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Qualitative behavior of integrodifferential systems with applications in reactor dynamics

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I. INTRODUCTION

The purpose of this thesis is to study properties of the solutions of integrodifferential equations of the following types:

$$x'(t) = (A + C(t))x(t) + \int_0^t B(t-s)x(s)ds + f(t); \qquad x(0) = x_0 \tag{L}$$

$$x'(t) = (A + C(t))x(t) + \int_0^t B(t-s)x(s)ds + f(t) + g(x)(t); \qquad x(0) = x_0 \tag{N}$$

where $t \geq 0$, $x' = \frac{dx}{dt}$, $x(t) = \mathrm{Col}(x_1(t),\ldots,x_n(t))$, $x_i(t)$ is a real valued function for $i = 1,\ldots,n$, A is a constant $n \times n$ matrix, $C(t)$ and $B(t)$ are $n \times n$ matrix functions, $f(t) = \mathrm{Col}(f_1(t),\ldots,f_n(t))$ where $f_i(t)$ is a real valued function for $i = 1,\ldots,n$, and $g(x)(t) = \mathrm{Col}(g_1(x)(t),\ldots,g_n(x)(t))$ where $g_i(x)(t)$ is a nonlinear functional of x for $i = 1,\ldots,n$. In addition to a general analysis of equation (N) this thesis also includes an application. This application is to the point kinetic model of a coupled core nuclear reactor.

The results presented are concerned with the integrability and boundedness of the solutions of (N) and (L) on the half line $R^+ = \{t : t \geq 0\}$ and with the asymptotic behavior of these solutions for large t. Equation (L) is treated as a perturbation of the equation

$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t);\; x(0) = x_0 \qquad \text{(E)}$$

The perturbation term is $C(t)x$. If $C(t)$ is a matrix having entries which are continuous and bounded on R^+ and tend to zero as t tends to infinity it is shown that the solution of (L) behaves much like that of (E). This is helpful since (E) can be analyzed by Laplace transform techniques. One fundamental result of this kind is given by Grossman and Miller in [4]: "Suppose in equation (E), $B(t)$ is Lebesgue integrable on R^+ and $\det[sI - A - B^*(s)] \neq 0$ for $\operatorname{Re} s \geq 0$ where $B^*(s)$ is the Laplace transform of $B(t)$. Then:

a. $x(t)$, the solution of (E), is continuous and Lebesgue integrable on R^+ if $f(t)$ is continuous and Lebesgue integrable on R^+.

b. $x(t)$ is a bounded continuous function on R^+ if $f(t)$ is bounded and continuous on R^+."

In this thesis similar results are obtained for equation (L) and the above result of Grossman and Miller is extended to the case where $B(t)$ is the sum of a Lebesgue integrable matrix and a constant matrix of a given class.

Necessary conditions are established for the local stability of (N). By local stability we mean $x(t)$, the solution of (N), remains small for small initial data and small forcing function $f(t)$. This treatment of (N) parallels that of Grossman and Miller [3]. However, the form of the nonlinear term, $g(x)$, is more general than that in [3] to accommodate the application that follows.

The application deals with a system of integro-differential equations of type (N) that occurs in nuclear reactor dynamics. The equations were derived by H. Plaza and W. H. Kohler [11] and represent the point kinetics model of a reactor containing M fuel elements or cores:

$$\frac{dP_j}{dt} = \frac{\rho_j(t) - \epsilon_j - \beta_j}{\Lambda_j} P_j(t) + \sum_{i=1}^{N} \lambda_{ij} C_{ij}(t)$$

$$+ \frac{1}{\Lambda_j} \sum_{k=1}^{M} \epsilon_{kj} \int_0^{\infty} P_k(t-\tau) h_{kj}(\tau)\, d\tau \qquad 1.1.a$$

$$\frac{dC_{ij}}{dt} = \frac{\beta_{ij}}{\Lambda_j} P_j(t) - \lambda_{ij} C_{ij} \qquad 1.1.b$$

$$j = 1,\ldots,M$$

$$i = 1,\ldots,N$$

P_j denotes the power of the jth core. Power is an indication of the neutron density in a core and, hence, an indication of the amount of energy being released in that core. These equations relate the rate of neutron production to the neutrons present in the reactor. Neutrons are produced in two ways directly by fission and by the decay of precursors created by fission. C_{ij} denotes the effective concentration of the ith precursor in the jth core. There are N of these precursors. β_j is a constant denoting the effective loss of neutrons in the jth core due to production of precursors. β_{ij} is the effective loss due to production of the ith precursor in the jth core. The relation $\beta_j = \sum_{i=1}^{N} \beta_{ij}$ holds. λ_{ij} is the decay constant of the ith precursor in the jth core. ϵ_{kj} is the coupling coefficient from the kth to the jth core and $h_{kj}(\tau)$ is the coupling function from the kth to the jth core. $h_{kj}(\tau)d\tau$ is the probability that a neutron produced in core k at time t' enters reactor j at time t after a delay of time $\tau = t - t'$. It is assumed the cores are separated by a nonmultiplying medium. Also, $\int_0^\infty h_{kj}(\tau)d\tau = 1$ for $k,j = 1,\ldots,M$. ϵ_k and ϵ_{kj} are positive constants for $j,k = 1,\ldots,M$.

It is assumed that the reactor has an equilibrium state. At equilibrium the power of each core is constant. This constant is denoted by $P_{jo}, j = 1,\ldots,M$. $\rho_j(t)$ is the reactivity of the jth core as measured from equilibrium; that is, $\rho_j = 0$ for $j = 1,\ldots,M$ when the reactor is at equilibrium. The equations as stated are a good approximation near the equilibrium state. In this discussion we shall only consider the problem where

$P_k(t) \equiv P_{ko}$ for $t < 0$, $k = 1,\ldots,M$ and a small perturbation is introduced at $t = 0$. Also, it is assumed that the reactivity, $\rho_j(t)$, is dependent only on the temperature of the jth core. (There are other factors which can affect reactivity such as position of control rods and concentration of poisons in the cores. See Akcasu [1]). This dependence is expressed in the form

$$\rho_j(t) = -\int_{G_j} \alpha^j(x) T^j(x,t)\,dx$$

where G_j is a region containing the jth core, $T^j(x,t)$ denotes the temperature of the jth core at the point x as measured from the equilibrium temperature, and $\alpha^j(x)$ is the heat coefficient at x. Here x is a space variable taking values in G^j. The temperature depends on the power in the following way:

$$\frac{\partial T^j(x,t)}{\partial t} = L_x(T(x,t)) + q^j(x)T^j(x,t) + n^j(x)(P_j(t) - P_{jo})$$

$$T^j(x,0) = h^j(x) \quad \text{for} \quad x \in G^j \qquad 1.2$$

$$\text{and} \quad s^j(x)T^j(x,t) + r^j(x)\frac{\partial T^j(x,t)}{\partial n} = 0 \quad \text{for} \quad x \in \Gamma^j, t \geq 0$$

$$j = 1,\ldots,M$$

where Γ^j is the boundary of G^j, $\eta^j(x)$, $r^j(x)$ and $s^j(x)$ are prescribed functions, $h^j(x)$ is the initial temperature distribution in core j, $\frac{\partial T^j}{\partial n}$ is the normal derivative of $T^j(x,t)$ for $x \in \Gamma^j$, and $L_x(\cdot)$ is a partial differential operator. Note the above equations do not allow for the transfer of heat between cores. This is a reasonable assumption since the cores are usually surrounded by coolant.

We want to study the stability of the coupled system (1.1) and (1.2). By stable we mean stability with respect to perturbations in power, precursor concentration and temperature introduced at time $t = 0$. In [7], Levin and Nohel studied a single core reactor in the case

$L_x(\cdot) = \frac{\partial^2(\cdot)}{\partial x^2}$ with $G = [0,\pi]$ and with $G = (-\infty, +\infty)$.

With certain assumptions on α, η, and h they obtained global stability results by using Lyapunov functionals. Helliwell in [5] considered a single core but a more general operator $L_x(\cdot)$ and the region G to be in R^n. Local results were obtained by Laplace transform techniques. In this thesis we shall first study the multi-core reactor in the simpler case where $L_x^j(\cdot) = \frac{\partial^2 \cdot}{\partial x^2}$ and $G^j = [0,\pi]$ and then in a more general setting as in Helliwell [5]. In both cases the system (1.1) - (1.2) is shown to be locally stable.

In Chapter II of the thesis some background material is introduced. Chapter III contains the analysis of equations (L) and (N). The reactor problem is studied in Chapter IV.

II. PRELIMINARY MATERIAL

The following notation will be used in the work that follows.

R^n is real Euclidean n-space.

$|\cdot|$ is the Euclidean norm in R^n. $|x| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$ for $x = \mathrm{Col}(x_1,\ldots,x_n)$. C is the set of all continuous functions with domain R^+ and range R^n. BC is the subset of C containing all bounded functions

and $\quad \|f\|_o = \sup_{t\in R^+} |f(t)|$ is the norm on BC.

L^p is the usual Lebesgue space of measurable function f such that

$$\|f\|_p = \left\{\int_0^\infty |f(t)|^p dt\right\}^{1/p} < +\infty,\ 1 \le p < \infty.$$

L^∞ is the Lebesgue space of measurable functions $f(t)$ such that for some $M \ge 0, |f(t)| \le M$ for all $t \in R^+$ except possibly on a set of measure zero.

LL^p is the set of all functions which are locally of class L^p on R^+; that is, f is in LL^p if and only if

$$\left(\int_0^T |f(t)|^p dt\right)^{1/p} < +\infty \quad \text{for any} \quad T \geq 0,\ 1 \leq p < +\infty.$$

Let $A = (a_{ij})$ be an n by n matrix with entries in R^1. Define the norm of A by $|A| = \sum_{i,j=1}^{n} |a_{ij}|$. We say that an n by n function $A(t) = (a_{ij}(t))$ is in a space X if $a_{ij}(t)$ is in X for $i, j = 1,\ldots,n$. If $A(t)$ is an n by n matrix in a Banach space X, with norm $\|\ \|_X$, by $\|A\|_X$ we mean

$$\|A\|_X = \sum_{i,j=1}^{n} \|a_{ij}\|_X.$$

An important concept in the study of integrodifferential equations is that of the resolvent. Consider a general linear integrodifferential equation of the form

$$x'(t) = A(t)x(t) + \int_0^t B(t,s)x(s)\,ds + f(t); \qquad 2.1$$

$$x(0) = x_o$$

where $x(t) = \mathrm{Col}(x_1(t),\ldots,x_n(t))$, $A(t)$ is an n by n

matrix in C, $B(t,s)$ is an n by n matrix that is locally integrable in both variables, and $f(t)$ is in C. Grossman and Miller in [3] show that the solution, $x(t)$, of (2.1) can be expressed as

$$x(t) = R(t,0)x_o + \int_o^t R(t,s)f(s)ds \tag{2.2}$$

where $R(t,s)$ is an $n \times n$ matrix that is continuous in (t,s) for $0 \leq s \leq t$ and satisfies

$$\begin{aligned} \frac{\partial R(t,s)}{\partial s} &= - A(s)R(t,s) - \int_s^t B(t,u)R(u,s)du \\ R(t,t) &= I \quad \text{for} \quad 0 \leq s \leq t . \end{aligned} \tag{2.3}$$

$R(t,s)$ is called the resolvent of equation (2.1). In the special case $A(t)$ is a constant matrix and $B(t,s) = B(t-s)$, (2.3) reduces to $R(t,s) = R(t-s)$ and

$$R'(t) = AR(t) + \int_o^t B(t-s)R(s)ds; \; R(0) = I . \tag{2.4}$$

In this case equation (2.1) is said to be of convolution type. But in either case the solution of (2.1) can, for a given x_o and $f(t)$, be expressed in terms of $R(t,0)$ and a map ρ defined by

$$\rho(f)(t) = \int_o^t R(t,s)f(s)ds; \quad t \geq 0.$$

Definition 2.1. A Frechet space is a complete linear topological space with a metric d that is additively invariant. That is, $d(x,y) = d(x-y,0)$ for all x and y in the space.

We note here that LL^1 is a Frechet space.

Definition 2.2. Let $\mathfrak{F}$ be a Frechet subspace of LL' with metric d. Then the metric topology on $\mathfrak{F}$ is stronger than the topology on $\mathfrak{F}$ inherited from LL^1 if and only if x_n, $x \in \mathfrak{F}$ and $d(x_n,x) \to 0$ as $n \to \infty$ imply that $x_n \to x$ in LL'.

The following results concerning this map ρ may be found in Grossman and Miller [3].

Theorem 2.1. Let X and Y be Frechet subspaces of LL^1 both having a topology stronger than LL^1. If $\rho(X) \subset Y$ then ρ is continuous as a mapping from X into Y.

Definition 2.3. If X and Y are Frechet spaces we say that ρ is an admissible map from X into Y if $\rho(X)$ is contained in Y and ρ is continuous as a map from X into Y. The set of all admissible maps from X into Y is denoted by $\mathfrak{a}(X,Y)$.

Interesting examples of Frechet subspaces of LL^1 are C, BC, BC_ℓ - the set of all functions in BC having a limit at infinity, BC_o - set of functions in BC_ℓ that have limit zero at ∞, and $L^p \cap BC_o$ - set of functions in both L^p and BC_o $(1 \leq p < \infty)$. BC_o and BC_ℓ are Banach spaces with the supremum norm, $L^p \cap BC_o$ is a Banach space with norm $\| \|_{L^p \cap BC_o} = \| \|_o + \| \|_p$ for $1 \leq p < \infty$. The following theorem due to Corduneanu characterizes admissible maps from BC to BC. The proof may be found in Miller [10], page 261.

Theorem 2.2. Let ρ be a continuous map from C into C, $\rho(f)(t) = \int_0^t R(t,s) f(s) ds$. Then ρ is in $\mathfrak{a}(BC,BC)$ if and only if

$$\sup_{t \in R^+} \int_0^t |R(t,s)| ds \leq M \quad \text{for some } M > 0.$$

For equations of convolution type we have the following result by Grossman and Miller [4].

Theorem 2.3. Suppose in equation (E) that $B(t)$ is Lebesgue integrable on R^+. If $\det[sI - A - B^*(s)] \neq 0$ for all complex numbers s such that $\mathrm{Re}\, s \geq 0$ where $B^*(s)$ is the Laplace transform of $B(t)$, then $R(t)$ the resolvent of (E) is in $L^p \cap BC_o$ and $R'(t)$ is in $L^p \cap BC_o$ for all p in $[1,\infty)$.

The resolvent is also helpful in dealing with the non-linear equation (N) as demonstrated by the following theorem again due to Grossman and Miller [3].

Theorem 2.4. Suppose in equation (N) that $A(t)$ is continuous, $B(t,s)$ is locally integrable in both variables, and g maps LL^1 into itself. Then, for t in R^+ a function $x(t)$ solves (N) if and only if $x(t)$ solves the equation

$$x(t) = R(t,0)x_o + \int_o^t R(t,s)f(s)ds + \int_o^t R(t,s)g(x)(s)ds \qquad 2.5$$

for $0 \leq t < +\infty$.

There are corresponding results for pure Volterra integral equations. Consider the equation

$$x(t) = f(t) + \int_0^t A(t-s)x(s)\,ds \tag{V}$$

where $t \geq 0$, $x(t)$ and $f(t)$ are functions from R^+ into R^n, and $A(t)$ is an $n \times n$ matrix in LL^1. The solution, $x(t)$, of (V) can be written as

$$x(t) = f(t) - \int_0^t r(t-s)f(s)\,ds \qquad 2.6$$

where $r(t)$ is an $n \times n$ matrix in LL^1 and solves

$$r(t) = -A(t) + \int_0^t A(t-s)r(s)\,ds \qquad 2.7$$

for $t \geq 0$. $r(t)$ is called the integral resolvent of (V). We state the following result of Paley and Wiener.

