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Quasigroups, right quasigroups, coverings, and representations

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INTRODUCTION

The theory of quasigroup modules was laid out clearly in [S2]. The theory has also been described as quasigroup representation theory, and depends heavily on the construction of the universal multiplication group. The theory of universal multiplication group has been studied for some time now, e.g. [P]. In [S2], it has been shown that quasigroup representation theory is equivalent to the representation theory of stabilizers in the universal multiplication group. To every knot, there has been associated an invariant called the fundamental group. In general, we have Poincaré's fundamental group functor from a topological space. A generalization of this fundamental group is the so-called fundamental groupoid [B].

It is a fact that covering spaces of topological spaces can be classified by their fundamental groupoids. Another generalization of fundamental groupoids and coverings in the directed graph case are given in [H]. The strongest invariant of a knot, the so-called knot quandle, was found and studied by [J]. The concept of algebroid and left algebroid modules has been described in [M] and [R]. In [M], an algebroid has been defined as a category equipped with module structure on each hom set such that the composition is bilinear. In [R] an algebroid modules has been defined as a k-functor from an algebroid to the category of modules.

Chapter 1 of this dissertation explains the theory of group modules which is the background of the theory of quasigroup modules. Chapter 2 includes the theory of quasigroup modules with some new results, e.g. interpreting quasigroup modules as representations of the fundamental groupoid of the Cayley diagram of the quasigroup in the category of abelian groups. The equivalent coverings will also be described here. Chapter 3 is concerned with the theory of right quasigroup modules. This consists of generalization of the new results obtained in Chapter 2. Chapter 4 is about the theory of right quasigroup modules in a variety over a fixed field. Here we will need the concept of left-algebroid module. A new category will also be defined, the so-called category of ordered modules in a fixed variety. In particular, modules over quandles are studied here.

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CHAPTER 1

GROUP MODULES

Representations of a group G are usually considered as abelian groups $(M, +)$ on which G acts, i.e. there is a group morphism $G \to \mathrm{rm\ Aut}(M, +)$. Categorically, this means that there is a functor from G to rm $\mathrm{Aut}(M, +)$. However, there is another approach to looking at modules M, for a group G. This consists of building the split extension $M \,]\, G$, together with its projection $\pi \colon M\,]\,G \to G$. The underlying set of $M\,]\,G$ is $M \times G$, equipped with the group operation.

$$(m_1, g_1)(m_2, g_2) = (m_1 + m_2 g_1^{-1}, g_1 g_2).$$

The map $\pi \colon M\,]\,G \to G$; $(m, g) \mapsto g$ is a group homomorphism [in fact even onto]. We can think of $\pi^{-1}(g)$ as the fibre of $g \in G$, isomorphic to M [or in other words, "M located at g"].

A G-module morphism $\varphi \colon M \to M'$ is an abelian group homomorphism such that $(m\varphi)g = (mg)\varphi$. This induces a group homomorphism $\overline{\varphi} \colon M\,]\,G \to M'\,]\,G$; $(m, g) \mapsto (m\varphi, g)$ such that the following diagram commutes.

$$\begin{tikzcd} M\,]\,G \arrow[r, "\overline{\varphi}"] \arrow[d, "\pi"'] & M'\,]\,G \arrow[d, "\pi"] \\ G \arrow[r, "1_G"'] & G \end{tikzcd}$$

On the other hand, a group homomorphism $f \colon M\,]\,G \to M'\,]\,G$ induces a module homomorphism $f' \colon M \to M'$ defined by $(mf', 1) = (m, 1)f$. One can then think about the category $\mathfrak{G}/G$ of groups over G with the above structure.

Let D be the graph

$$\overset{A_1}{\bullet} \xrightarrow{\ a_1\ } \overset{A_0}{\bullet} \xleftarrow{\ a_2\ } \overset{A_2}{\bullet}.$$

Then the pull back of D is the left limit L

$$\begin{tikzcd} L \arrow[r, "f_1"] \arrow[d, "f_2"'] & A_1 \arrow[d, "a_1"] \\ A_2 \arrow[r, "a_2"'] & A_0 \end{tikzcd}$$

Analogously, the pull back of

$$G' \xrightarrow{\pi'} G \xleftarrow{\pi''} G''$$

is the group H defined by

$$H = G' \times_G G'' = \{(g', g'') \in G' \times G'' \mid g'\pi' = g''\pi''\}.$$

From each object in this category $\mathfrak{G}/G$, there exists a morphism to $1\colon G \to G$, which makes 1 the terminal object of $\mathfrak{G}/G$.

Formulating everything in the category $\underline{\underline{\mathrm{Set}}}$ of sets, an abelian group A is a set A with maps $0\colon \{1\} = A^0 \to A$, $-\colon A \to A$, and $+\colon A^2 \to A$ such that various identities are satisfied.

These identities can be written as commuting diagrams in the category $\underline{\underline{\mathrm{Set}}}$ of sets. Generalizing, define an abelian group A in $\mathfrak{G}/G$ to be an object A of $\mathfrak{G}/G$ equipped with $\mathfrak{G}/G$-morphisms $0\colon A^0 \to A$, $-\colon A \to A$, and $+\colon A^2 \to A$. The diagrams representing abelian group identities, now interpreted as diagrams in $\mathfrak{G}/G$, are required to commute. For example the "abelian" property is the commuting diagram

$$\begin{tikzcd} A \times_Q A \arrow[r, "\tau"] \arrow[d, "+"'] & A \times_Q A \arrow[d, "+"] \\ A \arrow[r, "1_A"'] & A \end{tikzcd}$$

An example of an abelian group in $\mathfrak{G}/G$ is $M\,]\,G \xrightarrow{\pi} G$, where

$$0_G\colon G \to M\,]\,G\,;\; g \mapsto (0, g)$$

$$-\colon M\,]\,G \to M\,]\,G\,;\; (m, g) \mapsto (-m, g)$$

$$+\colon M\,]\,G \times_G M\,]\,G \to M\,]\,G\,;\; ((m_1, g), (m_2, g)) \mapsto (m_1 + m_2, g).$$

Hence each G-module M gives an element $M]G \to G$ of $\mathfrak{G}/G$.

Suppose we have an abelian group $H \xrightarrow{\pi} G$ in $\mathfrak{G}/G$. Clearly π is an epimorphism in $\mathfrak{G}$, with kernel congruence α. Identify H^α with G via the natural isomorphism. Consider the kernel of $-\colon H \times_Q H \xrightarrow{1\times(-)} H \times_Q H \xrightarrow{+} H$, say $(\alpha|\alpha)$. Then $(\alpha|\alpha)$ satisfies certain properties:

(C0) $(a,b)(\alpha|\alpha)(c,d) \implies a-b=c-d \implies a^\alpha = (a-b)^\alpha = (c-d)^\alpha = c^\alpha$.

(C1) For fixed (a,b) in α, the mapping $(a,b)^{(\alpha|\alpha)} \to a^\alpha$; $(c,d) \mapsto c$ has the two-sided inverse $c \mapsto (c, c-(a-b))$.

(C2) : (RR) If $a^\alpha = c^\alpha$, then $a-a = a^\alpha 0_Q = c^\alpha 0_Q = c-c$. So that $(a,a)(\alpha|\alpha)(c,c)$.

(RS) $(a,b)(\alpha|\alpha)(c,d)$

$\implies a-b=c-d \implies b-a=d-c \implies (b,a)(\alpha|\alpha)(d,c)$

(RT) $(a,b)(\alpha|\alpha)(c,d)$ and $(b,e)(\alpha|\alpha)(d,f)$

$\implies a-b=c-d$ and $b-e=d-f$

$\implies a-e=c-f \implies (a,e)(\alpha|\alpha)(c,f)$.

We also will have an isomorphism from $\alpha^{(\alpha|\alpha)} \to H^\alpha \cong G$ by defining

$$f\colon \alpha^{(\alpha|\alpha)} \to H^\alpha;\ (a,b)^{(\alpha|\alpha)} \mapsto a^\alpha \text{ to } H \xrightarrow{\pi} G$$

by the mutually inverse $\mathfrak{G}/G$-morphisms $j\colon H \to \alpha^{(\alpha|\alpha)}$; $a \mapsto (a, a^\alpha 0_G)^{(\alpha|\alpha)}$ and $k\colon \alpha^{(\alpha|\alpha)} \to H$; $(a,b)^{(\alpha|\alpha)} \mapsto a-b$.

We will say that the congruence $(\alpha|\alpha)$ on α is the centreing congruence by which α centralizes itself.

CHAPTER 2

REPRESENTATIONS OF QUASIGROUPS

A quasigroup can be considered either as a not-necessarily finite Latin square or as a not-necessarily associative group (not necessarily containing an identity element).

Definition 2.1. [S2, 116]. A *quasigroup* Q is a set Q with three binary operations $\cdot$, $/$, and $\backslash$ called respectively *multiplication*, *right division*, and *left division* such that these operations satisfy the following axioms:

$$(ER)\colon (x/y)\cdot y = x \quad ;$$
$$(UR)\colon (x\cdot y)/y = x \quad ;$$
$$(EL)\colon x\cdot (x\backslash y) = y \quad ;$$
$$(UL)\colon x\backslash (x\cdot y) = y \quad .$$

If $(Q,\cdot,/,\backslash)$ is a quasigroup, then a new quasigroup structure $(Q,\cdot_{\mathrm{op}},\backslash_{\mathrm{op}},/_{\mathrm{op}})$ on Q is obtained by defining $x \cdot_{\mathrm{op}} y = y\cdot x$. This new quasigroup Q^{op} is called the opposite or flip of $(Q,\cdot)$. If a certain general property of quasigroup has been established then the observation that the property also applies to the flip is called the flipping argument.

Definition 2.2. A *right quasigroup* Q is a set Q with two binary operations $\cdot$, and $/$, satisfying (ER) and (UR).

The names (ER), (UR), (EL), and (UL) stand respectively for E̲xistence of a solution involving R̲ight division, U̲niqueness of the solution involving R̲ight division, and similarly for L̲eft division.

Definition 2.3. Let C be a category and let c be an object of C. The *comma category* of C over c has as its objects all C-morphisms $f\colon c' \to c$ and as its morphisms from $f: c' \to c$ to $g: c'' \to c$ all C-morphisms $\theta: c' \to c''$ such that the diagram

$$\begin{tikzcd} c' \arrow[r, "\theta"] \arrow[d, "f"'] & c'' \arrow[d, "g"] \\ c \arrow[r, "1_c"'] & c \end{tikzcd}$$

commutes. This category will be denoted by C/c.

An example of a comma category is $\mathfrak{Q}/Q$, the variety of all quasigroups $\mathfrak{Q}$ over a fixed quasigroup Q.

Definition 2.4. [S2, 314] A *Q-module* in $\mathfrak{Q}$ is an abelian group in $\mathfrak{Q}/Q$ (the comma category of $\mathfrak{Q}$ over Q), i.e., an object $A \to Q$ of $\mathfrak{Q}/Q$ equipped with $\mathfrak{Q}/Q$-morphisms $O_Q: Q \to A, -: A \to A$, and $+: A \times_Q A \to A$ such that the abelian group identity diagrams commute. A Q-module morphism $f: A \to B$ between Q-modules in $\mathfrak{Q}$ is a $\mathfrak{Q}/Q$-morphism such that $+f = (f \times_Q f)+, -f = f-$, and $0_Q f = 0_Q$. The category $\mathfrak{A} \otimes (\mathfrak{Q}/Q)$ of Q-modules in $\mathfrak{Q}$ has Q-modules in $\mathfrak{Q}$ as its objects and Q-morphisms between them as its morphisms.

For example, the identity $x + (-x) = 0$ in the category $\underline{\underline{\text{Set}}}$ of sets is now interpreted as the commuting:

$$\begin{tikzcd} A \arrow[r, "1\times(-)"] \arrow[d] & A \times_Q A \arrow[d] \\ A^0 \arrow[r, "0"'] & A \end{tikzcd} .$$

Definition 2.5. [S2,13] Given a variety $\mathfrak{V}$ of algebras (A, Ω), and a congruence β on an algebra (A, Ω) in $\mathfrak{V}$, let γ be a second congruence on (A, Ω) in $\mathfrak{V}$, and let $(\gamma|\beta)$ be a congruence on the algebra (β, Ω). Then γ is said to *centralize* β by means of the centreing congruence $(\gamma|\beta)$ if the following conditions are satisfied:

(C0) $(x, y)(\gamma|\beta)(x', y') \to x \, \gamma \, x'$

(C1) $\forall (x,y) \in \beta$, $\pi^0 \colon (x,y)^{(\gamma|\beta)} \to x^\gamma$; $(x',y') \mapsto x'$ bijects.

(C2) The following conditions are satisfied:

(RR): $\forall (x,y) \in \gamma$, $(x,x)(\gamma|\beta)(y,y)$;

(RS): $(x,y)(\gamma|\beta)(x',y') \implies (y,x)(\gamma|\beta)(y',x')$;

(RT): $(x,y)(\gamma|\beta)(x',y')$ and $(y,z)(\gamma|\beta)(y',t')$

$\implies (x,z)(\gamma|\beta)(x',z')$.

An object of $\mathfrak{A} \otimes (\mathfrak{Q}/Q)$ can be thought as a quasigroup A which has a self-centralizing congruence α such that $A^\alpha \cong Q$ (via a natural isomorphism) [S2, 317 & 318].

Another characterization of Q-modules is using the concept of universal multiplication group.

The structure of a quasigroup implies that the mappings $R_Q(q) \colon Q \to Q$; $x \mapsto xq$ and $L_Q(q) \colon Q \to Q$; $x \mapsto qx$ are permutations of the set Q for each $q \in Q$.

Definition 2.6. If P is a subquasigroup of the quasigroup Q, then the *relative multiplication group* $\mathrm{Mlt}_Q P$ of P in Q is the permutation group generated by $\{R_Q(p), L_Q(p) \mid p \in P\}$. In the case $P = Q$, we simply call the permutation group the *(combinatorial) multiplication group* $\mathrm{Mlt}\, Q$, and we may write $R(q)$ for $R_Q(q)$.

An important relative multiplication group is $\mathrm{Mlt}_Q \widetilde{Q}$, where $\widetilde{Q} = Q * I$, the coproduct of Q with the free quasigroup I on one generator x in the variety $\mathfrak{Q}$ of all quasigroups.

Definition 2.7. The *universal multiplication group* $U(Q,\mathfrak{Q})$ of Q in $\mathfrak{Q}$ is the relative multiplication group of Q in $\widetilde{Q}$. We will use the notation $\widetilde{R}(q)$ and $\widetilde{L}(q)$ respectively for $R_{\widetilde{Q}}(q)$ and $L_{\widetilde{Q}}(q)$ for $q \in Q$.

Clearly the assignment $U \colon \mathfrak{Q} \to Q$; $Q \mapsto U(Q,\mathfrak{Q})$ is a functor.

A word on n arguments $q_1, \ldots, q_n$ in $U(Q,\mathfrak{V})$, where $\mathfrak{V}$ is a variety of quasigroups, will be denoted by $\widetilde{E}(q_1, \ldots, q_n)$.

Lemma 2.8. [S2, 236] *If a relation*

$$x\widetilde{E}(p_1,\ldots,p_m)\widetilde{F}(q_1,\ldots,q_n) = x\widetilde{E}(p_1,\ldots,p_m)$$

holds, then $\widetilde{F}(q_1,\ldots,q_n) = 1$.

Proof. [S2, 236].

Definition 2.9. From x, y in Q, define $\tilde{\rho}(x,y) = \widetilde{R}(x \setminus x)^{-1}\widetilde{R}(x \setminus y)$ and $\rho(x,y) = R(x \setminus x)^{-1}R(x \setminus y)$. Clearly, $q\tilde{\rho}(x,y) = q\rho(x,y)$.

From [S2, §3] we see that the category of Q-modules is equivalent to the category of $\widetilde{G}_e$-modules, where $\widetilde{G}_e$ is the stabilizer of a fixed $e \in Q$ in the universal multiplication group $\widetilde{G} = U(Q,\mathfrak{Q})$. If $(\alpha|\alpha) = \operatorname{Ker}(-)\colon A \times_Q A \xrightarrow{1\times(-)} A \times_Q A \xrightarrow{+} A$, then $(\alpha|\alpha)$ is the centreing congruence by which α centralizes itself. The product $\cdot$, right division $/$, and left division $\setminus$ satisfy the following equations:

$$\begin{aligned} a \cdot b &= a\widetilde{R}(b\pi) + b\widetilde{L}(a\pi) \\ a/b &= \Big(a - b\widetilde{L}(a\pi/b\pi)\Big)\,\widetilde{R}(b\pi)^{-1} \\ a \setminus b &= \Big(b - a\widetilde{R}(a\pi/b\pi)\Big)\,\widetilde{L}(a\pi)^{-1}. \end{aligned}$$

If F is the functor from $\mathfrak{A} \otimes \mathfrak{Q}/Q$ to $\widetilde{G}_e$-mod that gives the equivalence (and F' is the other one), then an element $A \xrightarrow{\pi} Q \in \mathfrak{A} \otimes \mathfrak{Q}/Q$ under FF' will get sent to

$$\begin{aligned} \dot{\bigcup_{q\in Q}} M \otimes \tilde{\rho}(e,q) &\to Q \\ m \otimes \tilde{\rho}(e,q) &\mapsto q. \end{aligned}$$

Here $M = \pi^{-1}(e)$.

