TOKSÖZ, Sadik, 1930-
DRAIN SPACING FORMULAS AND NOMOGRAPHS
FOR STRATIFIED SOILS.

Iowa State University, Ph.D., 1969
Engineering, agricultural

DRAIN SPACING FORMULAS AND NOMOGRAPHS FOR STRATIFIED SOILS

by

Sadik Toksöz

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Water Resources

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa

1969

TABLE OF CONTENTS

INTRODUCTION AND REVIEW OF LITERATURE

Since 1940, the theory of flow of water into tile drains advanced rapidly. The theoretical developments involved both the transient flow concept (i.e., the water table is not in equilibrium with recharge but may be rising or falling) and the steady-state flow concept (i.e., water table is in equilibrium with recharge). Notable transient flow theories are given by Dumm (1954, 1964) and by Maasland (1959, 1961). For a recent and comprehensive review of numerous steady-state drainage theories, see Kirkham (1966) and subsequent discussions of his paper by Soliman (1966), Hammad, Amer, Youngs, Dagan and Warrick (1966) and a closure of discussion by Kirkham (1967). A characteristic of all the studies reported in the above references was that the flow medium was assumed to be a uniform, homogeneous and isotropic soil. However, under ordinary field conditions, the water flow into tile drains takes place through layered soils and therefore the flow medium, as a whole, can no longer be assumed as homogeneous and isotropic.

The research reported in this thesis is about a steady-state drainage problem in stratified soils. Although this type of drainage problem is a more common one, it is also a difficult problem to solve and the resulting mathematical expressions are more complicated. This is perhaps the main reason why there are only a few theories on drainage of stratified soils. Hooghoudt's equation (1940) can be written for a two-layered soil, but it is a highly special case because the interface of the two layers passes through the drain centers. Kirkham (1951, 1954) was the first to provide rigorous solutions to two drainage problems in a two-layered flow system. Kirkham's first paper was based on the assumption of a ponded water table over the soil surface. Kirkham's second paper (1954) utilized also the ponded water assumption but, in addition, he also considered upward seepage of artesian water into drains. Later, Swartzendruber (1962) showed how the "epsilon method" of Polubarinova-Kochina (1962) can be applied to complex but exact equations of Kirkham (1951) to obtain simpler but approximate results. Recently, Dagan (1965) has solved, by an approximate approach, the steady-state flow of water into tile drains in a two-layered soil. His solution will later be discussed at some length.

The first purpose of the research reported in this thesis is to give an exact and general steady-state theory of water flow into tile drains in stratified soils. First, this problem will be solved for a two-layered soil. Flow nets for a given flow geometry and for five values, including zero and infinity, of the hydraulic conductivity of the lower layer will be given. Drain spacing formulas will be obtained and a set of nomographs will be presented for drain spacing calculations. Expressions for errors in drain spacings resulting from neglecting the effect of the lower layer will be developed and discussed. Next, the problem will be solved for a three-layered soil. From the two and three-layered solutions one will be able to deduce how to solve problems for soils with more than three layers.

TWO-LAYERED PROBLEM

The geometry of the drainage problem to be solved is shown in Figure 1. A steady, uniform rainfall or irrigation recharge, R, is removed by an infinite array of equally spaced tile drains of diameter 2r. The drain spacing is 2s. The drains are assumed to be running half-full. The removal of the steady recharge by the drains results in a steady, arch-shaped water table, with H indicating the maximum height of the water table at the midpoint of the drains. The flow medium consists of two layers of soil. The hydraulic conductivity of the upper layer is K_1 and of the lower layer is K_2. However, each layer is assumed to be homogeneous and isotropic itself. The upper layer extends a distance "a" below the line connecting the centers of the drains. The lower layer terminates at an impermeable layer located at a finite distance of h below the drain centers. The flow is assumed to be two dimensional.

Formulation of Problem

First, we should observe that, because of symmetry, it is sufficient to consider only half of the flow medium between the two tile drains in Figure 1. We can then represent the field problem depicted in Figure 1 by an idealized geometry, as shown in Figure 2. In Figure 2, and hereafter in the text, the subscripts 1 and 2 refer to upper soil layer and lower soil layer, respectively. Next, in order to translate the field problem into a two dimensional boundary value problem, we shall make use of one assumption and two physical artifices. The assumption, which was also used by Hooghoudt (1940) and Kirkham (1958) is that the hydraulic head

Figure 1. Geometry for a steady-state tile drainage system for a two-layered flow medium terminated by an impermeable layer at a finite depth h below the drain centers.

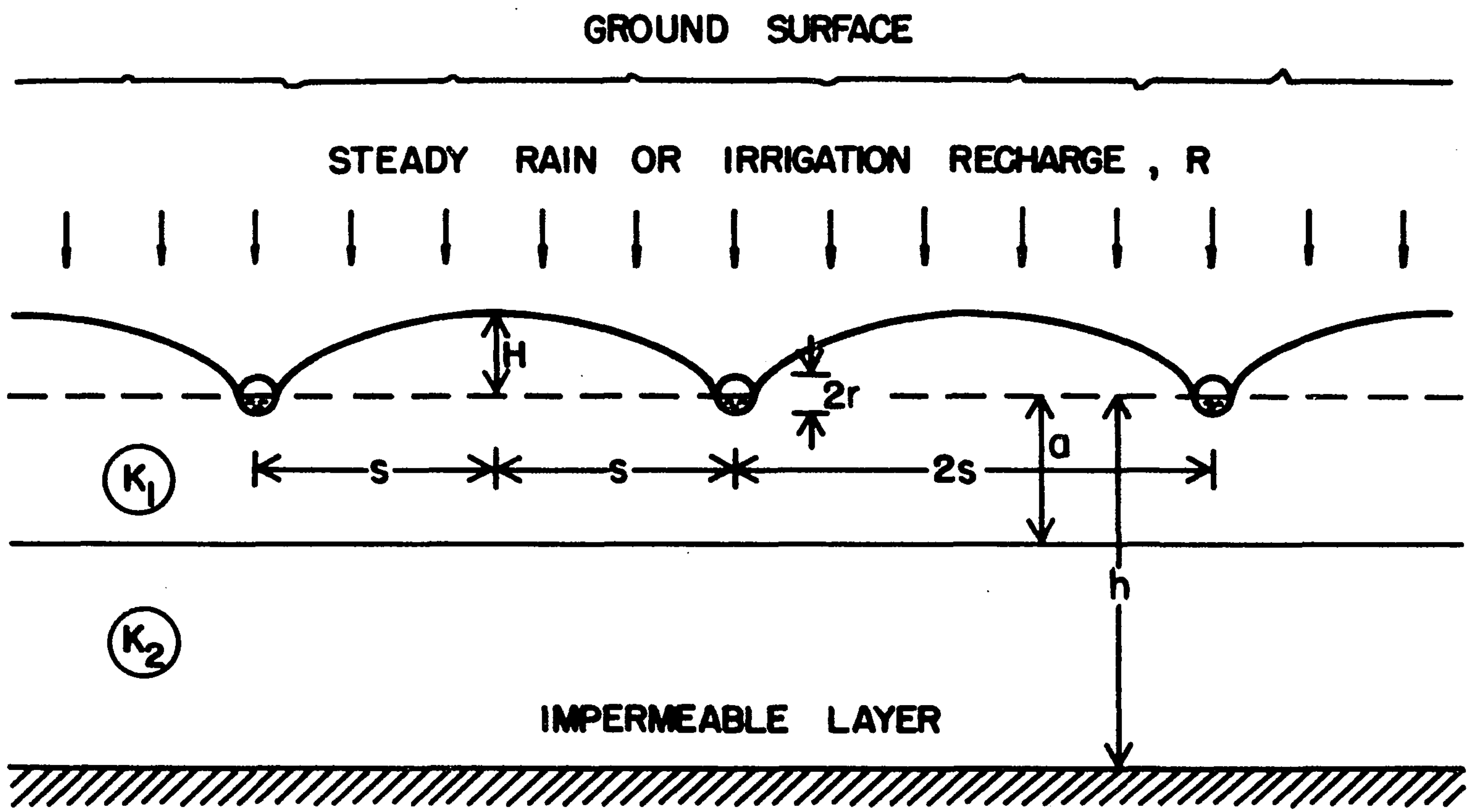
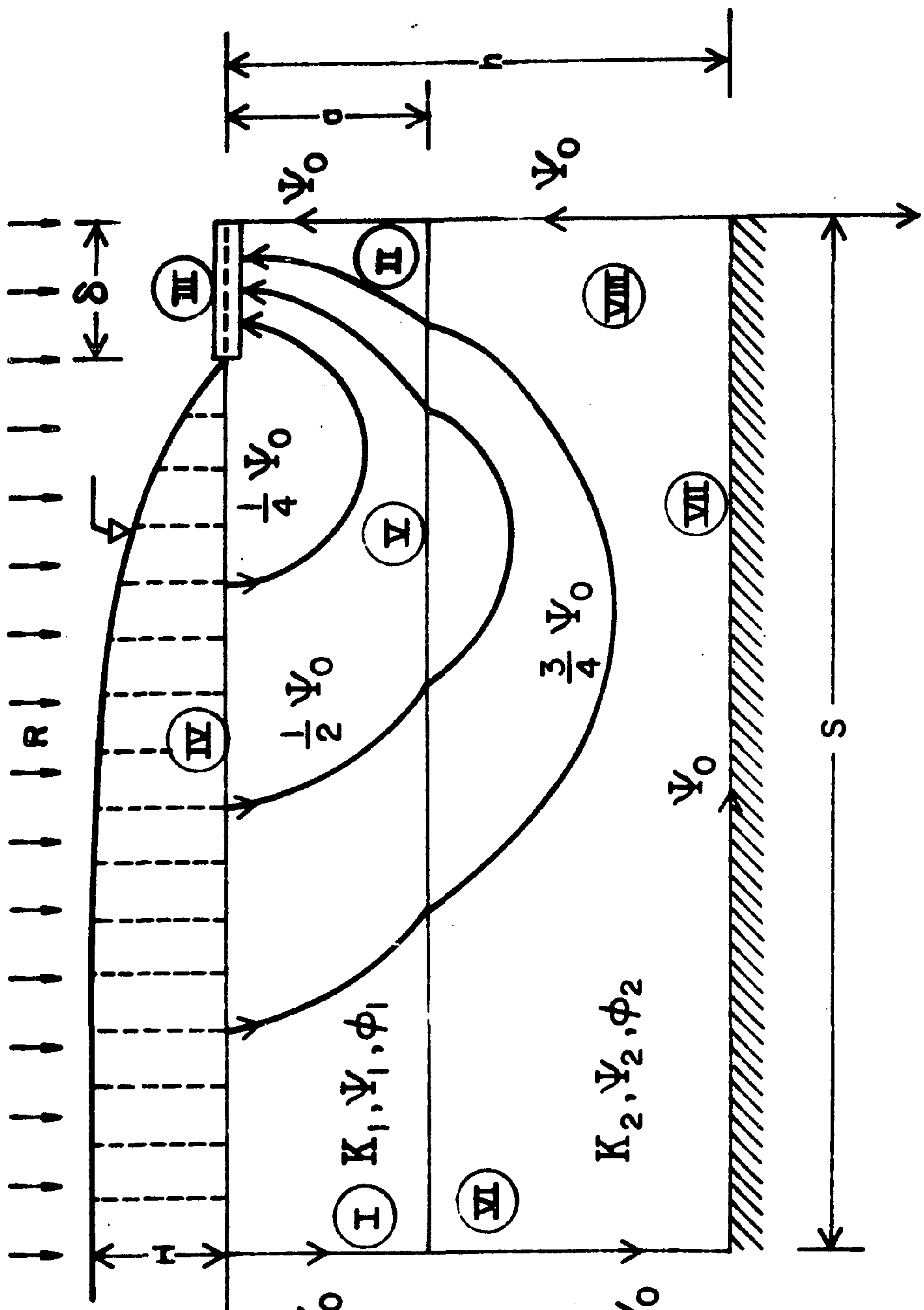

GROUND SURFACE
STEADY RAIN OR IRRIGATION RECHARGE, R
H
2r
a
s
s
2s
K1
h
K2
IMPERMEABLE LAYER

Figure 2. Idealized geometry of the steady-state drainage problem for a two-layered flow medium terminated by an impermeable layer at a finite depth h.


h
a
Ψ_0
δ
R
S
H
$\frac{1}{4}\Psi_0$
$\frac{1}{2}\Psi_0$
$\frac{3}{4}\Psi_0$
K_1, Ψ_1, ϕ_1
K_2, Ψ_2, ϕ_2
I
II
III
IV
V
VI
VII
VIII
X
Y

loss in the arched-region above the drains is negligible compared to the head loss for the remainder of the region. This assumption can be physically approximated by replacing the soil in the arched region with coarse gravel of effectively infinite conductivity so that there will be no head loss, and simultaneously introducing an infinite number of fictitious, rigid, and frictionless membranes, as indicated by the dotted lines in Figure 2. The membranes are needed to keep the curved shape of the water table. The membranes also serve as piezometers to measure the water pressure at their base which is the datum for hydraulic head. Without membranes, a curved water table cannot be maintained in a flow medium of infinite conductivity. In such a flow medium the water table would be flat. This "fictitious membranes" artifice was first used by Kirkham (1958). The vertical membranes replace the true streamlines in the arched region. Therefore, the rainfall or irrigation recharge will be forced to go vertically downward at a uniform rate and, as a result, the streamlines will be equally spaced, that is linearly distributed, along the line connecting the drain centers. The boundary condition IV, as marked in Figure 2, is a direct consequence of this assumption. We should mention here that, after obtaining expressions for potential functions, the gravel in the arched region will again be replaced by soil and the head loss which was assumed to be negligible will be taken into account as it was done by Kirkham (1961).

We note from Figure 2 that the circular drain is replaced by a slit drain of thickness zero and width δ which, later, will also be shrunk to zero. This "slit drain" artifice was also used by Kirkham (1958). It is assumed that the streamlines will be equally spaced, that is, linearly distributed as they enter into the slit drain. The boundary condition III, as marked in Figure 2 is a direct consequence of this assumption.

It is known that both the stream function and the potential function satisfy the Laplace's equation, assuming that the Darcy's linear flow equation and the equation of continuity for water flow are valid at all points of a flow medium. It should be observed that by combining the "fictitious membranes" and the "slit drain" artifices, the arched shaped portion of the flow medium can be excluded because the distribution of flow lines is now known along the line connecting the axes of the drains. Hence all the boundary conditions along the perimeter of the idealized flow geometry can be expressed in terms of half the drain discharge $\psi_o = Rs$, and because of this, we should attempt to solve our flow problem by first finding expressions for the stream functions $\psi_1(x,y)$ and $\psi_2(x,y)$ rather than potential functions $\Phi_1(x,y)$ and $\Phi_2(x,y)$. Hereafter, when referring to stream and potential functions, they will be written as ψ_1, ψ_2, Φ_1 and Φ_2, that is, the x's and y's of the functional notation will be dropped. Note that ϕ_1 and ϕ_2 in Figure 2 refer to hydraulic heads not to potential functions. Potential functions are defined as $\Phi_1 = K_1\phi_1$ and $\Phi_2 = K_2\phi_2$. The reference level for hydraulic head is the x axis, that is, the hydraulic head is measured upward from the x axis.

From the above explanation, it follows that our drainage flow problem should be formulated as the following boundary value problem: First, find expressions for ψ_1 and ψ_2 that will satisfy Laplace's equation and the relevant boundary conditions, as marked in Figure 2. Next, find expressions for $\bar{\phi}_1$ and $\bar{\phi}_2$ that will also satisfy Laplace's equation and the relevant boundary conditions, as marked in Figure 2. In mathematical terms, our task is to find expressions for ψ_1, ψ_2, $\bar{\phi}_1$ and $\bar{\phi}_2$ to satisfy the following equations, respectively,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{1}$$

$$\frac{\partial^2 \bar{\phi}}{\partial x^2} + \frac{\partial^2 \bar{\phi}}{\partial y^2} = 0 \tag{2}$$

subject to the following set of boundary conditions:

I.	$\psi_1 = \psi_0$	at $x = s$,	$0 < y < a$	For medium one
II.	$\psi_1 = \psi_0$	at $x = 0$,	$0 < y < a$	
III.	$\psi_1 = \frac{\delta - x}{\delta}\psi_0$	at $y = 0$,	$0 < x < \delta$	
IV.	$\psi_1 = \frac{x - \delta}{s - \delta}\psi_0$	at $y = 0$,	$\delta < x < s$	
Va.	$\psi_1 = \psi_2$	at $y = a$,	$0 < x < s$	At the interface
Vb.	$\phi_1 = \phi_2$	at $y = a$,	$0 < x < s$	
VI.	$\psi_2 = \psi_0$	at $x = s$,	$a < y < h$	For medium two
VII.	$\psi_2 = \psi_0$	at $y = h$,	$0 < x < s$	
VIII.	$\psi_2 = \psi_0$	at $x = 0$,	$a < y < h$	

Stream and Potential Functions

Laplace's equation is a second order, partial differential equation with an infinite number of solutions. However, we are after particular solutions of Laplaces equation for stream and potential functions that will not only satisfy, respectively, Equations 1 and 2 but also the set of boundary conditions. The following type of a general solution of Laplace's equation

$$F(x,y) = A + Bx + Cy + Dxy + \sum_{m=1}^{\infty} E_m \begin{matrix}\sinh\\ \text{or}\\ \cosh\end{matrix}\ \alpha_m\left(b \mp \begin{matrix}x\\ \text{or}\\ y\end{matrix}\right) \begin{matrix}\sin\\ \text{or}\\ \cos\end{matrix}\ \left(c \mp \begin{matrix}y\\ \text{or}\\ x\end{matrix}\right) \quad (3)$$

is very useful in building up expressions for stream and potential functions, as explained by Kirkham (1970). However, after Equation 5 of Kirkham (1958), we can write, by inspection, the expression for the stream function for medium one as:

$$\psi_1 = \psi_0 + \sum_{m=1}^{\infty} A_m \sin\frac{m\pi x}{s} \frac{\sinh[m\pi(a - y)/s]}{\sinh(m\pi a/s)}$$

$$+ \sum_{m=1}^{\infty} B_m' \sin\frac{m\pi x}{s} \frac{\sinh(m\pi y/s)}{\sinh(m\pi a/s)} \quad (4)$$

where A_m and B_m' are arbitrary constants. Hereafter, the sign Σ will mean $\sum_{m=1,2,\ldots}^{\infty}$, unless stated otherwise. Observe that Equation 4 satisfies boundary conditions I and II. For boundary conditions III and IV, where $y = 0$, Equation 4 reduces to

$$\psi_1 = \psi_0 + \Sigma A_m \sin\frac{m\pi x}{s} \quad (5)$$

In Equation 5, the arbitrary constant A_m can be expressed as a Fourier sine series:

$$A_m = \frac{2}{s} \int_0^s f(x) \sin \frac{m\pi x}{s} dx \qquad (6)$$

where f(x) will be defined as a function satisfying boundary conditions III and IV.

Boundary condition III, when applied to Equation 5, yields:

$$\frac{\delta - x}{\delta} \psi_0 = \psi_0 + \Sigma A_m \sin \frac{m\pi x}{s}, \qquad (0 < x < \delta)$$

which reduces to

$$-\frac{x}{\delta} \psi_0 = \Sigma A_m \sin \frac{m\pi x}{s}, \qquad (0 < x < \delta) \qquad (7)$$

Boundary condition IV, when applied to Equation 5, yields:

$$\frac{x - \delta}{s - \delta} \psi_0 = \psi_0 + \Sigma A_m \sin \frac{m\pi x}{s}, \qquad (\delta < x < s)$$

which reduces to

$$\frac{x - s}{s - \delta} \psi_0 = \Sigma A_m \sin \frac{m\pi x}{s} \qquad (\delta < x < s) \qquad (8)$$

From Equations 7 and 8, we can define f(x) as:

$$f(x) = \begin{cases} -\frac{x}{\delta} \psi_0, & 0 \leq x \leq \delta \\ \frac{x - s}{s - \delta} \psi_0, & \delta \leq x \leq s \end{cases} \qquad (9)$$

Inserting the above expressions of f(x) into Equation 6 gives us:

$$A_m = \frac{2}{s} \left[\left(-\frac{\psi_0}{\delta}\right) \int_0^{\delta} x \sin\frac{m\pi x}{s}\, dx + \frac{\psi_0}{s-\delta} \int_{\delta}^{s} (x - s) \sin\frac{m\pi x}{s}\, dx \right] \qquad (10)$$

Evaluation of the above definite integrals and, afterwards, taking the limit of the result as $\delta \rightarrow 0$, that is as the width of the slit drain shrinks to zero, yields:

$$A_m = -\frac{2\psi_0}{m\pi} \qquad (11)$$

Therefore, Equation 4 can now be written as:

$$\psi_1 = \psi_0 - \frac{2\psi_0}{\pi} \Sigma \left\{ \frac{1}{m} \sin\frac{m\pi x}{s} \frac{\sinh[m\pi(a - y)/s]}{\sinh(m\pi a/s)} \right.$$

$$\left. + B_m \sin\frac{m\pi x}{s} \frac{\sinh(m\pi y/s)}{\sinh(m\pi a/s)} \right\} \qquad (12)$$

where we have used $B_m' = -\frac{2\psi_0}{\pi} B_m$.

By inspection of Equation 4 and boundary conditions VI, VII, and VIII we can write the stream function for medium two as:

$$\psi_2 = \psi_0 - \frac{2\psi_0}{\pi} \Sigma\, C_m \sin\frac{m\pi x}{s} \frac{\sinh[m\pi(h - y)/s]}{\cosh[m\pi(h - a)/s]} \qquad (13)$$

where C_m is another arbitrary constant.

Because the expressions for the stream and potential functions are analytic functions, they satisfy the Cauchy-Riemann conditions. The Cauchy-Riemann conditions are

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad (14)$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad (15)$$

Therefore, if the expression for the stream function is known, the potential function can be found from Equations 14 and 15, either by integration or by inspection. Either way it follows that the potential functions for mediums one and two are

$$\bar{\Phi}_1 = -\frac{2\psi_0}{\pi}\Sigma\{\frac{1}{m}\cos\frac{m\pi x}{s}\frac{\cosh[m\pi(a-y)/s]}{\sinh(m\pi a/s)}$$

$$-B_m\cos\frac{m\pi x}{s}\frac{\cosh(m\pi y/s)}{\sinh(m\pi a/s)}\}+\frac{2\psi_0}{\pi}B_{om} \qquad (16)$$

and

$$\bar{\Phi}_2 = -\frac{2\psi_0}{\pi}\Sigma\{C_m\cos\frac{m\pi x}{s}\frac{\cosh[m\pi(h-y)/s]}{\cosh[m\pi(h-a)/s]}\}+\frac{2\psi_0}{\pi}C_{om} \qquad (17)$$

where B_{om} and C_{om} are also arbitrary constants.

