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Aspects of simultaneous inference

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John Aleong

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I. INTRODUCTION

A. Statement of the Problem

This dissertation is concerned with the multiple comparison problem. The model considered is the one-way classification with errors that are normally and independently distributed with zero mean and constant unknown variance σ_e^2 . The observations are denoted by X_{ij} and the model is

$$X_{ij} = u_i + e_{ij} \quad \begin{array}{l} i = 1, \dots, n \\ j = 1, \dots, r_i \end{array}$$

in which the u_i are unknown constants or parameters and the e_{ij} are the errors. The data may be reduced to the sample means \bar{X}_i , $i = 1, \dots, n$ and the pooled estimate of σ_e^2 , denoted by s^2 or $\hat{\sigma}_e^2$ where

$$\bar{X}_i = \frac{1}{r_i} \sum_j X_{ij} \quad \text{and} \quad s^2 = \frac{1}{R-n} \sum_{ij} (X_{ij} - \bar{X}_i)^2$$

with $R = \sum_i r_i$.

We want to give some substance to the notion of evidence given by $\bar{X}_1, \dots, \bar{X}_n$ and s^2 with respect to u_1, \dots, u_n and σ_e^2 . Some procedures which have been suggested in the literature will be reviewed in this chapter. These procedures differ considerably in the mode of approach.

It is our view that the choice of a procedure must be based on prior opinions about the true means, that is, the values of u_1, \dots, u_n . These opinions may be used to develop a procedure for making assertions about the true means and differences between them.

B. Review of the Literature with Minor Additions

A very extensive review of multiple comparison procedures in current use is given in Duncan (1965), Miller (1966), and O'Neill and Wetherill (1971). For this reason we will give only a brief discussion on the significant contributions. For quick reference and in order to keep this work as self-contained as possible we will describe concisely some of the procedures.

First, let us introduce some notation. A contrast of a set of p means is given by

$$\psi = \sum_{i=1}^p c_i u_i, \quad \text{with} \quad \sum c_i = 0. \quad (1.1)$$

ψ is estimated by

$$\hat{\psi} = \sum c_i \bar{X}_i. \quad (1.2)$$

The variance of \bar{X}_i , $i = 1, \dots, n$, is σ_e^2/r_i , which is estimated by $\hat{\sigma}_e^2/r_i$, where $\hat{\sigma}_e^2$ is based on v degrees of freedom with $v = R - n$. The procedures in current use give

significance tests and/or confidence intervals for sets of contrasts. The confidence intervals are of the form

$$\Psi \in \hat{\Psi} \pm \Delta$$

where the value of Δ is determined for each method.

The Least Significant Difference (LSD) is applied to pairwise differences and uses

$$\Delta = t_{\alpha/2}^v \hat{\sigma}_e \sqrt{\left(\frac{1}{r_i} + \frac{1}{r_j}\right)} \quad (1.3)$$

where $t_{\alpha/2}^v$ is the upper $100\alpha/2$ percent point of the Student's t distribution with v degrees of freedom.

Fisher (1935) proposed the Fisher Significant Difference (FSD) which uses, instead of $t_{\alpha/2}^v$, $t_{\alpha/2h}^v$ for an experiment with h comparisons. This modification of the LSD was suggested by Fisher to overcome the fact that the LSD gives too many false significances as the number of comparisons increase. If Fisher were interested, before looking at the data, in $m \leq h$ comparisons he would probably use $t_{\alpha/2m}^v$.

Scheffé' (1953, 1959) proposed the S-method. The S-method states that the probability is $1-\alpha$ that the values of all contrasts simultaneously satisfy the inequalities

$$\hat{\Psi} - S \leq \Psi \leq \hat{\Psi} + S \quad (1.4)$$

where $S = \hat{\sigma}_e \sqrt{(p-1)F_{(p-1),v}^{\alpha} \sum_i c_i^2/r_i}$, and $F_{(p-1),v}^{\alpha}$ is

the upper α point of the F distribution with parameters $p-1$ and v . This method is an inversion of the F -test. Even though the S -method is applicable to means based on samples of unequal sizes and also to means adjusted by covariates, it does not seem completely appropriate for selected comparisons, when the selection is made in the light of the data.

Recently Olshen (1973) considered regression problems where simultaneous confidence intervals of the S -method are used after a preliminary F -test rejects the null hypothesis that the regression parameters are zero. He showed that for significantly large critical values and at least two degrees of freedom for error, the probability of simultaneous coverage, conditioned on the rejection of the F -test, is always smaller than the unconditional probability. In other words, in practice, we usually make simultaneous confidence intervals after a significant F -test in the analysis of variance table. Each conditional probability of simultaneous coverage is less than $1-\alpha$ for all values of the unknown parameter when the F -test is made at level α . It appears that the same result extends to the Tukey T -method which will be described next.

Many writers, for example Newman (1939), Keuls (1952), Tukey (1953), and Duncan (1955) developed procedures based on the studentized range statistic for testing means with equal number of replications.

Tukey (1953) suggested the T-method which for balanced data, states that the probability is $1-\alpha$ that all the $\frac{n(n-1)}{2}$ differences $(u_i - u_j)$, $i \neq j$, simultaneously satisfy

$$\hat{u}_i - \hat{u}_j - T\hat{\sigma}_e \leq u_i - u_j \leq \hat{u}_i - \hat{u}_j + T\hat{\sigma}_e \quad (1.5)$$

where
$$T = q_{p,v}^{\alpha_p} / \sqrt{r} \quad , \quad (1.6)$$

$p=n$, $\alpha_p=\alpha$, and $q_{p,v}^{\alpha_p}$ is the upper $100\alpha_p$ percent point of the studentized range distribution with parameters p and v .

The problem of making statements about differences of the u_i has also been approached in a multistage way. The nature of multistage procedures is as follows. First an overall test is made to make a judgment of whether the whole set of means exhibits significant differences. If the result is nonsignificant, no further steps are taken, while if the result is significant, statistical tests are applied to subsets of the means. The overall test and the subsequent tests are based on the studentized range statistic. The Duncan D-method and the Student-Newman-Keuls (SNK) method both use Equation (1.5) but differ in the choice of α . If at some stage in the procedure a set of p -means is being compared in which at the beginning of the sequential procedure p equals n , and later p is less than n , the D-method uses

$\alpha_p = 1 - (1-\alpha)^{p-1}$ while the SNK method uses $\alpha_p = \alpha$ throughout.

In this multistage approach to the overall problem as any of the single stage approaches, the procedures depend intrinsically on the choice of the so-called error rates α_p or α . No logic is given on how these are to be determined. In our view this is a critical aspect of the overall problem. The experimenter when using any of the procedures is assumed to be able to specify these "error rates." In fact, the contrary seems to be the case. In simple one stage statistical testing situations, the problem of choosing an α can be evaded by giving instead the so-called P-values, leaving the problem of choice of a cut-off value to the discretion of the experimenter. Even in the case of a single test, for whether there are differences among the whole set of n means, this problem arises. It is common sense that if the data set is small in some sense, a test must not use a very small error rate, because then the power or sensitivity of the test will be very low. One approach in this simple case is to consider the power properties of tests with different null hypothesis error rates, α , and then to make a subjective judgment as to which test, determined by the value of α , has the most appeal.

Modifications were given for multiple range tests, for testing means with unequal number of replications, by Kramer

(1956, 1957), Duncan (1957) and Bancroft (1968).

Kramer (1956, 1957) suggested that for the D-method we should replace (1.6) by

$$T_1 = q_{p,v}^{\alpha} / \left(\frac{1}{2} \left(\frac{1}{r_i} + \frac{1}{r_j} \right) \right),$$

where $\alpha_p = 1 - (1-\alpha)^{p-1}$ and $p = 2, \dots, n$. Bancroft (1968) suggested that for the T-method we use

$$T_2 = q_{n,v}^{\alpha} / \bar{n}$$

where $\frac{1}{\bar{n}} = \frac{1}{n} \sum_i \left(\frac{1}{r_i} \right)$.

In these proposed modifications all the distributional properties, significance levels, etc., are lost.

Recently Spjøtvoll and Stoline (1973) and Hochberg (1973) gave an extension of the T-method to include the case with unequal sample sizes. The method is applicable to any set of heteroscedastic uncorrelated means. In these extensions the distributional properties are preserved. Spjøtvoll and Stoline (1973) proved

$$P \left\{ \sum_{i=1}^n x_i u_i \in \sum_i x_i \hat{u}_i \pm q_{n,v}^{\alpha} \hat{\sigma}_e \cdot L_d, \forall x_i \in \mathcal{L}_{lc} \right\} = 1-\alpha \quad (1.7)$$

where $L_d = \max \left\{ \sum_{i=1}^n (a_i x_i)^+, - \sum_{i=1}^n (a_i x_i)^- \right\},$

$$(a_i \ell_i)^+ = \max(0, a_i \ell_i) \quad , \quad (a_i \ell_i)^- = \min(0, a_i \ell_i) \quad ,$$

$a_i^2 = \frac{1}{r_i}$, and \mathcal{L}_{ℓ_c} = linear space of all linear combinations $\{\ell_1, \dots, \ell_n : \ell_1, \dots, \ell_n \text{ arbitrary}\}$.

In particular, for all pairwise comparisons of the population means u_1, \dots, u_n , we may make the following statement

$$P \left\{ (u_i - u_j) \in (\bar{X}_i - \bar{X}_j) \pm \hat{\sigma}_e T_3 \quad , \quad i \neq j \right\} \geq 1 - \alpha \quad (1.8)$$

$$\text{where } T_3 = q_{p, \nu}^{\alpha} / \min(\sqrt{r_i}, \sqrt{r_j}) \quad , \quad (1.9)$$

$\alpha_p = \alpha$ and $p = n$. Here $q_{n, \nu}^{\alpha}$ is the upper 100 α percent point of the studentized augmented range distribution with parameters n and ν . The studentized augmented range with parameters n and ν is the random variable

$$Q_{n, \nu}' = \max \left\{ |M|_n \quad , \quad R_\nu \right\} / \sqrt{(\chi_\nu^2 / \nu)}$$

where

$$|M|_n = \max_i \left\{ |X_i| \right\} \quad , \quad R_n = \max_{i, i'} \left\{ |X_i - X_{i'}| \right\} \quad ,$$

and X_1, \dots, X_n are independent, standard normal random variables with χ_ν^2 , an independent χ^2 variable on ν degrees of freedom. An equivalent method of defining the studentized augmented range is

$$Q'_{n,v} = \max_{i,i'=0,1,\dots,n} \left\{ |X_i - X_{i'}| \right\} / \sqrt{(\chi^2_v/v)}$$

where X_i , $i=1,\dots,n$ are given as before and X_0 is an independent random variable with $X_0 = 0$.

Although unmentioned in the literature, extensions of the D-method and the SNK procedure to group means with unequal number of replications can be constructed. The D-method would use (1.8) and (1.9) with $\alpha_p = 1 - (1 - \alpha)^{p-1}$ and $p = 2, \dots, n$, while the SNK would use $\alpha_p = \alpha$ and $p = 2, \dots, n$.

To use these new extensions of the D-method and the SNK procedure we need the upper α point of the studentized augmented range which is not tabulated. However, Tukey (1953) shows for $n > 2$ and $\alpha \leq .05$, that it differs from the corresponding upper α point of the studentized range by a practically negligible amount. For $n=2$, Tukey (1953) also shows that we may obtain the upper α point of the studentized maximum modulus by using the following relations;

$$Q'_{2,v}(x) = \frac{1}{2}V_{2,v}(x) + \frac{1}{2}V_{2,v}(x/\sqrt{2})$$

where $Q'_{2,v}(x)$ is the cumulative distribution function (cdf) of the studentized augmented range with parameters 2 and v , and $V_{2,v}(x)$ is the cdf of the studentized maximum

modulus with parameters 2 and v .

Most of the multiple comparison procedures discussed so far do not give an unambiguous grouping of the means.

Tukey (1949) suggested that a procedure based on the distribution of the studentized maximum gap statistic may help the experimenter to find the pattern or determine the grouping of the means.

Murphy (1973) explored the consequences of using this statistic for finding the grouping in a set of means which are normally distributed with a common variance. He gave only the exact null distribution of the maximum gap between adjacent means when there are four or five means. Some approximations were also given when the number of means is greater than five and less than twenty.

Here we develop an exact expression for the distribution of the maximum gap which involves n means from any population which has a continuous form. Though the derivation is straightforward, the formula is not simple for the normal distribution.

Let X_i , $i=1, \dots, n$ be distributed with cdf F and pdf f with $X_{(1)}, \dots, X_{(n)}$, the order statistics. It is required to find the distribution of

$$G_\ell = \max_{2 \leq i \leq n} \left(g_i \right) \text{ with } g_i = X_{(i)} - X_{(i-1)}$$

$$\text{and } \ell = n-1.$$

The joint density of all n order statistics is

$$= \begin{cases} n! f(x_1)f(x_2)\dots f(x_n) & \text{if } x_1 < x_2 < \dots < x_n \\ 0 & \text{otherwise.} \end{cases} \quad (1.10)$$

Let $\bar{g} = (g_2, \dots, g_n)$. Then by a linear transformation in (1.10) we obtain

$$f_{\bar{g}}(g_2, \dots, g_n) = \begin{cases} n! \int_{-\infty}^{\infty} \prod_{i=2}^n f(x + g_2 + g_3 + \dots + g_i) dx, & \text{if } g_i > 0, (2 \leq i \leq n) \\ 0, & \text{otherwise.} \end{cases} \quad (1.11)$$

Also from (1.11) we get

$$\begin{aligned} f_{g_i}(y) &= \int_{-\infty}^{\infty} f_{(X_{(i)}, X_{(i-1)})}(x, x+y) dx \\ &= \frac{n!}{(i-2)!(n-i)!} \int_{-\infty}^{\infty} \{F(x)\}^{i-2} \{1 - F(x+y)\}^{n-i} \\ &\quad \cdot f(x)f(x+y) dx. \end{aligned} \quad (1.12)$$

This expression was given in another context by Pyke (1965).

Before we give the distribution of G_ℓ let us give the Boole formula for the n events A_1, \dots, A_n . The probability of at least one of the A_i is

$$P \left(\bigcup_{i=1}^n A_i \right) = \sum_i P(A_i) - \sum_{i < j} P(A_i A_j) + \dots (-1)^{n-1} P(A_1 A_2 \dots A_n) \quad (1.13)$$

Let A_i be the event $\{g_i > g\}$. Then (1.13) may now be expressed as

$$P(G_\ell > g) = \sum_{i=2}^n P(g_i > g) - \sum_{i < j, i \neq 1} P(g_i > g, g_j > g) + \dots (-1)^{n-2} P(g_2 > g, g_3 > g, \dots, g_n > g) \quad (1.14)$$

Therefore if F is the normal distribution we may obtain the distribution of G_ℓ by substitution in (1.14). This general technique of finding the upper percentage points by the use of the Boole formula was described by David (1970). A special case was given by Fisher (1929) who used in our notation the first term of Boole formula for finding the upper percentage points.

Using the first term of (1.14), we obtain

$$P(G_\ell > g) \doteq \sum_{i=2}^n P(g_i > g) \quad (1.15)$$

An upper bound $g^{(1)}$ to $G_{\ell, \alpha}$ which is the upper $\alpha\%$ point of G_ℓ is obtained by solving (1.15) for g . With the existing computer software, it is not very difficult to solve (1.15).

Let us define the studentized maximum gap by

$R_{\ell, \nu} = G_{\ell}/s$ with s^2 an unbiased estimate of σ_e^2 with ν d.f. and independent of the order statistics.

The cumulative probability integral of the studentized maximum gap is given by

$$P(R_{\ell, \nu} \leq r) = \int_0^{\infty} P(G_{\ell} \leq sr) f(sr) ds$$

$$= \frac{2(\frac{1}{2}\nu)^{1/2\nu}}{\Gamma(\frac{1}{2}\nu)} \int_0^{\infty} s^{\nu-1} \exp(-\frac{1}{2}\nu s^2) P(G_{\ell} \leq sr) ds. \quad (1.16)$$

With the above expression we may calculate the distribution of the studentized maximum gap and the various upper percentage points of the distribution for different ℓ and ν .

These multiple comparison procedures are based on different principles and there are large differences between the solutions offered by these procedures. A major source of difference lies in the choice of the probability of a Type I error α . Some procedures control the per-comparison error rate which is the number of erroneous inferences divided by the number of inferences, while other procedures control the experimentwise error rate which is defined as the number of experiments with one or more erroneous inferences divided by the number of experiments. A basic question is how the

choice of α should be made, for example, in terms of a comparisonwise α or an experimentwise α or on some other basis. This is a question which has not been addressed.

Another inadequacy in the area of simultaneous inference is the choice of a multiple comparison procedure. Since the S-method and the LSD are based on the F statistic, while the D-method, the SNK and the T-method use the studentized range statistic, it seems that the choice of a multiple comparison procedure is related to the choice of a pivotal quantity. This aspect of the problem has not been considered in the literature.

Let us consider the work that was done on the logic for the choice of error rates. For the simple hypothesis vs the simple alternative Lehmann (1958) said that any experiment gives a convex curve passing through (0,1) and (1,0) of the admissible values of α and β , the probabilities of errors of the first and second kind. To pick α he says that we may specify a series of indifference curves on the (α, β) plane and use this to pick an optimum α . But he abandons this approach because of the difficulty of specifying the indifference map and suggests that we fix α and select a test which minimizes β . This recipe does not tell us how to select α .

Lindley and Savage in a series of papers, for example, Savage et al., (1962), and Edwards et al., (1963) showed under

certain assumptions for the simple hypothesis vs the simple alternative that the indifference curves are parallel straight lines whose slope is the prior odds-ratio when we assume a zero-one loss function. They gave a logic for the choice of α and a Bayes rule for the simple hypothesis vs the simple alternative. In this dissertation we will give the Lindley-Savage argument and extend it for the multiple comparison problem.

Lindley (1961) considered the composite hypothesis vs the composite alternative and showed that the critical values depend on the loss function, the prior distribution and the sample size. This important result shows that α should decrease with increasing sample size. We will review the Lindley (1961) work and also extend his arguments in this dissertation.

For the symmetric multiple comparison problem, the major contributions were Duncan (1961, 1965) and Waller and Duncan (1969). They developed a Bayes rule for the symmetric multiple comparison problem. Duncan (1965) showed that when the F ratio in the analysis of variance table for the one-way classification is small, for example, $F \leq 2.5$, the Bayes rule has the same character as the experimentwise α -rule, and when the F ratio is large, the Bayes rule has the same form as the comparisonwise α -rule. Waller and Duncan (1969) considered the one-way classification model with equal number of replications and variances unknown but equal. Using

a comparisonwise approach to the multiple comparison problem, they developed a Bayes rule and claimed to have a method for the choice of α . They used priors which are functions of the data. These priors have two difficulties in spite of the analytical convenience that they offer. They do not extend obviously to the case of unequal sample sizes and they depend on the data. The authors developed a Bayes Least Significant Difference. We will discuss the Waller and Duncan (1969) rule in detail in this dissertation.

The work in this dissertation represents an improvement over Waller and Duncan (1969) because we are using priors which are not functions of the data and also because we have considered the one-way classification model with unequal number of replications and with a common unknown variance.

II. GENERAL FRAMEWORK FOR A BAYESIAN APPROACH TO THE MULTIPLE COMPARISON PROBLEM

A. Introduction

The main objective of this chapter is to present a general framework for a Bayesian approach to the multiple comparison problem.

First, the probability space and Bayes theorem are discussed. Next the decision theoretic formulation of the multiple comparison problem for the one-way classification model is developed. This involves a discussion of a linear loss function used by Waller and Duncan (1969). Finally de Finetti's ideas of exchangeability are presented and it is shown how we can characterize our prior knowledge or beliefs by exchangeable priors.

B. Probability Space and Bayes Theorem

Consider the probability space $(\Omega, \mathcal{F}, P_\theta : \theta \in \Theta)$, where $\{P_\theta : \theta \in \Theta\}$ is a family of probability measures defined on the measurable space (Ω, \mathcal{F}) . Consider X as a \mathcal{F} -measurable mapping of Ω onto \mathcal{X} , i.e.,

$$(\Omega, \mathcal{F}, P_\theta) \xrightarrow{X} (\mathcal{X}, \mathcal{B}, P_\theta^X)$$

$$\vdash P_\theta^X(B) = P_\theta(X \in B) = P_\theta(X^{-1}(B)) \quad B \in \mathcal{B}.$$

In addition, consider a decision space D and a loss function, $L : \Theta \times D \rightarrow [0, \infty)$. We may also define a class of decision rules, $\Delta^* = \{\delta : \mathcal{X} \rightarrow D\}$. The risk function $R(\delta, \theta)$ is given by

$$R(\delta, \theta) = E_{\theta}[L(\theta, \delta(x))] .$$

Suppose Θ supports a σ -field and a probability measure over it. Then we may find a Bayes rule by minimizing the expected risk. Let $\Pi = \{\pi : \pi \text{ is a probability measure over } (\Theta, \mathcal{C}), \text{ where } \mathcal{C} \text{ includes single points}\}$, then the Bayes risk of δ with respect to π is given by

$$B(\pi, \delta) = \int R(\delta, \theta) d\pi(\theta) .$$

Now δ_0 is a Bayes rule with respect to π if

$$B(\pi, \delta_0) = \inf_{\delta \in \Delta^*} B(\pi, \delta) ,$$

where Δ^* is the class of randomized decision rules. Lehmann (1959) said it is convenient to consider this class. He said 'actually the introduction of randomized procedures leads to an important mathematical simplification by enlarging the class of risk functions so that it becomes convex. In addition, there are problems in which some features of the risk function such as its maximum can be improved by using a randomized procedure.' The notion of using randomized procedures is unappealing from the viewpoint of the evidential

content of data. However, in the situations to be discussed this will not be involved.

We will now characterize the measure on Θ by means of some density $\pi(\theta)$ with respect to some dominating σ -finite measure. The famous Bayes theorem is given by

$$\pi(\theta|x) = p(x|\theta)\pi(\theta)/p(x) ,$$

where
$$p(x) = \int p(x|\theta)\pi(\theta)d\theta .$$

The Bayesians go a step further with the introduction of a utility function $U[d(x),\theta]$, which is bounded, real-valued and defined over $D \times \Theta$. It has become standard in decision theory problems to use the negative of the utility, and to call this number the loss.

Bayesians claim that the decision problem is solved by maximizing the posterior expected utility which is given by

$$\int U[d(x),\theta]p(x|\theta)\pi(\theta)dx d\theta \tag{2.1}$$

where θ may be a vector. If $\pi(\theta)$ is a proper prior distribution then (2.1) is finite. Now by Fubini's theorem (2.1) is

$$\int dx \{ \int d\theta U[d(x),\theta]p(x|\theta)\pi(\theta) \} .$$

Therefore the best decision is the one which maximizes for each x , the quantity

$$\int d\theta U[d(x),\theta]p(x|\theta)\pi(\theta) . \tag{2.2}$$

For a further discussion on the decision and inference problem the reader may refer to Lindley (1971a).

In this development utility theory is a cornerstone of much decision theory. The reader must be aware of the fact that although many able statisticians use utility theory there are deep obscurities. Luce and Raiffa (1957) discuss some of the difficulties. They said that the problem of inter-personal comparisons of utility has not been solved and that reported preferences of individuals never satisfy the axioms but are usually intransitivities. They also said "There can be no question that it is extremely difficult to determine a person's utility function even under the most ideal and idealized experimental conditions, one can almost say that it has yet to be done." Therefore one may ask the following question; Can one base statistical methods on utility theory? For further discussion see Kempthorne (1972).

C. Decision Theoretic Formulation of the Problem

From the one-way classification model let

$$\bar{X}_i \sim \text{NID}(u_i, \sigma_e^2/r_i), \quad i=1, \dots, n.$$

The multiple comparisons among the means are reduced to a set of separate comparisons between the means u_i and u_j . A comparison of one mean u_i to a given mean u_j may be considered as the problem $P(i,j)$, which allows two

decisions

$d_{ij}^+ : \bar{X}_i$ is significantly larger than \bar{X}_j ,

or $u_i > u_j$

$d_{ij}^0 : \bar{X}_i$ is not significantly larger than \bar{X}_j

or u_i is unranked relative to u_j . (2.3)

If we consider the complementary problem $P(j,i)$ we obtain with a combination of $P(i,j)$ and $P(j,i)$ the familiar three-decision problem $Q(i,j)$, allowing the decisions

$d_{ij}^1 = d_{ij}^+ \cap d_{ji}^0 : \bar{X}_i$ is significantly larger than \bar{X}_j

$d_{ij}^2 = d_{ij}^0 \cap d_{ji}^0 : \bar{X}_i$ is not significantly different from \bar{X}_j

$d_{ij}^3 = d_{ij}^0 \cap d_{ji}^+ : \bar{X}_i$ is significantly less than \bar{X}_j . (2.4)

Note we have ruled out the decision

$d_{ij}^+ \cap d_{ji}^+ : \bar{X}_i$ is significantly larger than \bar{X}_j
and \bar{X}_j is significantly larger than \bar{X}_i .

[See Lehmann 1957a and 1957b] .

We have now reduced the multiple comparison problem to one of a three-decision problem $Q(i,j)$ for all of the

$\frac{n(n-1)}{2}$ combinations of (u_i, u_j) taken simultaneously.

For the single component problem $P(i, j)$, an appropriate loss table may be given by Figure 1.1 .

		Acts	
		d_{ij}^+	d_{ij}^0
States of Nature	$\delta_{ij} > 0$	Gain	Loss
	$\delta_{ij} \leq 0$	Loss	Gain which depends on δ_{ij}

Figure 1.1 Loss Table

Here we may represent gain by negative loss and $\delta_{ij} = u_i - u_j$.

In particular for the single component problem $P(i, j)$, we will consider a simple linear loss function used by Waller and Duncan (1969) .

$$L(d_{ij}^+, \theta) = \begin{cases} k_1 |\delta_{ij}| & \delta_{ij} \leq 0 \\ 0 & \delta_{ij} > 0 \end{cases} \quad L(d_{ij}^0, \theta) = \begin{cases} 0 & \delta_{ij} \leq 0 \\ k_2 \delta_{ij} & \delta_{ij} > 0 \end{cases} \quad (2.5)$$

where $\theta = (u_1, \dots, u_n, \sigma_e^2)$, $\delta_{ij} = u_i - u_j$, and k_1 and k_2 are positive. The numbers k_1 and k_2 are very important in this approach because they represent the relative

seriousness of the Type I and Type II errors. The solution depends on k_1 and k_2 through $k = k_1/k_2$.

Kempthorne and Folks (1971) say that in most real situations there is great difficulty in specifying $L(d, \theta)$ because the consequences of terminal decisions extend into the indefinite future. Here we assume this difficulty has been solved by the users of this procedure.

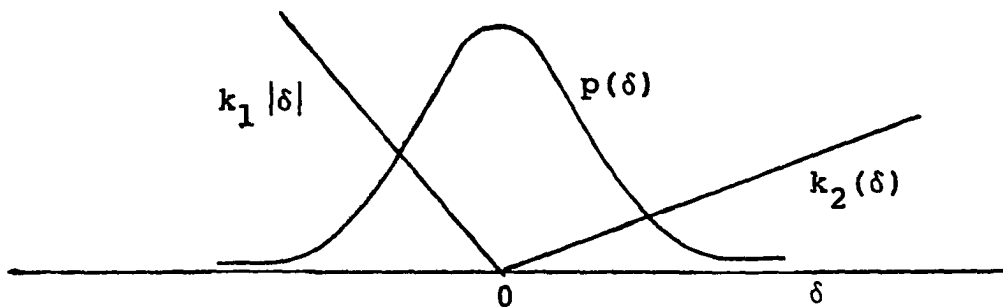


Figure 1.2 Loss Function and Prior Distribution

We have used the above convenient loss function (Figure 1.2) and do not claim that this loss function is appropriate to every problem that an experimenter faces in practice, nor of the appropriateness of the assumption of normality of the distribution of the data and the parameters. Hence our solution is not exactly optimal for any practical problem. We also have symmetry and monotonicity of the loss function.

This loss function is also unbounded but this property does not hinder us from interchanging the order of the

integrals in the expression for the Bayes risk in our search for a Bayes solution. Here we appeal to Tonelli's theorem which allows us to interchange the order of the integrals in Equation 2.1. A statement of the theorem is given below.

Theorem (Tonelli) [Royden (1968)].

Let (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) be two σ -finite measure spaces, and let f be a nonnegative measurable function on $X \times Y$. Then

- i for almost all x the function f_x defined by $f_x(y) = f(x, y)$ is a measurable function on Y ,
- i' for almost all y the function f_y defined by $f_y(x) = f(x, y)$ is a measurable function on X ,
- ii $\int_Y f(x, y) d\nu(y)$ is a measurable function on X ,
- ii' $\int_X f(x, y) d\mu(x)$ is a measurable function on Y ,
- iii
$$\begin{aligned} \int_X \left[\int_Y f d\nu \right] d\mu &= \int_{X \times Y} f d(\mu \times \nu) \\ &= \int_Y \left[\int_X f d\mu \right] d\nu . \end{aligned}$$

If d^i 's are the component decisions of the form of (2.3) then for a multiple comparison decision d where

$d = \bigcap_{i=1}^p d^i$ we obtain under the linear additive loss model

$$L(d, \theta) = \sum_{i=1}^p L(d^i, \theta) \quad . \quad (2.6)$$

In this dissertation we have as a matter of convenience considered only an additive loss structure, but we can envisage cases where the loss structure is not additive. In certain cases we may be able to transform a non-additive loss structure to an additive loss structure. At the moment it is unclear how one deals with non-additive loss functions in general.

D. Exchangeability

De Finetti introduced the idea of exchangeability in 1931 and 1937. The random variables X_1, \dots, X_n are exchangeable if the $n!$ permutations $(X_{k_1}, \dots, X_{k_n})$ have the same n dimensional probability structure. An excellent translation of his paper is given in de Finetti (1964) where he shows for random variables taking the values 0 or 1, that their distribution could be represented as the weighted average of probabilities obtained by coin tossing processes. He also extended this characterization to general random variables. A proof of this general result is given in this

chapter. Hewitt and Savage (1955) studied this theorem for random variables in abstract topological spaces.

This major result shows that if exchangeability is assumed for every n in $\theta_1, \dots, \theta_n$, then a mixture seems to be the only way to generate an exchangeable distribution. An alternative restatement is that an exchangeable sequence of random variables behaves like a random sample from some distribution with a (prior) distribution over the sampled distribution. Lindley (1971b) says that exchangeability can be a substitute for the concept of randomness and that exchangeability is an easier condition to check than the concept of independence which is involved in the concept of randomness. Here he means that one can easily see whether the joint distribution has the property of exchangeability, i.e., invariance under permutation of the suffices. The reader must be aware that this concept of exchangeability is not always valid, and that in some cases one may have to modify the idea of exchangeability and talk about between and within exchangeability. A simple example which demonstrates the inapplicability of this idea is given by the one-way classification model where some of the treatments are experimental varieties and some are controls. We cannot assume here that the treatment means are all exchangeable; we may suppose from a particular structure of the treatments that we have exchangeability between the controls and between the experimental varieties. In other cases our beliefs about

the structure of the treatments may show that the assumption of exchangeability is not applicable.

Lindley (1971a, 1971b), Smith (1973a, 1973b), and Lindley and Smith (1972) used the idea of exchangeability to generate prior distributions in the estimation problem. We hope in this thesis to use this idea to generate prior distributions for an analysis of the multiple comparison problem.

With the assumption of exchangeability we are imposing some form of structure in our prior information or beliefs. Also on account of the difficulty of assigning priors in higher dimensions in any meaningful way, by using exchangeable priors we are able to reduce the dimensionality of the problem. This point will be illustrated later by an example.

All the results given in this thesis are valid if we drop de Finetti's idea of exchangeability, but we may regard exchangeability as a convenient way to generate prior distributions. This advice is given because the Hewitt and Savage (1955) theorem is true only in an infinite dimensional space while Lindley and Smith in their work quoted earlier appear to be using it for a finite dimensional space [Godambe, 1971]. When Lindley says that if exchangeability is assumed for every n in $\theta_1, \dots, \theta_n$, then a mixture is the only way to generate an exchangeable distribution, it seems that he is using the theorem for a finite dimensional space.

Let us restate these ideas on exchangeability by giving a formal definition of exchangeability and by reproducing a result which is given in Løve (1963). But before we do this we need some notation. Let (Ω, \mathcal{A}, P) be the sample space. Define X_1, X_2, \dots as random variables on this space with distribution function $F_{k_1 \dots k_m}$ and with conditional distribution function $F_{k_1 \dots k_m}^{\mathcal{B}}$ where \mathcal{B} is a sub σ -field contained in \mathcal{A} .

Definition: The X 's given \mathcal{B} are conditionally independent if

$$F_{k_1 \dots k_m}^{\mathcal{B}} = F_{k_1}^{\mathcal{B}} \dots F_{k_m}^{\mathcal{B}}$$

where the subscripts form an arbitrary finite subset. Now if we take the expectation by integrating with respect to $P_{\mathcal{B}}$ where

$$P_{\mathcal{B}}(B) = P(B) \quad , \quad B \in \mathcal{B} \quad ,$$

and using the definition of conditional expectation we obtain

$$F_{k_1 \dots k_m} = E(F_{k_1}^{\mathcal{B}} \dots F_{k_m}^{\mathcal{B}}) \quad .$$

Note also if $\mathcal{B} = \mathcal{A}$, X_1, X_2, \dots are always conditionally independent, so we assume $\mathcal{B} \neq \mathcal{A}$.

Suppose the r.v's are conditionally independent with some common conditional distribution function $F^{\mathcal{G}}$; then we have as above

$$F_{k_1 \dots k_m}(x_1, \dots, x_m) = E(F^{\mathcal{G}}(x_1) \cdots F^{\mathcal{G}}(x_m)) . \quad (2.7)$$

So the joint distributions of any m r.v's do not depend upon their subscripts but only upon the number m . This leads to de Finetti's definition of exchangeability which may be restated

Definition: The random variables are exchangeable if the probability that X_{i_1}, \dots, X_{i_m} satisfy a given condition is the same no matter how the distinct indices i_1, \dots, i_m are chosen.

Theorem (de Finetti-Loève).

The concept of exchangeability is equivalent to that of conditional independence with a common conditional distribution function.

Proof (Loève, 1963) (Given in detail): ← This is true from (2.7) where the joint distributions of any m of the r.v's do not depend upon the subscripts but only on m . So the r.v's are exchangeable.