<u>Theorem 2.5</u>. Suppose that in equation (V) $A(t)$ is an $n \times n$ matrix in L^1. Then the resolvent of (V) is of class L^1 if and only if the determinant

$$\det\left[I - \int_0^\infty e^{-st}A(t)\,dt\right] \neq 0 \quad \text{for}$$

all complex numbers s satisfying $\operatorname{Re} s \geq 0$.

The proof of this theorem and a discussion of the integral resolvent can be found in Miller [10].

III. MAIN RESULTS

A. Perturbation Theorems

There are a number of results concerning equation (E). Analysis of (E) is usually done by Laplace transform techniques as in Theorem 2.3. In this section equation (L) is examined as a perturbation of (E). It is shown that (L) inherits much of the behavior of (E) for an appropriate perturbation term $C(t)x$. The resolvent of (E) or (L) is denoted by R_E and R_L respectively. Similarly, ρ_E and ρ_L denote the maps defined by R_E and R_L. The admissibility result included in the following theorem is useful later when studying equation (N).

Theorem 3.1. Suppose in equation (L) that $B(t)$ is in L^p for some p satisfying $1 \leq p \leq \infty$, and that $R_E(t)$ is in $L^1 \cap BC$. If $C(t)$ is a matrix function in BC_o and $f(t)$ is a function in BC then, $x(t)$, the solution of (L) is in BC and ρ_L is in $\mathcal{a}(BC,BC)$.

Proof of Theorem 3.1. From equation (2.2) we see that one can express, $x(t)$, the solution of (L) in the following two ways

$$x(t) = R_L(t,0)x_o + \int_o^t R_L(t,s)f(s)ds \qquad 3.1$$

and

$$x(t) = R_E(t)x_o + \int_o^t R_E(t-s)f(s)ds + \int_o^t R_E(t-s)C(s)x(s)ds \qquad 3.2$$

This last equation follows from writing (L) as

$$x'(t) = Ax + \int_o^t B(t-s)x(s)ds + \tilde{f}(t); \quad x(0) = x_o$$

where $\tilde{f}(t) = C(t)x(t) + f(t)$.

For the moment we assume $C(t)$ is small. By small we mean $\|C\|_o < (\|R_E\|_1)^{-1}$. Then from (3.2), for any $T > 0$

$$\begin{aligned}\|x\|_{[0,T]} &= \sup_{t\in[0,T]} |x(t)| \\ &\leq \|R_E\|_o|x_o| + \sup_{t\in[0,T]} \int_o^t |R_E(t-s)||f(s)|ds \\ &\quad + \sup_{t\in[0,T]} \int_o^t |R_E(t-s)||C(s)||x(s)|ds \\ &\leq \|R_E\|_o|x_o| + \|R_E\|_1\|f\|_o + \|R_E\|_1\|C\|_o\|x\|_{[0,T]}\end{aligned}$$

Hence,

$$\|x\|_{[0,T]} \leq (\|R_E\|_o |x_o| + \|R_E\|_1 \|f\|_o)(1 - \|R_E\|_1 \|C\|_o)^{-1}$$

Thus, $x(t)$ is in BC and the theorem is true for small $C(t)$.

Now let $C(t)$ be an arbitrary matrix in BC_o. Then there is a $T > 0$ such that $\|C_T(t)\|_o < (\|R_E\|_1)^{-1}$ where $C_T(t) = C(t+T)$. Since $x(t)$ is a continuous function on $[0,T]$ there exists a constant $K_T > 0$ such that

$$\|x\|_{[0,T]} = \sup_{t\in[0,T]} |x(t)| \leq K_T.$$

For $t \geq T$, $x(t)$ still solves (L) and this may be expressed by translating (L) and replacing t by $t + T$

$$x'(t+T) = (A + C(t+T))x(t+T) + \int_o^{t+T} B(t+T-s)x(s)ds$$
$$+ f(t+T); \quad x(0+T) = x(T)$$

where now $t \geq 0$. Writing $x(t+T)$ as $x_T(t)$ we have

$$x_T'(t) = (A + C_T(t))x_T(t) + \int_0^{t+T} B(t+T-s)x(s)\,ds + f_T(t);$$

$$x_T(0) = x(T).$$

Performing a change of variable, $u = s - T$, inside the integral we get

$$x_T'(t) = (A + C_T(t))x_T(t) + \int_{-T}^{t} B(t-u)x_T(u)\,du + f_T(t);$$

$$x_T(0) = x(T)$$

or

$$x_T'(t) = (A + C_T(t))x_T(t) + \int_0^{t} B(t-u)x_T(u)\,du + F(t); \qquad 3.3$$

$$x_T(0) = x(T)$$

where

$$F(t) = \int_{-T}^{0} B(t-u)x_T(u)\,du + f_T(t) \qquad 3.4$$

We now show that $F(t)$ is in BC. Since $B(t)$ is in L^p for some p in $[1,\infty]$ it is locally Lebesgue integrable.

This implies $\int_{-T}^{0} B(t-u)x_T(u)\,du$ is continuous as a function of t for $t \in R^+$. We refer to Royden [12], page 90.

Now if $B(t)$ is in L^∞, then there exists an $M > 0$ such that $|B(t)| \leq M$ for all $t \in R^+$ except possibly a set of measure zero. Hence,

$$\left|\int_{-T}^{0} B(t-u)x_T(u)\,du\right| \leq \int_{-T}^{0} |B(t-u)|\,du\; K_T$$

$$\leq T\, M\, K_T \quad \text{for} \quad t \in R^+$$

where $K_T = \sup\limits_{u\in[-T,0]} |x_T(u)|$. So $\|F\|_0 \leq T\,M\,K_T + \|f\|_0$.

If $B(t)$ is in L^p for $1 < p < \infty$ then by Holder's Inequality it follows that

$$\left|\int_{-T}^{0} B(t-u)x_T(u)\,du\right| \leq \int_{-T}^{0} |B(t-u)|\,du\; K_T$$

$$= \int_{0}^{T} |B(t+s)|\,ds\; K_T$$

$$\leq \left(\int_{0}^{T} |B(t+s)|^p ds\right)^{1/p} \left(\int_{0}^{T} 1\, ds\right)^{1/q} K_T$$

Hence,

$$\left| \int_{-T}^{0} B(t-u)\,x_T(u)\,du \right| \leq \|B\|_p \; T^{1/q} \; K_T$$

and

$$\|F\|_0 \leq \|B\|_p \; T^{1/q} \; K_T + \|f\|_0 .$$

If $B(t)$ is in L^1 then

$$\left| \int_{-T}^{0} B(t-u)\,x_T(u)\,du \right| \leq \int_{-T}^{0} |B(t-u)|\,du \; K_T$$

$$\leq \|B\|_1 \; K_T .$$

So

$$\|F\|_0 \leq \|B\|_1 \; K_T + \|f\|_0 .$$

Thus, we conclude $F(t)$ is a function in BC.

Now $x_T(t)$ solves the equation (3.3) and in this equation $C_T(t)$ is small, that is, $\|C_T\|_0 < (\|R_E\|_1)^{-1}$. Then, from the first portion of the proof, x_T is in BC. It follows that $x(t)$ is in BC.

We now show that ρ_L is in $\mathfrak{a}(BC,BC)$. Let $x_0 = 0$ in (L) so

$$\rho_L(f)(t) = x(t) = \int_0^t R_L(t,s)f(s)ds$$

where $x(t)$ is the solution of (L) with $x_o = 0$. Since $x(t)$ is in BC if $f(t)$ is in BC, ρ_L maps BC into BC. BC is a Banach subspace of LL^1 with a stronger topology so by Theorem 2.1 we see ρ_L is in $\mathcal{G}(BC,BC)$.

Q.E.D.

We note in the proof of Theorem 3.1 the hypothesis that $C(t)$ have limit zero was stronger than was necessary. It would have been sufficient if

$$\lim_{t\to+\infty} \text{Sup}|C(t)| < (\|R_E\|_1)^{-1}.$$

This fact is useful if it is possible to get an upper bound on $\|R_E\|_1$. For instance if $B(t)$ is in L^1 and the entries of $R_E(t)$ are of the same sign on R^+, then

$$\int_0^\infty |R(t)|dt = \left| \int_0^\infty R_E(t)dt \right| = |R_E^*(0)|$$

where $R_E^*(s) = \int_0^\infty e^{-st}R(t)dt$ the Laplace transform of

$R_E(t)$. From equation (2.2) one may calculate

$$R_E^*(s) = [sI - A - B^*(s)]^{-1}$$

provided this inverse exists for s a complex number. Then $R_E^*(0) = -[A + B^*(0)]^{-1}$.

Using this theorem it is possible to obtain a number of similar results by further restricting $R_E(t)$ and $f(t)$. To do this the following lemmas concerning the convolution product are needed.

<u>Lemma 3.1</u>. Suppose $A(t)$ is an n by n matrix function in L^1. If $b(t)$ is a function in BC_ℓ then the convolution product of A and b defined by

$(A * b)(t) = \int_0^t A(t-s)b(s)\,ds$ is a function in BC_ℓ and

$$\lim_{t\to+\infty} (A * b)(t) = \int_0^\infty A(s)\,ds \cdot b(\infty)$$

where $b(\infty) = \lim_{t\to+\infty} b(t)$.

<u>Proof of Lemma 3.1</u>. Let A be an $n \times n$ matrix in L^1 and b a function in BC_ℓ then

$$\left|\int_0^t A(t-s)b(s)\,ds - \int_0^\infty A(s)\,ds \cdot b(\infty)\right|$$

$$\leq \left|\int_0^t A(t-s)b(s)\,ds - \int_0^t A(t-s)b(\infty)\,ds\right|$$

$$+ \left|\int_0^t A(t-s)b(\infty)\,ds - \int_0^\infty A(s)\,ds \cdot b(\infty)\right|$$

$$\leq \left|\int_0^t A(t-s)\,(b(s) - b(\infty))\,ds\right| + \left|\int_t^\infty A(s)\,ds \cdot b(\infty)\right| .$$

The last term has limit zero as t tends to infinity. This is also true of the first term. Define $f(s) = b(s) - b(\infty)$ then f is in BC_o and

$$\int_0^t A(t-s)f(s)\,ds = \int_0^t A(s)f(t-s)\,ds$$

$$= \int_0^\infty A(s)F(t,s)\,ds$$

where
$$F(t,s) = \begin{cases} f(t-s) & \text{for } 0 \leq s \leq t \\ 0 & \text{for } s > t \end{cases} .$$

Now $|A(s)F(t,s)| \leq |A(s)|\,\|f\|_o$ and $|A(s)|\,\|f\|_o$ is integrable on R^+. Finally, for any $s \in R^+$ $\lim_{t\to+\infty} F(t,s) = 0$ so the Lebesgue Dominated Convergence Theorem applies. Hence

$$\lim_{t\to\infty} \int_0^t A(t-s)f(s)ds = 0.$$

So, $$\lim_{t\to+\infty} \left| \int_0^t A(t-s)b(s)ds - \int_0^\infty A(s)ds \cdot b(\infty) \right| = 0.$$

Q.E.D.

<u>Lemma 3.2</u>. If $A(t)$ is an $n \times n$ matrix in LL^1, $b(t)$ is an n vector in LL^p $(1 \leq p < \infty)$, and $h(t)$ is defined by

$$h(t) = \int_0^t A(t-s)b(s)ds$$

then h is an n vector in LL^p and for any $K > 0$

$$\|h\|_{p[0,K]} \equiv \left(\int_0^K |h(s)|^p ds \right)^{1/p} \leq \left(\int_0^K |A(s)|ds \right) \left(\int_0^K |b(s)|^p ds \right)^{1/p}.$$

For a proof of Lemma 3.2 see Miller [10], page 167.

Corollary 3.1. In equation (L), suppose $B(t)$ is in L^p for some $p \in [1,\infty]$, $R_E(t)$ is a function of $L^1 \cap BC_o$, and $C(t)$ is in BC_o. If f is in BC_ℓ then $x(t)$, the solution of (L), is in BC_ℓ and

$x(\infty) = \lim_{t\to\infty} x(t) = \int_0^\infty R_E(s)\,ds \cdot f(\infty)$. Furthermore, ρ_L is in $\mathcal{G}(BC_\ell, BC_\ell)$.

Proof of Corollary 3.1. From equation (3.2), $x(t)$ satisfies

$$x(t) = R_E(t)x_o + \int_0^t R_E(t-s)f(s)\,ds + \int_0^t R_E(t-s)C(s)x(s)\,ds$$

and is a bounded continuous function by Theorem 3.1. Now all the terms on the right hand side of this equation have limits as $t \to +\infty$. It is clear that

$$\lim_{t\to\infty} R_E(t)x_o = 0.$$

By Lemma 3.1, the second term has a limit

$$\lim_{t\to+\infty} \int_0^t R_E(t-s)f(s)\,ds = \int_0^\infty R_E(s)\,ds \cdot f(\infty).$$

Finally,

$$\lim_{t\to+\infty} \int_0^t R_E(t-s)C(s)x(s)\,ds = 0$$

since $\lim_{t\to\infty} C(t)x(t) = 0$. Thus,

$$\lim_{t\to+\infty} x(t) = \int_0^\infty R_E(s)\,ds \cdot f(\infty).$$

Now we argue that ρ_L is in $\mathcal{G}(BC_\ell, BC_\ell)$. Letting $x_0 = 0$ in (L), for $f \in BC_\ell$

$$\rho_L(f) = \int_0^t R_L(t,s)f(s)\,ds = x(t)$$

where $x(t)$ is the solution of (L) and is in BC_ℓ. Hence, ρ_L maps BC_ℓ into BC_ℓ and BC_ℓ is a Banach subspace of LL^1 with a stronger topology. Theorem 2.1 implies $\rho_L \in \mathcal{G}(BC_\ell, BC_\ell)$.

Q.E.D.

Corollary 3.2. Suppose in equation (L) that $B(t)$ is in L^p for some p satisfying $1 \leq p < \infty$. If $R_E(t)$ is in $L^1 \cap BC_0$, $C(t)$ is in BC_0, and f is in BC_0 then $x(t) \in BC_0$ and ρ_L is in $\mathcal{G}(BC_0, BC_0)$.

Proof of Corollary 3.2. From Corollary 3.1, $x(t)$ is in BC_ℓ and

$$x(\infty) = \lim_{t\to\infty} x(t) = \int_0^\infty R_E(s)\,ds \cdot f(\infty).$$

But $f(\infty) = \lim_{t\to+\infty} f(t) = 0$ so $x(t) \in BC_o$. The fact that ρ_L is in $\mathcal{G}(BC_o, BC_o)$ follows by the same argument used in Corollary 3.1.

Q.E.D.

Corollary 3.3. Suppose in equation (L) that $B(t)$ is in L^q for some $q \in [1,\infty]$ and that $R_E(t)$ is in $L^1 \cap BC_o$. For any fixed $p \in [1,\infty)$, if $C(t)$ is in $L^p \cap BC_o$ and $f(t)$ is in $L^p \cap BC_o$ then $x(t)$ is in $L^p \cap BC_o$ and ρ_L is in $\mathcal{G}(L^p \cap BC_o, L^p \cap BC_o)$.

Proof of Corollary 3.3. From equation (3.2) the solution, $x(t)$, of equation (L) satisfies

$$x(t) = R_E(t)x_o + \int_0^t R_E(t-s)f(s)\,ds + \int_0^t R_E(t-s)C(s)x(s)\,ds$$

Let p be a fixed element of $[1,\infty)$. Now $R_E(t)x_o$ is in both in L^1 and in BC_o so $R_E(t)x_o$ is in L^p. Hence,

$R_E(t)x_o$ is also in $L^p \cap BC_o$. Since the convolution of an L^1 function with an L^p function is an L^p function (see Lemma 3.2), $R_E * f$ is in L^p. It also follows from Lemma 3.1 that $R_E * f$ is in BC_o. Hence, $R_E * f$ is in $L^p \cap BC_o$. We know from Theorem 3.1 that $x(t)$ is bounded. Thus, $C(t)x(t)$ is in $L^p \cap BC_o$. Hence, $R * Cx$ is in $L^p \cap BC_o$. It follows then that $x(t)$ is in $L^p \cap BC_o$.