Other preparatory lemmas are needed here.

Lemma 2.10. *Let Q be a quasigroup appearing as a subquasigroup of a quasigroup A in a variety $\mathfrak{V}$. Then*

$$r_A\colon \widetilde{G} \to \operatorname{Mlt}_A Q\ ;\ \widetilde{F}(q_1,q_2,\ldots,q_n) \mapsto F_A(q_1,\ldots,q_n)$$

is a group epimorphism of $\widetilde{G} = U(Q, \mathfrak{V})$ *onto the relative multiplication group of* Q *in* A.

Proof. [S2, 334].

Lemma 2.11. [S2, 338] *Let* Q *be a quasigroup with element* e*, and* $\widetilde{G} = U(Q, \Omega)$*. Then for each quasigroup word* $x_1 \dots x_n w$ *with* n *arguments, there is an* n*-vector* $\underline{\widetilde{W}}\colon Q^n \to (ZG_e)$ *such that*

$$(m_1, q_1) \dots (m_n, q_n)w = \Big(m_1 \widetilde{W}_1(q_1 \dots q_n) \\ + \dots + m_n \widetilde{W}_n(q_1, \dots, q_n), q_1 q_2 \dots q_n w\Big)$$

in $Z\widetilde{G}_e \times Q$.

Another important property of the universal multiplication group $\widetilde{G} = U(Q, \Omega)$ is its freeness on $\{\widetilde{R}(q), \widetilde{L}(q) \mid q \in Q\}$.

Theorem 2.12. *Given a quasigroup* Q*, the universal multiplication group* $\widetilde{G} = U(Q, \Omega)$ *is the free group on* $\{\widetilde{R}(q), \widetilde{L}(q) | q \in Q\}$.

Before proving this theorem, we need some basic terminologies as introduced by [E].

Definition 2.13. The major components of the word $u \cdot v$ are u and v, and the length of $u \cdot v$ is defined by $\ell(u \cdot v) = \ell(u) + \ell(v)$, where the length of a generator is taken as 1.

Definition 2.14. Let $r_i(a, b, c, \dots) = r_i'(a, b, c, \dots)$ $(i = 1, 2, \dots)$ be a set of equations between words in $a, b, c, \dots$. We shall call two words in $a, b, c, \dots$ *equivalent* if we can transform one into the other by a finite sequence of applications of the axioms of quasigroups and the equations $r_i = r_i'$. The resulting equivalence classes form a quasigroup, if we define $\{u\} \circ \{v\}$ to be $\{u \circ v\}$, where $\{w\}$ denotes the equivalence class containing the word w. We call this the quasigroup generated by $a, b, c, \dots$, with relations $r_i(a, b, c, \dots) = r_i'(a, b, c, \dots)$.

Definition 2.15. Let Q be a quasigroup generated by $a, b, c, \ldots$. We say that its relations are in *closed form* if they satisfy the following conditions:

(i) Every one of the relations is of the form $x \circ y = z$, where x, y and z are generators.

(ii) If one of the three $x \cdot y = z$, $x = z/y$, $y = x \setminus z$ is a relation then so are the other two.

(iii) No two relations occur such as $x \cdot y = z$, $x \cdot y = z'$, identifying two generators z, z'.

Definition 2.16. Let Q be a quasigroup defined by a closed set of relations. Let w be a word in the generators $a, b, c, \ldots$ of Q. By an *elementary reduction* of w, we mean the replacing of component of w of the form

(i) $u \cdot (u \setminus v)$ by v	(ii) $(v/u) \cdot u$ by v
(iii) $u \setminus (u \cdot v)$ by v	(iv) $(v \cdot u)/u$ by v
(v) $u/(v \setminus u)$ by v	(vi) $(u/v) \setminus u$ by v

where u, v are words. (vii) $x \circ y$ by z if $x \circ y = z$ is one of the defining relations of Q. A word in the generators of Q is said to be normal if no elementary reduction of the word is possible.

Another concept which is needed in proving Theorem 2.11 is the Cayley graph Cay Q of a quasigroup Q.

Definition 2.17. [S2, 213] The Cayley graph Cay(Q) of a quasigroup Q is a directed graph with vertex set Q. For each x, y in Q there is an arrow $(x, R(y), x \cdot y)$ from x to $x \cdot y$, and an arrow $(x, L(y), y \cdot x)$ from x to $y \cdot x$.

Proof of Theorem 2.12. In the Cayley graph $Cay(Q)$, consider the subgraph $(x\widetilde{G})$ consisting of all vertices lying in the orbit $x\widetilde{G}$ of x under $\widetilde{G}$, and of all arcs between these vertices labelled $\widetilde{R}(q)$ or $\widetilde{L}(q)$ for some q in Q. Note that $(x\widetilde{G})$ is (weakly) connected. If there is a circuit in $(x\widetilde{G})$ starting at a vertex $x\widetilde{E}(p_1, \cdots, p_m)$, its labels form a product $\widetilde{F}(q_1, \cdots, q_n)$ s.t. $x\widetilde{E}(p_1, \cdots, p_m)\widetilde{F}(q_1, \cdots, q_n) = x\widetilde{E}(P_1, \cdots, p_m)$. By Lemma 2.8, it follows that $\widetilde{F}(q_1, \cdots, q_n) = 1$.

Because $\widetilde{F}(q_1, \cdots, q_n)$ is a product of labels, we can assume $\widetilde{F}(q_1, \cdots, q_n) = \prod_{i=1}^{S} \widetilde{m_{j_i k_i}}(h_i)$ where $h_i \in \{q_1, \cdots, q_n), j_i = R$ or $L, k_i = \pm 1$, with the following conventions:

$$\text{if } j_i = R, k_i = 1, \textit{then } \tilde{m}_{R1} = \widetilde{R}(h_i);$$
$$\text{if } j_i = R, k_i = -1, \textit{then } \tilde{m}_{R-1} = \widetilde{R}^{-1}(h_i);$$
$$\text{if } j_i = L, k_i = 1, \textit{then } \tilde{m}_{L1} = \widetilde{L}(h_i);$$
$$\text{if } j_i = L, k_i = -1, \textit{then } \tilde{m}_{L-1} = \widetilde{L}^{-1}(h_i).$$

Suppose the circuit is not trivial. Then we can assume further that $S \geq 1$ (since $S = 0 \Rightarrow \widetilde{F} = 1$), and $\widetilde{F}$ is in "reduced form", i.e., there is no i such that $j_i = j_{i+1}, k_i = -k_{i+1}$, and $h_i = h_{i+1}$. From $\widetilde{F} = 1$, we have $x\widetilde{F}(q_1, \cdots, q_n) = x$. Now x is in "normal form" [E, 2.1] in $Q * I$, so that there exists a reduction chain

$$U = x\widetilde{F} \to U_1 \to U_2 \to \cdots \to U_k = x$$

[E, T2.2]. Now, $Q = \langle q \in Q | q_1 b q_2 = q_3$ if $q_1 b q_2 = q_3$ in $Q\rangle$ is a set of closed relations for Q with $b \in \{\cdot, /, \backslash\}$ (in the sense of) [E, 1.3], so that $Q * I = \langle q \in Q, x | q_1 b q_2 = q_3$ if $q_1 b q_2 = q_3$ in $Q\rangle$.

Define a relation $\leq$ on the set of "components"[E, 1.2] of $x\widetilde{F}$, by $z_1 \leq z_2$ if z_1 is a component of z_2. Let m be the "minimal" component of $x\widetilde{F}$ such that the elementary operation $U \to U_1$ occurs within it. Since the "reduced" form of $\widetilde{F}$ is not 1, the length of $m, \ell(m) \geq 2$ (if $\ell(m) = 1$, m is a generator, so that the elementary operation is on a generator, a contradiction). So the operation occurs at $m = x \prod_{i=1}^{t} \widetilde{m_{j_i k_i}}(h_i)$, $j \geq 1$, not at $x \prod_{i=1}^{t-1} \widetilde{m_{j_i k_i}}(h_i)$, i.e., involving h_t. In cases (i)-(iv) of elementary reductions [E, 2.1], we will have a contradiction since $j_{t-1} = j_t$ and $k_{t-1} = -k_t$. Cases (v) and (vi) are out of consideration since h_j is already in normal form. In case (vii), if we replace $(x\prod_{i=1}^{t-1}\widetilde{m_{j_ik_i}}(h_i))\cdot h_t$ by z, then $z=q$ for some $q\in Q$ so that $x\widetilde{F}(q_1,\cdots,q_n)=q\prod_{i=\ell+1}^{S}\widetilde{m_{j_ik_i}}(h_i)\in Q$, but $x\in Q$, a contradiction. Hence the circuit is trivial.

A left action of $\widetilde{G}$ on $(x\widetilde{G})$ is defined by letting $\widetilde{F}(q_1,\cdots,q_n)$ in $\widetilde{G}$ send the arc $(x\widetilde{E}(p_1,\cdots,p_m),\widetilde{D}(q),x\widetilde{E}(p_1,\cdots,p_m)\widetilde{D}(q))$ to $(x\widetilde{F}(q_1,\cdots,q_n)^{-1}\widetilde{E}(p_1,\cdots,p_m),$ $\widetilde{D}(q),x\widetilde{F}(q_1,\cdots,q_n)^{-1}\widetilde{E}(p_1,\cdots,p_m)\widetilde{D}(q))$, where $\widetilde{D}(q)$ denotes $\widetilde{R}(q)$ or $\widetilde{L}(q)$. Suppose that a vertex $x\widetilde{E}(p_1,\cdots,p_m)$ is fixed by an element $\widetilde{F}(q_1,\cdots,q_n)$ of $\widetilde{G}$. Then $x\widetilde{E}(p_1,\cdots,p_m)=x\widetilde{F}(q_1,\cdots,q_n)^{-1}\widetilde{E}(p_1,\cdots,p_n)$, whence $x\widetilde{F}(q_1,\cdots,q_n)=x$. By [S2, 236] with $m=0$, we have $\widetilde{F}(q_1,\cdots,q_n)=1$. Thus no non-identity element of $\widetilde{G}$ leaves a vertex of $(x\widetilde{G})$ fixed.

Now suppose that an arc $(x\widetilde{E}(p_1,\cdots,p_m),\widetilde{R}(q),x\widetilde{E}(p_1,\cdots,p_m)\widetilde{R}(q))$ of $(x\widetilde{G})$ is inverted by $\widetilde{F}(q_1,\cdots,q_n)$ in $\widetilde{G}$, so that $x\widetilde{F}(q_1,\cdots,q_n)^{-1}\widetilde{E}(p_1,\cdots,p_m)=x\widetilde{E}(p_1,\cdots,p_m)\widetilde{R}(q)$ and $x\widetilde{F}(q_1,\cdots,q_n)^{-1}\widetilde{E}(p_1,\cdots,p_m)\widetilde{R}(q)=x\widetilde{E}(p_1,\cdots,p_m)$. Then $x\widetilde{E}(p_1,\cdots,p_m)\widetilde{R}(q)^2=x\widetilde{E}(p_1,\cdots,p_m)$, whence $\widetilde{R}(q)^2=1$ by Lemma 2.8. In particular $xq\cdot q=x$. Consider the quasigroup $(\mathcal{Q},\cdot,/,\backslash)$ defined on the set of rationals $\mathcal{Q}$ by $r\cdot s=2r+s, r/s=(r-s)/2$, and $r\backslash s=s-2r$ for r,s in $\mathcal{Q}$. Define $f\colon Q\to\mathcal{Q}; q\mapsto 0$. Since $\{0\}$ is a subquasigroup of $(\mathcal{Q},\cdot,/,\backslash)$, f is a quasigroup morphism. The image of $x=(xq)q$ in $\widetilde{Q}$ under $f*(x\mapsto 1);\widetilde{Q}\to Q$ is $1=(2.0)\cdot 0=4$, an impossibility. Thus no arc of $(x\widetilde{G})$ labelled $\widetilde{R}(q)$ is inverted by an element of $\widetilde{G}$. A flipping argument [S2, 115] shows that no arc labelled $\widetilde{L}(q)$ is inverted. Thus $\widetilde{G}$ acts freely on $(x\widetilde{G})$ (in the sense of [ST, I.3.3]). The quotient graph $\widetilde{G}\setminus(x\widetilde{G})$ is a bouquet of circles labelled with the elements of $\widetilde{R}(Q)U\widetilde{L}(Q)$. By the Reidemeister Theorem [ST, Theorem 2.4] it follows that $\widetilde{G}$ is the free group on $\widetilde{R}(Q)U\widetilde{L}(Q)$. $\square$

Quasigroups can be considered as generalizations of groups. Another generalization of a group in the categorical sense is a groupoid. Almost all the terminologies here are adapted from [H].

Definition 2.18. A *groupoid* is a category such that all its morphisms are invertible.

Definition 2.19. The *fundamental groupoid* on a directed graph X, denoted $\pi(X)$, is the free groupoid on the graph X, i.e., the codomain of a graph map $i\colon X \to \pi(X)$, such that for every groupoid G and graph map $j\colon X \to G$, there exists a unique groupoid map $\tau\colon \pi(X) \to G$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "j"'] & \pi(X) \arrow[d, "\tau"] \\ G \arrow[r, "1_G"'] & G \end{tikzcd}$$

commutes.

An easy characterization of the fundamental groupoid $\pi(X)$ on a graph X has been given in [H, Ch.8].

For a graph X, we can consider the coverings of the fundamental groupoid on the graph X. These coverings arise naturally in analogy with coverings of a topological space.

Definition 2.20. Let G and G' be two groupoids. Let $V(G)$ denote the set of objects (vertex set) of G, and let G_{j*} denote the set of all morphisms in G with source $j \in V(G)$. G' *covers* G if there exists a category map $\varphi\colon G' \to G$ such that for every $i \in V(G')$, the restriction $\varphi_i\colon G'_{i*} \to G_{i\varphi *}$ of φ is bijective.

Definition 2.21. An *abelian covering* of the fundamental groupoid on the Cayley diagram of a quasigroup Q is a covering map $\varphi\colon E \to \pi \operatorname{Cay} Q$ such that:

(i) For every q in Q, the inverse image of q under φ in E, viz. E^q, is an abelian group.

(ii) For every morphism a from q to s in $\pi \operatorname{Cay} Q$, the map $E^a : E^q \to E^s; v \mapsto w$ (here w is the target of the unique cover of a with starting point v) is an isomorphism.

Definition 2.22. The *category of abelian coverings* of the fundamental groupoid on the Cayley diagram of a fixed quasigroup Q is the subcategory of the comma category of groupoids $\mathfrak{G}$ over $\pi\,\mathrm{Cay}\,Q$ with objects all abelian covers of $\pi\,\mathrm{Cay}\,Q$ and morphisms all $\mathfrak{G}/\pi\,\mathrm{Cay}\,Q$ morphisms $\theta\colon E_1 \to E_2$ from $\varphi_1\colon E_1 \to \pi\,\mathrm{Cay}\,Q$ to $\varphi_2\colon E_2 \to \pi\,\mathrm{Cay}\,Q$ such that for every q in Q, the restriction $\theta|_{E_1{}^q}$ of θ to $E_1{}^q$ is a homomorphism of abelian groups into $E_2{}^q$. We will denote this category by $Ab\,\mathrm{Cov}\,Q$.

An easy characterization of an element of $Ab\,\mathrm{Cov}\,Q$ is given by the following proposition.

Proposition 2.23. *Let Q be a non-empty quasigroup. Suppose we are given a covering $\varphi\colon E \to \pi\,\mathrm{Cay}\,Q$ of $\pi\,\mathrm{Cay}\,Q$ such that the following statements are true:*

- (i) *there exists an r in Q such that the inverse image of r under φ in E, namely $E^r = \{v \in E | v\varphi = r\}$, is an abelian group;*
- (ii) *for every morphism a from r to r in $\pi\,\mathrm{Cay}\,Q$, the map $E^a : E^r \to E^r; v \mapsto w$ (here w is the target of the unique cover of a with starting point v) is a homomorphism. Then φ is an object of $Ab\,\mathrm{Cov}\,Q$.*

Proof.

(i) We claim that $\forall q \in Q, E^q$ is an abelian group. Let $b \in [\pi\,\mathrm{Cay}\,Q]_{rq}$, then $E^b E^{b^{-1}} : E^r \to E^r$. If $\bar{b}$ is the unique cover of b with starting point $v \in E^r$ and target $v' \in E^q$, while $\overline{b^{-1}}$ is the unique cover of b^{-1} with starting point v' and target $v'' \in E^r$, then $\bar{b}\overline{b^{-1}}$ is the unique cover of $bb^{-1} = 1_r = 1_v\varphi$, so that $v'' = v$, i.e., E^b is a bijection. Defining $vE^b \cdot wE^b = (v \cdot w)E^b$ in E^q makes E^q an abelian group. This product is well-defined, since if $v_1E^b = v_2E^c, w_1E^b = w_2E^c$ for $c \in [\pi\,\mathrm{Cay}\,Q]_{rq}$, then $(v_1 \cdot w_1)E^b = v_1E^b \cdot w_1E^b = v_2E^c \cdot w_2E^c = (v_2 \cdot w_2)E^c$, which proves (i).