Let

$$\Rightarrow G_m = F_{k_1 \dots k_m} \text{ and } \forall x \in R$$

define

$$S_n(x) = \frac{1}{n} \sum_{j=1}^n I_{[X_{k_j} < x]} \quad .$$

From exchangeability

$$E[I_{[X_i < x]}] = P[X_i < x] = G_1(x) = m_1$$

and

$$E[I_{[X_i < x]} I_{[X_j < x]}] = G_2(x, x) = m_2 \quad i \neq j \quad .$$

Now for $m < n$

$$\begin{aligned} E[S_m(x) - S_n(x)]^2 &= E\left[\frac{1}{m} \sum_{j=1}^m I_{[X_{k_j} < x]} - \frac{1}{n} \sum_{j=1}^n I_{[X_{k_j} < x]}\right]^2 \\ &= \frac{1}{m^2} E\left[\sum_{j=1}^m I_{[X_{k_j} < x]}\right]^2 + \frac{1}{n^2} E\left[\sum_{j=1}^n I_{[X_{k_j} < x]}\right]^2 \\ &\quad - \frac{2}{mn} E\left[\sum_{j=1}^m I_{[X_{k_j} < x]} \sum_{j=1}^n I_{[X_{k_j} < x]}\right] \\ &= \frac{1}{m^2} [mm_1 + m(m-1)m_2] + \frac{1}{n^2} [nm_1 + n(n-1)m_2] \\ &\quad - \frac{2}{mn} [mm_1 + m(n-1)m_2] \\ &= m_1 \left[\frac{1}{m} + \frac{1}{n} - \frac{2}{n} \right] + m_2 \left[\frac{m-1}{m} + \frac{n-1}{n} - \frac{2(n-1)}{n} \right] \end{aligned}$$

$$\begin{aligned}
&= m_1 \left[\frac{1}{m} - \frac{1}{n} \right] + m_2 \left[\frac{1}{n} - \frac{1}{m} \right] \\
&= (m_1 - m_2) \left(\frac{1}{m} - \frac{1}{n} \right) ,
\end{aligned}$$

and as $m, n \rightarrow \infty$ $E[S_m(x) - S_n(x)]^2 \rightarrow 0$. This shows that $S_n(x)$ is a Cauchy sequence which implies \exists

$S(x) \rightarrow E[S_n(x) - S(x)]^2 \rightarrow 0$. But convergence in mean square \implies convergence in probability [Tucker (1967)] .
Hence

$$S_n(x) \xrightarrow{P} S(x) .$$

Since $S_n(x)$ is bounded by 1 , we have by the Bounded Convergence Theorem and the a.s invariance under finite permutations of X 's of $B \in \mathcal{B} = \mathcal{B}(S(x), x \in R)$

$$E(S_n(x_1) \cdots S_n(x_m) I_B) \rightarrow E([S(x_1) \cdots S(x_m)] I_B) .$$

The LHS = $P^{\mathcal{B}}[X_1 < x_1, \dots, X_m < x_m]$ while

$$\text{RHS} = S(x_1) \cdots S(x_m) ,$$

so $P^{\mathcal{B}}[X_1 < x_1, \dots, X_m < x_m] = S(x_1) \cdots S(x_m)$ a.s .

Now $S_n(x)$ is a step function and is a d.f in X . Therefore $S(x)$ has a.s the properties of a d.f in X , and from the above can be replaced by a conditional distribution function i.e., convergence in probability \implies convergence in

distribution. Note that theorems on the existence of conditional distributions are given in Loève (1963).

This major result by de Finetti shows that the p.d.f.'s of a class of exchangeable random variables are averages of independent random variables. It gives us also a method of obtaining exchangeable distributions.

In the example given next we show how exchangeable priors are used in estimating the mean from a normal distribution with a known variance.

Example 1.1. Suppose $X_i \sim \text{NID}(u_i, \sigma^2)$ $i=1, \dots, n$ where σ^2 is known.

Assume also the distribution of $u_i \forall n$ is exchangeable. By our theorem

$$p(\underline{u}) = \int \prod_{i=1}^n p(u_i | \theta) dQ(\theta)$$

where $p(u_i | \theta) \forall \theta$ and $Q(\theta)$ are arbitrary p.d.f.'s. Note $p(\underline{u})$ is a mixture weighting by $Q(\theta)$ of i.i.d. distributions given θ .

By exchangeability we have $E(u_i)$ is a constant, $\theta, \forall i$. In particular, let u_i be i.i.d. $N(\theta, \tau^2)$ with τ^2 known.

In addition, we assume θ has a distribution which can be supposed diffuse and the variance for θ tends to infinity. It is instructive at this point to mention that by assuming an exchangeable prior we have reduced the dimension of the

parameter space from n to one.

From the above assumptions we have the elementary result

$$E(u_i | x) = \frac{X_i/\sigma^2 + X_./\tau^2}{1/\sigma^2 + 1/\tau^2} \quad \text{where } X_. = \sum X_i/n \quad .$$

As mentioned, before one applies the assumption of exchangeability it is necessary to check whether it is practically realistic.

III. ATTEMPTS OF A LOGIC OF CHOICE OF ERROR RATES

A. Introduction

In this chapter we review in detail some of the major contributions on the choice of α , the probability of a Type I error. First, the assumptions of a "rational" decision maker are given and using these assumptions we state a result due to Raiffa and Schlaifer (1961) which shows that a decision maker's indifference surfaces must be parallel hyperplanes.

The Lehmann argument is discussed and the Raiffa and Schlaifer result is then used to reproduce the Lindley-Savage argument on the choice of α . Hypothesis testing within a Bayesian framework is then discussed with the simple hypothesis versus the simple alternative. Lindley's argument on the choice of α for a composite hypothesis versus a composite alternative is also given with an extension of the argument to the multiple comparison problem. A critical discussion of the Waller-Duncan argument is also given.

B. Assumptions of a "Rational" Decision Maker

Raiffa and Schlaifer (1961) showed that under "three basic assumptions concerning logically consistent behavior," a decision maker's indifference surfaces must be parallel hyperplanes. In this discussion we will use a decision space, a bounded utility function, and a decision will be selected

which maximizes the expected utility.

A basic assumption is that the parameter space Θ is finite. The three basic assumptions which will characterize a logically consistent behavior are

- i) **Sure-thing Principle:** Suppose $\Theta = \{\theta_1, \dots, \theta_r\}$ and let $\underline{a} = \{a_1, \dots, a_r\}$ be the utility for decision d_1 , and $\underline{b} = \{b_1, \dots, b_r\}$ the utility for decision d_2 . Then $d_1 \succ d_2$ (d_1 is preferred to d_2) if $a_i \geq b_i$ for all i , and $a_i > b_i$ for some i .
- ii) **Continuous Substitutability:** Indifference surfaces extend smoothly from boundary to boundary of a region R in r -space in the sense that if \underline{a} is a point on the indifference surface, and if we make a small change in any $(r-1)$ coordinates, then by making a small compensating change in the remaining coordinates, we can obtain a new point on the same indifference surface as \underline{a} .
- iii) Suppose there are three decisions d_1 , d_2 , and d_3 such that $d_1 \sim d_2$ (d_1 is indifferent to d_2). Then a mixed strategy which selects d_1 with probability p and d_3 with probability $1-p$ is indifferent to a mixed strategy which selects d_2 with probability p and d_3 with probability $(1-p)$.

Theorem:

Under these three assumptions, the decision maker's indifference surfaces must be parallel hyperplanes.

Proof: See Raiffa and Schlaifer (1961) pg. 25-27.

C. The Lehmann Argument

Lehmann (1958) in his discussion on the choice of error rates considered X as distributed as

$$dP_{\theta, v}^X(x) = C(\theta, v) \exp[\theta U(x) + \sum_{i=1}^r v_i T_i(x)] d\mu(x)$$

with $(\theta, v) \in \Theta$,

$$v = (v_1, \dots, v_r) \text{ and } T = (T_1, \dots, T_r).$$

Consider the problem of testing the hypothesis $H_0: \theta = \theta_0$ against the alternative $H_1: \theta > \theta_0$. By the Neyman-Pearson theory, the uniformly most powerful unbiased test is given by

$$\phi(u, t) = \begin{cases} 1 & \text{if } u > C(t) \\ \gamma(t) & \text{if } u = C(t) \\ 0 & \text{if } u < C(t) \end{cases} \quad (3.1)$$

with the functions C and γ determined by

$$E_{\theta_0} [\phi(U, T) | T=t] = \alpha$$

and

$$E_{\theta_0}[U\phi(U,T)|T=t] = \alpha E_{\theta_0}[U|T=t] \quad \text{for all } t.$$

Let \mathcal{C} be the class of tests satisfying (3.1). We want to select a specific test from \mathcal{C} .

For $r=0$ we have no nuisance parameters. Let α and β be the error probabilities associated with testing $H_0:\theta=\theta_0$ against $H_1:\theta=\theta_1$. The attainable pairs (α,β) form a convex set, the lower boundary of which corresponds to the admissible test (3.1). This lower boundary S is a convex curve connecting $(0,1)$ and $(1,0)$. We need a method to select a point on this curve. Lehmann considers an indifference map as shown in Figure 3.1(a).

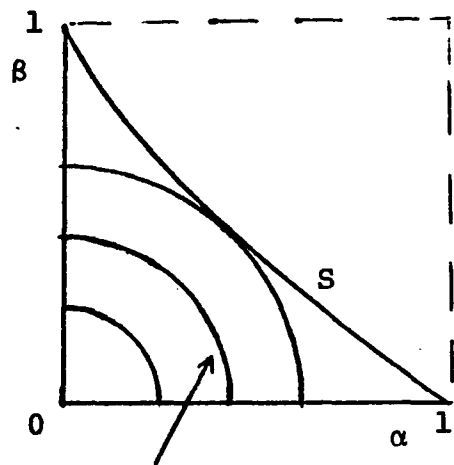
He then suggests that the optimum test would be given by that point of S lying on the indifference curve closest to the origin.

However, he abandons this approach by commenting on its complexity and by saying that the indifference map may be any family of curves running in a north-westerly to south-easterly direction.

Some other suggestions have been given. Consider fixing α without regard to power. Let $L = \{\alpha|\alpha=\alpha'\}$ as shown in Figure 3.1(b). The required test is given by the intersection of L and S , which is the point (α',β') .

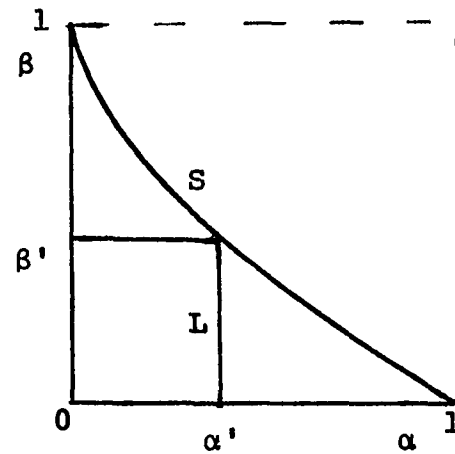
We may also consider, as illustrated in Figure 3.1(c) the intersection of S and C , where $C = \{\beta|\beta=f(\alpha)\}$ and f is a continuous strictly increasing function with $f(0)=0$.

Figure 3.1 Diagrams for selecting α .

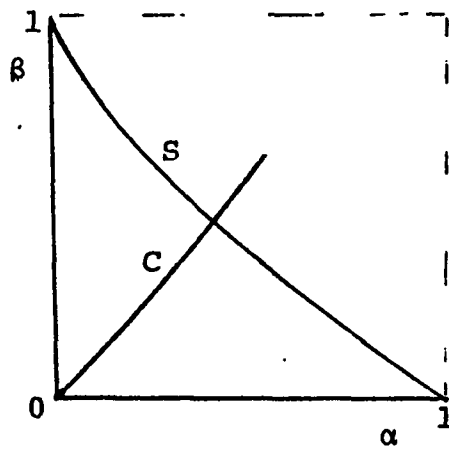


Indifference
Curves

(a)



(b)



(c)

In particular we may take $\beta = k\alpha$. Since for all admissible tests $\beta \leq 1 - \alpha$, we have $\alpha \leq k/(k+1)$ and $1/(k+1)$ is an upper bound for α . The problem seems to be how to determine the function f .

Lehmann also considers the case for $r > 0$ and gave some discussion on the choice of α and β .

D. The Lindley-Savage Argument

Consider the case of a simple hypothesis and a simple alternative. Savage et al., (1962) said our choice of (α, β) must be made subjectively.

Let us consider the unit square with axes α , $0 \leq \alpha \leq 1$, and β , $0 \leq \beta \leq 1$. By the principle of admissibility we will restrict our choice of (α, β) to the south-west portion of the square. Any experiment will give a convex curve passing through the two corners $(1,0)$ and $(0,1)$ and the statistician will select a point on this curve as the best in some sense. We know that the admissible tests on this curve are the likelihood ratio tests.

Let us consider a family of indifference curves in the (α, β) plane. By the argument given in Section IIIB these curves should be a family of parallel straight lines. So the question is how to pick the slope of these lines which is simply the rate at which one is willing to increase β per unit decrease in α , or the rate of exchange.

Since any decision will amount to the choice of a slope, we will arrive at a decision by being Bayesians with utilities and subjective probabilities. By Bayes theorem for A and $D \in \mathcal{F}$

$$P(A|D) = \frac{P(D|A)P(A)}{P(D)} \quad \text{provided } P(D) \neq 0.$$

$$\text{Also } P(\bar{A}|D) = \frac{P(D|\bar{A})P(\bar{A})}{P(D)} \quad \text{with } P(D) \neq 0.$$

$$\text{So } \frac{P(A|D)}{P(\bar{A}|D)} = \frac{P(D|A)}{P(D|\bar{A})} \frac{P(A)}{P(\bar{A})}.$$

The left-hand side is the posterior odds for A over \bar{A} and $P(A)/P(\bar{A})$ is the prior odds, so posterior odds = likelihood ratio \times prior odds i.e., $\Omega(A|D) = L(A:D) \Omega(A)$, where $\Omega(A|D) = P(A|D)/P(\bar{A}|D)$, $L(A:D) = P(D|A)/P(D|\bar{A})$ and $\Omega(A) = P(A)/P(\bar{A})$.

Now consider the following payoff matrix,

	Correctly	Incorrect
A	I	0
\bar{A}	J	0

and the expected cash matrix,

	Correctly	Incorrect
A	IP(A)	0
\bar{A}	JP(\bar{A})	0

We will guess A iff $IP(A) > JP(\bar{A})$. After an examination of the data we will guess A iff $\Omega(A|D) > J/I$ iff $L(A:D) > J/(I\Omega(A)) = \Lambda$, where Λ is the critical likelihood ratio. Therefore the indifference curves will have then slope $-[J/(I\Omega(A))]$. So once we specify our prior odds and our utilities we can find our slope and our choice of (α, β) in the unit square. In particular, if $J = I = 1$ then the slope of the indifference curve is $-P(\bar{A})/P(A)$.

Lindley (1971b) showed the unsoundness of the minimax method which is sometimes employed in the selection of α .

The Lindley-Savage argument provides a logic for the choice of α in the simple hypothesis versus the simple alternative. Using a Bayesian proof of the Neyman-Pearson lemma we restate formally the Lindley-Savage argument for determining α .

E. Bayesian Hypothesis Testing

1. Simple vs simple

Let $\Theta = (\theta_0, \theta_1)$, $D = (d_0, d_1)$, where d_0 accepts the hypothesis while d_1 rejects it. Also let

$$L(d_i, \theta_j) = \begin{cases} 0 & i=j \\ a_j & i \neq j \end{cases} \quad (3.2)$$

such that $a_0 > 0$ and $a_1 > 0$, $\alpha(\delta) = p(\delta = d_1 | \theta_0)$ and $\beta(\delta) = p(\delta = d_0 | \theta_1)$. Consider a prior distribution over Θ which is given by $p(\theta = \theta_0) = \pi$ \dagger $\pi \in (0, 1)$ and $p(\theta = \theta_1) = 1 - \pi$. Now the Bayes risk

$$\begin{aligned} R(\pi, \delta) &= a_0 \pi \alpha(\delta) + a_1 (1 - \pi) \beta(\delta) \\ &= a \alpha(\delta) + b \beta(\delta) \end{aligned} \quad (3.3)$$

such that $a = a_0 \pi$ and $b = a_1 (1 - \pi)$. For $i = 0, 1$ let f_i be the pdf of the observation X when $\theta = \theta_i$.

Given the above we may restate the Neyman-Pearson lemma for $H_0: \theta = \theta_0$ versus $H_1: \theta = \theta_1$.

Lemma (DeGroot (1970)). For any constants $a > 0$ and $b > 0$, let δ^* be a decision such that

$$\delta^*(x) = d_0 \quad \text{if} \quad af_0(x) > bf_1(x)$$

and

$$\delta^*(x) = d_1 \quad \text{if} \quad af_0(x) < bf_1(x). \quad (3.4)$$

The value of $\delta^*(x)$ may be either d_0 or d_1 if $af_0(x) = bf_1(x)$. Then for any other δ

$$a\alpha(\delta^*) + b\beta(\delta^*) \leq a\alpha(\delta) + b\beta(\delta) .$$

Proof. (See DeGroot (1970)).

This lemma says that we should pick

$$\begin{aligned} \delta^*(x) &= d_0 \quad \text{if} \quad \frac{f_1(x)}{f_0(x)} < \frac{a}{b} = \frac{a_0\pi}{a_1(1-\pi)} \\ &= d_1 \quad \text{if} \quad \frac{f_1(x)}{f_0(x)} > \frac{a}{b} = \frac{a_0\pi}{a_1(1-\pi)} . \end{aligned} \quad (3.5)$$

The above result shows that in the simple hypothesis versus the simple alternative our choice of a decision function depends on our prior distribution, our losses or utilities and the likelihood ratio $f_1(x)/f_0(x)$.

For the above test we know that α is the probability of rejecting H_0 when H_0 is true. Therefore in this case if we specify our prior probabilities and our losses or negative utilities we can calculate this probability and hence obtain the required value for α .

By the above argument we have not only found a logic for the choice of error rates but have also found a method for determining whether to reject or accept the hypothesis under discussion.

As illustrations of the suggested process we give two examples.

Example 3.1. Continuous case

$$\text{Let } H_0: f_0(x) = 1 \quad x \in [0,1]$$

$$H_1: f_1(x) = 3x^2 \quad x \in [0,1] \quad .$$

$$\text{Now } L(x) = f_1(x)/f_0(x) = 3x^2 \quad x \in [0,1] \quad .$$

By the above Neyman-Pearson lemma,

$$\text{reject } H_0 \text{ if } L(x) > \pi a_0 / [(1-\pi)a_1]$$

$$\text{or accept } H_0 \text{ if } L(x) < \pi a_0 / [(1-\pi)a_1] \quad .$$

For $\lambda \leq 3$,

$$\begin{aligned} \alpha &= P_{\theta_0} \{ x: L(x) > \lambda \} = P_{\theta_0} \{ x: 3x^2 > \lambda \} \\ &= 1 - (\lambda/3)^{1/2} \quad . \end{aligned}$$

$$\begin{aligned} \text{Also } \beta &= P_{\theta_1} \{ x: L(x) < \lambda \} = P_{\theta_1} \{ x: x < (\lambda/3)^{1/2} \} \\ &= (\lambda/3)^{3/2} = (1-\alpha)^3 \quad . \end{aligned}$$

Suppose the loss associated with decision d_i is a_i and

$P(H_0) = \pi$, then for $\pi = 3/4$, $a_0 = 3/2$, and $a_1 = 3$

$$\lambda = \pi a_0 / [(1-\pi)a_1] = 3/2 \quad .$$

$$\text{So } \alpha = 1 - (\lambda/3)^{1/2} = 1 - \sqrt{2}/2 = .293 \quad .$$

Example 3.2. Discrete case

Consider the discrete r.v x } the probability under H_i is p_i , $i=0,1$ with

x	0	1	2	3
$p_0(x)$.1	.2	.3	.4
$p_1(x)$.2	.1	.4	.3

Now $L(x) = p_1(x)/p_0(x)$.

x	1	3	2	0
$L(x)$	1/2	3/4	4/3	2

Now let $a_0 = a_1 = 1$ and $\pi = .6$.

$$\text{So } \lambda = \frac{\pi a_0}{(1-\pi)a_1} = \frac{.6}{.4} = \frac{3}{2} .$$

$$\alpha = P_{\theta_0} \{ x: L(x) > \lambda \} = P_{\theta_0} \{ x=0 \} = .1$$

$$\beta = P_{\theta_1} \{ x: L(x) < \lambda \} = P_{\theta_1} \{ x=1 \text{ or } x=2 \text{ or } x=3 \} = .8$$

so $\alpha = .1$ and $\beta = .8$.

2. Composite vs composite

Here we will review the logic of Lindley (1961) for the choice of α when one has the following hypothesis testing problem:

$$H_0: \theta = 0 \text{ versus } H_1: \theta \neq 0 \text{ where } \theta \text{ is a nuisance parameter.}$$

Lindley discussed the use of prior probability distributions in statistical inference and decisions. He was concerned with the large sample problem and showed how the effect of the

prior distribution is minimal. He obtained results which are comparable to the large sample theory of testing. Let $D = (d_0, d_1)$ where d_0 accepts H_0 while d_1 rejects H_0 . Let us make the following assumptions about the utility function.

$$U(d_0; 0, \emptyset) > U(d_1; 0, \emptyset)$$

and
$$U(d_0; \theta, \emptyset) < U(d_1; \theta, \emptyset) .$$

We note that the optimum decision which is obtained by maximizing (2.1), is not affected by subtracting any function of $\underline{\theta}$ from $U(d, \underline{\theta}) = U(d(x), \underline{\theta})$ where $\underline{\theta} = (\theta, \emptyset)$. With the same notation the above inequalities are replaced by

$$U(d_0; \theta, \emptyset) = 0 \quad \theta \neq 0$$

$$U(d_1; 0, \emptyset) = 0$$

and
$$U(d_1; \theta, \emptyset) > 0 \quad \text{otherwise} . \quad (3.6)$$

Following Jeffreys (1961) we will assume that along the line $\theta=0$ there is a density $\pi_0(\emptyset)$ with respect to Lebesgue measure on this line, while over the rest of the parameter space we will assume a density $\pi_1(\theta, \emptyset)$ with respect to Lebesgue area over $\underline{\theta} = (\theta, \emptyset)$.

By definition the prior-odds in favour of H_0 is

$$\int \pi_0(\emptyset) d\emptyset / \iint \pi_1(\theta, \emptyset) d\theta d\emptyset .$$

From the expected utility let

$$\lambda(d_0, x) = \int U(d_0; 0, \theta) p(x|0, \theta) \pi_0(\theta) d\theta \quad (3.7)$$

$$\text{and } \lambda(d_1, x) = \iint U(d_1; \theta, \theta) p(x|\theta, \theta) \pi_1(\theta, \theta) d\theta d\theta. \quad (3.8)$$

Let us approximate the integrals (3.7) and (3.8). First let us introduce some notation. Let

$$w_0(\theta) = U(d_0; 0, \theta) \pi_0(\theta), \quad (3.9)$$

$$\text{and } w_1(\theta, \theta) = U(d_1; \theta, \theta) \pi_1(\theta, \theta). \quad (3.10)$$

For $\underline{x} = (x_1, \dots, x_n)$ \dagger $x_i \sim$ i.i.d. with $p(x_i|\theta)$, let

$$\bar{p}(\underline{x}|\theta, \theta) = \prod_{i=1}^n p(x_i|\theta, \theta)$$

and

$$\Delta = \det(nc_{ij}(\hat{\underline{\theta}})) \quad \text{with } nc_{ij}(\hat{\underline{\theta}}) = - \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log \prod_{i=1}^n p(x_i|\hat{\underline{\theta}}),$$

where $\hat{\underline{\theta}}$ is the maximum likelihood estimate (mle) of $\underline{\theta}$

while $\tilde{\theta}$ is the mle of θ when $\theta=0$.

For the case of n independent and identically distributed observations Lindley (1961) under suitable regularity conditions approximated (3.7) by

$$\lambda(d_0, \underline{x}) \doteq \sqrt{(2\pi)w_0(\tilde{\theta})} \bar{p}(\underline{x}|0, \tilde{\theta}) [nc_{22}(0, \tilde{\theta})]^{-.5} \quad (3.11)$$

and (3.8) by

$$\lambda(d_1, \underline{x}) \doteq (2\pi)w_1(\hat{\theta}, \hat{\theta}) \bar{p}(\underline{x}|\hat{\theta}, \hat{\theta}) \Delta^{-.5} \quad (3.12)$$

Assuming that $w_0(\theta)$ and $w_1(\theta, \theta)$ are bounded away from

zero near the mle values, the rule is to accept H_0 if

$$\lambda(d_0, \underline{x}) > \lambda(d_1, \underline{x}),$$

that is, if

$$\frac{\bar{p}(\underline{x}|0, \tilde{\theta})}{\bar{p}(\underline{x}|\hat{\theta}, \hat{\theta})} > \frac{w_1(\hat{\theta}, \tilde{\theta})}{w_0(\tilde{\theta})} \left[\frac{2\pi n c_{22}(0, \tilde{\theta})}{\Delta} \right]^{.5}. \quad (3.13)$$

This test is the usual likelihood ratio test as shown by Lindley (1961). Let a_n be the value of the right-hand side of (3.13). Now a_n is obtained from the prior probabilities and utilities evaluated at the mle values and the sample size.

We can get a better estimate of a_n by including more terms in the expansion of (3.7) and (3.8). Let

$$\Lambda(x) = \bar{p}(\underline{x}|0, \tilde{\theta}) / \bar{p}(\underline{x}|\hat{\theta}, \hat{\theta})$$

then (3.13) becomes

$$-2\log \Lambda(x) < -\log \left[\frac{w_1^2(\hat{\theta}, \tilde{\theta})}{w_0^2(\tilde{\theta})} \frac{2\pi c_{22}(0, \tilde{\theta})}{n^{-2}\Delta} \right] + \log n. \quad (3.14)$$

Under $H_0: \theta=0$, the left-hand side of (3.14) is distributed as a χ^2 with one degree of freedom. The left-hand side is $O(1)$. With $n \rightarrow \infty$ the right-hand side in square brackets tends to a finite limit and the difference between it and the limit is $O(1/\sqrt{n})$. Lindley then replaced

the expression (3.14) by its limit which is obtained by replacing $\hat{\theta}$ by $\theta=0$ and $\hat{\theta}$ and $\tilde{\theta}$ by θ_0 . Therefore we may write (3.14) as

$$x^2 < A(\theta_0) + \log n \quad (3.15)$$

with $x \sim N(0,1)$ and

$$A(\theta_0) = - \log \left[\frac{w_1^2(0, \theta_0)}{w_0^2(\theta_0)} \frac{2\pi c_{22}(0, \theta_0)}{n^{-2} \Delta} \right] .$$

The right-hand side of (3.14) is a random variable. For the simple hypothesis versus the simple alternative the corresponding expression is a constant, so to make the right-hand side of (3.14) a constant Lindley replaced it by its limit. In practice we do not know θ_0 the true value.

For this problem

$$\alpha = 2 \int_{u_n}^{\infty} 1/\sqrt{2\pi} \exp(-t^2/2) dt \quad (3.16)$$

where $u_n = [A(\theta_0) + \log n]^{.5}$. Integrating (3.16) by parts we are able to bound (3.16) and show that α is asymptotically

$$[(\pi/2 \exp(A(\theta_0))n \log n)^{-.5} \quad . \quad (3.17)$$

Here α depends on θ_0 , the true value, which is unknown. It appears that with the mle of θ we may compute (3.17). With (3.17) Lindley concludes that α is asymptotically proportional to $[n \log n]^{-.5}$, so as n increases α should decrease.

He also remarked that an extension of this argument can be made without difficulty for a finite number of decisions but we will adopt a slightly different point of view.

In this test of a composite versus a composite hypothesis as proposed by Lindley (1961) there are many interesting questions one may ask. What is the required nature of a test here in the presence of a nuisance parameter? Does the procedure have the needed frequency properties? These questions are outside the scope of this study and will not be discussed here.

$C(\alpha)$ tests were also proposed by Neyman (1959 and 1969) and Neyman and Scott (1965) for the test of a hypothesis in the presence of a nuisance parameter. Extensions of Neyman's $C(\alpha)$ test were given by Bartoo and Puri (1967) and Bühler and Puri (1966).

3. Extension of the Lindley argument

In our review of Lindley's work we mentioned the fact that the techniques given may be extended to include the case where there is a finite number of decisions, such as in

the analysis of variance situation. Lindley said that we may find approximations to $\lambda(d_i, \underline{x})$ for each d_i .

Here we will not use this suggestion but will incorporate some of his basic results in our extension.

In the one-way classification model with k treatments we may want to test the following hypothesis from the analysis of variance.

$$H_0: u_1 = u_2 = \dots = u_k$$

or equivalently

$$H_0: u_i - u_j = 0 \quad i \neq j$$

vs $H_1: u_i - u_j \neq 0$.

Let $\delta_{ij} = u_i - u_j$, so we have

$$H_0: \delta_{ij} = 0 \quad i \neq j$$

vs $H_1: \delta_{ij} \neq 0$

and \emptyset is a nuisance parameter. As we have indicated before a comparison between two means u_i and u_j , $i \neq j$ is considered as the problem $P(i, j)$. By formulating the problem in this manner we can reduce the problem of comparison of k means to $h = \binom{k}{2}$ component decision problems where we compare two means at a time. By using the same utility and prior

structure as Lindley (1961) we accept H_0 if $\lambda(d_0, \underline{x}) > \lambda(d_1, \underline{x})$ iff (3.13) holds. So for the component problem which involves the means u_i and u_j , and two decisions d_0 and d_1 we have α^* which is similar in form to (3.16) and is given by

$$\alpha^* = 2 \int_0^\infty \frac{1/\sqrt{(2\pi)} \cdot \exp\{-t^2/2\} dt}{[A(\theta_0) + \log n]^{.5}}$$

$$= [\pi/2 \exp\{A(\theta_0)\} \cdot n \log n]^{-1/2}$$

$$\text{where } A(\theta_0) = -\log \left[\frac{w_1^2(0, \theta_0)}{w_0^2(\theta_0)} \frac{2\pi c_{22}(0, \theta_0)}{n^{-2} \Delta} \right],$$

$\hat{\delta}_{ij}$ is replaced by $\delta_{ij}=0$, and $\hat{\theta}$ and $\tilde{\theta}$ are replaced by θ_0 . Also $w_0(\theta)$ and $w_1(0, \theta)$ are defined in (3.9) and (3.10).

So $1-\alpha^* = P[|x| < [A(\theta_0) + \log n]^{1/2}]$ where

$$x \sim N(0,1).$$

Since for k -treatments there are $h = \frac{k(k-1)}{2}$

component problems, by an application of a Bonferroni inequality, we achieve an α for the overall problem by using $\alpha^* = \alpha/h^*$ for the component problem where $h^* = h$.

Hence from our previous calculations

$$\alpha^* = \alpha/h = [(\pi/2 \exp\{A(\theta_0)\} n \log n)^{-.5}].$$

Therefore $\alpha = h[\pi/2 \exp\{A(\theta_0)\}n \log n]^{-.5}$. As before we will use the mle value of θ_0 in determining α . Therefore as in the two-decision problem, we see in the multiple comparison problem that α depends on the utility function, the prior distributions, the sample sizes, and the number of samples. The reader may wish to use also an improved Bonferroni inequality. Here the $\alpha^* = \frac{h-1+\alpha}{h}$.

Another recommendation may be that one uses α^* irrespective of the number of treatments the experimenter may have to compare.

It is interesting to note that the above recommendations depend on the choice of the priors, loss function, and the sample sizes.

F. The Waller-Duncan Approach to the Multiple Comparison Problem

Waller and Duncan (1969) gave a Bayesian approach to the symmetric multiple comparison problem. Here we will give the main ideas in this approach. They gave a Bayes rule for the symmetric multiple comparison problem and claimed to have a logic for the choice of error rates.

Consider the one-way classification random effect model

$$x_{ij} = u_i + e_{ij} \quad , \quad \begin{array}{l} i=1, \dots, n, \\ j=1, \dots, r, \end{array}$$

where $e_{ij} \sim \text{NID}(0, \sigma_e^2)$ and $u_i \sim N(0, \sigma_B^2)$. Now

$\bar{X}_i \sim \text{NID}(u_i, \sigma_e^2/r)$ for $i=1, \dots, n$ and s_{eE}^2 is an independent estimate of σ_e^2 such that $f_E s_{eE}^2 / \sigma_e^2 \sim \chi^2(f_E)$.

In Section IIC we reduced the multiple comparison problem to one of a three-decision problem $Q(i, j)$ for all of the $n(n-1)/2$ combinations of (u_i, u_j) considered simultaneously. For the $P(i, j)$ problem we also gave a linear loss function.

For $\underline{X} = (\bar{X}_1, \dots, \bar{X}_n)$, let $y = O\underline{X}$ where O is the Helmert orthogonal matrix with $E(y) = \eta$ $\eta' = (\eta_1, \dots, \eta_{n-1})$.

Consider a normal prior on η_i with mean zero and variance equals σ_B^2 . On the unknown expected mean squares σ_e^2 and $\sigma_e^2 + r\sigma_B^2 = \sigma_T^2$, the authors considered two independent prior distributions which are of the form

$$P_2(\sigma_T^2, \sigma_e^2 | s_{TP}^2, s_{eP}^2, q_P, f_P) = K^{-1} P_3(\sigma_T^2 | s_{TP}^2, q_P) P_3(\sigma_e^2 | s_{eP}^2, f_P) \\ \text{for } 0 < \sigma_e^2 < \sigma_T^2 \quad (3.18)$$

where

$$P_3(\sigma^2 | s^2, m) = \left[2^{m/2} \Gamma(m/2) \right]^{-1} (ms^2/\sigma^2)^{m/2} \exp(-ms^2/(2\sigma^2)) (1/\sigma^2) \\ \text{for } \sigma^2 \in R^+$$

and

$$K = \int_0^\infty \int_{\sigma_e^2}^\infty P_3(\sigma_T^2 | s_{TP}^2, q_P) P_3(\sigma_e^2 | s_{eP}^2, f_P) d\sigma_T^2 d\sigma_e^2.$$

Note the joint density of σ_T^2 and σ_e^2 is derived by considering the joint distribution of σ_T^2 and σ_e^2 obtained from the joint distribution of $\mu_T = (q_p s_{TP}^2)/\sigma_T^2$ and $\mu_e = (f_p s_{eP}^2)/\sigma_e^2$ which are independently distributed as χ^2 with q_p and f_p degrees of freedom respectively.

Now we may combine all these prior distributions over the parameter space $\underline{\gamma}' = (\eta', \sigma_T^2, \sigma_e^2)$ to obtain the prior distribution

$$\lambda(\underline{\gamma}) = P_1(\eta | (\sigma_T^2 - \sigma_e^2)/r) P_2(\sigma_T^2, \sigma_e^2 | s_{TP}^2, s_{eP}^2, q_p, f_p) .$$

Since y_i is distributed normally with mean η_i and variance σ_e^2/r and $(f_E s_{eE}^2)/\sigma_e^2$ is distributed independently of y_i as a Chi Square variable with f_E d.f., we may now write the distribution of $z = (y_1, \dots, y_{n-1}, s_{eE}^2)$

$$f(z | \underline{\gamma}) = \left[\frac{r}{2\pi\sigma_e^2} \right]^{(n-1)/2} \exp \left[- \sum_1^{(n-1)} \frac{(y_i - \eta_i)^2}{2\sigma_e^2/r} \right]$$

$$\cdot \frac{\left(\frac{f_E}{2} \right)^{\frac{f_E}{2}} s_{eE}^{(f_E-2)}}{\Gamma\left(\frac{f_E}{2}\right) \sigma^{f_E}} \exp \left[- \frac{f_E s_{eE}^2}{2\sigma_e^2} \right] .$$

For the two-decision problem $P(i,j)$, an application of Bayes theorem gives the Bayes risk proportional to

$$g_{12}(z) = \int [L(d_{12}^+, \underline{Y}) - L(d_{12}^0, \underline{Y})] f(z|\underline{Y}) \lambda(\underline{Y}) d\underline{Y} .$$

So one would take decision d_{12}^+ if $g_{12}(z) < 0$ and d_{12}^0 if $g_{12}(z) \geq 0$. Using the loss function which is given in (2.5) we have

$$0 > g_{12}(z) = \int_{w^-} k_1 |\delta_{12}| f(z|\underline{Y}) \lambda(\underline{Y}) d\underline{Y} - \int_{w^+} k_2 \delta_{12} f(z|\underline{Y}) \cdot \lambda(\underline{Y}) d\underline{Y}$$

where $w^- = \{\underline{Y} : \delta_{12} \leq 0\}$ and $w^+ = \{\underline{Y} : \delta_{12} > 0\}$. From the above equation we obtain

$$\frac{\int_{w^+} \delta_{12} f(z|\underline{Y}) \lambda(\underline{Y}) d\underline{Y}}{\int_{w^-} |\delta_{12}| f(z|\underline{Y}) \lambda(\underline{Y}) d\underline{Y}} > \frac{k_1}{k_2} = k .$$

Waller (1967) shows that integrating w.r.t.