We will now evaluate the arbitrary constants B_m and C_m. By definition, $\phi_1 = \bar{\Phi}_1/K_1$ and $\phi_2 = \bar{\Phi}_2/K_2$. Using these definitions, boundary condition Vb and Equations 16 and 17, we get

$$\frac{1}{K_1}\{\Sigma[\frac{1}{m}\frac{1}{\sinh(m\pi a/s)}-B_m\coth(m\pi a/s)]\cos\frac{m\pi x}{s}$$

$$-B_{om}\} = \frac{1}{K_2}[\Sigma C_m\cos\frac{m\pi x}{s}-C_{om}] \qquad (18)$$

from which we get, by equating the coefficients of the term $\cos\frac{m\pi x}{s}$ and dropping the Σ sign, the results

$$C_m = \frac{K_2}{K_1}[\frac{1}{m}\frac{1}{\sinh(m\pi a/s)}-B_m\coth\frac{m\pi a}{s}] \qquad (19)$$

and

$$C_{om} = \frac{K_2}{K_1}B_{om} \qquad (20)$$

Similarly, boundary condition Va states that $\psi_1 = \psi_2$ at $y = a$, which results in, after cancelling certain terms,

$$\Sigma B_m \sin \frac{m\pi x}{s} = \Sigma C_m \tanh \frac{m\pi(h - a)}{s} \sin \frac{m\pi x}{s} \tag{21}$$

from which we get, by equating the coefficients of $\sin \frac{m\pi x}{s}$ and dropping the Σ sign, the result

$$B_m = C_m \tanh \frac{m\pi(h - a)}{s}. \tag{22}$$

Substituting this value of B_m into Equation 19, and after rearranging, we get

$$C_m = \frac{1}{m} \frac{1}{\sinh(m\pi a/s)} \frac{1}{(K_1/K_2) + \tanh[m\pi(h - a)/s] \coth(m\pi a/s)} \tag{23}$$

and

$$B_m = \frac{1}{m} \frac{1}{\sinh(m\pi a/s)} \frac{1}{(K_1/K_2) \coth[m\pi(h - a)/s] + \coth(m\pi a/s)} \tag{24}$$

We shall now evaluate B_{om} and C_{om}. First we write the identity:

$$\frac{\cosh[m\pi(a - y)/s]}{\sinh(m\pi a/s)} = e^{-(m\pi y/s)} + \frac{e^{-(m\pi a/s)} \cosh(m\pi y/s)}{\sinh(m\pi a/s)} \tag{25}$$

We use this identity in Equation 16 to obtain

$$\Phi_1 = -\frac{2\psi_0}{\pi} \Sigma \{\frac{1}{m} \cos \frac{m\pi x}{s} e^{-(m\pi y/s)} + \frac{e^{-(m\pi a/s)} \cosh(m\pi y/s)}{\sinh(m\pi a/s)}$$

$$- B_m \cos \frac{m\pi x}{s} \frac{\cosh(m\pi y/s)}{\sinh(m\pi a/s)}\} - \frac{2\psi_0}{\pi} B_{om} \tag{26}$$

Next, we use another identity

$$\Sigma \frac{1}{m} e^{-(m\pi y/s)} \cos \frac{m\pi x}{s} = -\frac{1}{2} \ln[2 e^{-(\pi y/s)}(\cosh \frac{\pi y}{s} - \cos \frac{\pi x}{s})] \qquad (27)$$

So that Equation 26 reduces to

$$\bar{\phi}_1 = -\frac{2\psi_0}{\pi} \{-\frac{1}{2} \ln[2 e^{-(\pi y/s)}(\cosh \frac{\pi y}{s} - \cos \frac{\pi x}{s})]$$

$$+ \Sigma [\frac{1}{m} \cos \frac{m\pi x}{s} e^{-(m\pi a/s)} \frac{\cosh(m\pi y/s)}{\sinh(m\pi a/s)}$$

$$- B_m \cos \frac{m\pi x}{s} \frac{\cosh(m\pi y/s)}{\sinh(m\pi a/s)}] - B_{om}\} \qquad (28)$$

Remember that $\phi_1 = \bar{\phi}_1/K_1$ and; remember further that we had assumed our tile drains to be flowing half-full, that is $\phi_1(x=r, y=0) = 0$. By using these two conditions in Equation 28, we obtain

$$B_{om} = -\frac{1}{2} \ln[2(1 - \cos \frac{\pi r}{s})]$$

$$+ \Sigma [\frac{1}{m} \cos \frac{m\pi r}{s} \frac{e^{-(m\pi a/s)}}{\sinh(m\pi a/s)} - B_m \cos \frac{m\pi r}{s} \frac{1}{\sinh(m\pi a/s)}] \qquad (29)$$

where we observe that

$$-\frac{1}{2} \ln[2(1 - \cos \frac{\pi r}{s}] = -\frac{1}{2} \ln[2 \ (2) \ (\sin \frac{\pi r}{2s})^2]$$

$$= -\frac{1}{2} \ln[2 \sin \frac{\pi r}{2s}]^2 = \ln \frac{1}{2 \sin(\pi r/2s)}$$

so that

$$B_{om} = \ln \frac{1}{2 \sin(\pi r/2s)} + \Sigma [\frac{1}{m} \cos \frac{m\pi r}{s} (-1 + \coth \frac{m\pi a}{s})$$

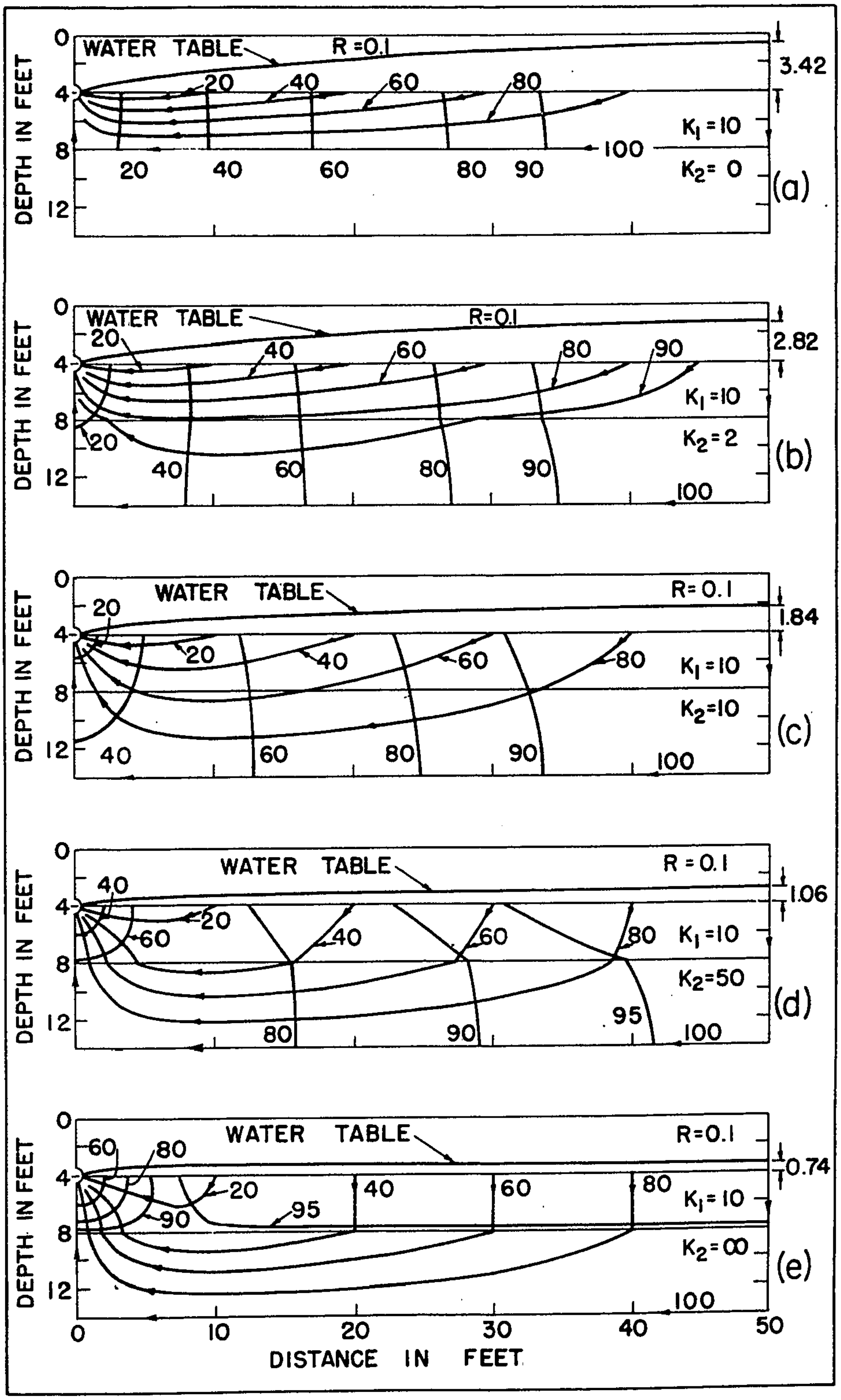
$$- B_m \cos \frac{m\pi r}{s} \frac{1}{\sinh(m\pi a/s)}] \qquad (30)$$

where we have used the identity $e^{-(m\pi a/s)}/\sinh(m\pi a/s) = -1 + \coth(m\pi a/s)$. Because B_m is defined by Equation 24, B_{om} is now known. Furthermore, when B_{om} is known, C_{om} is also known from Equation 20. Hence we have determined all the coefficients for ψ_1, ψ_2, $\bar{\phi}_1$ and $\bar{\phi}_2$, and in turn ϕ_1 and ϕ_2.

Flow Nets

By using Equations 12, 13, 16, 17, 20, 23, 24, and 30, flow nets can be drawn for any given set of soil and hydrologic parameters. Figure 3 shows five flow nets, labeled from a through e. These flow nets were prepared for the following set of dimensionless variables: $a/2s = 1/25$, $a/h = 2/5$, $a/2r = 4$, $R/K_1 = 1/100$ and $K_1/K_2 =$ infinity, 5, 1, 1/5 and zero, for the cases a, b, c, d and e, respectively. However, to facilitate quantitative discussion, it was assumed that the drains were placed at a depth of four feet below the ground surface with $a = 4$ feet, $h = 10$ feet, $2s = 100$ feet, $2r = 1$ foot, $R = 0.1$ inch per day, $K_1 = 10$ inches per day. It was also assumed that the hydraulic conductivity of the lower layer, K_2, would vary as follows: zero, 2 inches per day 10 inches per day, 50 inches per day and infinity, for the cases a, b, c, d, and e, respectively. As one may observe, these flow nets were prepared to show the effect of the hydraulic conductivity of the lower soil layer on the flow lines, the equal hydraulic head lines (equipotentials), and on the maximum height of water table above the drains. Equations 12, 13, 23 and 24 were used to compute the streamlines. The streamlines were expressed as a percentage of half the drain discharge, $\psi_0 = Rs$, that is, as $100(\psi_1/\psi_0)$ and as

Figure 3. Flow nets for the dimensionless parameters $a/2s = 1/25$, $a/h = 2/5$, $a/2r = 4$, $R/K_1 = 1/100$ and $K_1/K_2 = \infty$, 5, 1, 1/5 and zero for the cases a through e respectively. Depth and distance in feet are shown for purposes of quantitative discussion.


DEPTH IN FEET
WATER TABLE
R = 0.1
3.42
K1 = 10
K2 = 0
(a)
2.82
K2 = 2
(b)
1.84
K2 = 10
(c)
1.06
K2 = 50
(d)
0.74
K2 = ∞
(e)
20
40
60
80
90
95
100
0
4
8
12
0
10
20
30
40
50
DISTANCE IN FEET

$100(\psi_2/\psi_0)$. Equations 16, 17, 20, 23, 24 and 30 were used to compute the equipotentials. The equipotentials were expressed as a percentage of the maximum hydraulic head, $\phi_1(s,0)$, that is as $100[\phi_1(x,y)/\phi_1(s,0)]$ and as $100[\phi_2(x,y)/\phi_1(s,0)]$. The water table was plotted from $\phi_1(x,0)$. Because of slow convergence of some of the series found in stream and potential functions, a digital computer was used in computations. The following is a summary of the formulas used for preparing the flow nets for cases a through e of Figure 3.

Case a: $K_1 = 10 \quad K_2 = 0 \quad K_1/K_2 = \infty$

The problem reduces to the single layered problem with an impermeable layer at a depth a below the drains. Equations 12 and 16 were used, but observe that $K_1/K_2 = \infty$ results in $B_m = 0$ and,

$$B_{om} = \ln \frac{1}{2\sin(\pi r/2s)} + \Sigma \frac{1}{m} \cos \frac{m\pi r}{s} \left(-1 + \coth \frac{m\pi a}{s}\right) \tag{31}$$

In view of the above results, Equation 12 reduces to

$$\psi_1 = \psi_0 - \frac{2\psi_0}{\pi} \Sigma \frac{1}{m} \sin \frac{m\pi x}{s} \frac{\sinh[m\pi(a-y)/s]}{\sinh(m\pi a/s)} \tag{32}$$

and Equation 16 reduces to

$$\phi_1 = -\frac{2\psi_0}{\pi K_1} \Sigma \left\{\frac{1}{m} \cos \frac{m\pi x}{s} \frac{\cosh[m\pi(a-y)/s]}{\sinh(m\pi a/s)}\right\} + \frac{2\psi_0}{\pi} B_{om} \tag{33}$$

Case b: $K_1 = 10 \quad K_2 = 2 \quad K_1/K_2 = 5$

Equations 12, 13, 16, 17, 20, 23, 24 and 30 were used.

Case c: $K_1 = K_2 = 10 \quad K_1/K_2 = 1$

The problem again reduces to the single-layered problem, but in this case, the impermeable layer is at a depth of h below the drains. Observe that $K_1 = K_2$ implies that $a \rightarrow h$ which, because of $\coth[m\pi(h-h/s] =$

∞, results in $B_m = 0$. Therefore, Equations 31, 32 and 33 were used after replacing the symbol a in these equations with the symbol h.

Case d: $K_1 = 10 \quad K_2 = 50 \quad K_1/K_2 = 1/5$

Equations 12, 13, 16, 17, 20, 23, 24 and 30 were used.

Case e: $K_1 = 10 \quad K_2 = \infty \quad K_1/K_2 = 0$

Observe that $K_1/K_2 = 0$ results in

$$B_m = \frac{1}{m}\frac{1}{\cosh(m\pi a/s)} \tag{34}$$

$$C_m = \frac{1}{m}\frac{1}{\tanh[m\pi(h - a)/s]\cosh(m\pi a/s)} \tag{35}$$

While using Equations 12, 13, 16, 17, 20 and 30, one should insert values of B_m and C_m as given by Equations 34 and 35.

A comparison between the cases c and d of Figure 3 shows that a five-fold increase in the hydraulic conductivity of the lower layer - a rather common observation under field conditions - would result in a decrease of $[(1.84 - 1.06)/1.84]100 = 42$ percent in maximum height of water table. Furthermore, one observes from Figure 3 a through e that as K_2 increases from zero to infinity, the 40, 60, 80 percent streamlines in the upper soil layer start deviating from their somewhat horizontal directions toward a vertical direction. These streamlines pass through the lower layer somewhat horizontally but they converge rapidly in the vicinity of the drain. At the interface of the soil layers, both the streamlines and the equipotentials obey to the well known laws of refraction. The angles the streamlines in the upper and lower layers make with the normal to the interface, that is α_1 and α_2, respectively, can be found from $\tan \alpha_1/\tan \alpha_2 = K_1/K_2$. The angles the equipotentials in the upper and lower layers make with the normal to the interface, that is γ_1 and γ_2, respectively, can be found from $\cot \gamma_1/\cot \gamma_2 = K_1/K_2$.

Drain Spacing Formulas

Remembering that $\phi_1 = \bar{\phi}_1/K_1$, $\psi_0 = Rs$ and the identity $e^{-(m\pi a/s)} e^{(m\pi a/s)} = 1$, and substituting B_m and B_{om} from the Equations 24 and 30 into Equation 28, we can rewrite Equation 28 as

$$\begin{aligned}
\phi_1 = \frac{2Rs}{\pi K_1} \{ & \frac{1}{2} \ln[2\, e^{-(\pi y/s)}(\cosh \frac{\pi y}{s} - \cos \frac{\pi x}{s})] \\
& - \Sigma \frac{1}{m} \cos \frac{m\pi x}{s}\, e^{-(m\pi a/s)} \frac{\cosh(m\pi y/s)}{\sinh(m\pi a/s)} \\
& + \Sigma \frac{1}{m} \frac{1}{\sinh(m\pi a/s)} \frac{e^{-(m\pi a/s)}\; e^{(m\pi a/s)}}{(K_1/K_2) \coth[m\pi(h-a)/s] + \coth(m\pi a/s)} \\
& \quad \cos \frac{m\pi x}{s} \frac{\cosh(m\pi y/s)}{\sinh(m\pi a/s)} \\
& - \Sigma \frac{1}{m} \frac{1}{\sinh(m\pi a/s)} \frac{e^{-(m\pi a/s)}\; e^{(m\pi a/s)}}{(K_1/K_2) \coth[m\pi(h-a)/s] + \coth(m\pi a/s)} \\
& \quad \cos \frac{m\pi r}{s} \frac{1}{\sinh(m\pi a/s)} \\
& + \Sigma [\frac{1}{m} \cos \frac{m\pi r}{s} \frac{e^{-(m\pi a/s)}}{\sinh(m\pi a/s)}] + \ln \frac{1}{2 \sin(\pi r/2s)} \}
\end{aligned} \tag{36}$$

After using the identity $e^{-(m\pi a/s)}/\sinh(m\pi a/s) = -1 + \coth(m\pi a/s)$, and after rearranging Equation 36, we get

$$\phi_1 = \frac{2Rs}{\pi K_1} \{\ln \frac{1}{2 \sin(\pi r/2s)} + \frac{1}{2} \ln[2\ e^{-(\pi y/s)}(\cosh \frac{\pi y}{s} - \cos \frac{\pi x}{s})]$$

$$+ \Sigma \frac{1}{m} (-1 + \coth \frac{m\pi a}{s})(\cos \frac{m\pi r}{s} - \cos \frac{m\pi x}{s} \cosh \frac{m\pi y}{s})$$

$$[1 - \frac{e^{(m\pi a/s)}}{\sinh(m\pi a/s)} \frac{1}{(K_1/K_2) \coth[m\pi(h-a)/s] + \coth(m\pi a/s)}]\} \quad (37)$$

Now, by definition, $\phi_1(s,0) = H$, the maximum height of the water table midway between the drains. Inserting $y = 0$ and $x = s$ and using the relation $\cos \pi = -1$ and using the identity

$$\ln \frac{1}{2 \sin(\pi r/2s)} + \frac{1}{2} \ln[2(1 - \cos \pi)] = \ln \frac{1}{\sin(\pi r/2s)} \quad (38)$$

one sees that Equation 37 reduces to

$$H = \frac{2Rs}{\pi K_1} \{\ln \frac{1}{\sin(\pi r/2s)} + \Sigma [\frac{1}{m} (-1 + \coth \frac{m\pi a}{s})(\cos \frac{m\pi r}{s} - \cos m\pi)$$

$$- \frac{1}{m} (-1 + \coth \frac{m\pi a}{s})(\cos \frac{m\pi r}{s} - \cos m\pi) \frac{e^{(m\pi a/s)}}{\sinh(m\pi a/s)}$$

$$\frac{1}{(K_1/K_2) \coth[m\pi(h-a)/s] + \coth(m\pi a/s)}]\} \quad (39)$$

Equation 39 is the general formula relating all relevant design variables for a two-layered drainage problem.

We can distinguish seven limiting cases of the general formula given by Equation 39. The first two cases result from the limiting values of the thicknesses of two soil layers and the remaining five cases from the limiting values of hydraulic conductivities.

Case 1: $h \rightarrow \infty$ While a = finite and $a/h = 0$

Because of $\coth[m\pi(h-a)/s] \rightarrow 0$ as $h \rightarrow \infty$, Equation 39 reduces to

$$H = \frac{2Rs}{\pi K_1} \{\ln \frac{1}{\sin(\pi r/2s)} + \Sigma \frac{1}{m} (-1 + \coth \frac{m\pi a}{s})(\cos \frac{m\pi r}{s} - \cos m\pi)$$
$$[1 - \frac{e^{(m\pi a/s)}}{(K_1/K_2) \sinh(m\pi a/s) + \cosh(m\pi a/s)}]\} \qquad (40)$$

Observe that in Equation 40, the parameter h does not appear.

Case 2: $a \rightarrow \infty$ While $h > a$ and K_1 = finite

Because of $\coth(m\pi a/s) - 1 = 0$ as $a \rightarrow \infty$, Equation 39 reduces to the following simple form:

$$H = \frac{2Rs}{\pi K_1} \ln \frac{1}{\sin(\pi r/2s)}, \quad (\text{for } a \rightarrow \infty) \qquad (41)$$

Observe that in Equation 41 the parameters a, h, and K_2 do not appear.

Case 3: $K_2 = 0$ While K_1 = finite

Our two-layered problem would reduce to a single-layered problem and the impermeable layer would be at a depth a below the drains. Equation 39 reduces to

$$H = \frac{2Rs}{\pi K_1} [\ln \frac{1}{\sin(\pi r/2s)} + \Sigma \frac{1}{m} (-1 + \coth \frac{m\pi a}{s})(\cos \frac{m\pi r}{s} - \cos m\pi)] \qquad (42)$$

To see this, one should observe that $K_2 = 0$ implies $K_1/K_2 = \infty$ which causes the following term in Equation 39

$$\frac{1}{(K_1/K_2) \coth[m\pi(h-a)/s] + \coth(m\pi a/s)}$$

to be zero.

Case 4: $K_2 = K_1$

Our two-layered problem would again reduce to a single-layered problem, but here, the impermeable layer would be at a depth h below the drains. Equation 39 would again reduce to

$$H = \frac{2Rs}{\pi K_1} [\ln \frac{1}{\sin(\pi r/2s)} + \Sigma \frac{1}{m} (-1 + \coth \frac{m\pi h}{s})(\cos \frac{m\pi r}{s} - \cos m\pi)] \quad (43)$$

To see this, one should observe that $K_2 = K_1$ implies $a \rightarrow h$, and in turn, the term $\coth[m\pi(h-a)/s] \rightarrow \infty$ which causes the following term in Equation 39

$$\frac{1}{(K_1/K_2) \coth[m\pi(h-a)/s] + \coth(m\pi a/s)}$$

to be zero. Observe that Equations 42 and 43 are identical except that the symbol a in Equation 42 is replaced by the symbol h in Equation 43, or vice versa.

