oint_{ABCD} h(z)\,dz = \frac{\pi^3 n 2^{-n}(1+2^{-2n})}{2K^2(1-k_0)^2}. \tag{3.27}$$

Equating (3.22) and (3.27) and solving for $A_n(k)$ gives the desired result.

Theorem 3.7 *If $t_\theta(z)$ is given by (1.4) then*

$$A_n(t_\theta) = \frac{\pi^2 \sin n\alpha(\theta)}{4K^2\sqrt{k_0}\,(1-2^{-2n})\sin\theta(1+k_0^2-2k_0\cos 2\theta)^{1/2}},\quad 0<\theta\le\frac{\pi}{2},\quad (n=1,2,\cdots)$$

where $\alpha(\theta)$ is as in Theorem 3.1.

Proof: Integrate the function $h(z) = t_\theta(\varphi(\cos z))e^{inz}$ over the rectangle $PQRS$ with vertices at the points $-\pi$, π, $-\pi+\pi\tau$, and $-\pi+\pi\tau$. The poles of $h(z)$ occur at the points z for which

$$\varphi(\cos z) = e^{-i\theta},\qquad 0<\theta\le\frac{\pi}{2} \tag{3.28}$$

and

$$\varphi(\cos z) = e^{i\theta},\qquad 0<\theta\le\frac{\pi}{2} \tag{3.29}$$

We found in Theorem 3.1 that solutions of (3.28) inside $PQRS$ are $\alpha(\theta)+\frac{\pi\tau}{4}$ and $-\alpha(\theta)+\frac{3\pi\tau}{4}$. Similarly, solutions of (3.29) inside $PQRS$ are found to be $-\alpha(\theta)+\frac{\pi\tau}{4}$ and $\alpha(\theta)+\frac{3\pi\tau}{4}$. So by the residue theorem

$$\oint_{PQRS} h(z)\,dz = 2\pi i(\mathrm{Res}_{\alpha(\theta)+\frac{\pi\tau}{4}} + \mathrm{Res}_{-\alpha(\theta)+\frac{3\pi\tau}{4}} + \mathrm{Res}_{-\alpha(\theta)+\frac{\pi\tau}{4}} + \mathrm{Res}_{\alpha(\theta)+\frac{3\pi\tau}{4}}). \tag{3.30}$$

By periodicity of $h(z)$ integrals over QR and SP cancel each other. We have

$$\int_{PQ} h(z)\,dz = 2\int_0^\pi t_\theta(\varphi(\cos x))\cos nx\,dx \tag{3.31}$$

and

$$\int_{RS} h(z)\,dz = \int_\pi^{-\pi} t_\theta(\varphi(\cos(x+\pi\tau)))e^{in(x+\pi\tau)}\,dx. \tag{3.32}$$

Using (3.3) in (3.32) gives

$$\int_{RS} h(z)\,dz = -2\cdot 2^{-4n}\int_0^\pi t_\theta(\varphi(\cos x))\cos nx\,dx. \tag{3.33}$$

Adding (3.31) and (3.33) yields

$$\oint_{PQRS} h(z)\,dz = 2(1-2^{-4n})\int_0^\pi t_\theta(\varphi(\cos x))\cos nx\,dx. \tag{3.34}$$

It follows from (3.17)

$$t_\theta(\sqrt{k_0}\,\mathrm{sn}\,(u+u_0)) = \frac{1}{2\sin\theta(1+k_0^2-2k_0\cos 2\theta)^{1/2}} + \cdots$$

where u_0 is given by (3.8). Hence

$$\mathrm{Res}_{\alpha(\theta)+\frac{\pi\tau}{4}} = -\frac{\pi 2^{-n} e^{in\alpha(\theta)}}{4K\sin\theta(1+k_0^2-2k_0\cos 2\theta)^{1/2}}. \tag{3.35}$$

To find $\mathrm{Res}_{-\alpha(\theta)+\frac{3\pi\tau}{4}}$ expand the function $t_\theta(\sqrt{k_0}\,\mathrm{sn}\,(v+v_0))$ about $v=0$, where

$$v_0 = \frac{2K}{\pi}\left(\frac{\pi}{2}+\alpha(\theta)-\frac{3\pi\tau}{4}\right).$$