$\gamma_2 = \eta_2, \dots, \gamma_{n-1} = \eta_{n-1}$, the critical region is given by

$$I_+(t_{12}, F, q, f) / [I_-(t_{12}, F, q, f)] > k$$

where $t_{12} = (\bar{X}_1 - \bar{X}_2) / s_d$, $F = s_T^2 / s_e^2$,

$$s_T^2 = [(n-1)s_{TE}^2 + q_P s_{TP}^2] / q \quad \text{and} \quad s_e^2 = (f_E s_{eE}^2 + f_P s_{eP}^2) / f$$

with $q = (n-1) + q_p$, $f = f_E + f_p$ where

$$s_{TE}^2 = r \sum_1^{n-1} y_i^2 / (n-1) \quad \text{and} \quad s_d^2 = 2s_e^2 / r .$$

Therefore the rule is

\bar{X}_i is significantly greater than \bar{X}_j if $t_{ij} > t^*$

\bar{X}_i is not significantly greater than \bar{X}_j if $t_{ij} \leq t^*$

where for $i, j=1, 2$, $t^*=t(k, F, q, f)$ is the solution for t_{12} of the equation

$$I_+(t_{12}, F, q, f) / [I_-(t_{12}, F, q, f)] = k .$$

We find that the critical t value t^* is the same for all the $P(i, j)$ problems because of the symmetry of the loss function and the prior density.

On a simultaneous application of the above rule to the $P(i, j)$ and the $P(j, i)$ problems we can derive the three decision rule for the $Q(i, j)$ problem which is given as follows

\bar{X}_i is significantly greater than \bar{X}_j if $t_{ij} > t^*$

\bar{X}_i is not significantly different from \bar{X}_j if $|t_{ij}| \leq t^*$

\bar{X}_i is significantly smaller than \bar{X}_j if $t_{ij} < -t^*$.

By the simultaneous application of this three-decision rule to all the $\binom{n}{2}$ pairs of treatments we obtain the Bayes rule for this symmetric multiple comparison problem which is given as follows

\bar{X}_i is significantly greater than \bar{X}_j if $\bar{X}_i - \bar{X}_j > \text{BLSD}$

\bar{X}_i is not significantly different from \bar{X}_j if

$$|\bar{X}_i - \bar{X}_j| \leq \text{BLSD}$$

\bar{X}_i is significantly smaller than \bar{X}_j if $\bar{X}_i - \bar{X}_j < -\text{BLSD}$

where the Bayes Least Significant Difference

$$(\text{BLSD}) = s_d t(k, t, q, f) .$$

The cut-off points $t(k, F, q, f)$ depend on the F ratio, k , q , and f . So they depend on the data. The α used is a random variable. One would like, it seems, an α determined a priori by one's prior opinions and one's loss structure as in the case of a simple hypothesis versus a simple alternative.

We view Waller and Duncan's work as limited and unsatisfactory for the following additional reasons. They considered the case of a one-way classification where the groups are equally replicated. In addition for σ_T^2 they used a density which is proportional to $\sigma_T^2 = \sigma_e^2 + r\sigma_B^2$, but this prior has two difficulties in spite of the analytical convenience that it offers: (i) it does not extend obviously

to the case of unequal sample sizes, (ii) it depends on the sample size r .

There is no reason for an experimenter's prior opinions to depend on the data, though one's prior opinion may well influence the choice of the sample size. Concisely, a prior is chosen to represent the knowledge or beliefs of an experimenter before an experiment. It should not depend on what he plans to undertake next. Therefore, we adopt the view that a prior should be independent of the sample size and ease of integration should not be the motivating force in the choice of a prior dependent on the sample size, in view of the availability of high speed computers.

IV. A BAYES RULE FOR THE MULTIPLE COMPARISON PROBLEM

A. Introduction

The background for this work is the view that the choice of a testing procedure must necessarily be based on prior ideas or partial beliefs about the nature of the true means. Characterizing our prior beliefs, rather than our prior ignorance and using the Waller-Duncan decision theoretic formulation of the problem which is given in Section IIC we propose a solution to the multiple comparison problem. As mentioned, one convenient method of characterizing our prior ideas or partial beliefs is the use of de Finetti's ideas of exchangeability.

Consider the one-way classification model with a common unknown variance and where there are no control treatments. The posterior distribution of $\underline{u} | z$ is derived and estimates of \underline{u} are given. The Bayes rule for the multiple comparison problem is then found with an algorithm for the computation of the critical t values.

B. The Posterior Distribution

Consider the one-way classification model with a common unknown variance and with no control treatments. In the practical situation the variance components are unknown. The Bayesian process requires the assignment of priors to these

components. One method of doing this is the use of exchangeable priors, but as mentioned in Chapter II, the results given in here may be obtained without the use of exchangeable priors.

The observations

$$X_{ij} \sim N(u_i, \sigma_e^2) , \quad i=1, \dots, n ,$$

$$\text{and } j=1, \dots, r_i .$$

For the u_i which are unknown, we will assume conditional on u_0 and σ_B^2 that

$$u_i \sim \text{NID}(u_0, \sigma_B^2) .$$

As a prior for u_0 , we shall use the improper uniform distribution over $(-\infty, \infty)$. It is surmised that this will have little effect, in that one could use a proper normal prior with very large variance. As prior distributions for σ_B^2 and σ_e^2 , which are the unknown variance components for the one-way classification model with a common unknown variance, we use the conjugate inverse- χ^2 family. For given q_2, f_2, s_{e2}^2 , and s_{B2}^2 , we assume that

$$(q_2 s_{B2}^2) / \sigma_B^2 \sim \chi^2(q_2) \quad \text{and} \quad (f_2 s_{e2}^2) / \sigma_e^2 \sim \chi^2(f_2)$$

independently. Here we are using one mode of quantifying our beliefs or prognosis which can also be expressed in other

terms.

From the additive loss structure as discussed in Chapter II the Bayes risk for the multiple comparison problem is the sum of the Bayes risk for the $\frac{n(n-1)}{2}$ component problems. Therefore a minimization of the Bayes risk for the multiple comparison problem reduces to the minimization of the Bayes risk for each of the component problems. It is then necessary only to find the Bayes rule for the component problem $P(i,j)$ and a simultaneous application of this rule to all the component problems gives us the Bayes rule for the overall problem.

To find the Bayes rule for the component problem let us consider the decision problem $P(i,j)$. The experimenter who is minimizing Bayes risk will pick the rule

$$d_{ij}^+ \quad \text{if } g_{i,j}(z) < 0, \quad i \neq j$$

or

$$d_{ij}^0 \quad \text{if } g_{i,j}(z) \geq 0$$

where $g_{i,j}(z)$ is an $(n+3)$ -fold integral and is given by

$$g_{i,j}(z) = \int [L(d_{ij}^+, \gamma) - L(d_{ij}^0, \gamma)] f(z|\gamma) \lambda(\gamma) d\gamma \quad (4.1)$$

with $\lambda(\gamma)$ the prior distribution, $f(z|\gamma)$ the density of the data, $\gamma = (u_0, \underline{u}, \sigma_B^2, \sigma_e^2)$ and $z = (\bar{X}_1, \dots, \bar{X}_n, s_{e1}^2)$. With the

loss function given in (2.5)

$$g_{i,j}(z) = \int_{\omega^-} k_1 |\delta_{ij}| f(z|\gamma) \lambda(\gamma) d\gamma \\ - \int_{\omega^+} k_2 \delta_{ij} f(z|\gamma) \lambda(\gamma) d\gamma$$

where $\omega^- = \{\gamma: \delta_{ij} \leq 0\}$, $\omega^+ = \{\gamma: \delta_{ij} > 0\}$

and $\delta_{ij} = u_i - u_j$. Therefore, we will pick d_{ij}^+ if

$$\frac{\int_{\omega^+} \delta_{ij} f(z|\gamma) \lambda(\gamma) d\gamma}{\int_{\omega^-} |\delta_{ij}| f(z|\gamma) \lambda(\gamma) d\gamma} > k_1/k_2 = k \quad (4.2)$$

So the Bayes rule depends on the solution of (4.2). In this chapter we will solve (4.2).

Let us consider the numerator of the right-hand side of (4.2) which is equal to

$$\int_{\omega^+} (u_i - u_j) \prod_{i=1}^n \left(r_i / (2\pi\sigma_e^2) \right)^{1/2} \exp \left[- \sum (\bar{X}_i - u_i)^2 / (2\sigma_e^2 r_i) \right] \\ \cdot f_1^{f_1/2} s_{e1}^2 (f_1/2)^{-1} \frac{1}{\Gamma(f_1/2) 2^{f_1/2} (\sigma_e^2)^{f_1/2}} \exp \left[- f_1 s_{e1}^2 / (2\sigma_e^2) \right] \\ \cdot 1 / (2\pi\sigma_B^2)^{n/2} \exp \left[- \sum (u_i - u_0)^2 / (2\sigma_B^2) \right] \\ \cdot \left(q_2 s_{B2}^2 / \sigma_B^2 \right)^{q_2/2} (1/\sigma_B^2) \frac{1}{\Gamma(q_2/2) 2^{q_2/2}} \exp \left[- q_2 s_{B2}^2 / (2\sigma_B^2) \right]$$

$$\begin{aligned}
 & \cdot \left(f_2 s_{e2}^2 / \sigma_e^2 \right)^{f_2/2} (1/\sigma_e^2) \frac{1}{\Gamma(f_2/2) 2^{f_2/2}} \exp \left[-f_2 s_{e2}^2 / (2\sigma_e^2) \right] \\
 & \cdot du_0 d\underline{u} d\sigma_e^2 d\sigma_B^2 \quad . \quad (4.3)
 \end{aligned}$$

Now (4.3) is proportional to

$$\int_{\omega} (u_i - u_j) p(u_0, \underline{u}, \sigma_B^2, \sigma_e^2 | z) du_0 d\underline{u} d\sigma_e^2 d\sigma_B^2$$

where $p(u_0, \underline{u}, \sigma_B^2, \sigma_e^2 | z)$ is the posterior distribution of $\gamma | z$.

We find

$$\begin{aligned}
 p(u_0, \underline{u}, \sigma_B^2, \sigma_e^2 | z) & \propto (\sigma_e^2)^{-1/2(n + f_1 + f_2 + 2)} \\
 & \cdot (\sigma_B^2)^{-1/2(n + q_2 + 2)} \\
 & \cdot \exp \left[-\frac{1}{2\sigma_e^2} \left(\sum_{i=1}^n r_i (\bar{X}_i - u_i)^2 + f_1 s_{e1}^2 + f_2 s_{e2}^2 \right) \right] \\
 & \cdot \exp \left[-\frac{1}{2\sigma_B^2} \left(\sum_i (u_i - u_0)^2 + q_2 s_{B2}^2 \right) \right] \quad . \quad (4.4)
 \end{aligned}$$

Using

$$\frac{\sum (u_i - u_0)^2}{\sigma_B^2} = \frac{\sum (u_i - \underline{u})^2 + \sum (u_0 - \underline{u})^2}{\sigma_B^2}$$

where $\underline{u} = \sum u_i/n$, we integrate (4.4) with respect to u_0 . This integration replaces $\sum (u_i - u_0)^2$ in (4.4) by $\sum (u_i - \underline{u})^2$ and multiplies the expression by σ_B , so (4.4) becomes

$$\begin{aligned}
 & (\sigma_e^2)^{-\frac{1}{2}(n+f+2)} \cdot (\sigma_B^2)^{-\frac{1}{2}(n+q_2+1)} \\
 & \cdot \exp \left[-\frac{1}{2\sigma_e^2} \left(\sum_{i=1}^n r_i (\bar{X}_i - u_i)^2 + fs_e^2 \right) \right] \\
 & \cdot \exp \left[-\frac{1}{2\sigma_B^2} \left(\sum_i (u_i - \underline{u})^2 + q_2 s_{B2}^2 \right) \right] ,
 \end{aligned}$$

where $f_1 + f_2 = f$ and $f_1 s_{e1}^2 + f_2 s_{e2}^2 = fs_e^2$. Now integrating with respect to σ_e^2 and σ_B^2 and using

$$\int_0^\infty (\sigma^2)^{-k} \exp(-A/\sigma^2) d\sigma^2 \propto A^{-(k-1)} ,$$

we obtain

$$\begin{aligned}
 p(\underline{u}|z) & \propto \left[\sum_i r_i (\bar{X}_i - u_i)^2 + fs_e^2 \right]^{-\frac{1}{2}[n+f]} \\
 & \cdot \left[\sum_i (u_i - \underline{u})^2 + q_2 s_{B2}^2 \right]^{-\frac{1}{2}(q_2+n-1)} .
 \end{aligned}$$

So the posterior distribution of \underline{u} is a product of two multivariate t-distributions. This distribution is similar to the one obtained by Lindley (1971a). Now

$$p(\underline{u}|z) \propto \left[1 + \frac{\sum_{i=1}^n r_i (\bar{X}_i - u_i)^2}{f s_e^2} \right]^{-\frac{1}{2}(n+f)} \cdot \left[1 + \frac{\sum_i (u_i - u.)^2}{q_2 s_{B2}^2} \right]^{-\frac{1}{2}(n-1+q_2)} \quad (4.5)$$

It is instructive at this stage to note that distributions of this type have been discussed by Tiao and Zellner (1964) and Fisher (1941, 1961a,b) in connection with the Behren's integral. We propose to use Fisher's methods which were applied also by Tiao and Zellner (1964) to find asymptotic expressions for the mean and variance of the above posterior distribution. In the Appendix we give some of the asymptotic expressions for the posterior distribution and the marginal distribution.

C. Moments of the Posterior Distribution

The moments of the posterior distribution may be found by expanding (4.5) as a double inverse power series in f and $q=q_2$ which are the degrees of freedom for the distribution. Let

$$v_1 = f, \quad v_2 = q, \quad s_{B2}^2 = s_B^2,$$

$$\tilde{u}_i = \left(\frac{r_i}{s_e^2} + \frac{1}{s_B^2} \right)^{-1} \left(\frac{r_i \bar{X}_i}{s_e^2} + \frac{u.}{s_B^2} \right),$$

$$v_{ii}^{-1} = \left(\frac{r_i}{s_B^2} + \frac{1}{s_B^2} \right), \quad \omega_i = \frac{v_{ii} r_i}{s_e^2}$$

$$\xi_i = \frac{r_i}{s_e^2} (\tilde{u}_i - \bar{X}_i)^2, \quad \gamma_i = \frac{1}{s_B^2} v_{ii},$$

$$\rho_i = \frac{1}{s_B^2} (\tilde{u}_i - u.)^2, \quad s = n-1$$

and

$$f(\underline{u}; \tilde{u}_i, v_{ii}) = \left[\prod_{i=1}^n v_{ii}^{-1} \right]^{1/2} (2\pi)^{-n/2} \exp \left[-\frac{1}{2} \left[\sum_{i=1}^n (u_i - \tilde{u}_i)^2 v_{ii}^{-1} \right] \right]. \quad (4.6)$$

As shown in the Appendix we may write the posterior distribution which is given in (9.51) as

$$p(\underline{u}|\underline{z}) = f(\underline{u}; \tilde{u}_i, v_{ii}) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d_{ij} v_1^{-i} v_2^{-j}$$

for $-\infty < \underline{u} < \infty$, where the quantities d_{ij} are given in (9.20) to (9.25).

Using the first three terms of the power series in v_1^{-1} and v_2^{-1} we have

$$p(\underline{u}|\underline{z}) = f(\underline{u}; \tilde{\underline{u}}_i, v_{ii}) (1 + v_1^{-1} d_{10} + v_2^{-1} d_{01}) + o(1) + o(1) \quad (4.7)$$

where

$$d_{10} = \frac{1}{4} \left[Q_1^2 - 2nQ_1 - 2 \left(\sum_{i=1}^n (\omega_i^2 + 2\omega_i \xi_i) \right) - \left(\sum_{i=1}^n (\omega_i + \xi_i) \right)^2 + 2n \left(\sum_{i=1}^n (\omega_i + \xi_i) \right) \right]$$

and

$$d_{01} = \frac{1}{4} \left[Q_2^2 - 2(n-1)Q_2 - 2 \left(\sum_{i=1}^n (\gamma_i^2 + 2\rho_i \gamma_i) \right) - \left(\sum_{i=1}^n (\gamma_i + \rho_i) \right)^2 + 2(n-1) \left(\sum_{i=1}^n (\gamma_i + \rho_i) \right) \right],$$

$$\text{with } Q_1 = \sum_{i=1}^n r_i (\bar{X}_i - u_i)^2 / s_e^2 \quad \text{and} \quad Q_2 = \sum_{i=1}^n (u_i - u.)^2 / s_B^2.$$

The posterior distribution is the product of a multivariate normal distribution and a power series in v_1^{-1} and v_2^{-1} . As v_1 and v_2 get very large, all terms of the power series except the leading one vanishes so that the posterior distribution is asymptotically distributed

$$N_n(\tilde{\underline{u}}, M^{-1}) \quad \text{where} \quad \tilde{\underline{u}} = (\tilde{u}_1, \dots, \tilde{u}_n) \quad \text{and}$$

$$M = \text{diag} \left(\frac{r_i}{s_e^2} + \frac{1}{s_B^2} \right).$$

The terms in the power series may be interpreted for finite values of v_1 and v_2 as the corrections in the normal approximations to the posterior distribution $p(\underline{u}|z)$.

From the posterior distribution given in (4.7) we will give the marginal distribution of \underline{u}_ℓ where $\underline{u} = (\underline{u}_\ell; \underline{u}_r)$. This distribution is derived in the Appendix and is given by (9.71) as

$$p(\underline{u}_\ell|z) = f(\underline{u}_\ell; \tilde{u}_i, v_{ii}) \left[1 + v_1^{-1} \delta_{10} + v_2^{-1} \delta_{01} \right] + o(1) + o(1) \quad (4.8)$$

where

$$\begin{aligned} \delta_{10} = \frac{1}{4} & \left[-2 \sum_{i=1}^{\ell} (\omega_i^2 + 2\omega_i \xi_i) - \left\{ \sum_{i=1}^{\ell} (\omega_i + \xi_i) - \sum_{i=1}^{\ell} (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right\} \right. \\ & \cdot \left. \left\{ \sum_{i=1}^{\ell} (\omega_i + \xi_i) + 2 \sum_{i=\ell+1}^n (\omega_i + \xi_i) + \sum_{i=1}^{\ell} (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} - 2n \right\} \right] \end{aligned}$$

and

$$\begin{aligned} \delta_{01} = \frac{1}{4} & \left[-2 \sum_{i=1}^{\ell} (\gamma_i^2 + 2\rho_i \gamma_i) - \left\{ \sum_{i=1}^{\ell} (\gamma_i + \rho_i) - \sum_{i=1}^{\ell} (u_i - u_{\cdot})^2 / s_B^2 \right\} \right. \\ & \cdot \left. \left\{ \sum_{i=1}^{\ell} (\gamma_i + \rho_i) + 2 \sum_{i=\ell+1}^n (\gamma_i + \rho_i) + \sum_{i=1}^{\ell} (u_i - u_{\cdot})^2 / s_B^2 - 2s \right\} \right] . \end{aligned}$$

This distribution is a polynomial in u_i .

As shown in the Appendix by (9.80), we may obtain the joint density of the i -th and the j -th mean, $i \neq j$, for all $i, j = 1, \dots, n$, given the data. This joint density of u_i and u_j given the data is denoted by

$$\begin{aligned}
p(u_i, u_j | z) &= f(u_i, u_j; \tilde{u}_i, v_{ii}) \\
&\cdot \left[1 + \frac{1}{4} v_1^{-1} \left[-2(\omega_i^2 + 2\omega_i \xi_i + \omega_j^2 + 2\omega_j \xi_j) - (\psi_i' + \psi_j' - (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right. \right. \\
&\quad - (u_j - \bar{X}_j)^2 \frac{r_j}{s_e^2}) \left(\psi_i' + \psi_j' + 2 \sum_{\substack{k=1 \\ k \neq i, j}}^n \psi_k' + (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} + (u_j - \bar{X}_j)^2 \frac{r_j}{s_e^2} \right. \\
&\quad \left. \left. - 2n \right) \right] \\
&+ \frac{1}{4} v_2^{-1} \left[-2(\gamma_i^2 + 2\gamma_i \rho_i + \gamma_j^2 + 2\gamma_j \rho_j) - (\psi_i'' + \psi_j'' - (u_i - u.)^2 \frac{1}{s_B^2} \right. \\
&\quad - (u_j - u.)^2 \frac{1}{s_B^2}) \cdot \left(\psi_i'' + \psi_j'' + 2 \sum_{\substack{k=1 \\ k \neq i, j}}^n \psi_k'' + (u_i - u.)^2 \frac{1}{s_B^2} \right. \\
&\quad \left. \left. + (u_j - u.)^2 \frac{1}{s_B^2} - 2(n-1) \right) \right] + o(1) + o(1), \tag{4.9}
\end{aligned}$$

where $\psi_i' = \omega_i + \xi_i$ and $\psi_i'' = \gamma_i + \rho_i$.

To obtain the posterior density of $u_i | z$ we integrate with respect to u_j , $j \neq i$ and using Lemma (9.3) from the Appendix we obtain

$$p(u_i|z) = f(u_i; \tilde{u}_i, v_{ii})$$

$$\begin{aligned} & \cdot \left[1 + \frac{1}{4} v_1^{-1} \left[-2(\omega_i^2 + 2\omega_i \xi_i) - \left(\psi_i' - (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right) \left(\psi_i' + 2 \sum_{\substack{j=1 \\ j \neq i}}^n \psi_j' \right) \right. \right. \\ & \quad \left. \left. + (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} - 2n \right) \right] \\ & + \frac{1}{4} v_2^{-1} \left[-2(\gamma_i^2 + 2\rho_i \gamma_i) - \left(\psi_i'' - (u_i - u.)^2 / s_B^2 \right) \left(\psi_i'' + 2 \sum_{\substack{j=1 \\ j \neq i}}^n \psi_j'' \right) \right. \\ & \quad \left. \left. + (u_i - u.)^2 / s_B^2 - 2(n-1) \right) \right] \Bigg] + o(1) + o(1). \end{aligned} \quad (4.10)$$

Using the expressions for the moments of a normal variable, we found the asymptotic expression for the moments of $p(u_i|z)$ in the Appendix. The moments are given by (9.88) as

$$\begin{aligned} E(u_i|z) = & \tilde{u}_i + \frac{1}{4} v_1^{-1} \left[\left(\frac{r_i}{s_e^2} \right)^2 \left(-3 v_{ii}^2 \tilde{u}_i + 15 v_{ii} \tilde{u}_i + 4 v_{ii} \tilde{u}_i^3 \right. \right. \\ & \left. \left. - 12 \bar{X}_i v_{ii}^2 - 12 \bar{X}_i \tilde{u}_i^2 v_{ii} + 12 \bar{X}_i^2 \tilde{u}_i v_{ii} - 4 \bar{X}_i^3 v_{ii} \right) + \left(\frac{r_i}{s_e^2} \right) \right] \end{aligned}$$

$$\begin{aligned}
& \cdot \left(\sum_{\substack{j=1 \\ j \neq i}}^n \frac{r_j}{s_e^2} (v_{jj} + (\tilde{u}_j - \bar{X}_j)^2) \right) \left(4\tilde{u}_i v_{ii} - 4\bar{X}_i v_{ii} \right) + \left(2n \frac{r_i}{s_e^2} \right) \\
& \cdot \left(-2v_{ii} \tilde{u}_i + 2\bar{X}_i v_{ii} \right) \left] + \frac{1}{4} v_2^{-1} \left[\left(\frac{1}{s_B^2} \right)^2 \right. \\
& \left(-3v_{ii}^2 \tilde{u}_i + 15v_{ii} \tilde{u}_i + 4v_{ii} \tilde{u}_i^3 - 12u \cdot v_{ii}^2 - 12u \cdot \tilde{u}_i^2 v_{ii} \right. \\
& \left. \left. + 12u^2 \tilde{u}_i v_{ii} - 4u^3 v_{ii} \right) + \left(\frac{1}{s_B^2} \right) \right. \\
& \cdot \left(\sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{s_B^2} (v_{jj} + (\tilde{u}_j - u)^2) \right) \left(4\tilde{u}_i v_{ii} - 4u \cdot v_{ii} \right) \\
& \left. + \left(\frac{2n}{s_B^2} \right) \left(-2v_{ii} \tilde{u}_i + 2u \cdot v_{ii} \right) \right] + o(1) + o(1) . \quad (4.11)
\end{aligned}$$

To find the variance of $u_i | z$ we need to find the second raw moment from which we may calculate the variance using the fact that

$$\text{Var}(u_i | z) = E(u_i^2 | z) - E^2(u_i | z) .$$

Therefore as given in the Appendix by (9.89), the

$$\begin{aligned}
E(u_i^2 | z) = & (v_{ii} + \tilde{u}_i^2) + \frac{1}{4} v_1^{-1} \left[-2 \left(\frac{r_i}{s_e^2} \right)^2 (v_{ii}^2 + 2v_{ii}(\tilde{u}_i - \bar{X}_i)^2) (v_{ii} + \tilde{u}_i^2) \right. \\
& - (v_{ii} + \tilde{u}_i^2) \psi_i' \left(\psi_i' + 2 \sum_{\substack{j=1 \\ j \neq i}}^n \psi_j' - 2n \right) + \left(\frac{r_i}{s_e^2} \right) \\
& \cdot \left(2 \sum_{\substack{j=1 \\ j \neq i}}^n \psi_j' - 2n \right) \\
& \cdot \left(3v_{ii}^2 + 6\tilde{u}_i^2 v_{ii} + \tilde{u}_i^4 - 2\bar{X}_i \tilde{u}_i (3v_{ii} + \tilde{u}_i^2) + \bar{X}_i^2 (v_{ii} + \tilde{u}_i^2) \right) \\
& + \left(\frac{r_i}{s_e^2} \right)^2 \left(15v_{ii}^3 + 45v_{ii}^2 \tilde{u}_i^2 + 15v_{ii} \tilde{u}_i^4 + \tilde{u}_i^6 - 4\bar{X}_i \right. \\
& \cdot (15v_{ii} \tilde{u}_i + 10v_{ii} \tilde{u}_i^3 + \tilde{u}_i^5) + 6\bar{X}_i^2 (3v_{ii}^2 + 6\tilde{u}_i^2 v_{ii} + \tilde{u}_i^4) \\
& \left. \left. - 4\bar{X}_i^3 (3v_{ii} \tilde{u}_i + \tilde{u}_i^3) + \bar{X}_i^4 (v_{ii} + \tilde{u}_i^2) \right) \right] \\
& + \frac{1}{4} v_2^{-1} [\text{coefficient}] + o(1) + o(1) \tag{4.12}
\end{aligned}$$

where the coefficient of v_2^{-1} is easily obtained. From the above two expressions we may find the variance of $(u_i | z)$.

Similarly we may obtain the higher moments of $(u_i|z)$. It is informative to note that Lindley (1971a) using the same hierarchial form of prior structure derived a Bayes estimate of $(u_i|z)$. In our notation he found the mode for the posterior distribution given by (4.5). Consider Equation (4.5),

$$p(\underline{u}|z) \propto \left[f s_e^2 + \sum_{i=1}^n r_i (u_i - \bar{X}_i)^2 \right]^{-1/2(n+f)}$$

$$\left[q_2 s_{B2}^2 + \sum_i (u_i - u.)^2 \right]^{-1/2(n-1+q_2)} .$$

Let $q_2 = q$ and $s_{B2}^2 = s_B^2$. To find the mode of this distribution let us differentiate it with respect to u_i or let us differentiate its logarithm and set the result to zero. Then for

$$s_E^2 = \left[f s_e^2 + \sum_{i=1}^n r_i (\bar{X}_i - u_i)^2 \right] / (n+f) \quad (4.13)$$

$$\text{and } s_{B*}^2 = \left[q s_B^2 + \sum_{i=1}^n (u_i - u.)^2 \right] / (n-1+q) \quad (4.14)$$

which are the modal estimates of σ_e^2 and σ_B^2 respectively, we obtain

$$u_i^L = \left(\frac{u_{\cdot}}{s_{B*}^2} + \frac{r_i \bar{X}_i}{s_E^2} \right) \left(\frac{r_i}{s_E^2} + \frac{1}{s_{B*}^2} \right)^{-1} \quad (4.15)$$

where $u_{\cdot} = \sum w_i \bar{X}_i / \sum w_i$ (4.16)

with $w_i = r_i / (r_i s_{B*}^2 + s_E^2)$.

If we consider the first term in (4.11) as an estimate of u_i we find

$$\tilde{u}_i = \left(\frac{u_{\cdot}}{s_B^2} + \frac{r_i \bar{X}_i}{s_e^2} \right) \left(\frac{r_i}{s_e^2} + \frac{1}{s_B^2} \right)^{-1} \quad (4.17)$$

where $u_{\cdot} = \sum w_i \bar{X}_i / \sum w_i$ and $w_i = r_i / (r_i s_B^2 + s_e^2)$.

It is interesting to note that by considering the first term of (4.11) as an estimate of u_i we find that \tilde{u}_i and Lindley's estimate differ in their definitions of the estimates of σ_e^2 and σ_B^2 . The estimates have the same structure, i.e., a weighted average of \bar{X}_i and the overall mean, but the weights are different.

The reader must note that Lindley's estimates of σ_e^2 , σ_B^2 , and u_i are functions of the u_i , so that it is extremely difficult to obtain estimates of u_i , but \tilde{u}_i depends on the data and the prior information of the experimenter and is very easy to calculate.

Since Lindley's estimates have the same structure as the first term of our estimate (4.11) in this sense, we may consider Lindley's estimate as a special case of ours. If we apply our method of analysis to Lindley and Smith (1972) and Smith (1973a, 1973b), we may obtain general estimates of the parameters of interest and show that the estimates of Lindley and Smith are only special cases. Our estimate, though not simpler than Lindley's gives more information for the estimation of u_i . In the age of computers these estimates are easy to compute and as mentioned by Lindley (1971a) are more accurate than the modal estimates. Lindley's estimates are only likely to be good if the samples are fairly large and the resulting posterior distributions are approximately normal, whereas our estimate can be used for small samples also.

D. A Bayes Rule

As was shown before to find the Bayes rule for the multiple comparison problem, it is required to reduce (4.2). Integrating the numerator and denominator of (4.2) with respect to $u_0, \sigma_e^2, \sigma_B^2$ and u_ℓ , where $\ell = 1, \dots, n$ and $\ell \neq i$ and j , we obtain

$$\frac{\int_{\{\delta_{ij} > 0\}} \delta_{ij} p(u_i, u_j | z) d\delta_{ij}}{\int_{\{\delta_{ij} \leq 0\}} |\delta_{ij}| p(u_i, u_j | z) d\delta_{ij}} > \frac{k_1}{k_2} = k \quad (4.18)$$

where $\delta_{ij} = (u_i - u_j)$ for $i \neq j$.

$$\text{Let } I_+(z) = \int_{\{\delta_{ij} > 0\}} \delta_{ij} p(u_i, u_j | z) d\delta_{ij} \quad (4.19)$$

$$\text{and } I_-(z) = \int_{\{\delta_{ij} \leq 0\}} |\delta_{ij}| p(u_i, u_j | z) d\delta_{ij} \quad (4.20)$$

Then we will make decision

$$d_{ij}^+ : u_i > u_j \quad \text{if} \quad \frac{I_+(z)}{I_-(z)} > k$$

or $d_{ij}^0 : u_i$ is unranked relative to u_j if $\frac{I_+(z)}{I_-(z)} \leq k$.

We will reduce (4.18) by first reducing (4.19) which will also reduce (4.20) after a few manipulations

From the symmetry of the loss function and the prior distributions, it is evident that once a Bayes solution is obtained for the $P(1,2)$ problem we will be able to give a Bayes solution for the $P(i,j)$ problem with $i \neq j$, $i, j = 1, \dots, n$. In other words, from the symmetry in the multiple comparison problem, it is only necessary to consider in detail the $P(1,2)$ problem.

Before we attempt to integrate (4.19), it will be informative to find the distribution of $p(\delta | z)$ from

$p(u_1, u_2 | z)$ where $\delta = u_1 - u_2$.

From (4.8) with $\beta=2$, we may write the distribution of $p(u_1, u_2 | z)$ which is given by

$$\begin{aligned}
 p(u_1, u_2 | z) &= f(u_1, u_2; \tilde{u}_i, v_{ii}) \\
 &\cdot \left[1 + \frac{1}{4} v_1^{-1} \left[-2 \sum_{i=1}^2 (\omega_i^2 + 2\omega_i \xi_i) \right. \right. \\
 &\quad \left. \left. - \left(\sum_{i=1}^2 (\omega_i + \xi_i) - \sum_{i=1}^2 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right) \right. \right. \\
 &\quad \left. \left. \cdot \left(\sum_{i=1}^2 (\omega_i + \xi_i) + 2 \sum_{i=3}^n (\omega_i + \xi_i) + \sum_{i=1}^2 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} - 2n \right) \right] \right. \\
 &\quad \left. + \frac{1}{4} v_2^{-1} \left[-2 \sum_{i=1}^2 (\gamma_i^2 + 2\rho_i \gamma_i) - \left(\sum_{i=1}^2 (\gamma_i + \rho_i) \right. \right. \right. \\
 &\quad \left. \left. - \sum_{i=1}^2 (u_i - u.)^2 / s_B^2 \right) \right. \\
 &\quad \left. \left. \cdot \left(\sum_{i=1}^2 (\gamma_i + \rho_i) + 2 \sum_{i=3}^n (\gamma_i + \rho_i) \right. \right. \right. \\
 &\quad \left. \left. \left. + \sum_{i=1}^2 (u_i - u.)^2 / s_B^2 - 2(n-1) \right) \right] \right] \\
 &+ o(1) + o(1) \quad . \tag{4.21}
 \end{aligned}$$

Let us make the transformation from (u_1, u_2) to (δ, u_2) where $\delta = u_1 - u_2$. The Jacobian of this transformation is 1. The

distribution of $p(\delta|z)$ is obtained by making the required transformation and integrating the distribution with respect to u_2 . The computations are given in the Appendix leading to Equation (9.110).