Case 5: $K_2 = \infty$ While K_1 = finite

This implies that $K_1/K_2 = 0$ and Equation 39 reduces to

$$H = \frac{2Rs}{\pi K_1} \{\ln \frac{1}{\sin(\pi r/2s)}$$

$$+ \Sigma \frac{1}{m} (-1 + \coth \frac{m\pi a}{s})(\cos \frac{m\pi r}{s} - \cos m\pi)[1 - \frac{e^{(m\pi a/s)}}{\cosh(m\pi a/s)}]\} \quad (44)$$

Observe that in Equation 44, K_2 and h do not appear.

Case 6: $K_1 = 0$ While K_2 = finite

One may deduce from Equation 39 that $H \rightarrow \infty$. There would be no flow into drains that are placed in an impermeable layer. Therefore, the steady recharge would cause the water table to build up and to reach, eventually, a theoretical height of infinity.

Case 7: $K_1 = \infty$ While K_2 = finite

One may deduce from Equation 39 that $H \rightarrow$ zero. No hydraulic head would be needed for water to flow into drains when the drains are placed in an infinitely conducting medium. Therefore, the water table would be flat and at the axis of the drains.

We should now recall an assumption that was made earlier, under the subheading "Formulation of Problem". This assumption was: The hydraulic head loss in the arched region above the drains is negligible compared to the head loss for the remainder of the region. However, Kirkham (1961) has shown that multiplication of the right hand side of Equation 39 by the factor $[1 - (R/K_1)]^{-1}$ will take this neglected head loss into account. This results in

$$H = \frac{2Rs}{\pi K_1} \frac{1}{1 - (R/K_1)} \{\ln \frac{1}{\sin(\pi r/2s)}$$

$$+ \Sigma \, [\frac{1}{m} (-1 + \coth \frac{m\pi a}{s})(\cos \frac{m\pi r}{s} - \cos m\pi)$$

$$- \frac{1}{m} (-1 + \coth \frac{m\pi a}{s})(\cos \frac{m\pi r}{s} - \cos m\pi) \frac{e^{(m\pi a/s)}}{\sinh(m\pi a/s)}$$

$$\frac{1}{(K_1/K_2) \coth[m\pi(h-a)/s] + \coth(m\pi a/s)}]\} \qquad (45)$$

Notice that Equation 45 is exactly the same of Equation 39 except for the factor $[1 - (R/K_1)]^{-1}$ which takes into account the neglected head loss in the arched region above the drains. Therefore, Equation 45 rather than Equation 39 will hereafter be called "the general formula". In Equation 45, the recharge, R, sometimes also called "the drainage coefficient", reflects the duration, intensity and frequency of either the rainfall or irrigation applications and determined accordingly. The soil parameters K_1, K_2, a and h are determined through field tests and borings. The maximum height of water table over the drains, H, is mainly a function of the rooting habits of crops, among other factors. Normally, all of these parameters are "given" quantities. In other words, these parameters can be determined, within reasonable margins, based on information collected during investigation and planning activities. The designer then selects a tile diameter, 2r, and proceeds to compute the drain spacings, 2s – the quantity he is really interested to know. However, Equation 45 is not of too much help to him in achieving his task because, the drain spacing, 2s, is not given explicitly by this equation. This difficulty can be overcome by using a procedure outlined by Toksöz and Kirkham (1961).

Let us define three functions, i.e., $E(\frac{2s}{a}, \frac{a}{2r})$, $F(\frac{2s}{a}, \frac{a}{2r})$ and $G(\frac{2s}{a}, \frac{a}{2r}, \frac{a}{h}, \frac{K_1}{K_2})$ as follows

$$E(\frac{2s}{a}, \frac{a}{2r}) = \frac{1}{\pi} \ln \frac{1}{\sin[(\pi/2)(2r/a)(a/2s)]} \tag{46}$$

$$F(\frac{2s}{a}, \frac{a}{2r}) = \frac{1}{\pi} \frac{1}{m} (-1 + \coth \frac{m\pi a}{s})(\cos m\pi \frac{2r}{a} \frac{a}{2s} - \cos m\pi) \tag{47}$$

and

$$G\left(\frac{2s}{a}, \frac{a}{2r}, \frac{a}{h}, \frac{K_1}{K_2}\right) = \frac{e^{(2m\pi a/2s)}}{\sinh(2m\pi a/2s)}$$

$$\frac{1}{(K_1/K_2)\coth 2m\pi[(h/a)(a/2s)-(a/2s)] + \coth(2m\pi a/2s)} \quad (48)$$

By using these functions in Equation 45 and by dividing both sides of it by the symbol a, and after rearranging it, we can rewrite the general formula as

$$\frac{H}{a}\left(\frac{K_1}{R} - 1\right) = \frac{2s}{a}\left[E\left(\frac{2s}{a}, \frac{a}{2r}\right) + \Sigma F\left(\frac{2s}{a}, \frac{a}{2r}\right) - \Sigma F\left(\frac{2s}{a}, \frac{a}{2r}\right)G\left(\frac{2s}{a}, \frac{a}{2r}, \frac{a}{h}, \frac{K_1}{K_2}\right)\right] \quad (49)$$

Similarly, Equations 40, 41, 42, 43, and 44, which correspond to the first five limiting cases, can also be rewritten. Equation 40, for $h \rightarrow \infty$ and $a/h = 0$ becomes

$$\frac{H}{a}\left(\frac{K_1}{R} - 1\right) = \frac{2s}{a}\left\{E\left(\frac{2s}{a}, \frac{a}{2r}\right) + \Sigma F\left(\frac{2s}{a}, \frac{a}{2r}\right)\left[1 - \frac{e^{(2m\pi a/2s)}}{(K_1/K_2)\sinh(2m\pi a/2s) + \cosh(2m\pi a/2s)}\right]\right\} \quad (50)$$

Equation 41, for $a \rightarrow \infty$, becomes

$$\frac{H}{a}\left(\frac{K_1}{R} - 1\right) = \frac{2s}{a} E\left(\frac{2s}{a}, \frac{a}{2r}\right) \quad (51)$$

Equation 42, for $K_2 = 0$, becomes

$$\frac{H}{a}\left(\frac{K_1}{R} - 1\right) = \frac{2s}{a}\left[E\left(\frac{2s}{a}, \frac{a}{2r}\right) + \Sigma F\left(\frac{2s}{a}, \frac{a}{2r}\right)\right] \quad (52)$$

Equation 43, for $K_2 = K_1$, becomes

$$\frac{H}{h}(\frac{K_1}{R} - 1) = \frac{2s}{h} [E(\frac{2s}{h}, \frac{h}{2r}) + \Sigma F(\frac{2s}{h}, \frac{h}{2r})] \tag{53}$$

Equation 44, for $K_2 = \infty$, becomes

$$\frac{H}{a}(\frac{K_1}{R} - 1) = \frac{2s}{a} \{E(\frac{2s}{a}, \frac{a}{2r}) + \Sigma F(\frac{2s}{a}, \frac{a}{2r}) [1 - \frac{e^{(2m\pi a/2s)}}{\cosh(2m\pi a/2s)}]\} \tag{54}$$

which, if desired, may be reduced to

$$\frac{H}{a}(\frac{K_1}{R} - 1) = \frac{2s}{a} [E(\frac{2s}{a}, \frac{a}{2r}) - \Sigma F(\frac{2s}{a}, \frac{a}{2r}) \tanh(2m\pi a/2s)] \tag{54a}$$

and, further, be reduced to

$$\frac{H}{a}(\frac{K_1}{R} - 1)$$

$$= \frac{2s}{a} [E(\frac{2s}{a}, \frac{a}{2r}) + \Sigma (-1)^m \ln \frac{\cosh(4m\pi a/2s) + 1}{\cosh(4m\pi a/2s) - \cos(2r/a)(a/2s)}] \tag{54b}$$

Equations 49 through 54 are the drain spacing formulas for a two-layered drainage problem, covering the general as well as the limiting cases. One can see that the left hand sides of all of these drain spacing formulas are common, consisting of a given set of parameters, and also are known by the designer. Therefore, if the right hand sides of these drain spacing formulas can be calculated for a given set of the dimensionless parameters a/h, K_1/K_2, $a/2r$ and $2s/a$, then nomographs similar to those of Toksöz and Kirkham (1961) can be prepared, and by using such nomographs, the drain spacing, 2s, can be explicitly calculated.

Nomographs for Drain Spacing Calculations

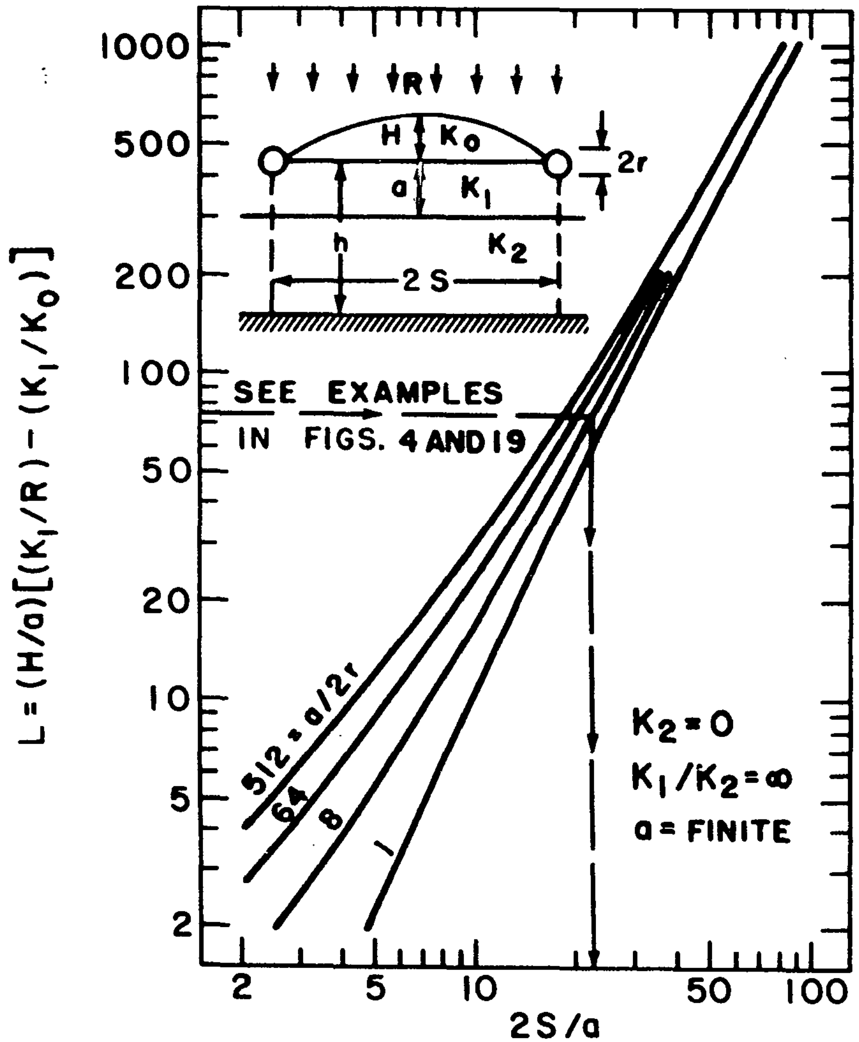
First, let us observe, as Wesseling (1964) pointed out, that Kirkham (1961) derived the factor $[1 - (R/K_1)]^{-1}$ by using the properties of the soil in the arched region only. Hence, we can consider the soil in the arched region as a separate soil layer having a hydraulic conductivity of K_0. As a result, the factor becomes $[1 - (R/K_0)]^{-1}$ and our two-layered drainage problem can thus be extended to a special case of a three-layered problem. The use of the new factor $[1 - (R/K_0)]^{-1}$ would only change the left hand side of Equations 49 through 54 to $(H/a)[(K_1/R) - (K_1/K_0)]$. Note that, when the soil in the arched region above the drains extends to a depth a below the drains, i.e., when $K_0 = K_1$, then $K_1/K_0 = 1$ and $(H/a)[(K_1/R) - (K_1/K_0)]$ would reduce to $(H/a)[(K_1/R) - 1]$.

Next, let us also observe that for our Equations 51 and 53, which correspond to our limiting cases 2 and 4, the drain spacing nomographs have already been given by Toksöz and Kirkham (1961), as their figures 2 and 1, respectively. One should note that if the captions in the ordinate axis of figures 2 and 1 of Toksöz and Kirkham (1961) are replaced by $1.364[(K_1/R) - (K_1/K_0)]$ and by $(H/h)[(K_1/R) - (K_1/K_0)]$, respectively, these figures may also be used for a special case of a two-layered problem with the interface of the soil layers passing through the drain centers, as Wesseling (1964) pointed out. Notice that in preparing these nomographs, Toksöz and Kirkham (1961) made use of the following assumption: $\ln[1/\sin(\pi r/2s)] = \ln(2s/\pi r)$ when $s >> r$. This assumption is perfectly valid for most practical purposes. In reality, we need only figure 1 of Toksöz and Kirkham (1961) because when the impermeable layer is located at a depth greater than half the drain spacing, i.e., when $h > s$, the effect of impermeable layer on drain spacing becomes negligible, as one may calculate from their figure. Hence our limiting case 2, that is $h = \infty$, is purely a theoretical case. When h is large but finite, the problem can still be solved by figure 1 of Toksöz and Kirkham (1961). Observe that our Equations 52 and 53 which correspond to our limiting cases 3 and 4, respectively, are similar. Therefore, figure 1 of Toksöz and Kirkham (1961) can also be used for our limiting case 3, that is for our Equation 52, provided that the symbol h in the figure is replaced by the symbol a. Such a figure is given as our Figure 4.

So far we have demonstrated that the drain spacing nomograph shown in our Figure 4 can be used to solve our Equations 52 and 53, corresponding to our limiting cases 3 and 4. We have also indicated that our limiting case 2, corresponding to our Equation 51 is a theoretical case and practical problems involving large s, that is, $h > s$ can still be solved by our Figure 4. Figures 5 through 18 are the drain spacing nomographs for the general case and the limiting case 1, corresponding to Equations 49 and 50, respectively. Figure 19 is the nomograph for the limiting case 5, corresponding to Equation 54. Notice that our limiting case 5, that is $K_2 = \infty$ may be thought to represent a soil layer overlying a coarse gravel bed that rests on top of an impermeable barrier such that no natural outlet exists for the drainage of gravel layer.

To prepare these nomographs, the right hand sides of Equations 49, 50 and 54 have been calculated by using a digital computer. The calculations have been made in terms of the dimensionless parameters a/h, K_1/K_2,

Figure 4. Drain spacing nomograph for $K_2 = 0$. This nomograph can also be used for $K_2 = K_1$ by replacing the symbol a in the nomograph by the symbol h.



EXAMPLE:

GIVEN:

$K_2 = 0$

$K_1 \doteq K_0 = 1.2\,m/day$

$R = 0.006\,m/day$

$H = 0.6\,m$

$a = 1.6\,m$

$2r = 0.2\,m$

$a/2r = 8$

$L = 74.6$

FROM GRAPH:

$2S/a = 22$

ANSWER:

$2S = (22)(1.6)$
$= 35.2\,m$

Figure 5. Drain spacing nomographs for $K_1/K_2 = 50$ and $a/h = 0$ and 0.2.

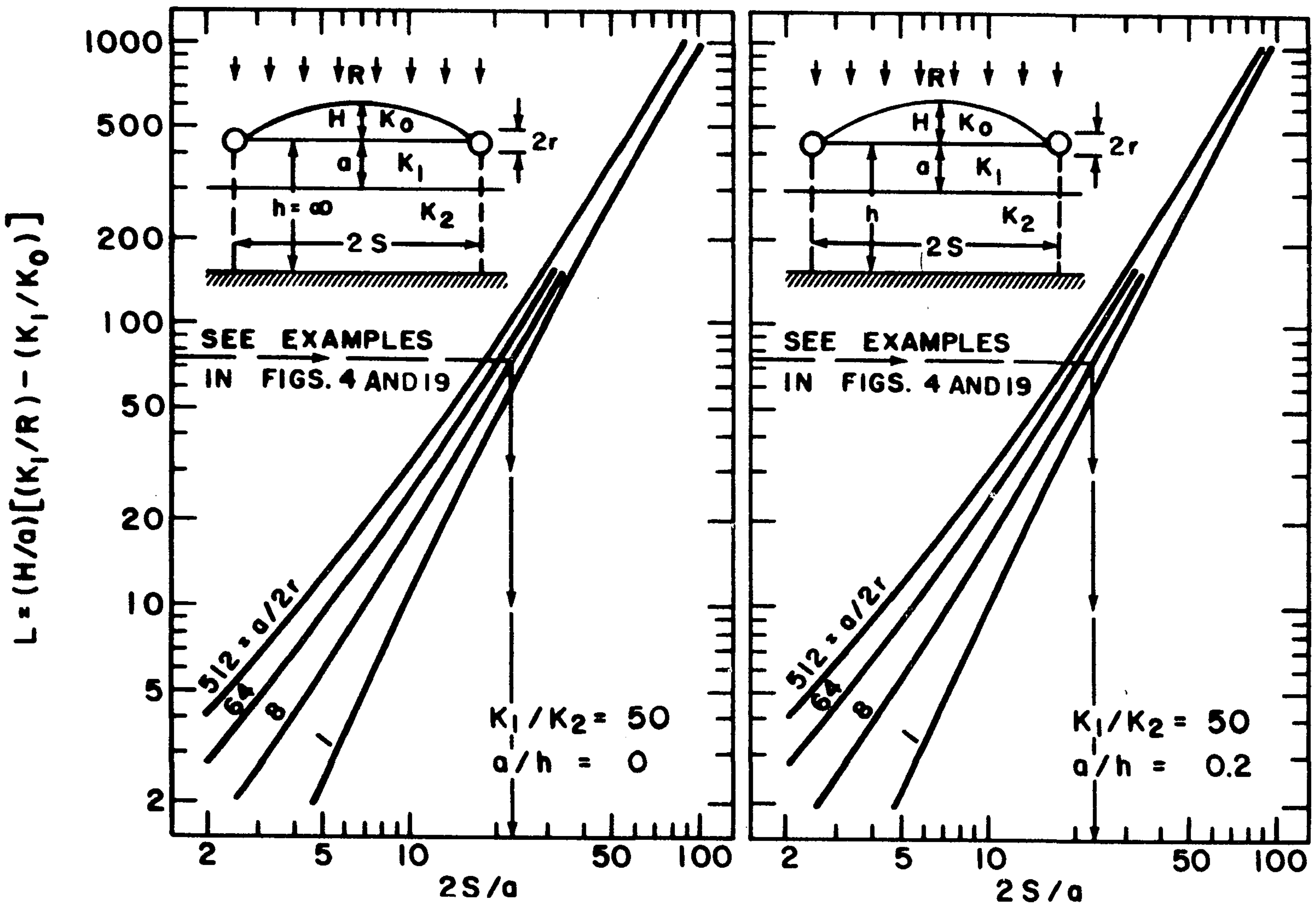

R
H
K_0
a
K_1
2r
h = ∞
K_2
2S
SEE EXAMPLES
IN FIGS. 4 AND 19
512 = a/2r
64
8
1
$K_1/K_2 = 50$
$a/h = 0$
h
$K_1/K_2 = 50$
$a/h = 0.2$
$L = (H/a)[(K_1/R) - (K_1/K_0)]$
2S/a
1000
500
200
100
50
20
10
5
2
2
5
10
50
100

Figure 6. Drain spacing nomographs for $K_1/K_2 = 50$ and $a/h = 0.4$ and 0.8.

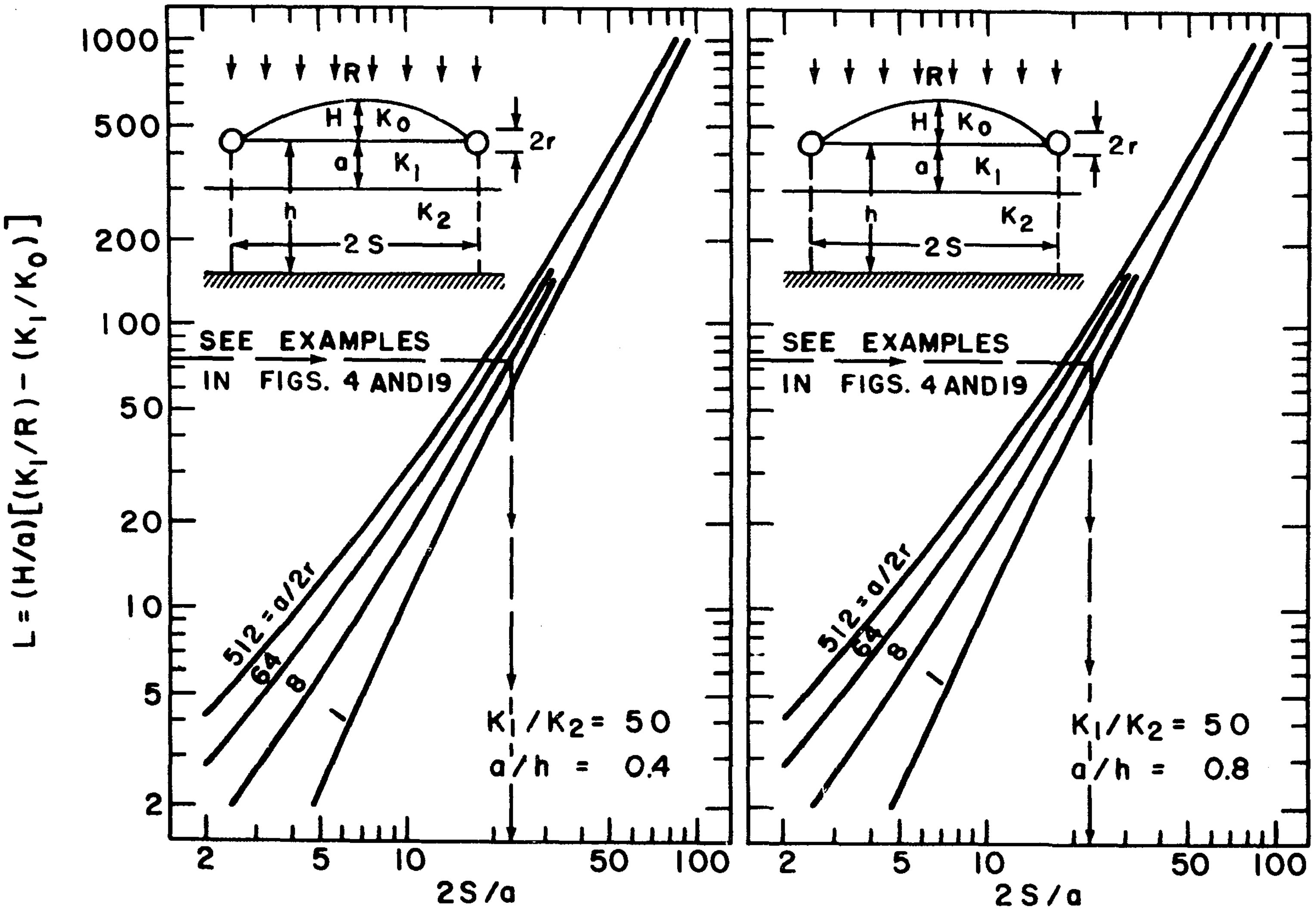

L = (H/a)[(K1/R) - (K1/K0)]
1000
500
200
100
50
20
10
5
2
R
H
K0
a
K1
h
K2
2S
2r
SEE EXAMPLES
IN FIGS. 4 AND 19
512 = a/2r
64
8
1
K1/K2 = 50
a/h = 0.4
a/h = 0.8
2
5
10
50
100
2S/a

Figure 7. Drain spacing nomographs for $K_1/K_2 = 10$ and $a/h = 0$ and 0.2.