The addition formula for the Jacobi elliptic sine function gives

$$\sqrt{k_0}\,\mathrm{sn}\,(v+v_0) = \frac{\sqrt{k_0}\,(\mathrm{sn}\,v\,\mathrm{cn}\,v_0\,\mathrm{dn}\,v_0 + \mathrm{sn}\,v_0\,\mathrm{cn}\,v\,\mathrm{dn}\,v)}{1-k_0^2\,\mathrm{sn}^2 v_0\,\mathrm{sn}^2 v}. \tag{3.36}$$

We have

$$\sqrt{k_0}\,\mathrm{sn}\,v_0 = e^{i\theta}, \qquad 0<\theta\le\frac{\pi}{2}.$$

By using addition formulas for $\mathrm{cn}\,u$ and $\mathrm{dn}\,u$ we can easily show that

$$Re\left\{\mathrm{cn}\left(x-\frac{3iK'}{2}\right)\right\}\ge 0$$

and

$$Re\left\{\mathrm{dn}\left(x-\frac{3iK'}{2}\right)\right\}\le 0$$

for $K<x\le 2K$. Hence it follows from (3.11) and (3.12) that

$$\mathrm{cn}\,v_0 = \sqrt{1-\frac{e^{-2i\theta}}{k_0}}$$

and

$$\operatorname{dn} v_0 = -\sqrt{1 - k_0 e^{-2i\theta}}.$$

Therefore (3.13) results in

$$\sqrt{k_0}\operatorname{cn} v_0 \operatorname{dn} v_0 = -ie^{-i\theta}(1 + k_0^2 - 2k_0\cos 2\theta)^{1/2}. \tag{3.37}$$

Using (3.14), (3.15), (3.16), and (3.37) in (3.36) and doing some manipulation gives

$$\sqrt{k_0}\operatorname{sn}(v + v_0) = e^{-i\theta} - ie^{-i\theta}(1 + k_0^2 - 2k_0\cos 2\theta)^{1/2} v + \cdots.$$

Thus

$$I_\theta(\sqrt{k_0}\operatorname{sn}(v + v_0)) = -\frac{1}{2\sin\theta(1 + k_0^2 - 2k_0\cos 2\theta)^{1/2} v} + \cdots.$$

As a result

$$Res_{-\alpha(\theta)+\frac{3\pi r}{4}} = \frac{\pi 2^{-3n} e^{-in\alpha(\theta)}}{4K\sin\theta(1 + k_0^2 - 2k_0\cos 2\theta)^{1/2}}. \tag{3.38}$$

Choosing a principal branch we obtain

$$\sqrt{k_0 - e^{2i\theta}}\sqrt{1 - k_0 e^{2i\theta}} = -ie^{i\theta}(1 + k_0^2 - 2k_0\cos 2\theta)x^{1/2}. \tag{3.39}$$

Then using the above argument with (3.39) gives

$$Res_{-\alpha(\theta)+\frac{\pi r}{4}} = \frac{\pi 2^{-n} e^{-in\alpha(\theta)}}{4K\sin\theta(1 + k_0^2 - 2k_0\cos 2\theta)^{1/2}} \tag{3.40}$$

and

$$Res_{-\alpha(\theta)+\frac{3\pi r}{4}} = -\frac{\pi 2^{-3n} e^{in\alpha(\theta)}}{4K\sin\theta(1 + k_0^2 - 2k_0\cos 2\theta)^{1/2}}. \tag{3.41}$$

Substituting (3.35), (3.38), (3.40), and 3.41) into (3.30) yields

$$\oint_{PQRS} h(z)\,dz = \frac{\pi^2 2^{-n}(1 + 2^{-2n})\sin n\alpha(\theta)}{K\sin\theta(1 + k_0^2 - 2k_0\cos 2\theta)^{1/2}}. \tag{3.42}$$

Equating (3.34) and (3.42) gives the desired result.

Theorem 3.8 *If $t_\theta(z)$ is given by (2.1) then*

$$A_n(t_\theta)=\frac{(-1)^{n-1}\pi^2\sin n[\alpha(\pi-\theta)]}{K\sin\theta(1+k_0^2-2k_0\cos 2\theta)^{1/2}},\quad \frac{\pi}{2}\le\theta\le\pi,\quad (n=1,2,\cdots)$$

where $\alpha(\theta)$ is as in Theorem 3.1.

Proof: Similar to the proof of Theorem 3.7.