To prove (ii) of Definition 2.2, from (i) it is clear that E^b is an isomorphism $\forall b \in [\pi\,\mathrm{Cay}\,Q]_{rq}$. Let $a = b^{-1}d$ where $b \in [\pi\,\mathrm{Cay}\,Q]_{rq}$, and $d \in [\pi\,\mathrm{Cay}\,Q]_{rs}$. Then

$E^a = E^{b^{-1}d} = E^{b^{-1}}E^d$ is an isomorphism, since $E^{b^{-1}}$ and E^d are. $\square$

Notice also that $a\varphi^{-1} = \{(m,n) \in E^q \times E^s | mE^a = n\}$ is an abelian group by the isomorphism E^a, i.e., these pairs form a subgroup of $E^q \times E^r$ isomorphic with E^q and E^r.

Before stating the main result in this section, the definition of representations in the categorical sense will be introduced.

Definition 2.24. Given two categories C_1 and C_2, the *category* ${C_1}^{C_2}$ *of representations* of C_2 into C_1 has all functors $P\colon C_2 \to C_1$ as its objects and all natural transformations between them as its morphisms.

If C_2 is a groupoid, then groupoid representations of C_2 into C_1 are the same as category representations of C_2 into C_1. For example, the category $\mathfrak{A}^{\pi\,\mathrm{Cay}\,Q}$ is the category of all representations of the fundamental groupoid on the Cayley diagram of Q into the category $\mathfrak{A}$ of abelian groups. Another example is the category $\mathfrak{A}^G$ of representations of a group G into the category of abelian groups, in particular $\mathfrak{A}^{\widetilde{G}_e}$, where $\widetilde{G}_e$ is the stabilizer of e in the universal multiplication group $U(\Omega, Q)$.

Theorem 2.25. *The following categories are equivalent:*
(i) $\mathfrak{A} \otimes (\Omega/Q)$; *(ii)* $\mathfrak{A}^{\widetilde{G}_e}$; *(iii)* $\mathfrak{A}^{\pi CayQ}$; *(iv)* $AbCovQ$.

The proof will be given in some steps.

Proposition 2.26. *The stabilizer* $\widetilde{G}_e$ *is the vertex group of* $\pi CayQ$ *at the vertex* e.

Proof. Take $y = [y_1, y_2, \cdots, y_n] \in [\pi\ \mathrm{Cay}\,Q]_{ee}$, a loop at e.

Then

$$y_i = \begin{cases} \langle e_i, R(e_i \backslash e_{i+1}), e_{i+1}\rangle & \text{or} \\ \langle e_i, R^{-1}(e_{i+1} \backslash e_i), e_{i+1}\rangle & \text{or} \\ \langle e_i, L(e_{i+1}/e_i), e_{i+1}\rangle & \text{or} \\ \langle e_i, L^{-1}(e_i/e_{i+1}), e_{i+1}\rangle, & \end{cases}$$

where $e_1 = e_{n+1} = e$.

We can denote y_i by $\langle e_i, m_{j_ik_i}(e_i,e_{i+1}), e_{i+1}\rangle$, where $j_i = R$ or L and $k_i = \pm 1$, with the following conventions:

$$\begin{aligned}
&\text{if } j_i = R,\ k_i = 1,\ \text{ then } m_{j_ik_i}(e_i,e_{i+1}) &&= R(e_i \setminus e_{i+1});\\
&\text{if } j_i = R, k_i = -1, \text{ then } m_{j_ik_i}(e_i,e_{i+1}) &&= R^{-1}(e_{i+1} \setminus e_i);\\
&\text{if } j_i = L, k_i = 1,\ \text{ then } m_{j_ik_i}(e_i,e_{i+1}) &&= L(e_{i+1}/e_i);\\
&\text{if } j_i = L, k_i = -1, \text{ then } m_{j_ik_i}(e_i,e_{i+1}) &&= L^{-1}(e_i/e_{i+1}).
\end{aligned}$$

Let $f\colon [\pi\,\mathrm{Cay}\,Q]_{ee} \to \widetilde{G}$ be defined by $f([y_1,y_2,\cdots,y_n]) = \prod_{i=1}^{n} \widetilde{m_{j_ik_i}}(e_i,e_{i+1})$, where $\widetilde{m_{j_ik_i}}(e_i,e_{i+1}) \in \widetilde{G}$ with the following conventions:

$$\begin{aligned}
&\text{if } j_i = R,\ k_i = 1,\ \text{ then } \widetilde{m_{j_ik_i}}(e_i,e_{i+1}) &&= \widetilde{R}(e_i \setminus e_{i+1});\\
&\text{if } j_i = R, k_i = -1, \text{ then } \widetilde{m_{j_ik_i}}(e_i,e_{i+1}) &&= \widetilde{R}^{-1}(e_{i+1} \setminus e_i);\\
&\text{if } j_i = L, k_i = 1,\ \text{ then } \widetilde{m_{j_ik_i}}(e_i,e_{i+1}) &&= \widetilde{L}(e_{i+1}/e_i);\\
&\text{if } j_i = L, k_i = -1, \text{ then } \widetilde{m_{j_ik_i}}(e_i,e_{i+1}) &&= \widetilde{L}^{-1}(e_i/e_{i+1}).
\end{aligned}$$

Clearly f is a well-defined mapping, since if $[y_1,y_2,\cdots,\hat{y}_i,\hat{y}_{i+1},y_{i+2},\cdots,y_n]$ is a simple reduction (i.e., $y_i = y_{i+1}^{-1}$) of y (where $\hat{y}_i$ means y_i is omitted), then $y_i = \langle e_i, m_{j_ik_i}(e_i,e_{i+1}),\ e_{i+1}\rangle$, and $y_{i+1} = \langle e_{i+1}, m_{j_{i+1}k_{i+1}}(e_{i+1},e_{i+2}), e_{i+2}\rangle$ where $e_{i+1} = e_i$ and $m_{j_ik_i}(e_i,e_{i+1}) = m_{j_{i+1}k_{i+1}}(e_{i+1},e_{i+2})^{-1}$. Hence $f([y_1,y_2,\cdots,y_i,y_{i+1},y_{i+2},\cdots,y_n]) = \prod_{t\in\{1,2,\cdots,\hat{i},\widehat{i+1},i+2,\cdots,n\}} \widetilde{m_{j_tk_t}}(e_t,e_{t+1})$ $= \prod_{t=1}^{n} \widetilde{m_{j_tk_t}}(e_t,e_{t+1}) = f(y)$, where $\widetilde{m_{j_ik_i}}(e_i,e_{i+1}) = \widetilde{m_{j_{i+1}k_{i+1}}}(e_{i+1},e_{i+2})^{-1}$. We can restrict the codomain of f to be $\widetilde{G}_e$, since $e\prod_{i=1}^{n} \widetilde{m_{j_ik_i}}(e_i,e_{i+1}) = e$. Now f is also a homomorphism, since

$$\begin{aligned} f(x\cdot y) &= f([x_1,x_2,\cdots,x_m][y_1,y_2,\cdots,y_n]) \\ &= \prod_{i=1}^{m} \widetilde{n_{j_ik_i}}(e_i,e_{i+1}) \prod_{i=1}^{n} \widetilde{m_{j_{m+i}k_{m+i}}}(e_{m+i},e_{m+i+1}) \\ &= (\prod_{i=1}^{m} \widetilde{n_{j_ik_i}}\,(e_i,e_{i+1})) \left(\prod_{i=1}^{n} \widetilde{m_{j_{m+i}k_{m+i}}}(e_{m+i},e_{m+i+1}) \right) \\ &= f(x)f(y), \end{aligned}$$

where

$$x_i = \langle e_i, n_{j_ik_i}(e_i,e_{i+1}), e_{i+1}\rangle$$

and

$$y_i = \langle e_{m+i}, m_{j_{m+i}k_{m+i}}(e_{m+i},e_{m+i+1}), e_{m+i+1}\rangle, \text{with } e_1 = e_{m+1} = e_{m+n+1}.$$

Now f is one to one: Suppose that x is a reduced path in $[\pi\,\mathrm{Cay}\,Q]_{ee}$ and $f(x) = f([x_1,x_2,\cdots,x_m]) = 1_{\widetilde{Q}}$. Then $\prod_{i=1}^{n} \widetilde{n_{j_ik_i}}(e_i,e_{i+1}) = 1_{\widetilde{Q}}$ implies, by the freeness of $\widetilde{F}$ on $\{\widetilde{R}(q),\widetilde{L}(q)|q\in Q\}$ [Theorem 2.12], that $\exists i \in \{1,2,\cdots,n\}$. $\widetilde{n_{j_ik_i}}(e_i,e_{i+1}) = \widetilde{n_{j_{i+1}k_{i+1}}}(e_{i+1},e_{i+2})^{-1}$. Applying r_Q [Lemma 2.10], we obtain $n_{j_ik_i}(e_i,e_{i+1}) = n_{j_{i+1}k_{i+1}}(e_{i+1},e_{i+2})^{-1}$, i.e., $x_i = x_{i+1}^{-1}$, contradicting the reducedness of x. Hence $x = 1_{ee}$. The homomorphism f is also onto, since $\widetilde{G}_e = \langle \widetilde{T}_e(q), \widetilde{R}_e(q,r), \widetilde{L}_e(q,r)\rangle$ [S2, 244], and $\widetilde{T}_e(q) = f[\langle e, R(e\setminus q), q\rangle, \langle q, L^{-1}(q/e), e\rangle]$;
$\widetilde{R}_e(q,r) = f[\langle e, R(e\setminus q), q\rangle, \langle q, R(r), qr\rangle, \langle qr, R(e\setminus qr)^{-1}, e\rangle]$;
$\widetilde{L}_e(q,r) = f\left[\langle e, L(q/e), q\rangle, \langle q, L(r), rq\rangle, \langle rq, L(rq/e)^{-1}, e\rangle\right]$. Hence we conclude $\widetilde{G}_e \cong [\pi\,\mathrm{Cay}\,Q]_{ee}$. $\square$

Suppose we have a representation δ from $\widetilde{G_e}$ to $\mathfrak{A}$. Then we will define an element $\delta\alpha = P\colon \pi\,\mathrm{Cay}\,Q \to \mathfrak{A}$ of $\mathfrak{A}^{\pi\,\mathrm{Cay}\,Q}$ as follows:

$$\tilde{\rho}(e,q)\alpha\colon \delta(e) = M \to M\otimes\tilde{\rho}(e,q); m \mapsto m\otimes\tilde{\rho}(e,q).$$

If $a \in [\pi CayQ]_{qr}$, then a is written uniquely in the form $\tilde{\rho}(e,q)^{-1}g_e\tilde{\rho}(e,r)$ by taking $g_e = \tilde{\rho}(e,q)a\tilde{\rho}(e,r)^{-1} \in \widetilde{G}_e = [\pi CayQ]_{ee}$ by Proposition 2.24, so that

$$a\alpha\colon M \otimes \tilde{\rho}(e,q) \to M \otimes \tilde{\rho}(e,r),$$

$$a\alpha = [\tilde{\rho}(e,q)\alpha]^{-1}(g_e\alpha)[\tilde{\rho}(e,r)\alpha], \quad \text{and}$$

$$g_e\alpha = g_e.$$

On objects, $qP = M \otimes \tilde{\rho}(e,q)$, so that $eP = M \otimes \tilde{\rho}(e,e) = M \otimes 1 = M$, hence P is well-defined.

If $f\colon \delta_1 \to \delta_2$ is a morphism in $\mathfrak{A}^{\widetilde{\mathfrak{G}}_e}$, i.e., a $\widetilde{G}_e$-module homomorphism, then $f\alpha\colon \delta_1\alpha \to \delta_2\alpha$ is

$$M_1 \otimes \tilde{\rho}(e,q) \to M_2 \otimes \tilde{\rho}(e,q); m_1 \otimes \tilde{\rho}(e,q) \mapsto m_1 f \otimes \tilde{\rho}(e,q)$$

with the property that for every $g_e \in \widetilde{G}_e$,

$$\begin{aligned}(m_1 \otimes \tilde{\rho}(e,q))f\alpha g_e\delta_2 &= (m_1 f \otimes \tilde{\rho}(e,q))g_e\delta_2 = (m_1 f)g_e \otimes \tilde{\rho}(e,q)\\ &= (m_1 g_e \otimes \tilde{\rho}(e,q))f\alpha = (m_1 \otimes \tilde{\rho}(e,q))g_e\delta_1 f\alpha,\end{aligned}$$

that is $f\alpha g_e\delta_2 = g_e\delta_1 f\alpha$, so that $f\alpha$ is a natural transformation from $\delta_1\alpha$ to $\delta_2\alpha$. This leads us to the following proposition.

Proposition 2.27. *The category map α gives a functor from $\mathfrak{A}^{\widetilde{G}_e}$ to $\mathfrak{A}^{\pi CayQ}$.*

Given a representation $P\colon \pi\,\mathrm{Cay}\,Q \to \mathfrak{A}$, define $P\beta = P|_{[\pi\,\mathrm{Cay}\,Q]_{ee}}$.

Proposition 2.28. *The map β can be extended to a functor from $\mathfrak{A}^{\pi CayQ}$ to $\mathfrak{A}^{\widetilde{G}_e}$.*

Proof. Take a morphism $f\colon P_1 \to P_2$ in $\mathfrak{A}^{\pi CayQ}$ under β to $f\beta = f|_{P_1\beta}$. Clearly $f\beta$ is a morphism in $\mathfrak{A}^{\widetilde{G}_e}$. $\square$

Lemma 2.29. *For each representation P of $\pi CayQ$ into $\mathfrak{A}$, there is a natural $\mathfrak{A}^{\pi CayQ}$-isomorphism h_p: $P(\beta\alpha) \to P$, a collection of $\mathfrak{A}$-morphisms, one for each object q of $\pi CayQ$:*

$$\begin{array}{ccccc} qPh_p\colon M \otimes \tilde{\rho}(e,q) & \xrightarrow{[\tilde{\rho}(e,q)\alpha]^{-1}} & M & \xrightarrow{\tilde{\rho}(e,q)P} & A_q \\ \\ m \otimes \tilde{\rho}(e,q) & \longrightarrow & m & \longrightarrow & m\tilde{\rho}(e,q)P. \end{array}$$

Proof. For every $a \in [\pi CayQ]_{qr}$, the following diagram commutes:

$$\begin{array}{ccc} qP\beta\alpha & \xrightarrow{qPh_p} & qP \\ {\scriptstyle aP\beta\alpha}\big\downarrow & & \big\downarrow{\scriptstyle aP} \\ rP\beta\alpha & \xrightarrow[rPh_p]{} & rP \end{array},$$

i.e.,

$$\begin{array}{ccc} M \otimes \tilde{\rho}(e,q) & \xrightarrow{qPh_p} & Aq \\ {\scriptstyle a\alpha=[\tilde{\rho}(e,q)\alpha]^{-1}(g_e\alpha)[\tilde{\rho}(e,q)\alpha]}\big\downarrow & & \big\downarrow{\scriptstyle a=[\tilde{\rho}(e,q)P]^{-1}g_eP[\tilde{\rho}(e,r)P]} \\ M \otimes \tilde{\rho}(e,r) & \xrightarrow[rPh_p]{} & A_r \end{array}$$

where $a = \tilde{\rho}(e,q)^{-1}g_e\tilde{\rho}(e,r)$, since $[m\otimes\tilde{\rho}(e,q)]qPh_pa = (m\tilde{\rho}(e,q)P)(\tilde{\rho}(e,q)P)^{-1}g_eP\tilde{\rho}(e,r)P$ $= mg_eP\tilde{\rho}(e,r)P = mg_e\alpha\tilde{\rho}(e,r)P = m(g_e\alpha)(\tilde{\rho}(e,r)\alpha)[\tilde{\rho}(e,r)\alpha]^{-1}\tilde{\rho}(e,r)P =$ $[m \otimes \tilde{\rho}(e,q)]a\beta\alpha[rPh_p]$.