Again since $L^p \cap BC_o$ is a Banach subspace of LL^1 with a stronger topology and ρ_L maps $L^p \cap BC_o$ into $L^p \cap BC_o$, ρ_L is in $\mathcal{G}(L^p \cap BC_o, L^p \cap BC_o)$.

Q.E.D.

The next theorem is similar to Corollary 3.3 except the hypothesis on $B(t)$ is strengthened and that on $C(t)$ is weakened but the result is the same.

Theorem 3.2. Let p be a fixed number satisfying $1 \leq p < \infty$. Suppose in equation (L) that $B(t)$ is in L^p and $R_E(t)$ is a function in $L^1 \cap BC$. If $C(t)$ is in BC_o and $f(t)$ is in $L^p \cap BC$ then $x(t)$, the solution of (L), is in $L^p \cap BC$ and ρ_L is in $\mathcal{G}(L^p \cap BC, L^p \cap BC)$.

The proof requires the following lemma.

<u>Lemma 3.3</u>. Suppose $b(t)$ is a scalar valued function in L^p for some $p \in [1,\infty)$ and T is a positive real number. Then

$$g(t) = \int_0^T b(t+s)\,ds$$

is a function in $L^p \cap BC$.

<u>Proof of Lemma 3.3</u>.

Case 1. $p = 1$

Since $b(t)$ is L^1, $g(t)$ is continuous for t in $[0,\infty)$ (See Royden [12], page 90.). Then for any $A > 0$

$$\int_0^A \left|g(t)\right| dt = \int_0^A \left| \int_0^T b(t+s)\,ds \right| dt$$

$$\leq \int_0^A \int_0^T |b(t+s)|\,ds\ dt$$

$$= \int_0^T \int_0^A |b(t+s)|\,dt\ ds$$

This last equality follows from Fubini's Theorem. Now since $b(t)$ is in L^1, for any $A > 0$ and all $s \in [0,T]$, $\int_0^A |b(t+s)|\,dt$ exists. Thus,

$\lim_{A\to\infty} \int_0^A |b(t+s)|dt = \int_0^\infty |b(t+s)|dt$ exists and is a continuous function of s for $s \in [0,T]$. Fubini's Theorem implies

$$\int_0^T \int_0^\infty |b(t+s)|dt\, ds = \int_0^\infty \int_0^T |b(t+s)|ds\, dt.$$

Also, $|g(t)| \leq \int_0^T |b(t+s)|ds \leq \int_0^\infty |b(s)|ds$. So $g(t)$ is a function in $L^1 \cap BC$.

Case II. $1 < p < \infty$

If $b(t)$ is in L^p for some $p \in (1,\infty)$, then $b(t)$ is locally L^1 and from Holder's Inequality

$$|g(t)| \leq \int_0^T |b(t+s)|ds \leq \left(\int_0^T |b(t+s)|^p ds\right)^{1/p}\left(\int_0^T |1|^q ds\right)^{1/q}$$

$$\leq \|b\|_p \cdot T^{1/q}$$

where $1 = 1/p + 1/q$ and $t \geq 0$. Thus,

$g(t) = \int_0^T |b(t+s)|ds$ is a continuous bounded function for t in $[0.\infty)$. We claim that $g(t)$ is also in L^p for $1 \leq p < \infty$. For $A > 0$,

$$\int_0^A |g(t)|^p dt = \int_0^A \left| \int_0^T b(t+s)\,ds \right|^p dt$$

$$\leq \int_0^A \left(\int_0^T |b(t+s)|\,ds \right)^p dt$$

and

$$\left(\int_0^T |b(t+s)|\,ds \right)^p \leq \left(\int_0^T |b(t+s|^p ds \right) T^{p/q};\ 1/p + 1/q = 1$$

by Holder's Inequality. Then

$$\int_0^A |g(t)|^p dt \leq \int_0^A \left(\int_0^T |b(t+s)|^p ds \right) dt\ T^{p/q}.$$

Interchanging the order of integration we get

$$\int_0^A |g(t)|^p dt \leq \int_0^T \int_0^A |b(t+s)|^p dt\ ds\ T^{p/q}.$$

Since $b(t)$ is in L^p, $\int_0^\infty |b(t+s)|^p dt$ exists and is continuous for $s \in [0,T]$. Hence, $\int_0^T \int_0^\infty |b(t+s)|^p dt\ ds$ exists and is equal to $\int_0^\infty \int_0^T |b(t+s)^p ds\ dt$ by Fubini's

Theorem. Thus, $g(t)$ is in $L^p \cap BC$.

Q.E.D.

Proof of Theorem 3.2. The method of proof is the same as that used in Theorem 3.1. The theorem will first be proved for small $C(t)$. Let $p \in [1,\infty)$.

Suppose that $\|C(t)\|_o < (\|R_E\|_1)^{-1}$. From equation (3.2), $x(t)$, the solution of (L) satisfies

$$x(t) = R_E(t)x_o + \int_o^t R_E(t-s)f(s)\,ds + \int_o^t R_E(t-s)C(s)x(s)\,ds.$$

Then, if $A > 0$

$$\begin{aligned}\|x\|_{p[0,A]} &= \left(\int_o^A |x(t)|^p dt\right)^{1/p} \\ &\leq \|R_E(t)x_o\|_{p[0,A]} + \left\|\int_o^t R_E(t-s)f(s)\,ds\right\|_{p[0,A]} \\ &\quad + \left\|\int_o^t R_E(t-s)C(s)x(s)\,ds\right\|_{p[0,A]}\end{aligned}$$

by Minkowski's inequality. Now since $R_E(t)$ is in $L^1 \cap BC$, $R_E(t)$ is also in $L^p \cap BC$. Using this fact and Lemma 3.2 we get

$$\|x\|_{p[0,A]} \leq \|R_E\|_p |x_o| + \|R_E\|_1 \|f\|_p$$

$$+ \|R_E\|_1 \|C\|_o \|x\|_{p[0,A]} .$$

Thus,

$$\|x\|_{p[0,A]} \leq (\|R_E\|_p |x_o| + \|R_E\|_1 \|f\|_p)(1 - \|C\|_o \|R_E\|_1)^{-1}$$

so $x(t)$ is in L^p. Since $x(t)$ solves (L) we see from equation (3.2) that $x(t)$ is continuous. It is also clear from equation (3.2) that if $x(t)$ is in L^p it is also bounded. Hence, $x(t)$ is in $L^p \cap BC$.

Now let $C(t)$ be an arbitrary function in BC_o. Then there is a $T > 0$ such that $\|C_T\|_o < (\|R_E\|_1)^{-1}$ where $C_T(t) = C(t+T)$. The solution $x(t)$ is continuous on $[0,T]$ and, so, is bounded by some real number $K > 0$ in this interval. For $t > 0$, $x_T(t) = x(t+T)$ solves

$$x_T'(t) = (A + C_T(t))x_T(t) + \int_0^t B(t-u)x_T(u)\,du + F(t);$$

$$x_T(0) = x(T)$$

where $F(t) = \int_{-T}^{0} B(t-u)x_T(u)\,du + f_T(t)$. See equation (3.3) and (3.4) in the proof of Theorem 3.1. We claim that $F(t)$ is in $L^p \cap BC$. Now $\sup_{t\in[0,T]} |x(t)| \leq K$ for some $K > 0$, so that

$$\left| \int_{-T}^{0} B(t-u)x_T(u)\,du \right| \leq \int_{0}^{T} |B(t+u)|\,du\; K.$$

From Lemma 3.3; $\int_{0}^{T} |B(t+u)|\,du$ is in $L^p \cap BC$. Hence, $F(t)$ is in $L^p \cap BC$. Thus, $x_T(t)$ is in $L^p \cap BC$ by the above proof for small $C(t)$. Therefore, $x(t)$ is in $L^p \cap BC$.

It is now a routine matter to prove ρ in $\mathcal{G}(L^p \cap BC, L^p \cap BC)$. For if $f \in L^p \cap BC$, let $x_o = 0$ in (L) then $\rho_L(f)(t) = x(t)$ and $x(t)$ is in $L^p \cap BC$. So ρ_L maps $L^p \cap BC$ into itself. $L^p \cap BC$ is a Banach subspace of LL^1 with a stronger topology. Hence Theorem 2.1 implies ρ_L is in $\mathcal{G}(L^p \cap BC, L^p \cap BC)$.

Q.E.D.

The previous theorems were restricted to perturbations of integral differential equations of the convolution type.

If we require $C(t)$ to be a matrix in $L^1 \cap BC$ it is possible to prove perturbation theorems for more general types of integral equations. Consider the equations

$$x'(t) = A(t)x(t) + \int_0^t B(t,s)x(s)ds + f(t); \quad x(0) = x_o \tag{3.5}$$

and

$$x'(t) = (A(t)+C(t))x(t) + \int_0^t B(t,s)x(s)ds + f(t); \quad x(0) = x_o \tag{3.6}$$

where $x(t)$ is in R^n, $B(t,s)$ is an n by n matrix that is locally Lebesgue integrable in both variables, $A(t)$ and $C(t)$ are n by n matrices in C, and f is an n vector in C. From equations (2.2) and (2.3) above, we see that both equations (3.5) and (3.6) have continuous solutions and resolvents. Let $R_5(t,s)$ and $R_6(t,s)$ be the resolvents of equations (3.5) and (3.6) respectively.

<u>Theorem 3.3</u>. Suppose that in equation (3.6) $C(t)$ and $f(t)$ are in $L^1 \cap BC$. If $R_5(t,s)$ satisfies $|R_5(t,s)| \leq M$ for $0 \leq s \leq t < +\infty$ and some $M > 0$ then

$x(t)$, the solution of equation (3.6), is in BC and ρ_6, defined by

$$\rho_6(f)(t) = \int_0^t R_6(t,s)f(s)ds, \quad \text{is in} \quad \mathcal{G}(L^1 \cap BC, BC).$$

There are theorems analogous to this in ordinary differential equations. In fact, the proof of this theorem is accomplished by using Gronwall's inequality.

Proof of Theorem 3.3. From equation (2.2), $x(t)$, the solution of equation (3.6) satisfies

$$x(t) = R_5(t,0)x_0 + \int_0^t R_5(t,s)f(s)ds + \int_0^t R_5(t,s)C(s)x(s)ds$$

for $t \geq 0$, so

$$|x(t)| \leq M|x_0| + M\|f\|_1 + M\int_0^t |C(s)||x(s)|ds.$$

By using Gronwall's inequality we see

$$|x(t)| \leq M(|x_0| + \|f\|_1)\exp[M\int_0^t |C(s)|ds]$$

and since $C(t)$ is in L^1, $x(t)$ is in BC.

Now let $f \in L^1 \cap BC$ and consider equation (3.6) with $x(0) = x_o = 0$. Then

$$\rho_6(f)(t) = \int_o^t R_6(t,s)f(s)ds = x(t).$$

So ρ_6 maps $L^1 \cap BC$ into BC. Both $L \cap BC$ and BC are Banach subspaces of LL^1 with stronger topologies. Hence, ρ_6 is in $\mathcal{Q}(L' \cap BC, BC)$.

Q.E.D.

B. A System of Convolution Type

This section deals with system (E). We seek to extend the result of Grossman and Miller [4] (see Theorem 2.3) to the case where $B(t)$ is the sum of a matrix that is Lebesgue integrable on R^+ and a constant matrix of a given class. Such a result is needed in the analysis of the example in reactor dynamics that follows in Chapter IV. Shea and Wainger in [13] using different techniques than those used here obtained similar results for a scalar equation of type (E).

Consider the equation

$$x'(t) = Ax(t) + \int_0^t [B(t-s)+D]x(s)\,ds + f(t); \qquad x(0) = x_o \tag{G}$$

where A and D are constant n by n matrices, $B(t)$ is a matrix of size n in L^1, and $x(t)$ and $f(t)$ are column vectors. Equation (G) is of convolution type so $x(t)$ the solution of (G) can be written

$$x(t) = R_G(t)x_o + \int_0^t R_G(t-s)f(s)\,ds$$

where R_G is the resolvent of equation (G). The resolvent $R_G(t)$ satisfies

$$R_G'(t) = A\,R_G(t) + \int_0^t (B(t-s)+D)R_G(s)\,ds; \quad R_G(0) = I.$$

If $B'(t)$ exists and is in L^1 this equation may be differentiated

$$R_G''(t) = A\,R_G'(t) + (B(0)+D)R_G(t) + \int_0^t B'(t-s)R_G(s)\,ds;$$

$$R_G(0) = I; \quad R'_G(0) = A.$$

Letting $y_1(t) = R_G(t)$ and $y_2(t) = R_G'(t)$ we can write this as the $2n$ dimensional system

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & I \\ B(0)+D & A \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 & 0 \\ B'(t-s) & 0 \end{pmatrix}\begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds$$

$$y_1(0) = I, \quad y_2(0) = A.$$

Then from Theorem 2.3, $y_1(t) = R_G(t)$ and $y_2(t) = R_G'(t)$ are in $L^1 \cap BC_o$ if

$$\mathrm{Det}\left(sI - \begin{pmatrix} 0 & I \\ B(0)+D & A \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ B'^*(s) & 0 \end{pmatrix}\right) =$$

$$\mathrm{Det}\begin{pmatrix} sI & -I \\ -sB^*(s)+D & sI-A \end{pmatrix} \neq 0 \quad \text{for} \quad \mathrm{Re}\, s \geq 0.$$

Recall that $B'^*(s) = sB^*(s) - B(0)$ where $B^*(s)$ is the Laplace transform of $B(t)$.

However, in some cases it is not convenient to assume $B'(t)$ exists or is in L^1. We prove the following theorem concerning (G).

<u>Theorem 3.4</u>. The resolvent, $R_G(t)$; of equation (G) is in $L^1 \cap BC_0$ if

(i) $\mathrm{Det}[s^2 I - sA - sB^*(s) - D] \neq 0$ for $\mathrm{Re}\ s = 0$ where $B^*(s)$ is the Laplace transform of $B(t)$.

(ii) There exists an n by n matrix $\Phi(t)$, such that $\Phi(t)$, $\Phi'(t)$, and $\int_t^\infty \Phi(u)du$ are in $L \cap BC_0$, $\Phi(0) = I$, $\mathrm{Det}\ \Phi^*(s) \neq 0$ for $\mathrm{Re}\ s \geq 0$, and $\int_0^\infty \Phi(u)du \cdot D$ is a matrix with eigenvalues having negative real parts. ($\Phi^*(s)$ denotes the Laplace transform of $\Phi(t)$).

Hypothesis (i) is predictable in view of Theorem 2.3. However, hypothesis (ii) is perhaps best explained by proceeding with the proof of the theorem. The technique of proof is the same as that used in the proof of Theorem 2.3 in Grossman and Miller [4] but there are added complications. We remark here that if D has eigenvalues with negative real part one may take $\Phi(t) = e^{-t}I$. The following lemma concerning Volterra integral equations is needed for the proof.

Lemma 3.4. Let $B(t)$ be an $n \times n$ matrix in $L^1 \cap BC_o$ and A and $n \times n$ matrix with all eigenvalues having negative real part. If $\operatorname{Det}[sI - sB^*(s) - A] \neq 0$ for $\operatorname{Re} s \geq 0$ where s is a complex number then the resolvent, $r(t)$, of the integral equation

$$x(t) = f(t) + \int_0^t (B(t-s) + A)x(s)\,ds$$

is in $L^1 \cap BC_o$.