In the following three theorems we obtain sharp bounds for the Faber coefficients of functions in the classes $C(E)$, $P(E)$, and $T(E)$. We first need the following two lemmas:

Lemma 3.1 *If $\alpha(\theta)$ is given by (3.2) then $\alpha'(\theta)$ decreases as θ increases from 0 to $\pi/2$.*

Lemma 3.2 *If $\alpha(\theta)$ is given by (3.2) then*

$$\frac{|\sin n\alpha(\theta)|}{\sin\theta}\le\frac{\pi n}{2K(1-k_0)},\qquad 0\le\theta\le\frac{\pi}{2}.$$

Proof of Lemma 3.1: We have from (3.2)

$$\frac{2K}{\pi}\left(\frac{\pi}{2}-\alpha(\theta)-\frac{\pi\tau}{4}\right)=\mathrm{sn}^{-1}\left(\frac{e^{-i\theta}}{\sqrt{k_0}}\right)=\int_0^{e^{-i\theta}/\sqrt{k_0}}\frac{dt}{\sqrt{1-t^2}\sqrt{1-k_0^2t^2}}.$$

Hence we obtain

$$\alpha'(\theta)=\frac{ie^{-i\theta}}{\sqrt{k_0}}\frac{\pi}{2K}\frac{1}{\sqrt{1-\dfrac{e^{-2i\theta}}{k_0}}\sqrt{1-k_0e^{-2i\theta}}}.$$

Thus it follows from (3.13) that

$$\alpha'(\theta)=\frac{\pi}{2K[(1-k_0)^2+4k_0\sin^2\theta]^{1/2}},\quad 0\le\theta\le\frac{\pi}{2}.$$

Hence

$$\alpha''(\theta) = -\frac{2k_0\pi \sin 2\theta}{K[(1-k_0)^2 + 4k_0 \sin^2\theta]^{3/2}} \le 0$$

since $0 \le \theta \le \frac{\pi}{2}$, and Lemma 3.1 follows.

Proof of Lemma 3.2: Let

$$g(\theta) = \alpha'(0)\sin\theta - \sin\alpha(\theta), \quad 0 \le \theta \le \frac{\pi}{2}.$$

It follows from Lemma 1

$$g'(\theta) = \alpha'(0)\cos\theta - \alpha'(\theta)\cos\alpha(\theta) \ge \alpha'(0)(\cos\theta - \cos\alpha(\theta)). \tag{3.43}$$

Since $\alpha(\theta)$ increases from 0 to $\frac{\pi}{2}$ as θ increases from 0 to $\frac{\pi}{2}$ and $\alpha'(\theta)$ decreases as θ increases from 0 to $\frac{\pi}{2}$ we have

$$\alpha(\theta) \ge \theta, \quad 0 \le \theta \le \pi/2. \tag{3.44}$$

Thus it follows from (3.43) and (3.44) that $g'(\theta) \ge 0$, i.e.

$$\frac{\sin\alpha(\theta)}{\sin\theta} \le \alpha'(0) = \frac{\pi}{2K(1-k_0)}. \tag{3.45}$$

Hence (3.45) yields

$$\frac{|\sin n\alpha(\theta)|}{\sin\theta} = \frac{|\sin n\alpha(\theta)|}{\sin\alpha(\theta)}\frac{\sin\alpha(\theta)}{\sin\theta} \le \frac{n\pi}{2K(1-k_0)}$$

which completes the proof of Lemma 2.

Theorem 3.9 *If* $c(z) = c_0(z) = \frac{z}{1-z}$ *and* $f \in C$, *then*

$$|A_n(f)| \le A_n(c) = \frac{\pi^2}{4K^2\sqrt{k_0}\,(1-k_0)(1+2^{-2n})}, \quad (n = 0, 1, 2, \cdots).$$

Proof: Using (3.1) with $\mathcal{F} = C$ and Theorem 1.3 it is enough to show that

$$\max_{0\le\theta<2\pi} |A_n(c_\theta)| = A_n(c).$$

We will give the proof for only $\theta \in \left[0, \frac{\pi}{2}\right]$. For other values of θ proofs may be given by using either Theorem 3.2 or Theorem 3.3 or Theorem 3.4 depending on the quadrant of θ. Using Theorem 3.1 it suffices to show

$$\frac{(1-2^{-2n})^2 + 4\cdot 2^{-2n}\sin^2 n\alpha(\theta)}{(1-k_0)^2 + 4k_0\sin^2\theta} \le \frac{(1-2^{-2n})^2}{(1-k_0)^2}$$

or equivalently

$$\frac{\sin^2 n\alpha(\theta)}{\sin^2\theta} \le \frac{k_0 2^{2n}(1-2^{-2n})^2}{(1-k_0)^2}, \quad (n = 1, 2, \cdots). \tag{3.46}$$