This posterior distribution may be written as

$$p(\delta|z) = \left[\sqrt{(2\pi)} / \left(\sum_{i=1}^2 v_{ii} \right) \right]^{-1} \exp \left[-(\delta - (\tilde{u}_1 - \tilde{u}_2))^2 / (2 \sum_{i=1}^2 v_{ii}) \right] \\ \cdot \left[1 + \frac{1}{4} v_1^{-1} (g_{10} + \delta g_{11} + \delta^2 g_{12} + \delta^3 g_{13} + \delta^4 g_{14}) \right. \\ \left. + \frac{1}{4} v_2^{-1} (g_{20} + \delta g_{21} + \delta^2 g_{22} + \delta^3 g_{23} + \delta^4 g_{24}) \right] \\ + o(1) + o(1) , \quad (4.22)$$

$$\text{where} \quad \tilde{u}_i = v_{ii} \left(\frac{r_i \bar{X}_i}{s_e^2} + \frac{u_i}{s_B^2} \right) \quad (4.23)$$

$$\text{with} \quad v_{ii}^{-1} = \left(\frac{r_i}{s_e^2} + \frac{1}{s_B^2} \right) ,$$

$$g_{10} = - \sum_{i=1}^2 (\omega_i^2 + 2\xi_i \omega_i) - \sum_{i=1}^2 \psi_i' \left(\sum_{i=1}^2 \psi_i' + 2 \sum_{i=3}^n \psi_i' - 2n \right) \\ + \left(\sum_{i=3}^n \psi_i' - n \right) \left(\left(2 \frac{r_1}{s_e^2} \right) \cdot \left(\bar{X}_1^2 + b^2 - av_{11} - 2b\bar{X}_1 \right) + \left(2 \frac{r_2}{s_e^2} \right) \right. \\ \left. (b^2 - av_{11} - 2\bar{X}_2 b + \bar{X}_2^2) \right) + 2 \frac{r_1 r_2}{s_e^4} \left[3a^2 v_{11}^2 - 6b^2 av_{11} \right]$$

$$\begin{aligned}
& + b^4 - 2(\bar{X}_1 + \bar{X}_2)(b^3 - 3abv_{11}) + (\bar{X}_1^2 + 4\bar{X}_2\bar{X}_1 + \bar{X}_2^2)(b^2 - av_{11}) \\
& - 2b(\bar{X}_1\bar{X}_2^2 + \bar{X}_1^2\bar{X}_2) + \bar{X}_2^2\bar{X}_1^2 \Big] + \left(\frac{r_2}{s_e^2}\right)^2 \Big[3a^2v_{11}^2 - 6b^2av_{11} + b^4 \\
& - 4\bar{X}_2(b^2 - 3abv_{11}) + 6\bar{X}_2^2b^2 - 6\bar{X}_2^2av_{11} - 4\bar{X}_2^3b + \bar{X}_2^4 \Big] \\
& + \left(\frac{r_1}{s_e^2}\right) \Big[3a^2v_{11}^2 - 6b^2av_{11} + b^4 - 4\bar{X}_1b^2 + 12\bar{X}_1abv_{11} + 6\bar{X}_1^2b^2 \\
& - 6\bar{X}_1^2av_{11} - 4b\bar{X}_1^3 + \bar{X}_1^3 \Big] , \tag{4.24}
\end{aligned}$$

$$\begin{aligned}
g_{11} = & \left(\sum_{i=3}^n \psi_i' - n \right) \left(2 \frac{r_1}{s_e^2} (-2\bar{X}_1 + 2ab + 2b - 2a\bar{X}_1) + 2 \frac{r_2}{s_e^2} (2ab - 2a\bar{X}_2) \right) \\
& + 2 \frac{r_1 r_2}{s_e^4} \Big[-12a^2bv_{11} + 4ab^3 + 2b^3 - 6abv_{11} - 2(\bar{X}_1 + \bar{X}_2) \\
& \cdot (-3a^2v_{11} + 3ab^2) - 2\bar{X}_1(b^2 - av_{11}) + 2ab(\bar{X}_1^2 + 4\bar{X}_2\bar{X}_1 + \bar{X}_2^2) \\
& - 4\bar{X}_2(b^2 - av_{11}) - 2a\bar{X}_1\bar{X}_2^2 - 2a\bar{X}_1^2\bar{X}_2 + 2b\bar{X}_2^2 + 4b\bar{X}_1\bar{X}_2 - 2\bar{X}_2^2\bar{X}_1 \Big] \\
& + \left(\frac{r_2}{s_e^2}\right)^2 \Big[-12a^2bv_{11} + 4ab^3 - 4\bar{X}_2(-3a^2v_{11} + 3ab^2) \\
& + 12\bar{X}_2^2ab - 4\bar{X}_2^3a \Big] \\
& + \left(\frac{r_1}{s_e^2}\right)^2 \Big[-12a^2bv_{11} + 4ab^3 + 4b^3 - 12abv_{11} + 4\bar{X}_1(3a^2v_{11} - 3ab^2)
\end{aligned}$$

$$-12\bar{X}_1(b^2 - av_{11}) + 12\bar{X}_1^2 ab - 4a\bar{X}_1^3 + 12b\bar{X}_1^2 - 4\bar{X}_1^3 \Big], \quad (4.25)$$

$$g_{12} = \left(\sum_{i=3}^n \psi_i^! - n \right) \left[2 \frac{r_1}{s_e^2} (1 + a^2 + 2a) + 2a^2 \frac{r_2}{s_e^2} \right] + 2 \frac{r_1 r_2}{s_e^4} \\ \cdot \left[-6a^2 v_{11} + 6a^2 b^2 - 6a^2 v_{11} + 6ab^2 - 2(\bar{X}_1 + \bar{X}_2)(2a^2 b + ba^2) \right. \\ \left. + b^2 - av_{11} - 4\bar{X}_1 ab + a^2(\bar{X}_1^2 + 4\bar{X}_2 \bar{X}_1 + \bar{X}_2^2) - 8ab\bar{X}_2 + 2a\bar{X}_2^2 \right. \\ \left. + 4a\bar{X}_1 \bar{X}_2 - 2b\bar{X}_2 + \bar{X}_2^2 \right] + \left(\frac{r_2}{s_e^2} \right)^2 \left[-6a^3 v_{11} + 6a^2 b^2 \right. \\ \left. - 4\bar{X}_2(2a^2 b + ba^2) + 6a^2 \bar{X}_2^2 \right] + \left(\frac{r_1}{s_e^2} \right)^2 \left[-6a^3 v_{11} + 6a^2 b^2 \right. \\ \left. + 4(-3a^2 v_{11} + 3ab^2) - 4\bar{X}_1(2a^2 b + ba^2) + 6b^2 - 6av_{11} \right. \\ \left. - 24\bar{X}_1 ab + 6\bar{X}_1^2 a^2 + 12a\bar{X}_1^2 - 12b\bar{X}_1 + 6\bar{X}_1^2 \right] , \quad (4.26)$$

$$g_{13} = 2 \frac{r_1 r_2}{s_e^4} (4a^3 b + 4a^2 b + 2ba^2 - 2(\bar{X}_1 + \bar{X}_2)a^3 + 2ab - 2\bar{X}_1 a^2 - 4\bar{X}_2 a^2 \\ - 2a\bar{X}_2) + \left(\frac{r_2}{s_e^2} \right)^2 \cdot (4a^3 b - 4\bar{X}_2 a^3) + \left(\frac{r_1}{s_e^2} \right)^2 (4a^3 b + 8a^2 b \\ + 4ba^2 - 4\bar{X}_1 a^3 + 12ab - 12\bar{X}_1 a^2 - 12a\bar{X}_1 + 4b - 4\bar{X}_1) ; \quad (4.27)$$

$$\begin{aligned}
g_{14} = & 2 \frac{r_1 r_2}{s_e^4} (a^4 + 2a^3 + a^2) + \left(\frac{r_2}{s_e^2} \right)^2 a^4 \\
& + \left(\frac{r_1}{s_e^2} \right)^2 (a^4 + 4a^3 + 6a^2 + 4a + 1) , \quad (4.28)
\end{aligned}$$

$$\begin{aligned}
g_{20} = & -2 \sum_{i=1}^2 (\gamma_i^2 + 2\rho_i \gamma_i) - \sum_{i=1}^2 \psi_i'' \left(\sum_{i=1}^2 \psi_i'' + 2 \sum_{i=3}^n \psi_i'' - 2(n-1) \right) \\
& + \frac{2}{s_B^2} \left(\sum_{i=3}^n \psi_i'' - (n-1) \right) \left[2u^2 + 2b^2 - 2av_{11} - 4u \cdot b \right] + \frac{2}{s_B^4} \\
& \cdot \left[3a^2 v_{11}^2 - 6b^2 av_{11} + b^4 - 4u \cdot b^3 + 12u \cdot abv_{11} + 6u^2 b^2 \right. \\
& \left. - 6u^2 av_{11} - 4u^3 b + u^4 \right] + \frac{1}{s_B^4} \left[3a^2 v_{11}^2 - 6b^2 av_{11} + b^4 \right. \\
& \left. - 4u \cdot b^3 + 12u \cdot abv_{11} + 6u^2 b^2 - 6u^2 av_{11} - 4u^3 b + u^4 + 3a^2 v_{11}^2 \right. \\
& \left. - 6b^2 av_{11} + b^4 - 4u \cdot b^2 + 12u \cdot abv_{11} + 6u^2 b^2 - 6u^2 av_{11} \right. \\
& \left. - 4bu^3 + u^3 \right] , \quad (4.29)
\end{aligned}$$

$$\begin{aligned}
g_{21} = & \frac{2}{s_B^2} \left(\sum_{i=3}^n \psi_i'' - (n-1) \right) (-2u \cdot + 4ab + 2b - 4u \cdot a) + \frac{2}{s_B^4} \left[-12a^2 bv_{11} \right. \\
& \left. + 4ab^3 + 2b^3 - 6abv_{11} + 12u \cdot a^2 v_{11} - 12u \cdot ab^2 - 2u \cdot b^2 + 2u \cdot av_{11} \right.
\end{aligned}$$

$$\begin{aligned}
& + 12abu^2 - 4u.b^2 + 4u.av_{11} - 4au^3 + 2bu^2 + 4bu^2 - 2u^3 \Big] \\
& + \frac{1}{s_B^4} \Big[-12a^2bv_{11} + 4ab^3 + 12u.a^2v_{11} - 12u.ab^2 + 12u^2ab \\
& - 4u^3a - 12a^2bv_{11} + 4ab^3 + 4b^3 - 12abv_{11} + 12u.a^2v_{11} \\
& - 12u.ab^2 - 12u.b^2 + 12u.av_{11} + 12u^2ab - 4au^3 + 12bu^2 \\
& - 12u^3 \Big] , \tag{4.30}
\end{aligned}$$

$$\begin{aligned}
g_{22} = & \frac{2}{s_B^2} \left(\sum_{i=3}^n \psi_i'' - (n-1) \right) (1 + 2a^2 + 2a) + \frac{2}{s_B^4} \Big[-6a^3v_{11} + 6a^2b^2 \\
& - 6a^2v_{11} + 6ab^2 - 8u.a^2b - 4u.ba^2 + b^2 - av_{11} - 4u.ab + 6u^2a^2 \\
& - 8u.ab + 2au^2 + 4au^2 - 2bu. + u^2 \Big] + \frac{1}{s_B^4} \Big[-6a^3v_{11} + 6a^2b^2 \\
& - 4u. (2a^2b + ba^2) + 6u^2a^2 - 6a^3v_{11} + 6a^2b^2 - 12a^2v_{11} + 12ab^2 \\
& - 8u.a^2b - 4u.ba^2 + 6b^2 - 6av_{11} - 24u.ab + 6u^2a^2 + 12au^2 \\
& - 12bu. + 6u^2 \Big] , \tag{4.31}
\end{aligned}$$

$$\begin{aligned}
g_{23} = & \frac{2}{s_B^4} (4a^3b + 4a^2b + 2ba^2 - 4u.a^3 + 2ab - 2u.a^2 - 4u.a^2 - 2au.) \\
& + \frac{1}{s_B^4} (4a^3b - 4u.a^3 + 4a^3b + 8a^2b + 4ba^2 - 4u.a^3 + 12ab - 12u.a^2 \\
& - 12au. + 4b - 4u.) , \tag{4.32}
\end{aligned}$$

and

$$g_{24} = \frac{1}{s_B^4} (4a^4 + 8a^3 + 8a^2 + 4a + 1) \tag{4.33}$$

$$\text{where } a = -v_{11}^{-1} (v_{11}^{-1} + v_{22}^{-1})^{-1} \tag{4.34}$$

$$\text{and } b = (\tilde{u}_2 v_{22}^{-1} + \tilde{u}_1 v_{11}^{-1}) (v_{11}^{-1} + v_{22}^{-1})^{-1} . \tag{4.35}$$

We have now found an asymptotic expression for $p(\delta|z)$ which is given in (4.22). As mentioned before it is required to find

$$I_+(z) = \int_{\{\delta > 0\}} \delta p(\delta|z) d\delta . \tag{4.36}$$

Before computing (4.36) we note from Lemma 9.1 in the Appendix

that for $\delta \sim N(u^*, \sigma^{*2})$ where $u^* = \tilde{u}_1 - \tilde{u}_2$ and $\sigma^{*2} = \sum_{i=1}^2 v_{ii}$

that

$$\begin{aligned}
 A_k &= \frac{1}{\sqrt{(2\pi)\sigma^*}} \int_0^\infty \delta^k \exp\left[-\frac{1}{2}(\delta-u^*)^2 / (\sigma^*)^2\right] d\delta \\
 &= \frac{1}{\sqrt{(2\pi)}} \exp\left(-\frac{u^{*2}}{2\sigma^{*2}}\right) \sum_{i=0}^\infty \frac{(u^*/\sigma^*)^i}{i!} 2^{\frac{k+i-1}{2}} \\
 &\quad \cdot \left(\frac{k+i-1}{2}\right)! (\sigma^{*2})^{\frac{k+i}{2}} .
 \end{aligned} \tag{4.37}$$

Using (4.37) we see that (4.36) is given by

$$\begin{aligned}
 I_+(z) &= A_1 + \frac{1}{4} v_1^{-1} [A_1 g_{10} + A_2 g_{11} + A_3 g_{12} + A_4 g_{13} + A_5 g_{14}] \\
 &\quad + \frac{1}{4} v_2^{-1} [A_1 g_{20} + A_2 g_{21} + A_3 g_{22} + A_4 g_{23} + A_5 g_{24}] .
 \end{aligned} \tag{4.38}$$

It is also required to find

$$I_-(z) = \int_{-\infty}^0 |\delta| p(\delta|z) d\delta = - \int_{-\infty}^0 \delta p(\delta|z) d\delta . \tag{4.39}$$

Let $\tau = -\delta$. Then

$$I_-(z) = - \int (-\tau) p(-\tau|z) d\tau = \int_0^\infty \tau p(-\tau|z) d\tau .$$

Now substitute $\delta = -\tau$ in (4.22). Then

$$\begin{aligned}
p(-\tau|z) &= \left[\sqrt{(2\pi)} \sqrt{\left[\sum_{i=1}^2 v_{ii} \right]} \right]^{-1} \exp \left[-(\tau - (\tilde{u}_1 - \tilde{u}_2))^2 / (2 \sum_{i=1}^2 v_{ii}) \right] \\
&\cdot \left[1 + \frac{1}{4} v_1^{-1} \left(g_{10} - \tau g_{11} + \tau^2 g_{12} - \tau^3 g_{13} + \tau^4 g_{14} \right) \right. \\
&\quad \left. + \frac{1}{4} v_2^{-1} \left(g_{20} - \tau g_{21} + \tau^2 g_{22} - \tau^3 g_{23} + \tau^4 g_{24} \right) \right] \quad (4.40)
\end{aligned}$$

Now let

$$\begin{aligned}
B_k &= \frac{1}{\sqrt{(2\pi)} \sigma^*} \int_0^\infty \tau^k \exp \left[-\frac{1}{2} \left(\frac{\tau + u^*}{\sigma^*} \right)^2 \right] dz \\
&= \frac{1}{\sqrt{(2\pi)}} \exp \left[-u^{*2} / (2\sigma^{*2}) \right] \sum_{i=0}^\infty \frac{(-u^* / \sigma^{*2})^i}{i!} \frac{k+i-1}{2} \\
&\quad \cdot \left(\frac{k+i-1}{2} \right)! (\sigma^{*2})^{\frac{k+i}{2}},
\end{aligned}$$

so

$$\begin{aligned}
I_-(z) &= \int_0^\infty \tau p(-\tau|z) d\tau \\
&= B_1 + \frac{1}{4} v_1^{-1} [B_1 g_{10} - B_2 g_{11} + B_3 g_{12} - B_4 g_{13} + B_5 g_{14}] \\
&\quad + \frac{1}{4} v_2^{-1} [B_1 g_{20} - B_2 g_{21} + B_3 g_{22} - B_4 g_{23} + B_5 g_{24}] \cdot (4.41)
\end{aligned}$$

From the decision theoretic formulation we see for the component problem $P(i,j)$ that the Bayes rule

$$\bar{X}_i \text{ is significantly greater than } \bar{X}_j \text{ if } \frac{I_+(z)}{I_-(z)} > k \quad (4.42)$$

for $i,j=1,2$. Now let

$$\phi_{12} = \frac{I_+(z)}{I_-(z)} \quad \text{where} \quad (4.43)$$

$$\phi_{12} = \phi_{12}(r_i, r_j, s_e^2, s_B^2, \bar{X}_i, \bar{X}_j, v_1, v_2) \quad (4.44)$$

Once we obtain from the experiment the value of ϕ_{12} we can compare it with our error-weight ratio k and make a decision on \bar{X}_i, \bar{X}_j for $i=1,2$. Waller and Duncan (1969) defined for for $r_i=r$

$$t = \frac{(\bar{X}_i - \bar{X}_j)}{s_e \sqrt{(2/r)}} \quad , \quad F = \frac{s_e^2 + rs_B^2}{s_e^2}$$

and found

$$\phi_{12} = \phi_{12}(t, F, v_1, v_2) \quad (4.45)$$

so that they found a critical value t^* which is the solution of (4.43) and gave the following rule:

\bar{X}_i is significantly greater than \bar{X}_j if $t_{ij} > t^*$

or \bar{X}_i is not significantly greater than \bar{X}_j if $t_{ij} \leq t^*$.

A further discussion of their results is given in Chapter II.

We have found the Bayes solution for the component problem $P(1,2)$ which is given by

\bar{X}_i is significantly greater than \bar{X}_j if $\phi_{12} > k$

or \bar{X}_i is not significantly greater than \bar{X}_j if $\phi_{12} \leq k$

where $i, j=1,2$. Since we picked a symmetric loss function and symmetric prior distributions, we are able to give the Bayes solution for the component problem $P(i,j) \forall i, j=1, \dots, n$. The symmetry in the multiple comparison problem permits us to generalize some of the expressions given before which are essential ingredients for the general solution of the multiple comparison problem.

In general the posterior distribution of $\delta|z$ where now $\delta = u_i - u_j$ is given by

$$p(\delta|z) = \frac{1}{\sqrt{(2\pi)\sigma_{ij}}} \exp\left[-\frac{1}{2\sigma_{ij}^2} (\delta - (\tilde{u}_i - \tilde{u}_j))^2 \right]$$

$$\cdot \left[1 + \frac{1}{4} v_1^{-1} \left(\gamma_{10} + \delta \gamma_{11} + \delta^2 \gamma_{12} + \delta^3 \gamma_{13} + \delta^4 \gamma_{14} \right) \right]$$

$$\begin{aligned}
& + \frac{1}{4} v_2^{-1} \left(\gamma_{20} + \delta \gamma_{21} + \delta^2 \gamma_{22} + \delta^3 \gamma_{23} + \delta^4 \gamma_{24} \right) \Big] \\
& + o(1) + o(1) \quad . \quad (4.46)
\end{aligned}$$

where

$$\sigma_{ij}^2 = v_{ii} + v_{jj} \quad ,$$

$$\tilde{u}_{ij} = \tilde{u}_i - \tilde{u}_j \quad ,$$

$$N = \{i, j\} \quad ,$$

$$\begin{aligned}
\gamma_{10} = & - \sum_{k \in N} (\omega_k^2 + 2\xi_k \omega_k) - \sum_{k \in N} \psi'_k \left(\sum_{k \in N} \psi'_k + 2 \sum_{\substack{k=1 \\ k \neq i, j}}^n \psi'_k - 2n \right) \\
& + \left(\sum_{\substack{k=1 \\ k \neq i, j}}^n \psi'_k - n \right) \cdot \left[2 \frac{r_i}{s_e^2} (\bar{X}_i^2 + b^2 - av_{ii} - 2b\bar{X}_i) + 2 \frac{r_j}{s_j^2} \right. \\
& \cdot (b^2 - av_{ii} - 2\bar{X}_j b + \bar{X}_j^2) \Big] + 2 \frac{r_i r_j}{s_e^4} \left[3a^2 v_{ii}^2 - 6b^2 av_{ii} + b^4 \right. \\
& - 2(\bar{X}_i + \bar{X}_j) (b^2 - 3abv_{ii}) + (\bar{X}_i^2 + 4\bar{X}_j \bar{X}_i + \bar{X}_j^2) (b^2 - av_{ii}) \\
& - 2b(\bar{X}_i \bar{X}_j^2 + \bar{X}_i^2 \bar{X}_j) + \bar{X}_j^2 \bar{X}_i^2 \Big] + \left(\frac{r_j}{s_e^2} \right)^2 \left[3a^2 v_{ii}^2 - 6b^2 av_{ii} + b^4 \right. \\
& - 4\bar{X}_j (b^2 - 3abv_{ii}) + 6\bar{X}_j^2 b^2 - 6\bar{X}_j^2 av_{ii} - 4\bar{X}_j^3 b + \bar{X}_j^4 \Big]
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{r_i}{s_e^2} \right)^2 \left[3a^2 v_{ii}^2 - 6b^2 a v_{ii} + b^4 - 4\bar{X}_i b^2 + 12\bar{X}_i a b v_{ii} \right. \\
& \left. + 6\bar{X}_i^2 b^2 - 6\bar{X}_i^2 a v_{ii} - 4b\bar{X}_i^3 + \bar{X}_i^3 \right] , \tag{4.47}
\end{aligned}$$

$$\begin{aligned}
\gamma_{11} = & \left(\sum_{\substack{k=1 \\ k \neq i, j}}^n \psi_i^k - n \right) \left(2 \frac{r_i}{s_e^2} (-2\bar{X}_i + 2ab + 2b - 2a\bar{X}_i) + 2 \frac{r_j}{s_e^2} \right. \\
& \cdot (2ab - 2a\bar{X}_j) \left. \right) + 2 \frac{r_i r_j}{s_e^4} \left[-12a^2 b v_{ii} + 4ab^3 + 2b^3 - 6ab v_{ii} \right. \\
& - 2(\bar{X}_i + \bar{X}_j) (-3a^2 v_{ii} + 3ab^2) - 2\bar{X}_i (b^2 - a v_{ii}) + 2ab \\
& \cdot (\bar{X}_i^2 + 4\bar{X}_j \bar{X}_i + \bar{X}_j^2) - 4\bar{X}_j (b^2 - a v_{ii}) - 2a\bar{X}_i \bar{X}_j^2 - 2a\bar{X}_i^2 \bar{X}_j \\
& + 2b\bar{X}_j^2 + 4b\bar{X}_i \bar{X}_j - 2\bar{X}_j^2 \bar{X}_i \left. \right] + \left(\frac{r_j}{s_e^2} \right)^2 \left[-12a^2 b v_{ii} + 4ab^3 \right. \\
& - 4\bar{X}_j (-3a^2 v_{ii} + 3ab^2) + 12\bar{X}_j^2 ab - 4\bar{X}_j^3 a \left. \right] + \left(\frac{r_i}{s_e^2} \right)^2 \\
& \cdot \left[-12a^2 b v_{ii} + 4ab^3 + 4b^3 - 12ab v_{ii} + 4\bar{X}_i (3a^2 v_{ii} - 3ab^2) \right. \\
& \left. - 12\bar{X}_i (b^2 - a v_{ii}) + 12\bar{X}_i^2 ab - 4a\bar{X}_i^3 + 12b\bar{X}_i^2 - 4\bar{X}_i^3 \right] , \tag{4.48}
\end{aligned}$$

$$\begin{aligned}
\gamma_{12} = & \left(\sum_{\substack{k=1 \\ k \neq i, j}}^n \psi'_k - n \right) \left[2 \frac{r_i}{s_e^2} (1 + a^2 + 2a) + 2a^2 \frac{r_j}{s_e^2} \right] + 2 \frac{r_i r_j}{s_e^4} \\
& \cdot \left[-6a^3 v_{ii} + 6a^2 b^2 - 6a^2 v_{ii} + 6ab^2 - 2(\bar{X}_i + \bar{X}_j) (2a^2 b + ba^2) \right. \\
& + b^2 - av_{ii} - 4\bar{X}_i ab + a^2 (\bar{X}_i^2 + 4\bar{X}_j \bar{X}_i + \bar{X}_j^2) - 8ab\bar{X}_j + 2a\bar{X}_j^2 \\
& + 4a\bar{X}_i \bar{X}_j - 2b\bar{X}_j + \bar{X}_j^2 \left. \right] + \left(\frac{r_j}{s_e^2} \right)^2 \left[-6a^3 v_{ii} + 6a^2 b^2 - 4\bar{X}_j \right. \\
& \cdot (2a^2 b + ba^2) + 6a^2 \bar{X}_j^2 \left. \right] + \left(\frac{r_i}{s_e^2} \right)^2 \left[-6a^3 v_{ii} + 6a^2 b^2 \right. \\
& + 4(-3a^2 v_{ii} + 3ab^2) - 4\bar{X}_i (2a^2 b + ba^2) + 6b^2 - 6av_{ii} \\
& \left. - 24\bar{X}_i ab + 6\bar{X}_i^2 a^2 + 12a\bar{X}_i^2 - 12b\bar{X}_i + 6\bar{X}_i^2 \right] , \tag{4.49}
\end{aligned}$$

$$\begin{aligned}
\gamma_{13} = & 2 \frac{r_i r_j}{s_e^4} (4a^3 b + 4a^2 b + 2ba^2 - 2(\bar{X}_i + \bar{X}_j) a^3 + 2ab - 2\bar{X}_i a^2 - 4\bar{X}_j a^2 \\
& - 2a\bar{X}_j) + \left(\frac{r_j}{s_e^2} \right)^2 (4a^3 b - 4\bar{X}_j a^3) + \left(\frac{r_i}{s_e^2} \right)^2 (4a^3 b + 8a^2 b \\
& + 4ba^2 - 4\bar{X}_i a^3 + 12ab - 12\bar{X}_i a^2 = 12a\bar{X}_i + 4b - 4\bar{X}_i) , \tag{4.50}
\end{aligned}$$

$$\gamma_{14} = 2 \frac{r_i r_j}{s_e^4} (a^4 + 2a^3 + a^2) + \left(\frac{r_j}{s_e^2} \right)^2 a^4 + \left(\frac{r_i}{s_e^2} \right)^2 \cdot (a^4 + 4a^3 + 6a^2 + 4a + 1) , \quad (4.51)$$

$$\begin{aligned} \gamma_{20} = & -2 \sum_{k \in N} (\gamma_k^2 + 2\rho_k \gamma_k) - \sum_{k \in N} \psi_k'' \left(\sum_{k \in N} \psi_k'' + 2 \sum_{\substack{k=1 \\ k \neq i, j}}^n \psi_k'' - 2(n-1) \right) \\ & + \frac{2}{s_B^2} \left(\sum_{\substack{k=1 \\ k \neq i, j}}^n \psi_k'' - (n-1) \right) \left[2u^2 + 2b^2 - 2av_{ii} - 4u \cdot b \right] \\ & + \frac{2}{s_B^4} \left[3a^2 v_{ii}^2 - 6b^2 av_{ii} + b^4 - 4u \cdot b^3 + 12u \cdot abv_{ii} + 6u^2 b^2 \right. \\ & \left. - 6u^2 av_{ii} - 4u^3 b + u^4 \right] + \frac{1}{s_B^4} \left[3a^2 v_{ii}^2 - 6b^2 av_{ii} + b^4 - 4u \cdot b^3 \right. \\ & \left. + 12u \cdot abv_{ii} - 4u^3 b + u^4 + 6u^2 b^2 - 6u^2 av_{ii} + 3a^2 v_{ii}^2 - 6b^2 av_{ii} \right. \\ & \left. + b^4 - 4u \cdot b^2 + 12u \cdot abv_{ii} + 6u^2 b^2 - 6u^2 av_{ii} - 4bu^3 + u^3 \right] , \quad (4.52) \end{aligned}$$

$$\gamma_{21} = \frac{2}{s_B^2} \left(\sum_{\substack{k=1 \\ k \neq i, j}}^n \psi_k'' - (n-1) \right) (-2u \cdot + 4ab + 2b - 4u \cdot a) + \frac{2}{s_B^4} \left[-12a^2 bv_{ii} \right.$$

$$\begin{aligned}
& + 4ab^3 + 2b^3 - 6abv_{ii} + 12u.a^2v_{ii} - 12u.ab^2 - 2u.b^2 \\
& + 2u.av_{ii} + 12abu^2 - 4u.b^2 + 4u.av_{ii} - 4au^3 + 2bu^2 + 4bu^2 \\
& - 2u^3 \Big] + \frac{1}{s_B^4} \Big[-12a^2bv_{ii} + 4ab^3 + 12u.a^2v_{ii} - 12u.ab^2 \\
& + 12u^2ab - 4u^3a - 12a^2bv_{ii} + 4ab^3 + 4b^3 - 12abv_{ii} + 12u.a^2v_{ii} \\
& - 12u.ab^2 - 12u.b^2 + 12u.av_{ii} + 12u^2ab - 4au^3 + 12bu^2 \\
& - 12u^3 \Big] , \tag{4.53}
\end{aligned}$$

$$\begin{aligned}
\gamma_{22} = & \frac{2}{s_B^2} \left(\sum_{\substack{k=1 \\ k \neq i, j}}^n \psi_k^n - (n-1) \right) (1 + 2a^2 + 2a) + \frac{2}{s_B^4} \Big[-6a^3v_{ii} + 6a^2b^2 \\
& - 6a^2v_{ii} + 6ab^2 - 8u.a^2b - 4u.ba^2 + b^2 - av_{ii} - 4u.ab \\
& + 6u^2a^2 - 8u.ab + 2au^2 + 4au^2 - 2bu. + u^2 \Big] + \frac{1}{s_B^4} \Big[-6a^3v_{ii} \\
& + 6a^2b^2 - 4u.(2a^2b + ba^2) + 6u^2a^2 - 6a^3v_{ii} + 6a^2b^2 - 12a^2v_{ii} \\
& + 12ab^2 - 8u.a^2b - 4u.ba^2 + 6b^2 - 6av_{ii} - 24u.ab + 6u^2a^2
\end{aligned}$$

$$+ 12au^2 - 12bu. + 6u^2 \Big] , \quad (4.54)$$

$$\begin{aligned} \gamma_{23} = & \frac{2}{s_B} (4a^3b + 4a^2b + 2ba^2 - 4u.a^3 + 2ab - 2u.a^2 - 4u.a^2 - 2au.) \\ & + \frac{1}{s_B} (4a^3b - 4u.a^3 + 4a^3b + 8a^2b + 4ba^2 - 4u.a^3 + 12ab - 12u.a^2 \\ & - 12au. + 4b - 4u.) , \end{aligned} \quad (4.55)$$

and

$$\gamma_{24} = \frac{1}{s_B} (4a^4 + 8a^3 + 8a^2 + 4a + 1) \quad (4.56)$$

where a and b are now defined by

$$a = -v_{ii}^{-1} \left(v_{ii}^{-1} + v_{jj}^{-1} \right)^{-1} \quad (4.57)$$

$$\text{and } b = \left(\tilde{u}_j v_{jj}^{-1} + \tilde{u}_i v_{ii}^{-1} \right) \left(v_{ii}^{-1} + v_{jj}^{-1} \right)^{-1} . \quad (4.58)$$

Therefore $I_+(z)$ and $I_-(z)$ are given by

$$\begin{aligned} I_+(z) = & A_1 + \frac{1}{4} v_1^{-1} \left[A_1 \gamma_{10} + A_2 \gamma_{11} + A_3 \gamma_{12} + A_4 \gamma_{13} + A_5 \gamma_{14} \right] \\ & + \frac{1}{4} v_2^{-1} \left[A_1 \gamma_{20} + A_2 \gamma_{21} + A_3 \gamma_{22} + A_4 \gamma_{23} + A_5 \gamma_{24} \right] \end{aligned} \quad (4.59)$$

and

$$I_-(z) = B_1 + \frac{1}{4} v_1^{-1} [B_1 \gamma_{10} - B_2 \gamma_{11} + B_3 \gamma_{12} - B_4 \gamma_{13} + B_5 \gamma_{14}] \\ + \frac{1}{4} v_2^{-1} [B_1 \gamma_{20} - B_2 \gamma_{21} + B_3 \gamma_{22} - B_4 \gamma_{23} + B_5 \gamma_{24}] \quad (4.60)$$

where

$$A_k = \frac{1}{\sqrt{(2\pi)}} \exp \left[- \tilde{u}_{ij}^2 / (2\sigma_{ij}^2) \right] \sum_{s=0}^{\infty} \frac{(\tilde{u}_{ij}/\sigma_{ij}^2)^s}{s!} \\ \cdot 2^{\frac{k+s-1}{2}} \left(\frac{k+s-1}{2} \right)! \left(\sigma_{ij}^2 \right)^{\frac{k+s}{2}} \quad (4.61)$$

and

$$B_k = \frac{1}{\sqrt{(2\pi)}} \exp \left[- \tilde{u}_{ij}^2 / (2\sigma_{ij}^2) \right] \sum_{s=0}^{\infty} \frac{(-\tilde{u}_{ij}/\sigma_{ij}^2)^s}{s!} \\ \cdot 2^{\frac{k+s-1}{2}} \left(\frac{k+s-1}{2} \right)! \left(\sigma_{ij}^2 \right)^{\frac{k+s}{2}} \quad (4.62)$$

The Bayes rule for the component problem $P(i, j)$ is \bar{X}_i is significantly greater than \bar{X}_j if

$$\Phi(r_i, r_j, s_e^2, s_B^2, \bar{X}_i, \bar{X}_j, v_1, v_2) > k$$

or

\bar{X}_i is not significantly greater than \bar{X}_j if

$$\Phi(r_i, r_j, s_e^2, s_B^2, \bar{X}_i, \bar{X}_j, v_1, v_2) \leq k$$

where $i, j=1, \dots, n$ and

$$\phi(r_i, r_j, s_e^2, s_B^2, \bar{X}_i, \bar{X}_j, v_1, v_2) = \frac{I_+(z)}{I_-(z)} \quad (4.63)$$

This Bayes rule says that given a multiple comparison problem and the beliefs or experience of the experimenter, we can make a decision on the treatment means u_i and u_j for all $i, j=1, \dots, n$ by an application of the above rule.

This multiple comparison procedure, unlike those mentioned earlier uses the prior information which the experimenter has in the form of past experiments and the results of other workers.

This procedure may not seem easy for someone to use with a desk calculator. But with the use of high speed computers it is not difficult to program this procedure, so that when an experimenter walks into the consulting statisticians room with his data and his experience, we calculate (4.63) and tell him the various decision on his means.

E. Normal Approximation

In this section we will examine the main results obtained in the earlier sections of this chapter and determine their behaviour as we consider a normal approximation to the posterior distribution. We saw that the posterior distribution of $\underline{u}|z$ is product of a normal distribution and

a double power series in v_1^{-1} and v_2^{-1} .

In (4.7) we found the posterior distribution of $\underline{u}|z$ which is given by

$$p(\underline{u}|z) = f(\underline{u}; \tilde{u}_i, v_{ii}) (1 + v_1^{-1} d_{10} + v_2^{-1} d_{01}) + o(1) + \bar{o}(1) \quad (4.64)$$

where d_{10} and d_{01} are given in (4.7). If we consider the first term in the double power series, the posterior distribution of $(\underline{u}|z)$ is

$$p(\underline{u}|z) = f(\underline{u}; \tilde{u}_i, v_{ii}) . \quad (4.65)$$

From (4.7) the marginal distribution is given by

$$p(u_i|z) = f(u_i; \tilde{u}_i, v_{ii}) . \quad (4.66)$$

Therefore $u_i|z$ is distributed $N(\tilde{u}_i, v_{ii})$ where

$$\tilde{u}_i = \left(\frac{r_i}{s_e^2} + \frac{1}{s_B^2} \right)^{-1} \left(\frac{r_i \bar{X}_i}{s_e^2} + \frac{u_{\cdot}}{s_B^2} \right) \quad (4.67)$$

$$\text{and } v_{ii}^{-1} = \left(\frac{r_i}{s_e^2} + \frac{1}{s_B^2} \right) \text{ for all } i . \quad (4.68)$$

As was pointed out in Section IVC Lindley (1971a) obtained posterior estimate of u_i using a hierarchical prior

structure. As mentioned before Lindley's modal estimates now have the same structure as our estimates of the mean of the distribution of $u_i|z$ which are given in (4.67).

Let us also derive the Bayes decision rule for the multiple comparison problem by considering only the first term of the double power series in v_1^{-1} and v_2^{-1} . From (4.46), using the first term of the double power series in v_1^{-1} and v_2^{-1} we have for

$$p(\delta_{ij}|z) = \frac{1}{\sqrt{(2\pi)\sigma_{ij}^2}} \exp \left[-\frac{1}{2\sigma_{ij}^2} (\delta_{ij} - \tilde{u}_{ij})^2 \right] \quad (4.69)$$

where $\sigma_{ij}^2 = v_{ii} + v_{jj}$

and $\tilde{u}_{ij} = \tilde{u}_i - \tilde{u}_j$.