Proof of Lemma 3.4. Recall from Chapter 2 above that $r(t)$ is a matrix of size n in LL^1 satisfying

$$r(t) = -A - B(t) + \int_0^t (B(t-s) + A)r(s)\,ds.$$

Let $g(t) = -A - B(t) + \int_0^t B(t-s)r(s)\,ds$. Then $r(t)$ solves

$$r(t) = g(t) + \int_0^t A\, r(s)\,ds.$$

This is an integral equation and may be solved in terms of the resolvent of the kernel A. If r_A denotes the resolvent of A, r_A solves

$$r_A(t) = -A + \int_o^t A\, r_A(s)\,ds.$$

Hence, $r_A(t) = -A\, e^{At}$. From (2.6) we can express $r(t)$ in terms of $r_A(t)$ as follows

$$r(t) = g(t) - \int_o^t r_A(t-s)\,g(s)\,ds.$$

Substituting for $g(t)$ we get

$$r = -A - B + B * r - r_A * (-A - B + B * r)$$

where $*$ denotes convolution. Since $r_A = -A + A * r_A$ this may be reduced to

$$r = r_A - (B - r_A * B) + (B - r_A * B) * r$$

which may be rewritten as

$$r(t) = h(t) + (H * r)(t) \qquad 3.7$$

where

$$h(t) = r_A - (B - r_A * B)$$

and

$$H(t) = B - r_A * B.$$

Now (3.7) is an integral equation and the solution $r(t)$ can be expressed in terms of the resolvent of (3.7). Denote this resolvent by r_H. Hence,

$$r(t) = h(t) - \int_0^t r_H(t-s)h(s)ds.$$

Since all the eigenvalues of A have negative real parts, r_A is in $L^1 \cap BC_0$. It follows from Lemmas 3.1 and 3.2 that $h(t)$ and $H(t)$ are in $L^1 \cap BC_0$. Also, $r(t)$ is in $L^1 \cap BC_0$ if r_H is in L^1. From Theorem 2.5, r_H is in L^1 if and only if

$$\det[I - H^*(s)] \neq 0 \quad \text{for} \quad \text{Re } s \geq 0.$$

But

$$H^*(s) = B^*(s) - (r_A * B)^*(s)$$

$$= B^*(s) - r_A^*(b)B^*(s)$$

$$= B^*(s) + A(sI - A)^{-1}B^*(s).$$

So

$$\det[I - H^*(s)] = \det[I - B^*(s) - A(sI - A)^{-1}B^*(s)].$$

Since A has eigenvalues with real parts negative $\det[sI - A] \neq 0$ for $\operatorname{Re} s \geq 0$ so that if

$$\det[sI - A]\det(I - B^*(s) - A(sI - A)^{-1}B^*(s)]$$

$$= \det(sI - A - sB^*(s) - A\,B^*(s) + A\,B^*(s)]$$

$$= \det[sI - A - sB^*(s)] \neq 0 \quad \text{for } \operatorname{Re} s \geq 0$$

then $\det[I - H^*(s)] \neq 0$ for $\operatorname{Re} s \geq 0$. Hence, r_H is in L^1 implying $r(t)$ is in $L^1 \cap BC_o$.

Q.E.D.

Proof of Theorem 3.4. The proof is done by converting equation (G) to a Volterra integral equation and using Lemma 3.5. In equation (G) if $f(t) \equiv 0$ the solution $x(t)$ satisfies $x(t) = R_G(t)x_o$. So it is sufficient to prove that $x(t)$ is a function in $L^1 \cap BC_o$. We convolution multiply equation (G) by $\Phi(t)$ and let $f(t) \equiv 0$. We get

$$\Phi * x' = \Phi * Ax + \Phi * B * x + \Phi * D * x$$

but

$$(\Phi * x')(t) = \int_o^t \Phi(t-s)x'(s)ds$$

$$= [\Phi(t-s)x(s)]_o^t - \int_o^t \Phi'(t-s)x(s)ds$$

$$= x(t) - \Phi(t)x_o - (\Phi' * x)(t)$$

where $\Phi' = \frac{d\Phi}{dt}$. Thus,

$$x(t) = \Phi(t)x_o + (-\Phi' + \Phi A + \Phi * B + \Phi * D) * x.$$

The expression in parentheses can be written as an L^1 function plus a constant by decomposing $\Phi * D$

$$\Phi * D = \int_0^t \Phi(s)\,ds \cdot D = -\int_t^\infty \Phi(s)\,ds \cdot D + \int_0^\infty \Phi(s)\,ds \cdot D.$$

So that

$$x(t) = \Phi(t)x_o + (K+M) * x)(t) \tag{3.8}$$

where

$$K = -\Phi' + \Phi A + \Phi * B - \int_t^\infty \Phi(s)\,ds \cdot D$$

and

$$M = \int_0^\infty \Phi(s)\,ds \cdot D.$$

Then $K(t)$ is in $L^1 \cap BC_o$ and M is a constant matrix with all of its eigenvalues having negative real parts. We may express $x(t)$ as

$$x(t) = \Phi(t)x_o - \int_0^t r_{K+M}(t-s)\Phi(s)x_o\,ds$$

where r_{K+M} is the resolvent of the integral equation (3.8). By Lemma 3.4, r_{K+M} is in $L^1 \cap BC_o$ if

$$\det[sI - sK^*(s) - M] \neq 0 \quad \text{for} \quad \mathrm{Re}\ s \geq 0.$$

Now

$$K^*(s) = -s\Phi^*(s) + I + \Phi^*(s)A + \Phi^*(s)B^*(s) - \frac{1}{s}[M - \Phi^*(s)D]$$

so

$$\begin{aligned} sI - sK^*(s) - M &= sI + s^2\Phi^*(s) - sI - s\Phi^*(s)A \\ &\qquad - s\Phi^*(s)B\ (s) - \Phi^*(s)D \\ &= \Phi^*(s)[s^2I - sA - sB^*(s) - D] \end{aligned}$$

Since $\det \Phi^*(s) \neq 0$ for $\operatorname{Re} s \geq 0$, then

$$\operatorname{Det}[sI - sK^*(s) - M] \neq 0 \quad \text{for} \quad \operatorname{Re} s \geq 0$$

if

$$\operatorname{Det}[s^2I - sA - sB^*(s) - D] \neq 0 \quad \text{for} \quad \operatorname{Re} s \geq 0.$$

This last statement is hypothesis (ii). Thus, r_{K+M} is in $L^1 \cap BC_o$ and since

$$x(t) = \Phi(t)x_o - \int_o^t r_{K+M}(t-s)\Phi(s)x_o\, ds,$$

$x(t)$ is in $L^1 \cap BC_o$ (see Lemmas 3.1 and 3.2 concerning convolution).

Q.E.D.

Hypothesis (ii) of Theorem 3.4 is a technical hypothesis. As previously mentioned if the eigenvalues of D all have negative real part one may take $\Phi(t) = e^{-t}I$. There are other matrices for which such a $\Phi(t)$ can be found. For instance, if the eigenvalues of D have positive real parts and large imaginary parts. For example, let

$$R = \begin{pmatrix} -1 & -3 \\ 3 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} -1 & +1 \\ -1 & -1 \end{pmatrix}.$$

Then R, U, and V have eigenvalues $\lambda = -2 \pm 6i$, $\lambda = -1 \pm 2i$ and $\lambda = -1 \pm i$, respectively. Also, $UV = R$. So, if $D = -R$, the eigenvalues of D are $2 \pm 6i$. But if $\Phi(t) = e^{Ut}$ then $\Phi(t)$, $\Phi'(t)$ and $\int_t^\infty \Phi(s)ds$ are in $L^1 \cap BC_o$ and $\det[sI - U] \neq 0$ for $\operatorname{Re} s \geq 0$. Furthermore, $\int_0^\infty e^{Ut}dt = U^{-1}$ so

$$\int_0^\infty e^{Ut} \cdot D = -U^{-1}D = -U^{-1}(-R) = U^{-1}UV = V$$

and V has eigenvalues that have negative real parts.

C. Nonlinear Systems

In this section we shall consider equation (N). Suppose in equation (N) that $C(t)$ is continuous, $B(t)$ is locally integrable and g maps LL^1 into itself. We see from Theorem 2.4 that if (N) has a solution $x(t)$ for $t \in R^+$ then for $t \in R^+$, $x(t)$ satisfies

$$x(t) = R_L(t,0)x_o + \int_o^t R_L(t,s)f(s)ds + \int_o^t R_L(t,s)g(x)(s)ds$$

where $R_L(t,s)$ is the resolvent of (L). We shall use this equation plus the principle of contraction maps to prove local results for equation (N). The theorem that follows is similar to one in Grossman and Miller [3] but considers a more general functional $g(x)$.

<u>Definition 3.1</u>. A functional h is of higher order in a Banach subspace X of LL^1 if h maps X into X, $h(0) = 0$, and for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|h(\varphi_1) - h(\varphi_2)\|_X < \epsilon \|\varphi_1 - \varphi_2\|_X$$

where $\| \|_X$ is the norm defined on X and φ_1, φ_2 are in X and satisfy $\|\varphi_1\|_X < \delta$, $\|\varphi_2\|_X < \delta$.

Theorem 3.5. Suppose in equation (N) that $C(t)$ is continuous, $B(t)$ is in LL^1, and $g(x) = g_1(x) + g_2(x)$ where g_1 is of higher order with respect to X a Banach subspace of LL^1 and g_2 maps X into X and satisfies $\|g_2(\varphi_1) - g_2(\varphi_2)\|_X \leq L\|\varphi_1 - \varphi_2\|_X$ for $\varphi_1, \varphi_2 \in X$, and some $L > 0$. If f is in X, $R_L(t,0)$ is in X and ρ_L is in $\mathfrak{a}(X,X)$ then for each $\epsilon > 0$ there is a $\eta > 0$ such that if $|x_o| < \eta$, $\|f\|_X < \eta$, and $L < \eta$; equation (N) has a unique solution $x(t)$ in X with $\|x\|_X \leq \epsilon$.

Proof of Theorem 3.5. For any φ in X define

$$T(\varphi)(t) = R_L(t,0)x_o + \int_o^t R_L(t,s)f(s)\,ds + \int_o^t R_L(t,s)g_1(\varphi)(s)\,ds$$
$$+ \int_o^t R_L(t,s)g_2(\varphi)(s)\,ds$$

for $0 \leq t < \infty$. Clearly T maps X into X.

Since $\rho_L \in \mathfrak{a}(X,X)$ there exists an $M > 0$ such that

$$\|\rho_L(\varphi)\|_X = \| \int_o^t R_L(t,s)\varphi(s)\,ds\|_X \leq M\|\varphi\|_X.$$

Now g_1 is of higher order in X so there exists a $\delta > 0$ such that

$$\|g_1(\varphi_1) - g_1(\varphi_2)\|_X \leq \frac{1}{3M}\|\varphi_1 - \varphi_2\|_X$$

if $\|\varphi_1\|_X, \|\varphi_2\|_X < \delta$.

Given $\epsilon > 0$ define $\epsilon_o = \mathrm{Min}\{\delta, \epsilon, 1\}$ and let $\eta = \mathrm{Min}\left\{\delta, \dfrac{\epsilon_o}{6\|R_L(t,0)\|_X}, \dfrac{\epsilon_o}{6M}\right\}$. Also, define $s(0,\epsilon_o) = \{\varphi \in X : \|\varphi\|_X < \epsilon_o\}$. For any $\varphi \in s(0,\epsilon_o)$,

$$\|T(\varphi)\|_X \leq \|R_L(t,0)\|_X |x_o| + M\|f\|_X + M\|g_1(\varphi)\|_X + M\|g_2(\varphi)\|_X.$$

So if $|x_o| < \eta$, $\|f\| < \eta$, and $L < \eta$ then

$$\|T(\varphi)\|_X \leq \frac{\epsilon_o}{6} + \frac{\epsilon_o}{6} + \frac{\epsilon_o}{3} + \frac{\epsilon_o^2}{6} \leq \frac{5}{6}\epsilon_o.$$

Hence, $T(\varphi)(t)$ is in $s(0,\epsilon_o)$. And for $\varphi_1, \varphi_2 \in s(0,\epsilon_o)$,

$$\|T(\varphi_1) - T(\varphi_2)\|_X \leq \|\rho_L(g_1(\varphi_1)) - \rho_L(g_1(\varphi_2))\|_X$$

$$+ \|\rho_L(g_2(\varphi_1)) - \rho_L(g_2(\varphi_2))\|_X$$

$$\leq M\|g_1(\varphi_1) - g_1(\varphi_2)\|_X + M\|g_2(\varphi_1) - g_2(\varphi_2)\|_X$$

$$\leq \frac{1}{3}\|\varphi_1 - \varphi_2\|_X + \frac{1}{6}\|\varphi_1 - \varphi_2\|_X$$

$$\leq \frac{1}{2}\|\varphi_1 - \varphi_2\|_X$$

Thus, T is a contraction map on $s(0,\epsilon_o)$ so has a unique fixed point $x(t)$. We conclude from Theorem 2.4 that this fixed point is a solution of (N). Finally, since $x(t) \in s(0.\epsilon_o)$, $\|x(t)\|_X \leq \epsilon_o$.

Q.E.D.

The condition that g_2 satisfy $\|g_2(\varphi_1) - g_2(\varphi_2)\|_X \leq L\|\varphi_1 - \varphi_2\|_X$ where L is small is not as artifical as it may seem. We have in mind a situation where L is a function of $|x_o|$ and $\|f\|_X$ and decreases to zero as $|x_o|$ and $\|f\|_X$ tend to zero. This arises in an application in Chapter IV.

IV. AN APPLICATION TO REACTOR DYNAMICS

In this chapter we shall study a system of equations of type (N) that occur in reactor dynamics. The system was given in the Introduction, equation (1.1), and is repeated here.

$$\frac{dP_j}{dt} = \frac{\rho_j - \epsilon_j - \beta_j}{\Lambda_j} P_j(t) + \sum_{i=1}^{N} \lambda_{ij} C_{ij}(t)$$

$$+ \frac{1}{\Lambda_j} \sum_{k=1}^{M} \epsilon_{kj} \int_0^{\infty} P_k(t-\tau) h_{kj}(\tau)\, d\tau \qquad 4.1.a$$

$$\frac{dC_{ij}}{dt} = \frac{\beta_{ij}}{\Lambda_j} P_j(t) - \lambda_{ij} C_{ij}(t) \qquad 4.1.b$$

$$j = 1,\ldots,M$$

$$i = 1,\ldots,N$$

M is the number of cores in the reactor and N is the number of neutron precursors. We shall now write (4.1) in a more convenient form.