It follows from Lemma 3.2 that

$$\frac{\sin^2 n\alpha(\theta)}{\sin^2\theta} \le \frac{\pi^2 n^2}{4K^2(1-k_0)^2}, \quad (n = 1, 2, \cdots).$$

Thus the proof is completed if we show

$$\frac{\pi^2 n^2}{4K^2(1-k_0)^2} \le \frac{k_0 2^{2n}(1-2^{-2n})^2}{(1-k_0)^2}, \quad (n = 1, 2, \cdots)$$

or

$$\frac{\pi}{2K\sqrt{k_0}} \le \frac{2^n(1-2^{-2n})}{n}, \quad (n = 1, 2, \cdots).$$

The sequence $\left\{\dfrac{2^n(1-2^{-2n})}{n}\right\}_{n=1}^{\infty}$ is an increasing sequence. Hence

$$\frac{2^n(1-2^{-2n})}{n} \ge \frac{3}{2} > \frac{\pi}{2K\sqrt{k_0}}, \quad (n = 1, 2, \cdots)$$

which completes the proof for $n = 1, 2, \cdots$. We may include $n = 0$ since (3.46) holds trivially in this case.

Theorem 3.10 *If* $p(z) = p_0(z) = \dfrac{1+z}{1-z}$ *and* $f \in P$, *then*

$$|A_n(f)| \leq A_n(p) = 2A_n(c), \qquad (n = 0, 1, 2, \cdots).$$

Proof: Using (3.1) with $\mathcal{F} = P$ and Theorem 1.2 it suffices to show that

$$\max_{0 \leq \theta < 2\pi} |A_n(p_\theta)| = A_n(p_0) = A_n(p).$$

Hence Theorem 3.10 follows from Theorems 3.5 and 3.9.

Theorem 3.11 *If* $k(z)$ *is the Koebe function and* $f \in T$, *then*

$$|A_n(f)| \leq A_n(k) = \frac{\pi^3 n}{8K^3\sqrt{k_0}\,(1-k_0)^2(1-2^{-2n})}, \qquad (n = 1, 2, \cdots).$$

(Note the proof for $n = 0$ *is given in Theorem 2.2.)*

Proof: Using (3.1) with $\mathcal{F} = T$ and Theorem 1.4 it is enough to show that

$$\max_{0 \leq \theta \leq \pi} |A_n(t_\theta)| = A_n(t_0) = A_n(k).$$

We obtain from Theorem 3.7 and Lemma 3.2

$$\begin{aligned} |A_n(t_\theta)| &\leq \frac{\pi^3 n}{8K^3\sqrt{k_0}\,(1-k_0)(1-2^{-2n})(1+k_0^2-2k_0\cos 2\theta)^{1/2}}, \quad 0 < \theta \leq \frac{\pi}{2} \\ &\leq \frac{\pi^3 n}{8K^3\sqrt{k_0}\,(1-k_0)^2(1-2^{-2n})}. \end{aligned}$$

Hence

$$|A_n(t_\theta)| \leq A_n(k), \qquad 0 < \theta \leq \frac{\pi}{2}.$$

Proof for $\frac{\pi}{2} \leq \theta \leq \pi$ follows from Theorem 3.8 in the same way.

Note that here we also showed that

$$\lim_{\theta \to 0} A_n(t_\theta) = A_n(k).$$

Theorems 3.9 and 3.10 show that Conjectures 2 and 3 made in Chapter 2 are true. Theorem 3.6 implies that Conjecture 1 may be written explicitly as

$$|A_n(f)| \leq \frac{\pi^3 n}{8K^3\sqrt{k_0}\,(1-k_0)^2(1-2^{-2n})}, \quad (n = 1, 2, \cdots), \quad f \in S$$

and

$$A_0(f) \leq A_0(k) = 10.5984, \qquad f \in S.$$

Proof of this conjecture for the cases $n = 0, 1, 2$ are given in Theorems 2.1, 2.2, and 2.3, respectively.

To replace Conjecture 4 by an explicit conjecture we evaluate $A_n(o)$ where $o(z)$ is given by (2.4).

Theorem 3.12 *If $o(z)$ is given by (2.4), then*

$$A_{2n-1}(o) = \frac{\pi^2}{4K^2\sqrt{k_0}\,(1-k_0)(1+2^{-4n+2})}, \qquad (n = 1, 2, \cdots).$$

(Note we showed in Corollary 2.2 that if $f \in S^{(2)}$ then $A_{2n}(f) = 0$.)