Therefore by (4.59)

$$I_+(z) = \int_{\{\delta_{ij} > 0\}} \delta_{ij} p(\delta_{ij}|z) d\delta_{ij} = A_1$$

where A_1 is given in (4.61) and

$$I_-(z) = \int_{\{\delta_{ij} < 0\}} (-\delta_{ij}) p(\delta_{ij}|z) d\delta_{ij} = B_1$$

where B_1 is given in (4.62).

So the full Bayes rule is

$$d_{ij}^+ : \bar{X}_i \text{ is significantly larger than } \bar{X}_j, \text{ or } u_i > u_j \\ \text{if } \frac{A_1}{B_1} > k$$

$$d_{ij}^0 : \bar{X}_i \text{ is not significantly larger than } \bar{X}_j \text{ or } u_i \text{ is} \\ \text{unranked relative to } u_j \text{ if } \frac{A_1}{B_1} \leq k .$$

Let $\phi = \frac{A_1}{B_1}$. Then from the decision theoretic

formulation we see for the component problem $P(i,j)$ that the Bayes rule is

$$\bar{X}_i \text{ is significantly greater than } \bar{X}_j \text{ if } \phi > k$$

$$\text{or } \bar{X}_i \text{ is not significantly greater than } \bar{X}_j \text{ if } \phi \leq k . \quad (4.70)$$

Then in an experiment, we compute the r_i, s_e^2, s_B^2 , and k , and we get our Bayes rule

$$u_i > u_j \text{ if } \phi > k . \quad (4.71)$$

From our decision theoretic set-up it seems that this is as far as one can go in the solution of the multiple comparison problem. But here we will go a little further and try to follow the method of Waller and Duncan (1969).

F. The Equal Replication Case

Let $r_i = r$, $t_{ij} = t = \frac{\bar{X}_i - \bar{X}_j}{s_e \sqrt{2/r}}$ and $F = \frac{s_e^2 + rs_B^2}{s_e^2}$.

Then $\ddot{u}_i = \left(\frac{r}{s_e^2} + \frac{1}{s_B^2} \right)^{-1} \left(\frac{r\bar{X}_i}{s_e^2} + \frac{u.}{s_B^2} \right)$

and $v_{ii}^{-1} = \left(\frac{r}{s_e^2} + \frac{1}{s_B^2} \right)$.

So
$$\frac{(\ddot{u}_i - \ddot{u}_j)}{(v_{ii} + v_{jj})^{1/2}} = \left(\frac{r}{s_e^2} + \frac{1}{s_B^2} \right)^{-1} \frac{r}{s_e^2} (\bar{X}_i - \bar{X}_j) / \left[2 \left(\frac{r}{s_e^2} + \frac{1}{s_B^2} \right)^{-1} \right]^{1/2}$$

$$= \left(\frac{s_e^2}{rs_B^2 + s_e^2} \right)^{1/2} \frac{(\bar{X}_i - \bar{X}_j)}{s_e \sqrt{2/r}} \sqrt{r} \frac{s_B}{s_e}$$

$$= \frac{1}{F^{1/2}} t \sqrt{r} \frac{s_B}{s_e}$$

also
$$\frac{\ddot{u}_i - \ddot{u}_j}{v_{ii} + v_{jj}} = \left(\frac{r}{s_e^2} + \frac{1}{s_B^2} \right)^{-1} \frac{r}{s_e^2} (\bar{X}_i - \bar{X}_j) / 2 \left(\frac{r}{s_e^2} + \frac{1}{s_B^2} \right)^{-1}$$

$$= \frac{r}{s_e^2} \left(\frac{\bar{X}_i - \bar{X}_j}{2} \right)$$

$$= \sqrt{(r/2)} (t/s_e),$$

so expanding (4.69) we obtain

$$p(\delta_{ij}|z) = \frac{(v_{ii}+v_{jj})^{-1/2}}{(2\pi)^{1/2}} \exp \left[-\frac{1}{2} \frac{(\delta_{ij}^2 - 2\delta_{ij}(\tilde{u}_i - \tilde{u}_j) + (\tilde{u}_i - \tilde{u}_j)^2)}{(v_{ii} + v_{jj})} \right] .$$

Let $\sigma_{ij}^2 = v_{ii} + v_{jj} = \sigma^2$,

$$\text{so } p(\delta_{ij}|t) = \frac{1}{\sqrt{(2\pi\sigma^2)}} \exp \left[-\frac{1}{2} \frac{\delta_{ij}^2}{\sigma^2} + \delta_{ij} \sqrt{\frac{r}{2}} \frac{t}{s_e} - \frac{1}{2} \frac{t^2}{F} r \frac{s_B^2}{s_e^2} \right]$$

$$= \frac{1}{\sqrt{(2\pi\sigma^2)}} \exp \left[-\frac{1}{2} \frac{t^2}{F} r \frac{s_B^2}{s_e^2} \right] \exp \left[-\frac{1}{2} \frac{\delta_{ij}^2}{\sigma^2} \right]$$

$$\cdot \sum_{i=0}^{\infty} \left[\delta_{ij} \sqrt{\frac{r}{2}} \frac{t}{s_e} \right]^i / i! .$$

As before to obtain the Bayes rule it is required to solve (4.18). Let $\delta_{ij} = \delta$. Then by Lemma (9.1)

$$I_+(t) = \int_0^{\infty} \delta p(\delta|t) d\delta$$

$$= \frac{1}{\sqrt{(2\pi)}} \exp \left[-\frac{1}{2} \frac{t^2}{F} r \frac{s_B^2}{s_e^2} \right] \sum_{i=0}^{\infty} \left[\sqrt{\frac{r}{2}} \frac{t}{s_e} \right]^i / i!$$

$$\cdot \Gamma\left(\frac{2+i}{2}\right) (\sigma^2)^{\frac{1+i}{2}} 2^{i/2} . \quad (4.72)$$

Similarly

$$\begin{aligned}
 I_-(t) &= \int_{\{\delta_{ij} < 0\}} (-\delta_{ij}) p(\delta_{ij}|t) d\delta_{ij} \\
 &= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2} \frac{t^2}{F} + r \frac{s_B^2}{s_e^2}\right] \sum_{i=0}^{\infty} \left[-\sqrt{\frac{r}{2}} t/s_e\right]^i / i! \\
 &\quad \cdot \Gamma\left(\frac{2+i}{2}\right) (\sigma^2)^{\frac{1+i}{2}} 2^{i/2} .
 \end{aligned} \tag{4.73}$$

Therefore from the decision theoretic point of view we see for the component problem $P(i,j)$ that the Bayes rule is

\bar{X}_i is significantly larger than \bar{X}_j if $t_{ij} > t^*$

or \bar{X}_i is not significantly larger than \bar{X}_j if $t_{ij} \leq t^*$,

where t^* is the solution of the equation

$$\frac{I_+(t)}{I_-(t)} = k . \tag{4.74}$$

Here we note that to obtain t^* one must specify the following values r , s_e^2 , s_B^2 , and k . We will give a further discussion on the solution of (4.74).

On a simultaneous application of the above rule to the $P(i,j)$ and the $P(j,i)$ problems we can derive the three-decision rule for $Q(i,j)$ problem which is given as follows

\bar{X}_i is significantly greater than \bar{X}_j if $t_{ij} > t^*$

\bar{X}_i is not significantly different from \bar{X}_j if $|t_{ij}| \leq t^*$

\bar{X}_i is significantly smaller than \bar{X}_j if $t_{ij} < -t^*$

where $t^* = t(k, s_e^2, s_B^2, r)$.

By the simultaneous application of this three-decision rule to all the $\binom{n}{2}$ pairs of treatments we obtain the Bayes rule for the symmetric multiple comparison problem, which is given as follows

\bar{X}_i is significantly larger than \bar{X}_j if $\bar{X}_i - \bar{X}_j > \text{BLSD}$

\bar{X}_i is not significantly different from \bar{X}_j if

$$|\bar{X}_i - \bar{X}_j| \leq \text{BLSD}$$

\bar{X}_i is significantly smaller than \bar{X}_j if $\bar{X}_i - \bar{X}_j < -\text{BLSD}$

where the Bayes Least Significant Difference

$$(\text{BLSD}) = s_e \sqrt{\frac{2}{r}} t(k, s_e^2, s_B^2, r) .$$

For the case of comparing two out of n means we saw the critical value depends on k, s_e^2, s_B^2 , and r .

We now present a step by step outline of an iterative solution of equation (4.74) which is

$$\frac{I_+(t)}{I_-(t)} = k$$

- 1) Specify $\{k, s_e^2, s_B^2, \text{ and } r\}$ and note that

$$F = \frac{s_e^2 + r s_B^2}{s_e^2} .$$

- 2) Choose an initial value of t . A first choice may be

$$t = t_0 = \sqrt{F} .$$

- 3) Calculate $I_+(t), I_-(t)$, $K^*(t) = I_+(t)/I_-(t)$

$$\text{and } K'(t) = \frac{d}{dt} (K^*(t)) .$$

- 4) Compare $K^*(t)$ and k

- (i) If $|K^*(t) - k| \leq \beta$, β some specified precision level

then $t^* = t$ is the Bayes significant t value and is the solution of (4.74) for given values of k, s_e^2, s_B^2 , and r , and we are done.

- (ii) If $|K^*(t) - k| > \beta$, then a better approximation for t using Newton's method is given by

$$t_1 = t + (k - K^*(t))/K'(t)$$

and we return to step 3.

The above steps calculates the Bayes critical values.

V. EXTENSIONS

A. Introduction

The derivation of the Bayes rule in Chapter IV may be extended for other designs and for different loss or prior structures. Here we sketch some of the main arguments that may be used in such extensions. First, we consider the one-way classification model with response surface priors and no control treatments. Then we include a control treatment. We also investigate the effect of exchangeable and response surface priors on data which arose from a randomized block experiment. In the cases investigated we derive the posterior distributions. With these distributions, we may find the moments, estimates of variance components, and a Bayes rule for the pairwise comparisons of the means.

B. The One-way Classification

1. Response surface priors and no control treatments

As before the model is

$$X_{ij} = u_i + e_{ij}, \quad i=1, \dots, n, \quad j=1, \dots, r_i \quad (5.1)$$

where X_{ij} are the observations, u_i are the unknown constants or parameters and e_{ij} are the errors which are normally and independently distributed with zero mean and a constant unknown variance σ^2 . We assume also that the treatment effects

$\underline{u}=(u_1, \dots, u_n)$ correspond to the levels $y_1 < \dots < y_n$ of a single factor and that the effects lie approximately on some polynomial response curve of degree s when $s < n-1$. For example, if y_i represents the levels of a certain fertilizer with u_i the corresponding treatment effect, we may say that the response of the treatment to different levels of the fertilizer is linear or quadratic. A Bayesian formulation for the situation where σ^2 is known has been advanced by Smith (1973a). It is assumed that, conditional on $\tau^*=(\tau_0^*, \dots, \tau_s^*)$ and σ_u^2 , the u_i are independent and

$$u_i \sim N \left(\sum_{j=0}^s \tau_j^* y_i^j, \sigma_u^2 \right). \quad (5.2)$$

Now (5.2) gives a probability density proportional to

$$(\sigma_u^2)^{-n/2} \exp \left[-(\underline{u} - P\tau^*)' (\underline{u} - P\tau^*) / (2\sigma_u^2) \right] \quad (5.3)$$

where $E\underline{u} = P\tau^*$ and the i -th row of P is $(1, y_i, \dots, y_i^s)$.

Smith (1973a) called (5.2) a response surface prior. The idea is to allow for lack of fit, by the presence of the variance term σ_u^2 . Therefore though the relationship between the u_i 's is approximately a polynomial of degree s , we are not certain about its actual numerical specification. As a prior for each $\tau_0^*, \dots, \tau_s^*$, we shall use the improper uniform distribution over

$(-\infty, \infty)$. It is commonly thought that this will have little effect in that one could use a proper prior with a very large variance, but there may be difficulties in this point of view, with regard to the existence of modes and means. Novick (1972) raises the possibility of bimodality of posterior distributions with particular choices of prior distributions.

For the unknown σ^2 and σ_u^2 we follow the usual path of using the conjugate inverse χ^2 -family as the prior distributions; that is for given $\nu, \lambda, \nu_u, \lambda_u$, we assume independently that

$$(\nu\lambda/\sigma^2) \sim \chi_\nu^2 \quad \text{and} \quad (\nu_u\lambda_u/\sigma_u^2) \sim \chi_{\nu_u}^2. \quad (5.4)$$

We may rewrite (5.3) as

$$\begin{aligned} P(\underline{u}|\tau^*, \sigma_u^2) &\propto (\sigma_u^2)^{-(n-s)/2} \exp \left[-(\underline{u} - P\tau)'(\underline{u} - P\tau)/(2\sigma_u^2) \right] \\ &\cdot (\sigma_u^2)^{-s/2} \exp \left[-(\tau^* - \tau)'P'P(\tau^* - \tau)/(2\sigma_u^2) \right] \end{aligned} \quad (5.5)$$

where τ is given such that $P'P\tau = P'\underline{u}$.

We rewrite (5.1) in general matrix notation to cover this and other cases, as

$$X = Au + e.$$

Then, as given by Smith (1973a)

$$P(X|\underline{u}, \sigma^2) \propto (\sigma^2)^{-R/2} \exp \left[- \left(S^2 + (\underline{u} - \hat{\underline{u}})'A'A(\underline{u} - \hat{\underline{u}}) \right) / (2\sigma^2) \right] \quad (5.6)$$

where $R = \sum_{i=1}^n r_i$, $S^2 = (X - A\hat{u})'(X - A\hat{u})$, where \hat{u} is given

by $A'A\hat{u} = A'X$.

Combining (5.4), (5.5), and (5.6) and integrating with respect to τ^* , we obtain the posterior distribution of \underline{u} , σ^2 , and σ_u^2 which is given by

$$\begin{aligned} p(\underline{u}, \sigma^2, \sigma_u^2 | X) \propto & (\sigma_u^2)^{-(n-s)/2} \exp \left[-(\underline{u} - P\tau)'(\underline{u} - P\tau) / (2\sigma_u^2) \right] \\ & \cdot (\sigma_u^2)^{-(v_u+2)/2} \exp \left[-v_u \lambda_u / (2\sigma_u^2) \right] \\ & \cdot (\sigma^2)^{-(v+2)/2} \exp \left[-v\lambda / (2\sigma^2) \right] \\ & \cdot (\sigma^2)^{-R/2} \exp \left[-\left(S^2 + (\underline{u} - \hat{u})'A'A(\underline{u} - \hat{u}) \right) / (2\sigma^2) \right] \end{aligned}$$

which is

$$\begin{aligned} \propto & (\sigma^2)^{-(R+v+2)/2} \exp \left[-\left(S^2 + (\underline{u} - \hat{u})'A'A(\underline{u} - \hat{u}) \right. \right. \\ & \quad \left. \left. + v\lambda \right) / (2\sigma^2) \right] \\ & \cdot (\sigma_u^2)^{-(v_u+2+n-s)/2} \exp \left[-\left((\underline{u} - P\tau)'(\underline{u} - P\tau) \right. \right. \\ & \quad \left. \left. + v_u \lambda_u \right) / (2\sigma_u^2) \right] \quad (5.7) \end{aligned}$$

Integrating with respect to σ_u^2 and σ^2 , we obtain the marginal distribution of $\underline{u} | X$ which is

$$\begin{aligned} p(\underline{u} | X) \propto & \left[S^2 + (\underline{u} - \hat{u})'A'A(\underline{u} - \hat{u}) + v\lambda \right]^{-(R+v)/2} \\ & \cdot \left[(\underline{u} - P\tau)'(\underline{u} - P\tau) + v_u \lambda_u \right]^{-(v_u+n-s)/2} \end{aligned}$$

$$\propto \left[1 + \frac{S^2}{v\lambda} + \frac{(\underline{u}-\hat{u})' A' A (\underline{u}-\hat{u})}{v\lambda} \right]^{-(R+v)/2} \cdot \left[1 + \frac{(\underline{u}-P\tau)' (\underline{u}-P\tau)}{v_u \lambda_u} \right]^{-v''/2} \quad (5.8)$$

where $v'' = v_u + n - s$.

This posterior distribution (5.8) is similar to (4.5) which was obtained by using exchangeable priors. It can be shown that the above density is the product of two multivariate t densities. If in this expression (5.8) P is the matrix with all elements zero, we obtain

$$p(\underline{u}|X) \propto \left[1 + \frac{S^2}{v\lambda} + \frac{(\underline{u}-\hat{u})' A' A (\underline{u}-\hat{u})}{v\lambda} \right]^{-(R+v)/2} \cdot \left[1 + (\underline{u}'\underline{u}) / (v_u \lambda_u) \right]^{-(v_u+n)/2}$$

which is of the form of (4.5) obtained earlier.

As shown in Chapter IV we can expand this distribution as the product of a normal distribution and a double inverse power series in the degrees of freedom. Then we can find the moments of this distribution and a Bayes rule for the pairwise comparisons of the treatment means, as in Chapter IV.

2. A control treatment

Consider the model (5.1). Suppose we have one control variety and $i=2, \dots, n$ are the experimental varieties. We want to compare each experimental variety with the control to determine whether any of the experimental varieties differ from the control. Dunnett (1955, 1964) considered this problem and a good account of his solution is also given in Miller (1966). Here we will consider this problem from a Bayesian viewpoint.

Here for the unknown u_i , the exchangeability of the u_i for all i is inappropriate because the first variety is a control and the remaining $(n-1)$ are experimental varieties. But we will modify the assumption to one of exchangeability within the experimental varieties. It might be reasonable to say that our prior knowledge of the experimental varieties u_i , $i=2, \dots, n$ is exchangeable, and that this group of treatments is independent of the control treatment.

Suppose

$$u_1 \sim N(0, \sigma_c^2) \quad ,$$

$$u_i \sim N(0, \sigma_e^2) \quad , \quad i=2, \dots, n$$

and these distributions are independent. Since σ_c^2 and σ_e^2 are unknown, we will specify priors for σ_c^2 and σ_e^2 . We will use the conjugate family of priors which in this situation is the inverse χ^2 ; that is given $\nu, \lambda, \nu_c, \lambda_c, \nu_e$, and λ_e , we

assume that $(v\lambda/\sigma^2) \sim \chi_v^2$, $(v_c\lambda_c)/\sigma_c^2 \sim \chi_{v_c}^2$ and

$(v_e\lambda_e)/\sigma_e^2 \sim \chi_{v_e}^2$, and that these variables are independent.

Combining the likelihood and the priors by Bayes theorem we obtain the posterior distribution which is proportional to

$$\begin{aligned}
 & \left[\prod_{i=1}^n (2\pi\sigma^2)^{r_i/2} \right]^{-1} \exp \left[- \frac{1}{(2\sigma^2)} \sum_{ij} (x_{ij} - u_i)^2 \right] \\
 & \cdot \left(\frac{v\lambda}{\sigma^2} \right)^{v/2} \frac{1}{\sigma^2} \frac{\exp(-v\lambda/(2\sigma^2))}{\Gamma\left(\frac{v}{2}\right) 2^{v/2}} [2\pi\sigma_c^2]^{-1/2} \exp \left[- \frac{1}{(2\sigma_c^2)} u_1^2 \right] \\
 & (v_c\lambda_c/\sigma_c^2)^{v_c/2} (\sigma_c^2)^{-1} \exp(-v_c\lambda_c/(2\sigma_c^2)) \left[\Gamma(v_c/2) 2^{v_c/2} \right]^{-1} \\
 & \cdot (2\pi\sigma_e^2)^{-(n-1)/2} \exp \left[- \sum_{i=2}^n u_i^2 / (2\sigma_e^2) \right] (v_e\lambda_e/\sigma_e^2)^{v_e/2} (\sigma_e^2)^{-1} \\
 & \exp(-v_e\lambda_e/(2\sigma_e^2)) \left[\Gamma(v_e/2) 2^{v_e/2} \right]^{-1} \\
 \propto & (\sigma^2)^{-(R+v+2)/2} \exp \left[- \left(\sum_{ij} (x_{ij} - u_i)^2 + v\lambda \right) / (2\sigma^2) \right] \\
 & \cdot [\sigma_c^2]^{-(v_c+3)/2} \exp \left[- (u_1^2 + v_c\lambda_c) / (2\sigma_c^2) \right] \\
 & \cdot [\sigma_e^2]^{-(n+v_e+1)/2} \exp \left[- \left(\sum_{i=2}^n u_i^2 + v_e\lambda_e \right) / (2\sigma_e^2) \right] \quad (5.9)
 \end{aligned}$$

where $R = \sum r_i$.

We find by integrating (5.9) with respect to σ^2 , σ_c^2 , and σ_e^2 that the posterior distribution is

$$\begin{aligned}
& \propto \left[\sum_{ij} (x_{ij} - u_i)^2 + v\lambda \right]^{-(R+v)/2} \left[u_1^2 + v_c \lambda_c \right]^{-(1+v_c)/2} \\
& \quad \cdot \left[\sum_{i=2}^n u_i^2 + v_e \lambda_e \right]^{-(n-1+v_e)/2} \\
\text{so } p(\underline{u} | x_{ij}) & \propto \left[1 + \sum_{ij} \frac{(x_{ij} - u_i)^2}{v\lambda} \right]^{-(R+v)/2} \\
& \quad \cdot \left[1 + u_1^2 / (v_c \lambda_c) \right]^{-(1+v_c)/2} \\
& \quad \cdot \left[1 + \frac{\sum_{i=2}^n u_i^2}{v_e \lambda_e} \right]^{-(n-1+v_e)/2} \quad (5.10)
\end{aligned}$$

This density is the product of three multivariate t densities. From this distribution with an appropriate loss function, we may find estimates of u_i , $i=1, \dots, n$ and a Bayes rule for the $(n-1)$ component problems, where a component problem is a $P(1, i)$ problem which involves a comparison between a u_i and u_1 for $i=2, \dots, n$. We can use the methods of Chapter IV to obtain the Bayes rule.

C. Randomized Blocks

Suppose the observations in a randomized block experiment arose from the model

$$X_{ij} = \mu + u_i + \beta_j + e_{ij}, \quad i=1, \dots, a, \quad j=1, \dots, b \quad (5.11)$$

where μ , u_i , and β_j are unknown constants such that

$$\sum u_i = \sum \beta_j = 0 \quad \text{and} \quad e_{ij} \sim \text{NID}(0, \sigma^2).$$

To find a Bayes rule for the pairwise comparisons of the treatment means we will first assign priors to the unknown parameters or constants and use some appropriate loss function. But before we find a Bayes rule, it is instructive to find the posterior distribution of $\theta = (\mu, \underline{u}', \beta')$.

For convenience we will assume that the block effects are exchangeable and independently that a response surface prior is suitable for the treatment effects. As discussed before a response surface prior reduces to an exchangeable prior if we consider the polynomial to be of zero degree.

We may write the above model in general matrix notation as

$$X = A\theta + e \quad (5.12)$$

Then, as given by Smith (1973a),

$$p(X|\theta, \sigma^2) \propto (\sigma^2)^{-1/(2n)} \exp \left[- \frac{1}{(2\sigma^2)} \left(S^2 + (\theta - \hat{\theta})' A' A (\theta - \hat{\theta}) \right) \right] \quad (5.13)$$

where n = total number of observations, $S^2 = (X - A\hat{\theta})' (X - A\hat{\theta})$

and $\hat{\theta}$ is the solution of the equations

$$A'A\hat{\theta} = A'X$$

$$H\hat{\theta} = 0 \quad \text{where} \quad H' = \begin{bmatrix} 0 & 0 \\ 1_a & 0 \\ 0 & 1_b \end{bmatrix}$$

The priors for β and \underline{u} are given by

$$p(\beta | \sigma_\beta^2) \propto (\sigma_\beta^2)^{-1/2(b-1)} \exp \left[-\frac{1}{(2\sigma_\beta^2)} \beta' \beta \right]$$

$$p(\underline{u} | \tau^+, \sigma_u^2) \propto (\sigma_u^2)^{-1/2(a-1)} \exp \left[-\frac{1}{(2\sigma_u^2)} (\underline{u} - P\tau^+) ' (\underline{u} - P\tau^+) \right]$$

where $E\underline{u} = \sum_{j=0}^S \tau_j^+ \xi_j = P\tau^+$ with $P = [\xi_1, \dots, \xi_S]$ where the

ξ_j 's are normalized orthogonal polynomials over the set of numbers $\{y_1, \dots, y_a\}$. Here it is assumed, as in Section B, that the treatment effect $\underline{u} = (u_1, \dots, u_a)$ correspond to the levels $y_1 < \dots < y_a$ of a single factor. Now we take the priors for μ and τ^+ as uniform over $(-\infty, \infty)$. The prior on τ^+ depends on y_i which may render the results rather suspect.

As before we use for $\sigma^2, \sigma_\beta^2, \sigma_u^2$ the conjugate inverse χ^2 -family, i.e., for given $\nu, \lambda, \nu_u, \lambda_u, \nu_\beta, \lambda_\beta$ we assume, independently, that

$$\frac{v\lambda}{\sigma^2} \sim \chi_v^2, \quad \frac{v_u \lambda_u}{\sigma_u^2} \sim \chi_{v_u}^2 \quad \text{and} \quad \frac{v_\beta \lambda_\beta}{\sigma_\beta^2} \sim \chi_{v_\beta}^2.$$

Combining the likelihood and the priors by Bayes' theorem and integrating out $\tau^+, u, \sigma^2, \sigma_u^2$, and σ_β^2 from the posterior distribution we obtain the marginal posterior for θ which is given by

$$\begin{aligned} p(\theta|x) &\propto \left[v\lambda + s^2 + (\theta - \hat{\theta})' A' A (\theta - \hat{\theta}) \right]^{-1/2 (n+v)} \\ &\cdot \left[v_u \lambda_u + (\underline{u} - P\tau)' (\underline{u} - P\tau) \right]^{-1/2 (a+v_u-s-1)} \\ &\cdot \left[v_\beta \lambda_\beta + \beta' \beta \right]^{-1/2 (b+v_\beta-1)} \end{aligned} \quad (5.14)$$

where $H\theta = 0$ and $n = ab$. So the posterior density of θ is a product of three multivariate t densities. This distribution was obtained by Lindley and Smith (1972) and Smith (1973a).

We have already discussed in the one-way classification model how to obtain a Bayes rule for the pairwise comparisons of means when the posterior distribution of θ is a product of multivariate t distributions. These same ideas may be extended.

From expression (5.14) we may examine what are the consequences if our prior knowledge of the β_i is diffuse. If $v_\beta = 0$ then $p(\beta) \propto d\beta$ and it seems that the third factor of

(5.14) should be replaced by unity. The posterior distribution of θ will be given then as the product of two multivariate t distributions which was discussed before.

From (5.14) we may find the mean and the variance of the posterior distribution of $\theta = (\mu, \underline{u}', \beta')$. In a randomized block design the experimenter is usually interested in treatment differences rather than block differences. So we can express the posterior distribution of $\theta = (\mu, \underline{u}', \beta')$ in terms of a triple inverse power series in v, v_u , and v_β which are the degrees of freedom. Though the details of this expression have not been given before, it seems that it is a natural extension of the idea of expressing a product of two multivariate t distributions as a double inverse power series in the degrees of freedom.

We can then find, by integrating out the unwanted parameters, the marginal posterior distribution of $\underline{u}|x$, i.e., $p(\underline{u}|x)$. This distribution can be used to find a Bayes rule for the pairwise comparisons of the treatment means using the methods of Chapter IV.

VI. SUMMARY

In this dissertation we considered the multiple comparison problem. The model investigated is the one-way classification with errors that are normally and independently distributed with zero mean and common unknown variance σ_e^2 . The observations are denoted by X_{ij} and the model is

$$X_{ij} = u_i + e_{ij} \quad \begin{array}{l} i=1, \dots, n \\ j=1, \dots, r_i \end{array}$$

where the u_i are unknown true means or parameters and e_{ij} are the errors. From the sample mean $(\bar{X}_1, \dots, \bar{X}_n)$ and the sample variance s^2 , we want to give some substance to the notion of evidence with respect to u_1, \dots, u_n and σ_e^2 .

A critical review of the existing multiple comparison procedures which have been suggested in the literature is given. These procedures differ considerably in the mode of approach. It is our belief that the choice of a procedure must be based in some way on prior opinions or guesses about the true means. These opinions may be used to develop a procedure for making assertions about the true means and differences between them.

The currently available non-Bayesian procedures use a concept of error rate, α , of assertions derived from the data with regard to parameter values. It was hoped to develop some partial logic for the choice of error rates by the incorporation

of prior opinions, represented by prior distributions on parameter values. This led to the development of Bayes rules for the making of assertions on differences among the true means.

In Chapter I a review of the multiple comparison procedures in current use is given. An extension of the Duncan and the Student-Newman-Keuls procedure is proposed. The Duncan method would use (1.8) and (1.9) with $\alpha_p = 1 - (1-\alpha)^{p-1}$ and $p = 2, \dots, n$, while the Student-Newman-Keuls method would use $\alpha_p = \alpha$ and $p = 2, \dots, n$. This extension is based on the upper percentage points of the studentized augmented range distribution.

We also gave the exact distribution of the studentized maximum gap statistic. Let X_i , $i = 1, \dots, n$ be distributed with cdf F and pdf f with $X_{(1)}, \dots, X_{(n)}$, the order statistics. Consider

$$R_{\ell, v} = G_{\ell} / s$$

where $G_{\ell} = \max_{2 \leq i \leq n} g_i$ with $g_i = X_{(i)} - X_{(i-1)}$,

$\ell = n-1$ and s^2 an unbiased estimate of σ_e^2 with v d.f.

and independent of the order statistics. We may assume

$X_i \sim N(\mu_i, \sigma_e^2)$. Then

$$P(R_{\ell, v} \leq r) = 2 \frac{(v/2)^{v/2}}{\Gamma(v/2)} \int_0^\infty s^{v-1} \exp(-vs^2/2) P(G_\ell \leq sr) ds$$

where $P(G_\ell \leq sr)$ is given in (1.14). The studentized maximum gap statistic may be used to develop a grouping procedure.

In Chapter II we gave a general framework for a Bayes approach to the multiple comparison problem. The decision theoretic formulation of the multiple comparison problem for the one-way classification is given.

In Chapter III we reviewed in detail some of the major contributions on the choice of α , the probability of a Type I error. We considered the two-decision problem and gave the Lindley-Savage argument for the choice of α in the simple hypothesis versus the simple alternative. Under certain assumptions of a "rational" decision maker they showed that his indifference curves are straight parallel lines whose slopes are the prior-odds with a zero-one loss function. Using this system of indifference curves and our admissible tests we can select an α which is a function of our prior information and our losses. In the composite hypothesis versus the composite alternative, we reviewed the Lindley (1961) argument which shows that α is a function of our priors and losses. In particular α decreases as the sample size increases. This idea was extended for the multiple comparison problem using a comparisonwise approach. We found that α depends on the

utility function, the prior distributions, the sample sizes and the number of samples.

A critical review of the Waller and Duncan (1969) argument is also given. Their work uses priors which are functions of the data. These priors, even though they are convenient analytically do not extend obviously to the case of unequal sample sizes. The authors found a Bayes rule for the multiple comparison problem and claimed to have a logic for the choice of α , but the cut-off points for their rule depend on the data. In the context of a theory of testing hypothesis, it seems that the error rate should be determined a priori by one's prior opinions and loss function as in the case of a simple hypothesis versus a simple alternative. Bayesian arguments do not lead to a logic for choosing this error rate.

In Chapter IV we tried to extend the Lindley-Savage argument, using a Waller-Duncan type decision theoretic formulation of the multiple comparison problem. Characterizing our beliefs or prognosis and using an additive linear loss function we attempted a logic for the choice of error rates. But we were only able to give an improved Bayes rule for the multiple comparison problem in the one-way classification model, with a common unknown variance and no control treatments. We first derived the posterior distribution of $\underline{u}|z$, (4.7); then we partitioned $\underline{u} = (\underline{u}_0 : \underline{u}_r)$ and found the distribution of $\underline{u}_0|z$, (4.8). The moments of $p(u_i|z)$ were given by (4.11)

and (4.12). If we consider only the first term of our estimate of u_i we obtain an estimate which has a structure similar to those of Stein (1962) and Lindley (1971a), though with different weights. In effect we have derived a general estimate of u_i based on posterior means. In our derivation of the posterior distribution of $u|z$ we expanded the distribution of $u|z$ as a double power series in v_1^{-1} and v_2^{-1} which are the degrees of freedom for the distribution. A detailed account of these expansions is given in the Appendix. We also obtained the distribution of $\delta|z$ where $\delta = u_i - u_j$ and derived the following Bayes rule for the symmetric multiple comparison problem. Using the definition of $I_+(z)$ given by (4.59) and $I_-(z)$ given by (4.60) we proposed the following Bayes rule for the component problem $P(i,j)$, $i,j=1,\dots,n$;

\bar{X}_i is significantly greater than \bar{X}_j if

$$\Phi(r_i, r_j, s_e^2, s_B^2, \bar{X}_i, \bar{X}_j, v_1, v_2) > k$$

\bar{X}_i is not significantly greater than \bar{X}_j if

$$\Phi(r_i, r_j, s_e^2, s_B^2, \bar{X}_i, \bar{X}_j, v_1, v_2) \leq k$$

where

$$\Phi(r_i, r_j, s_e^2, s_B^2, \bar{X}_i, \bar{X}_j, v_1, v_2) = I_+(z)/I_-(z) .$$

Therefore when an experimenter enters the consulting statisticians room with his data and his prognosis, we can calculate $\phi(\cdot)$ by a computer program and tell him the appropriate decisions on his means.

We then examined the consequences of the main results which were derived earlier in this chapter as the degrees of freedom get large. We obtained the posterior distribution of $\underline{u}|\underline{z}$ and found estimates of u_i which have the same form as the Lindley estimates but with different weights. We also gave a Bayes rule for the comparison of u_i and u_j $i \neq j$. For the case where

$$(i) \quad r_i = r$$

$$\text{and} \quad (ii) \quad v_1 \rightarrow \infty \quad \text{and} \quad v_2 \rightarrow \infty ,$$

we were able to obtain a Bayes rule which depends on $t = (\bar{X}_i - \bar{X}_j)/(s_e \sqrt{(2/r)})$, which is like the standard t statistic. The Bayes rule for the symmetric multiple comparison problem then takes the following form:

\bar{X}_i is significantly larger than \bar{X}_j if

$$\bar{X}_i - \bar{X}_j > \text{BLSD} ,$$

\bar{X}_i is not significantly different from \bar{X}_j if

$$|\bar{X}_i - \bar{X}_j| \leq \text{BLSD} ,$$

\bar{X}_i is significantly smaller than \bar{X}_j if

$$\bar{X}_i - \bar{X}_j < - \text{BLSD} ,$$

where the Bayes Least Significant Difference

$$\text{BLSD} = s_e \sqrt{(2/r)t(k, s_e^2, s_B^2, r)}$$

This rule for this special case appears to be similar to the Waller-Duncan rule. It would be interesting to compare the two rules. We also gave an algorithm for the computation of the critical values for the Bayes rule which is derived in this chapter.

In Chapter V we extended the results of Chapter IV to other cases and for different prior structures. First we considered the one-way classification with a common unknown variance, no control treatments and response surface priors. Then we looked at the one-way classification model with a control treatment. Finally we investigated the case where data arise from a complete balanced block design.

Approaches to the multiple comparison problem that are commonly used involve the choice of a pivotal function and a choice of an error rate. This study started with the idea that it should be possible to obtain a logic for this process, both with regard to the pivotal function and the choice of α . The underlying idea is that prior opinions of the nature of a

guess expressed in probability terms of the true situation with a specification of a loss function would enable the development of a procedure for deciding the existence of differences between the treatment means.

Bayesian arguments can be used to develop a decision procedure as shown in this dissertation.

The whole process may be questioned from the standpoint that the necessary ingredients in terms of prior distributions and loss functions will not be available in many situations. If an experiment is the first of its kind, a Bayesian proposal is to use what is called a vague prior, but there may be difficulties associated with this process. If experiments like the previous one have been performed before, the results of the previous experiments may be useful in suggesting appropriate priors for the various parameters in the present problem. It is not altogether easy to specify an appropriate loss function because one does not know the consequences of terminal decisions which extend into the indefinite future. The use of an additive loss structure is a strong defect of the development.