On the other hand, the following diagram also commutes

$$\begin{array}{ccc} qP & \xrightarrow{qPh_p^{-1}} & qP\beta\alpha \\ {\scriptstyle aP}\big\downarrow & & \big\downarrow{\scriptstyle aP\beta\alpha} \\ rP & \xrightarrow[rPh_p^{-1}]{} & rP\beta\alpha \end{array}$$

i.e.,

$$\begin{array}{ccc} Aq & \xrightarrow{qPh_p^{-1}} & M \otimes \tilde{\rho}(e,q) \\ {\scriptstyle aP=\tilde{\rho}(e,q)^{-1}g_eP\tilde{\rho}(e.r)P}\big\downarrow & & \big\downarrow{\scriptstyle a\alpha=[\tilde{\rho}(e,q)\alpha]^{-1}(g_e\alpha)[\tilde{\rho}(e,r)\alpha]} \\ A_r & \xrightarrow[rPh_p^{-1}]{} & M \otimes \tilde{\rho}(e,r) \end{array}$$

$$
\begin{aligned}
tqPh_p^{-1}a\beta\alpha &= t\,[\tilde{\rho}(e,q)P]^{-1}\,\tilde{\rho}(e,q)\alpha\,[\tilde{\rho}(e,q)\alpha]^{-1} \\
(g_e\alpha)\,[\tilde{\rho}(e,r)\alpha] &= t\,[\tilde{\rho}(e,q)P]^{-1}\,g_eP\cdot[\tilde{\rho}(e,r)P]\,[\tilde{\rho}(e,r)P]^{-1}\,[\tilde{\rho}(e,r)\alpha] \\
&= taPrPh_p^{-1}.
\end{aligned}
$$

Theorem 2.30. *The functors α and β give an equivalence between $\mathfrak{A}^{\tilde{G}_e}$ and $\mathfrak{A}^{\pi CayQ}$.*

Proof. It is obvious that $\alpha\beta = 1$. To show that $\beta\alpha$ is equivalent to 1, suppose $f\colon P_1 \to P_2$ is an $\mathfrak{A}^{\pi CayQ}$ morphism, i.e., $\forall q \in Q$, $qf\colon A_{1q} \to A_{2q}$.

Then the following diagram commutes:

$$
\begin{array}{ccc}
P_1\beta\alpha & \xrightarrow{P_1h_{P_1}} & P_1 \\
{\scriptstyle f\beta\alpha}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
P_2\beta\alpha & \xrightarrow{P_2h_{P_2}} & P_2
\end{array}
\quad ,
$$

since $[m\otimes\tilde{\rho}(e,q)](qf\beta\alpha)(qP_2h_{P_2}) = [m\tilde{\rho}(e,q)P_1(qf)(\tilde{\rho}(e,q)P_2)^{-1}\otimes\tilde{\rho}(e,q)]qP_2h_{P_2} = m\tilde{\rho}(e,q)\,P_1(qf)(\tilde{\rho}(e,q)P_2)^{-1}\tilde{\rho}(e,q)P_2 = m\tilde{\rho}(e,q)P_1(qf) = [m\otimes\tilde{\rho}(e,q)](qP_1h_{P_1})(qf)$.

Analogously, the following diagram commutes:

$$
\begin{array}{ccc}
P_1 & \xrightarrow{P_1h_{P_1}^{-1}} & P_1\beta\alpha \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f\beta\alpha} \\
P_2 & \xrightarrow[P_2h_{P_2}^{-1}]{} & P_2\beta\alpha
\end{array}
\quad .
$$

Since

$$
\begin{aligned}
n(qf)\left(P_2h_{p_2}^{-1}\right) &= n(qf)\,[\tilde{\rho}(e,q)P_2]^{-1}\otimes\tilde{\rho}(e,q) \\
&= n\,[\tilde{\rho}(e,q)P_1]^{-1}\,\tilde{\rho}(e,q)P_1(qf)\,(\tilde{\rho}(e,q)P_2)^{-1}\otimes\tilde{\rho}(e,q) \\
&= n\left[(\tilde{\rho}(e,q)P_1)^{-1}\otimes\tilde{\rho}(e,q)\right][qf\beta\alpha] \\
&= n\,[\tilde{\rho}(e,q)P_1]^{-1}\,\tilde{\rho}(e,q)\alpha\,[qf\beta\alpha] \\
&= n\left[qP_1h_{p_1}^{-1}\right](qf\beta\alpha). \qquad \square
\end{aligned}
$$

Proof of Theorem 2.25. The equivalence between $\mathfrak{A} \otimes (\Omega/Q)$ and $\mathfrak{A}^{\widetilde{G_e}}$ is given in [S2, 336]. Theorem 2.30 gives the equivalence between $\mathfrak{A}^{\widetilde{G_e}}$ and $\mathfrak{A}^{\pi\,\mathrm{Cay}\,Q}$. For the equivalence between $\mathfrak{A}^{\pi\,\mathrm{Cay}\,Q}$ and $Ab\,\mathrm{Cov}\,Q$, we just have to use the corresponding functors $\Lambda\colon \mathfrak{A}^{\pi\,\mathrm{Cay}\,Q} \to Ab\,\mathrm{Cov}\,Q$ and $\Lambda'\colon Ab\,\mathrm{Cov}\,Q \to \mathfrak{A}^{\pi\,\mathrm{Cay}\,Q}$ [H, 13.30] as follows:

Let $P \in \mathfrak{A}^{\pi\,\mathrm{Cay}\,Q}$, then by forgetting the structure, we can consider $P\colon \pi\,\mathrm{Cay}\,Q \to \mathfrak{A}$, then $P\Lambda\colon E \to \pi\,\mathrm{Cay}\,Q$ is defined by

$$v(E) = \coprod_{q\in Q} E^q = \{(v,q)\,|\,v \in E^q\}.$$

The morphisms of E from (v,q) to (w,r) are all pairs (a,v) where $a \in [\pi\,\mathrm{Cay}\,Q]_{qr}$ and $vaP = w$. $(v,q)P\Lambda = q$ and $(a,v)P\Lambda = a$. Clearly $\forall q \in Q$, E^q is an abelian group and $\forall a \in [\pi\,\mathrm{Cay}\,Q]_{qs}$, the map $E^a\colon E^q \to E^s$ is an isomorphism. Hence, the object part of Λ has been defined. If $f\colon P_1 \to P_2$ is a morphism in $\mathfrak{A}^{\pi\,\mathrm{Cay}\,Q}$, i.e., $\forall q \in Q$ $qf\colon E_1^q = qP_1 \to qP_2 = E_2^q$ s.t. for every $a \in [\pi\,\mathrm{Cay}\,Q]_{qr}$, the following diagram commutes:

$$\begin{array}{ccc} qP_1 & \xrightarrow{qf} & qP_2 \\ {\scriptstyle aP_1}\big\downarrow & & \big\downarrow{\scriptstyle aP_2} \\ rP_1 & \xrightarrow[rf]{} & rP_2 \end{array}.$$

Then $\{qf\,|\,q \in Q\}$ induces a map $\bar{f}\colon E_1 \to E_2$ s.t. the following diagram commutes

$$\begin{array}{ccc} E_1 & \xrightarrow{\bar{f}} & E_2 \\ {\scriptstyle P_1\Lambda}\big\downarrow & & \big\downarrow{\scriptstyle P_2\Lambda} \\ \pi\,\mathrm{Cay}\,Q & \xrightarrow[1]{} & \pi\,\mathrm{Cay}\,Q \end{array}.$$

Here,

$$(v,q)\bar{f} = (vf,q)$$
$$(a,v)\bar{f} = (a,vf).$$

Clearly $\bar{f}$ is a groupoid map, hence $\bar{f}$ is a morphism in $Ab\,\mathrm{Cov}\,Q$, and $f\Lambda' = \bar{f}$.

Let $\varphi \in Ab\,\mathrm{Cov}\,Q$. Considering φ as an element of coverings of $\pi\,\mathrm{Cay}\,Q$, then

$$\begin{aligned}\varphi\Lambda' \colon \pi\,\mathrm{Cay}\,Q \to \delta,;&\\ q &\mapsto E^q = \{v \in E \mid v\varphi = q\}\\ a &\mapsto (E^a \colon E^q \to E^r\,;\, v \mapsto w)\end{aligned}$$

where w is the target of the unique cover (of a) with source v. Clearly, by considering the structure of φ, $\varphi\Lambda' \in Ab\,\mathrm{Cov}\,Q$. If $f\colon \varphi_1 \to \varphi_2$ is a morphism in $Ab\,\mathrm{Cov}\,Q$, then

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \varphi_2} \\ \pi\,\mathrm{Cay}\,Q & \xrightarrow[1]{} & \pi\,\mathrm{Cay}\,Q \end{array}$$

commutes and $E_1^q f \subseteq E_2^q$ and $vE_1^a f = vfE_2^a$ for $a \in [\pi\,\mathrm{Cay}\,Q]_{qr}$. Hence f induces a morphism in $\mathfrak{A}^{\pi\,\mathrm{Cay}\,Q}$ from $\varphi_1\Lambda'$ to $\varphi_2\Lambda'$, i.e., a collection of $\mathfrak{A}$ morphisms, one for each $q \in Q$, $qf\colon E_1^q \to E_2^q$ s.t. the following diagram commutes for $a \in [\pi\,\mathrm{Cay}\,Q]_{qr}$:

$$\begin{array}{ccc} E_1^q & \xrightarrow{qf} & E_2^q \\ {\scriptstyle E_1^a}\downarrow & & \downarrow{\scriptstyle E_2^a} \\ E_1^r & \xrightarrow[rf]{} & E_2^r \end{array}.$$

We claim that $\Lambda\Lambda'$ is equivalent to 1. Define a natural transformation $\rho\colon 1 \to \Lambda\Lambda'$, i.e., a family of $Ab\,\mathrm{Cov}\,Q$ morphisms, one for each object φ of $Ab\,\mathrm{Cov}\,Q$; $\varphi\rho$ has object part $\coprod_{q\in Q} (E^q \to E^q\,;\, v \mapsto (v,q))$ and morphism part $\bar{a}_v \mapsto (a,v)$ where $\bar{a}_v$ is the cover (of $a \in [\pi\,\mathrm{Cay}\,Q]_{qr}$) with starting point v.

The following diagram commutes

$$\begin{array}{ccc} \varphi_1 & \xrightarrow{\varphi_1\rho} & \varphi_1\Lambda'\Lambda \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\Lambda'} \\ \varphi_2 & \xrightarrow[\varphi_2\rho]{} & \varphi_2\Lambda'\Lambda \end{array}$$

for every $Ab\,\mathrm{Cov}\,Q$ morphism $f\colon \varphi_1 \to \varphi_2$ since

$$\begin{tikzcd} v \arrow[r, "\varphi_1\rho"] \arrow[d, "f"'] & (v,q) \arrow[d, "f\Lambda\Lambda'"] \\ vf \arrow[r, "\varphi_2\rho"'] & (vf,q) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \bar{a}_v \arrow[r, "\varphi_1\rho"] \arrow[d, "f"'] & (a,v) \arrow[d, "f\Lambda'\Lambda"] \\ \bar{a}_v f = \bar{a}_{vf} \arrow[r, "\varphi_2\rho"'] & (a,vf) \end{tikzcd}$$

commute.

Define $\rho'\colon \Lambda'\Lambda \to 1$ by

$$\varphi\rho'\colon \coprod E^q \to E\ ;$$
$$(v,q) \mapsto v \in E^q\ ,$$
$$(a,v) \mapsto \bar{a}_v\ .$$

Then the following diagram commutes:

$$\begin{tikzcd} \varphi_1\Lambda'\Lambda \arrow[r, "\varphi_1\rho'"] \arrow[d, "f\Lambda'\Lambda"'] & \varphi_1 \arrow[d, "f"] \\ \varphi_2\Lambda'\Lambda \arrow[r, "\varphi_2\rho'"'] & \varphi_2 \end{tikzcd}$$

$$\begin{tikzcd} (v,q) \arrow[r, "\varphi_1\rho'"] \arrow[d, "f\Lambda'\Lambda"'] & v \in E_1^q \arrow[d, "f"] \\ (vf,q) \arrow[r, "\varphi_2\rho'"'] & vf \in E_2^q \end{tikzcd} \qquad \begin{tikzcd} (a,v) \arrow[r, "\varphi_1\rho'"] \arrow[d, "f\Lambda\Lambda'"'] & \bar{a}_v \arrow[d, "f"] \\ (a,vf) \arrow[r, "\varphi_2\rho'"'] & \bar{a}_{vf} = \bar{a}_v f \end{tikzcd}$$

Hence $\Lambda'\Lambda$ is equivalent to 1.

Define a natural transformation $\Delta\colon 1 \to \Lambda'\Lambda$, i.e., a family of $\mathfrak{A}^{\pi\,\mathrm{Cay}\,Q}$ morphisms, one for each object P of $\mathfrak{A}^{\pi\,\mathrm{Cay}\,Q}$ where $P\Delta$ is a collection of $\mathfrak{A}$ morphisms, one for each object q of $\pi\,\mathrm{Cay}\,Q$:

$$qP\Delta\colon qP \to qP\Lambda\Lambda'$$

$$Mq = qP \ni v \mapsto (v,q) \in E^q = \{(v,q) \mid v \in M_q\}.$$

$P\Delta$ is an $\mathfrak{A}^{\pi \operatorname{Cay} Q}$ morphism since for all $a \in [\pi \operatorname{Cay} Q]_{qr}$ the following diagrams commute:

$$\begin{tikzcd} qP \arrow[r, "qP\Delta"] \arrow[d, "aP"'] & qP\Lambda\Lambda' \arrow[d, "aP\Lambda\Lambda'"] \\ rP \arrow[r, "rP\Delta"'] & rP\Lambda\Lambda' \end{tikzcd}$$

Also the following diagram commutes for every $\mathfrak{A}^{\pi \operatorname{Cay} Q}$ morphism $f\colon P_1 \to P_2$:

$$\begin{tikzcd} P_1 \arrow[r, "P_1\Delta"] \arrow[d, "f"'] & P_1\Lambda\Lambda' \arrow[d, "f\Lambda\Lambda'"] \\ P_2 \arrow[r, "qP_2\Delta"'] & P_2\Lambda\Lambda' \end{tikzcd},$$

since

$$\begin{tikzcd} qP_1 \arrow[r, "qP_1\Delta"] \arrow[d, "qf"'] & qP_1\Lambda\Lambda' \arrow[d, "qf\Lambda\Lambda'"] \\ qP_2 \arrow[r, "qP_2\Delta"'] & qP_2\Lambda\Lambda' \end{tikzcd}$$

commutes.

Define a natural transformation

$$\Delta'\colon \Lambda\Lambda' \to 1, \quad \text{by}$$
$$qP\Delta'\colon qP\Lambda\Lambda' \to qP$$
$$E^q \ni (v,q) \mapsto v \in M_q\ .$$

Then $P\Delta'$ is an $\mathfrak{A}^{\pi \operatorname{Cay} Q}$ morphism since for $a \in [\pi \operatorname{Cay} Q]_{qr}$, the diagram

$$\begin{tikzcd} qP\Lambda\Lambda' \arrow[r, "qP\Delta'"] \arrow[d, "aP\Lambda\Lambda'"'] & qP \arrow[d, "aP"] \\ rP\Lambda\Lambda' \arrow[r, "rP\Delta'"'] & rP \end{tikzcd}$$

commutes.

For every $\mathfrak{A}^{\pi\,\mathrm{Cay}\,Q}$ morphism $f\colon P_1 \to P_2$, the following diagram commutes:

$$\begin{array}{ccc} qP_1\Lambda\Lambda' & \xrightarrow{qP_1\Delta'} & qP_1 \\ {\scriptstyle qf\Lambda'\Lambda}\downarrow & & \downarrow{\scriptstyle qf} \\ qP_2\Lambda\Lambda' & \xrightarrow[qP_2\Delta']{} & qP_2 \end{array} .$$

Clearly $\Delta\Delta' = 1$ and $\Delta'\Delta = 1$, hence $\Lambda\Lambda'$ is equivalent to 1.

Since both $\Lambda'\Lambda$ and $\Lambda\Lambda'$ are equivalent to 1, we concude that $\mathfrak{A}^{\pi\,\mathrm{Cay}\,Q}$ is equivalent to $Ab\,\mathrm{Cov}\,Q$. □

If we start with an abelian group $A \to Q$ in $\mathfrak{Q}/Q$, then we will get a right $\widetilde{G}_e$-module $M = \pi^{-1}(e), e \in Q$. Applying α, we will get a representation $P\colon \pi CayQ \to \mathfrak{A}$ where $A_q = M \otimes \tilde{\rho}(e,q)$, and $\tilde{\rho}(e,q)\alpha\colon M \to M \otimes \tilde{\rho}(e,q)$; $m \mapsto m \otimes \tilde{\rho}(e,q)$. If $a \in [\pi CayQ]_{qr}$, $a\alpha\colon M \otimes \tilde{\rho}(e,q) \to M \otimes \tilde{\rho}(e,r)$. Applying Λ, we have a covering groupoid $\varphi\colon E \to \pi CayQ$, where $V(E) = \coprod_{q\in Q} M_q$ $= \{(v,q) | v \in M_q\} = \{(m \otimes \tilde{\rho}(e,q), q) | m \in \pi^{-1}(e)\}$. This vertex set is exactly the set $A = \overset{\circ}{\bigcup_{q\in Q}} M \otimes \tilde{\rho}(e,q)$, of [S2, 336], and $V(\varphi)\colon V(E) \to Q$ is $\pi\colon A \to Q$ with operations analogous to that of those given in the remark after Definition 2.9.

CHAPTER 3

REPRESENTATIONS OF RIGHT QUASIGROUPS

The structure of a right quasigroup implies that the mapping $R_Q(q)$ given after Definition 2.5 is a permutation of the underlying set Q of a right quasigroup Q. However, the corresponding $L_Q(q)$ need not be a permutation on Q. Let SQ denote the monoid of all mappings from the set Q into Q with composition as the binary operation.