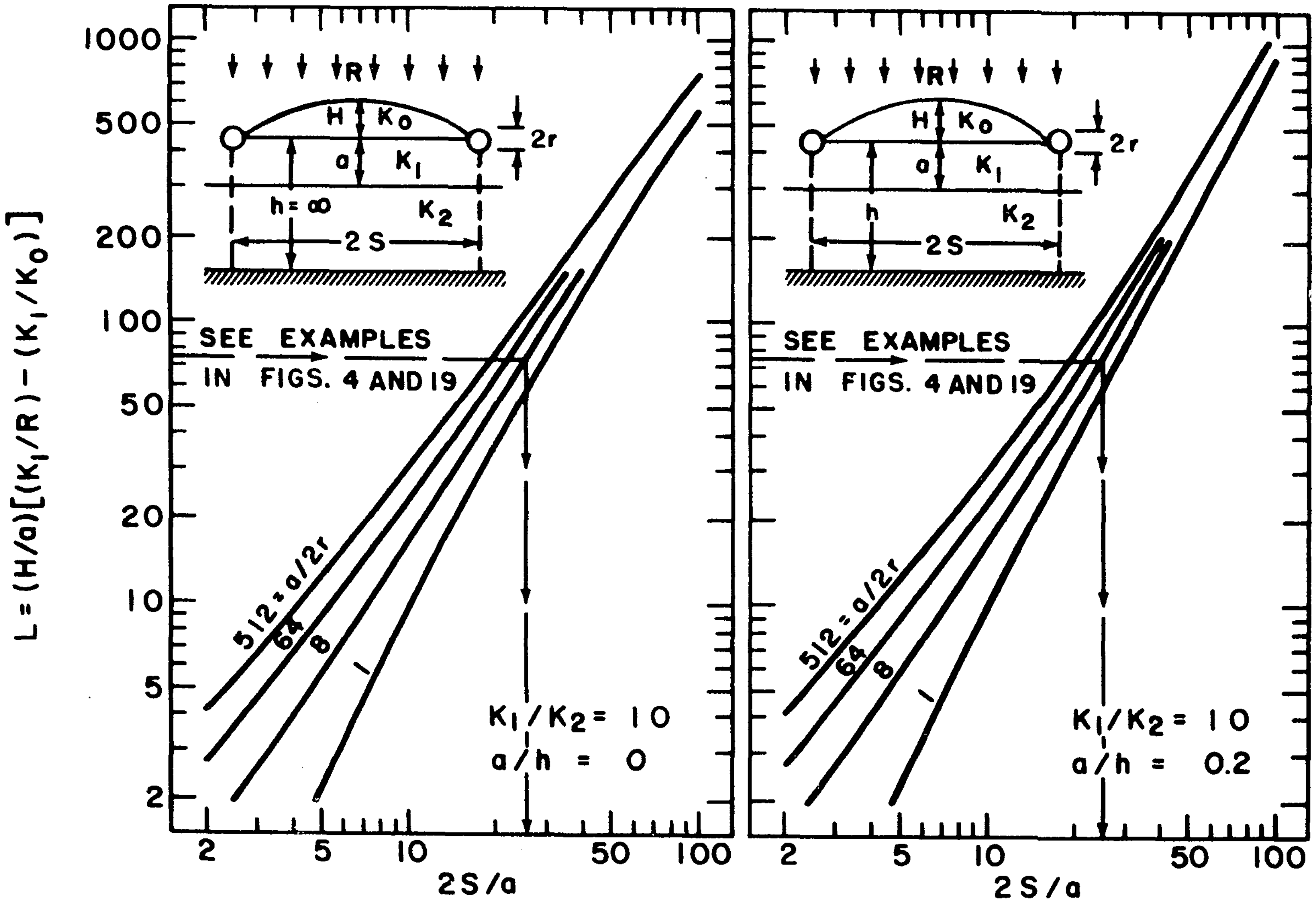

L = (H/a)[(K1/R) - (K1/K0)]
1000
500
200
100
50
20
10
5
2
R
H
K0
2r
a
K1
h = ∞
K2
2S
SEE EXAMPLES
IN FIGS. 4 AND 19
512 = a/2r
64
8
1
K1/K2 = 10
a/h = 0
2
5
10
50
100
2S/a
R
H
K0
2r
a
K1
h
K2
2S
SEE EXAMPLES
IN FIGS. 4 AND 19
512 = a/2r
64
8
1
K1/K2 = 10
a/h = 0.2
2
5
10
50
100
2S/a

Figure 8. Drain spacing nomographs for $K_1/K_2 = 5$ and $a/h = 0$ and 0.2.

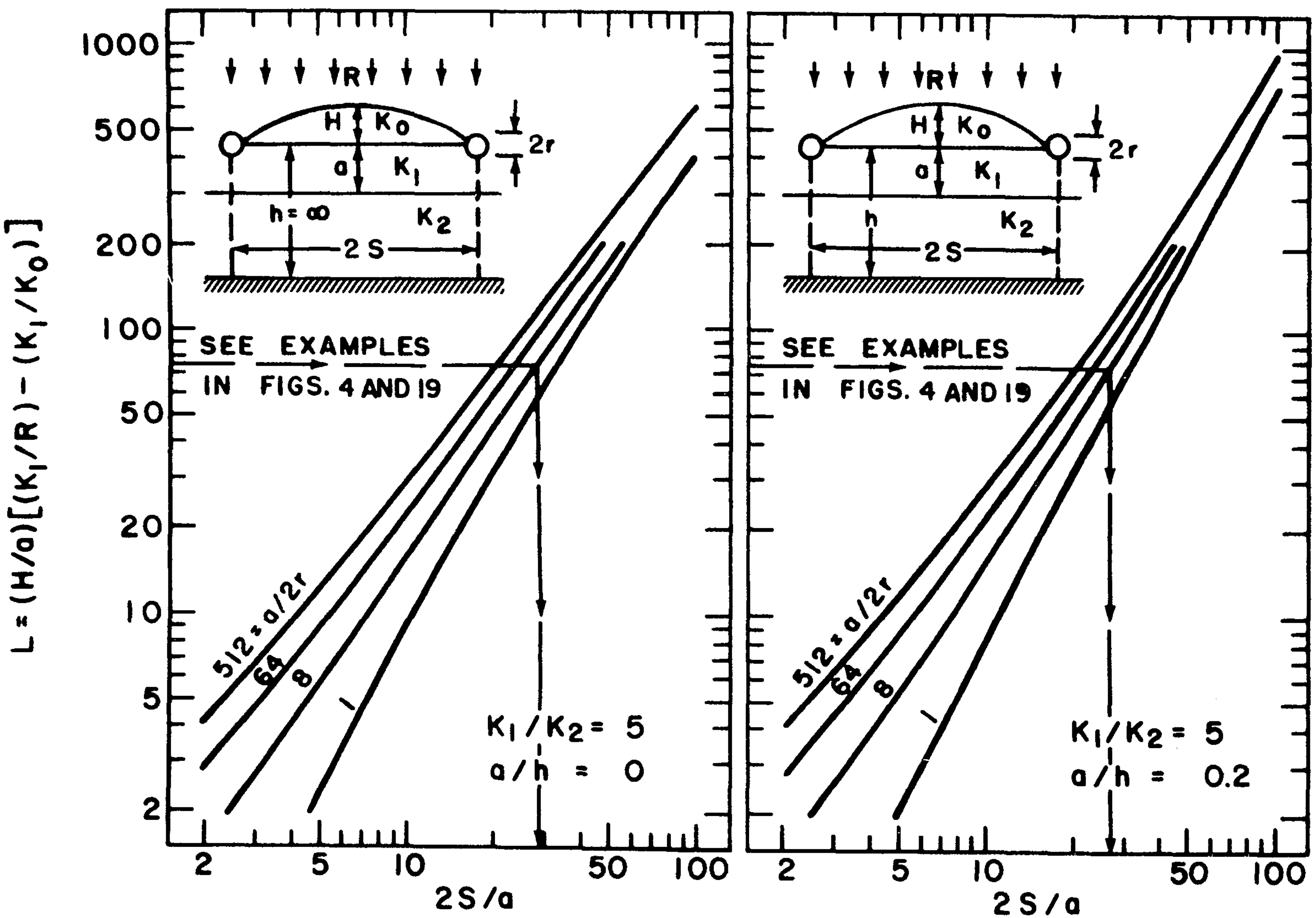

R
H
K_0
2r
a
K_1
h = ∞
K_2
2 S
1000
500
200
100
50
20
10
5
2
L = (H/a)[(K₁/R) − (K₁/K₀)]
SEE EXAMPLES
IN FIGS. 4 AND 19
512 = a/2r
64
8
1
$K_1/K_2 = 5$
a/h = 0
2
5
10
50
100
2S/a
h
$K_1/K_2 = 5$
a/h = 0.2

Figure 9. Drain spacing nomographs for $K_1/K_2 = 2$ and $a/h = 0$ and 0.2.

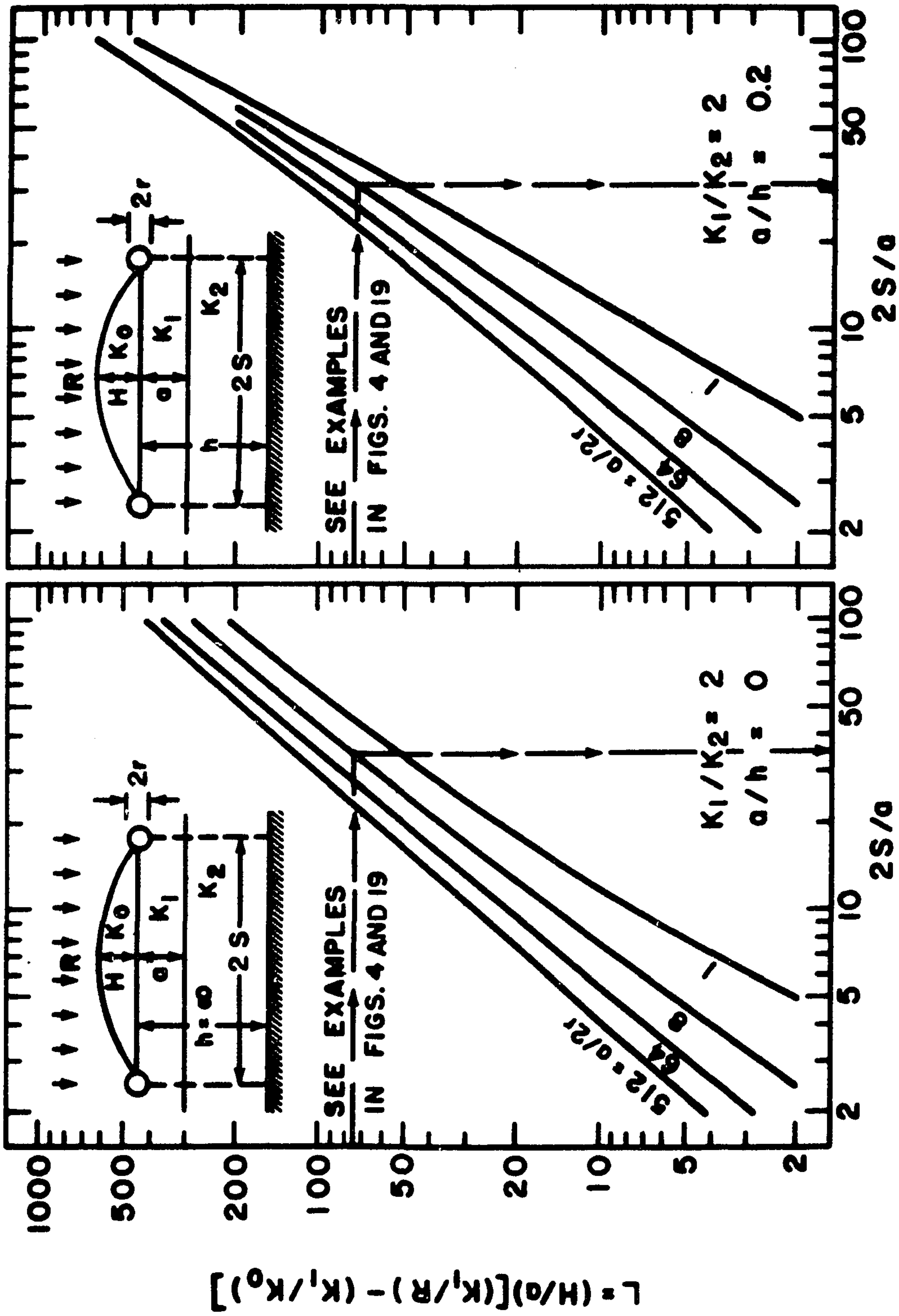

L = (H/a)[(K1/R) - (K1/K0)]
1000
500
200
100
50
20
10
5
2
2
5
10
50
100
2S/a
K1/K2 = 2
a/h = 0
K1/K2 = 2
a/h = 0.2
SEE EXAMPLES
IN FIGS. 4 AND 19
512 = a/2r
64
8
1
H
R
K0
a
K1
K2
2S
h = ∞
h
2r

Figure 10. Drain spacing nomographs for $K_1/K_2 = 2$ and $a/h = 0.4$ and 0.8.

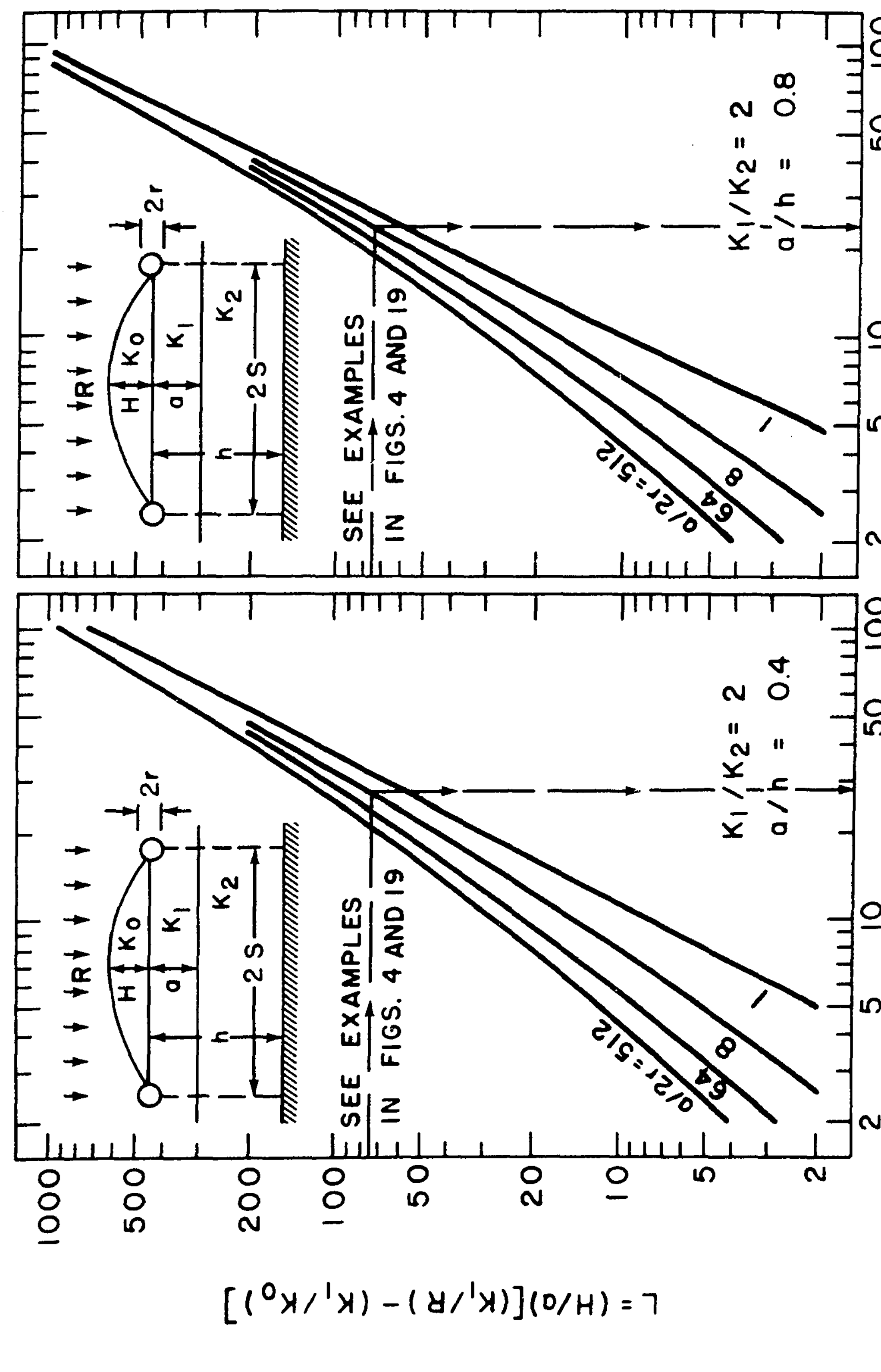
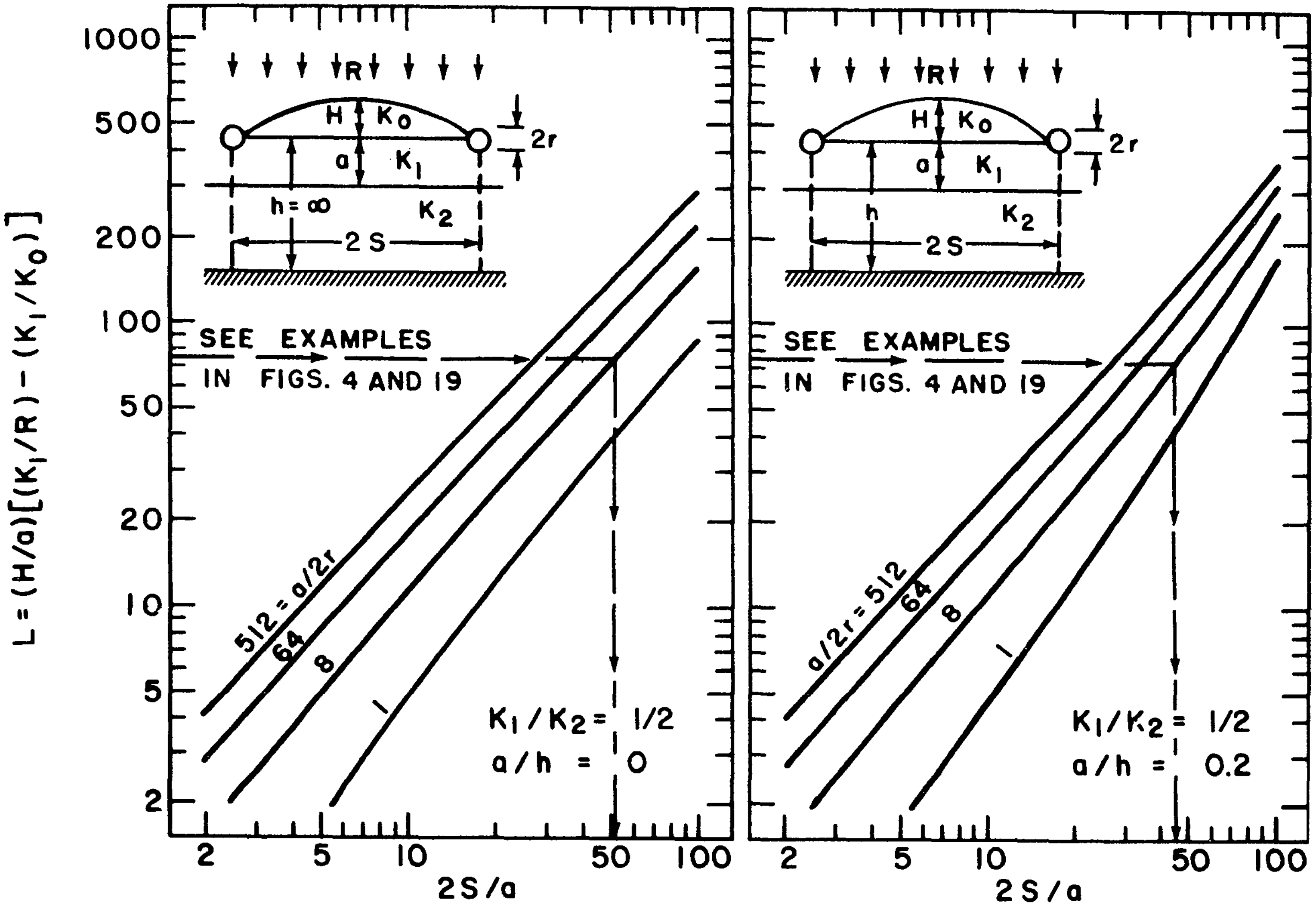

L = (H/a)[(K1/R) - (K1/K0)]
1000
500
200
100
50
20
10
5
2
2
5
10
50
100
2S/a
R
H
K0
a
K1
h
K2
2S
2r
SEE EXAMPLES
IN FIGS. 4 AND 19
a/2r=512
64
8
1
K1/K2 = 2
a/h = 0.4
a/h = 0.8

Figure 11. Drain spacing nomographs for $K_1/K_2 = 1/2$ and $a/h = 0$ and 0.2.



R
H
K_0
a
K_1
2r
h = ∞
h
K_2
2 S
SEE EXAMPLES IN FIGS. 4 AND 19
$L = (H/a)[(K_1/R) - (K_1/K_0)]$
1000
500
200
100
50
20
10
5
2
2S/a
2
5
10
50
100
512 = a/2r
a/2r = 512
64
8
1
$K_1/K_2 = 1/2$
$a/h = 0$
$a/h = 0.2$

Figure 12. Drain spacing nomographs for $K_1/K_2 = 1/2$ and $a/h = 0.4$ and 0.8.