It is assumed that $P_j(t) = P_{jo}$ for $t < 0$ and $j = 1,\ldots,M$. That is, the reactor is in an equilibrium state until perturbed at $t = 0$. At equilibrium we see from (4.1.b) that

$$\frac{\beta_{ij}}{\Lambda_j} P_{jo} = \lambda_{ij} C_{ijo} \tag{4.2}$$

where C_{ijo} is the equilibrium concentration of the ith precursor in the jth core. Then from the relation $\beta_j = \sum_{i=1}^{N} \beta_{ij}$ it follows that

$$\frac{\beta_j}{\Lambda_j} P_{jo} = \sum_{i=1}^{N} \frac{\beta_{ij}}{\Lambda_j} P_{jo} = \sum_{i=1}^{N} \lambda_{ij} C_{ijo}.$$

Recall that at equilibrium $\rho_j \equiv 0$ so substituting $P_j(t) \equiv P_{jo}$ into (4.1.a) we get

$$\begin{aligned} & -\frac{\epsilon_j}{\Lambda_j} P_{jo} + \frac{1}{\Lambda_j} \sum_{k=1}^{M} \epsilon_{kj} \int_0^{\infty} h_{kj}(\tau) P_{jo} d\tau \\ & = -\frac{\epsilon_j}{\Lambda_j} P_{jo} + \frac{1}{\Lambda_j} \sum_{k=1}^{M} \epsilon_{kj} P_{jo} = 0 \end{aligned} \tag{4.3}$$

Now consider the change of variables

$$p_j(t) = \frac{P_j(t) - P_{jo}}{P_{jo}} \; ; \; c_{ij}(t) = \frac{\lambda_{ij}\Lambda_j}{P_{jo}\beta_{ij}} (C_{ij}(t) - C_{ijo})$$

or

$$P_j(t) = p_j(t)P_{jo} + P_{jo}; \; C_{ij}(t) = \frac{P_{jo}\beta_{ij}}{\lambda_{ij}\Lambda_j} c_{ij}(t) + C_{ijo}$$

Note that at equilibrium $p_j(t) \equiv 0$ and $c_{ij}(t) \equiv 0$. Using (4.2) and (4.3) and making the above change of variables, equation (4.1) becomes

$$\frac{dp_j}{dt} = \frac{\rho_j - \epsilon_j - \beta_j}{\Lambda_j} p_j(t) + \frac{\rho_j}{\Lambda_j}$$

$$+ \sum_{i=1}^{N} \frac{\beta_{ij}}{\Lambda_j} c_{ij}(t) + \frac{1}{\Lambda_j} \sum_{k=1}^{M} \epsilon_{kj} \frac{P_{ko}}{P_{jo}} \int_0^t h_{kj}(t-s) p_k(s)\, ds \qquad 4.4.a$$

$$\frac{dc_{ij}}{dt} = \lambda_{ij}[p_j(t) - c_{ij}(t)] \qquad 4.4.b$$

$i = 1,\ldots,N$

$j = 1,\ldots,M.$

Now $\rho_j(t)$ is defined by the equation

$$\rho_j(t) = -\int_{G_j} \alpha^j(x) T^j(x,t)\,dx; \quad j = 1,\ldots,M \tag{4.5}$$

where G_j is the region containing the jth core, x is the space variable varying over G_j, $\alpha^j(x)$ the temperature coefficient of reactivity at x, and $T^j(x,t)$ the temperature at the point x at time t in the jth core as measured from equilibrium. $T^j(x,t)$ is identically zero in the equilibrium state. We shall first consider a simplified reactor where each core is a slab of height π, $G_j = [0,\pi]$ for $j = 1,\ldots,M$. And for $j = 1,\ldots,M$; $T^j(x,t)$ solves

$$\begin{aligned} T^j_t(x,t) &= T^j_{xx}(x,t) + \eta^j(x) p_j(t) P_{jo} \\ T^j(x,0) &= h^j(x) \end{aligned} \tag{4.6}$$

and $$T^j_x(0,t) = T^j_x(\pi,t) = 0$$

Here $T_x(x,t) = \dfrac{\partial T(x,t)}{\partial x}$ and $T_t(x,t) = \dfrac{\partial T(x,t)}{\partial t}$. Physically these boundary conditions correspond to the faces of each core being insulated. This model has been studied by Levin and Nohel [7] for a reactor with a single core.

Now we make the following assumptions:

$$\alpha^j, \eta^j, \frac{d\eta^j}{dx}, h^j, \text{ and } \frac{dh^j}{dx} \text{ are in } L^2[0,\pi] \text{ and } \eta^j, h^j \text{ satisfy the boundary conditions of (4.6)} \tag{4.7}$$

Here $L^2[0,\pi]$ denotes the set of all square integrable functions on $[0,\pi]$.

We now formally solve (4.6) for $T^j(x,t)$ explicitly in terms of $p_j(t)$ and substitute this into (4.4) via equation (4.5). Referring to Weinberger [14], Section 29, we see that if (4.7) holds the solution of (4.6) is given by

$$T^j(x,t) = P_{jo} \sum_{n=o}^{\infty} \int_o^t e^{-n^2(t-s)} p_j(s)\,ds\ \eta_n^j \cos n\,x + \sum_{n=o}^{\infty} h_n^j e^{-n^2 t} \cos n\,x. \tag{4.8}$$

where

$$h_o^j = \frac{1}{\pi}\int_o^{\pi} h^j(x)\,dx$$

$$h_n^j = \frac{2}{\pi}\int_o^{\pi} h^j(x)\,\mathrm{cox}\ n\,x\,dx, \quad n \geq 1$$

$$\eta_o^j = \frac{1}{\pi}\int_o^{\pi} \eta^j(x)\,dx$$

$$\eta_n^j = \frac{2}{\pi}\int_o^{\pi} \eta^j(x)\cos n\,x\,dx, \quad n \geq 1$$

Then,

$$\mu_j(t) = -\int_o^{\pi} \alpha^j(x)\,T^j(x,t)\,dx$$

$$= -P_{jo}\sum_{n=o}^{\infty}\int_o^t e^{-n^2(t-s)}p_j(s)\,ds\cdot\eta_n^j\alpha_n^j$$

$$-\sum_{n=o}^{\infty}\alpha_n^j h_n^j e^{-n^2 t}$$

where

$$\alpha_o^j = \frac{1}{\pi}\int_o^{\pi}\alpha^j(x)\,dx$$

$$\alpha_n^j = \frac{2}{\pi}\int_o^{\pi}\alpha^j(x)\,\mathrm{cox}\;n\;s\,dx, \quad n \geq 1$$

for $j = 1,\ldots,M$. Define

$$a_j(t) = \sum_{n=o}^{\infty}\eta_n^j\alpha_n^j e^{-n^2 t}$$

$$b_j(t) = \sum_{n=o}^{\infty}\alpha_n^j h_n^j e^{-n^2 t} \qquad 4.9$$

so $\rho_j(t) = -P_{jo}\int_0^t a_j(t-s)p_j(s)\,ds - b_j(t)$. Now substitute this into (4.4)

$$\frac{dp_j}{dt} = -\frac{P_{jo}}{\Lambda_j}\,p_j(t)\int_0^t a_j(t-s)p_j(s)\,ds - \frac{b_j(t)}{\Lambda_j}\,p_j(t)$$

$$-\frac{\epsilon_j+\beta_j}{\Lambda_j}\,p_j(t) - \frac{P_{jo}}{\Lambda_j}\int_0^t a_j(t-s)p_j(s)\,ds - \frac{b_j(t)}{\Lambda_j} \qquad 4.10.a$$

$$+\sum_{i=1}^{N}\frac{\beta_{ij}}{\Lambda_j}\,c_{ij}(t) + \frac{1}{\Lambda_j}\sum_{i=1}^{M}\epsilon_{kj}\,\frac{P_{ko}}{P_{jo}}\int_0^t h_{kj}(t-s)p_k(s)\,ds$$

$$\frac{dc_{ij}}{dt} = \lambda_{ij}[p_j(t) - c_{ij}(t)] \qquad 4.10.b$$

$$j = 1,\ldots,M; \quad i = 1,\ldots,N.$$

Finally, we eliminate $c_{ij}(t)$ by solving (4.10.b) and substituting into (4.10.a).

$$c_{ij}(t) = \lambda_{ij}\int_0^t e^{-\lambda_{ij}(t-s)}\,p_j(s)\,ds + c^o_{ij}e^{-\lambda_{ij}t} \qquad 4.11$$

where $c^o_{ij} = \dfrac{\lambda_{ij}\Lambda_j}{P_{jo}\beta_{ij}}\,C_{ij}(0) - C_{ijo}$, $C_{ij}(0)$ is the initial concentration of C_{ij} at $t = 0$. Substituting this into (4.10.a) we obtain an equation solely in $p_j(t)$

$$\frac{dp_j}{dt} = -\left[\frac{\alpha_j + \beta_j}{\Lambda_j} + \frac{b_j(t)}{\Lambda_j}\right]p_j(t) - \frac{P_{jo}}{\Lambda_j}\int_0^t a_j(t-s)p_j(s)\,ds$$

$$+ \frac{1}{\Lambda_j}\sum_{k=1}^{M} \epsilon_{kj}\frac{P_{ko}}{P_{jo}}\int_0^t h_{kj}(t-s)p_j(s)\,ds$$

$$+ \sum_{i=1}^{M}\frac{\beta_{ij}}{\Lambda_j}\lambda_{ij}\int_0^t e^{-\lambda_{ij}(t-s)}p_j(s)\,ds - \frac{b_j(t)}{\Lambda_j}$$

$$+ \sum_{i=1}^{N} c_{ij}^{o}e^{-\lambda_{ij}t} - \frac{P_{jo}}{\Lambda_j}p_j(t)\int_0^t a_j(t-s)p_j(s)\,ds$$

$$j = 1,2,\ldots,M.$$

Let $p(t) = \mathrm{Col}(p_1(t),\ldots,p_M(t))$. This equation plus initial data has the form

$$\frac{dp}{dt} = A\,p + \int_0^t B(t-s)p(s)\,ds + f(t) + k(t)p(t)$$

$$+ g(p)(t); \quad p(0) = p^o \qquad 4.12$$

where p^o is the initial vector; A, $k(t)$, and $B(t)$

are matrices defined by

$$A_{jj} = -\frac{\epsilon_j + \beta_j}{\Lambda_j}; \quad A_{ij} = 0 \quad i \neq j \tag{4.13.a}$$

$$B_{jj}(t) = -\frac{P_{jo}}{\Lambda_j} a_j(t) + \sum_{i=1}^{N} \frac{\beta_{ij}}{\Lambda_j} \lambda_{ij} e^{-\lambda_{ij} t} + \frac{\epsilon_{jj}}{\Lambda_j} h_{jj}(t)$$

$$B_{ij}(t) = \frac{\epsilon_{ji}}{\Lambda_i} \frac{P_{jo}}{P_{io}} h_{ji}(t) \quad i \neq j \tag{4.13.b}$$

$$k_{jj}(t) = -\frac{b_j(t)}{\Lambda_j} + \sum_{i=1}^{N} c_{ij}^{o} e^{-\lambda_{ij} t}; \quad k_{ij} = 0 \quad i \neq j \tag{4.13.c}$$

and the entries of M-vectors f and g are as follows

$$f_j(t) = \sum_{i=1}^{N} c_{ij}^{o} e^{-\lambda_{ij} t} - \frac{1}{\Lambda_j} b_j(t) \tag{4.13.d}$$

$$g_j(p)(t) = -\frac{P_{jo}}{\Lambda_j} p_j(t) \int_{o}^{t} a_j(t-s) p_j(s)\, ds. \tag{4.13.e}$$

<u>Definition 4.1</u>. A solution of equations (4.12), (4.11), and (4.6) is a set of functions $p_j(t)$, $c_{ij}(t)$, and $T^j(x,t)$ for $i = 1,\ldots,N$ and $j = 1,\ldots,M$ such that

(i) $p_j'(t)$, $c_{ij}'(t)$ are in C for $i = 1,\ldots,N$ and $j = 1,\ldots,M$.

(ii) $T^j(x,t)$, $T^j_t(x,t)$, and $T^j_{xx}(x,t)$ are continuous in (x,t) for $x \in [0,\pi]$, $t \in R^+$ and $j = 1,\ldots,M$.

(iii) $\lim_{t\to o^+} T^j(x,t) = h^j(x)$ for $x \in [0,\pi]$ and $T^j_x(0,t) = T^j_x(\pi,t) = 0$ for $t \geq 0$; $j = 1,\ldots,M$.

(iv) $p_j(t)$, $c_{ij}(t)$ and $T^j(x,t)$ satisfy (4.12), (4.11) and (4.6) for $t \in R^+$ and $x \in [0,\pi]$.

Now the intention is to show (4.12) is stable for small perturbations about equilibrium, that is, perturbations from equilibrium power, precursor concentration, and core temperature. By stable we mean in the sense of Theorem 3.5, $p(t)$ is small in some Banach space X if $p(0) = p^o$ is small in R^n and $f(t)$ is small in X. The claim is then that $f(t)$ is small if the initial precursor concentrations c^o_{ij} are small in R and the initial temperature distributions, $h^j(x)$, are small in $L^2[0,\pi]$. This follows from equation (4.13.d):

$$|f_j(t)| = \left| \sum_{i=1}^{N} c_{ij}^o e^{-\lambda_{ij} t} - \frac{1}{\Lambda_j} b_j(t) \right|$$

$$\leq \sum_{i=1}^{N} |c_{ij}^o| e^{-\lambda_{ij} t} + \frac{1}{\Lambda_j} \left| \sum_{n=o}^{\infty} \alpha_n^j h_n^j e^{-n^2 t} \right|$$

$$\leq \sum_{i=1}^{N} |c_{ij}^o| e^{-\lambda_{ij} t} + \left| \frac{\alpha_o^j h_o^j}{\Lambda_j} \right| + \sum_{n=1}^{\infty} |\alpha_n^j h_n^j| e^{-t}.$$

So, by Schwarz's inequality,

$$|f_j(t)| = \sum_{i=1}^{N} |c_{ij}^o| e^{-\lambda_{ij} t} + \left| \frac{\alpha_o^j h_o^j}{\Lambda_j} \right| + e^{-t} \left(\sum_{n=1}^{\infty} |\alpha_n^j|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |h_n^j|^2 \right)^{\frac{1}{2}}. \tag{4.14}$$

Using Parseval's relation, $\frac{2}{\pi} \int_o^{\pi} |h^j(x)|^2 dx = \sum_{n=o}^{\infty} |h_n^j|^2.$

It is clear then that in such Banach spaces as BC_o, BC_ℓ or $L^1 \cap BC_o$, $f(t)$ is small in norm if c_{ij}^o and $h^j(x)$ are sufficiently small for $i = 1,\ldots,N$ and $j = 1,\ldots,M$.

<u>Definition 4.2</u>. Equation (4.12) is stable with respect to a Banach space X if for any $\epsilon > 0$, there exists a

$\delta > 0$ such that if $|p^o| < \delta$, $\sum_{i=1}^{N} |c_{ij}^o| < \delta$ and $\|h^j\|_2 < \delta$ for $j = 1,\ldots,M$, then (4.12) has a unique solution, $p(t)$, in X satisfying $\|p\|_X < \epsilon$.

After the behavior of $p(t)$ is determined from (4.12) one can use equations (4.11) and (4.8) to study $c_{ij}(t)$ and $T^j(x,t)$.

Theorem 4.1. Consider equations (4.6), (4.11) and (4.12). Assume (4.7) holds and that λ_{ij}, β_{ij}, P_{jo}, Λ_j and C_{ijo} are positive for $i = 1,\ldots,N$ and $j = 1,\ldots,M$.

(i) Suppose in (4.9) that $\alpha_o^j \eta_o^j = \alpha_o^j h_o^j = 0$ for $j = 1,\ldots,M$ and that $\det[sI - A - B^*(s)] \neq 0$ for $\operatorname{Re} s \geq 0$. Then:

a. Equation (4.12) is stable with respect to $L^1 \cap BC_o$.

b. $c_{ij}(t)$ is in $L^1 \cap BC_o$ and $\|c_{ij}\|_{L^1 \cap BC_o} \leq \|p_j\|_{L^1 \cap BC_o} + (1 + \lambda_{ij}^{-1})|c_{ij}^o|$ for $i = 1,\ldots,N$; $j = 1,\ldots,M$.

c. Equation (4.6) has a unique solution, $T^j(x,t)$, such that

$$\lim_{t\to+\infty} T^j(x,t) = P_{jo}\int_0^\infty p_j(s)\,ds \cdot \eta_o^j + h_o^j$$

uniformly for $x \in [0,\pi]$, $j = 1,\ldots,M$.

(ii) If in (4.9), $h_o^j = 0$ and $\alpha_o^j\eta_o^j > 0$ for $j = 1,\ldots,M$ and $\det[s^2I - sA - sB^*(s)] \neq 0$ for $\mathrm{Re}\, s \geq 0$, then:

a. Equation (4.12) is stable with respect to $L^1 \cap BC_o$.

b. $c_{ij}(t)$ is in $L^1 \cap BC_o$ and $\|c_{ij}\|_{L^1\cap BC_o} \leq \|p_j\|_{L^1\cap BC_o} + (1+\lambda_{ij}^{-1})|c_{ij}^o|$ for $i = 1,\ldots,N$; $j = 1,\ldots,M$.

c. Equation (4.6) has a unique solution $T^j(x,t)$ such that

$$\lim_{t\to+\infty} T^j(x,t) = P_{jo}\int_0^\infty p_j(s)\,ds \cdot \eta_o^j$$

uniformly for $x \in [0,\pi]$, $j = 1,\ldots,M$.