The classical procedures have known or developable operating characteristics. For some workers in statistics, knowledge of the operating characteristics of a statistical procedure is considered important. We have no knowledge of the operating characteristics of the procedures developed

herein and whether these procedures have reasonable operating characteristics is unknown. Whether they lead to error rates which are nearly constant and independent of nuisance parameters is unknown. Carmer and Swanson (1973) in an evaluation of ten pairwise multiple comparison procedures by Monte Carlo methods recommended the use of the Waller and Duncan (1969) procedure and the use of a preliminary F test with the LSD since these procedures are more sensitive in detecting real differences. But the procedures suggested in this dissertation are improvements over the Waller and Duncan (1969) procedure, so it is surmised that the procedures contained herein would have reasonable operating characteristics.

In this dissertation we attempted a logic for the choice of α by extending the Lindley-Savage argument which was given for the simple hypothesis versus the simple alternative, to the multiple comparison problem. We found that even though we used priors which do not depend on the data, our critical values for the Bayes rule depend on the data. This shows that an extension of the Lindley-Savage argument does not give us a logic for the choice of α .

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IX. APPENDIX

A. Asymptotic Expansion of the Posterior Distribution

In (4.5) we found the distribution of $\underline{u}|z$,

$$p(\underline{u}|z) \propto \left[1 + \sum_{i=1}^n \frac{r_i (u_i - \bar{X}_i)^2}{f s_e^2} \right]^{-1/2(n+f)} \cdot \left[1 + \sum_i \frac{(u_i - u.)^2}{q_2 s_{B2}^2} \right]^{-1/2(n-1+q_2)} \quad (9.1)$$

$$\text{Let } Q_1 = \sum_{i=1}^n \frac{r_i (u_i - \bar{X}_i)^2}{s_e^2}, \quad v_1 = f, \quad M_1 = \text{diag} \left(\frac{r_i}{s_e^2} \right),$$

$$s_B^2 = s_{B2}^2, \quad Q_2 = \sum_{i=1}^n \frac{1}{s_B^2} (u_i - u.)^2, \quad v_2 = q_2,$$

$$M_2 = \text{diag} \left(\frac{1}{s_B^2} \right), \quad \theta = \underline{u}, \quad k = n, \quad k' = n-1, \quad \hat{\theta}_1' = [\bar{X}_1, \dots, \bar{X}_n]'$$

$$\text{and } \hat{\theta}_2' = [u., \dots, u.]', \quad u. = \sum w_i \bar{X}_i / \sum w_i \text{ with } w_i = \frac{r_i}{r_i s_B^2 + s_e^2}.$$

Then for $k = k'$ we may write $p(\underline{u}|z)$ as

$$p(\theta|z) = c^{-1} g(Q_1, Q_2), \quad -\infty < \theta < \infty$$

where

$$c = \int_{-\infty < \theta < \infty} g(Q_1, Q_2) d\theta \quad (9.2)$$

$$\text{and } g(Q_1, Q_2) = \prod_{i=1}^2 \left[1 + \frac{Q_i}{v_i} \right]^{-1/2(v_i+k)} \quad (9.3)$$

Box and Tiao (1973) in another context gave an asymptotic expansion of the multivariate "double t" probability density. This expansion which is an extension of Fisher (1926) also appeared in Tiao and Zellner (1964). Here the main ideas are given.

Expanding (9.3) as a double power series in v_1^{-1} and v_2^{-2} which are the degrees of freedom for the distribution we have

$$g(Q_1, Q_2) = \exp(-1/(2Q_1)) \exp(-1/(2Q_2)) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_i q_j v_1^{-i} v_2^{-j} \quad (9.4)$$

where

$$\left. \begin{aligned} p_0 &= 1, \\ p_1 &= \frac{1}{4} (Q_1^2 - 2kQ_1), \\ p_2 &= \frac{1}{96} [3Q_1^4 - 4(3k+4)Q_1^3 + 12k(k+2)Q_1^2] \end{aligned} \right\} \quad (9.5)$$

$$\left. \begin{aligned} q_0 &= 1, \\ q_1 &= \frac{1}{4} (Q_2^2 - 2kQ_2) \\ \text{and } q_2 &= \frac{1}{96} [3Q_2^4 - 4(3k+4)Q_2^3 + 12k(k+2)Q_2^2] \end{aligned} \right\} \quad (9.6)$$

For vectors x, a, b , and c of length n and matrices A and B of dimension $n \times n$ it is evident that

$$(x-a)'A(x-a) + (x-b)'B(x-b) = (x-c)'(A+B)(x-c) + (a-b)'A(A+B)^{-1}B(a-b) \quad (9.7)$$

where $c = (A+B)^{-1}(Aa + Bb)$.

Substituting (9.7) into (9.4) we obtain

$$p(\theta|z) = c^{-1} g(Q_1, Q_2) = w^{-1} h(\theta) \quad (9.8)$$

$$\text{where } h(\theta) = \frac{|M|^{1/2}}{(2\pi)^{k/2}} \exp\left(-\frac{1}{2}Q\right) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_i q_j v_1^{-i} v_2^{-j}$$

$$\text{for } Q = (\theta - \tilde{\theta})' M (\theta - \tilde{\theta}), \quad M = M_1 + M_2,$$

$$\tilde{\theta} = M^{-1}(M_1 \hat{\theta}_1 + M_2 \hat{\theta}_2) \quad \text{and}$$

$$w = \int_{-\infty < \theta < \infty} h(\theta) d\theta. \quad (9.9)$$

To evaluate (9.9), first, we find the joint cumulant generating function of Q_1 and Q_2 which is defined as

$$\kappa(t_1, t_2) = \log \int_{-\infty < \theta < \infty} \frac{|M|^{1/2}}{(2\pi)^{k/2}} \exp(t_1 Q_1 + t_2 Q_2 - \frac{1}{2}Q) d\theta. \quad (9.10)$$

It can be shown that the cumulants are given by

$$\left. \begin{aligned} \kappa_{10} &= \text{tr } M^{-1}M_1 + \eta_1' M_1 \eta_1 \\ \kappa_{01} &= \text{tr } M^{-1}M_2 + \eta_2' M_2 \eta_2 \end{aligned} \right\} \quad (9.11)$$

and

$$\begin{aligned} \kappa_{rs} &= 2^{r+s-1} (r+s-2)! \left[(r+s-1) \text{tr } M^{-1}G^{rs} + (r\eta_1 + s\eta_2)' G^{rs} \right. \\ &\quad \left. \cdot (r\eta_1 + s\eta_2) - r\eta_1' G^{rs} \eta_1 - s\eta_2' G^{rs} \eta_2 \right] \quad r+s \geq 2 \end{aligned} \quad (9.12)$$

where $G^{rs} = M(M^{-1}M_1)^r (M^{-1}M_2)^s$

and $\eta_i = (\tilde{\theta} - \hat{\theta}_i)$ for $i=1,2$. (9.13)

Cook (1951) derived formulae giving bivariate population moment-coefficients in terms of cumulants and cumulants in terms of moment coefficients up to the sixth order. Using Cook's inversion formulae we may write (9.9) as

$$w = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij} v_1^{-i} v_2^{-j} \quad (9.14)$$

where

$$b_{00} = 1 \quad ,$$

$$b_{10} = \frac{1}{4} \left[\kappa_{20} + \kappa_{10}^2 - 2\kappa_{10} \right] \quad , \quad (9.15)$$

$$b_{01} = \frac{1}{4} \left[\kappa_{02} + \kappa_{01}^2 - 2\kappa_{01} \right] \quad , \quad (9.16)$$

$$\begin{aligned}
b_{11} = \frac{1}{16} & \left[\kappa_{22} + 2\kappa_{21}\kappa_{01} + \kappa_{20}\kappa_{01}^2 + \kappa_{20}\kappa_{02} + 2\kappa_{12}\kappa_{10} + 2\kappa_{11}^2 \right. \\
& + 4\kappa_{11}\kappa_{10}\kappa_{01} + \kappa_{10}^2\kappa_{01}^2 + \kappa_{10}^2\kappa_{02} \\
& - 2k(\kappa_{12} + \kappa_{21} + \kappa_{02}\kappa_{10} + \kappa_{20}\kappa_{01} + 2\kappa_{11}\kappa_{01} + 2\kappa_{11}\kappa_{10} \\
& \left. + \kappa_{01}^2\kappa_{10} + \kappa_{10}^2\kappa_{01}) + 4k^2(\kappa_{11} + \kappa_{10}\kappa_{01}) \right] . \quad (9.17)
\end{aligned}$$

$$\begin{aligned}
b_{20} = \frac{1}{96} & \left[3(\kappa_{40} + 4\kappa_{30}\kappa_{10} + 3\kappa_{20}^2 + 6\kappa_{20}\kappa_{10}^2 + \kappa_{10}^4) \right. \\
& - 4(3k+4)(\kappa_{30} + 3\kappa_{20}\kappa_{10} + \kappa_{10}^3) \\
& \left. + 12k(k+2)(\kappa_{20} + \kappa_{10}^2) \right] , \quad (9.18)
\end{aligned}$$

and

$$\begin{aligned}
b_{02} = \frac{1}{96} & \left[3(\kappa_{04} + 3\kappa_{02}^2 + 4\kappa_{03}\kappa_{01} + 6\kappa_{02}\kappa_{01}^2 + \kappa_{01}^4) \right. \\
& - 4(3k+4)(\kappa_{03} + 3\kappa_{02}\kappa_{01} + \kappa_{01}^3) \\
& \left. + 12k(k+2)(\kappa_{02} + \kappa_{01}^2) \right] .
\end{aligned}$$

We can now substitute (9.14) into (9.8) to obtain an asymptotic expression for the posterior distribution of θ which is given by

$$p(\theta|z) = \frac{|M|^{1/2}}{(2\pi)^{k/2}} \exp \left[-\frac{1}{2} (\theta - \tilde{\theta})' M (\theta - \tilde{\theta}) \right] \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d_{ij} v_1^{-i} v_2^{-j}$$

(9.19)

$$\text{where } d_{00} = 1 , \quad (9.20)$$

$$d_{10} = p_1 - b_{10} , \quad (9.21)$$

$$d_{01} = q_1 - b_{01} , \quad (9.22)$$

$$d_{11} = (p_1 - b_{10})(q_1 - b_{01}) + b_{10}b_{01} - b_{11} , \quad (9.23)$$

$$d_{20} = p_2 - b_{20} + b_{10}^2 - p_1b_{10} , \quad (9.24)$$

$$\text{and } d_{02} = q_2 - b_{02} + b_{01}^2 - q_1b_{01} . \quad (9.25)$$

We have now expressed the posterior distribution as the product of a multivariate normal distribution and a power series in v_1^{-1} and v_2^{-1} . When v_1 and v_2 get very large, all terms of the power series except the leading one vanishes so that the posterior distribution is asymptotically distributed $N(\tilde{\theta}, M^{-1})$. The terms in the power series can be interpreted for finite values of v_1 and v_2 as the corrections in the normal approximation to the distribution $p(\theta|z)$.

From our posterior distribution, we will now calculate the marginal distribution. Let

$$\theta' = (\theta'_\ell \quad \theta'_r) \quad \text{and} \quad M = \begin{bmatrix} M_{\ell\ell} & M_{\ell r} \\ M_{r\ell} & M_{rr} \end{bmatrix}$$

$$\text{with} \quad M^{-1} = \begin{bmatrix} V_{\ell\ell} & V_{\ell r} \\ V_{r\ell} & V_{rr} \end{bmatrix}.$$

After integrating the unwanted parameters θ_r the marginal posterior density is given by

$$p(\theta_\ell | z) = \frac{|V_{\ell\ell}^{-1}|^{1/2}}{(2\pi)^{\ell/2}} \exp \left[-\frac{1}{2} (\theta_\ell - \tilde{\theta}_\ell)' V_{\ell\ell}^{-1} (\theta_\ell - \tilde{\theta}_\ell) \right] f(\theta_\ell)$$

$$\text{with} \quad -\infty < \theta < \infty \quad (9.26)$$

where

$$f(\theta_\ell) = \frac{|M_{rr}|^{1/2}}{(2\pi)^{r/2}} \int_{-\infty < \theta_r < \infty} \exp \left[-\frac{1}{2} (\theta_r - \bar{\theta}_r)' M_{rr} (\theta_r - \bar{\theta}_r) \right] \cdot \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d_{ij} v_1^{-i} v_2^{-j} d\theta_r \quad (9.27)$$

$$\text{and} \quad \bar{\theta}_r = \tilde{\theta}_r - M_{rr}^{-1} M_{r\ell} (\theta_\ell - \tilde{\theta}_\ell).$$

As before let us partition the following matrices

$$\hat{\theta}'_1 = (\hat{\theta}'_{1\ell} \quad \hat{\theta}'_{1r}), \quad \hat{\theta}'_2 = (\hat{\theta}'_{2\ell} \quad \hat{\theta}'_{2r}),$$

$$M_1 = \begin{bmatrix} B_{\ell\ell} & B_{\ell r} \\ B_{r\ell} & B_{rr} \end{bmatrix} \quad M_1^{-1} = \begin{bmatrix} E_{\ell\ell} & E_{\ell r} \\ E_{r\ell} & E_{rr} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} C_{\ell\ell} & C_{\ell r} \\ C_{r\ell} & C_{rr} \end{bmatrix} \quad \text{and} \quad M_2^{-1} = \begin{bmatrix} F_{\ell\ell} & F_{\ell r} \\ F_{r\ell} & F_{rr} \end{bmatrix}.$$

The mixed cumulants of Q_1 and Q_2 are given by

$$\omega_{10} = \text{tr } M_{rr}^{-1} B_{rr} + \gamma_1' B_{rr} \gamma_1 + (\theta_\ell - \hat{\theta}_{1\ell})' E_{\ell\ell}^{-1} (\theta_\ell - \hat{\theta}_{1\ell}), \quad (9.28)$$

$$\omega_{01} = \text{tr } M_{rr}^{-1} C_{rr} + \gamma_2' C_{rr} \gamma_2 + (\theta_\ell - \hat{\theta}_{2\ell})' F_{\ell\ell}^{-1} (\theta_\ell - \hat{\theta}_{2\ell}) \quad (9.29)$$

and

$$\begin{aligned} \omega_{rs} = 2^{r+s-1} (r+s-2)! & \left[(r+s-1) \text{tr } M_{rr}^{-1} H^{rs} + (r\gamma_1 + s\gamma_2)' H^{rs} \right. \\ & \left. \cdot (r\gamma_1 + s\gamma_2) - r\gamma_1' H^{rs} \gamma_1 - s\gamma_2' H^{rs} \gamma_2 \right], r+s \geq 2, \end{aligned} \quad (9.30)$$

where

$$H^{rs} = M_{rr} (M_{rr}^{-1} B_{rr})^r (M_{rr}^{-1} C_{rr})^s, \quad (9.31)$$

$$\gamma_1 = (\tilde{\theta}_r - \hat{\theta}_{1r}) + (B_{rr}^{-1} B_{r\ell} - M_{rr}^{-1} M_{r\ell}) (\theta_\ell - \hat{\theta}_{1\ell}) \quad (9.32)$$

$$\text{and} \quad \gamma_2 = (\tilde{\theta}_r - \hat{\theta}_{2r}) + (C_{rr}^{-1} C_{r\ell} - M_{rr}^{-1} M_{r\ell}) (\theta_\ell - \hat{\theta}_{2\ell}) \quad (9.33)$$

Now from the above results the distribution of $\theta_\ell | z$ is

$$p(\theta_\ell | z) = \frac{|v_{\ell\ell}|^{-1/2}}{(2\pi)^{\ell/2}} \exp \left[-\frac{1}{2} (\theta_\ell - \tilde{\theta}_\ell)' v_{\ell\ell}^{-1} (\theta_\ell - \tilde{\theta}_\ell) \right] \cdot \sum_{i,j=0}^{\infty} \delta_{ij} v_1^{-i} v_2^{-j} \quad -\infty < \theta_\ell < \infty \quad (9.34)$$

where as before we may show that the quantities δ_{ij} are similar to the quantities d_{ij} which are given by

$$\delta_{00} = 1 \quad , \quad (9.35)$$

$$\delta_{10} = g_{10} - b_{10} \quad , \quad (9.36)$$

$$\delta_{01} = g_{01} - b_{01} \quad , \quad (9.37)$$

$$\delta_{11} = g_{11} - b_{11} - g_{10}b_{01} - g_{01}b_{10} + 2b_{01}b_{10} \quad , \quad (9.38)$$

$$\delta_{20} = g_{20} - b_{20} - g_{10}b_{10} + b_{10}^2 \quad , \quad (9.39)$$

$$\delta_{02} = g_{02} - b_{02} - g_{01}b_{01} + b_{01}^2 \quad , \quad (9.40)$$

$$g_{00} = 1 \quad , \quad (9.41)$$

$$g_{10} = \frac{1}{4} (\omega_{20} + \omega_{10}^2 - 2k\omega_{10}) \quad \text{and} \quad (9.42)$$

$$g_{01} = \frac{1}{4} (\omega_{02} + \omega_{01}^2 - 2k\omega_{01}) \quad . \quad (9.43)$$

It is interesting to note that for $i=0, j=0$ in (9.34) that $p(\theta_{\ell}|z)$ is a multivariate normal distribution with mean $\tilde{\theta}_{\ell}$, and variance-covariance matrix $V_{\ell\ell}$.

B. Some Useful Results in Integration

In this section we give some lemmas which will be applied later in this Appendix. The simple proofs will be omitted.

Lemma 9.1

Let $X \sim N(\mu, \sigma^2)$ then

$$\begin{aligned} B_k &= \frac{1}{\sqrt{(2\pi\sigma^2)}} \int_0^{\infty} x^k \exp\left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma}\right)^2\right] dx \\ &= \frac{1}{\sqrt{(2\pi)}} \exp[-\mu^2/(2\sigma^2)] \sum_{i=0}^{\infty} \frac{(\mu/\sigma^2)^i}{i!} 2^{\frac{k+i-1}{2}} \\ &\quad \cdot \left(\frac{k+i-1}{2}\right)! (\sigma^2)^{\frac{k+i}{2}}. \end{aligned}$$

Proof: Consider

$$\begin{aligned} B'_n &= \int_0^{\infty} x^n \exp\left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma}\right)^2\right] dx \\ &= \exp[-\mu^2/(2\sigma^2)] \int_0^{\infty} x^n \exp[-x^2/(2\sigma^2)] \exp[x\mu/\sigma^2] dx \end{aligned}$$

$$\begin{aligned}
&= \exp[-\mu^2/(2\sigma^2)] \sum_{i=0}^{\infty} \frac{(\mu/\sigma^2)^i}{i!} \int_0^{\infty} x^{n+i} \\
&\quad \cdot \exp[-x^2/(2\sigma^2)] dx \\
&= \exp[-\mu^2/(2\sigma^2)] \sum_{i=0}^{\infty} \frac{(\mu/\sigma^2)^i}{i!} \frac{1}{2} \Gamma(n+i+1)/2 \\
&\quad \cdot (2\sigma^2)^{(n+i+1)/2} .
\end{aligned}$$

From B'_n , we may find B_k .

Lemma 9.2

Let $X \sim N(a, \sigma^2)$ then

$$\mu'_1 = a$$

$$\mu'_2 = \sigma^2 + a^2$$

$$\mu'_3 = a(3\sigma^2 + a^2)$$

$$\mu'_4 = 3\sigma^4 + 6a^2\sigma^2 + a^4 \quad (9.44)$$

where μ'_r is the r -th moment of X .

Lemma 9.3

Let $u_i \sim N(\tilde{u}_i, v_{ii})$ then

$$a) \quad \psi'_i = \omega_i + \xi_i = \left(\frac{r_i}{s_e^2} \right) E(u_i - \bar{X}_i)^2$$

and

$$b) \quad 3\omega_i^2 + \xi_i^2 + 6\xi_i\omega_i = \left(\frac{r_i}{s_e^2} \right)^2 E(u_i - \bar{X}_i)^4 ,$$

$$\text{where } \omega_i = \frac{r_i}{s_e^2} \left(\frac{r_i}{s_e^2} + \frac{1}{s_B^2} \right)^{-1} = \frac{r_i}{s_e^2} v_{ii}$$

$$\begin{aligned} \text{and } \xi_i &= \frac{r_i}{s_e^2} \left[\left(\frac{r_i}{s_e^2} + \frac{1}{s_B^2} \right)^{-1} \left(\frac{r_i \bar{X}_i}{s_e^2} + \frac{u_i}{s_B^2} \right) - \bar{X}_i \right]^2 \\ &= \frac{r_i}{s_e} [\tilde{u}_i - \bar{X}_i]^2 . \end{aligned}$$

$$\begin{aligned} \text{Proof: } a) \quad E(u_i - \bar{X}_i)^2 &= \mu_2' - 2\mu_1 \bar{X}_i + \bar{X}_i^2 \\ &= v_{ii} + \tilde{u}_i^2 - 2\tilde{u}_i \bar{X}_i + \bar{X}_i^2 \\ &= v_{ii} + (\tilde{u}_i - \bar{X}_i)^2 \\ &= \frac{s_e^2}{r_i} (\omega_i + \xi_i) . \end{aligned}$$

$$\begin{aligned}
b) \quad E(u_i - \bar{X}_i)^4 &= \mu_4' - 4\bar{X}_i\mu_3' + 6\bar{X}_i^2\mu_2' - 4\bar{X}_i^3\mu_1' + \bar{X}_i^4 \\
&= 3v_{ii}^2 + 6\tilde{u}_i^2v_{ii} + \tilde{u}_i^4 - 4\bar{X}_i\tilde{u}_i(3v_{ii} + \tilde{u}_i^2) \\
&\quad + 6\bar{X}_i^2(v_{ii} + \tilde{u}_i^2) - 4\bar{X}_i^3\tilde{u}_i + \bar{X}_i^4 \\
&= 3v_{ii}^2 + 6\tilde{u}_i^2v_{ii} + \tilde{u}_i^4 - 12\bar{X}_i\tilde{u}_iv_{ii} \\
&\quad - 4\bar{X}_i\tilde{u}_i^3 + 6\bar{X}_i^2v_{ii} + 6\bar{X}_i^2\tilde{u}_i^2 \\
&\quad - 4\bar{X}_i^3\tilde{u}_i + \bar{X}_i^4 \\
&= 3\omega_i^2 + \xi_i^2 + 6\xi_i\omega_i \quad .
\end{aligned}$$

Lemma 9.4

Let $X \sim N_n(\mu, W)$ then for any vector g and any positive definite symmetric matrix W

$$(2\pi)^{n/2} |W|^{1/2} \exp\left(\frac{1}{2}g'Wg\right) = \int \exp\left(-\frac{1}{2}X'W^{-1}X + g'X\right) dX \quad .$$

where $g' = \mu'W^{-1}$.

C. Moments of the Posterior Distribution

In Chapter IV we gave the first and second moments of the posterior distribution. We also gave a Bayes rule for the comparison of the treatment means in a one-way classification where there are no control treatments. In this section we

derive the particular series expansion of the posterior distribution and the moments of the distribution.

In Equation (4.5) we derived the posterior distribution,

$$p(\underline{u}|z) \propto \left[1 + \sum_{i=1}^n \frac{r_i (\bar{X}_i - u_i)^2}{f s_e^2} \right]^{-1/2(n+f)} \cdot \left[1 + \sum_i \frac{(u_i - u.)^2}{q_2 s_{B2}^2} \right]^{-1/2(n-1+q_2)} \quad (9.45)$$

Using the results of Section A and B we will now expand (9.45) as a double inverse power series in the degrees of freedom f and q_2 . In (9.45) let $q = q_2$, $s_{B2}^2 = s_B^2$,

$$\sum_{i=1}^n \frac{r_i (\bar{X}_i - u_i)^2}{s_e^2} = Q_1 \quad \text{with} \quad v_1 = f, \quad (9.46)$$

$$\sum_{i=1}^n \frac{(u_i - u.)^2}{s_B^2} = Q_2 \quad \text{with} \quad v_2 = q, \quad (9.47)$$

and $s = n-1$. Then

$$p(\underline{u}|z) \propto \left[1 + \frac{Q_1}{v_1} \right]^{-1/2(n+v_1)} \left[1 + \frac{Q_2}{v_2} \right]^{-1/2(s+v_2)}.$$

From (9.46) and (9.47) we have

$$M_1 = \text{diag} \left(\frac{r_i}{s_e^2} \right)$$

and

$$M_2 = \text{diag} \left(\frac{1}{s_B^2} \right) .$$

With $M = M_1 + M_2$, $\hat{\theta}_1' = [\bar{X}_1, \dots, \bar{X}_n]'$ and $\hat{\theta}_2' = [u, \dots, u.]'$,

we have by using

$$\tilde{\theta} = M^{-1} (M_1 \hat{\theta}_1 + M_2 \hat{\theta}_2) ,$$

$$\tilde{\theta} = (\tilde{u}_1, \dots, \tilde{u}_n) \quad (9.48)$$

where

$$\tilde{u}_i = \left(\frac{r_i}{s_e^2} + \frac{1}{s_B^2} \right)^{-1} \left(\frac{r_i \bar{X}_i}{s_e^2} + \frac{u}{s_B^2} \right) . \quad (9.49)$$

Also $M = \text{diag} (v_{ii}^{-1})$

where $v_{ii}^{-1} = \left(\frac{r_i}{s_e^2} + \frac{1}{s_B^2} \right) .$

Let

$$f(\underline{u}; \tilde{u}_i, v_{ii}) = \left[\prod_{i=1}^n v_{ii}^{-1} \right]^{1/2} (2\pi)^{-n/2} \exp \left[-\frac{1}{2} \left(\sum_{i=1}^n (u_i - \tilde{u}_i)^2 v_{ii}^{-1} \right) \right] . \quad (9.50)$$

From (9.19) we may write the posterior distribution of $\underline{u}|z$

with $\underline{u} = \theta$ as

$$p(\underline{u}|\underline{z}) = f(\underline{u}; \tilde{u}_i, v_{ii}) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d_{ij} v_1^{-i} v_2^{-j} \quad (9.51)$$

where $-\infty < \underline{u} < \infty$ and the quantities d_{ij} are given in (9.20) to (9.25).

Let us find the first three terms of the double power series in v_1^{-1} and v_2^{-1} . From (9.21)

$$d_{10} = p_1 - b_{10}.$$

From (9.5) and (9.15)

$$d_{10} = \frac{1}{4}(Q_1^2 - 2kQ_1) - \frac{1}{4}(\kappa_{20} + \kappa_{10}^2 - 2k\kappa_{10}).$$

From (9.11), (9.13), (9.48), and (9.49)

$$\begin{aligned} \kappa_{10} &= \text{tr } M^{-1}M_1 + \eta_1' M_1 \eta_1 \\ &= \sum_{i=1}^n \left(v_{ii} \frac{r_i}{s_e^2} \right) + \sum_{i=1}^n \frac{r_i}{s_e^2} \{ \tilde{u}_i - \bar{X}_i \}^2 \end{aligned}$$

From (9.12)

$$\begin{aligned} \kappa_{20} &= 2 \left[\text{tr}(M^{-1}M_1)^2 + 2\eta_1' M(M^{-1}M_1)^2 \eta_1 \right] \\ &= 2 \left[\sum_{i=1}^n \left(v_{ii} \frac{r_i}{s_e^2} \right)^2 + \sum_{i=1}^n \left(v_{ii} \frac{r_i}{s_e^2} \right)^2 (\tilde{u}_i - \bar{X}_i)^2 \right]. \end{aligned}$$

$$\text{Let } \omega_i = \frac{r_i}{s_e^2} v_{ii} \quad (9.52)$$

$$\text{and} \quad \xi_i = \frac{r_i}{s_e^2} (\tilde{u}_i - \bar{x}_i)^2 \quad . \quad (9.53)$$

$$\text{Then} \quad \kappa_{10} = \sum_{i=1}^n (\omega_i + \xi_i) \quad (9.54)$$

$$\text{and} \quad \kappa_{20} = 2 \left[\sum_{i=1}^n (\omega_i^2 + 2\omega_i \xi_i) \right] \quad . \quad (9.55)$$

Similarly if we let

$$\gamma_i = \frac{1}{s_B^2} v_{ii} \quad (9.56)$$

$$\text{and} \quad \rho_i = \frac{1}{s_B^2} (\tilde{u}_i - u.)^2 \quad , \quad (9.57)$$

we have from (9.11) and (9.12)

$$\kappa_{01} = \sum_{i=1}^n (\gamma_i + \rho_i) \quad (9.58)$$

$$\text{and} \quad \kappa_{02} = 2 \left[\sum_{i=1}^n (\gamma_i^2 + 2\rho_i \gamma_i) \right] \quad . \quad (9.59)$$

Therefore with $n=k$

$$\begin{aligned} d_{10} &= \frac{1}{4} (Q_1^2 - 2nQ_1 - \kappa_{20} - \kappa_{10}^2 + 2n\kappa_{10}) \\ &= \frac{1}{4} \left[Q_1^2 - 2nQ_1 - 2 \left[\sum_{i=1}^n (\omega_i^2 + 2\omega_i \xi_i) \right] - \left[\sum_{i=1}^n (\omega_i + \xi_i) \right]^2 \right. \\ &\quad \left. + 2n \left(\sum_{i=1}^n (\omega_i + \xi_i) \right) \right] \quad . \quad (9.60) \end{aligned}$$

Also with $s=k$

$$\begin{aligned}
 d_{01} &= \frac{1}{4} (Q_2^2 - 2sQ_2 - \kappa_{02} - \kappa_{01}^2 + 2s\kappa_{01}) \\
 &= \frac{1}{4} \left[Q_2^2 - 2sQ_2 - 2 \left(\sum_{i=1}^n (\gamma_i^2 + 2\rho_i \gamma_i) \right) - \left(\sum_{i=1}^n (\gamma_i + \rho_i) \right)^2 \right. \\
 &\quad \left. + 2s \left(\sum_{i=1}^n (\gamma_i + \rho_i) \right) \right]. \quad (9.61)
 \end{aligned}$$

We may rewrite (9.51) as

$$p(\underline{u}|z) = f(\underline{u}; \tilde{\underline{u}}_i, v_{ii}) (1 + v_1^{-1} d_{10} + v_2^{-1} d_{01}) + o(1) + o(1). \quad (9.62)$$

From the posterior distribution given in (9.51) we may derive the marginal distribution of \underline{u}_ℓ where $\underline{u}' = (\underline{u}_\ell'; \underline{u}_r')$. By (9.34) with $\underline{u}=\underline{\theta}$ and $\tilde{\underline{u}}=\tilde{\underline{\theta}}$

$$\begin{aligned}
 p(\underline{u}_\ell | z) &= |v_{\ell\ell}|^{-1/2} (2\pi)^{-1/2\ell} \exp \left[-\frac{1}{2} (\underline{u}_\ell - \tilde{\underline{u}})' v_{\ell\ell}^{-1} (\underline{u}_\ell - \tilde{\underline{u}}) \right] \\
 &\cdot \sum_{i,j=0}^{\infty} \delta_{ij} v_1^{-i} v_2^{-j} \quad \text{for } -\infty < \underline{u}_\ell < \infty, \quad (9.63)
 \end{aligned}$$

where the quantities δ_{ij} are given in (9.35) to (9.43).

Let us find the first three terms of the power series in v_1^{-1} and v_2^{-1} . From (9.36)

$$\delta_{10} = g_{10} - b_{10}.$$

Using (9.15) and (9.42) with $n=k$, we have

$$\delta_{10} = \frac{1}{4} (\omega_{20} + \omega_{10}^2 - 2n\omega_{10} - \kappa_{20} - \kappa_{10}^2 + 2n\kappa_{10}) \quad .$$

From (9.28)

$$\omega_{10} = \text{tr } M_{rr}^{-1} B_{rr} + \gamma_1' B_{rr} \gamma_1 + (\underline{u}_\ell - \hat{\underline{u}}_{1\ell})' E_{\ell\ell}^{-1} (\underline{u}_\ell - \hat{\underline{u}}_{1\ell})$$

where $M_{rr}, B_{rr}, \gamma_1', E_{\ell\ell}$ and $\hat{\underline{u}}_{1\ell}$ are defined in Section IXA .

Making the necessary substitutions, we find

$$\begin{aligned} \omega_{10} = & \sum_{i=\ell+1}^n \left(v_{ii} \frac{r_i}{s_e^2} \right) + \sum_{i=\ell+1}^n \frac{r_i}{s_e^2} (\tilde{u}_i - \bar{X}_i)^2 \\ & + \sum_{i=1}^{\ell} (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \quad . \end{aligned}$$

Using (9.52) and (9.53) we have

$$\omega_{10} = \sum_{i=\ell+1}^n (\omega_i + \xi_i) + \sum_{i=1}^{\ell} (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \quad . \quad (9.64)$$

Also from (9.30)

$$\begin{aligned} \omega_{20} = & 2 \left[\text{tr } (M_{rr}^{-1} B_{rr})^2 + 2\gamma_1' M_{rr} (M_{rr}^{-1} B_{rr})^2 \gamma_1 \right] \\ = & 2 \left[\sum_{i=\ell+1}^n \left(v_{ii} \frac{r_i}{s_e^2} \right)^2 + 2 \sum_{i=\ell+1}^n v_{ii} \left(\frac{r_i}{s_e^2} \right)^2 (\tilde{u}_i - \bar{X}_i)^2 \right] \end{aligned}$$

Using (9.52) and (9.53) we have

$$\omega_{20} = 2 \left[\sum_{i=\ell+1}^n (\omega_i^2 + 2\omega_i \xi_i) \right] . \quad (9.65)$$

Similarly with (9.29), (9.30), (9.56), and (9.57) we may show

$$\omega_{01} = \sum_{i=\ell+1}^n (\gamma_i + \rho_i) + \sum_{i=1}^{\ell} (u_i - \bar{u})^2 / s_B^2 \quad (9.66)$$

$$\text{and } \omega_{02} = 2 \left[\sum_{i=\ell+1}^n (\gamma_i^2 + 2\gamma_i \rho_i) \right] . \quad (9.67)$$

Therefore

$$\begin{aligned} \delta_{10} &= \frac{1}{4} \left[\omega_{20} + \omega_{10}^2 - 2n\omega_{10} - \kappa_{20} - \kappa_{10}^2 + 2n\kappa_{10} \right] \\ &= \frac{1}{4} \left[2 \sum_{i=\ell+1}^n (\omega_i^2 + 2\omega_i \xi_i) + \left(\sum_{i=\ell+1}^n (\omega_i + \xi_i) \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^{\ell} (u_i - \bar{X}_i)^2 \frac{r_i^2}{s_e^2} \right)^2 \right. \\ &\quad \left. - 2n \left(\sum_{i=\ell+1}^n (\omega_i + \xi_i) + \sum_{i=1}^{\ell} (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right) \right. \\ &\quad \left. - 2 \sum_{i=1}^n (\omega_i^2 + 2\omega_i \xi_i) - \left(\sum_{i=1}^n (\omega_i + \xi_i) \right)^2 \right. \\ &\quad \left. + 2n \sum_{i=1}^n (\omega_i + \xi_i) \right] \quad (9.68) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \left[-2 \sum_{i=1}^{\ell} (\omega_i^2 + 2\omega_i \xi_i) + \left(\sum_{i=\ell+1}^n (\omega_i + \xi_i) \right. \right. \\
&\quad \left. \left. + \sum_{i=1}^{\ell} (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right)^2 - \left(\sum_{i=1}^n (\omega_i + \xi_i) \right)^2 \right. \\
&\quad \left. + 2n \left(\sum_{i=1}^n (\omega_i + \xi_i) - \sum_{i=\ell+1}^n (\omega_i + \xi_i) - \sum_{i=1}^{\ell} (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right) \right] \\
&= \frac{1}{4} \left[-2 \sum_{i=1}^{\ell} (\omega_i^2 + 2\omega_i \xi_i) + \left(- \sum_{i=1}^{\ell} (\omega_i + \xi_i) + \sum_{i=1}^{\ell} (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right) \right. \\
&\quad \cdot \left(\sum_{i=1}^{\ell} (\omega_i + \xi_i) + 2 \sum_{i=\ell+1}^n (\omega_i + \xi_i) + \sum_{i=1}^{\ell} (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right) \\
&\quad \left. + 2n \left(\sum_{i=1}^{\ell} (\omega_i + \xi_i) - \sum_{i=1}^{\ell} (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right) \right] \\
&= \frac{1}{4} \left[-2 \sum_{i=1}^{\ell} (\omega_i^2 + 2\omega_i \xi_i) - \left(\sum_{i=1}^{\ell} (\omega_i + \xi_i) - \sum_{i=1}^{\ell} (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right) \right. \\
&\quad \cdot \left(\sum_{i=1}^{\ell} (\omega_i + \xi_i) + 2 \sum_{i=\ell+1}^n (\omega_i + \xi_i) + \sum_{i=1}^{\ell} (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} - 2n \right) \left. \right].
\end{aligned}$$

(9.69)

Also

$$\begin{aligned}
\delta_{01} &= g_{01} - b_{01} \\
&= \frac{1}{4} \left[\omega_{02} + \omega_{01}^2 - 2s\omega_{01} - \kappa_{02} - \kappa_{01}^2 + 2s\kappa_{01} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \left[2 \sum_{i=\ell+1}^n (\gamma_i^2 + 2\gamma_i \rho_i) + \left(\sum_{i=\ell+1}^n (\gamma_i + \rho_i) \right. \right. \\
&\quad \left. \left. + \sum_{i=1}^{\ell} (u_i - u.)^2 / s_B^2 \right)^2 \right. \\
&\quad \left. - 2s \left(\sum_{i=\ell+1}^n (\gamma_i + \rho_i) + \sum_{i=1}^{\ell} (u_i - u.)^2 / s_B^2 \right)^2 \right. \\
&\quad \left. - 2 \sum_{i=1}^n (\gamma_i^2 + 2\rho_i \gamma_i) \right. \\
&\quad \left. - \left(\sum_{i=1}^n (\gamma_i + \rho_i) \right)^2 + 2s \sum_{i=1}^n (\gamma_i + \rho_i) \right] \\
&= \frac{1}{4} \left[-2 \sum_{i=1}^{\ell} (\gamma_i^2 + 2\rho_i \gamma_i) - \left(\sum_{i=1}^{\ell} (\gamma_i + \rho_i) - \sum_{i=1}^{\ell} (u_i - u.)^2 / s_B^2 \right) \right. \\
&\quad \left. \left(\sum_{i=1}^{\ell} (\gamma_i + \rho_i) + 2 \sum_{i=\ell+1}^n (\gamma_i + \rho_i) + \sum_{i=1}^{\ell} (u_i - u.)^2 / s_B^2 - 2s \right) \right].
\end{aligned}
\tag{9.70}$$

Now substituting (9.69) and (9.70) into (9.63) we may write (9.63) using only the first three terms of the power series in v_1^{-1} and v_2^{-1} as

$$\begin{aligned}
p(\underline{u}_\ell | z) &= f(\underline{u}_\ell; \tilde{u}_i, v_{ii}) \left[1 + v_1^{-1} \delta_{10} + v_2^{-1} \delta_{01} \right] \\
&\quad + o(1) + o(1),
\end{aligned}
\tag{9.71}$$

where δ_{10} and δ_{01} are given by (9.69) and (9.70)

respectively. Note that (9.71) is a polynomial in u_i .