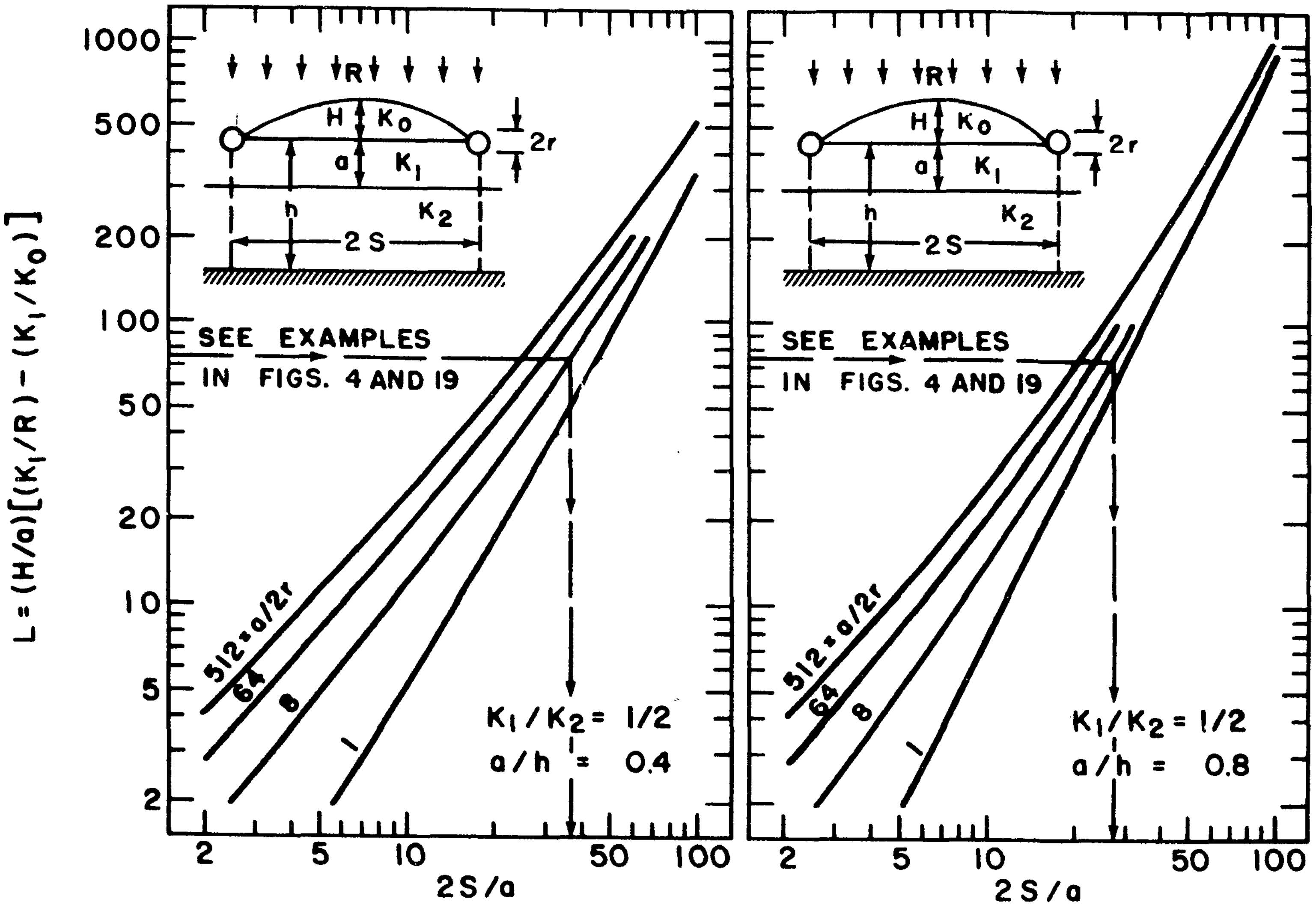
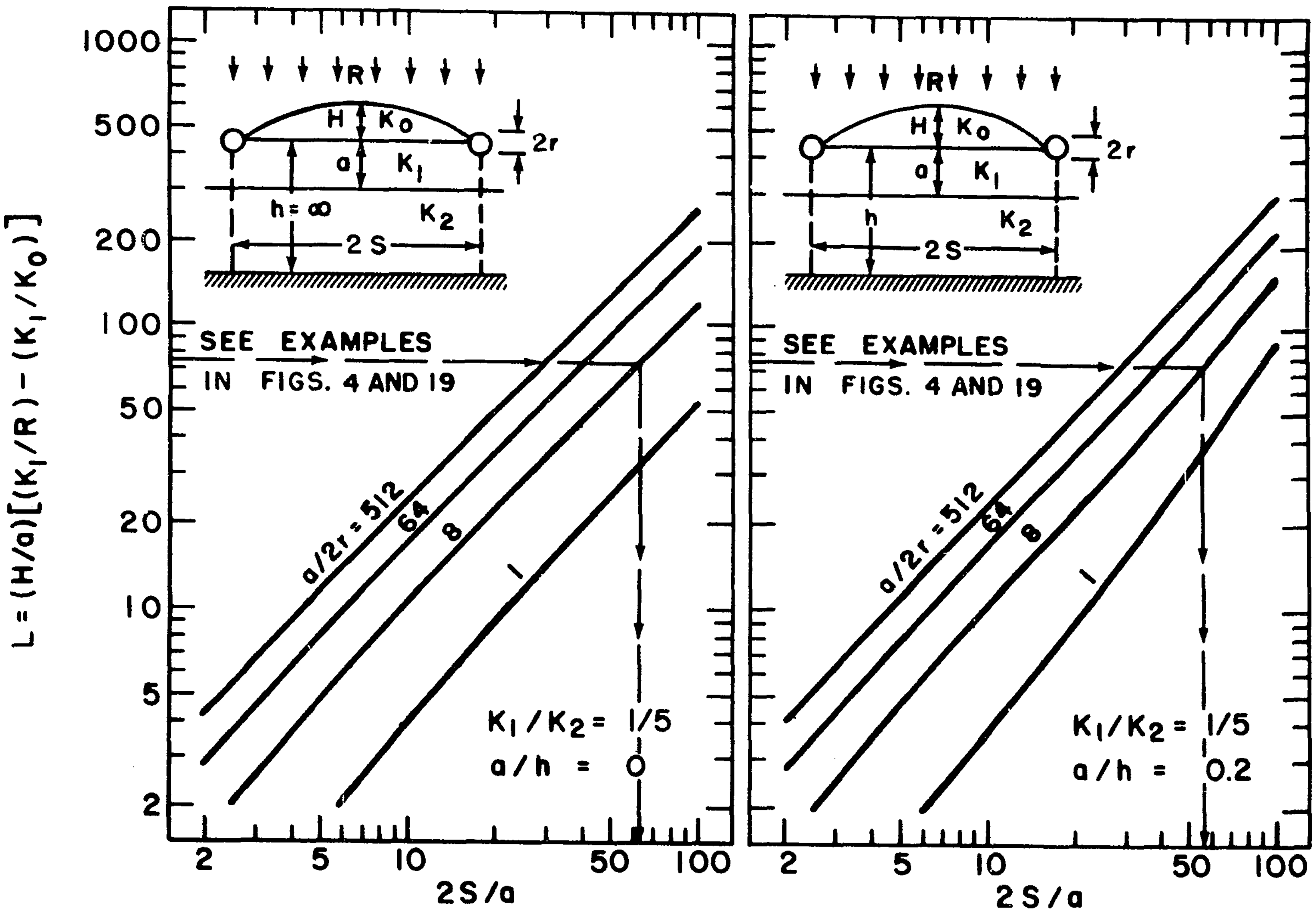
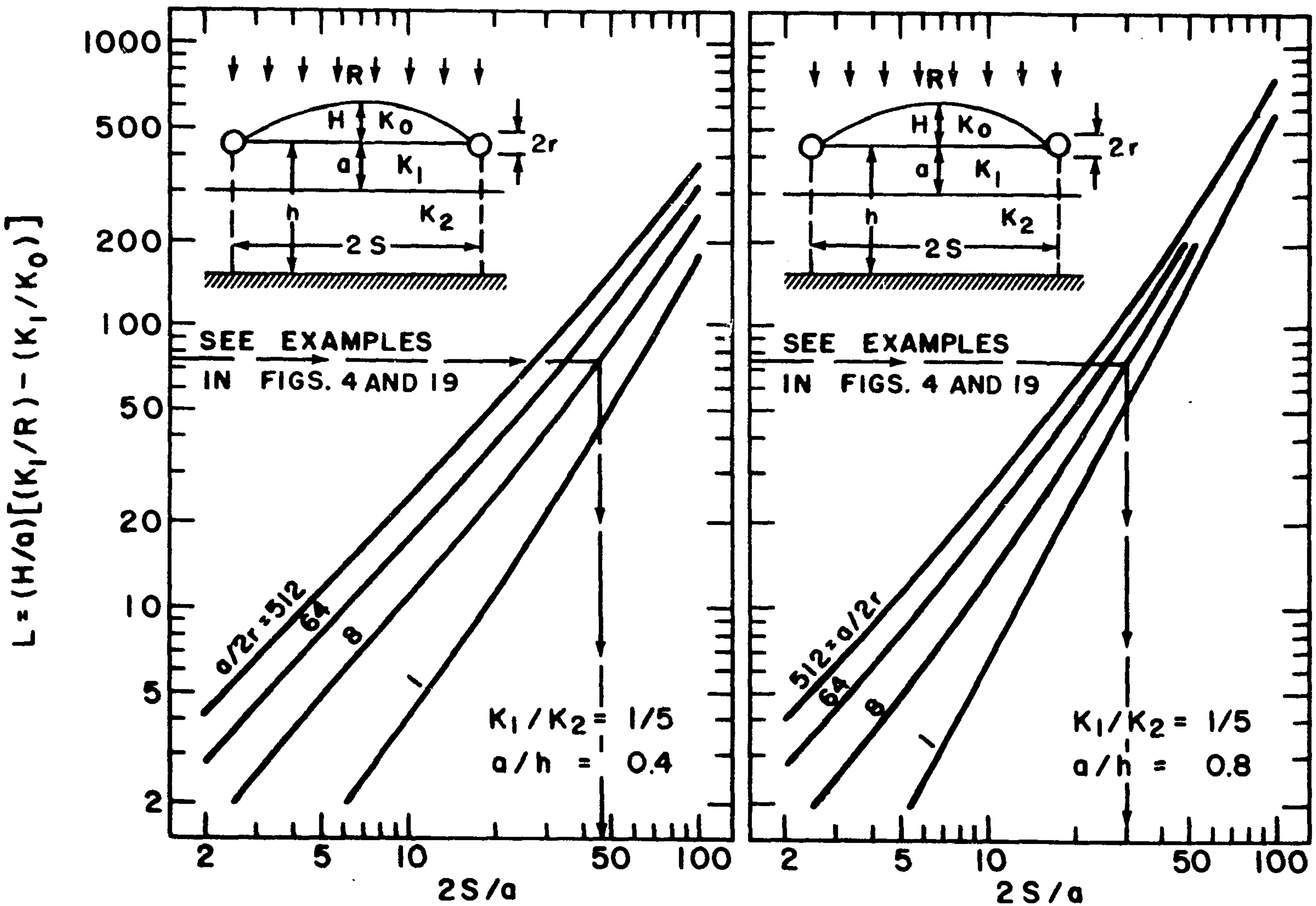

L = (H/a)[(K₁/R) - (K₁/K₀)]
1000
500
200
100
50
20
10
5
2
R
H
K₀
2r
a
K₁
h
K₂
2S
SEE EXAMPLES
IN FIGS. 4 AND 19
512 = a/2r
64
8
1
K₁/K₂ = 1/2
a/h = 0.4
K₁/K₂ = 1/2
a/h = 0.8
2
5
10
50
100
2S/a
2S/a

Figure 13. Drain spacing nomographs for $K_1/K_2 = 1/5$ and $a/h = 0$ and 0.2.



R
H
K_o
2r
a
K_1
h = ∞
K_2
2 S
SEE EXAMPLES
IN FIGS. 4 AND 19
a/2r = 512
64
8
1
$K_1/K_2 = 1/5$
$a/h = 0$
$L = (H/a)[(K_1/R) - (K_1/K_0)]$
2S/a
1000
500
200
100
50
20
10
5
2
2
5
10
50
100
R
H
K_o
2r
a
K_1
h
K_2
2 S
SEE EXAMPLES
IN FIGS. 4 AND 19
a/2r = 512
64
8
1
$K_1/K_2 = 1/5$
$a/h = 0.2$
2S/a
2
5
10
50
100

Figure 14. Drain spacing nomographs for $K_1/K_2 = 1/5$ and $a/h = 0.4$ and 0.8.



R
H
K_0
a
K_1
h
K_2
2r
2 S
SEE EXAMPLES IN FIGS. 4 AND 19
$L = (H/a)[(K_1/R) - (K_1/K_0)]$
a/2r = 512
64
8
1
K_1/K_2 = 1/5
a/h = 0.4
512 = a/2r
K_1/K_2 = 1/5
a/h = 0.8
2S/a
1000
500
200
100
50
20
10
5
2
2
5
10
50
100

Figure 15. Drain spacing nomographs for $K_1/K_2 = 1/10$ and $a/h = 0.4$ and 0.8.

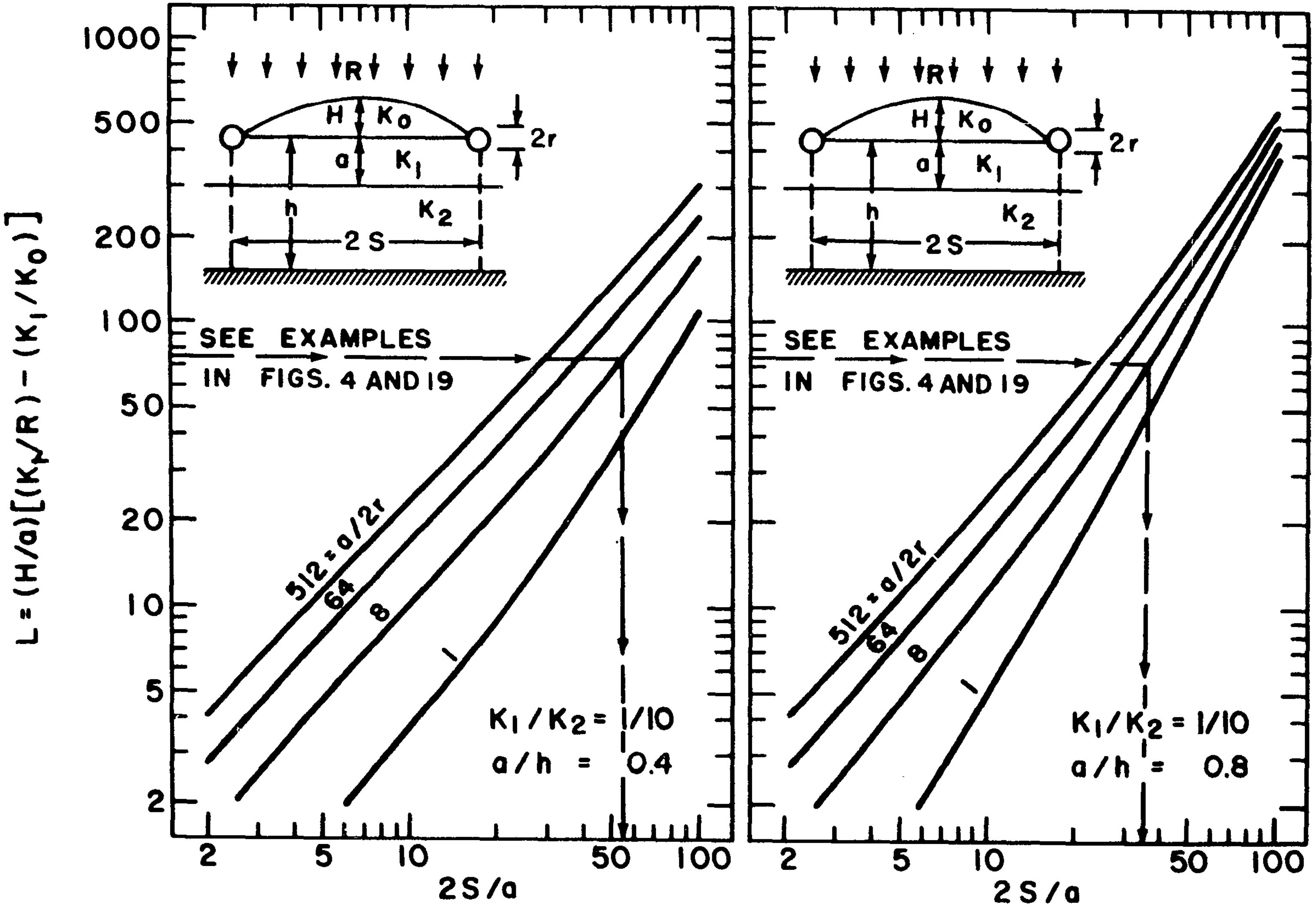
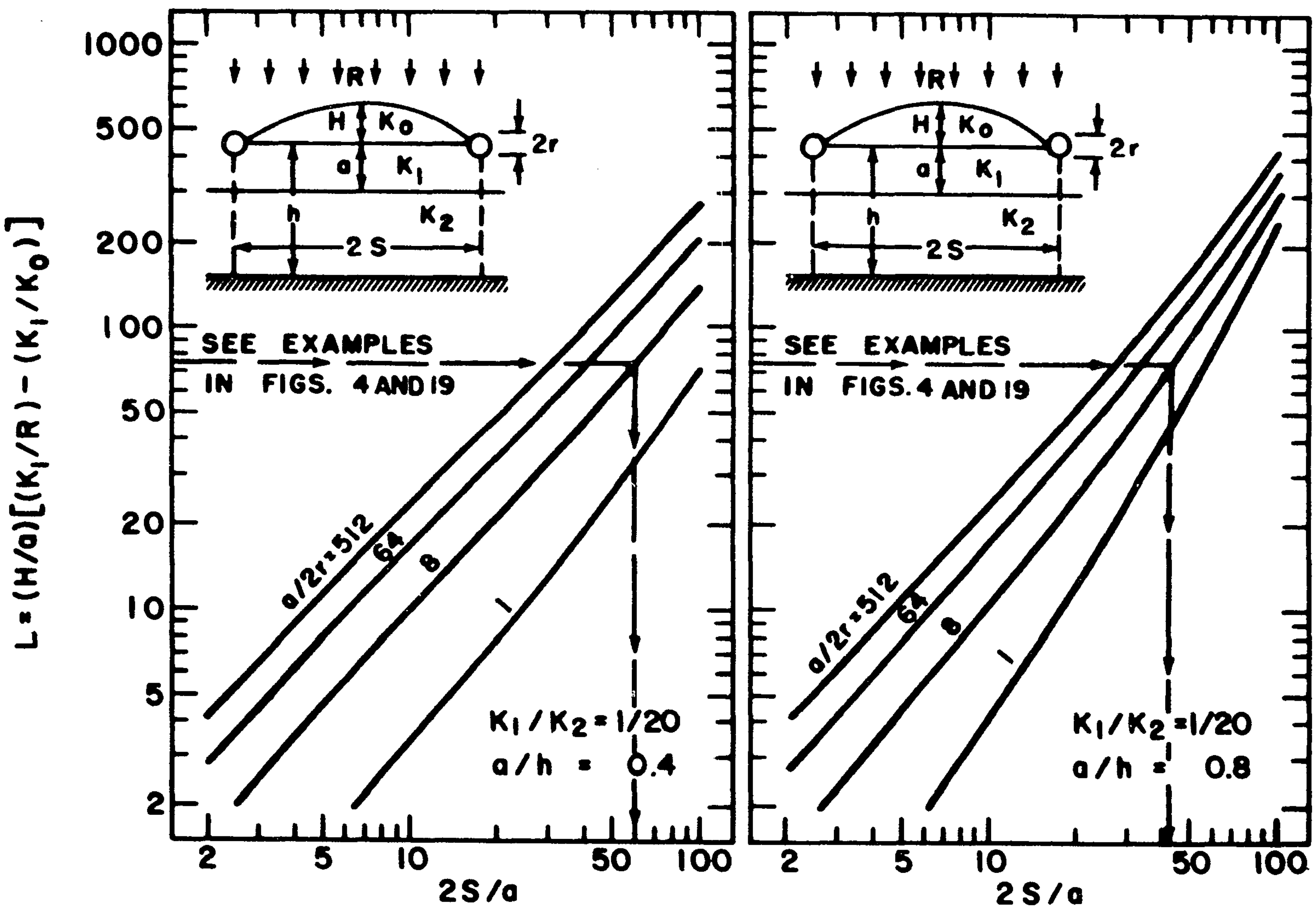
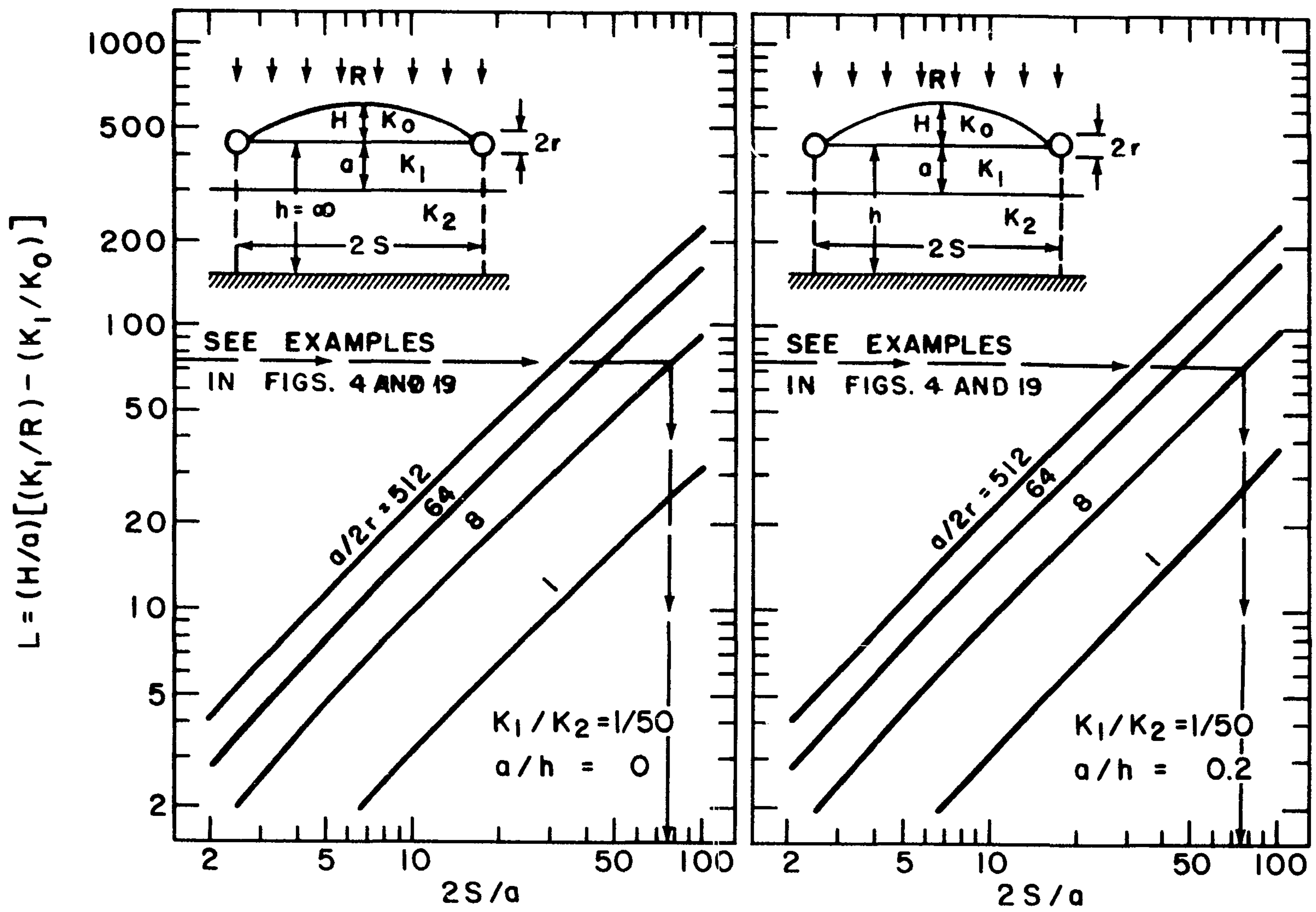

L = (H/a)[(K₁/R) − (K₁/K₀)]
1000
500
200
100
50
20
10
5
2
R
H
K₀
a
K₁
2r
h
K₂
2 S
SEE EXAMPLES
IN FIGS. 4 AND 19
512 = a/2r
64
8
1
K₁ / K₂ = 1/10
a / h = 0.4
a / h = 0.8
2
5
10
50
100
2 S / a

Figure 16. Drain spacing nomographs for $K_1/K_2 = 1/20$ and $a/h = 0.4$ and 0.8.