(iii) Suppose in (4.9), $\alpha_o^j\eta_o^j = 0$ for $j = 1,\ldots,M$. If $\eta^{j''} = \dfrac{d^2\eta^j}{dx^2}$ is in $L^2[0,\pi]$, $h'_{ij}(t)$ exists and is in L^1 for i, $j = 1,\ldots,M$, and

$$\det\begin{bmatrix} sI & -I \\ -sB^*(s) & sI - A \end{bmatrix} \neq 0 \quad \text{for} \quad \operatorname{Re} s \geq 0$$

then:

a. Equation (4.12) is stable with respect to $L^1 \cap BC_o$.

b. $c_{ij}(t)$ is in $L^1 \cap BC_o$ and $\|c_{ij}\|_{L^1 \cap BC_o} \leq \|p_j\|_{L^1 \cap BC_o} + (1 + \lambda_{ij}^{-1})\,|c_{ij}^o|$ for $i = 1,\ldots,N$; $j = 1,\ldots,M$.

c. Equation (4.6) has a unique solution, $T^j(x,t)$, such that

$$\lim_{t \to +\infty} T^j(x,t) = P_{jo} \int_0^\infty p_j(s)\,ds \cdot \eta_o^j$$

and this limit is uniform for $x \in [0,\pi]$.

(iv) If in (4.9), $\alpha_o^j \eta_o^j = 0$ for $j = 1,\ldots,M$ and if $\det[sI - B^*(s) - A] \neq 0$ for $\operatorname{Re} s \geq 0$, then:

a. Equation (4.12) is stable with respect to BC_ℓ and $p(\infty) = \lim_{t \to +\infty} p(t)$ is a solution of the equation

$$-[A + B^*(0)]x = f(\infty) + k(\infty)x + G\,x^2 \qquad 4.15$$

where $x = \mathrm{Col}(x_1,\ldots,x_M)$, $f(\infty) = \lim_{t\to+\infty} f(t)$, $k(\infty) = \lim_{t\to+\infty} k(t)$,

G is an M by M diagonal matrix with $G_{jj} = \int_0^\infty a_j(t)\,dt$,

and $x^2 = \mathrm{Col}(x_1^2, x_2^2, \ldots, x_M^2)$.

b. $c_{ij}(t)$ is in BC_ℓ with $\|c_{ij}\|_o \le \|p_j\|_o + |c_{ij}^o|$ and $\lim_{t\to+\infty} c_{ij}(t) = p_j(\infty)$ for $i = 1,\ldots,N$; $j = 1,\ldots,M$.

c. Equation (4.6) has a unique solution, $T^j(x,t)$, such that if $\eta_o^j = 0$ for $j = 1,\ldots,M$ then

$$\lim_{t\to+\infty} T^j(x,t) = p_j(\infty)P_{jo} \sum_{n=1}^{\infty} \eta_n^j n^{-2} \cos nx + h_o^j$$

uniformly for $x \in [0,\pi]$; $j = 1,\ldots,M$. If $\eta_o^j \neq 0$ for some $j = 1,\ldots,M$, then $\lim_{t\to+\infty} T^j(x,t)$ may not exist.

Parts (i), (ii) and (iii) of Theorem 4.1 are similar to results of Levin and Nohel [7] for a single core reactor. Their results, however, were global and involved a certain positivity condition on the Fourier coefficients of η, α and h. Here this condition is replaced by requiring a certain determinant not vanish. We remark here that positive Fourier coefficients correspond to negative reactivity (see (4.9)) and, hence, to a stable reactor.

In proving this theorem we shall write (4.12) as a system of the form

$$\frac{dp}{dt} = Ap(t) + \int_0^t B(t-s)p(s)ds + f(t) + G(p)(t)$$

where $G(p)(t) = k(t)p(t) + g(p)(t)$. We shall use Theorem 3.5 to prove Theorem 4.1; $g(p)(t)$ is treated as a higher order term and $k(t)p$ as a small linear term. Hereafter, $R(t)$ will denote the resolvent of equation (4.12). Neglecting the local integrability of $B(t)$, there are essentially four hypothesis to be established in Theorem 3.5:

1. The resolvent $R(t)$ is in X and the map ρ_R defined by $\rho_R(f)(t) = \int_0^t R(t-s)f(s)ds$ is in $\mathcal{G}(X,X)$.

2. g is of higher order in X.

3. f is in X.

4. $k(t)p$ satisfies $\|k(t)p_1 - k(t)p_2\|_X \leq L\|p_1 - p_2\|_X$ where $p_1, p_2 \in X$ and L is small.

In each case (3) and (4) present no particular problem,

(1) will be insured by requiring the determinant of a certain matrix not vanish. However, (2) is more difficult to establish so here we include the following lemma.

Lemma 4.1. Let $g(x)(t) = x(t)\int_0^t a(t-s)x(s)\,ds$ where $a(t)$ and $x(t)$ are scalar functions. If $a(t)$ is in L^1 then g is of higher order in BC_ℓ and in BC_0. If $a(t)$ is in BC then g is of higher order in $L^1 \cap BC_0$.

Proof of Lemma 4.1. Suppose $a(t)$ is in L^1 and $x(t)$ is in BC. Then it is clear that $g(x)(t)$ is a continuous function of t for $t \in R^+$.

Let $\|\varphi\|_{[0,T]} = \sup_{t\in[0,T]} |\varphi(t)|$ for $\varphi \in C$ and $T > 0$.

Then

$$\|g(x)\|_{[0,T]} = \|x(t)\int_0^t a(t-s)x(s)\,ds\|_{[0,T]}$$

$$\leq \|x\|_{[0,T]}\ \|\int_0^t a(t-s)x(s)\,ds\|_{[0,T]}$$

$$\leq \|x\|_{[0,T]}\|x\|_{[0,T]}\|a\|_1 .$$

So g maps BC into BC. Also, if $x \in BC_\ell$ then

$\lim_{t\to\infty} x(t) = x(\infty)$. From Lemma 3.1 it is clear that

$$\lim_{t\to+\infty} g(x)(t) = \lim_{t\to+\infty} x(t) \int_0^t a(t-s)x(s)\,ds$$

$$= \int_0^\infty a(s)\,ds \cdot x^2(\infty) .$$

Thus, $g(x)(t)$ is in BC_ℓ.

Now let x_1, x_2 be in BC_ℓ and write

$g(x) = x(t)(a * x)(t) = x(t) \int_0^t a(t-s)x(s)\,ds$. Then

$$\|g(x_1) - g(x_2)\|_0 = \|x_1 a * x_1 - x_2 a * x_2\|_0$$

$$\leq \|x_1 a * x_1 - x_2 a * x_1\|_0 + \|x_2 a * x_1 - x_2 a * x_2\|_0$$

$$\leq \|x_1 - x_2\|_0 \|a * x_1\|_0 + \|x_2\|_0 \|a * x_1 - a * x_2\|_0$$

$$\leq \|x_1 - x_2\|_0 \|a\|_1 \|x\|_0 + \|x_2\|_0 \|a\|_1 \|x_1 - x_2\|_0 .$$

So for any $\epsilon > 0$, let $\delta = \frac{\epsilon}{2}(\|a\|_1)^{-1}$ then if

$\|x_1\|_0, \|x_2\|_0 < \delta$,

$$\|g(x_1) - g(x_2)\|_o \leq \frac{\epsilon}{2}\|x_1 - x_2\|_o + \frac{\epsilon}{2}\|x_1 - x_2\|_o$$

$$\leq \epsilon\|x_1 - x_2\|_o$$

which shows g is of higher order in BC_ℓ. If $x(t)$ is in BC_o, $x(\infty) = \lim_{t\to+\infty} x(t) = 0$ so by the same argument as above it follows that $g(x)$ is of higher order in BC_o.

Now suppose $a(t)$ is in BC. We want to show g is of higher order in $L^1 \cap BC_o$. Recall that for $x \in L^1 \cap BC_o$, $\|x\|_{L^1 \cap BC_o} = \|x\|_o + \|x\|_1$. Let $x \in L^1 \cap BC_o$. Then for $T > 0$

$$\|g(x)\|_{1[0,T]} \equiv \int_0^T |g(x)(t)|dt$$

$$= \|x(t) \int_0^t a(t-s)x(s)ds\|_{1[0,T]}$$

$$\leq \|x\|_1 \| \int_0^t a(t-s)x(s)ds\|_{[0,T]}$$

$$\leq \|x\|_1\|a\|_o\|x\|_1$$

and

$$\|g(x)\|_{[0,T]} = \|x(t) \int_0^t a(t-s)x(s)ds\|_{[0,T]}$$

$$\leq \|x\|_o\|a\|_o\|x\|_1.$$

Hence, $\|g(x)\|_{L^1 \cap BC_o} = \|g(x)\|_1 + \|g(x)\|_o < +\infty$. Also,

$$|g(x)(t)| = |x(t)\int_o^t a(t-s)x(s)ds|$$

$$\leq |x(t)|\ \|a\|_o\|x\|_1 .$$

So, $\lim\limits_{t\to+\infty} g(x)(t) = \lim\limits_{t\to+\infty} |x(t)|\ \|a\|_o\|x\|_1 = 0$. Thus, $g(x)$ is in $L^1 \cap BC_o$ if $x \in L^1 \cap BC_o$. Now let x_1, x_2 be in $L^1 \cap BC_o$

$$\|g(x_1) - g(x_2)\|_1 = \|x_1 a * x_1 - x_2 a * x_2\|_1$$

$$\leq \|x_1 a * x_1 - x_2 a * x_1\|_1 + \|x_2 a * x_1 - x_2 a * x_2\|_1$$

$$\leq \|x_1 - x_2\|_1\|a * x_1\|_o + \|x_2\|_1\|a * x_1 - a * x_2\|_o$$

$$\leq \|x_1 - x_2\|_1\|a\|_o\|x_1\|_1 + \|x_2\|_1\|a\|_o\|x_1 - x_2\|_1$$

and

$$\|g(x_1) - g(x_2)\|_o = \|x_1 a * x_1 - x_2 a * x_2\|_o$$

$$\leq \|x_1 a * x_1 - x_2 a * x_1\|_o + \|x_2 a * x_1 - x_2 a * x_2\|_o$$

$$\leq \|x_1 - x_2\|_o\|a * x_1\|_o + \|x_2\|_o\|a * x_1 - a * x_2\|_o$$

$$\leq \|x_1 - x_2\|_o \|a\|_o \|x_1\|_1 + \|x_2\|_o \|a\|_o \|x_1 - x_2\|_1$$

Then,

$$\|g(x_1) - g(x_2)\|_{L^1 \cap BC_o} = \|g(x_1) - g(x_2)\|_1 + \|g(x_1) - g(x_2)\|_o$$

$$\leq \|x_1 - x_2\|_1 \|a\|_o \|x_1\|_1 + \|x_1\|_1 \|a\|_o \|x_1 - x_2\|_1$$

$$+ \|x_1 - x_2\|_o \|a\|_o \|x_1\|_1 + \|x_2\|_o \|a\|_o \|x_1 - x_2\|_1$$

so for any $\epsilon > 0$ let $\delta = \frac{\epsilon}{4\|a\|_o}$. If $\|x_1\|_{L^1 \cap BC_o}$ and $\|x_2\|_{L^1 \cap BC_o}$ are less than δ,

$$\|g(x_1) - g(x_2)\|_{L^1 \cap BC_o} \leq \epsilon \|x_1 - x_2\|_{L^1 \cap BC_o}.$$

Thus, g is of higher order in $L^1 \cap BC_o$. This completes the proof of Lemma 4.1.

Q.E.D.

We now note the particular form of

$g(p), g_j(p) = g_j(p_j) = p_j(s) \int_o^t a_j(t-s) p_j(s) ds$. It follows that $g(p)$ is of higher order in BC_ℓ, BC_o, or $L^1 \cap BC_o$ if this is true for each component $g_j(p_j)$. This fact along with Lemma 4.1 will be useful in the proof of Theorem 4.1.

Proof of Theorem 4.1. The proof of each part is accomplished by showing the given conditions imply the hypothesis of Theorem 3.5. For this reason the proof of each result is divided into four parts labeled as on page 70 above according to the respective hypothesis to be established.

Proof of Theorem 4.1 (i). Conclusion (a) will be proved by using Theorem 3.5.

1. We claim that $R(t)$ is in $L^1 \cap BC_o$ and ρ_R is in $\mathcal{C}(L^1 \cap BC_o, L^1 \cap BC_o)$.

Since $\alpha_o^j \eta_o^j = 0$ then $a_j(t) = \sum_{n=1}^{\infty} \eta_n^j \alpha_n^j e^{-n^2 t}$ and by Schwarz's inequality

$$|a_j(t)| \leq e^{-t}\left(\sum_{n=1}^{\infty} |\eta_n^j|^2\right)^{1/2}\left(\sum_{n=1}^{\infty} |\alpha_n^j|^2\right)^{1/2}$$

for $t \geq 0$ so $a_j(t)$ is in $L^1 \cap BC_o$ for $j = 1,\ldots,M$. Referring to (3.12.b) we see $B(t)$ is in $L^1 \cap BC_o$. Since $\det[sI - A - B^*(s)] \neq 0$ for $\mathrm{Re}\ s \geq 0$, Theorem 2.3 implies $R(t)$ is in $L^1 \cap BC_o$. Then it also follows from the properties of the convolution product (see Lemma 3.1 and 3.2) and Theorem 2.1 that $\rho_R \in \mathcal{Q}(L^1 \cap BC_o, L^1 \cap BC_o)$

2. g is of higher order in $L^1 \cap BC_o$. This follows from Lemma 4.1 since $a_j(t) \in L^1 \cap BC_o$ for $j = 1,\ldots,M$ and $g_j(p)(t) = p_j(t) \int_o^t a_j(t-s) p_j(s)\,ds$.

3. $f(t)$ is in $L^1 \cap BC_o$. Since $\alpha_o^j h_o^j = 0$ for $j = 1,\ldots,M$, (4.14) implies $f \in L^1 \cap BC_o$.

4. $k(t)p$ is a small linear term in $L^1 \cap BC_o$. Let p_1 and p_2 be in $L^1 \cap BC_o$ then

$$\begin{aligned}
\|kp_1 - kp_2\|_{L^1 \cap BC_o} &= \|kp_1 - kp_2\|_o + \|kp_1 - kp_2\|_1 \\
&\leq \|k\|_o \|p_1 - p_2\|_o + \|k\|_o \|p_1 - p_2\|_1 \\
&\leq \|k\|_o \|p_1 - p_2\|_{L^1 \cap BC_o} .
\end{aligned}$$

From 4.13c, $k_{jj}(t) = -\frac{b_j(t)}{\Lambda_j} + \sum_{i=1}^{N} c_{ij} e^{-\lambda_{ij} t}$ and

$k_{ij}(t) = 0$, $i \neq j$. It is clear then from (4.14) that $\|k\|_o$ can be made arbitrarily small if $\|h^j\|_2$ and $\sum_{i=1}^{N} |c_{ij}^o|$ are sufficiently small for $j = 1,\ldots,M$.

Also, from (4.14) it is clear that $\|f\|_{L^1 \cap BC_o}$ can be made arbitrarily small by choosing $\sum_{i=1}^{N} |c_{ij}^o| < \delta$ and $\|h^j\|_2 < \delta$, then (4.12) has a unique solution, $p(t)$ satisfying $\|p\|_{L^1 \cap BC_o} < \epsilon$. Thus, (a) is true.