Definition 3.1. The submonoid of SQ generated by $\{R(q), R^{-1}(q), L(q) | q \in Q\}$ is called the *multiplication monoid* MQ, and the group generated by $\{R(q) | q \in Q\}$ contained in MQ is called the *right multiplication group* $RMltQ$.

Definition 3.2. If P is a right subquasigroup of a right quasigroup Q, then the *relative multiplication monoid* $M_Q P$ of P in Q is the submonoid of MQ generated by $\{R_Q(p), R_Q(p)^{-1}, L_Q(p) | p \in P\}$.

The analogous construction of the $\mathcal{R}$-universal multiplication monoid $UM(Q, \mathcal{R})$ of Q in the variety $\mathcal{R}$ of all right quasigroups is the relative multiplication monoid of Q in $\widetilde{Q} = Q * I$, the coproduct in $\mathcal{R}$ of Q with the free right quasigroup I on a generator x. We will also have the analogous category $\mathfrak{A} \otimes \mathcal{R}/Q$ of Q-modules in $\mathcal{R}$. Let $\pi \colon A \to Q$ be a Q-module in $\mathcal{R}$. Then there is an $\mathcal{R}/Q$-morphism $- \colon \alpha \to A$, called *subtraction*, defined as the following composition:

$$- \colon \alpha = A \times_Q A \xrightarrow{1 \times (-)} A \times_Q A \xrightarrow{+} A.$$

Again, the kernel of $- \colon \alpha \to A$ is a congruence $(\alpha | \alpha)$ on α which is a centreing congruence by which α centralizes itself (cf. [S2, 315]). Also, we will get analogies to [Lemma 2.10] and [S2, 336] as follows:

Proposition 3.3. *Let Q be a right quasigroup appearing as a right subquasigroup of a right quasigroup A in R. Then $r_A \colon UM(Q, \mathcal{R}) \to M_A Q$;*

$\widetilde{F}(q_1, \cdots, q_n) \mapsto F_A(q_1, \cdots, q_n)$ *is a monoid epimorphism from* $\widetilde{G} = UM(Q, \mathcal{R})$ *onto* $M_A Q$.

Definition 3.4. Let Q be a right quasigroup. Then the right Cayley diagram $R\,\mathrm{Cay}\,Q$ is a directed graph with vertex set Q and labelled arcs. For each x and y in Q, there is an arc $(x, R(y), xy)$ from x to xy labelled $R(y)$.

We will also have an analogue of Theorem 1.11 as follows:

Theorem 3.5. *Let* $\mathcal{R}$ *be the variety of all right quasigroups. Then for a right quasigroup* Q*, the universal right multiplication group* $\widetilde{RG} = UR(Q, \mathcal{R})$ *is the free group on* $\{\widetilde{R}(q) \mid q \in Q\}$.

Before proving this theorem, we need some preliminary theory about normal forms in a right quasigroup adapted from the quasigroup case. Definition of words carries over, with the restriction that $u \circ v$ means $u \cdot v$ or u/v. Also the definition of relations in closed form carries over, with the exception of axiom (ii): If one of the two $x \cdot y = z$, $x = z/y$ is a relation, so is the other. There are only three elementary reductions in a right quasigroup, that is replacing (i) $(v \cdot u)/u$ by v, (ii) $(v/u) \cdot u$ by v and (iii) $x \circ y$ by z if $x \circ y = z$ is one of the defining relations of Q.

Proof of Theorem 3.5. In the right Cayley diagram $R\,\mathrm{Cay}\,(Q * I)$, consider the subgraph $(xR\widetilde{G})$ consisting of all vertices lying in the orbit $xR\widetilde{G}$ of x under $R\widetilde{G}$ and of all arcs between these vertical labelled $\widetilde{R}(q)$ for some q in Q. Note that $(xR\widetilde{G})$ is (weakly) connected. If there is a circuit in $(xR\widetilde{G})$ starting at a vertex $x\widetilde{E}(p_1, \ldots, p_m)$, its labels form a product $\widetilde{F}(q_1, \ldots, q_n)$ such that $x\widetilde{E}\widetilde{F} = x\widetilde{E}$. By the analogue of Lemma 2.8., it follows that $\widetilde{F}(q_1, \ldots, q_n) = 1$. Because $\widetilde{F}$ is a product of labels we can assume $\widetilde{F}(q_1, \ldots, q_n) = \prod_{i=1}^{k} \widetilde{R}^{\varepsilon_i}(h_i)$ where $h_i \in \{q_1, \ldots, q_n\}$ and $\varepsilon_i = +1$ or -1.

Suppose the circuit is not trivial, then we can assume further that $\widetilde{F}$ is in 're-duced' form, i.e., there is not i s.t. $\varepsilon_i = -\varepsilon_{i+1}$, $h_i = h_{i+1}$. Also $k \geq 1$ since otherwise the 'reduced' form of $\widetilde{F}$ is 1, contradicting the non-trivialness. Since we have $\widetilde{F} = 1$, in particular $x\widetilde{F}(q_1, \ldots, q_n) = x$. Now x is in "normal form" in $Q * I = \langle q \in Q,\ x \mid q_1 \circ q_2 = q_3$ if $q_1 \circ q_2 = q_3$ in $Q\rangle$, where $Q = \langle q \in Q \mid q_1 \circ q_2 = q_3$ if $q_1 \circ q_2 = q_3$ in $Q\rangle$. Hence there exists a reduction chain

$$u = x\widetilde{F} \to u_1 \to u_2 \to \cdots \to u_k = x$$

[E, T3.2]. Define a relation $\leq$ on the set of components of $x\widetilde{F}$ by $z_1 \leq z_2$ if z_1 is a component of z_2. Let m be the "minimal" component of $x\widetilde{F}$ such that the elementary operation $u \to u_1$ happens in it. Since the "reduced" form of F is not 1, $\ell(m) \geq 2$. [If $\ell(m) = 1$, m is a generator, so that the elementary operation is on a generator, which is a contradiction.]

So the operation occurs at $m = x \prod_{i=1}^{t} \widetilde{R}^{\varepsilon_i}(h_i)$, $t \geq 1$, not at $x \prod_{i=1}^{t-1} \widetilde{R}^{\varepsilon_i}(h_i)$, i.e., involving h_t. In cases (i) and (ii) of the three elementary reductions, we will have a contradiction since $\varepsilon_{t-1} = -\varepsilon_t$, and $h_{t-1} = h_t$. In case (iii), we replace $\left(x \prod_{i=1}^{t-1} \widetilde{R}^{\varepsilon_i}(h_i)\right) \cdot h_t$ by z, then since the closed relations only involve $q \in Q$, we will have $z = q$ for some $q \in Q$ so that $x\widetilde{F}(q_1, \ldots, q_n) = q \prod_{i=t+1}^{k} \widetilde{R}^{\varepsilon_i}(h_i) \in Q$, but $x \notin Q$, a contradiction. Hence the circuit is trivial. This means that $(x\widetilde{RG})$ is a tree.

A left action of $R\widetilde{G}$ on $(xR\widetilde{G})$ is defined by letting $\widetilde{F}(q_1, \ldots, q_n)$ in $R\widetilde{G}$ send the arc $\left(x\widetilde{E}(p_1, \ldots, p_m), \widetilde{R}(q), x\widetilde{E}(p_1, \ldots, p_m)\widetilde{R}(q)\right)$ to $\left(x\widetilde{F}(q_1, \ldots, q_n)^{-1}\widetilde{E}(p_1, \ldots, p_m),\ \widetilde{R}(q), x\widetilde{F}(q_1, \ldots, q_n)^{-1}\widetilde{E}(p_1, \ldots, p_m)\widetilde{R}(q)\right)$. Suppose that a vertex $x\widetilde{E}(p_1, \ldots, p_m)$ is fixed by an element $\widetilde{F}(q_1, \ldots, q_n)$ of $R\widetilde{G}$. Then $x\widetilde{E}(p_1, \ldots, p_m) = x\widetilde{F}(q_1, \ldots, q_n)^{-1} \cdot E(p_1, \ldots, p_m)$ whence $x\widetilde{F}(q_1, \ldots, q_n) = x$. Applying Lemma 2.8. with $m = 0$, we have $\widetilde{F}(q_1, \ldots, q_n) = 1$. Thus no non-identity of $R\widetilde{G}$ leaves a vertex of $(x\widetilde{RG})$ fixed.

Now suppose that an arc $\left(x\widetilde{E}(p_1, \ldots, p_m), \widetilde{R}(q), x\widetilde{E}(p_1, \ldots, p_m)\widetilde{R}(q)\right)$ of $(xR\widetilde{G})$ is inverted by $\widetilde{F}(q_1, \ldots, q_n)$ in $R\widetilde{G}$, so that $x\widetilde{F}^{-1}(q_1, \ldots, q_n)\widetilde{E}(p_1, \ldots, p_m)$ $= x\widetilde{E}(p_1, \ldots, p_m)\widetilde{R}(q)$ and $x\widetilde{F}^{-1}(q_1, \ldots, q_n)\widetilde{E}(p_1, \ldots, p_m)\widetilde{R}(q) = x\widetilde{E}(p_1, \ldots, p_m)$, whence $\widetilde{R}(q)^2 = 1$ by Lemma 2.8. In particular $xq \cdot q = x$. Consider the right quasigroup $(\mathbb{Q}, \cdot, /)$ defined on the set of rationals $\mathbb{Q}$ by $r \cdot s = 2r + s$, $r/s = (r-s)/2$ for r, s in $\mathbb{Q}$. Define $f\colon Q \to \mathbb{Q};\ q \mapsto 0$. Since $\{0\}$ is a subright quasigroup of $(\mathbb{Q}, \cdot, /)$, f is a right quasigroup morphism. The image of $x = (xq)q$ in $\widetilde{Q}$ under $f * (x \mapsto 1)\colon \widetilde{Q} \to \mathbb{Q}$ is $1 = (1 \cdot 0) \cdot 0 = 4$, an impossibility. Thus no arc of $(xR\widetilde{G})$ is inverted by an element of $R\widetilde{G}$.

Thus $R\widetilde{G}$ acts freely on $(xR\widetilde{G})$ (in the sense of [ST, I.3.3]). The quotient graph $R\widetilde{G} \setminus (xR\widetilde{G})$ is a bouquet of circles labelled with elements of $\widetilde{R}(Q)$. By the Reidemeister Theorem [ST, Theorem I.4] it follows that $R\widetilde{G}$ is the free group on $\widetilde{R}(Q)$. □

A generalization of Proposition 2.24 will also be obtained as follows.

Proposition 3.6. *The stabilizer $[UR(Q, \mathcal{R})]_e$ of e in the universal right multiplication group of Q in $\mathcal{R}$ is isomorphic to the vertex group $[\pi R \operatorname{Cay}(Q)]_e$ at e of the fundamental groupoid of the right Cayley diagram of Q.*

Proof. Define $f\colon [\pi R \operatorname{Cay}(Q)]_e \to UR(Q, \mathcal{R})$ as follows: Take $y = [y_1, y_2, \ldots, y_n] \in [\pi R \operatorname{Cay}(Q)]_e$. Then

$$y_i = \begin{cases} \langle e_i, R(x_i), e_{i+1} \rangle & \text{where } e_{i+1} = e_i R(x_i) \quad \text{or} \\ \langle e_i, R^{-1}(x_i), e_{i+1} \rangle & \text{where } e_{i+1} = e_i R^{-1}(x_i) \end{cases}$$

and $e_1 = e = e_{n+1}$.

In general $y_i = \langle e_i, R^{\varepsilon_i}(x_i), e_{i+1} \rangle$ and $e_{i+1} = e_i R^{\varepsilon_i}(x_i)$.

Then
$$f([y_1, y_2, \ldots, y_n]) = \prod_{i=1}^{n} \widetilde{R}^{\varepsilon_i}(x_i).$$

Clearly f is a well-defined mapping, since if $[y_1, y_2, \ldots, \hat{y}_i, \hat{y}_{i+1}, \ldots, y_n]$ is a simple reduction (i.e., $y_i = y_{i+1}^{-1}$) of y (where $\hat{y}_i$ means y_i omitted) then $y_i = \langle e_i, R^{\varepsilon_i}(x_i), e_{i+1} \rangle$ and $y_{i+1} = \langle e_{i+1}, R^{-\varepsilon_i}(x_i), e_i \rangle$. Hence

$$\begin{aligned} f([y_1, y_2, \ldots, \hat{y}_i, \hat{y}_{i+1}, \ldots, y_n]) &= \prod \widetilde{R}^{\varepsilon_t}(x_t) \qquad t \in \{1, 2, \ldots, \hat{\imath}, \widehat{i+1}, \ldots, n\} \\ &= \prod_{t=1}^{n} \widetilde{R}^{\varepsilon_t}(x_t) = f(y). \end{aligned}$$

We can restrict the codomain of f to be $[UR(Q,\mathcal{R})]_e$, since $e\prod_{i=1}^{n}\widetilde{R}^{\varepsilon_i}(x_i)=e$. Now f is also a homomorphism, since

$$\begin{aligned} f(y\cdot z) &= ([y_1,\dots,y_n][z_1,\dots,z_m])\,f \\ &= \prod_{i=1}^{n}\widetilde{R}^{\varepsilon_{x_i}}(x_i)\prod_{i=1}^{m}\widetilde{R}^{\varepsilon_{t_i}}(t_i) \\ &= \left[\prod_{i=1}^{n}\widetilde{R}^{\varepsilon_{x_i}}(x_i)\right]\left[\prod_{i=1}^{m}\widetilde{R}^{\varepsilon_{t_i}}(t_i)\right] \\ &= f(y)f(z)\ , \end{aligned}$$

where $y_i=\langle e_i,R^{\varepsilon_{x_i}}(x_i),e_{i+1}\rangle$ and $z_i=\langle f_i,R^{\varepsilon_{t_i}}(t_i),f_{i+1}\rangle$, $e_1=e_{n+1}=f_1=f_{m+1}=e$.

Now f is $1-1$; suppose that y is a reduced path in $[\pi R\,\mathrm{Cay}\,Q]_e$ and $f(y)=f([y_1,y_2,\dots,y_n])=1_{Q[x]}$ implies, by the freeness of $UR(Q,\mathcal{R})$ on $\{\widetilde{R}(q)\mid q\in Q\}$ that there exists some $i\in\{1,\dots,n\}$ such that $\widetilde{R}^{\varepsilon_i}(x_i)=\widetilde{R}^{\varepsilon_{i+1}}(x_{i+1})^{-1}$. Applying $r_Q\colon UR(Q,\mathcal{R})\to \mathrm{Mlt}\,RQ$, we obtain $R^{\varepsilon_i}(x_i)=R^{\varepsilon_{i+1}}(x_{i+1})^{-1}$, i.e., $y_i=y_{i+1}^{-1}$, contradicting the reducedness of y.

The homomorphism f is also onto, since if $\prod_{i=1}^{n}\widetilde{R}^{\varepsilon_i}(x_i)\in[UR(Q,\mathcal{R})]_e$, s.t. $x_i=x_{i+1}\to\varepsilon_i=\varepsilon_{i+1}$, then $e\prod_{i=1}^{n}R^{\varepsilon_i}(x_i)=e$ by applying r_Q. Consider $y=[y_1,\dots,y_n]$ where $y_i=\langle e_i,R^{\varepsilon_i}(x_i),e_{i+1}\rangle$, $e_1=e_{n+1}=e$, then clearly $y\in[\pi R\,\mathrm{Cay}\,(Q)]_e$ by the fact that $e\prod_{i=1}^{n}R^{\varepsilon_i}(x_i)=e$, and $f(y)=\prod_{i=1}^{n}\widetilde{R}^{\varepsilon_i}(x_i)$. Hence we conclude that $[\pi R\,\mathrm{Cay}\,Q]_e\cong[UR(Q,\mathcal{R})]_e$. $\square$

Proposition 3.7.

$$\prod_{[e]\in\pi R\,\mathrm{Cay}\,Q}[UR(Q,\mathcal{R})]_e\cong\prod_{[e]\in\pi R\,\mathrm{Cay}\,Q}[\pi R\,\mathrm{Cay}\,Q]_e\ ,$$

where $[e]$ is the path component of $\pi R\,\mathrm{Cay}\,Q$ containing e.

Proof. From the preceding proposition, it is clear that there exists an isomorphism

f_e from $[\pi R\,\mathrm{Cay}\,Q]_e$ to $[UR(Q,\mathcal{R})]_e$. Hence $\prod_{[e]\in\pi R\,\mathrm{Cay}\,Q} f_e$ is an isomorphism from $\prod_{[e]\in\pi R\,\mathrm{Cay}\,Q} [\pi R\,\mathrm{Cay}\,Q]_e$ to $\prod_{[e]\in\pi R\,\mathrm{Cay}\,Q} [UR(Q,\mathcal{R})]_e$. □

The attempt to get an analogue of [S1, 244] fails, since it is possible to have a right quasigroup s.t. for $[e] \in [\pi R\,\mathrm{Cay}\,Q]$, for every $n \in \mathbb{N}$, there exists a $q \in [e]$ such that the minimal reduced path from e to q is of length n, for example the right quasigroup defined on $\{0,1,2,3,\dots\}$ by

$$x \cdot y = \begin{cases} x+1 & \text{if } x < y \\ 0 & \text{if } x = y \\ x & \text{if } x > y \end{cases}$$

$$x/y = \begin{cases} x-1 & \text{if } x < y \\ 0 & \text{if } x = y \\ x & \text{if } x > y \end{cases}.$$

A generalization of [S2, 336] which will be needed to prove the equivalences of categories will be given in Proposition 3.8.