L = (H/a)[(K1/R) - (K1/K0)]
1000
500
200
100
50
20
10
5
2
R
H
K0
a
K1
2r
h
K2
2 S
SEE EXAMPLES IN FIGS. 4 AND 19
a/2r = 512
64
8
1
K1 / K2 = 1/20
a / h = 0.4
a / h = 0.8
2
5
10
50
100
2 S / a

Figure 17. Drain spacing nomograph for $K_1/K_2 = 1/50$ and $a/h = 0$ and 0.2.



L = (H/a)[(K1/R) - (K1/K0)]
1000
500
200
100
50
20
10
5
2
R
H
K0
a
K1
2r
h = ∞
K2
2S
SEE EXAMPLES
IN FIGS. 4 AND 19
a/2r = 512
64
8
1
K1 / K2 = 1/50
a/h = 0
2
5
10
50
100
2S/a
R
H
K0
a
K1
2r
h
K2
2S
SEE EXAMPLES
IN FIGS. 4 AND 19
a/2r = 512
64
8
1
K1 / K2 = 1/50
a/h = 0.2
2
5
10
50
100
2S/a

Figure 18. Drain spacing nomograph for $K_1/K_2 = 1/50$ and $a/h = 0.4$ and 0.8.

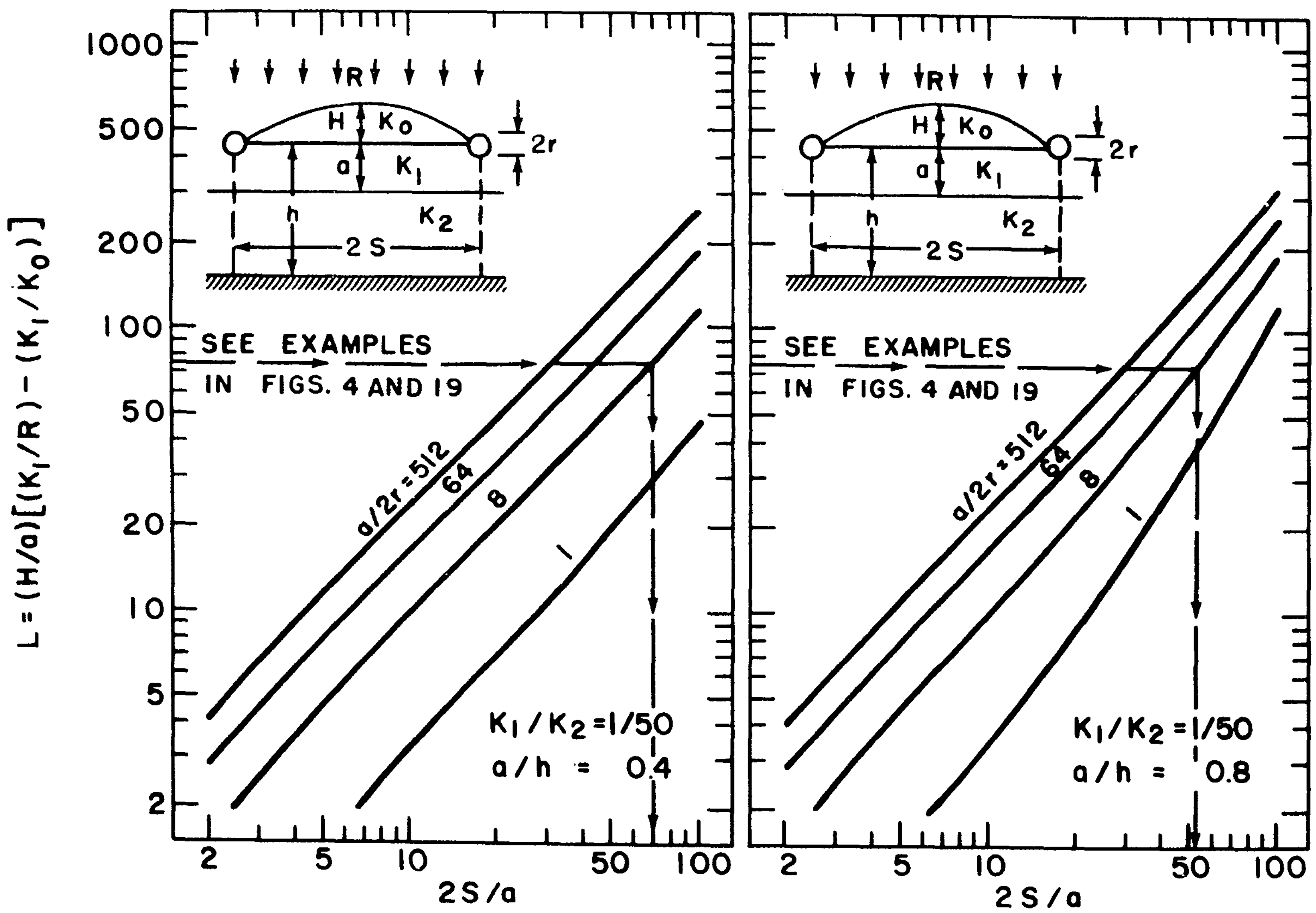
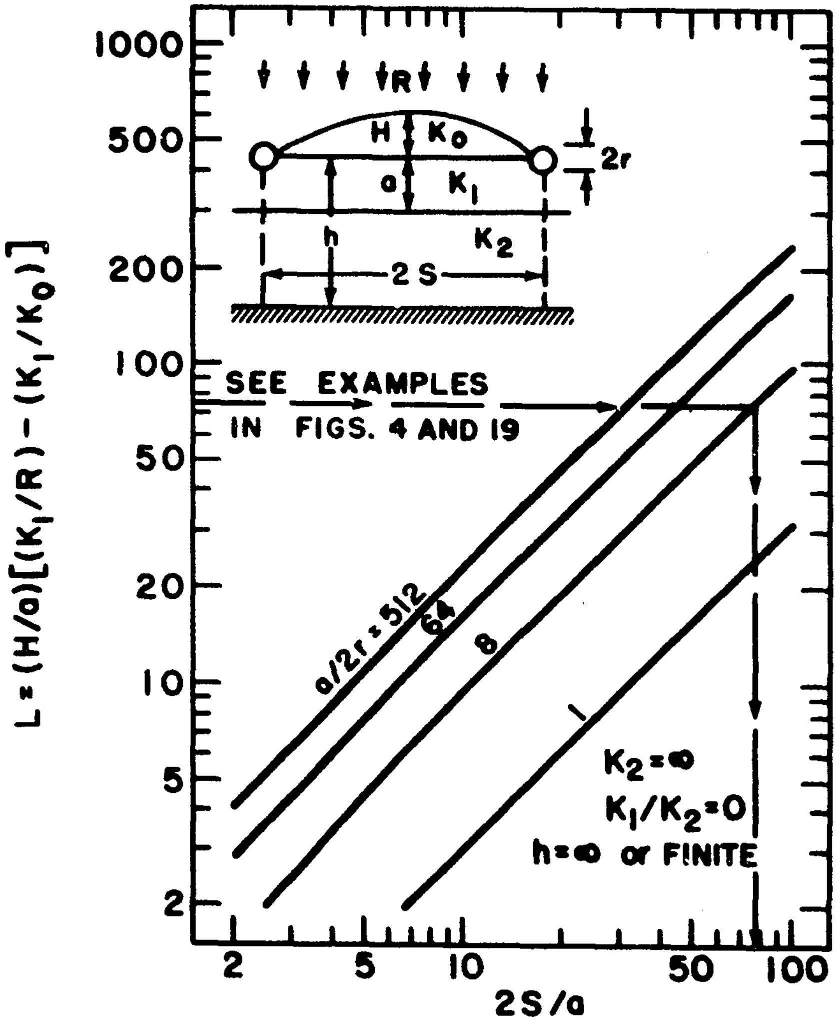

R
H
K_o
a
K_1
h
K_2
2S
2r
SEE EXAMPLES IN FIGS. 4 AND 19
a/2r = 512
64
8
1
$K_1/K_2 = 1/50$
a/h = 0.4
a/h = 0.8
$L = (H/a)[(K_1/R) - (K_1/K_o)]$
2S/a
1000
500
200
100
50
20
10
5
2
2
5
10
50
100

Figure 19. Drain spacing nomograph for $K_2 = \infty$.



EXAMPLE:

GIVEN:

$K_2 = \infty$

$K_1 = K_0 = 1.2$ m/day

R = 0.006 m/day

H = 0.6 m

a = 1.6 m

2r = 0.2 m

a/2r = 8

L = 74.6

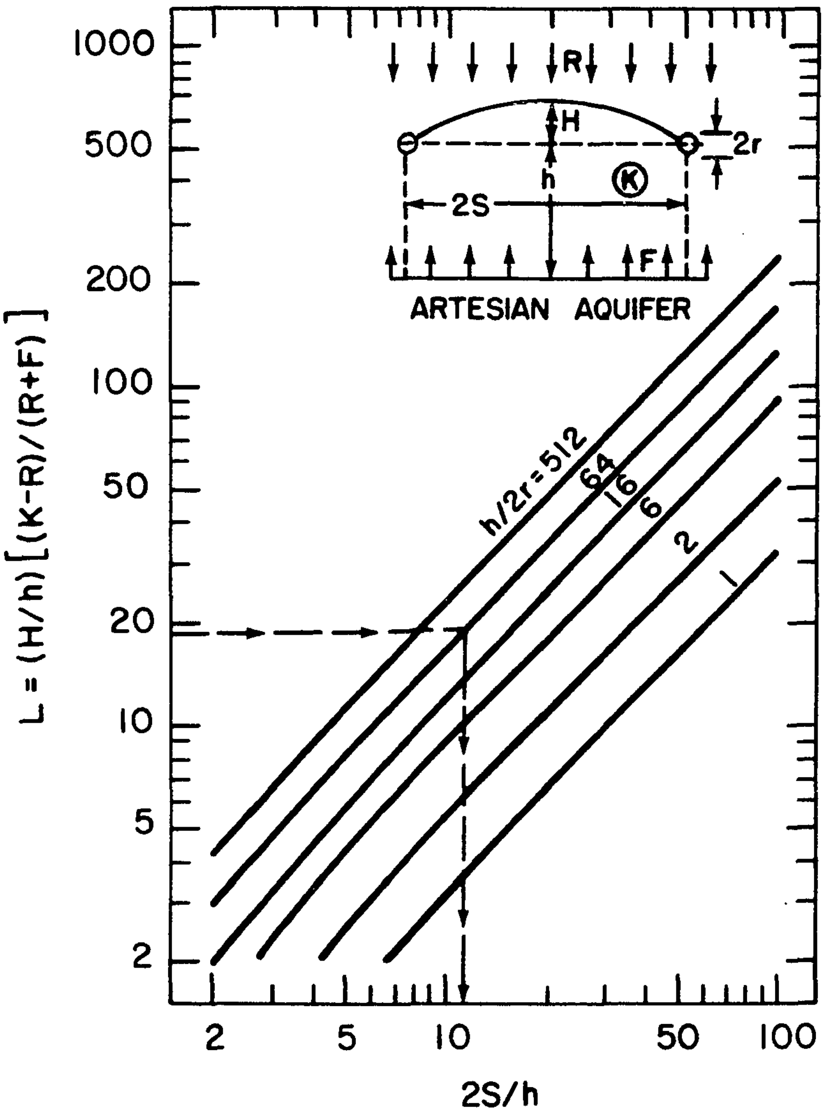
FROM GRAPH:

2S/a = 77

ANSWER:

2S = (77)(1.6)
= 123.2 m

Figure 20. Drain spacing nomograph for both surface recharge and artesian seepage. The nomograph is for solving Hinesly-Kirkham formula.



EXAMPLE

H=0.6 m
h=6.4 m
2r=0.1 m
R=4 mm/day
F=2 mm/day
K_l=1.2 m/day
h/2r=64
L=18.7
WE READ:
2S/h=11.6
2S=72.4 m

a/2r, and 2s/a. For Equation 49, the values of the dimensionless parameters were as follows: $a/h = 0.2$, 0.4, and 0.8; $K_1/K_2 = 50$, 20, 10, 5, 2, 1/2, 1/5, 1/10, 1/20 and 1/50; $a/2r = 1$, 8, 64, 512; $2s/a = 2$, 4, 8, 16, 32, 64, 128, and 256. For Equation 50: $a/h = 0$ whereas values for K_1/K_2, a/2r, and 2s/a were as for Equation 49. For Equation 54: $a/2s = 2.5$, 5, 10, 20, 40, 80, and 160; values for a/2r were as for Equation 49. The computer outputs resulted in values similar to those given by Table 3 of Toksöz and Kirkham (1961) and the drain spacing nomographs shown in Figures 5 through 19 were also prepared following the same steps used by them. For Equations 49 and 50, we have used four different values of a/h, and 10 different values of K_1/K_2. To represent the full array of these parameters, one would have needed $(4)(10) = 40$ nomographs. Notice that we have included only 28 of these 40 nomographs as our Figures 5 through 18. The main reason for excluding some of the nomographs was to save space while staying within reasonable limits of accuracy. More will be said about this later, under the subheading Discussion of Results.

An interesting and useful addition to the above nomographs is shown in Figure 20. This figure provides graphical solutions to drainage problems where both downward surface recharge, R, and upward artesian seepage, F, must be taken into account. This problem has already been solved by Hinesly and Kirkham (1966). Their equation 15 can be reduced to the following form

$$\frac{H}{h}\left(\frac{K-R}{R+F}\right) = \frac{2s}{h}\,\frac{2}{\pi}\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m}\,\frac{\cosh\{(m\pi/4)[\{2s/h)-(2r/h)]\}-1}{\sinh[(m\pi/4)(2s/h)]} \qquad (55)$$

for which Figure 20 has been prepared. See Appendix A for the required steps to reduce equation 15 of Hinesly and Kirkham (1966) to our Equation 55.

Discussion of Results

Let us write Equation 49 as

$$2s = H(\frac{K_1}{R} - 1)(\frac{1}{E + \Sigma F - \Sigma FG}) \qquad (56)$$

where, for brevity, we have dropped the arguments of the functions defined by Equations 46, 47 and 48. Notice that in Equations 46 and 47, the symbol h does not appear. In view of Equations 46, 47 and 48, one may deduce that the term ΣFG in Equation 56 reflects the effect of lower soil layer on the drain spacing, 2s. We will now consider the two conceivable types of errors that could be made in calculating the drain spacings.

The first type of error occurs when the hydraulic conductivity of the lower layer is assumed to be zero, that is $K_2 = 0$, while it is not zero. This assumption means $K_1/K_2 = \infty$ which yields $G = 0$. The drain spacings calculated on the basis of this assumption will always be smaller than the correct spacings, because if $G = 0$ then the term FG in Equation 56 would vanish. In reality, however, $K_2 \neq 0$ and also $G \neq 0$.

The percentage error in drain spacings resulting from the assumption $K_2 = 0$, will be

$$\frac{[1/(E + \Sigma F - \Sigma FG)] - [1/(E + \Sigma F)]}{[1/(E + \Sigma F - \Sigma FG)]} 100 = \frac{\Sigma FG}{E + \Sigma F} 100 \qquad (57)$$

One may observe, in view of Equations 46, 47 and 48, that such an error is not only a function of the soil parameters K_1/K_2, as it is commonly thought, but also a function of the geometrical parameters of the flow medium, i.e., of a, h, and 2r. It follows that statements like "when the hydraulic conductivity of the upper layer is five to 10 times greater than the hydraulic conductivity of the lower layer, then the lower layer can be assumed to be impermeable" may be misleading. Obviously, when the hydraulic conductivity of the lower layer is less than that of the upper layer, percentage errors in drain spacings resulting from the assumption $K_2 = 0$ would be smaller as compared to errors that would result when the hydraulic conductivity of the lower layer is higher than that of the upper layer. Furthermore, such errors will decrease as the thickness of the upper layer increases. Table 1 is prepared by using E, F and G values obtained from computer outputs and shows the expected errors for some selected values of $K_1/K_2 = a/2s$ and for $a/2r = 8$ and $a/h = 0.2$ and validates the preceding statements.

The second type of error results when the lower layer is completely ignored, that is when the upper layer is assumed to extend to a depth h, or simply when it is assumed $K_2 = K_1$. The drain spacing would be computed from Equation 53, rewritten in the form

$$2s = H\left(\frac{K_1}{R} - 1\right) \frac{1}{E_h + \Sigma F_h} \tag{58}$$

where E_h and F_h are defined by Equations 46 and 47 by replacing the symbol a in these equations by the symbol h. The correct drain spacing is of course given by Equation 56. The erroneous drain spacings, resulting from the assumption $K_2 = K_1$, would be larger if $K_1 > K_2$ and they will

Table 1. Percentage errors that would result in drain spacings when the hydraulic conductivity of the lower layer is assumed to be zero while it is not zero. The errors have been computed from Equation 57 for $a/2r = 8$ and $a/h = 0.2$, and for selected values of $2s/a$ and K_1/K_2, as indicated

$\frac{2s}{a}$	$\frac{K_1}{K_2}$	Percent Error	$\frac{K_1}{K_2}$	Percent Error
256	5/1	43	1/5	93
256	10/1	27	1/2	86
128	5/1	41	1/5	89
128	10/1	26	1/2	83
64	5/1	37	1/5	84
64	10/1	23	1/2	77
32	5/1	31	1/5	74
32	10/1	19	1/2	68
16	5/1	20	1/5	57
16	10/1	12	1/2	51
8	5/1	9	1/5	33
8	10/1	5	1/2	28

be smaller if $K_1 < K_2$. The absolute value of the percentage error will, for both cases, be

$$[1 - \frac{E + \Sigma F - \Sigma FG}{E_h + \Sigma F_h}]\ 100 \qquad (59)$$

Such errors will decrease as the thickness of the upper layer increases.

Table 2 shows a set of drain spacings, calculated for the following set of data: $H = 0.6m$, $a = 1.2m$, $2r = 0.2m$, $K_1 = K_0 = 1.2$ m/day, and $R = 0.006$ m/day. The parameters h and K_2 are assumed to vary, as indicated in table 2. Using the above data in the left hand side of our drain spacing formulas, that is in $L = (H/a)[(K_1/R) - (K_1/K_0)]$, yields a constant value of $L = 74.6$. This constant value has been used to calculate the spacings given in table 2. The arrows shown in Figures 4 through 19 refer to spacing calculations made for table 2, and therefore, each arrow indicates a specific example. See also Figures 4, 19, and 20 for detailed examples, showing the use of the nomographs. To save space, nomographs for the following cases are not included in Figures 4 through 19: $K_1/K_2 = 20$; $a/h = 0.4$ and 0.8 for $K_1/K_2 = 10$ and 5; $a/h = 0$ and 0.2 for $K_1/K_2 = 1/10$ and 1/20. However, with the given nomographs, drain spacings for the above missing cases can be calculated by interpolation. To minimize interpolation errors, a series of drain spacings should be plotted against the corresponding values of the parameter in question. The resulting points should then be connected with a smooth curve and this curve should be used to carry out the interpolation. Figure 21 describes,

Table 2. Calculated drain spacings in meters for $H = 0.6$m, $a = 1.6$m, $2r = 0.2$m, $K_1 = K_0 = 1.2$ m/day, and $R = 0.6$ cm/day. K_2 and h vary, as indicated

$\frac{K_1}{K_2}$	$\frac{a}{h} = 1.0$	$\frac{a}{h} = 0.8$	$\frac{a}{h} = 0.4$	$\frac{a}{h} = 0.2$	$\frac{a}{h} = 0$[a]
		→ → h increases → →			
∞[b]	36.8				
50 ↓		36.0	36.5	36.8	36.8
10 K_2		36.8	38.0	39.0	42.0
5 increases		36.8	40.0	42.0 (44.0)	46.0
2 ↓		36.8	45.0	50.0	56.0
1[c]	36.8[c]				
1/2		43.0	59.0	72.0	83.0
1/5		48.0	74.0	90.0 (88.0)	101.0
1/10		56.0	90.0	101.0	112.0
1/50		83.0	112.0	118.0	122.0
0[d]		123.2	123.2	123.2	123.2

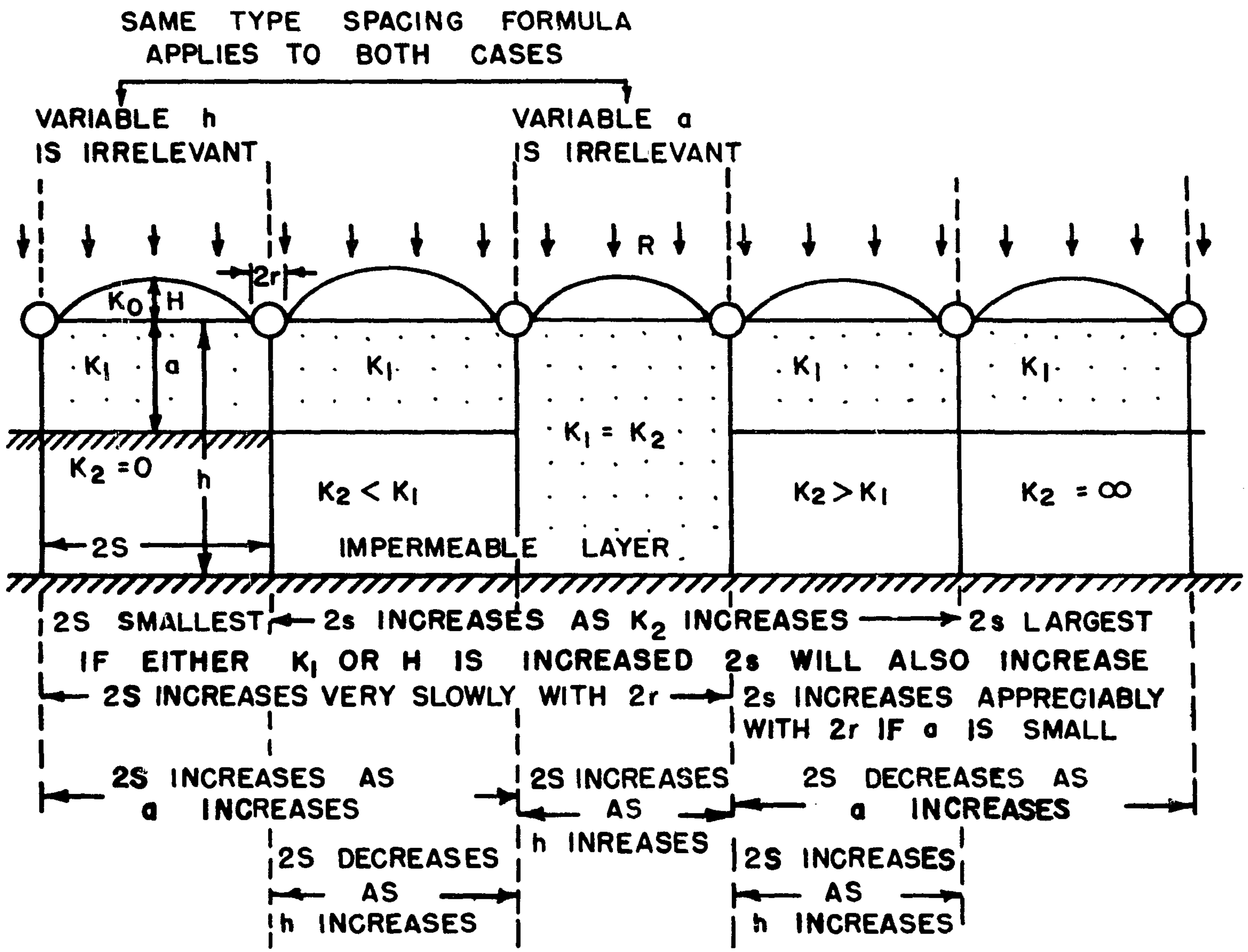
[a] $h = \infty$

[b] $K_2 = 0$

[c] $K_1 = K_2$ and $a = h$

[d] $K_2 = \infty$

Figure 21. Qualitative description of the effect of various parameters on drain spacing, 2s.