To prove (b) we see from (4.11) that

$$c_{ij}(t) = \lambda_{ij} \int_o^t e^{-\lambda_{ij}(t-s)} p_j(s)\,ds + c_{ij}^o e^{-\lambda_{ij} t}.$$

Then

$$\|c_{ij}\|_o \leq \lambda_{ij} \int_o^\infty e^{-\lambda_{ij} s} ds \|p_j\|_o + |c_{ij}^o|$$

$$\leq \lambda_{ij} \cdot \frac{1}{\lambda_{ij}} \|p_j\|_o + |c_{ij}^o|$$

and from Lemma 3.2

$$\|c_{ij}\|_1 \leq \lambda_{ij}\|e^{-\lambda_{ij}t} * p_j\|_1 + \|e^{-\lambda_{ij}t}\|_1 |c^o_{ij}|$$

$$\leq \lambda_{ij} \cdot \frac{1}{\lambda_{ij}} \|p_j\|_1 + \frac{1}{\lambda_{ij}} |c^o_{ij}|$$

So $\|c_{ij}\|_{L^1 \cap BC_o} \leq \|p_j\|_{L^1 \cap BC_o} + (1 + \frac{1}{\lambda_{ij}}) |c^o_{ij}|$ for $i = 1,\ldots,N$; $j = 1,\ldots,M$. This proves (b). Also, since $p_j(t)$ is unique and in $L^1 \cap BC_o$ it follows from hypothesis (4.7) and Weinberger [14], Section 29, that (4.6) has a unique solution given by (4.8). Then

$$\lim_{t\to+\infty} T^j(x,t) = \lim_{t\to+\infty} \Big\{ p_{jo} \sum_{n=o}^{\infty} \int_o^t e^{-n^2(t-s)} p_j(s)\,ds.$$

$$\eta^j_n \cos nx + \sum_{n=0}^{\infty} h^j_n e^{-n^2 t} \cos nx \Big\}$$

By hypothesis (4.7); η^j, $\eta^{j\,'}$, h^j, and $h^{j\,'}$ are in $L^2[0,\pi]$. Hence,

$$\sum_{n=1}^{\infty} |\eta^j_n| = \sum_{n=1}^{\infty} |\eta^j_n n \tfrac{1}{n}| \leq \Big(\sum_{n=1}^{\infty} |\eta^j_n n|^2 \Big)^{1/2} \Big(\sum_{n=1}^{\infty} \frac{1}{n^2} \Big)^{1/2}$$

so $\sum_{n=1}^{\infty} |\eta_n^j| < +\infty$ and, likewise, $\sum_{n=1}^{\infty} |h_n^j| < +\infty$. Then

$$\lim_{t\to+\infty} T^j(x,t) = p_{jo} \sum_{n=o}^{\infty} \left(\lim_{t\to+\infty} \int_o^t e^{-n^2(t-s)} p_j(s)\,ds\right)\eta_n^j \cos n\,x$$

$$+ \sum_{n=o}^{\infty} \left(\lim_{t\to+\infty} e^{-n^2 t}\right) h_n^j \cos n\,x$$

$$= p_{jo} \int_o^{\infty} p_j(s)\,ds \cdot \eta_o^j + h_o^j$$

and this limit is uniform for $x \in [0,\pi]$. This proves part (c) and concludes the proof of Theorem 4.1 (i).

<u>Proof of Theorem 4.1 (ii)</u>. We again use Theorem 3.5 to prove (a).

1. For $j = 1,\ldots,M$,

$$a_j(t) = \alpha_o^j \eta_o^j + \sum_{n=1}^{\infty} \alpha_n^j \eta_n^j e^{-n^2 t}.$$

Since $\alpha^j(x)$ and $r^j(x)$ are in $L^2[0,\pi]$, it follows by the same argument used in part (i) of this theorem that

$a_j(t) - \alpha_o^j \eta_o^j$ is a function in $L^1 \cap BC_o$. So, from (4.13.b), $B(t) = B_1 + B_2(t)$ where B_1 is a constant diagonal matrix with negative entries and $B_2(t)$ is in $L^1 \cap BC_o$. Now we apply Theorem 3.4 with $\Phi(t) = e^{-t}I$ to see that $R(t)$ is in $L^1 \cap BC_o$ since $\det[s^2 I - sA - sB^*(s)] = \det(s^2 I - sA - sB_2^*(s) - B_1] \neq 0$ for $\mathrm{Re}\ s \geq 0$. Then it is clear from Lemmas 3.1 and 3.2 that ρ_R maps $L^1 \cap BC_o$ into itself. Hence, Theorem 2.1 implies ρ_R is in $\mathcal{a}(L^1 \cap BC_o, L^1 \cap BC_o)$.

2. Since $a_j(t)$ is in BC it follows from Lemma 4.1 that $g(p)(t)$ is of higher order in $L^1 \cap BC_o$.

3. It is clear from (4.14) and the hypothesis $h_o^j = 0$ for $j = 1,\ldots,M$ that $f(t)$ is in $L^1 \cap BC_o$.

4. The proof that $k(t)p(t)$ is a small linear term in $L^1 \cap BC_o$ is identical to the proof of (4) in part (i).

Now Theorem 3.5 implies (4.12) is stable with respect to $L^1 \cap BC_o$.

The proof of (b) follows from (4.11) and Lemma 3.1 and 3.2 as it did in part (i).

Also, equation (4.6) has a unique solution $T^j(x,t)$ given by (4.8) and

$$\lim_{t\to+\infty} T^j(x,t) = p_{jo}\int_0^\infty p_j(s)\,ds \cdot \eta_j^o$$

uniformly for $x \in [0,\pi]$; $n = 1,\ldots,M$. This follows from the proof in part (i) plus the hypothesis $h_o^j = 0$. This completes the proof of Theorem 4.1 (ii).

Proof of (iii). This result is similar to (ii) in that we want to show (4.12) is stable with respect to $L^1 \cap BC_o$. However, the positivity condition on the first Fourier coefficients of $\alpha^j(x)$ and $\eta^j(x)$ is replaced by a stricter assumption on $h_{ij}(t)$ and a different determinant condition. The method of proof is the same.

1. $R(t)$ is a matrix in $L^1 \cap BC$ and ρ_R is in $\mathfrak{a}(L^1 \cap BC_o . L^1 \cap BC_o)$. We first show $B'(t)$ is in L^1. Looking at (4.13.b) we see

$$B_{jj}(t) = -\frac{P_{jo}}{\Lambda_j} a_j(t) + \sum_{i=1}^{N} \frac{\beta_{ij}}{\Lambda_j} \lambda_{ij} e^{-\lambda_{ij}t} + \frac{\epsilon_{jj}}{\Lambda_j} h_{jj}(t)$$

and

$$B_{ij}(t) = \frac{\epsilon_{ij}}{\Lambda_i} \frac{P_{jo}}{P_{io}} h_{ji}(t) \quad \text{for } i \neq j.$$

By hypothesis $h'_{ij}(t)$ is in L^1 for $i,j = 1,\dots,M$ and

$$a'_j(t) = -\sum_{n=1}^{\infty} \alpha_n^j \eta_n^j n^2 e^{-n^2 t}$$

so $a'_j(t)$ is in L^1 if $\sum_{n=1}^{\infty} |\alpha_n^j \eta_n^j n^2|$ converges. Using

Schwarz's inequality

$$\sum_{n=1}^{\infty} |\alpha_n^j \eta_n^j n^2| \le \left(\sum_{n=1}^{\infty} |\alpha_n^j|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} |\eta_n^j n^2|^2\right)^{1/2}$$

and both sums on the right exist since both $\alpha^j(x)$ and $\frac{d^2\eta^j}{dx^2}$ are in $L^2[0,\pi]$. Hence $B'(t)$ is in L^1. Then we can write $B(t)$ as the sum of a constant diagonal matrix B_1 and a matrix $B_2(t)$ where both $B_2(t)$ and $B_2'(t)$ are in L^1. Now by referring to the discussion proceeding Theorem 3.4 we see that $R(t)$ is in $L^1 \cap BC_o$ if

$$\det\begin{bmatrix} sI & -I \\ -sB^*(s) & sI-A \end{bmatrix} \ne 0 \quad \text{for} \quad \operatorname{Re} s \ge 0.$$

Hence, $R(t)$ is in $L^1 \cap BC_o$ and it follows that ρ_R is in $\mathcal{G}(L^1 \cap BC_o, L^1 \cap BC_o)$.

2. Since $a_j(t)$ is in BC it is clear from Lemma 4.1 that $g(p)(t)$ is of higher order in $L^1 \cap BC_o$.

3. $b_j(t) = \sum_{n=o}^{\infty} h_n^j \alpha_n^j e^{-n^2 t}$ and $h_o^j = 0$ for $j = 1,\ldots,M$ so from (4.13.d) and (4.14) it follows that $f(t)$ is in $L^1 \cap BC_o$.

4. The proof is the same as in part (i).

Theorem 3.5 again implies (4.12) is stable with respect to $L^1 \cap BC_o$.

Conclusion (b), also, follows from (4.11) as in part (i).

By the same reasoning as in part (i) we conclude equation (4.6) has a unique solution, $T^j(x,t)$, and

$$\lim_{t\to+\infty} T^j(x,t) = P_{jo} \int_o^{\infty} p_j(s)\,ds \cdot \eta_o^j$$

uniformly for $x \in [0,\pi]$; $j = 1,\ldots,M$. This completes the proof of Theorem 4.1 (iii).

<u>Proof of (iv)</u>. We again verify the hypothesis of Theorem 3.5.

1. If $\alpha_o^j \eta_o^j = 0$ for $j = 1,\ldots,M$ then

$a_j(t) = \sum_{n=1}^{\infty} \alpha_n^j \eta_n^j e^{-n^2 t}$ is in $L^1 \cap BC_o$. Thus $B(t)$ is in L^1 and since $\det[sI - B^*(s) - A] \neq 0$ for $\operatorname{Re} s \geq 0$, Theorem 2.3 implies $R(t)$ is in $L^1 \cap BC_o$. Hence, ρ_R is in $\Omega(\overset{BC_\ell}{\cancel{L^1 \cap BC_o}}, \overset{BC_\ell}{\cancel{L^1 \cap}\, BC_o})$.

2. Since $a_j(t)$ is in $L^1 \cap BC_o$ for $j = 1,\ldots,M$ it follows from Lemma 4.1 that $g(p)$ is of higher order in BC_ℓ.

3. From (4.14) we see $f(t)$ is in BC_ℓ.

4. If $p_1(t)$ and $p_2(t)$ are in BC_ℓ, then from (4.13.c) it follows $k(t)p_1(t)$ is in BC_ℓ and $\|kp_1 - kp_2\|_o \leq \|k\|_o \|p_1 - p_2\|_o$. But $\|k\|_o$ is small if $f(t)$ is small, hence $k(t)p(t)$ is a small linear term in BC_ℓ.

Then Theorem 3.5 implies (4.12) is stable with respect to BC_ℓ. Thus, $\lim_{t\to+\infty} p(t) = p(\infty)$ exists. Now $p(t)$ satisfies

$$p(t) = R(t)p^o + \int_o^t R(t-s)f(s)\,ds + \int_o^t R(t-s)k(s)p(s)\,ds$$
$$+ \int_o^t R(t-s)g(p)(s)\,ds$$

for $t \geq 0$. From Lemma 3.2, a BC_ℓ function convoluted with an L^1 function is in BC_ℓ. So all the terms on the right hand side of this equation have limits as $t \to +\infty$. Taking the limit as $t \to +\infty$, we have

$$p(\infty) = \int_0^\infty R(s)\,ds\ f(\infty) + \int_0^\infty R(s)\,ds \cdot k(\infty)\,p(\infty) + \int_0^\infty R(s)\,ds \cdot g(p)(\infty)$$

which can be written

$$p(\infty) = R^*(0)f(\infty) + R^*(0)k(\infty)p(0) + R^*(0)g(p)(\infty)$$

where $R^*(0) = -[A + B^*(0)]^{-1}$ and

$g_j(p)(\infty) = \int_0^\infty a_j(s)\,ds \cdot p_j^2(\infty)$. Recall that

$R^*(s) = [sI - A - B^*(s)]^{-1}$ from equation (2.4) and that $g_j(p)(\infty)$ was calculated in Lemma 4.1. Equation (4.15) is obtained by multiplying by $-[A + B^*(0)]$.

Since $p(t) \in BC_\ell$, it is clear from (4.11) and Lemma 3.1 that $c_{ij}(t) \in BC_\ell$ and

$$\lim_{t\to+\infty} c_{ij}(t) = \lim_{t\to+\infty} \lambda_{ij} \int_0^t e^{-\lambda_{ij}(t-s)} p_j(s)\,ds + c_{ij}^o e^{-\lambda_{ij}t}$$

$$= p_j(\infty)$$

Also, since $\int_0^\infty e^{-\lambda_{ij}t}\,dt = \lambda_{ij}^{-1}$,

$$\|c_{ij}\|_o \le \lambda_{ij}\|e^{-\lambda_{ij}t}\|_1\|p_j\|_o + |c_{ij}^o|$$

$$\le \|p_j\|_o + |c_{ij}^o| \quad \text{for} \quad i = 1,\dots,N;\ j = 1,\dots,M.$$

If $\eta_o^j = 0$ we can also calculate

$$\lim_{t\to+\infty} T^j(x,t) = \lim_{t\to+\infty}\Big\{P_{jo}\sum_{n=o}^{\infty}\int_o^t e^{-n^2(t-s)}p_j(s)\,ds\cdot\eta_n^j\cos n\,x$$

$$+ \sum_{n=o}^{\infty} h_n^j e^{-n^2 t}\cos n\,x\Big\}$$

$$= P_{jo}\sum_{n=1}^{\infty}\Big(\int_o^\infty e^{-n^2 s}ds\Big)p_j(\infty)\,\eta_n^j\cos n\,x + h_o^j$$

$$= P_{jo}p_j(\infty)\sum_{n=1}^{\infty}\eta_n^j j^{-2}\cos n\,x + h_o^j$$

for $j = 1,\dots,M$. This limit is uniform for $x \in [0,\pi]$ since $\sum_{n=1}^{\infty}|\eta_n^j| < +\infty$ and $\sum_{n=1}^{\infty}|h_n^j| < +\infty$. If $\eta_o^j \ne 0$ then

$$\lim_{t\to+\infty} T^j(x,t) = \lim_{t\to+\infty}\Big\{P_{jo}\int_o^t p_j(s)\,ds +$$

$$P_{jo}\sum_{n=1}^{\infty}\int_o^t e^{-n^2(t-s)}p_j(s)\,ds\ \eta_n^j\cos nx + \sum_{n=o}^{\infty} h_o^j e^{-n^2 t}\cos nx\Big\}$$

but $\lim_{t\to+\infty} P_{jo}\int_o^t p_j(s)\,ds$ does not exist unless $p_j(s)$ is in L^1. Certainly this limit does not exist if $p_j(\infty) = \lim_{t\to+\infty} p_j(\infty) \neq 0$. Now $p(\infty) = 0$ is not a solution of (4.15) if $f(\infty) \neq 0$. So if $f(\infty) \neq 0$ then $p_j(\infty) \neq 0$ for at least some j satisfying $1 \leq j \leq M$.

This completes the proof of Theorem 4.1.

Q.E.D.

We note that $\lim_{t\to+\infty} p(t) = 0$ and $\lim_{t\to+\infty} c_{ij}(t) = 0$ correspond to the reactor asymptotically returning to its equilibrium state. This was the case in parts (i), (ii) and (iii) of Theorem 4.1. Insulating the faces of the reactor results in a zero eigenvalue which accounts for the heat build-up in the cores. In (i), (ii), and (iii) this was of the same order of magnitude as $\|p\|_1$ and $\|f\|_o$, both of which are small. However, in Theorem 4.1 (iv) it was possible for the temperature to become unbounded. This is not reflected in equation (4.12) since

$$\rho_j(t) = -\int_0^{\pi} \alpha_j(x)\, T^j(x,t)\, dx$$

$$= - P_{jo} \sum_{n=1}^{\infty} \int_0^t \eta_n^j \alpha_n^j e^{-n^2(t-s)} p_j(s)\, ds + \sum_{n=o}^{\infty} \alpha_n^j h_n^j e^{-n^2 t}$$

The term $\eta_o^j \alpha_o^j P_{jo} \int_0^t p_j(s)\, ds$ vanishes since $\eta_o^j \alpha_o^j = 0$.