Proof: Integrate the function $h(z) = o(\varphi(\cos z))e^{inz}$ over the parallelogram with vertices at the points $\pi - \frac{\pi\tau}{2}$, $3\pi - \frac{\pi\tau}{2}$, $\pi + \frac{\pi\tau}{2}$, and $-\pi + \frac{\pi\tau}{2}$. Inside the parallelogram, there are four poles of $h(z)$ at the points $\frac{\pi\tau}{4}$, $\pi + \frac{\pi\tau}{4}$, $\pi - \frac{\pi\tau}{4}$ and $2\pi - \frac{\pi\tau}{4}$. The rest of the proof is similar to the proof of the Theorem 3.7.

We now restate Conjecture 4 as follows:

If $f \in S^{(2)}$ then

$$|A_{2n-1}(f)| \leq \frac{\pi^2}{4K^2\sqrt{k_0}\,(1-k_0)(1+2^{-4n+2})}, \qquad (n = 1, 2, \cdots).$$

Note that the proof of this conjecture for $n = 1$ is given in Theorem 2.2.

CHAPTER 4. CONCLUSION

Summary

In Chapter 2, we found sharp bounds for the Faber coefficients A_0, A_1, and A_2 of functions $F(z)$ in the classes $S(E)$, $C(E)$, and $P(E)$. We also found a sharp bound for the Faber coefficient A_1 of functions $F(z)$ in the class $S^{(2)}(E)$. In addition, we obtained sharp bounds for certain linear combinations of the Faber coefficients of functions $F(z)$ in the classes $S(E)$, $C(E)$, $P(E)$, and $S^{(2)}(E)$.

In Chapter 3, we obtained sharp bounds for the Faber coefficients A_n, $(n = 0, 1, 2, \cdots)$ of functions $F(z)$ in the classes $C(E)$, $P(E)$, and $T(E)$. Then we made conjectures for bounds of the Faber coefficients A_n, $(n = 0, 1, 2, \cdots)$ of functions $F(z)$ in the classes $S(E)$, and $S^{(2)}(E)$.

Future Work

Two conjectures made in Chapter 3 are future research problems. We will mention some other future work problems about "Faber transformations."

Let Ω be bounded, simply connected domain in $\mathbf{C}$ with analytic boundary and let $\{\Phi_n(\zeta)\}_{n=0}^{\infty}$ be the Faber polynomials associated with Ω. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $\Delta(0,1)$, then we define the Faber transformation of $f(z)$ by

$$T(f(z))(\xi) = F(\xi) = \sum_{n=0}^{\infty} a_n \Phi_n(\xi).$$

By Theorem 1.0.9, the Faber transformation of $f(z)$, $F(\xi)$, is analytic in Ω. It is clear that if $f(z)$ is a polynomial of degree n then so is $F(\xi)$. Ellacott [9] showed that if $f(z)$ is a rational function then so is $F(\xi)$. So it is natural to ask whether a property of $f(z)$ is preserved by the Faber transformation. Johnston [11] showed that if $f(z)$ is analytically continuable across a subarc of $|z| = 1$ then so is $F(\xi)$ analytically continuable across a subarc of $\partial\Omega$. Now we ask the following three future work problems:

1. If $f(z) \in H^p(\Delta(0,1))$, $(p > 0)$ (see [8]) then is there a $q > 0$ for which $F(\xi) \in H^q(\Omega)$? If so, what is the best value of q?

2. If $f(z)$ is differentiable on $\overline{\Delta(0,1)}$ then what can be said about differentiability of $F(\xi)$ on $\overline{\Omega}$?

3. If $f(z)$ is univalent in $\Delta(0,1)$ then what can be said about univalence of $F(\xi)$ in Ω?

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APPENDIX JACOBI ELLIPTIC FUNCTIONS

In this section we give a summary of some of the basic properties of the Jacobi elliptic functions. (See [13, Chapters 1 and 2])

We first need to introduce four theta functions:

Theta Functions

Definition *The first theta function $\theta_1(z|\tau)$ for $z \in \mathbf{C}$, $Im\{\tau\} > 0$ is defined by the series*

$$\theta_1(z|\tau) = -i \sum_{n=-\infty}^{\infty} (-1)^n \exp\{i[(n+\frac{1}{2})^2\pi\tau + (2n+1)z]\}.$$