SAME TYPE SPACING FORMULA APPLIES TO BOTH CASES
VARIABLE h IS IRRELEVANT
VARIABLE a IS IRRELEVANT
R
2r
K0
H
K1
a
K2 = 0
h
2S
K2 < K1
K1 = K2
IMPERMEABLE LAYER
K2 > K1
K2 = ∞
2S SMALLEST
2s INCREASES AS K2 INCREASES
2s LARGEST
IF EITHER K1 OR H IS INCREASED 2s WILL ALSO INCREASE
2S INCREASES VERY SLOWLY WITH 2r
2s INCREASES APPRECIABLY WITH 2r IF a IS SMALL
2S INCREASES AS a INCREASES
2S INCREASES AS h INREASES
2S DECREASES AS a INCREASES
2S DECREASES AS h INCREASES
2S INCREASES AS h INCREASES

in qualitative terms, the effect of various parameters on drain spacings, and should prove to be useful in assessing, at least, the general direction of such effects, and in explaining the interactions among various design parameters.

In solving the two-layered problem, we have used two physical artifices and thus "linearized" the streamlines along the line connecting the centers of the drains. Dagan (1965) has solved exactly the same problem by what he calls "an approximate approach". In his approximate approach, Dagan combined a mathematical linearization with the Dupuit-Forcheimer theory. Along the line connecting the centers of the drains, Dagan linearized the streamlines within the segment $0 \leq x \leq 2h$. In linearizing the streamlines, he did not, however, use any physical artifices. Instead, he started from the non-linear equation for the free surface, that is

$$\left(\frac{\partial \bar{\phi}}{\partial x}\right)^2 + \left(\frac{\partial \bar{\phi}}{\partial y}\right)^2 - (R + K)\frac{\partial \bar{\phi}}{\partial y} + KR = 0 \qquad (60)$$

For Equation 60, see Dagan (1964). By ignoring the quadratic terms as well as the term $R(\partial \bar{\phi}/\partial y)$, Equation 60 is linearized, and becomes

$$\frac{\partial \bar{\phi}}{\partial y} = R \qquad (61)$$

Outside the zone of linearization, that is within the segment $2h < x < s$, Dagan assumed the flow to be essentially <u>horizontal</u> and used the Dupuit-Forcheimer theory. Using the linearized theory and the Dupuit-Forcheimer theory, he developed two independent expressions for the water table height at a distance $x = 2h$ from the drain. He designates this water table height by the symbol h_1. The drain spacings from Dagan's formula are found by eliminating h_1 between the two expressions.

For the case $K_1 > K_2$, one would expect a fairly good agreement between the drain spacings calculated from Dagan's formula and from our Equation 49. This is because, as one can see from Figure 3b, when $K_1 > K_2$ the flow is somewhat horizontal, as Dagan assumed, and the Dupuit-Forcheimer theory can be used within the segment $2h < x < s$. On the other hand, one would also expect that the drain spacings calculated from Dagan's formula would deviate somewhat from the spacings obtained from our Equation 49, when $K_1 < K_2$. This is because, as one can see from Figure 3d, when $K_1 < K_2$, the streamlines are not anymore horizontal within the segment $2h < x < s$, as Dagan assumed. One may deduce, from an inspection of Figure 3d and e, that as K_2 increases while K_1 stays constant, that is as K_1/K_2 decreases, the streamlines tend to approach to a vertical direction - a fact that has been reported by Dumm (1966) - and the applicability of the Dupuit-Forcheimer theory becomes highly questionable.

Let us now return to Table 2. The two spacings given in parenthesis in Table 2 have been calculated from Dagan's formula. One sees that the agreement between the spacings obtained from his formula and from our nomographs agree well not only for the case $K_1/K_2 = 5$ but also for the case $K_1/K_2 = 1/5$, despite the fact that the applicability of the Dupuit-Forcheimer theory can be disputed on theoretical grounds. This paradox, however, can be explained. From Figure 3d, one sees that the maximum hydraulic head at $x = s$ is $H = 1.06$ feet. If one calculates further $\phi_1(2h,0) = 0.93$ foot, one sees that, $[\phi_1(2h,0)/H]100 = (0.93/1.06)100 = 88$ percent of the maximum hydraulic head has already been dissipated between a distance of $x = r$ and $x = 2h$. This means that the water table within the segment $2h < x < s$ is almost flat, as Dagan points out. Therefore, Dagan's formula works not because the flow is horizontal but because the major portion of the hydraulic head dissipation occurs within the segment $2h < x < s$ where this head dissipation is properly accounted for by the linearized theory. It should be pointed out that it is not possible to calculate drain spacings from Dagan's formula for, say, K_1/K_2 = either 10 or 1/10, because an essential graph for such calculations is available for the range $1/9 < K_1/K_2 < 9$ only. It should also be pointed out that his method of solution does not permit one to prepare flow nets.

The following approximations are true if $s >> h$, a, m, and r.

$$\ln \frac{1}{\sin(\pi r/2s)} = \ln \frac{2s}{\pi r}$$

$$\cos \frac{m\pi a}{s} = 1 \, ; \quad e^{(m\pi a/s)} = 1$$

$$\coth \frac{m\pi a}{s} = 1 = \coth \frac{m\pi a}{s} = \frac{s}{m\pi a}$$

$$e^{(m\pi a/s)} = 1$$

$$\sinh \frac{m\pi a}{s} = \frac{m\pi a}{s}$$

$$\coth \frac{m\pi(h-a)}{s} = \frac{s}{m\pi(h-a)}$$

Inserting these approximations into Equation 39 yields

$$H = \frac{2Rs}{\pi K_1} \{\ln \frac{2s}{\pi r} + \Sigma \frac{1}{m} (1 - \cos m\pi)(\frac{s}{m\pi a})$$

$$[1 - \frac{s}{m\pi a} \frac{1}{(K_1/K_2)[s/m\pi(h-a)] + (s/m\pi a)}]\} \tag{62}$$

If we define $d = h - a$ and observe that $(1 - \cos m\pi)/m^2 = \pi^2/4$, then Equation 62 reduces to

$$H = \frac{2Rs}{\pi K_1} \{\ln \frac{2s}{\pi r} + \frac{\pi s}{4a} [1 - \frac{1}{(K_1/K_2)(a/d) + 1}]\} \tag{63}$$

As another approximation, we can ignore the term $\ln(2s/\pi r)$ because due to its logarithmic nature it is small as compared to the second term. This yields

$$H = \frac{Rs^2}{K_1 a} \frac{1}{1 + (K_2/K_1)(d/a)} \tag{64}$$

a result that can be obtained by a formal application of Dupuit-Forcheimer theory. One should keep in mind that the spacings obtained from Equation 64 represent the lowest limit, because the Dupuit-Forcheimer theory neglects the head losses resulting from the convergence of stream lines. Therefore, one should be very cautious in using drain spacings obtained from the Dupuit-Forcheimer theory.

THREE-LAYERED PROBLEM

The geometry of the three-layered drainage problem is similar to the two-layered problem that has already been solved in the preceding chapters. However, the flow medium consists of not two but three layers of soil, an upper, a middle, and a lower layer, as shown in Figure 22. The upper layer extends a distance "a" and the middle layer a distance "b" below the drain centers. The lower layer terminates at an impermeable layer located at a finite distance h below the drains. K_1, K_2 and K_3 refer to hydraulic conductivities of the upper, middle, and lower soil layers, respectively.

Formulation of Problem

As in the two-layered problem, the head loss in the arched region above the drains is assumed to be negligible. Also, the two physical artifices, that is "fictitious membranes" and "slit drain" artifices, that were used in formulating the two-layered problem are also used in formulating the three-layered problem. Following the same line of reasoning that was used for the two-layered problem, our three-layered problem can be formulated as the boundary value problem shown below:

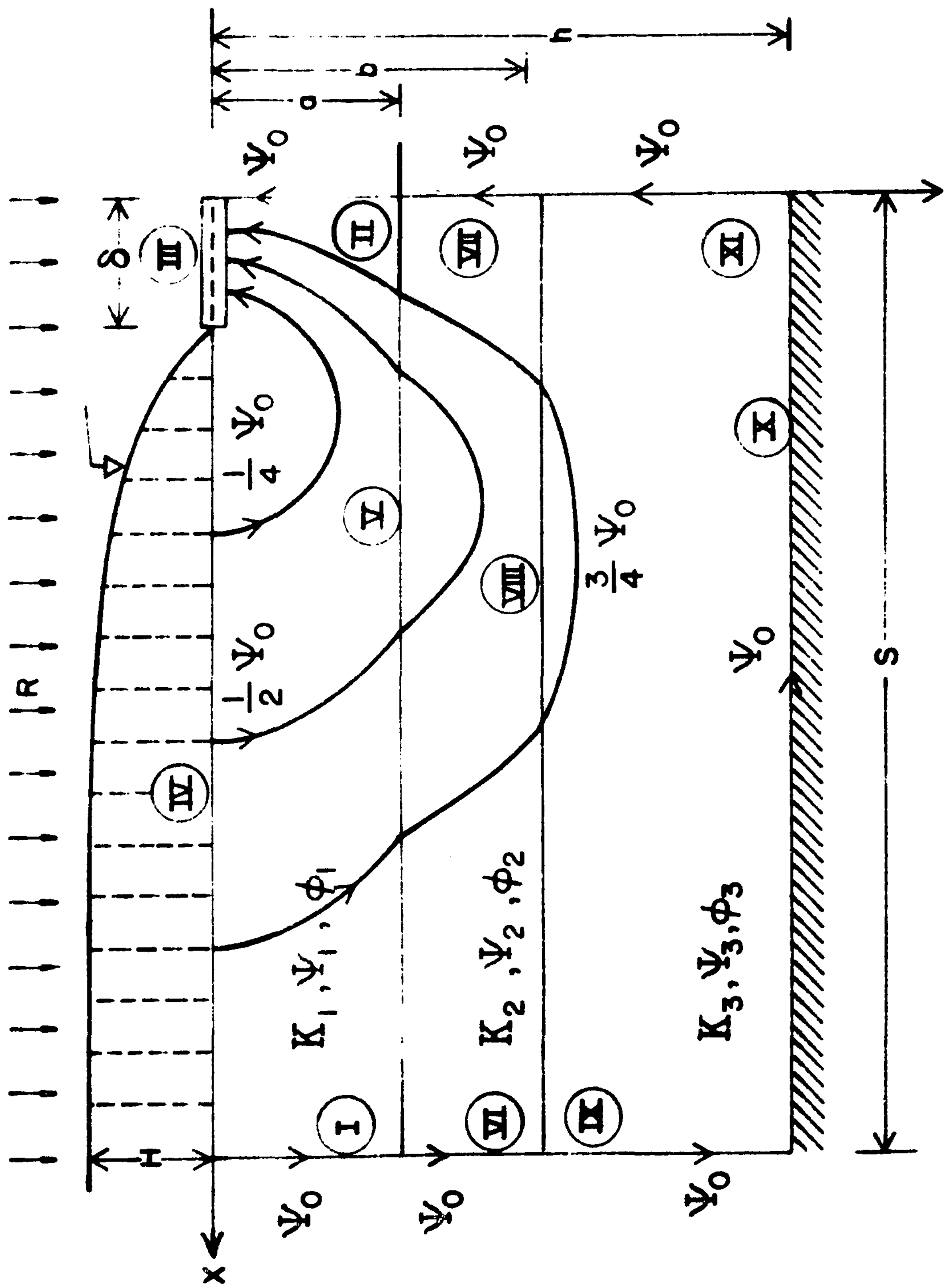
Find expressions for ψ_1, ψ_2, ψ_3, Φ_1, Φ_2 and Φ_3 to satisfy the equations

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \qquad (65)$$

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \qquad (66)$$

subject to the following set of boundary conditions which are shown in Figure 22:

Figure 22. Geometry for a steady-state tile drainage system for a three-layered flow medium.



h
b
a
δ
R
H
S
x
y
Ψ_0
$\frac{1}{4}\Psi_0$
$\frac{1}{2}\Psi_0$
$\frac{3}{4}\Psi_0$
K_1, Ψ_1, ϕ_1
K_2, Ψ_2, ϕ_2
K_3, Ψ_3, ϕ_3
I
II
III
IV
V
VI
VII
VIII
IX
X
XI

I.	$\psi_1 = \psi_0$	at $x = s$,	$0 < y < a$	For the upper layer
II.	$\psi_1 = \psi_0$	at $x = 0$,	$0 < y < a$	
III.	$\psi_1 = \frac{\delta - x}{\delta} \psi_0$	at $y = 0$,	$0 < x < \delta$	
IV.	$\psi_1 = \frac{x - \delta}{s - \delta} \psi_0$	at $y = 0$,	$\delta < x < s$	
Va.	$\phi_1 = \phi_2$	$y = a$	$0 < x < s$	At the upper interface
Vb.	$\psi_1 = \psi_2$	$y = a$	$0 < x < s$	
VI	$\psi_2 = \psi_0$	$x = s$	$a < y < b$	For the middle layer
VII.	$\psi_2 = \psi_0$	$x = 0$	$a < y < b$	
VIIIa.	$\phi_2 = \phi_3$	$y = b$	$0 < x < s$	At the lower interface
VIIIb.	$\psi_2 = \psi_3$	$y = b$	$0 < x < s$	
IX.	$\psi_3 = \psi_0$	$x = s$	$b < y < h$	For the lower layer
X	$\psi_3 = \psi_0$	$y = h$	$0 < x < s$	
XI.	$\psi_3 = \psi_0$	$x = 0$	$b < y < h$	

Stream and Potential Functions

The stream function for the upper layer will be identical to Equation 12, which is rewritten here as

$$\psi_1 = \psi_0 - \frac{2\psi_0}{\pi} \Sigma \left\{\frac{1}{m} \sin \frac{m\pi x}{s} \frac{\sinh[m\pi(a-y)/s]}{\sinh(m\pi a/s)} + B_m \sin \frac{m\pi x}{s} \frac{\sinh(m\pi y/s)}{\sinh(m\pi a/s)}\right\} \tag{67}$$

where B_m is an arbitrary constant. Equation 67 satisfies boundary conditions I and II and the term $(-2\psi_0/\pi)$ is obtained to satisfy boundary conditions III and IV, by following the same steps as previously explained by Equations 5 through 11.

The stream function for the middle layer should contain two arbitrary constants that will be selected to satisfy boundary conditions Vb and VIIIb. The first term of ψ_2 should be similar to Equation 13, but symbol h in Equation 13 should be replaced by symbol b. The second term of ψ_2 should be similar to the last term of Equation 67 but the denominator should be $\cosh(m\pi b/s)$ rather than $\sinh(m\pi a/s)$. The stream function for the middle layer thus is

$$\psi_2 = \psi_0 - \frac{2\psi_0}{\pi} \Sigma \{C_m \sin \frac{m\pi x}{s} \frac{\sinh[m\pi(b-y)/s]}{\cosh[m\pi(b-a)/s]} + D_m \sin \frac{m\pi x}{s} \frac{\sinh(m\pi y/s)}{\sinh(m\pi b/s)}\} \tag{68}$$

where C_m and D_m are arbitrary constants. Equation 68 satisfies boundary conditions VI and VII.

The stream function for the lower layer should be identical to Equation 13 but symbol a in Equation 13 should now be replaced by symbol b. The stream function for the lower layer thus is

$$\psi_3 = \psi_0 - \frac{2\psi_0}{\pi} \Sigma E_m \sin \frac{m\pi x}{s} \frac{\sinh[m\pi(h-y)/s]}{\cosh[m\pi(h-b)/s]} \tag{69}$$

where E_m is an arbitrary constant. Equation 69 satisfies boundary conditions IX, X, and XI.

Again in comparison with Equations 16 and 17, we can write down expressions for the potential functions.

$$\mathcal{J}_1 = -\frac{2\psi_0}{\pi}\,\Sigma\,\{\frac{1}{m}\cos\frac{m\pi x}{s}\,\frac{\cosh[m\pi(a-y)/s]}{\sinh(m\pi a/s)}$$

$$-\,B_m\cos\frac{m\pi x}{s}\,\frac{\cosh(m\pi y/s)}{\sinh(m\pi a/s)}\} + \frac{2\psi_0}{\pi}\,B_{om} \tag{70}$$

$$\mathcal{J}_2 = -\frac{2\psi_0}{\pi}\,\Sigma\,\{C_m\cos\frac{m\pi x}{s}\,\frac{\cosh[m\pi(b-y)/s]}{\cosh[m\pi(b-a)/s]}$$

$$-\,D_m\cos\frac{m\pi x}{s}\,\frac{\cosh(m\pi y/s)}{\sinh(m\pi b/s)}\} + \frac{2\psi_0}{\pi}\,C_{om} \tag{71}$$

$$\mathcal{J}_3 = -\frac{2\psi_0}{\pi}\,\Sigma\,\{E_m\cos\frac{m\pi x}{s}\,\frac{\cosh[m\pi(h-y)/s]}{\cosh[m\pi(h-b)/s]}\} + \frac{2\psi_0}{\pi}\,D_{om} \tag{72}$$

where B_{om}, C_{om} and D_{om} are arbitrary constants. Notice that Equations 67 through 77 satisfy the Cauchy-Riemann conditions. Notice further that we have satisfied all boundary conditions except Va, Vb, VIIIa, and VIIIb. By using these remaining boundary conditions, we shall now evaluate the arbitrary constants B_m, C_m, D_m and E_m.

By definition, $\phi_1 = \mathcal{J}_1/K_1$ and $\phi_2 = \mathcal{J}_2/K_2$. Boundary condition Va states that $\phi_1 = \phi_2$ at $y = a$. It follows that

$$\frac{1}{K_1}\,[\Sigma\,\frac{1}{m}\,\frac{1}{\sinh(m\pi a/s)} - \Sigma\,B_m\,\frac{\cosh(m\pi a/s)}{\sinh(m\pi a/s)}]\cos\frac{m\pi x}{s} - \frac{1}{K_1}\,B_{om} =$$

$$\frac{1}{K_2}\,[\Sigma\,C_m - \Sigma\,D_m\,\frac{\cosh(m\pi a/s)}{\sinh(m\pi b/s)}]\cos\frac{m\pi x}{s} - \frac{1}{K_2}\,C_{om} \tag{73}$$

By equating coefficients of $\cos \frac{m\pi x}{s}$, and after dropping the Σ signs, we obtain the following relations from Equation 73

$$C_m = \frac{K_2}{K_1}\left[\frac{1}{m}\frac{1}{\sinh(m\pi a/s)} - B_m \frac{\cosh(m\pi a/s)}{\sinh(m\pi a/s)}\right] + D_m \frac{\cosh(m\pi a/s)}{\sinh(m\pi b/s)} \tag{74}$$

$$C_{om} = \frac{K_2}{K_1} B_{om} \tag{75}$$

Boundary condition Vb states that $\psi_1 = \psi_2$ at $y = a$. It follows that

$$\Sigma\, B_m \sin \frac{m\pi x}{s} = \{\Sigma\, C_m \tanh[m\pi(b-a)/s] + D_m \frac{\sinh(m\pi a/s)}{\cosh(m\pi b/s)}\} \sin \frac{m\pi y}{s} \tag{76}$$

from which we get, by equating coefficients of $\sin \frac{m\pi x}{s}$ and after dropping the Σ sign, the result

$$B_m = C_m \tanh[m\pi(b-a)/s] + D_m \frac{\sinh(m\pi a/s)}{\cosh(m\pi b/s)} \tag{77}$$

Similarly, from boundary condition VIIIa, we obtain

$$D_{om} = \frac{K_3}{K_2} C_{om} \tag{78}$$

$$E_m = \frac{K_3}{K_2}\{C_m \frac{1}{\cosh[m\pi(b-a)/s]} - D_m \coth \frac{m\pi b}{s}\} \tag{79}$$

and from boundary condition VIIIb, we obtain

$$D_m = E_m \tanh \frac{m\pi(h-b)}{s} \tag{80}$$

Now, from Equations 74, 77, 79 and 80 we can solve for the coefficients B_m, C_m, D_m and E_m. Let us make the following substitutions:

$$\alpha_m = \frac{K_3}{K_2} \frac{1}{\cosh[m\pi(b-a)/s]}$$

$$\beta_m = \frac{K_3}{K_2} \coth \frac{m\pi b}{s}$$

$$\gamma_m = \tanh \frac{m\pi(h-b)}{s}$$

$$\delta_m = \tanh \frac{m\pi(b-a)}{s}$$

$$\eta_m = \frac{\sinh(m\pi a/s)}{\sinh(m\pi b/s)}$$

$$\rho_m = \frac{K_2}{K_1} \coth \frac{m\pi a}{s}$$

$$\mu_m = \frac{\cosh(m\pi a/s)}{\sinh(m\pi b/s)}$$

$$\epsilon_m = \frac{K_2}{K_1} \frac{1}{\sinh(m\pi a/s)}$$

Then Equations 74, 77, 79 and 80 can be written as

$$\rho_m B_m + C_m - \mu_m D_m = \frac{\varepsilon}{m} \tag{81}$$

$$-B_m + \delta_m C_m + \eta_m D_m = 0 \tag{82}$$

$$\alpha_m C_m - \beta_m D_m - E_m = 0 \tag{83}$$

$$-D_m + \gamma_m E_m = 0 \tag{84}$$

The solutions are

$$B_m = T_m[\delta_m + \gamma_m(\delta_m\beta_m + \alpha_m\eta_m)] \tag{85}$$

$$C_m = T_m(1 + \beta_m\gamma_m) \tag{86}$$

$$D_m = T_m\alpha_m\gamma_m \tag{87}$$

$$E_m = T_m\alpha_m \tag{88}$$

where T_m is given by

$$T_m = \frac{1}{m}\,\frac{\varepsilon_m}{(1 + \delta_m\rho_m)(1 + \beta_m\gamma_m) + \alpha_m\gamma_m(\rho_m\eta_m - \mu_m)} \tag{89}$$

We shall now evaluate the arbitrary constants B_{om}, C_{om} and D_{om}. If one follows the detailed steps given by Equations 25 through 30, one obtains the expression for B_{om}

$$B_{om} = \ln\frac{1}{2\sin(\pi r/2s)} + \Sigma\left[\frac{1}{m}\left(-1 + \coth\frac{m\pi a}{s}\right)\cos\frac{m\pi r}{s} - B_m\frac{\cos(m\pi r/s)}{\sinh(m\pi a/s)}\right] \tag{90}$$

where B_m is given by Equation 86. For C_{om} and D_{om}, we observe from Equations 75 and 78 that if B_{om} is known then C_{om} and D_{om} are also known. Because all arbitrary constants have now been evaluated, the stream functions given by Equations 67, 68 and 69, and the potential functions given by Equations 70, 71, and 72 are now defined.