Proposition 3.8. *Let $\pi\colon A \to Q$ be a Q-module. Identify Q with its image in A under O_Q. Then for elements a, b of A, one has:*

(i) $ab = a \cdot b\pi + a\pi \cdot b$;

(ii) $a/b = a/b\pi - [a\pi/b\pi \cdot b]/b\pi$.

Proof. (i) Is a generalization of [S2, 336].

(ii)
$$\begin{aligned} a\pi &= (a\pi/b) \cdot b \\ &= (a\pi/b + a\pi/b\pi) \cdot (b\pi + b) \\ &= (a\pi/b) \cdot b\pi + (a\pi/b\pi) \cdot b \end{aligned}$$

$$a\pi - (a\pi/b\pi) \cdot b = (a\pi/b) \cdot b\pi$$

$$[a\pi - (a\pi/b\pi) \cdot b]/b\pi = a\pi/b$$

$$a\pi/b\pi - [(a\pi/b\pi) \cdot b]/b\pi = a\pi/b$$

$$a\pi/b\pi - a\pi/b = a\pi/b\pi - [-(a\pi/b\pi) \cdot b]/b\pi.$$

Notice that $(a\pi, a)(\alpha|\alpha)(a\pi, a)$ and $(b\pi, b\pi)(\alpha|\alpha)(b, b)$ yield $(a\pi/b\pi, a/b\pi)(\alpha|\alpha)(a\pi/b, a/b)$ but $(a\pi/b\pi, a/b\pi)(\alpha|\alpha)\,(a\pi/b, a/b\pi - \{(a\pi/b\pi)\cdot b\}/b\pi)$ by above, so that

$$a/b = a/b\pi - [(a\pi/b\pi)\cdot b]\,/b\pi. \qquad \square$$

The next step is to define the Cayley diagram of a right quasigroup.

Definition 3.9. Let Q be a right quasigroup. The *Cayley diagram* $\operatorname{Cay} Q$ of Q is a directed graph with vertex set Q and labelled arcs. For each x, y in Q, there is an arc $(x, R(y), xy)$ from x to xy, an arc $(x, R^{-1}(y), x/y)$ from x to x/y, and an arc $(x, L(y), yx)$ from x to yx. These arcs are labelled $R(y), R^{-1}(y)$ and $L(y)$ respectively.

We are able to generate a category $\vec{P}\operatorname{Cay} Q$ from $\operatorname{Cay} Q$ by taking the set Q as objects and 'reduced' paths of $\operatorname{Cay} Q$ as morphisms. Here "reducedness" means applying all possible equations $\langle x, R(y), xy\rangle\langle xy, R^{-1}(y), x\rangle = 1$ and $\langle x, R^{-1}(y), x/y\rangle \cdot \langle x/y, R(y), x\rangle = 1$. Consider the groupoid generated by $\{(x, R(y), xy) | x, y \in Q\}$ which is a subcategory of $\vec{P}\operatorname{Cay} Q$. It consists of components C_i for i in an index set I. For each i, pick a representative $\bar{q}_i$ in $V(C_i)$. Then for each $q \in V(\vec{P}\operatorname{Cay} Q), q$ is connected to exactly one element $\bar{q}$ of the set of $\{\bar{q}_i | i \in I\}$ of representatives by a sequence of simple paths only using the R's. Let $\rho(\bar{q}, q)$ be the reduced path from $\bar{q}$ to q given by a sequence of labels $R^{\pm 1}(x_j)$, where $x_j \in Q$. Take $\rho(\bar{q}, \bar{q}) = 1$, the empty path at $\bar{q}$. Let $\tilde{\rho}(\bar{q}, q)$ be the element corresponding to $\rho(\bar{q}, q)$ in $UM(Q, R)$. Then given a Q-module $\pi\colon A \longrightarrow Q$ in $\mathcal{R}$, we will get a representation $P_A = \pi\alpha'$ of $\vec{P}\operatorname{Cay} Q$ into the category $\mathfrak{A}$ of abelian groups as follows:
$qP_A = \{(m, \bar{q}, q)\colon m \in \pi^{-1}(\bar{q})\}, \rho(\bar{q}, q)\alpha'\colon (m, \bar{q}, \bar{q}) \mapsto (m, \bar{q}, q)$. If $a \in [\vec{P}\operatorname{Cay} Q]_{qr}$, then a can be rewritten uniquely as $a = \rho(\bar{q}, q)^{-1}[\rho(\bar{q}, q)a\rho(\bar{r}, r)^{-1}]\rho(\bar{r}, r)$, so that $a\alpha' = \rho(\bar{q}, q)\alpha^{-1}[\tilde{\rho}(\bar{q}, q)\tilde{a}\tilde{\rho}(\bar{r}, r)^{-1}]\rho(\bar{r}, r)\alpha$, where $\tilde{\rho}(\bar{q}, q)\tilde{a}\tilde{\rho}(\bar{r}, r)^{-1} \in [UM(Q, R)]_{\bar{q}\bar{r}}$

acts via M_AQ as in Proposition 3.3. Clearly $\tilde{\rho}(\bar{q},q)\tilde{a}\tilde{\rho}(\bar{r},r)^{-1}$ is a homomorphism of abelian groups.

If $f\colon(\pi_1\colon A_1\to Q)\to(\pi_2\colon A_2\to Q)$ is a morphism in $\mathfrak{A}\otimes\mathcal{R}/Q$, then clearly $f[\pi_1^{-1}(\bar{q})]\subseteq\pi_2^{-1}(\bar{q})$, so that $f\alpha'\colon P_1=\pi_1\alpha'\to\pi_2\alpha'=P_2$, defined by $qP_1\to qP_2;(m,\bar{q},q)\mapsto(f(m),\bar{q},q)$, is an abelian group homomorphism.

Proposition 3.10. *The assignment α' is a functor from $\mathfrak{A}\otimes\mathcal{R}/Q$ to $\mathfrak{A}^{\vec{P}\,\mathrm{Cay}\,Q}$.*

Proof. Clearly for every $a\in[\vec{P}\,\mathrm{Cay}\,Q]_{qr}$, the diagram

$$\begin{tikzcd} qP_1 \arrow[r, "qf\alpha'"] \arrow[d, "aP_1"'] & qP_2 \arrow[d, "aP_2"] \\ rP_1 \arrow[r, "rf\alpha'"'] & rP_2 \end{tikzcd}$$

commutes, since

$$\begin{aligned}(m,\bar{q},q)(qf\alpha')(aP_2)&=(f(m),\bar{q},q)(aP_2)=(mf[\tilde{\rho}(\bar{q},q)\tilde{a}\tilde{\rho}(\bar{r},r)^{-1}]\pi_2,\bar{r},r)\\&=(m[\tilde{\rho}(\bar{q},q)\tilde{a}\tilde{\rho}(\bar{r},r)^{-1}]\pi_1f,\bar{r},r)\\&=(m[\tilde{\rho}(\bar{q},q)\tilde{a}\tilde{\rho}(\bar{r},r)^{-1}]\pi_1,\bar{r},r)(rf\alpha)\\&=(m,\bar{q},q)(aP_1)(rf\alpha).\end{aligned}$$

The other conditions to be a functor are satisfied by α' trivially. $\square$

Suppose on the other hand that we have a representation P from $\vec{P}\,\mathrm{Cay}\,Q$ into the category $\mathfrak{A}$ of abelian groups. Let $P\beta'\colon A\to Q$, where

$$\begin{aligned}V(A)&=\overset{\circ}{\bigcup_{\bar{q}\in\{\bar{q}_i|i\in I\}}}\coprod_{q\in[\bar{q}]}\bar{q}P\\&=\{(m,\bar{q},q)|q\in[\bar{q}],\bar{q}\in\{\bar{q}_i|i\in I\}\}\,.\end{aligned}$$

Here $[\bar{q}]$ is the path component of q in the groupoid generated by $\{(x,R(y),xy)|x,y\in Q\}$, and $(m,\bar{q},q)P\beta'=q$. Define

$$\begin{aligned}(m,\bar{q},q)\cdot(n,\bar{r},r)=\big[&m\rho(\bar{q},q)\langle q,R(r),qr\rangle\rho(\bar{q}\bar{r},qr)^{-1}\\&+n\rho(\bar{r},r)\langle r,L(q),qr\rangle\rho(\bar{q}\bar{r},qr)^{-1},\bar{q}\bar{r},qr\big]\end{aligned}$$

and

$$(m,\bar{q},q)/(n,\bar{r},r) = \Big([m\rho(\bar{q},q) - n\rho(\bar{r},r)\langle r, L(q/r), q\rangle] \\ \cdot \langle q, R^{-1}(r), q/r\rangle \rho(\overline{q/r}, q/r)^{-1}, \overline{q/r}, q/r\Big).$$

Also define maps $0_Q \colon Q \to A; q \mapsto (\bar{q},\bar{q},q)$ ("zero") , $- \colon A \to A$;
$(m,\bar{q},q) \mapsto (-m,\bar{q},q)$ ("negation") , and $+ \colon A \times_Q A \to A$;
$((m,\bar{q},q),(n,\bar{q},q)) \mapsto (m+n,\bar{q},q)$("addition"). Then it is easy to see that $P\beta'$ is a Q-module in $\mathcal{R}$.

Let $f \colon P_1 \to P_2$ be a morphism in $\mathfrak{A}^{\vec{P}\,\mathrm{Cay}\,Q}$, i.e., $\forall q \in Q, qf \colon qP_1 \to qP_2$ is given such that, for every $a \in [\vec{P}\,\mathrm{Cay}\,Q]_{qr}$, the diagram

$$\begin{tikzcd} qP_1 \arrow[r, "qf"] \arrow[d, "aP_1"'] & qP_2 \arrow[d, "aP_2"] \\ rP_1 \arrow[r, "rf"'] & rP_2 \end{tikzcd}$$

commutes. Define $f\beta' \colon (P_1\beta' = \pi_1 \colon A_1 \to Q) \to (P_2\beta' = \pi_2 \colon A_2 \to Q)$;
$(m,\bar{q},q) \ \mapsto \ (m\rho(\bar{q},q)P_1 qf\rho(\bar{q},q)^{-1}\ P_2,\bar{q},q)$.

Proposition 3.11. *The assignment β' gives a functor from $\mathfrak{A}^{\vec{P}\,\mathrm{Cay}\,Q}$ to $\mathfrak{A} \otimes \mathcal{R}/Q$.*

Proof. Here f is a right quasigroup morphism since

$$\begin{aligned}[(m,\bar{q},q)\cdot(n,\bar{r},r)]\,f\beta' &= \big[m\rho(\bar{q},q)P_1\langle q,R(r),qr\rangle\rho(\overline{qr},qr)^{-1}P_1 \\ &\qquad + n\rho(\bar{r},r)P_1\langle r,L(q),qr\rangle\rho(\overline{qr},qr)^{-1}P_1, \overline{qr}, qr\big]\,qrf\beta' \\ &= \big[m\rho(\bar{q},q)P_1\langle q,R(r),qr\rangle P_1 qrf\rho(\overline{qr},qr)^{-1}P_2 \\ &\qquad + n\rho(\bar{r},r)P_1\langle r,L(q),qr\rangle P_1 qrf\rho(\overline{qr},qr)^{-1}P_2, \overline{qr}, qr\big]\end{aligned}$$

$$= \big[m\rho(\bar{q},q)P_1 qf\langle q,R(r),qr\rangle P_2\rho(\overline{qr},qr)^{-1}P_2$$
$$+ n\rho(\bar{r},r)P_1\langle r,L(q),qr\rangle P_1 qrf\rho(\overline{qr},qr)^{-1}P_2,\overline{qr},qr\big]$$
$$= \big(m\rho(\bar{q},q)P_1 qf\rho(\bar{q},q)^{-1}P_2,\bar{q},q\big)$$
$$\cdot\big(n\tilde{\rho}(\bar{r},r)P_1 rf\tilde{\rho}(\bar{r},r)^{-1}P_2,\bar{r},r\big)$$
$$= (m,\bar{q},q)f\beta'\cdot(n,\bar{r},r)f\beta'$$

and

$$[(m,\bar{q},q)/(n,\bar{r},r)]\,f\beta'$$
$$= \big([m\rho(\bar{q},q)P_1 - n\rho(\bar{r},r)P_1\langle r,L(q/r),q\rangle P_1]$$
$$\langle q,R^{-1}(r),q/r\rangle P_1\rho(\overline{q/r},q/r)^{-1},\overline{q/r},q/r\big)\,f\beta'$$
$$= \big([m\rho(\bar{q},q)P_1 - n\rho(\bar{r},r)P_1\langle r,L(q/r),q\rangle P_1]$$
$$\langle q,R^{-1}(r),q/r\rangle P_1\rho(\overline{q/r},q/r)^{-1}P_1\rho(\overline{q/r},q/r)P_1$$
$$q/r\;f\rho(\overline{q/r},q/r)^{-1}P_2,\overline{q/r},q/r\big)$$
$$= \big([m\rho(\bar{q},q)P_1(qf) - n\rho(\bar{r},r)P_1(rf)\langle r,L(q/r),q\rangle P_2]$$
$$\langle q,R^{-1}(r),q/r\rangle P_2\rho(\overline{q/r},q/r)^{-1}P_2,\overline{q/r},q/r\big)$$
$$= \big(m\rho(\bar{q},q)P_1(qf)\rho(\bar{q},q)^{-1}P_2,\bar{q},q\big)\;/\;\big(n\rho(\bar{r},r)P_1(rf)\rho(\bar{r},r)P_2^{-1},\bar{r},r\big)$$
$$= (m,\bar{q},q)f\beta'/(n,\bar{r},r)f\beta'.$$

$f\beta'$ is a Q-morphism, since:

(i)
$$[(m,\bar{q},q)+(n,\bar{q},q)]\,f$$
$$= (m+n,\bar{q},q)f\beta'$$
$$= \big((m+n)\rho(\bar{q},q)P_1(qf)\rho(\bar{q},q)P_2^{-1},\bar{q},q\big)$$
$$= \big(m\rho(\bar{q},q)P_1(qf)\rho(\bar{q},q)P_2^{-1},\bar{q},q\big)$$
$$+ \big(n\rho(\bar{q},q)P_1(qf)\rho(\bar{q},q)P_2^{-1},\bar{q},q\big)$$
$$= (m,\bar{q},q)f\beta' + (n,\bar{q},q)f\beta'\;;$$

(ii)
$$(m,\bar{q},q)(-f\beta') = (-m,\bar{q},q)f\beta'$$
$$= \left(-m\rho(\bar{q},q)P_1 qf\rho(\bar{q},q)P_2^{-1},\bar{q},q\right)$$
$$= -\left(m\rho(\bar{q},q)P_1 qf\rho(\bar{q},q)P_2^{-1},\bar{q},q\right)$$
$$= (m,\bar{q},q)(f\beta'-)\;;$$

(iii)
$$qO_q f\beta' = (\bar{q},\bar{q},q)f\beta' = (\bar{q}_{\bar{q}}f,\bar{q},q)$$
$$= (\bar{q},\bar{q},q) = qO_Q.$$

Suppose $f\colon P_1 \to P_2$ and $g\colon P_2 \to P_3$ in $\mathfrak{A}^{\vec{P}\,\mathrm{Cay}\,Q}$, then

$$(m,\bar{q},q)(fg)\beta' = \left(m\rho(\bar{q},q)P_1 q(fg)\rho(\bar{q},q)^{-1}P_3,\bar{q},q\right)$$
$$= \left(m\rho(\bar{q},q)P_1(qf)\rho(\bar{q},q)P_2^{-1}\rho(\bar{q},q)P_2(qg)\rho(\bar{q},q)P_3,\bar{q},q\right)$$
$$= \left((m,\bar{q},q)f\beta'\right)g\beta'.$$

Clearly $(m,\bar{q},q)\,1\,\beta' = (m,\bar{q},q)$. So we conclude that β' is a functor. $\square$

Lemma 3.12. *For each Q-module $\pi\colon A \to Q$ in $\mathcal{R}$, there exists an $(\mathfrak{A}\otimes\mathcal{R}/Q)$-natural isomorphism $g'_\pi\colon(\pi\colon A \to Q) \to (\pi\alpha'\beta'\colon A' \to Q)$ given by restrictions $\pi^{-1}(q) \to \pi^{-1}(\bar{q})\times\{\bar{q}\}\times\{q\}; m \mapsto (m\rho(\bar{q},q)\pi^{-1},\bar{q},q)$. (Note that indices have been omitted from the notation here for clarity).*

Proof. The map $\pi g'_\pi$ is a right quasigroup homomorphism, since for $m \in \pi^{-1}(q)$ and $n \in \pi^{-1}(r)$ we have $mn \in \pi^{-1}(qr)$ and $m/n \in \pi^{-1}(q/r)$, so that $m\pi g'_\pi \cdot n\pi g'_\pi =$ $(m\rho(\bar{q},q)\pi^{-1},\bar{q},q)\cdot(n\rho(\bar{r},r)\pi^{-1},\bar{r},r) = [m\rho(\bar{q},q)\pi^{-1}\rho(\bar{q},q)\pi\alpha\langle q,R(r),qr\rangle\rho(\bar{qr},qr)^{-1}\pi\alpha$ $+n\rho(\bar{r},r)\pi^{-1}\rho(\bar{r},r)\ \ \pi\alpha\langle r,L(q),qr\rangle\rho(\bar{qr},qr)^{-1}\pi\alpha,\bar{qr},qr] =$ $([m\langle q,R(r),qr\rangle + n\langle r,L(q),qr\rangle]\rho(\bar{qr},qr)^{-1}\pi,\bar{qr},qr) = (mn\rho(\bar{qr},qr)\pi^{-1},\bar{qr},qr) =$ $(mn)\pi g'_\pi$, using Proposition 3.8.