Let q be defined by the equation $q = e^{i\pi\tau}$. Since $Im\{\tau\} > 0$ we have $|q| < 1$. The parameters τ and q are called the parameter and nome, respectively, of the $\theta_1(z|\tau)$. Dependence of $\theta_1(z|\tau)$ on the nome q is shown by

$$\theta_1(z,q) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{i(2n+1)z}. \tag{A.1}$$

The series on the right hand side of (A.1) converges uniformly for $\forall z \in \mathbf{C}$ since $|q| < 1$. Using the identity

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

we obtain from (A.1) another representation of $\theta_1(z,q)$:

$$\theta_1(z,q) = 2\sum_{n=0}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} \sin(2n+1)z.$$

It is clear that $\theta_1(z) = \theta_1(z|\tau)$ is an odd, entire, and periodic function of z with period 2π.

Definition *The second theta function, $\theta_2(z)$, is defined by*

$$\begin{aligned}\theta_2(z) = \theta_1(z + \frac{\pi}{2}) &= \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e^{(2n+1)iz} \\ &= \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} \cos(2n+1)z.\end{aligned}$$

Hence $\theta_2(z)$ is an even, entire, and periodic function of z with period 2π.

Definition *The fourth theta function, $\theta_4(z)$, is defined by*

$$\begin{aligned}\theta_4(z) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz} \\ &= 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz.\end{aligned}$$

Note that $\theta_4(z)$ is an even, entire, and periodic function of z with period π.

Definition *The third theta function, $\theta_3(z)$, is defined by replacing z by $z+\frac{\pi}{2}$ in $\theta_4(z)$, i.e.*

$$\begin{aligned}\theta_3(z) = \theta_4(z + \frac{\pi}{2}) &= \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz} \\ &= 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nz\end{aligned}$$

Note that $\theta_3(z)$ is also an even, entire, and periodic function of z with period π.

Because of the identity $\theta_1(z+\pi\tau) = -(qe^{2iz})^{-1}\theta_1(z)$, $\pi\tau$ is called a quasi-period of $\theta_1(z)$ with periodicity factor $-(qe^{2iz})^{-1}$. Writing $\lambda = qe^{2iz}$ and $\mu = q^{\frac{1}{4}}e^{iz}$ it is easy to verify the following identities:

$$\theta_1(z) = -\theta_1(z+\pi) = -\lambda\theta_1(z+\pi\tau) = \lambda\theta_1(z+\pi+\pi\tau). \tag{A.2}$$

$$\theta_2(z) = -\theta_2(z+\pi) = \lambda\theta_2(z+\pi\tau) = -\lambda\theta_2(z+\pi+\pi\tau).$$

$$\theta_3(z) = \theta_3(z+\pi) = \lambda\theta_3(z+\pi\tau) = \lambda\theta_3(z+\pi+\pi\tau),$$

$$\theta_4(z) = \theta_4(z+\pi) = -\lambda\theta_4(z+\pi\tau) = -\lambda\theta_4(z+\pi+\pi\tau), \tag{A.3}$$

$$\theta_1(z) = -\theta_2(z+\frac{\pi}{2}) = -i\mu\theta_4(z+\frac{\pi\tau}{2}) = -i\mu\theta_3(z+\frac{\pi}{2}+\frac{\pi\tau}{2}), \tag{A.4}$$

$$\theta_2(z) = \theta_1(z+\frac{\pi}{2}) = \mu\theta_3(z+\frac{\pi\tau}{2}) = \mu\theta_4(z+\frac{\pi}{2}+\frac{\pi\tau}{2}),$$

$$\theta_3(z) = \theta_4(z+\frac{\pi}{2}) = \mu\theta_2(z+\frac{\pi\tau}{2}) = \mu\theta_1(z+\frac{\pi}{2}+\frac{\pi\tau}{2}),$$

$$\theta_4(z) = \theta_3(z+\frac{\pi}{2}) = -i\mu\theta_1(z+\frac{\pi\tau}{2}) = -i\mu\theta_2(z+\frac{\pi}{2}+\frac{\pi\tau}{2}).$$

It follows from $\theta_1(0) = 0$ and (A.2) that the zeros of $\theta_1(z)$ occur at the points $z = m\pi + n\pi\tau$ where m and n are integers. The zeros of the other three theta functions can then be obtained from equations (A.4) as follows:

$$\theta_2(z) = 0 \quad \text{when} \quad z = \left(m+\frac{1}{2}\right)\pi + n\pi\tau, \tag{A.5}$$

$$\theta_3(z) = 0 \quad \text{when} \quad z = \left(m+\frac{1}{2}\right)\pi + (n+\frac{1}{2})\pi\tau, \tag{A.6}$$

$$\theta_4(z) = 0 \quad \text{when} \quad z = m\pi + \left(n+\frac{1}{2}\right)\pi\tau. \tag{A.7}$$