Flow Nets

Dimensionless flow nets for the three-layered drainage problem can be prepared by following exactly the same procedures previously explained in detail for the two-layered problem. Equations 67 through 72, 75, 78, and 85 through 90 should be used. Figure 23 shows a flow net that has been prepared for the following dimensionless variables: $a/2s = 1/25$, $a/h = 2/5$, $b/h = 3/5$, $a/2r = 4$, $R/K_1 = 100$, $K_1/K_2 = 1/10$, $K_1/K_3 = 1$, and $K_2/K_3 = 10$. The numerical values of $a = 4$ feet, $b = 6$ feet, $h = 10$ feet, $2s = 100$ feet and $2r = 1$ foot have been used in order to facilitate quantitative discussion. One observes from Figure 23 that the existence of a two-feet thick and 10 times more permeable middle layer resulted in a maximum water table height of 1.12 feet as compared to 1.84 feet of Figure 3c which represents a homogeneous soil. Furthermore, one sees that only about 10 percent of the flow passes through the lower layer in Figure 23, because the stream lines refract sharply when they reach the more permeable middle layer.

Drain Spacing Formulas

By definition $\phi_1(s,0) = H$ and from Equation 70, we can write the expression for H as

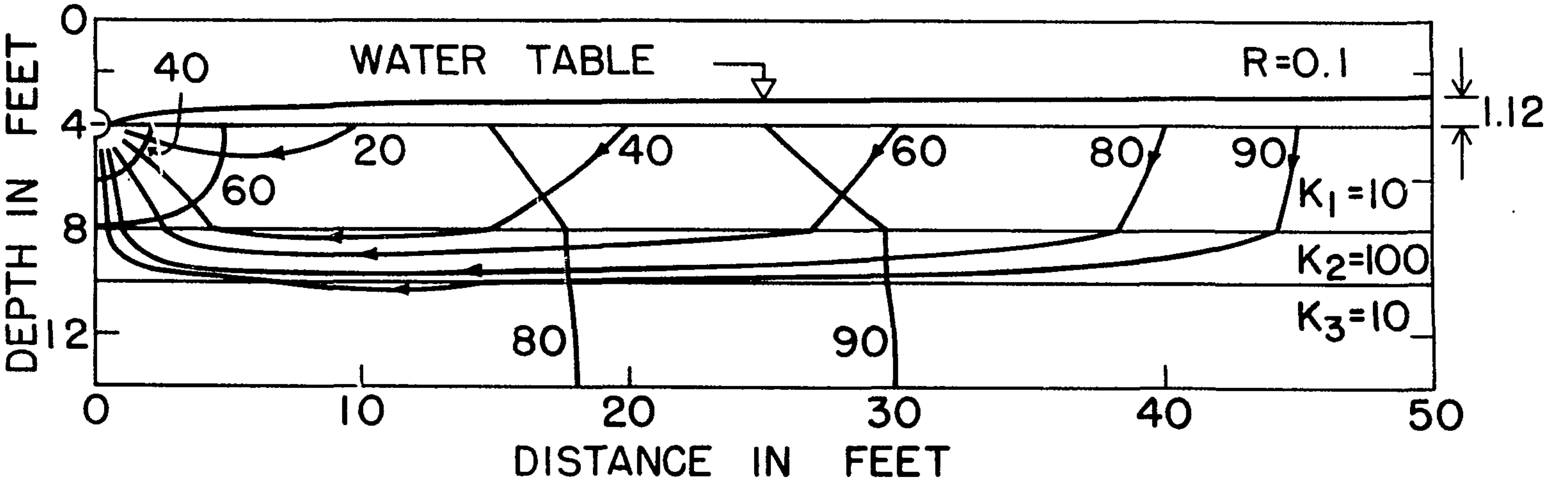
$$H = \frac{2Rs}{\pi K_1} \{B_{om} - \Sigma \,[\frac{1}{m} \cos m\pi \coth \frac{m\pi a}{s} - B_m \frac{\cos m\pi}{\sinh(m\pi a/s)}]\} \qquad (91)$$

After inserting the expression for B_{om} from Equation 90 into Equation 91, multiplying the right hand side of it by the factor $[1 - (R/K_1)]^{-1}$ in order to account for the head loss in the arched region, and after rearranging it, we obtain

$$H = \frac{2s}{\pi[(K_1/R)-1]} \{\ln \frac{1}{\sin(\pi r/2s)} + \Sigma \frac{1}{m} (-1 + \coth \frac{m\pi a}{s})$$

$$(\cos \frac{m\pi r}{s} - \cos m\pi)[1 - m\, B_m e^{(m\pi a/s)}]\} \qquad (92)$$

Figure 23. Flow net for the dimensionless parameters $a/2s = 1/25$, $a/h = 2/5$, $b/h = 3/5$, $a/2r = 4$, $R/K_1 = 100$, $K_1/K_2 = 1/10$, and $K_2/K_3 = 10$. Depth and distances in feet are shown for purposes of quantitative discussion.


WATER TABLE
R=0.1
1.12
K1=10
K2=100
K3=10
40
20
60
40
60
80
90
80
90
DEPTH IN FEET
0
4
8
12
0
10
20
30
40
50
DISTANCE IN FEET

85c

or, by inserting the expression for B_m from Equation 85, we get

$$H = \frac{2s}{\pi[(K_1/r)-1]} \{\ln \frac{1}{\sin(\pi r/2s)} + \Sigma \frac{1}{m} (-1 + \coth \frac{m\pi a}{s})(\cos \frac{m\pi r}{s} - \cos m\pi)$$

$$[1 - e^{(m\pi a/s)} T_m \delta_m - e^{(m\pi a/s)} T_m \gamma_m (\delta_m \beta_m + \alpha_m \eta_m)]\} \tag{93}$$

Equation 93 is the general formula for the three-layered drainage problem. When $K_3 = 0$, then α_m, $\beta_m = 0$ and Equation 93 reduces to Equation 45, that is, our three-layered problem reduces to two-layered problem. Similarly, when $K_2 = 0$, then $T_m = 0$ and Equation 93 reduces to Equation 42, that is, our three-layered problem reduces to single-layered problem. Therefore, all drain spacing formulas that have been previously obtained for the two-layered problem can be deduced from Equation 93 as special cases of the three-layered problem. Furthermore, by changing the term $[(K_1/R)-1]^{-1}$ by the term $[(K_0/R)-1]^{-1}$, our three-layered drainage problem can be transformed into a special case of a four-layered problem.

Discussion of Results

Let us define, in addition to the functions E and F given by Equations 46 and 47, two new functions I and J as follows:

$$I = e^{(m\pi a/s)} T_m \delta_m \tag{94}$$

$$J = e^{(m\pi a/s)} T_m \gamma_m (\delta_m \beta_m + \alpha_m \eta_m) \tag{95}$$

Then, Equation 93 can be rewritten as

$$2s = H (\frac{K_1}{R} - 1) \frac{1}{E + \Sigma F[1 - I - J]} \tag{96}$$

where functions I and J show, as mentioned previously, the contributions of the middle and lower layers, respectively, on the drain spacing 2s. For example, if the lower layer is erroneously assumed to be impermeable, then J is erroneously assumed to be zero and the percentage error in drain spacing would be

$$\frac{\Sigma\ FJ}{E + \Sigma\ F(1 - I)}\ 100 \tag{97}$$

Similarly, if K_2 is erroneously assumed to be zero, the resulting error in drain spacing, 2s, would be

$$\frac{\Sigma\ F(I + J)}{E + \Sigma\ F}\ 100 \tag{98}$$

Other combination of assumptions that would lead to such errors can easily be formulated by using Equation 96.

Let us observe that Kirkham (1958) solved the single layered problem by using five boundary conditions. Two and three-layered problems required nine and 13 boundary conditions, respectively. One can see that each additional layer increases the number of the boundary conditions by four. Therefore, the n-layered drainage problem can be formulated as a mathematical boundary value problem with (4n + 1) boundary conditions. The steps to be followed in solving such a boundary value problem are identical to those explained in this thesis. However, as the number of soil layers increases, the expressions for the arbitrary constants become more complicated. To see this, one need only to insert the values of α_m, β_m, δ_m, ε_m, γ_m, η_m, ρ_m and μ_m into Equation 85 and compare it with Equation 24. Yet, modern computers make numerical calculations, even with such complicated expressions, a relatively easy task, as it has been demonstrated by the flow net given in Figure 23.

SUMMARY AND CONCLUSIONS

The problem of steady drainage of two and three-layered soils has been solved by using and extending the methods and procedures developed by Kirkham (1958, 1961) for the steady drainage of a homogeneous soil. Five flow nets for the two-layered problem and one flow net for the three-layered problem have been prepared. The five flow nets for the two-layered problem show the effect of the variations in hydraulic conductivity of the lower layer on the flow lines and equipotentials as well as on the maximum height of the water table above the drain tubes. The general drain spacing formula for the two-layered problem is

$$H = \frac{2s}{\pi[(K_1/R)-1]} \{\ln \frac{1}{\sin(\pi r/2s)} + \Sigma \frac{1}{m} (-1 + \coth \frac{m\pi a}{s})(\cos \frac{m\pi r}{s} - \cos m\pi)$$

$$[\frac{e^{(m\pi a/s)}}{\sinh(m\pi a/s)} \frac{1}{(K_1/K_2) \coth[m\pi(h-a)/s] + \coth(m\pi a/s)}]\} \qquad (99)$$

where a and h are the distances the upper and the lower layers, respectively, extend from the centers of the drains; 2s is the drain spacing; H is the maximum water table height above the drain centers; r is the drain radius and K_1/K_2 are the hydraulic conductivities of the upper and the lower soil layers, respectively. A set of 16 nomographs have been prepared to solve explicitly for 2s, the drain spacing, for the two-layered problem. An additional nomograph has been prepared for a formula of Hinesly and Kirkham (1966) which takes into account both recharge and upward artesian seepage in homogeneous soils. The general drain spacing formula for the three-layered problem is

$$H = \frac{2s}{\pi[(K_1/R)-1]} \{\ln \frac{1}{\sin(\pi r/2s)} + \Sigma \frac{1}{m} (-1 + \coth \frac{m\pi a}{s})(\cos \frac{m\pi r}{s} - \cos m\pi)$$

$$[1 - e^{(m\pi a/s)} T_m(\delta_m + \gamma_m(\delta_m\beta_m + \alpha_m\eta_m)]\} \tag{100}$$

where the symbols T_m, δ_m, γ_m, β_m, α_m and η_m refer to algebraic substitutions given in the text. The parameters a, r, H, h, 2s, K_1, K_2 which were defined above as well as the parameters K_3, the hydraulic conductivity of the third layer, and b, the distance the middle layer extends below the drain centers, are involved in these substitutions. The nomographs for the three-layered problem have not been prepared for space limitations, but they can be prepared by following the same procedures developed for the nomographs of the two-layered problem.

If one neglects the effect of one of the soil layers, the resulting drain spacings would be in error. Expressions for calculating such errors have been developed and discussed. A solution of the two-layered problem as given by Dagan (1965) has also been discussed at some length.

It is concluded that:

1. A steady drainage problem in a stratified soil which consists of n layers, can be formulated as a mathematical boundary value problem. This problem is to find particular solutions for Laplace's equation subject to (4n + 1) boundary conditions. The single-layered problem has been solved by Kirkham (1958), nomographs for the single-layered problem have been given by Toksöz and Kirkham (1961). In this thesis, the problems for the two and three layers have been solved and extensive nomographs have been given for the two-layered problem.

The steady drainage problems with more than three layers can also be solved by following exactly the same methods and procedures developed in this thesis. Therefore, the method developed in this thesis can be considered as a general theory for the steady drainage of stratified soils;

2. For a two-layered soil, statements like "when the hydraulic conductivity of the upper layer is five to 10 times greater than that of the lower layer, then the lower layer can be assumed to be impermeable" are misleading. The drain spacings calculated on the basis of such statements will always be smaller than the correct drain spacings. For example, for $2s/a = 128$, $a/2r = 8$, $a/h = 0.2$, the error in drain spacings would be 41 percent for $K_1/K_2 = 5/1$ and 26 percent for $K_1/K_2 = 10/1$. If one neglects the effect of the lower layer when K_2 is larger than K_1 the errors would even be larger. Such errors would decrease as the thickness of the upper layer increases;
3. In designing a subsurface drainage system, the second soil layer should always be taken into account because it may have an appreciable effect on drain spacings. Spacing calculations for a two-layered soil can easily be made by using the drain spacing nomographs given in Figures 4 through 20. For a three-layered soil, drain spacings can be calculated from Equation 100;
4. As the number of soil layers increase, the contribution of the lowest layer on drain spacings decreases. However, if $K_3 >> K_2 > K_1$, the effect of the third layer may be appreciable, depending on the

geometry of the flow system and on the numerical values of the hydraulic conductivities;

5. For the two-layered problem, the drain spacings calculated from Dagan's (1965) formula agree well with those calculated from our nomographs. For the case $K_1 > K_2$ one would expect such an agreement. For the case $K_1 < K_2$ Dagan's formula still yields good results, but not because the flow is horizontal within the segment $2h < x < s$, as he has assumed, but because the major proportion of the hydraulic head loss occurs within the segment $0 < x < 2h$ (near the drain tube) where it has been properly taken care of by his linearized theory.

It is correct that the water table within the segment $2h < x < s$ (away from the drain) is almost flat, but it does not follow that the flow is horizontal in this segment. In fact, as one can see from the flow nets of Figure 3d and e, the flow is not at all horizontal but approaches to a vertical direction as K_2 increases. Dagan's analysis does not permit one to find expressions for the flow nets, and does not provide the analysis for soils of great depth.

LITERATURE CITED

Dagan, Gedeon. March, 1964. Spacing of drains by an approximate method. Journal of the Irrigation and Drainage Division, ASCE Proc. 90, No. IR1: 41-66.

________. September, 1965. Steady drainage of a two-layered soil. Journal of the Irrigation and Drainage Division, ASCE Proc. 91, No. IR3: 51-64.

Dumm, L. D. 1954. New formula for determining depth and spacing of subsurface drains in irrigated lands. Agricultural Engineering 35: 726-730.

________. 1964. Transient flow concept in subsurface drainage: its validity and use. ASAE Trans. 7, No. 2:142-146.

________. June, 1966. [Discussion of] Steady drainage of a two-layered soil. Journal of the Irrigation and Drainage Division, ASCE Proc. 92, No. IR2:102-103.

Hammad Y, Hammad, Abdel-wahab M. Amer, E. G. Youngs, Gedeon Dagan and A. W. Warrick. December, 1966. [Discussion of] Steady-state theories for drainage. Journal of the Irrigation and Drainage Division, ASCE Proc. 92, No. IR4:75-88.

Hinesly, T. D. and Don Kirkham. Third Quarter, 1966. Theory and flow nets for rain and artesian water seeping into soil drains. Water Resources Research 2, No. 3:497-511.

Hooghoudt, S. B. 1940. Bijdragen tot de kennis van eenige natuurkundige grootheden van den grond, 7, algemeene beschowing van het probleem van de detail ontwatering en de infiltratie door middel van parallel loopende drains, greppels, slooten en kanalen. Versl. Landbouwk. Ond. 46:517-707.

Kirkham, Don. 1951. Seepage into drain tubes in stratified soil. Trans. Amer. Geophys. Union 32:422-442.

________. 1954. Seepage of artesian and surface water into drain tubes in stratified soil. Trans. Amer. Geophys. Union 35:775-790.

________. 1958. Seepage of steady rainfall through soil into drains. Trans. Amer. Geophys. Union 39:892-908.

________. 1961. An upper limit for the height of the water table in drainage design formulas. Trans. 7th Internatl. Congress for Soil Science 1:486-492.

Kirkham, Don. March, 1966. Steady-state theories for drainage. Journal of the Irrigation and Drainage Division, ASCE, Proc. 92, No. IR1:19-39.

________. June, 1967. [Closure of Discussion of] Steady-state theories for drainage. Journal of the Irrigation and Drainage Division, ASCE Proc. 93, No. IR3:67-74.

________ and W. L. Powers. To be published circa 1970. Advanced soil physics. John Wiley and Sons, New York, N.Y.

Maasland, Marinus. 1959. Water table fluctuations induced by intermittent recharge. Jour. of Geophys. Res. 64:549-559.

________. 1961. Water table fluctuations induced by irrigation. Journal of the Irrigation and Drainage Division, ASCE Proc. 87, No. IR2:39-58.

Polubarinova-Kochina, P. Ya. 1962. Theory of the ground water movement. Princeton Univ. Press, Princeton, N. J. (Translated from the Russian by J. M. Roger de Wiest.)

Soliman, M. M. September, 1966. [Discussion of] Steady-state theories for drainage. Journal of the Irrigation and Drainage Division, ASCE Proc. 92, No. IR3:81-82.

Swartzendruber, Dale. 1962. Approximate water flow rates for tube drains in stratified soils. Journal of Geophys. Res. 67:2395-2402.

Toksöz, Sadik and Don Kirkham. 1961. Graphical solution and interpretation of a new drain-spacing formula. Jour. of Geophys. Res. 66:509-516.

Wesseling, Jans. 1964. A comparison of the steady-state drain spacing formulas of Hooghoudt and Kirkham in connection with design practice. Jour. of Hydrology 2:25-32.

APPENDIX A

We have, from Equation 15 of Hinesly and Kirkham (1966)

$$H\phi = -\frac{8h(R+F)}{\pi^2(b-a)} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^2} \left(\sin\frac{m\pi b}{2h} - \sin\frac{m\pi a}{2h}\right) \cos\frac{m\pi y}{2h}$$

$$\frac{\cosh[m\pi(s-x)/2h]}{\sin(m\pi s/2h)} + R(h-y) + KG \tag{101}$$

Hereafter, the sign Σ will refer to $\sum\limits_{m=1,3,5,\ldots}^{\infty}$.

From Equations 45 and 46 of Hinesly and Kirkham (1966), we get

$$\frac{h}{\pi(b-a)} \left(\sin\frac{m\pi b}{2h} - \sin\frac{m\pi a}{2h}\right) = \frac{m}{2} \cos\frac{m\pi c}{2h} \tag{102}$$

as $(b-a) \longrightarrow 0$. By using the last result, we obtain their Equation 48 as

$$K\phi = -\frac{4(R+F)s}{\pi} \Sigma \frac{1}{m} \cos\frac{m\pi c}{2h} \cos\frac{m\pi y}{2h} \frac{\cosh[m\pi(s-x)/2h]}{\sinh(m\pi s/2h)}$$

$$+ R(h-y) + KG \tag{103}$$

For a drain running half-full, c = 0 and Equation 103 reduces to

$$K\phi = -\frac{4(R+F)s}{\pi} \Sigma \frac{1}{m} \cos\frac{m\pi y}{2h} \frac{\cosh[m\pi(s-x)/2h]}{\sinh(m\pi s/2h)} + R(h-y) + KG \tag{104}$$

We evaluate KG by observing that $\phi(r,0) = 0$.

$$KG = \frac{4(R+F)s}{\pi} \left\{\Sigma \frac{1}{m} \frac{\cosh[m\pi(s-r)/2h]}{\sinh[m\pi s/2h]}\right\} - RH \tag{105}$$

By using Equation 105 in Equation 104, and by observing that $\phi(s,0) = H$, we obtain

$$H = \frac{4(R+F)s}{\pi K} \Sigma \frac{1}{m} \frac{-1 + \cosh[m\pi(s-r)/2h]}{\sinh(m\pi s/2h)} \tag{106}$$

To account for the neglected head loss in the arched region, we multiply the right hand side of Equation 106 by the factor $[(R/K)-1]^{-1}$, and after rearranging Equation 106, we get

$$\frac{H}{h}\left(\frac{K-R}{R+F}\right) = \frac{2s}{h}\,\Sigma\,\frac{2}{m\pi}\;\frac{-1+\coth[m\pi(s-r)/2h]}{\sinh(m\pi s/2h)} \tag{107}$$

which we can rewrite it as

$$\frac{H}{h}\left(\frac{K-R}{R+F}\right) = \frac{2s}{h}\,\Sigma\,\frac{2}{m\pi}\;\frac{-1+\cosh\{(m\pi/4)[(2s/h)-(2r/h)]\}}{\sinh[(m\pi/4)(2s/h)]}\,, \tag{108}$$

which is identical to our Equation 55.

ACKNOWLEDGEMENTS

This research was supported by project B-013-IA of the U.S. Department of Interior, Office of Water Resources Research, and by project 998 of the Iowa Agriculture and Home Economics Experiment Station. The writer wishes to express his gratitude to Dr. Don Kirkham for his continued encouragement.