Similarly,

$$\begin{aligned}
m\pi g'_\pi / n\pi g'_\pi &= \left(m\rho(\bar{q},q)\pi^{-1},\bar{q},q\right) / \left(n\rho(\bar{r},r)\pi^{-1},\bar{r},r\right) \\
&= \Big(\big[m\rho(\bar{q},q)\pi^{-1}\rho(\bar{q},q)\pi\alpha - n\rho(\bar{r},r)\pi^{-1}\rho(\bar{r},r)\pi\alpha\langle r, L(q/r), q\rangle\big] \\
&\qquad \langle q, R^{-1}(r), q/r\rangle \rho(\overline{q/r}, q/r)\pi\alpha^{-1}, \overline{q/r}, q/r\Big) \\
&= [m - n\langle r, L(q/r), q\rangle]\,\langle q, R^{-1}(r), q/r\rangle g'_\pi \\
&= (m/n)g'_\pi.
\end{aligned}$$

Thus $\pi g'_\pi$ is a right quasigroup homomorphism. Clearly $\pi g'_\pi$ is a Q-morphism, since if $m,n \in \pi^{-1}(q)$

$$\begin{aligned}
(m+n)\pi g'_\pi &= \left((m+n)\rho(\bar{q},q)\pi^{-1},\bar{q},q\right) \\
&= \left(m\rho(\bar{q},q)\pi^{-1} + n\rho(\bar{q},q)\pi^{-1},\bar{q},q\right) \\
&= \left(m\rho(\bar{q},q)\pi^{-1},\bar{q},q\right) + \left(n\bar{\rho}(\bar{q},q)\pi^{-1}\bar{q},q\right) \\
&= m\pi g'_\pi + n\pi g'_\pi, \\
m(-\pi g'_\pi) = (-m)\pi g'_\pi &= \left((-m)\rho(\bar{q},q)\pi^{-1},\bar{q},q\right) \\
&= \left(-(m\rho(\bar{q},q)\pi^{-1},\bar{q},q\right) = -\left(m\rho(\bar{q},q)\pi^{-1},\bar{q},q\right) \\
&= m(\pi g'_\pi -),
\end{aligned}$$

$$\begin{aligned}
q(O_Q\pi g'_\pi) = q\pi g'_\pi &= \left(q\rho(\bar{q},q)\pi^{-1},\bar{q},q\right) \\
&= (\bar{q},\bar{q},q) = qO_Q.
\end{aligned}$$

Hence g'_π is an $\mathcal{A}\otimes\mathcal{R}/Q$ morphism. Analogously, ${g'}_\pi^{-1}$ is also an $(\mathcal{A}\otimes\mathcal{R}/Q)$-morphism. □

Lemma 3.13. *Given a representation P from $\vec{P}$ Cay Q to the category $\mathfrak{A}$ of abelian groups, there exists a natural isomorphism $h'_p \colon P \to P\beta\alpha$, defined by* $qP \to \bar{q}P \times \{\bar{q}\} \times \{q\};\ m \mapsto (m\rho(\bar{q},q)P^{-1},\bar{q},q)$.

Proof. Suppose $a \in [\vec{P}\,\mathrm{Cay}\,Q]_{qr}$. Then $a = \rho(\bar{q},q)^{-1}[\rho(\bar{q},q)a\rho(\bar{r},r)^{-1}]\rho(\bar{r},r)$, so that $a\beta'\alpha' = \rho(\bar{q},q)\alpha^{-1}[\rho(\bar{q},q)a\rho(\bar{r},r)^{-1}]\rho(\bar{r},r)\alpha$. The diagram

$$\begin{tikzcd}
qP \arrow[r, "qPh'_p"] \arrow[d, "aP"'] & \bar{q}P \times \{\bar{q}\} \times \{q\} \arrow[d, "a\beta'\alpha'"] \\
rP \arrow[r, "rPh'_p"'] & \bar{r}P \times \{\bar{r}\} \times \{r\}
\end{tikzcd}$$

commutes, since $t(qPh'_p)(a\beta'\alpha') = (t\tilde{\rho}(\bar{q},q)P^{-1},\bar{q},q)(a\beta'\alpha') =$
$(t\tilde{\rho}(\bar{q},q)P^{-1}\tilde{\rho}(\bar{q},q)\ \ \tilde{a}\tilde{\rho}(\bar{r},r)^{-1},\bar{r},r) = (t\tilde{\rho}(\bar{q},q)P^{-1}(\tilde{\rho}(\bar{q},q)\tilde{a}\rho(\bar{r},r)^{-1})P,\bar{r},r)$
$= t(aP)(rPh'_p)$. The penultimate equality here holds since the action of $\tilde{\rho}(\bar{q},q)\tilde{a}\tilde{\rho}(\bar{r},r)^{-1}$ in $[UM(Q,R)]_{\bar{q}\bar{r}}$ is via $M_A Q$ by Proposition 3.3, which is the same as that of $\rho(\bar{q},q)a\rho(\bar{r},r)P$. Analogously, the diagram

$$\begin{tikzcd}
\bar{q}P \times \{\bar{q}\} \times \{\bar{q}\} \arrow[r, "qPh_p'^{-1}"] \arrow[d, "a\beta\alpha"'] & qP \arrow[d, "a"] \\
\bar{r}P \times \{\bar{r}\} \times \{\bar{r}\} \arrow[r, "rPh_p'^{-1}"'] & rP
\end{tikzcd}$$

commutes, since

$$\begin{aligned}
(t,\bar{q},q)(qPh'^{-1}_p)[aP] &= t\rho(\bar{q},q)P[aP] \\
&= t\tilde{\rho}(\bar{q},q)\tilde{a}\tilde{\rho}(\bar{r},r)^{-1}\rho(\bar{r},r)P \\
&= \left(t\rho(\bar{q},q)a\rho(\bar{r},r)^{-1},\bar{r},r\right)[rPh'_{p-1}] \\
&= (t,\bar{q},q)[a\beta\alpha][rPh'^{-1}_p]
\end{aligned}$$

by the same reasoning as before. $\square$

Theorem 3.14. *The functors α' and β' give an equivalence between $\mathfrak{A} \otimes (\mathcal{R}/Q)$ and $\mathfrak{A}^{\vec{P}\,\mathrm{Cay}\,Q}$.*

Proof. If $f\colon (\pi_1\colon A_1 \to Q) \to (\pi_2\colon A_2 \to Q)$ is a morphism in $\mathfrak{A} \otimes \mathcal{R}/Q$, then the diagram

$$\begin{array}{ccc} \pi_1 & \xrightarrow{\pi_1 g'_{\pi_1}} & \pi_1\alpha'\beta' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f\alpha\beta} \\ \pi_2 & \xrightarrow[\pi_2 g'_{\pi_2}]{} & \pi_2\alpha'\beta' \end{array}$$

commutes, since

$$\begin{aligned} m(\pi_1 g'\pi_1)[f\alpha'\beta'] &= (m\rho(\bar{q},q)\pi_1{}^{-1},\bar{q},q)[f\alpha'\beta'] \\ &= (m\rho(\bar{q},q)\pi_1{}^{-1}f_{\bar{q}},\bar{q},q) \\ &= (mf_q\rho(\overline{qf},qf)\pi_2{}^{-1},\bar{q},q) \\ &= (mf_q\rho(\bar{q},q)\pi_2{}^{-1},\bar{q},q) = mf(\pi_2 g'_{\pi_2}). \end{aligned}$$

Analogously, the diagram

$$\begin{array}{ccc} \pi_1\alpha'\beta' & \xrightarrow{g'_{\pi_1}{}^{-1}} & \pi_1 \\ {\scriptstyle f\alpha'\beta'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \pi_2\alpha'\beta' & \xrightarrow[g'_{\pi_2}{}^{-1}]{} & \pi_2 \end{array}$$

commutes, since

$$\begin{aligned} (m,\bar{q},q)\pi_1 g'^{-1}_{\pi_1} f &= m\rho(\bar{q},q)\pi_1 f = mf_{\bar{q}}\rho(\bar{q}f,qf)\pi_2 \\ &= mf_{\bar{q}}\rho(\bar{q},q)\pi_2 = (mf_{\bar{q}},\bar{q},q)\pi_2 g'_{\pi_2} \\ &= (m,\bar{q},q)f\alpha'\beta'\pi_2 g'^{-1}_{\pi_2}. \end{aligned}$$

Also for each morphism $f\colon P_1 \to P_2$, in $\mathfrak{A}^{\vec{P}\,\mathrm{Cay}\,Q}$, the diagram

$$\begin{array}{ccccc} P_1 & \xrightarrow{h'_{P_1}} & P_1\beta'\alpha' & \xrightarrow{h'^{-1}_{P_1}} & P_1 \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f\beta'\alpha'} & & \big\downarrow{\scriptstyle f} \\ P_2 & \xrightarrow[h'_{P_2}]{} & P_2\beta'\alpha' & \xrightarrow[h'^{-1}_{P_2}]{} & P_2 \end{array}$$

commutes, since

$$mh'_{p_1}(f\beta'\alpha') = m\rho(\bar{q},q)P_1^{-1}f\beta'\alpha' = mf\rho(\bar{q},q)P_2^{-1} = mfh'_{p_2}$$

and

$$\left(m\rho(\bar{q},q)P_1^{-1}\right)h'^{-1}_{p_1}f = mf = \left(m\rho(\bar{q},q)P_1^{-1}\right)(f\beta'\alpha')h'_{p_2^{-1}}. \qquad \square$$

Note that if Q is a quasigroup, and $\pi\colon A \to Q$ is a Q-module in $\mathbf{\Omega}$, then by forgetting the left division structure, we will have a Q-module $\pi\colon A \to Q$ in $\mathcal{R}$. The construction of α' and β' specializes in the case of quasigroups, in the sense that the following diagrams commute:

$$\begin{tikzcd}
\mathfrak{A}\otimes(\mathbf{\Omega}/Q) \arrow[r, "\tau\alpha"] \arrow[d, "F"'] & \mathfrak{A}^{\pi\,\mathrm{Cay}\,Q} \arrow[d, "F'"] \\
\mathfrak{A}\otimes(\mathcal{R}/Q) & \mathfrak{A}^{\vec{P}\,\mathrm{Cay}\,Q} \arrow[l, "\beta'"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{A}\otimes(\mathbf{\Omega}/Q) \arrow[d, "F"'] & \mathfrak{A}^{\pi\,\mathrm{Cay}\,Q} \arrow[l, "\beta\tau^{-1}"'] \arrow[d, "F'"] \\
\mathfrak{A}\otimes(\mathcal{R}/Q) \arrow[r, "\alpha'"'] & \mathfrak{A}^{\vec{P}\,\mathrm{Cay}\,Q}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{A}\otimes(\mathbf{\Omega}/Q) \arrow[r, "\tau\alpha"] \arrow[d, "F"'] & \mathfrak{A}^{\pi\,\mathrm{Cay}\,Q} \arrow[d, "F'"] \\
\mathfrak{A}\otimes(\mathcal{R}/Q) \arrow[r, "\alpha'"'] & \mathfrak{A}^{\vec{P}\,\mathrm{Cay}\,Q}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{A}\otimes(\mathbf{\Omega}/Q) \arrow[d, "F"'] & \mathfrak{A}^{\pi\,\mathrm{Cay}\,Q} \arrow[l, "\beta\tau^{-1}"'] \arrow[d, "F'"] \\
\mathfrak{A}\otimes(\mathcal{R}/Q) & \mathfrak{A}^{\vec{P}\,\mathrm{Cay}\,Q} \arrow[l, "\beta'"]
\end{tikzcd}\ .$$

Here τ is the functor that gives the equivalences of $\mathfrak{A}\otimes(\mathbf{\Omega}/Q)$ with $\mathfrak{A}^{\widetilde{Ge}}$, while F and F' are forgetful functors.

Another category which is equivalent to $\mathfrak{A}^{\vec{P}\,\mathrm{Cay}\,Q}$ is the category of ordered modules.

Definition 3.15. Let Ω be the category of "ordered modules", with objects collections of $\widetilde{RG_{\bar{e}}}$ modules $M_{\bar{e}}$, one for each $\bar{e}$, together with a collection of homomorphisms $\tilde{\rho}(\bar{e},e)\widetilde{L}(t)\tilde{\rho}(\bar{te},te)^{-1}$ from $M_{\bar{e}}$ to $M_{\bar{te}}$, one for each pair $(e,t)\in Q\times Q$.

Here $\widetilde{RG}_{\bar{e}}$ is the stabilizer of $\bar{e}$ in the right universal multiplication group $\widetilde{RG}$ of Q in $\mathcal{R}$. A morphism between two objects $\{M_{\bar{e}}\}$ and $\{M'_{\bar{e}}\}$ of Ω is just a collection $\{f_{\bar{e}}\}$ of module homomorphisms satisfying the "extension" module homomorphism condition, i.e., for $x \in M_{\bar{s}}$,

$$\begin{aligned} &x\tilde{\rho}(\bar{s}, s)\widetilde{L}(t)\tilde{\rho}(\overline{ts}, ts)^{-1} f_{\overline{ts}} \\ &\qquad = x f_{\bar{s}}\tilde{\rho}(\bar{s}, s)\widetilde{L}(t)\tilde{\rho}(\overline{ts}, ts)^{-1}. \end{aligned}$$

Suppose we have $A \xrightarrow{\pi} Q \in \mathfrak{A} \otimes \mathcal{R}/Q$. Let $M_e = \pi^{-1}(e)$. Then clearly $M_{\bar{e}}$ is a $\widetilde{RG}_e$ module. By taking $\tilde{\rho}(\bar{e}, e)\widetilde{L}(t)\tilde{\rho}(\overline{te}, te)^{-1} \in MQ$, we will get a homomorphism from $M_{\bar{e}}$ to $M_{\overline{te}}$. If $f\colon (A \to Q) \to (A' \to Q)$ is a Q-module homomorphism, clearly we can take $f_{\bar{e}} = f|M_{\bar{e}}$ and $f_{\bar{e}}$ will be a morphism in Ω.

Proposition 3.16. *The assignment Λ from $\mathfrak{A} \otimes \mathcal{R}/Q$ to Ω by taking $A \xrightarrow{\pi} Q$ to $\{M_{\bar{e}}\}$ and $f\colon A \to A'$ to $\{f_{\bar{e}}\}$ as described above is a functor.*

Suppose we have an object $\{M_{\bar{e}}\}$ of Ω. For every $s \in [\bar{e}]$, define $M_s = \{(x, s) \mid x \in M_{\bar{e}}\}$. Let $A = \bigcup_{s \in Q} M_s$ and let $\pi\colon A \to Q;\ (x, s) \mapsto s$.

Define

$$\begin{aligned} &\tilde{\rho}(\bar{s}, s)\pi\colon M_{\bar{s}} \to M_s\,;\ (x, \bar{s}) \mapsto (x, s) \\ &\widetilde{R}(t)\pi\colon M_s \to M_{st}\ ; \\ (x, s) \mapsto\ &(x, s)\tilde{\rho}(\bar{s}, s)\pi^{-1}\tilde{\rho}(\bar{s}, s)\widetilde{R}(t)\tilde{\rho}(\overline{st}, st)^{-1}\tilde{\rho}(\overline{st}, st)\pi \\ &\widetilde{L}(t)\pi\colon M_s \to M_{ts}\ ; \\ (x, s) \mapsto\ &(x, s)\tilde{\rho}(\bar{s}, s)\pi^{-1}\tilde{\rho}(\bar{s}, s)\widetilde{L}(t)\tilde{\rho}(\overline{ts}, ts)^{-1}\tilde{\rho}(\overline{ts}, ts)\pi. \end{aligned}$$

Clearly $\tilde{\rho}(\bar{s}, s)\pi$ and $\widetilde{R}(t)\pi$ are isomorphisms, while $\widetilde{L}(t)\pi$ is a homomorphism.