From Equation (9.71), with $\ell=3$ we have the joint distribution of u_1, u_2 , and u_3 given z which is

$$p(u_1, u_2, u_3 | z) = f(u_1, u_2, u_3; \tilde{u}_i, v_{ii}) \left[1 + v_1^{-1} \delta_{10} + v_2^{-1} \delta_{01} \right] + o(1) + o(1) \quad (9.72)$$

For $\ell=3$, we find the values of δ_{10} and δ_{01} . From (9.68) we have for $\ell=3$

$$\begin{aligned} \delta_{10} = \frac{1}{4} & \left[2 \sum_{i=4}^n (\omega_i^2 + 2\omega_i \xi_i) + \left(\sum_{i=4}^n (\omega_i + \xi_i) + \sum_{i=1}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right)^2 \right. \\ & - 2n \left(\sum_{i=4}^n (\omega_i + \xi_i) + \sum_{i=1}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right) - 2 \sum_{i=1}^n (\omega_i^2 + 2\omega_i \xi_i) \\ & \left. - \left(\sum_{i=1}^n (\omega_i + \xi_i) \right)^2 + 2n \sum_{i=1}^n (\omega_i + \xi_i) \right]. \end{aligned}$$

Let $\psi_i' = \omega_i + \xi_i$. Then we may rewrite

$$\begin{aligned} \delta_{10} = \frac{1}{4} & \left[2 \sum_{\substack{i=1 \\ i \neq 1,3}}^n (\omega_i^2 + 2\omega_i \xi_i) - 2\omega_2^2 - 4\omega_2 \xi_2 + \left[\sum_{\substack{i=1 \\ i \neq 1,3}}^n \psi_i' + \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right. \right. \\ & \left. \left. - \psi_2' + (u_2 - \bar{X}_2)^2 \frac{r_2}{s_e^2} \right]^2 - \left\{ 2n \left[\sum_{\substack{i=1 \\ i \neq 1,3}}^n \psi_i' + \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right] \right. \right. \\ & \left. \left. + 2n \left[-\psi_2' + (u_2 - \bar{X}_2)^2 \frac{r_2}{s_e^2} \right] \right\} - 2 \sum_{i=1}^n (\omega_i^2 + 2\omega_i \xi_i) \right] \end{aligned}$$

$$- \left[\sum_{i=1}^n \psi_i' \right]^2 + 2n \sum_{i=1}^n \psi_i' \right]. \quad (9.73)$$

From (9.73) consider

$$\begin{aligned} & \left[\sum_{\substack{i=1 \\ i \neq 1,3}}^n \psi_i' + \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} - \psi_2' + (u_2 - \bar{X}_2)^2 \frac{r_2}{s_e^2} \right]^2 \\ &= \left[\sum_{\substack{i=1 \\ i \neq 1,3}}^n \psi_i' + \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} - \psi_2' \right]^2 + (u_2 - \bar{X}_2)^4 \left(\frac{r_2}{s_e^2} \right)^2 + 2(u_2 - \bar{X}_2)^2 \frac{r_2}{s_e} \\ & \left[\sum_{\substack{i=1 \\ i \neq 1,3}}^n \psi_i' + \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} - \psi_2' \right] = \left[\sum_{\substack{i=1 \\ i \neq 1,3}}^n \psi_i' + \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right]^2 \\ & + \psi_2'^2 - 2\psi_2' \left[\sum_{\substack{i=1 \\ i \neq 1,3}}^n \psi_i' + \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right] + (u_2 - \bar{X}_2)^4 \left(\frac{r_2}{s_e^2} \right)^2 \\ & + 2(u_2 - \bar{X}_2)^2 \frac{r_2}{s_e} \cdot \left[\sum_{\substack{i=1 \\ i \neq 1,3}}^n \psi_i' + \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} - \psi_2' \right] \end{aligned} \quad (9.74)$$

We insert (9.74) in (9.73) and using Lemma 9.3 from Section IXB where

$$\psi_i' = \frac{r_i}{s_e^2} E(u_i - \bar{X}_i)^2 \quad \text{for all } i \quad \text{and} \quad (9.75)$$

$$(3\omega_i^2 + \xi_i^2 + 6\xi_i\omega_i) = \left(\frac{r_i}{s_e^2} \right)^2 E(u_i - \bar{X}_i)^4, \quad (9.76)$$

we combine (9.73) with the density $f(u_1, u_2, u_3; \tilde{u}_i, v_{ii})$ and integrate with respect to u_2 to obtain

$$\begin{aligned}
\delta'_{10} &= \frac{1}{4} \left[2 \sum_{\substack{i=1 \\ i \neq 1,3}}^n (\omega_i^2 + 2\omega_i \xi_i) - 2\omega_2^2 - 4\omega_2 \xi_2 + \left(\sum_{\substack{i=1 \\ i \neq 1,3}}^n \psi'_i \right. \right. \\
&\quad \left. \left. + \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right)^2 + \psi_2'^2 - 2\psi_2' \left(\sum_{\substack{i=1 \\ i \neq 1,3}}^n \psi'_i + \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right) \right. \\
&\quad \left. + 3\omega_2^2 + \xi_2^2 + 6\xi_2 \omega_2 + 2\psi_2' \left(\sum_{\substack{i=1 \\ i \neq 1,3}}^n \psi'_i + \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right) \right. \\
&\quad \left. - 2\psi_2'^2 - \left\{ 2n \left(\sum_{\substack{i=1 \\ i \neq 1,3}}^n \psi'_i + \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right) + 2n(-\psi_2' + \psi_2') \right\} \right. \\
&\quad \left. - 2 \sum_{i=1}^n (\omega_i^2 + 2\omega_i \xi_i) - \left(\sum_{i=1}^n \psi'_i \right)^2 + 2n \sum_{i=1}^n \psi'_i \right] \\
&= \frac{1}{4} \left[2 \sum_{\substack{i=1 \\ i \neq 1,3}}^n (\omega_i^2 + 2\omega_i \xi_i) + \left(\sum_{\substack{i=1 \\ i \neq 1,3}}^n \psi'_i + \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right)^2 \right. \\
&\quad \left. - 2n \left(\sum_{\substack{i=1 \\ i \neq 1,3}}^n \psi'_i + \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right) - 2 \sum_{i=1}^n (\omega_i^2 + 2\omega_i \xi_i) \right. \\
&\quad \left. - \left(\sum_{i=1}^n \psi'_i \right)^2 + 2n \sum_{i=1}^n \psi'_i \right] \\
&= \frac{1}{4} \left[-2(\omega_1^2 + 2\omega_1 \xi_1 + \omega_3^2 + 2\omega_3 \xi_3) - \left\{ \bar{\psi}_1 + \bar{\psi}_3 - \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right\} \right]
\end{aligned}$$

$$\cdot \left\{ \psi'_1 + \psi'_3 + 2 \sum_{\substack{i=1 \\ i \neq 1,3}}^n \psi'_i + \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} - 2n \right\} \quad (9.77)$$

Similarly, we may find with $\psi''_i = \gamma_i + \rho_i$,

$$\delta'_{01} = \frac{1}{4} \left[-2(\gamma_1^2 + 2\gamma_1\rho_1 + \gamma_3^2 + 2\gamma_3\rho_3) - \left(\psi''_1 + \psi''_3 - \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - u.)^2 / s_B^2 \right) \right. \\ \left. \cdot \left(\psi''_1 + \psi''_3 + 2 \sum_{\substack{i=1 \\ i \neq 1,3}}^n \psi''_i + \sum_{\substack{i=1 \\ i \neq 2}}^3 (u_i - u.)^2 / s_B^2 - 2(n-1) \right) \right] \quad (9.78)$$

Therefore using δ'_{10} and δ'_{01} as given in (9.77) and (9.78) we see by integrating out (9.72) with respect to u_2 that

$$p(u_1, u_3 | z) = f(u_1, u_3; \tilde{u}_i, v_{ii}) \left[1 + \frac{1}{4} v_1^{-1} \delta'_{10} + \frac{1}{4} v_2^{-1} \delta'_{01} \right] \\ + o(1) + o(1) \quad (9.79)$$

Comparing $p(u_1, u_3 | z)$ and $p(u_1, u_2 | z)$ which is easily obtained from (9.70) with $\ell=2$, we can see the symmetry in the distribution so that if we have the joint distribution of u_1 and u_2 we may write the distribution of u_i and u_j , $i \neq j$ for all $i, j=1, \dots, n$.

Therefore we may write the joint density of u_i and u_j given the data as follows

$$p(u_i, u_j | z) = f(u_i, u_j; \tilde{u}_i, v_{ii})$$

$$\begin{aligned} & \cdot \left[1 + \frac{1}{4} v_1^{-1} \left[-2(\omega_i^2 + 2\omega_i \xi_i + \omega_j^2 + 2\omega_j \xi_j) \right. \right. \\ & \quad \left. \left. - \left(\psi_i' + \psi_j' - (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} - (u_j - \bar{X}_j)^2 \frac{r_j}{s_e^2} \right) \right. \right. \\ & \quad \cdot \left(\psi_i' + \psi_j' + 2 \sum_{\substack{k=1 \\ k \neq i, j}}^n \psi_k' + (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} + (u_j - \bar{X}_j)^2 \frac{r_j}{s_e^2} - 2n \right) \left. \right. \\ & \quad + \frac{1}{4} v_2^{-1} \left[-2(\gamma_i^2 + 2\gamma_i \rho_i + \gamma_j^2 + 2\gamma_j \rho_j) - (\psi_i'' + \psi_j'' \right. \\ & \quad \left. - (u_i - u.)^2 / s_B^2 - (u_j - u.)^2 / s_B^2) \left(\psi_i'' + \psi_j'' + 2 \sum_{\substack{k=1 \\ k \neq i, j}}^n \psi_k'' \right. \right. \\ & \quad \left. \left. + (u_i - u.)^2 / s_B^2 + (u_j - u.)^2 / s_B^2 - 2(n-1) \right) \right] \left. \right] \\ & \quad + o(1) + o(1) \quad . \end{aligned} \tag{9.80}$$

To obtain the posterior density of $u_i | z$ we integrate out u_j , $j \neq i$ and using Lemma 9.3 from Section IXB we obtain

$$p(u_i | z) = f(u_i; \tilde{u}_i, v_{ii})$$

$$\cdot \left[1 + \frac{1}{4} v_1^{-1} \left[-2(\omega_i^2 + 2\omega_i \xi_i) - \left(\psi_i' - (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right) \right] \right]$$

$$\begin{aligned}
& \cdot \left(\psi_i' + 2 \sum_{\substack{j=1 \\ j \neq i}}^n \psi_j' + (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} - 2n \right) \Big] \\
& + \frac{1}{4} v_2^{-1} \left[-2(\gamma_i^2 + 2\rho_i \gamma_i) - \left(\psi_i'' - (u_i - u.)^2 / s_B^2 \right) \right. \\
& \cdot \left. \left(\psi_i'' + 2 \sum_{\substack{j=1 \\ j \neq i}}^n \psi_j'' + (u_i - u.)^2 / s_B^2 - 2(n-1) \right) \right] \Big] \\
& + o(1) + o(1) \tag{9.81}
\end{aligned}$$

Employing the expression for the moments of a Normal variable, we will now derive an asymptotic expression for the moments of $p(u_i | z)$.

In (9.81) consider the coefficients of $\frac{1}{4} v_1^{-1}$ and $\frac{1}{4} v_2^{-1}$ respectively in the power series in v_1^{-1} and v_2^{-1} . Therefore

$$\begin{aligned}
& \left[-2(\omega_i^2 + 2\omega_i \xi_i) - \left(\psi_i' - (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right) \left(\psi_i' + 2 \sum_{\substack{j=1 \\ j \neq i}}^n \psi_j' + (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} - 2n \right) \right] \\
& = \left[-2(\omega_i^2 + 2\omega_i \xi_i) - \left(\psi_i'^2 + 2\psi_i' \sum_{\substack{j=1 \\ j \neq i}}^n \psi_j' + \psi_i' (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right. \right. \\
& \quad \left. \left. - \psi_i' (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} - 2(u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \sum_{\substack{j=1 \\ j \neq i}}^n \psi_j' - (u_i - \bar{X}_i)^4 \left(\frac{r_i}{s_e^2} \right)^2 - 2n \psi_i' \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + 2n(u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \Big) \Big] \\
& = \left[-2(\omega_i^2 + 2\omega_i \xi_i) - \psi_i'^2 - 2\psi_i' \sum_{\substack{j=1 \\ j \neq i}}^n \psi_j' + 2n\psi_i' \right. \\
& \quad \left. + (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \left(2 \sum_{\substack{j=1 \\ j \neq i}}^n \psi_j' - 2n \right) + (u_i - \bar{X}_i)^4 \left(\frac{r_i}{s_e^2} \right)^2 \right].
\end{aligned} \tag{9.82}$$

Similarly the coefficient of $\frac{1}{4}v_2^{-1}$ is

$$\begin{aligned}
& = \left[-2(\gamma_i^2 + 2\gamma_i \rho_i) - \psi_i''^2 - 2\psi_i'' \sum_{\substack{j=1 \\ j \neq i}}^n \psi_j'' + 2(n-1)\psi_i'' \right. \\
& \quad \left. + \frac{(u_i - u.)^2}{s_B^2} \left(2 \sum_{\substack{j=1 \\ j \neq i}}^n \psi_j'' - 2(n-1) \right) + (u_i - u.)^4 \left(\frac{1}{s_B^2} \right)^2 \right].
\end{aligned} \tag{9.83}$$

If we consider the leading term of the power series in (9.81) we see that

$$u_i \sim N(\tilde{u}_i, v_{ii}) \tag{9.84}$$

Before we find the moments of (9.81), it will be instructive to evaluate the following expressions with respect to u_i which is distributed $N(\tilde{u}_i, v_{ii})$

$$\begin{aligned}
E[u_i (u_i - \bar{X}_i)^2] &= E[u_i (u_i^2 - 2u_i \bar{X}_i + \bar{X}_i^2)] \\
&= \mu_3' - 2\mu_2' \bar{X}_i + \bar{X}_i^2 \mu_1' \\
&= 3\tilde{u}_i v_{ii} + \tilde{u}_i^3 - 2\bar{X}_i v_{ii} - 2\bar{X}_i \tilde{u}_i^2 + \bar{X}_i^2 \tilde{u}_i .
\end{aligned} \tag{9.85}$$

The last line follows immediately from Lemma 9.2 which is given in Section IXB. Also

$$\begin{aligned}
E[u_i (u_i - \bar{X}_i)^4] &= [\mu_5' - 4\bar{X}_i \mu_4' + 6\bar{X}_i^2 \mu_3' - 4\bar{X}_i^3 \mu_2' + \bar{X}_i^4 \mu_1'] \\
&= 15v_{ii} \tilde{u}_i + 10v_{ii} \tilde{u}_i^3 + \tilde{u}_i^5 - 4\bar{X}_i (3v_{ii}^2 + 6\tilde{u}_i^2 v_{ii} + \tilde{u}_i^4) \\
&\quad + 6\bar{X}_i^2 \tilde{u}_i (3v_{ii} + \tilde{u}_i^2) - 4\bar{X}_i^3 (v_{ii} + \tilde{u}_i^2) + \bar{X}_i^4 \tilde{u}_i .
\end{aligned} \tag{9.86}$$

Now using (9.81) to (9.86) we have the mean of the posterior distribution is given by

$$\begin{aligned}
E(u_i | z) &= \tilde{u}_i + \frac{1}{4} v_1^{-1} \left[-2(\omega_i^2 + 2\omega_i \xi_i) \tilde{u}_i - \Psi_i'^2 \tilde{u}_i - 2\Psi_i' \tilde{u}_i \sum_{\substack{j=1 \\ j \neq i}}^n \Psi_j' \right. \\
&\quad + 2n\Psi_i' \tilde{u}_i + \frac{r_i}{s_e^2} \left(2 \sum_{\substack{j=1 \\ j \neq i}}^n \Psi_j' - 2n \right) \left(3\tilde{u}_i v_{ii} + \tilde{u}_i^3 - 2\bar{X}_i v_{ii} \right. \\
&\quad \left. \left. - 2\bar{X}_i \tilde{u}_i^2 + \bar{X}_i^2 \tilde{u}_i \right) + \left(\frac{r_i}{s_e^2} \right)^2 \left[15v_{ii} \tilde{u}_i + 10v_{ii} \tilde{u}_i^3 + \tilde{u}_i^5 - 4\bar{X}_i \right. \right. \\
&\quad \left. \left. \cdot (3v_{ii}^2 + 6\tilde{u}_i^2 v_{ii} + \tilde{u}_i^4) + 6\bar{X}_i^2 \tilde{u}_i (3v_{ii} + \tilde{u}_i^2) - 4\bar{X}_i^3 (v_{ii} + \tilde{u}_i^2) \right] \right]
\end{aligned}$$

$$\begin{aligned}
& + \bar{X}_i \tilde{u}_i \Big] \Big] + \frac{1}{4} v_2^{-1} [\text{coefficient}] \\
& = \tilde{u}_i + \frac{1}{4} v_1^{-1} \left[-2 \left(\frac{r_i}{s_e^2} \right)^2 \left(v_{ii}^2 + 2v_{ii} (\tilde{u}_i^2 - 2\tilde{u}_i \bar{X}_i + \bar{X}_i^2) \right) \tilde{u}_i \right. \\
& \quad - \left(\frac{r_i}{s_e^2} \right)^2 (v_{ii}^2 + 2v_{ii} (\tilde{u}_i - \bar{X}_i)^2 + (\tilde{u}_i - \bar{X}_i)^4) \tilde{u}_i \\
& \quad - 2\tilde{u}_i \Psi_i' \sum_{\substack{j=1 \\ j \neq i}}^n \Psi_j' + 2n \left(\frac{r_i}{s_e^2} \right) (v_{ii} + \tilde{u}_i^2 - 2\tilde{u}_i \bar{X}_i + \bar{X}_i^2) \tilde{u}_i \\
& \quad + \left(\frac{r_i}{s_e^2} \right) \left(2 \sum_{\substack{j=1 \\ j \neq i}}^n \Psi_j' - 2n \right) \left[3\tilde{u}_i v_{ii} + \tilde{u}_i^3 - 2\bar{X}_i v_{ii} - 2\bar{X}_i \tilde{u}_i^2 \right. \\
& \quad \left. + \bar{X}_i \tilde{u}_i \right] + \left(\frac{r_i}{s_e^2} \right)^2 (15v_{ii} \tilde{u}_i + 10v_{ii} \tilde{u}_i^3 + \tilde{u}_i^5 - 12\bar{X}_i v_{ii}^2 \\
& \quad - 24\bar{X}_i \tilde{u}_i^2 v_{ii} - 4\bar{X}_i \tilde{u}_i^4 + 18\bar{X}_i^2 \tilde{u}_i v_{ii} + 6\bar{X}_i^2 \tilde{u}_i^3 - 4\bar{X}_i^3 v_{ii} \\
& \quad \left. - 4\bar{X}_i^3 \tilde{u}_i^2 + \bar{X}_i^4 \tilde{u}_i) \right] + \frac{1}{4} v_2^{-1} [\text{coefficient}] . \tag{9.87} \\
& = \tilde{u}_i + \frac{1}{4} v_1^{-1} \left[\left(\frac{r_i}{s_e^2} \right)^2 (-2v_{ii}^2 \tilde{u}_i - 4v_{ii} \tilde{u}_i^3 + 8\tilde{u}_i^2 v_{ii} \bar{X}_i \right. \\
& \quad - 4v_{ii} \bar{X}_i^2 \tilde{u}_i - v_{ii}^2 \tilde{u}_i - 2v_{ii} \tilde{u}_i^3 + 4\tilde{u}_i^2 v_{ii} \bar{X}_i - 2v_{ii} \tilde{u}_i \bar{X}_i^2 \\
& \quad \left. - \tilde{u}_i^5 + 4\tilde{u}_i^4 \bar{X}_i - 6\tilde{u}_i^3 \bar{X}_i^2 + 4\tilde{u}_i^2 \bar{X}_i^3 - \bar{X}_i^4 \tilde{u}_i + 15v_{ii} \tilde{u}_i \right.
\end{aligned}$$

$$\begin{aligned}
& + 10v_{ii}\tilde{u}_i^3 + \tilde{u}_i^5 - 12\bar{X}_i v_{ii}^2 - 24\bar{X}_i \tilde{u}_i^2 v_{ii} - 4\bar{X}_i \tilde{u}_i^4 \\
& + 18\bar{X}_i^2 \tilde{u}_i v_{ii} + 6\bar{X}_i^2 \tilde{u}_i^3 - 4\bar{X}_i^3 v_{ii} - 4\bar{X}_i^3 \tilde{u}_i^2 + \bar{X}_i^4 \tilde{u}_i) \\
& + \left(\frac{r_i}{s_e^2} \right) \left(\sum_{\substack{j=1 \\ j \neq i}}^n \psi_j' \right) \left[-2\tilde{u}_i v_{ii} - 2\tilde{u}_i^3 + 4\tilde{u}_i^2 \bar{X}_i \right. \\
& \left. - 2\tilde{u}_i \bar{X}_i^2 + 6\tilde{u}_i v_{ii} + 2\tilde{u}_i^3 - 4\bar{X}_i v_{ii} - 4\bar{X}_i \tilde{u}_i^2 + 2\bar{X}_i^2 \tilde{u}_i \right] \\
& + 2n \frac{r_i}{s_e^2} \left[v_{ii} \tilde{u}_i + \tilde{u}_i^3 - 2\tilde{u}_i^2 \bar{X}_i + \bar{X}_i^2 \tilde{u}_i - 3\tilde{u}_i v_{ii} - \tilde{u}_i^3 \right. \\
& \left. + 2\bar{X}_i v_{ii} + 2\bar{X}_i \tilde{u}_i^2 - \bar{X}_i^2 \tilde{u}_i \right] + \frac{1}{4} v_2^{-1} [\text{coefficient}]. \\
= & \tilde{u}_i + \frac{1}{4} v_1^{-1} \left[\left(\frac{r_i}{s_e^2} \right)^2 \left(-3v_{ii}^2 \tilde{u}_i + 15v_{ii} \tilde{u}_i + 4v_{ii} \tilde{u}_i^3 - 12\bar{X}_i v_{ii}^2 \right. \right. \\
& \left. \left. - 12\bar{X}_i \tilde{u}_i^2 v_{ii} + 12\bar{X}_i^2 \tilde{u}_i v_{ii} - 4\bar{X}_i^3 v_{ii} \right) + \frac{r_i}{s_e^2} \left(\sum_{\substack{j=1 \\ j \neq i}}^n \left(\frac{r_i}{s_e^2} \right) \right. \right. \\
& \left. \left. [v_{jj} + (u_j - \bar{X}_j)^2] \right) \left(4\tilde{u}_i v_{ii} - 4\bar{X}_i v_{ii} \right) + \left(2n \frac{r_i}{s_e^2} \right) \right. \\
& \left. \left\{ -2v_{ii} \tilde{u}_i + 2\bar{X}_i v_{ii} \right\} + \frac{1}{4} v_2^{-1} \left[\left(\frac{1}{s_B^2} \right)^2 \left(-3v_{ii}^2 \tilde{u}_i + 15v_{ii} \tilde{u}_i \right. \right. \right. \\
& \left. \left. + 4v_{ii} \tilde{u}_i^3 - 12u_{ii} v_{ii}^2 - 12u_{ii} \tilde{u}_i^2 v_{ii} + 12u_{ii}^2 \tilde{u}_i v_{ii} - 4u_{ii}^3 v_{ii} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{s_B^2} \left(\sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{s_B^2} (v_{jj} + (\tilde{u}_i - u.)^2) \right) \{4\tilde{u}_i v_{ii} - 4u. v_{ii}\} \\
& + (2n/s_B^2) \{ -2v_{ii} \tilde{u}_i + 2u. v_{ii} \} \quad . \quad (9.88)
\end{aligned}$$

To find the variance of $u_i|z$ it is only required to find the second raw moment from which we may calculate the variance using the fact that

$$\text{Var}(u_i|z) = E(u_i^2|z) - E^2(u_i|z) .$$

$$\begin{aligned}
E(u_i^2|z) &= \mu_2' + \frac{1}{4} v_1^{-1} \left[-2(\omega_i^2 + 2\omega_i \xi_i) \mu_2' - \psi_i'^2 \mu_2' - 2\psi_i' \sum_{\substack{j=1 \\ j \neq i}}^n \psi_j' \mu_2' \right. \\
&+ 2n\psi_i' \mu_2' + \left(\frac{r_i}{s_e^2} \right) 2 \left(\sum_{\substack{j=1 \\ j \neq i}}^n \psi_j' - 2n \right) (\mu_4' - 2\mu_3' \bar{X}_i + \bar{X}_i^2 \mu_2') \\
&+ \left(\frac{r_i}{s_e^2} \right)^2 \left[\mu_6' - 4\bar{X}_i \mu_5' + 6\bar{X}_i^2 \mu_4' - 4\bar{X}_i^3 \mu_3' + \bar{X}_i^4 \mu_2' \right] \\
&+ \frac{1}{4} v_2^{-1} \left[\text{coefficient} \right] . \\
&= (v_{ii} + \tilde{u}_i^2) + \frac{1}{4} v_1^{-1} \left[-2 \left(\frac{r_i}{s_e^2} \right)^2 (v_{ii}^2 + 2v_{ii} (\tilde{u}_i - \bar{X}_i)^2) \right. \\
&\quad \left. (v_{ii} + \tilde{u}_i^2) - (v_{ii} + \tilde{u}_i^2) \psi_i' \left(\psi_i' + 2 \sum_{\substack{j=1 \\ j \neq i}}^n \psi_j' - 2n \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{r_i}{s_e^2} \left(2 \sum_{\substack{j=1 \\ j \neq i}}^n \psi_j' - 2n \right) \left(3v_{ii}^2 + 6\tilde{u}_i^2 v_{ii} + \tilde{u}_i^4 - 2\bar{X}_i \tilde{u}_i \right. \\
& \quad \cdot (3v_{ii} + \tilde{u}_i^2) + \bar{X}_i^2 (v_{ii} + \tilde{u}_i^2) \left. \right) + \left(\frac{r_i}{s_e^2} \right)^2 \left(15v_{ii}^3 + 45v_{ii}^2 \tilde{u}_i^2 \right. \\
& \quad + 15v_{ii} \tilde{u}_i^4 + \tilde{u}_i^6 - 4\bar{X}_i (15v_{ii} \tilde{u}_i + 10v_{ii} \tilde{u}_i^3 + \tilde{u}_i^5) \\
& \quad + 6\bar{X}_i^2 (3v_{ii}^2 + 6\tilde{u}_i^2 v_{ii} + \tilde{u}_i^4) - 4\bar{X}_i^3 (3v_{ii} \tilde{u}_i + \tilde{u}_i^3) \\
& \quad \left. + \bar{X}_i^4 (v_{ii} + \tilde{u}_i^2) \right) \left. \right] + \frac{1}{4} v_2^{-1} [\text{coefficient}] , \tag{9.89}
\end{aligned}$$

where the coefficient of v_2^{-1} may be obtained by symmetry.

To find the Bayes rule we need $p(\delta|z)$. Let us now find $p(\delta|z)$. From (9.71) let $l=2$ then

$$\begin{aligned}
p(u_1, u_2 | z) &= f(u_1, u_2; \tilde{u}_i, v_{ii}) \left[1 + v_1^{-1} \delta_{10} + v_2^{-1} \delta_{01} \right] \\
&\quad + o(1) + o(1) \tag{9.90}
\end{aligned}$$

where

$$\begin{aligned}
\delta_{10} &= \frac{1}{4} \left[-2 \sum_{i=1}^2 (\omega_i^2 + 2\omega_i \xi_i) - \left(\sum_{i=1}^2 (\omega_i + \xi_i) - \sum_{i=1}^2 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} \right) \right. \\
&\quad \cdot \left. \left(\sum_{i=1}^2 (\omega_i + \xi_i) + 2 \sum_{i=3}^n (\omega_i + \xi_i) + \sum_{i=1}^2 (u_i - \bar{X}_i)^2 \frac{r_i}{s_e^2} - 2n \right) \right] . \tag{9.91}
\end{aligned}$$

and

$$\delta_{01} = \frac{1}{4} \left[-2 \sum_{i=1}^2 (\gamma_i^2 + 2\rho_i \gamma_i) - \left(\sum_{i=1}^2 (\gamma_i + \rho_i) - \sum_{i=1}^2 (u_i - u_*)^2 \frac{1}{s_B^2} \right) \right]$$

$$\cdot \left(\sum_{i=1}^2 (\gamma_i + \rho_i) + 2 \sum_{i=3}^n (\gamma_i + \rho_i) + \sum_{i=1}^2 (u_i - u_2)^2 \frac{1}{s_B^2} - 2(n-1) \right) \Bigg] . \quad (9.92)$$

Perform the transformation of (u_1, u_2) to $\delta = u_1 - u_2$ and u_2 in (9.90). Then the leading term of the power series in (9.90) which is the joint density of u_1 and u_2 such that $u_i \sim N(\tilde{u}_i, v_{ii})$ for all i becomes after integrating with respect to u_2

$$f(\delta; \tilde{u}_i, v_{ii}) = \left[\sqrt{(2\pi)} \sqrt{\sum_{i=1}^2 v_{ii}} \right]^{-1} \exp \left[-\frac{1}{2 \sum_{i=1}^2 v_{ii}} (\delta - (\tilde{u}_1 - \tilde{u}_2))^2 \right] . \quad (9.93)$$

Therefore the leading term in the posterior distribution of $\delta|z$ is distributed normally with mean $(\tilde{u}_1 - \tilde{u}_2)$ and variance $(v_{11} + v_{22})$.

Let $\psi_i' = \omega_i + \xi_i$ and $\delta = u_1 - u_2$ then (9.91) becomes

$$\begin{aligned} \delta_{10} = \frac{1}{4} \Bigg[& -2 \sum_{i=1}^2 (\omega_i^2 + 2\omega_i \xi_i) - \left(\sum_{i=1}^2 \psi_i' - \frac{r_1}{s_e^2} (\delta + u_2 - \bar{X}_1)^2 \right. \\ & \left. - \frac{r_2}{s_e^2} (u_2 - \bar{X}_2)^2 \right) \left(\sum_{i=1}^2 \psi_i' + 2 \sum_{i=3}^n \psi_i' - 2n + \frac{r_1}{s_e^2} (\delta + u_2 - \bar{X}_1)^2 \right. \\ & \left. + \frac{r_2}{s_e^2} (u_2 - \bar{X}_2)^2 \right) \Bigg] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \left[-2 \sum_{i=1}^n (\omega_i^2 + 2\xi_i \omega_i) - \left(\sum_{i=1}^2 \psi_i^1 \left(\sum_{i=1}^2 \psi_i^1 + 2 \sum_{i=3}^n \psi_i^1 - 2n \right) \right. \right. \\
&\quad + \sum_{i=1}^2 \psi_i^1 \frac{r_1}{s_e^2} (\delta + u_2 - \bar{X}_1)^2 + \sum_{i=1}^2 \psi_i^1 \frac{r_2}{s_e^2} (u_2 - \bar{X}_2)^2 \\
&\quad - \left(\sum_{i=1}^2 \psi_i^1 + 2 \sum_{i=3}^n \psi_i^1 - 2n \right) \frac{r_1}{s_e^2} (\delta + u_2 - \bar{X}_1)^2 - \left(\frac{r_1}{s_e^2} \right)^2 (\delta + u_2 - \bar{X}_1)^4 \\
&\quad - 2 \frac{r_1 r_2}{s_e^4} (\delta + u_2 - \bar{X}_1)^2 (u_2 - \bar{X}_2)^2 - \frac{r_2}{s_e^2} (u_2 - \bar{X}_2)^2 \cdot \left\{ \sum_{i=1}^2 \psi_i^1 \right. \\
&\quad \left. + 2 \sum_{i=3}^n \psi_i^1 - 2n \right\} - \left(\frac{r_2}{s_e^2} \right)^2 (u_2 - \bar{X}_2)^4 \left. \right] \\
&= \frac{1}{4} \left[-2 \sum_{i=1}^n (\omega_i^2 + 2\xi_i \omega_i) - \sum_{i=1}^2 \psi_i^1 \left(\sum_{i=1}^2 \psi_i^1 + 2 \sum_{i=3}^n \psi_i^1 - 2n \right) \right. \\
&\quad + 2 \frac{r_1}{s_e^2} (\delta + u_2 - \bar{X}_1)^2 \left(\sum_{i=3}^n \psi_i^1 - n \right) + 2 \frac{r_2}{s_e^2} (u_2 - \bar{X}_2)^2 \left(\sum_{i=3}^n \psi_i^1 - n \right) \\
&\quad + 2 \frac{r_1 r_2}{s_e^4} (u_2 - \bar{X}_1)^2 (\delta + u_2 - \bar{X}_1)^2 + \left(\frac{r_2}{s_e^2} \right)^2 (u_2 - \bar{X}_2)^4 \\
&\quad \left. + \left(\frac{r_1}{s_e^2} \right)^2 (\delta + u_2 - \bar{X}_1)^4 \right] \quad (9.94)
\end{aligned}$$

Similarly with $\Psi_i'' = \gamma_i + \rho_i$ and $\delta = u_1 - u_2$ (9.92) becomes

$$\begin{aligned} \delta_{01} = \frac{1}{4} & \left[-2 \sum_{i=1}^n (\gamma^2 + 2\gamma_i \rho_i) - \sum_{i=1}^2 \Psi_i'' \left(\sum_{i=1}^2 \Psi_i'' + 2 \sum_{i=3}^n \Psi_i'' - 2(n-1) \right) \right. \\ & + \frac{2}{s_B^2} (\delta + u_2 - u.)^2 \left(\sum_{i=3}^n \Psi_i'' - (n-1) \right) + \frac{2}{s_B^2} (u_2 - u.)^2 \left(\sum_{i=3}^n \Psi_i'' - (n-1) \right) \\ & \left. + \frac{2}{s_B^4} (u_2 - u.)^2 (\delta + u_2 - u.)^2 + \frac{1}{s_B^4} (u_2 - u.)^4 + \frac{1}{s_B^4} (\delta + u_2 - u.)^4 \right] \end{aligned} \quad (9.95)$$

Therefore

$$p(\delta, u_2 | z) = f(\delta, u_2; \tilde{u}_i, v_{ii}) [1 + v_1^{-1} \delta_{10} + v_2^{-1} \delta_{01}] + o(1) + o(1) \quad (9.96)$$

where δ_{10} and δ_{01} are given by (9.94) and (9.95)

respectively. Here $f(\delta, u_2; \tilde{u}_i, v_{ii})$ is obtained from

$f(u_1, u_2; \tilde{u}_i, v_{ii})$ by performing the transformation of (u_1, u_2) to $\delta = u_1 - u_2$ and u_2 .