Thus, $\rho_j(t)$ is bounded on R^+.

The temperature appears in equation (4.1) only through the reactivity

$$\rho_j(t) = - \int_{G_j} \alpha^j(x)\, T^j(x,t)\, dx$$

For this reason the previous techniques are adaptable in the case of more general reactor geometries. We now follow Helliwell [5] and consider G_j to be a finite region in R^n. We suppose $T^j(x,t)$ solves:

$$T_t^j(x,t) = L_x(T^j(x,t)) + q^j(x)T^j(x,t) + \eta^j(x)P_{jo}p_j(t)$$

$$T^j(x,0) = h^j(x) \quad \text{for} \quad x \in G_j \tag{4.16}$$

$$T^j(x,t) + \gamma^j(x)\frac{\partial T(x,t)}{\partial n} = 0 \quad \text{for} \quad x \in \Gamma^j,\ t \geq 0$$

where $x = (x_1,\ldots,x_n) \in R^n$, Γ^j is the boundary of G_j, $\frac{\partial T^j}{\partial n}$ is the normal derivative of $T^j(x,t)$ at $x \in \Gamma^j$ and $L_x(\cdot)$ is an elliptic differential operator of the form

$$L_x(\cdot) = \frac{1}{\sqrt{a(x)}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\sqrt{a(x)}\ \frac{\partial(\cdot)}{\partial x_k}\right)$$

where $a_{jk}(x)$ are known functions and $a(x)$ is the determinant of the matrix formed by $a_{jk}(x)$.

<u>Definition 4.2</u>. A function $f(x)$ is Holder continuous of exponent α, $0 < \alpha < 1$, on a compact set $S \subset R^n$ if there exists an $M > 0$ such that

$$|f(x) - f(y)| \leq M|x-y|^{\alpha}$$

for all $x, y \in S$.

<u>Definition 4.3</u>. A function $f(x)$ defined on a compact set $S \subset R^n$ is in the class $C^m(s)$ if all of its first m partial derivatives exist and are continuous on S. If $f(x)$ is in $C^m(S)$ and all of its first m partial derivatives exist and are Holder continuous with exponent α then $f(x)$ is in $C^{m+\alpha}(S)$.

<u>Definition 4.4</u>. A surface Γ is in $C^m(\Gamma)$ (or $C^{m+\alpha}(\Gamma)$) if for each y in Γ there exists a neighborhood U_y and an x_i such that for x in U_y

$$x_i = h_y(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$$

where h_y is a function in $C^m(U_y)$ (or $C^{m+\alpha}(U_y)$).

We shall make the following assumptions concerning Γ^j and the given functions. Let $\bar{G}_j = G_j \cup \Gamma^j$.

<u>A-1</u> $\alpha^j(x), h^j(x) \in C(\bar{G}_j)$

$\eta^j(x), q^j(x) \in C^\alpha(\bar{G}_j)$

$q^j(x) \leq 0$; $h^j(x)$ satisfies the boundary conditions.

<u>A-2</u> $A_{ij}(x) \in C^{2+\alpha}(\bar{G}_j)$

there exists a constant $M > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}\zeta_i\zeta_j \geq M|\zeta|^2 \quad \text{for all} \quad x \in \bar{G}_j \quad \text{and for all}$$

real vectors ζ in R^n.

A-3 $\gamma^j(x) \geq \mu > 0$ or $\gamma^j(x) \equiv 0$

$\gamma^j(x) \in C^{2+\alpha}(\Gamma^j)$, Γ^j is of class $C^{2+\alpha}$.

We wish to solve (4.16) by using an eigenvalue expansion and obtaining a Green's function for the problem. Consider the homogeneous problem

$$\frac{\partial T^j(x,t)}{\partial t} = L_x(T^j(x,t)) + q^j(x)T^j(x,t)$$

$$T^j(x,0) = h^j(x) \quad \text{for} \quad x \in G_j$$

$$T^j(x,t) + \gamma^j(x)\,\frac{\partial T^j(x,t)}{\partial n} = 0 \quad \text{for} \quad x \in \Gamma^j;\ t \geq 0$$

$$j = 1,\ldots,M.$$

Let $T^j(x,t) = U^j(x)V^j(t)$. Then $U^j(x)$ and $V^j(t)$ solve

$$V^{j\,\prime}(t) = -\,\ell\, V^j(t) \qquad V^j(0) = V_o^j$$

$$L_x(U^j(x)) = -\,(\ell^j + q^j(x))U^j(x) \qquad 4.17$$

where V_o^j is determined by initial conditions and $U_j(x)$

satisfies $U^j(x) + \gamma^j(x) \frac{\partial U^j(x)}{\partial n} = 0$ for $x \in \Gamma^j$. We shall assume the following:

A-4 The eigenvalues of (4.17) can be arranged in a non-decreasing sequence, $0 \leq \ell_1^j \leq \ell_2^j \leq \dots \leq \ell_k^j \leq \dots$ where $\ell_k^j \to +\infty$ as $k \to +\infty$ for $j = 1,\dots,M$. All eigenvalues are of finite multiplicity and are repeated according to multiplicity.

A-5 The sequence of corresponding eigenfunctions $\{u_n^j(x)\}_{n=1}^{\infty}$ form a complete orthornormal set in $L^2[G_j]$ for $j = 1,\dots,M$.

Let h_n^j and η_n^j denote the Fourier coefficients of $h^j(x)$ and $\eta^j(x)$ respectively. That is,

$$h_n^j = \int_{G_j} h^j(x) u_n^j(x)\,dx, \quad n \geq 1$$

and

$$\eta_n^j = \int_{G_j} \eta^j(x) u_n^j(x)\,dx, \quad n \geq 1.$$

Then from Theorem 10, Ito [6] one can formally solve (4.6) for $T^j(x,t)$ in terms of $p_j(t)$ and obtain a solution of the form

$$T^j(x,t) = \sum_{n=1}^{\infty} e^{-\ell_n^j t} h_n^j u_n^j(x)$$

$$+ P_{jo} \sum_{n=1}^{\infty} e^{-\ell_n^j (t-s)} p_j(s)\,ds\ \eta_n^j u_n^j(x)$$

for $j = 1,\ldots,M.$

Thus,

$$\rho_j(t) = -\int_{G_j} \alpha^j(x) T^j(x,t)\,dx$$

$$= \sum_{n=1}^{\infty} \alpha_n^j h_n^j e^{-\ell_n^j t} + P_{jo} \sum_{n=1}^{\infty} \eta_n^j \alpha_n^j \int_0^t e^{-\ell_n^j (t-s)} p_j(s)\,ds$$

where $\alpha_n^j = \int_{G_j} \alpha^j(x) u_n(x)\,dx$, $n = 1,2,\ldots$. Letting

$$a_j(t) = \sum_{n=1}^{\infty} \alpha_n^j \eta_n^j e^{-\ell_n^j t} \qquad 4.19.a$$

and

$$b_j(t) = \sum_{n=1}^{\infty} \alpha_n^j h_n^j e^{-\ell_n^j t} \qquad 4.19.b$$

we can write

$$\rho_j(t) = -b_j(t) - P_{jo}\int_o^t a_j(t-s)p_j(s)ds.$$

If we substitute $\rho_j(t)$ into (4.4) and solve for $c_{ij}(t)$ (see (4.11)) we obtain the system

$$\frac{dp}{dt} = A\,p(t) + \int_o^t B(t-s)p(s)ds + f(t) + k(t)p(t) + g(p)(t)$$

$$p(0) = p^o \qquad 4.20$$

with A, B(t), k(t), f(t) and g(p)(t) defined as in (4.13) but with $a_j(t)$ and $b_j(t)$ defined as in (4.19).

<u>Definition 4.2</u>. A solution of system (4.16), (4.11), and (4.19) is a set of functions $p_j(t)$, $c_{ij}(t)$ and $T^j(x,t)$ such that

(i) $p_j'(t)$ and $c_{ij}'(t)$ are in C for $i = 1,\ldots,N$ and $j = 1,\ldots,M$.

(ii) $T^j(x,t)$, $T_t^j(x,t)$ and $T^j_{x_i x_j}(x,t)$ are continuous in (t,x) for $x \in G_j$, $t \in (0,+\infty)$ and $i,j = 1,\ldots,M$.

(iii) $\lim_{t\to o^+} T^j(x,t) = h^j(x)$ for $x \in G_j$, $j = 1,\ldots,M$ and

$$T^j(x,t) + \gamma^j(x)\frac{\partial T^j(x,t)}{\partial n} = 0 \quad \text{for} \quad x \in \Gamma^j,\ t > 0 \quad \text{and}$$

$j = 1,\ldots,M$.

(iv) $p_j(t)$, $c_{ij}(t)$ and $T^j(x,t)$ satisfy (4.16), (4.11) and (4.20) for $i = 1,\ldots,N$; $j = 1,\ldots,M$.

The analysis of (4.16) and (4.20) is done as in Theorem 4.1. We shall show the existence of a unique solution, $p(t)$, of (4.20) if p_j^o, c_{ij}^o and $h^j(x)$ is small for $i = 1,\ldots,N$; $j = 1,\ldots,M$. Then we will also determine $c_{ij}(t)$ and $T^j(x,t)$ in terms of $p_j(t)$.

In the theorem that follows we shall use the term stable as defined in Definition 4.2 but with reference to equation (4.20).

<u>Theorem 4.2</u>. Consider equations (4.11), (4.16) and (4.20). Assume that A-1 through A-5 hold and that B_{ij}, Λ_j, λ_{ij}, P_{jo} and C_{ijo} are positive numbers for $i = 1,\ldots,N$; $j = 1,\ldots,M$. If $\ell_1^j > 0$ for $j = 1,\ldots,M$ and $\det[sI - A - B^*(s)] \neq 0$ for $\mathrm{Re}\ s \geq 0$ then:

a. Equation (4.20) is stable with respect to $L^1 \cap BC_O$.

b. $c_{ij}(t)$ is in $L^1 \cap BC_o$ and $\|c_{ij}\|_{L^1 \cap BC_o} \leq \|p_j\|_{L^1 \cap BC_o} + (1+\lambda_{ij}^{-1})|c_{ij}^o|$ for $i = 1,\ldots,N$; $j = 1,\ldots,M$.

c. Equation (4.16) has a solution, $T^j(x,t)$, such that

$$\lim_{t\to+\infty} T^j(x,t) = 0 \quad \text{uniformly for } x \in \bar{G}_j.$$

Proof of Theorem 4.2. The proof is similar to that of Theorem 4.1. We first establish the four hypothesis of Theorem 3.5.

1. We claim that, $R(t)$, the resolvent associated with (4.20) is in $L^1 \cap BC_o$ and

$$\rho_R(f)(t) = \int_o^t R(t-s)f(s)\,ds$$

is in $\mathcal{L}(L^1 \cap BC_o, L^1 \cap BC_o)$. Now

$$|a_j(t)| \leq \left|\sum_{n=1}^{\infty} \eta_n^j \alpha_n^j e^{-\ell_n^j t}\right| \leq e^{-\ell_1^j t}\Big(\sum_{n=1}^{\infty} |\eta_n^j|^2\Big)^{1/2}\Big(\sum_{n=1}^{\infty} |\alpha_n^j|^2\Big)^{1/2}$$

So, $a_j(t)$ is in $L^1 \cap BC_o$, hence, $B(t)$ is in $L^1 \cap BC_o$ (see equation 4.13.b). Then, since $\det[sI - B^*(s) - A] \neq 0$ for $\operatorname{Re} s \geq 0$, Theorem 2.3 implies $R(t) \in L^1 \cap BC_o$. From Lemmas 3.1 and 3.2, ρ_R maps $L^1 \cap BC_o$ into itself. Hence, ρ_R is in $\mathcal{B}(L^1 \cap BC_o, L^1 \cap BC_o)$ by Theorem 2.1.

2. We have shown $a_j(t) \in L^1 \cap BC_o$, so, it follows from Lemma 4.1 that $g(p)(t)$ is of higher order in $L^1 \cap BC_o$.

3. $f(t)$ is in $L^1 \cap BC_o$. Since $\ell_1^j > 0$ for $j = 1,\ldots,M$ we have

$$|f_j(t)| \leq \sum_{i=1}^{N} |c_{ij}^o| e^{-\lambda_{ij} t} + \frac{1}{\Lambda_j} \sum_{n=1}^{\infty} |h_n^j \alpha_n^j e^{-\ell_n^j t}|$$

$$\leq \sum_{i=1}^{N} |c_{ij}^o| e^{-\lambda_{ij} t} + \frac{e^{-\ell_1^j t}}{\Lambda_j} \left(\sum_{n=1}^{\infty} |h_n^j|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} |\alpha_n^j|^2 \right)^{1/2}.$$

4. $k(t)p(t)$ is a small linear term in $L^1 \cap BC_o$. That is, $\|kp_1 - kp_2\|_{L^1 \cap BC_o} \leq L\|p_1 - p_2\|_{L^1 \cap BC_o}$ where p_1, p_2 are in $L^1 \cap BC_o$ and L is small.

We may take $L = \|k\|_o$ and this follows as in the proof of Theorem 4.1 (i).

Now the four hypothesis of Theorem 3.5 have been established. Finally, we note that

$$|f_j(t)| \leq \sum_{i=1}^{N} |c_{ij}^o| e^{-\lambda_{ij} t} + \frac{e^{-\ell_1^j t}}{\Lambda_j} \Big(\sum_{n=1}^{\infty} |h_n^j|^2 \Big)^{1/2} \Big(\sum_{n=1}^{\infty} |\alpha_n^j|^2 \Big)^{1/2}$$

and $\|h^j\|_2 = \Big(\sum_{n=1}^{\infty} |h_n^j|^2 \Big)^{1/2}$, $f(t)$ is small in $L^1 \cap BC_o$ if c_{ij}^o is small in R and h^j is small in $L^2(G_j)$ for $i = 1,\ldots,N$; $j = 1,\ldots,M$. Then it follows by Theorem 3.5 that (4.20) is stable with respect to $L^1 \cap BC_o$.

To prove (b) we use Lemmas 3.1 and 3.2 and the equation

$$c_{ij}(t) = \lambda_{ij} \int_o^t e^{-\lambda_{ij}(t-s)} p_j(s)\, ds + c_{ij}^o e^{-\lambda_{ij} t}.$$

From this it follows as in the proof of Theorem 4.1 that $c_{ij} \in L^1 \cap BC_o$ and

$$\|c_{ij}\|_{L^1 \cap BC_o} \leq \|p_j\|_{L^1 \cap BC_o} + (1 + \lambda_{ij}^{-1}) |c_{ij}|.$$

Now from Ito [6], Theorem 10 we see (4.16) has solution, $T^j(x,t)$, given by

$$T^j(x,t) = \sum_{n=1}^{\infty} e^{-\ell_n^j t} h_n^j u_n^j(x)$$

$$+ P_{jo} \sum_{n=1}^{\infty} \int_o^t e^{-\ell_n^j (t-s)} p_j(s)\,ds \;\eta_n^j u_n^j(x)$$

Furthermore, we can conclude from Theorem 1, page 157 and Theorem 4, page 167 of Friedman [2] that $\lim_{t\to+\infty} T^j(x,t) = 0$ uniformly for $x \in \bar{G}_j$.

Q.E.D.

It was assumed in Theorem 4.2 that the eigenvalues of (4.17) were positive. In the case of zero eigenvalues the problem is similar to the one dimensional reactor with insulated faces considered in Theorem 4.1. One can adapt the techniques used there and obtain similar results.

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