Jacobi's Elliptic Functions

Definition *The Jacobi's elliptic functions $sn\,u$, $cn\,u$, and $dn\,u$ are defined in terms of the theta functions as follows:*

$$sn\,u = \frac{\theta_3(0)\theta_1\left(\dfrac{u}{\theta_3^2(0)}\right)}{\theta_2(0)\theta_4\left(\dfrac{u}{\theta_3^2(0)}\right)},$$

$$cn\,u = \frac{\theta_4(0)\theta_2\left(\dfrac{u}{\theta_3^2(0)}\right)}{\theta_2(0)\theta_4\left(\dfrac{u}{\theta_3^2(0)}\right)},$$

$$dn\,u = \frac{\theta_4(0)\theta_3\left(\dfrac{u}{\theta_3^2(0)}\right)}{\theta_3(0)\theta_4\left(\dfrac{u}{\theta_3^2(0)}\right)}.$$

Definition *The modulus and complementary modulus of the Jacobi elliptic functions are given by the formulas* $k_0 = \dfrac{\theta_2^2(0)}{\theta_3^2(0)}$ *and* $k' = \dfrac{\theta_4^2(0)}{\theta_3^2(0)}$, *respectively.*

The following identities can be found in [13]:

$$k_0^2 + k'^2 = 1,$$

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1, \tag{A.8}$$

$$\mathrm{dn}^2 u + k_0^2\,\mathrm{sn}^2 u = 1, \tag{A.9}$$

$$\mathrm{dn}^2 u - k_0^2\,\mathrm{cn}^2 u = k'^2.$$

Definition *The constants* K *and* K', *are defined by the relations* $K = \frac{1}{2}\pi\theta_3^2(0)$ *and* $iK' = K\tau$.

Zeros of sn u, cn u, and dn u

Zeros of $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, and $\mathrm{dn}\,u$ occur at the zeros of $\theta_1\left(\dfrac{u}{\theta_3^2(0)}\right)$, $\theta_2\left(\dfrac{u}{\theta_3^2(0)}\right)$, and $\theta_3\left(\dfrac{u}{\theta_3^2(0)}\right)$, respectively. Thus it follows from (A.5), (A.6), and (A.7) that

$$\mathrm{sn}\,u = 0 \quad \text{when} \quad u = 2mK + 2inK'.$$

$$\mathrm{cn}\,u = 0 \quad \text{when} \quad u = (2m+1)K + 2inK'.$$

$$\operatorname{dn} u = 0 \quad \text{when} \quad u = (2m+1)K + i(2n+1)K'.$$

Poles of sn u, cn u, and dn u

The functions $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$ are analytic for all complex numbers u, except those satisfying $\theta_4\left(\frac{u}{\theta_3^2(0)}\right) = 0$. Hence we deduce from (A.7) that the poles of the functions $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$ occur at the points $u = 2mK + i(2n+1)K'$.

Double Periodicity of the Elliptic Functions

It follows from the definition of the function $\operatorname{sn} u$ and the periodicity of the functions $\theta_1(u)$ and $\theta_4(u)$ that

$$\operatorname{sn}(u+4K) = \frac{\theta_3(0)\theta_1\left(\dfrac{u+4K}{\theta_3^2(0)}\right)}{\theta_2(0)\theta_4\left(\dfrac{u+4K}{\theta_3^2(0)}\right)} = \frac{\theta_3(0)\theta_1\left(\dfrac{u}{\theta_3^2(0)}+2\pi\right)}{\theta_2(0)\theta_4\left(\dfrac{u}{\theta_3^2(0)}+2\pi\right)} = \operatorname{sn} u$$

In addition, using (A.2) and (A.3) gives

$$\operatorname{sn}(u+2iK') = \frac{\theta_3(0)\theta_1\left(\dfrac{u}{\theta_3^2(0)}+\pi\tau\right)}{\theta_2(0)\theta_4\left(\dfrac{u}{\theta_3^2(0)}+\pi\tau\right)} = \operatorname{sn} u.$$

Thus $\operatorname{sn} u$ has two periods given by $4K$ and $2iK'$.

It may be shown in a similar way that $\operatorname{cn} u$ has two periods given by $4K$ and $2K + 2iK'$ and $\operatorname{dn} u$ has also two periods, given by $2K$ and $4iK'$.