Before we integrate (9.96) with respect to u_2 to obtain $p(\delta | z)$ it is helpful to compute the following integrals. Let the leading term of (9.96) be

$$\begin{aligned} J = f(\delta, u_2; \tilde{u}_i, v_{ii}) &= \left[\sum_{i=1}^2 v_{ii}^{-1} \right]^{1/2} (2\pi)^{-1} \exp \left[-\frac{1}{2} (\delta + u_2 - \tilde{u}_1)^2 v_{11}^{-1} \right. \\ &\quad \left. - \frac{1}{2} (u_2 - \tilde{u}_2)^2 v_{22}^{-1} \right]. \end{aligned}$$

Also in (9.93) let $I = f(\delta; \tilde{u}_i, v_{ii})$.

$$\begin{aligned}
\text{Then } \int (\delta + u_2 - \bar{X}_1)^2 J du_2 \\
= \int \{(\delta - \bar{X}_1)^2 + u_2^2 + 2u_2(\delta - \bar{X}_1)\} J du_2 \\
= (\delta - \bar{X}_1)^2 I + \int u_2^2 J du_2 + 2(\delta - \bar{X}_1) \int u_2 J du_2 .
\end{aligned}$$

$$\text{Now } \int u_2 J du_2$$

$$\begin{aligned}
&= \left[\sum_{i=1}^2 v_{ii}^{-1} \right]^{1/2} (2\pi)^{-1} \exp \left[-\frac{1}{2} \{ (\delta - \tilde{u}_1)^2 v_{11}^{-1} + \tilde{u}_2^2 v_{22}^{-1} \} \right] \\
&\quad \int u_2 \exp \left[-\frac{1}{2} \{ u_2^2 [v_{11}^{-1} + v_{22}^{-1}] + 2 u_2 [(\delta - \tilde{u}_1) v_{11}^{-1} \right. \\
&\quad \left. - \tilde{u}_2 v_{22}^{-1}] \} \right] du_2 .
\end{aligned}$$

By Lemma 9.4 in Section IXB the above expression becomes

$$\begin{aligned}
&\left[\sum_{i=1}^2 v_{ii}^{-1} \right]^{1/2} (2\pi)^{-1} \exp \left[-\frac{1}{2} \{ (\delta - \tilde{u}_1)^2 v_{11}^{-1} + \tilde{u}_2^2 v_{22}^{-1} \} \right] (2\pi)^{1/2} \\
&\quad (v_{11}^{-1} + v_{22}^{-1})^{-1/2} \cdot \exp \left[\frac{1}{2} \{ \tilde{u}_2 v_{22}^{-1} - (\delta - \tilde{u}_1) v_{11}^{-1} \}^2 \right] (v_{11}^{-1} + v_{22}^{-1})^{-1} E(u_2)
\end{aligned}$$

where $u_2 \sim N(\mu_1, \mu_2)$

with $\mu_1 = - \left[(\delta - \tilde{u}_1) v_{11}^{-1} - \tilde{u}_2 v_{22}^{-1} \right] (v_{11}^{-1} + v_{22}^{-1})^{-1}$

and $\mu_2 = (v_{11}^{-1} + v_{22}^{-1})^{-1}$.

Therefore we may show $\int u_2 J du_2 = IE(u_2)$.

So $\int (\delta + u_2 - \bar{X}_1)^2 J du_2$

$$= (\delta - \bar{X}_1)^2 I + I(\mu_2') + 2I(\delta - \bar{X}_1)\mu_1' \quad . \quad (9.97)$$

Similarly

$$\int (u_2 - \bar{X}_2)^2 J du_2 = I(\mu_2' - 2\bar{X}_2\mu_1' + \bar{X}_2^2) \quad (9.98)$$

also $\int (u_2 - \bar{X}_2)^2 (\delta + u_2 - \bar{X}_1)^2 J du_2$

$$\begin{aligned} &= \int (u_2^2 - 2u_2\bar{X}_2 + \bar{X}_2^2) [(\delta - \bar{X}_1)^2 + u_2^2 + 2(\delta - \bar{X}_1)u_2] J du_2 \\ &= I \left[\mu_4' + 2[(\delta - \bar{X}_1) - \bar{X}_2]\mu_3' + [(\delta - \bar{X}_1)^2 - 4\bar{X}_2(\delta - \bar{X}_1) + \bar{X}_2^2]\mu_2' \right. \\ &\quad \left. + 2[(\delta - \bar{X}_1)\bar{X}_2^2 - (\delta - \bar{X}_1)^2\bar{X}_2]\mu_1' + \bar{X}_2^2(\delta - \bar{X}_1)^2 \right] \quad , \quad (9.99) \end{aligned}$$

$$\int (u_2 - \bar{X}_2)^4 J du_2 = I \left[\mu_4' - 4\bar{X}_2\mu_3' + 6\bar{X}_2^2\mu_2' - 4\bar{X}_2^3\mu_1' + \bar{X}_2^4 \right] \quad (9.100)$$

$$\begin{aligned} \text{and } \int (\delta - \bar{X}_1 + u_2)^4 J du_2 &= I \left[\mu_4' + 4(\delta - \bar{X}_1)\mu_3' + 6(\delta - \bar{X}_1)^2\mu_2' + 4(\delta - \bar{X}_1)^3\mu_1' \right. \\ &\quad \left. + (\delta - \bar{X}_1)^4 \right] \quad . \quad (9.101) \end{aligned}$$

Using (9.97) to (9.101) we may now write the integral of (9.96) w.r.t u_2 as

$$p(\delta|z) = f(\delta; \tilde{u}_i, v_{ii}) \left[1 + v_1^{-1} \delta_{10} + v_2^{-1} \delta_{01} \right] + o(1) + o(1) \quad (9.102)$$

where

$$\begin{aligned}
\delta_{10} = \frac{1}{4} & \left[-2 \sum_{i=1}^2 (\omega_i^2 + 2\omega_i \xi_i) - \sum_{i=1}^2 \psi_i' \left(\sum_{i=1}^2 \psi_i' + 2 \sum_{i=3}^n \psi_i' - 2n \right) \right. \\
& + \left(\sum_{i=3}^n \psi_i' - n \right) \cdot \left(2 \frac{r_1}{s_e^2} [(\delta - \bar{X}_1)^2 + \mu_2' + 2(\delta - \bar{X}_1) \mu_1'] \right. \\
& + 2 \frac{r_2}{s_e^2} [\mu_2' - 2\bar{X}_2 \mu_1' + \bar{X}_2^2] \\
& + 2 \frac{r_1 r_2}{s_e^4} \left(\mu_4' + 2[\delta - \bar{X}_1 - \bar{X}_2] \mu_3' + [(\delta - \bar{X}_1)^2 - 4\bar{X}_2(\delta - \bar{X}_1) \right. \\
& \quad \left. \left. + \bar{X}_2^2] \mu_2' \right. \right. \\
& + 2 [(\delta - \bar{X}_1) \bar{X}_2^2 - (\delta - \bar{X}_1)^2 \bar{X}_2] \mu_1' + \bar{X}_2^2 (\delta - \bar{X}_1)^2 \\
& + \left(\frac{r_2}{s_e^2} \right)^2 (\mu_4' - 4\bar{X}_2 \mu_3' + 6\bar{X}_2^2 \mu_2' - 4\bar{X}_2^3 \mu_1' + \bar{X}_2^4) \\
& + \left(\frac{r_1}{s_e^2} \right)^2 (\mu_4' + 4(\delta - \bar{X}_1) \mu_3' + 6(\delta - \bar{X}_1)^2 \mu_2' + 4(\delta - \bar{X}_1)^3 \mu_1' \\
& \quad \left. \left. + (\delta - \bar{X}_1)^4 \right) \right] \quad (9.103)
\end{aligned}$$

and

$$\begin{aligned}
\delta_{01} = \frac{1}{4} & \left[-2 \sum_{i=1}^2 (\gamma_i^2 + 2\rho_i \gamma_i) - \sum_{i=1}^2 \psi_i'' \left(\sum_{i=1}^2 \psi_i'' + 2 \sum_{i=3}^n \psi_i'' - 2(n-1) \right) \right. \\
& + \frac{2}{s_B^2} \left(\sum_{i=3}^n \psi_i'' - (n-1) \right) \left[(\delta - u.)^2 + \mu_2' + 2(\delta - u.) \mu_1' \right.
\end{aligned}$$

$$\begin{aligned}
& + (\mu_2' - 2u \cdot \mu_1' + u^2) \Big] + \frac{2}{s_B} (\mu_4' + 2(\delta - 2u) \mu_3' \\
& + [(\delta - u)^2 - 4u \cdot (\delta - u) + u^2] \mu_2' + 2[(\delta - u)u^2 - (\delta - u)^2 u] \\
& \cdot \mu_1' + u^2 (\delta - u)^2 + \frac{1}{s_B} (\mu_4' - 4u \cdot \mu_3' + 6u^2 \mu_2' - 4u^3 \mu_1' + u^4) \\
& + \frac{1}{s_B} (\mu_4' + 4(\delta - u) \mu_3' + 6(\delta - u)^2 \mu_2' + 4(\delta - u)^3 \mu_1' \\
& + (\delta - u)^4) \Big] \cdot (9.104)
\end{aligned}$$

It is instructive to note that

$$\mu_r' = E(u_2^r) \quad \text{where} \quad u_2 \sim N(\mu_1', \mu_2') \quad ,$$

$$\mu_1' = - \left[(\delta - \tilde{u}_1) v_{11}^{-1} - \tilde{u}_2 v_{22}^{-1} \right] \cdot \left[v_{11}^{-1} + v_{22}^{-1} \right]^{-1}$$

and

$$\mu_2' = \left[v_{11}^{-1} + v_{22}^{-1} \right]^{-1} .$$

$$\text{For } \mu_1' = \delta a + b \quad \text{where} \quad a = - v_{11}^{-1} \left[v_{11}^{-1} + v_{22}^{-1} \right]^{-1} \quad (9.105)$$

$$\text{and} \quad b = \left[\tilde{u}_2 v_{22}^{-1} + \tilde{u}_1 v_{11}^{-1} \right] \left[v_{11}^{-1} + v_{22}^{-1} \right]^{-1} \quad \text{we have}$$

$$\mu_2 = -a v_{11} \quad .$$

From Lemma 9.2 in Section IXB we have

$$\begin{aligned}
\mu_2' &= -av_{11} + (\delta a + b)^2 \\
&= a^2\delta^2 + 2\delta ab + b^2 - av_{11}
\end{aligned} \tag{9.106}$$

also

$$\begin{aligned}
\mu_3' &= (\delta a + b) [3(-av_{11}) + (\delta a + b)^2] \\
&= a^3\delta^3 + \delta^2(2a^2b + ba^2) + \delta(-3a^2v_{11} + 3ab^2) + b^3 \\
&\quad - 3abv_{11}
\end{aligned} \tag{9.107}$$

$$\begin{aligned}
\mu_4' &= 3(-av_{11})^2 + 6(\delta a + b)^2(-av_{11}) + (\delta a + b)^4 \\
&= \delta^4 a^4 + 4\delta^3 a^3 b + \delta^2(-6a^3 v_{11} + 6a^2 b^2) + \delta(-12a^2 b v_{11} \\
&\quad + 4ab^3) + 3a^2 v_{11}^2 - 6b^2 a v_{11} + b^4
\end{aligned} \tag{9.108}$$

Substituting (9.105) to (9.108) in (9.102) we obtain

$$\begin{aligned}
p(\delta|z) &= I \left[1 + \frac{1}{4} v_1^{-1} \left[-2 \sum_{i=1}^2 (\omega_i^2 + 2\xi_i \omega_i) - \sum_{i=1}^2 \psi_i' \right. \right. \\
&\quad \cdot \left(\sum_{i=1}^2 \psi_i' + 2 \sum_{i=3}^n \psi_i' - 2n \right) + \left(\sum_{i=3}^n \psi_i' - n \right) \\
&\quad \cdot \left(2 \frac{r_1}{s_e^2} [\delta^2 - 2\delta \bar{X}_1^2 + \bar{X}_1^2 + a^2 \delta^2 + 2\delta ab + b^2 - av_{11} + 2(\delta - \bar{X}_1) \right. \\
&\quad \cdot (\delta a + b)] + 2 \frac{r_2}{s_e^2} [a^2 \delta^2 + 2\delta ab + b^2 - av_{11} - 2\bar{X}_2(\delta a + b) + \bar{X}_2^2] \left. \right]
\end{aligned}$$

$$\begin{aligned}
& + 2 \frac{r_1 r_2}{s_e^4} \left[\delta^4 a^4 + 4\delta^3 a^3 b + \delta^2 (-6a^3 v_{11} + 6a^2 b^2) \right. \\
& + \delta (-12a^2 b v_{11} + 4ab^3) + 3a^2 v_{11}^2 - 6b^2 a v_{11} + b^4 \\
& + 2(\delta - \bar{X}_1 - \bar{X}_2) (a^3 \delta^3 + \delta^2 (2a^2 b + ba^2) + \delta (-3a^2 v_{11} + 3ab^2) \\
& + b^3 - 3ab v_{11}) + (\delta^2 - 2\delta \bar{X}_1 + \bar{X}_1^2 - 4\bar{X}_2 \delta + 4\bar{X}_2 \bar{X}_1 + \bar{X}_2^2) \\
& \cdot (a^2 \delta^2 + 2\delta ab + b^2 - a v_{11}) + 2(\delta \bar{X}_2^2 - \bar{X}_1 \bar{X}_2^2 - \delta^2 \bar{X}_2 \\
& + 2\delta \bar{X}_1 \bar{X}_2 - \bar{X}_1^2 \bar{X}_2) (\delta a + b) + \bar{X}_2^2 \delta^2 - 2\bar{X}_2^2 \delta \bar{X}_1 + \bar{X}_2^2 \bar{X}_1^2 \left. \right] \\
& + \left(\frac{r_2}{s_e^2} \right)^2 \cdot (a^4 \delta^4 + 4\delta^3 a^3 b + \delta^2 (-6a^3 v_{11} + 6a^2 b^2) \\
& + \delta (-12a^2 b v_{11} + 4ab^3) + 3a^2 v_{11}^2 - 6b^2 a v_{11} + b^4 \\
& - 4\bar{X}_2 (a^3 \delta^3 + \delta^2 (2a^2 b + ba^2) + \delta (-3a^2 v_{11} + 3ab^2) \\
& + b^3 - 3ab v_{11}) + 6\bar{X}_2^2 (a^2 \delta^2 + 2\delta ab + b^2 - a v_{11}) \\
& - 4\bar{X}_2^3 (\delta a + b) + \bar{X}_2^4) + \left(\frac{r_1}{s_e^2} \right)^2 \cdot (a^4 \delta^4 + 4\delta^3 a^3 b + \delta^2 \\
& \cdot (-6a^3 v_{11} + 6a^2 b^2) + \delta (-12a^2 b v_{11} + 4ab^3) \\
& + 3a^2 v_{11}^2 - 6b^2 a v_{11} + b^4 + 4(\delta - \bar{X}_1) (a^3 \delta^3 + \delta^2 (2a^2 b + ba^2)
\end{aligned}$$

$$\begin{aligned}
& + \delta(-3a^2v_{11} + 3ab^2) + b^3 - 2abv_{11}) + (6\delta^2 - 12\delta\bar{X}_1 + 6\bar{X}_1^2) \\
& \cdot (a^2\delta^2 + 2\delta ab + b^2 - av_{11}) + 4(\delta^3 - 3\delta^2\bar{X}_1 + 3\delta\bar{X}_1^2 - \bar{X}_1^3) \\
& \cdot (\delta a + b) + \delta^4 - 4\delta^3\bar{X}_1 + 6\delta^2\bar{X}_1^2 - 4\delta\bar{X}_1^3 + \bar{X}_1^3 \Big] \\
& + \frac{1}{4}v_2^{-1} [\text{coefficient}] \Big] + o(1) + o(1) \quad .
\end{aligned}$$

Simplifying we obtain

$$\begin{aligned}
p(\delta|z) &= I \left[1 + \frac{1}{4}v_1^{-1} \left[-2 \sum_{i=1}^2 (\omega_i^2 + 2\xi_i \omega_i) - \sum_{i=1}^2 \psi_i' \right. \right. \\
& \quad \cdot \left(\sum_{i=1}^2 \psi_i' + 2 \sum_{i=3}^n \psi_i' - 2n \right) + \left(\sum_{i=3}^n \psi_i' - n \right) \\
& \quad \left(2 \frac{r_1}{s_e^2} \left[\delta^2(1 + a^2 + 2a) + \delta(-2\bar{X}_1 + 2ab + 2b - 2a\bar{X}_1) \right. \right. \\
& \quad \left. \left. + \bar{X}_1^2 + b^2 - av_{11} - 2b\bar{X}_1 \right] \right. \\
& \quad \left. + 2 \frac{r_2}{s_e^2} \left[\delta^2 a^2 + 2\delta(ab - a\bar{X}_2) + b^2 - av_{11} - 2\bar{X}_2 b + \bar{X}_2^2 \right] \right) \\
& \quad \left. + 2 \frac{r_1 r_2}{s_e^4} \left[\delta^4(a^4 + 2a^3 + a^2) + \delta^3(4a^3 b + 4a^2 b + 2ba^2) \right. \right. \\
& \quad \left. \left. - 2(\bar{X}_1 + \bar{X}_2)a^3 + 2ab - 2\bar{X}_1 a^2 - 4\bar{X}_2 a^2 - 2a\bar{X}_2 \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \delta^2 (-6a^3 v_{11} + 6a^2 b^2 - 6a^2 v_{11} + 6ab^2 - 2(\bar{X}_1 + \bar{X}_2)(2a^2 b + ba^2) \\
& + b^2 - av_{11} - 4\bar{X}_1 ab + a^2(\bar{X}_1^2 + 4\bar{X}_2 \bar{X}_1 + \bar{X}_2^2) - 8ab\bar{X}_2 + 2a\bar{X}_2^2 + 4a\bar{X}_1 \bar{X}_2 \\
& - 2b\bar{X}_2 + \bar{X}_2^2) + \delta(-12a^2 b v_{11} + 4ab^3 + 2b^3 - 6abv_{11} - 2(\bar{X}_1 + \bar{X}_2) \\
& \cdot (-3a^2 v_{11} + 3ab^2) - 2\bar{X}_1(b^2 - av_{11}) + 2ab(\bar{X}_1^2 + 4\bar{X}_2 \bar{X}_1 + \bar{X}_2^2) \\
& - 4\bar{X}_2(b^2 - av_{11}) - 2a\bar{X}_1 \bar{X}_2^2 - 2a\bar{X}_1^2 \bar{X}_2 + 2b\bar{X}_2^2 + 4b\bar{X}_1 \bar{X}_2 - 2\bar{X}_2^2 \bar{X}_1) \\
& + 3a^2 v_{11}^2 - 6b^2 av_{11} + b^4 - 2(\bar{X}_1 + \bar{X}_2)(b^3 - 3abv_{11}) \\
& + (\bar{X}_1^2 + 4\bar{X}_2 \bar{X}_1 + \bar{X}_2^2)(b^2 - av_{11}) + 2b(-\bar{X}_1 \bar{X}_2^2 - \bar{X}_1^2 \bar{X}_2) + \bar{X}_2^2 \bar{X}_1^2] \\
& + \left(\frac{r_2}{s_e^2}\right)^2 [a^4 \delta^4 + \delta^3(4a^3 b - 4\bar{X}_2 a^3) + \delta^2(-6a^3 v_{11} + 6a^2 b^2 - 4\bar{X}_2 \\
& \cdot (2a^2 b + ba^2) + 6a^2 \bar{X}_2^2) + \delta(-12a^2 b v_{11} + 4ab^3 - 4\bar{X}_2 \\
& \cdot (-3a^2 v_{11} + 3ab^2) + 12\bar{X}_2^2 ab - 4\bar{X}_2^3 a) + 3a^2 v_{11}^2 - 6b^2 av_{11} \\
& + b^4 - 4\bar{X}_2(b^2 - 3abv_{11}) + 6\bar{X}_2^2(b^2 - av_{11}) - 4\bar{X}_2^3 b + \bar{X}_2^4] \\
& + \left(\frac{r_1}{s_e^2}\right)^2 [\delta^4(a^4 + 4a^3 + 6a^2 + 4a + 1) + \delta^3(4a^3 b + 4(2a^2 b + ba^2)
\end{aligned}$$

$$\begin{aligned}
& - 4\bar{X}_1 a^3 + 12ab - 12\bar{X}_1 a^2 - 12a\bar{X}_1 + 4b - 4\bar{X}_1) \\
& + \delta^2 (-6a^3 v_{11} + 6a^2 b^2 + 4(-3a^2 v_{11} + 3ab^2) \\
& - 4\bar{X}_1 (2a^2 b + ba^2) + 6b^2 - 6av_{11} - 24\bar{X}_1 ab + 6\bar{X}_1^2 a^2 \\
& + 12a\bar{X}_1^2 - 12b\bar{X}_1 + 6\bar{X}_1^2) + \delta (-12a^2 b v_{11} + 4ab^3 + 4b^3 \\
& - 12ab v_{11} - 4\bar{X}_1 (-3a^2 v_{11} + 3ab^2) - 12\bar{X}_1 (b^2 - av_{11}) \\
& + 12\bar{X}_1^2 ab - 4a\bar{X}_1^3 + 12b\bar{X}_1 - 4\bar{X}_1^3) + 3a^2 v_{11}^2 - 6b^2 av_{11} \\
& + b^4 - 4\bar{X}_1 (b^3 - 3ab v_{11}) + 6\bar{X}_1^2 (b^2 - av_{11}) \\
& - 4b\bar{X}_1^3 + \bar{X}_1^3) \Big] + \frac{1}{4} v_2^{-1} \left[\text{coefficient} \right] \\
& + o(1) + o(1) \quad . \quad (9.109)
\end{aligned}$$

The coefficient of $\frac{1}{4} v_2^{-1}$ is easily determined from our previous calculations. By a rearrangement of the terms in (9.109) we have the posterior distribution of $\delta|z$ is given as

$$p(\delta|z) = f(\delta; \tilde{u}_i, v_{ii})$$

$$\begin{aligned}
& \cdot \left[1 + \frac{1}{4} v_1^{-1} (g_{10} + \delta g_{11} + \delta^2 g_{12} + \delta^3 g_{13} + \delta^4 g_{14}) \right. \\
& \quad \left. + \frac{1}{4} v_2^{-1} (g_{20} + \delta g_{21} + \delta^2 g_{22} + \delta^3 g_{23} + \delta^4 g_{24}) \right] \quad (9.110)
\end{aligned}$$

where

$$\begin{aligned}
g_{10} = & - \sum_{i=1}^2 (\omega_i^2 + 2\xi_i \omega_i) - \sum_{i=1}^2 \psi_i' \left(\sum_{i=1}^2 \psi_i' + 2 \sum_{i=3}^n \psi_i' - 2n \right) \\
& + \left(\sum_{i=3}^n \psi_i' - n \right) \left(2 \frac{r_1}{s_e^2} (\bar{X}_1^2 + b^2 - av_{11} - 2b\bar{X}_1) \right. \\
& + 2 \frac{r_2}{s_e^2} (b^2 - av_{11} - 2\bar{X}_2 b + \bar{X}_2^2) \left. \right) + 2 \frac{r_1 r_2}{s_e^4} \\
& \cdot [3a^2 v_{11}^2 - 6b^2 av_{11} + b^4 - 2(\bar{X}_1 + \bar{X}_2)(b^3 - 3abv_{11}) \\
& + (\bar{X}_1^2 + 4\bar{X}_2 \bar{X}_1 + \bar{X}_2^2)(b^2 - av_{11}) - 2b(\bar{X}_1 \bar{X}_2^2 + \bar{X}_1^2 \bar{X}_2) + \bar{X}_2^2 \bar{X}_1^2] \\
& + \left(\frac{r_2}{s_e^2} \right)^2 [3a^2 v_{11}^2 - 6b^2 av_{11} + b^4 - 4\bar{X}_2(b^2 - 3abv_{11}) \\
& + 6\bar{X}_2^2 b^2 - 6\bar{X}_2^2 av_{11} - 4\bar{X}_2^3 b + \bar{X}_2^4] + \left(\frac{r_1}{s_e^2} \right)^2 [3a^2 v_{11}^2 - 6b^2 av_{11} \\
& + b^4 - 4\bar{X}_1 b^2 + 12\bar{X}_1 abv_{11} + 6\bar{X}_1^2 b^2 - 6\bar{X}_1^2 av_{11} \\
& - 4b\bar{X}_1^3 + \bar{X}_1^3] \quad (9.111)
\end{aligned}$$

$$\begin{aligned}
g_{11} = & \left(\sum_{i=3}^n \psi_i' - n \right) \left(2 \frac{r_1}{s_e^2} (-2\bar{X}_1 + 2ab + 2b - 2a\bar{X}_1) + 2 \frac{r_2}{s_e^2} \right. \\
& \cdot (2ab - 2a\bar{X}_2) + 2 \frac{r_1 r_2}{s_e^4} [-12a^2 bv_{11} + 4ab^3 + 2b^3 - 6abv_{11}
\end{aligned}$$

$$\begin{aligned}
& - 2(\bar{X}_1 + \bar{X}_2)(-3a^2v_{11} + 3ab^2) - 2\bar{X}_1(b^2 - av_{11}) + 2ab \\
& \cdot (\bar{X}_1^2 + 4\bar{X}_2\bar{X}_1 + \bar{X}_2^2) - 4\bar{X}_2(b^2 - av_{11}) - 2a\bar{X}_1\bar{X}_2^2 - 2a\bar{X}_1^2\bar{X}_2 \\
& + 2b\bar{X}_2^2 + 4b\bar{X}_1\bar{X}_2 - 2\bar{X}_2^2\bar{X}_1 + \left(\frac{r_2}{s_e^2}\right)^2 \left[-12a^2bv_{11} + 4ab^3 \right. \\
& \left. - 4\bar{X}_2(-3a^2v_{11} + 3ab^2) + 12\bar{X}_2^2ab - 4\bar{X}_2^3a \right] + \left(\frac{r_1}{s_2}\right)^2 \\
& \left[-12a^2bv_{11} + 4ab^3 + 4b^3 - 12abv_{11} + 4\bar{X}_1(3a^2v_{11} - 3ab^2) \right. \\
& \left. - 12\bar{X}_1(b^2 - av_{11}) + 12\bar{X}_1^2ab - 4a\bar{X}_1^3 + 12b\bar{X}_1^2 - 4\bar{X}_1^3 \right] \quad (9.112)
\end{aligned}$$

$$\begin{aligned}
g_{12} = & \left(\sum_{i=3}^n \psi_i' - n \right) \left(2 \frac{r_1}{s_e^2} (1 + a^2 + 2a) + 2a^2 \frac{r_2}{s_e^2} \right) + 2 \frac{r_1 r_2}{s_e^4} \\
& \cdot \left(-6a^3v_{11} + 6a^2b^2 - 6a^2v_{11} + 6ab^2 - 2(\bar{X}_1 + \bar{X}_2)(2a^2b + ba^2) \right. \\
& + b^2 - av_{11} - 4\bar{X}_1ab + a^2(\bar{X}_1^2 + 4\bar{X}_2\bar{X}_1 + \bar{X}_2^2) - 8ab\bar{X}_2 + 2a\bar{X}_2^2 \\
& + 4a\bar{X}_1\bar{X}_2 - 2b\bar{X}_2 + \bar{X}_2^2 \left. \right) + \left(\frac{r_2}{s_e^2}\right)^2 \left(-6a^3v_{11} + 6a^2b^2 - 4\bar{X}_2 \right. \\
& \cdot (2a^2b + ba^2) + 6a^2\bar{X}_2^2 \left. \right) + \left(\frac{r_1}{s_e^2}\right)^2 \left(-6a^3v_{11} + 6a^2b^2 \right. \\
& + 4(-3a^2v_{11} + 3ab^2) - 4\bar{X}_1(2a^2b + ba^2) + 6b^2 - 6av_{11}
\end{aligned}$$

$$- 24\bar{X}_1 ab + 6\bar{X}_1^2 a^2 + 12a\bar{X}_1^2 - 12b\bar{X}_1 + 6\bar{X}_1^2 \Big) \quad (9.113)$$

$$g_{13} = 2 \frac{r_1 r_2}{s_e^4} (4a^3 b + 4a^2 b + 2ba^2 - 2(\bar{X}_1 + \bar{X}_2) a^3 + 2ab - 2\bar{X}_1 a^2$$

$$- 4\bar{X}_2 a^2 - 2a\bar{X}_2) + \left(\frac{r_2}{s_e^2} \right)^2 (4a^3 b - 4\bar{X}_2 a^3) + \left(\frac{r_1}{s_e^2} \right)^2$$

$$(4a^3 b + 8a^2 b + 4ba^2 - 4\bar{X}_1 a^3 + 12ab - 12\bar{X}_1 a^2 - 12a\bar{X}_1$$

$$+ 4b - 4\bar{X}_1) \quad (9.114)$$

$$g_{14} = 2 \frac{r_1 r_2}{s_e^4} (a^4 + 2a^3 + a^2) + \left(\frac{r_2}{s_e^2} \right)^2 a^4 + \left(\frac{r_1}{s_e^2} \right)^2$$

$$(a^4 + 4a^3 + 6a^2 + 4a + 1) \quad (9.115)$$

$$g_{20} = -2 \sum_{i=1}^2 (\gamma_i^2 + 2\rho_i \gamma_i) - \sum_{i=1}^2 \psi_i'' \left(\sum_{i=1}^2 \psi_i'' + 2 \sum_{i=3}^n \psi_i'' - 2(n-1) \right)$$

$$+ \frac{2}{s_B^2} \left(\sum_{i=3}^n \psi_i'' - (n-1) \right) \left[2u^2 + 2b^2 - 2av_{11} - 4u.b \right] + \frac{2}{s_B^4}$$

$$\left[3a^2 v_{11}^2 - 6b^2 av_{11} + b^4 - 4u.b^3 + 12u.abv_{11} + 6u^2 b^2 \right.$$

$$\left. - 6u^2 av_{11} - 4u^3 b + u^4 \right] + \frac{1}{s_B^4} \left[3a^2 v_{11}^2 - 6b^2 av_{11} + b^4 \right.$$

$$\begin{aligned}
& - 4u.b^3 + 12u.abv_{11} + 6u^2b^2 - 6u^2av_{11} - 4u^3b + u^4 \\
& + 3a^2v_{11}^2 - 6b^2av_{11} + b^4 - 4u.b^2 + 12u.abv_{11} + 6u^2b^2 \\
& - 6u^2av_{11} - 4bu^3 + u^3] \quad (9.116)
\end{aligned}$$

$$\begin{aligned}
g_{21} = \frac{2}{s_B^2} \left(\sum_{i=3}^n \Psi_i'' - (n-1) \right) (-2u. + 4ab + 2b - 4u.a) + \frac{2}{s_B^4} \\
\cdot \left[-12a^2bv_{11} + 4ab^3 + 2b^3 - 6abv_{11} + 12u.a^2v_{11} - 12u.ab^2 \right. \\
- 2u.b^2 + 2u.av_{11} + 12abu^2 - 4u.b^2 + 4u.av_{11} - 4au^3 \\
+ 2bu^2 + 4bu^2 - 2u^3 \left. \right] + \frac{1}{s_B^4} \left[-12a^2bv_{11} + 4ab^3 + 12u.a^2v_{11} \right. \\
- 12u.ab^2 + 12u^2ab - 4u^3a - 12a^2bv_{11} + 4ab^3 + 4b^3 - 12abv_{11} \\
+ 12u.a^2v_{11} - 12u.ab^2 - 12u.b^2 + 12u.av_{11} + 12u^2ab - 4au^3 \\
+ 12bu^2 - 12u^3 \left. \right] \quad (9.117)
\end{aligned}$$

$$\begin{aligned}
g_{22} = \frac{2}{s_B^2} \left(\sum_{i=3}^n \Psi_i'' - (n-1) \right) (1 + 2a^2 + 2a) + \frac{2}{s_B^4} \left[-6a^3v_{11} + 6a^2b^2 \right. \\
- 6a^2v_{11} + 6ab^2 - 8u.a^2b - 4u.ba^2 + b^2 - av_{11} - 4u.ab \\
+ 6u^2a^2 - 8u.ab + 2au^2 + 4au^2 - 2bu. + u^2 \left. \right] + \frac{1}{s_B^4} \left[-6a^3v_{11} \right.
\end{aligned}$$

$$\begin{aligned}
& + 6a^2b^2 - 4u.(2a^2b + ba^2) + 6u^2a^2 - 6a^3v_{11} + 6a^2b^2 \\
& - 12a^2v_{11} + 12ab^2 - 8u.a^2b - 4u.ba^2 + 6b^2 - 6av_{11} \\
& - 24u.ab + 6u^2a^2 + 12au^2 - 12bu. + 6u^2] \quad (9.118)
\end{aligned}$$

$$\begin{aligned}
g_{23} = & \frac{2}{s_B} (4a^3b + 4a^2b + 2ba^2 - 4u.a^3 + 2ab - 2u.a^2 - 4u.a^2 - 2au.) \\
& + \frac{1}{s_B} (4a^3b - 4u.a^3 + 4a^3b + 8a^2b + 4ba^2 - 4u.a^3 + 12ab \\
& - 12u.a^2 - 12au. + 4b - 4u.) \quad (9.119)
\end{aligned}$$

$$\begin{aligned}
g_{24} = & \frac{1}{s_B} (a^4 + 2a^3 + a^2) + \frac{1}{s_B} (2a^4 + 4a^3 + 6a^2 + 4a + 1) \\
= & \frac{1}{s_B} (4a^4 + 8a^3 + 8a^2 + 4a + 1) \quad (9.120)
\end{aligned}$$