Period Parallelogram

Suppose an elliptic function $f(z)$ has two periods $2w_1$ and $2w_2$. Let $\Omega_{m,n} = 2mw_1 + 2nw_2$ where m and n are integers. A parallelogram with vertices at the points $\Omega_{m,n}$, $\Omega_{m+1,n}$, $\Omega_{m+1,n+1}$, and $\Omega_{m,n+1}$ is called a period parallelogram. Two points z and w

with $z \equiv w \pmod{2w_1, 2w_2}$ are called congruent points. Thus for each pair of congruent points z and w we have $f(z) = f(w)$. Hence $f(z)$ is completely determined by its values inside and on a pair of adjacent sides of a period parallelogram.

Addition Theorems

We state the following addition theorems for $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$. Proofs of these identities can be found in [13]:

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k_0^2 \operatorname{sn}^2 u \operatorname{sn}^2 v},$$

$$\operatorname{cn}(u+v) = \frac{\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k_0^2 \operatorname{sn}^2 u \operatorname{sn}^2 v},$$

and

$$\operatorname{dn}(u+v) = \frac{\operatorname{dn} u \operatorname{dn} v - k_0^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k_0^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}.$$

Maclaurin Expansions of sn u, cn u, and dn u

The following formulas for the derivatives of the functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ can be found in [13]:

$$\frac{d}{du} \operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u, \tag{A.10}$$

$$\frac{d}{du} \operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u, \tag{A.11}$$

$$\frac{d}{du} \operatorname{dn} u = -k_0^2 \operatorname{sn} u \operatorname{cn} u. \tag{A.12}$$

Repeated applications of (A.10), (A.11), and (A.12) and using $\operatorname{sn}(0) = 0$, $\operatorname{cn}(0) = 1$, $\operatorname{dn}(0) = 1$ yield

$$\operatorname{sn} u = u - \frac{1}{3!}(1 + k_0^2)u^3 + \frac{1}{5!}(1 + 14k_0^2 + k_0^4)u^5 - \cdots,$$

$$\operatorname{cn} u = 1 - \frac{1}{2!}u^2 + \frac{1}{4!}(1 + 4k_0^2)u^4 - \cdots,$$

and

$$\operatorname{dn} u = 1 - \frac{1}{2!}k_0^2 u^2 + \frac{1}{4!}k_0^2(4 + k_0^2)u^4 - \cdots.$$

Elliptic Functions with Imaginary Argument

The formulas given below can be found in [13]:

$$\operatorname{sn}(iu, k_0) = i\,\frac{\operatorname{sn}(u, k')}{\operatorname{cn}(u, k')}, \quad \operatorname{sn}(u, k') = -i\,\frac{\operatorname{sn}(iu, k_0)}{\operatorname{cn}(iu, k_0)},$$

$$\operatorname{cn}(iu, k_0) = \frac{1}{\operatorname{cn}(u, k')}, \quad \operatorname{cn}(u, k') = \frac{1}{\operatorname{cn}(iu, k_0)},$$

$$\operatorname{dn}(iu, k_0) = \frac{\operatorname{dn}(u, k')}{\operatorname{cn}(u, k')}, \quad \operatorname{dn}(u, k') = \frac{\operatorname{dn}(iu, k_0)}{\operatorname{cn}(iu, k_0)}.$$

Inverse Jacobi Elliptic Functions

The restricted function

$$y = \operatorname{sn}(x, k_0), \quad 0 \leq x \leq K$$

is 1-1, and hence, will have an inverse. Writing

$$w = \operatorname{sn}^{-1}(x, k_0) \text{ implies } \operatorname{sn} w = x, \quad 0 \leq x \leq K, \quad 0 \leq x \leq 1.$$

Hence (A.10), (A.8), and (A.9) yield

$$\frac{dx}{dw} = \operatorname{cn} w \operatorname{dn} w = \sqrt{(1 - x^2)(1 - k_0^2 x^2)}.$$

Thus

$$\operatorname{sn}^{-1}(x, k_0) = w = \int_0^x \frac{dt}{\sqrt{1 - t^2}\sqrt{1 - k_0^2 t^2}}, \quad 0 \leq x \leq 1.$$

In a similar way we obtain

$$\operatorname{cn}^{-1}(x, k_0) = \int_x^1 \frac{dt}{\sqrt{1 - t^2}\sqrt{k'^2 + k_0^2 t^2}}, \quad 0 \leq x \leq 1.$$