Define

$$\begin{aligned} (y, t) \cdot (x, q) &= (y, t)\widetilde{R}(q)\pi + (x, q)\widetilde{L}(t)\pi \quad \text{and} \\ (y, t)/(x, q) &= (y, t)\widetilde{R}(q)\pi^{-1} - \left[(x, q)\widetilde{L}(t/q)\pi\right] \widetilde{R}(q)\pi^{-1}. \end{aligned}$$

With the above definition, A is a right quasigroup since

$$\begin{aligned}
[(y,t)\cdot(x,q)]\,/(x,q) &= \left[(y,t)\widetilde{R}(q)\pi + (x,q)\widetilde{L}(t)\pi\right]/(x,q)\\
&= \left[(y,t)\widetilde{R}(q)\pi + (x,q)\widetilde{L}(t)\pi\right]\widetilde{R}(q)\pi^{-1}\\
&\quad - \left[(x,q)\widetilde{L}(tq/q)\pi\right]\widetilde{R}(q)\pi^{-1} = (y,t) \quad \text{and}\\
[(y,t)/(x,q)]\cdot(x,q) &= \left[(y,t)\widetilde{R}(q)\pi^{-1} - \{(x,q)\widetilde{L}(t/q)\pi\}\widetilde{R}(q)\pi^{-1}\right]\cdot(x,q)\\
&= \left[(y,t)\widetilde{R}(q)\pi^{-1} - \{(x,q)\widetilde{L}(t/q)\pi\}\widetilde{R}(q)\pi^{-1}\right]\widetilde{R}(q)\pi\\
&\quad + (x,q)\widetilde{L}(t/q)\pi = (y,t).
\end{aligned}$$

Also, it is clear that $\pi\colon A \to Q$ is an object of $\mathcal{R}/Q$.

Define

$$\begin{aligned}
&O_Q\colon Q \to A;\ q \mapsto (0,q)\ ,\\
&-\colon A \to A;\ (x,s) \mapsto (-x,s)\ ,\\
&+\colon A \times_Q A \to A;\ ((x,s),(x',s)) \mapsto (x+x',s).
\end{aligned}$$

Then

$$\begin{aligned}
&-[(x,s)\cdot(y,s')]\\
&= -\left(x\tilde{\rho}(\bar{s},s)\widetilde{R}(s')\tilde{\rho}(\overline{ss'},ss')^{-1} + y\tilde{\rho}(\overline{s'},s')\widetilde{L}(s)\tilde{\rho}(\overline{ss'},ss')^{-1}, ss'\right)\\
&= \left(-\left[x\tilde{\rho}(\bar{s},s)\widetilde{R}(s')\tilde{\rho}(\overline{ss'},ss')^{-1} + y\tilde{\rho}(\overline{s'},s')\widetilde{L}(s)\tilde{\rho}(\overline{ss'},ss')^{-1}\right], ss'\right)\\
&= \left((-x)\tilde{\rho}(\bar{s},s)\widetilde{R}(s')\tilde{\rho}(\overline{ss'},ss')^{-1} + (-y)\tilde{\rho}(\overline{s'},s')\widetilde{L}(s)\tilde{\rho}(\overline{ss'},ss')^{-1}, ss'\right)\\
&= (-x,s)\cdot(-y,s')\\
&= [-(x,s)]\cdot[-(y,s')]
\end{aligned}$$

$$\begin{aligned}
&[(x,s)+(x',s)]\cdot[(y,s')+(y',s')]\\
&\quad=(x+x',s)\cdot(y+y',s)\\
&\quad=\Big((x+x')\tilde{\rho}(\bar{s},s)\widetilde{R}(s')\tilde{\rho}(\overline{ss'},ss')^{-1}+(y+y')\tilde{\rho}(\overline{s'},s')\widetilde{L}(s)\tilde{\rho}(\overline{ss'},ss')^{-1},ss'\Big)\\
&\quad=\Big(x\tilde{\rho}(\bar{s},s)\widetilde{R}(s')\tilde{\rho}(\overline{ss'},ss')^{-1}+y\tilde{\rho}(\overline{s'},s')\widetilde{L}(s)\tilde{\rho}(\overline{ss'},ss')^{-1},ss'\Big)\\
&\qquad\quad+\Big(x'\tilde{\rho}(\bar{s},s)\widetilde{R}(s')\tilde{\rho}(\overline{ss'},ss')^{-1}+y'(\tilde{\rho}(\overline{s'},s')\widetilde{L}(s)\tilde{\rho}(\overline{ss'},ss')^{-1},ss'\Big)\\
&\quad=(x,s)\cdot(y,s')+(x',s)\cdot(y',s').
\end{aligned}$$

Clearly

$$O_Q(q\cdot r)=(0,q\cdot r)=(0,q)\cdot(0,r)=O_Q(q)\cdot O_Q(r).$$

Also, the abelian group identity diagrams commute, for example

$$(x,s)+(x',s)=(x+x',s)=(x'+x,s)=(x',s)+(x,s),$$

and

$$(x,s)+(-x,s)=(x-x,s)=(0,s)=O_Q(s).$$

If $\{f_{\bar{e}}\}$ is a morphism from $\{M_{\bar{e}}\}$ to $\{M'_{\bar{e}}\}$ in Ω, then let $f\colon A\to A'$; $(x,s)\mapsto(f_{\bar{s}}(x),s)$.

Clearly, $+f=f+$, $-f=f-$, and $O_Qf=O_Q$. Now f is an $\mathcal{R}/Q$ morphism since

$$\begin{aligned}
[(x,s)\cdot(y,s')]\,f&=\Big(x\tilde{\rho}(\bar{s},s)\widetilde{R}(s')\tilde{\rho}(\overline{ss'},ss')^{-1}f\overline{ss'}\\
&\qquad+y\tilde{\rho}(\overline{s'},s')\widetilde{L}(s)\tilde{\rho}(\overline{ss'},ss')^{-1}f_{\overline{ss'}},ss'\Big)\\
&=\Big(xf_{\bar{s}}\tilde{\rho}(\bar{s},s)\widetilde{R}(s)\tilde{\rho}(\overline{ss'},ss')^{-1}\\
&\qquad+\,yf_{\overline{s'}}\tilde{\rho}(\overline{s'},s)\widetilde{L}(s)\tilde{\rho}(\overline{ss'},ss')^{-1},ss'\Big)\\
&=(xf_{\bar{s}},s)\cdot(yf_{\overline{s'}},s')=(x,s)f\cdot(y,s')f.
\end{aligned}$$

Analogously, $[(x,s)/(y,s')]\,f=(x,s)f/(y,s')f$.

The above derivation leads us to the following propositions.

Proposition 3.17. *The assignment* $\{M_{\bar{e}}\} \mapsto \{M_{\bar{e}}\}\Lambda' = A$, *and* $\{f_{\bar{e}}\} \mapsto \{f_{\bar{e}}\}\Lambda' = f$ *above gives us a functor* Λ' *from* Ω *to* $\mathfrak{A} \otimes \mathcal{R}/Q$.

Proof. Clearly $1\Lambda' = 1$ and $(fg)\Lambda' = f\Lambda' \cdot g\Lambda'$. $\square$

Proposition 3.18. *For each* $A \xrightarrow{\pi} Q$ *in* $\mathfrak{A} \otimes \mathcal{R}/Q$, *there is a natural* $\mathfrak{A} \otimes \mathcal{R}/Q$-*isomorphism* $i_A \colon A \to A\Lambda\Lambda'$; $\pi^{-1}(s) \ni x \mapsto \big(x\tilde{\rho}(\bar{s}, s)^{-1}, s\big)$.

Proof. We have

$$\begin{aligned}
(x \cdot y)i_A &= \big(xy\tilde{\rho}(\overline{ss'}, ss')^{-1}, ss'\big) \\
&= \Big(\Big(x\widetilde{R}(s') + y\widetilde{L}(s)\Big)\,\tilde{\rho}\left(\overline{ss'}, ss'\right)^{-1}, ss'\Big) \\
&= \Big(x\tilde{\rho}(\bar{s}, s)^{-1}\tilde{\rho}(\bar{s}, s)\widetilde{R}(s')\tilde{\rho}(\overline{ss'}, ss')^{-1} \\
&\qquad + y\tilde{\rho}(\overline{s'}, s')^{-1}\tilde{\rho}(\overline{s'}, s')\widetilde{L}(s)\tilde{\rho}(\overline{ss'}, ss')^{-1}, ss'\Big) \\
&= \big(x\tilde{\rho}(\bar{s}, s)^{-1}, s\big) \cdot \big(y\tilde{\rho}(\overline{s'}, s')^{-1}, s'\big) \\
&= xi_A \cdot yi_A.
\end{aligned}$$

Also

$$\begin{aligned}
(x/y)i_A &= \Big((x/y)\tilde{\rho}(\overline{s/s'}, s/s')^{-1}, s/s'\Big) \\
&= \Big(\Big(x\widetilde{R}^{-1}(s') + y\widetilde{L}(s/s')\widetilde{R}^{-1}(s')\Big)\,\tilde{\rho}(\overline{s/s'}, s/s')^{-1}, s/s'\Big) \\
&= \Big(x\tilde{\rho}(\bar{s}, s)^{-1}\tilde{\rho}(\bar{s}, s)\widetilde{R}^{-1}(s')\tilde{\rho}(\overline{s/s'}, \overline{s/s'})^{-1} \\
&\qquad + y\tilde{\rho}(\overline{s'}, s')^{-1}\tilde{\rho}(\overline{s'}, s')\widetilde{L}(s/s')\widetilde{R}^{-1}(s')\tilde{\rho}(\overline{s/s'}, s/s')^{-1}, s/s'\Big) \\
&= \big(x\tilde{\rho}(\bar{s}, s)^{-1}, s\big) \,/\, \big(y\tilde{\rho}(\overline{s'}, s')^{-1}, s'\big) \\
&= xi_A/yi_A.
\end{aligned}$$

Clearly $+i_A = (i_A \times i_A)+$, $-i_A = i_A-$, and $0_Q i_A = 0_Q$, and the following diagram commutes for every $(A \xrightarrow{\pi} Q) \xrightarrow{f} (A' \xrightarrow{\pi'} Q)$:

$$\begin{array}{ccc} A & \xrightarrow{i_A} & A\Lambda\Lambda' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f\Lambda\Lambda'} \\ A' & \xrightarrow[i_{A'}]{} & A'\Lambda\Lambda' \end{array},$$

since

$$\begin{aligned} xi_A(f\Lambda\Lambda') &= \left(x\tilde{\rho}(\bar{s},s)^{-1}, s\right)(f\Lambda\Lambda') = \left(x\tilde{\rho}(\bar{s},s)^{-1}f_{\bar{s}}, s\right) \\ &= \left(xf_s\tilde{\rho}(\bar{s},s)^{-1}, s\right) = xfi_{A'}. \end{aligned}$$

It is also clear that:

$$\begin{aligned} ((x,s)\cdot(y,s'))\, i_A^{-1} &= \Big(x\tilde{\rho}(\bar{s},s)\widetilde{R}(s')\tilde{\rho}(\overline{ss'},ss')^{-1} \\ &\qquad + y\tilde{\rho}(\overline{s'},s')\widetilde{L}(s)\tilde{\rho}(\overline{ss'},ss')^{-1}, ss'\Big)\, i_A^{-1} \\ &= x\tilde{\rho}(\bar{s},s)\widetilde{R}(s') + y\tilde{\rho}(\overline{s'},s')\widetilde{L}(s) \\ &= (x,s)i_A^{-1}\cdot(y,s')i_A^{-1}. \end{aligned}$$

Analogously, $((x,s)/(y,s'))\, i_A^{-1} = (x,s)i_A^{-1}/(y,s')i_A^{-1}$. For a morphism $f\colon A \to A'$ in $\mathfrak{A}\otimes\mathcal{R}/Q$, the following diagram commutes:

$$\begin{array}{ccc} A\Lambda\Lambda' & \xrightarrow{i_A^{-1}} & A \\ {\scriptstyle f\Lambda\Lambda'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ A'\Lambda\Lambda' & \xrightarrow[i_{A'}^{-1}]{} & A' \end{array},$$

since

$$\begin{aligned} (x,s)i_A^{-1}f &= x\tilde{\rho}(\bar{s},s)f = xf\tilde{\rho}(\bar{s},s) \\ &= (xf,s)i_{A'}^{-1} = (x,s)(f\Lambda\Lambda')i_A^{-1}. \qquad \square \end{aligned}$$

Proposition 3.19. *For each $\{M_{\bar{s}}\}$ in Ω, there is a natural Ω-isomorphism*

$$J_M\colon \{M_{\bar{s}}\} \to \{M_{\bar{s}}\}\Lambda'\Lambda = \{M'_{\bar{s}}\}\ ;\ x \xrightarrow{J_s} (x,\bar{s}).$$

Proof. Since $((x+y)\tilde{g}_{\bar{s}})\,J_M = (x\tilde{g}_{\bar{s}} + y\tilde{g}_{\bar{s}})\,j_M = (x\tilde{g}_{\bar{s}}+y\tilde{g}_{\bar{s}},\bar{s}) = (x,\bar{s})\tilde{g}_{\bar{s}}+(y,\bar{s})\tilde{g}_{\bar{s}} = xJ_M\tilde{g}_{\bar{s}} + yJ_M\tilde{g}_{\bar{s}}$, we have proved that J_M is a $\widetilde{G}_{\bar{s}}$- module homomorphism. Also

$$\begin{aligned} x\left(\tilde{\rho}(\bar{s},s)\widetilde{L}(t)\tilde{\rho}(\overline{st},st)^{-1}\right) J_M &= \left(x\tilde{\rho}(\bar{s},s)\widetilde{L}(t)\tilde{\rho}(\overline{st},st)^{-1},\overline{st}\right)\\ &= (x,\bar{s})\tilde{\rho}(\bar{s},s)\widetilde{L}(t)\tilde{\rho}(\overline{st},st)^{-1}\\ &= xJ_M\tilde{\rho}(\bar{s},s)\widetilde{L}(t)\tilde{\rho}(\overline{st},st)^{-1}, \end{aligned}$$

so that the extended module homomorphism condition has been satisfied. Furthermore, the following diagram commutes:

$$\begin{tikzcd} \{M_{\bar{s}}\} \arrow[r, "\{J_s\}"] \arrow[d, "\{f_s\}"'] & \{M_{\bar{s}}\}\Lambda'\Lambda \arrow[d, "\{f_s\}\Lambda'\Lambda"] \\ \{M'_{\bar{s}}\} \arrow[r, "\{J'_s\}"'] & \{M'_{\bar{s}}\}\Lambda'\Lambda \end{tikzcd},$$

since $x\{j_{\bar{s}}\}\{f_{\bar{s}}\}\Lambda'\Lambda = (x,\bar{s})\{f_{\bar{s}}\}\Lambda'\Lambda - (xf_{\bar{s}},\bar{s}) = x\{f_{\bar{s}}\}\{j_{\bar{s}}\}$. J_M^{-1} is also an extended module homomorphism since $(x,\bar{s})\tilde{g}_{\bar{s}}J_M^{-1} = (x\tilde{g}_{\bar{s}},\bar{s})J_M^{-1} = x\tilde{g}_{\bar{s}} = (x,\bar{s})J_M^{-1}\tilde{g}_{\bar{s}}$, and

$$\begin{aligned} &(x,s)\left(\tilde{\rho}(\bar{s},s)\widetilde{L}(t)(\tilde{\rho}(\overline{ts},ts)^{-1}\right) J_M^{-1}\\ &= \left(x\tilde{\rho}(\bar{s},s)\widetilde{L}(t)\tilde{\rho}(\overline{ts},ts)^{-1}\right) J_M^{-1}\\ &= x\tilde{\rho}(\bar{s},s)\widetilde{L}(t)\tilde{\rho}(\overline{ts},ts)^{-1}\\ &= (x,\bar{s})J_M^{-1}\tilde{\rho}(\bar{s},s)\widetilde{L}(t)\tilde{\rho}(\overline{ts},ts)^{-1}. \end{aligned}$$

Here the identity

$$[(x,\bar{s}) + (x',\bar{s})]\,J_M^{-1} = (x,\bar{s})J_M^{-1} + (x',\bar{s})J_M^{-1}$$

is satisfied vacuously.

We will also have the commuting diagram

$$\begin{array}{ccc} \{M_{\bar{s}}\}\Lambda'\Lambda & \xrightarrow{J_{\bar{s}}^{-1}} & \{M_{\bar{s}}\} \\ \Big\downarrow{\scriptstyle \{f_{\bar{s}}\}\Lambda'\Lambda} & & \Big\downarrow{\scriptstyle \{f_{\bar{s}}\}} \\ \{M'_{\bar{s}}\}\Lambda'\Lambda & \xrightarrow[J'^{-1}_{\bar{s}}]{} & \{M'_{\bar{s}}\} \end{array},$$

since $(x,\bar{s})J_{\bar{s}}^{-1}\{f_{\bar{s}}\} = xf_{\bar{s}} = (xf_{\bar{s}},\bar{s})J_{\bar{s}}^{-1} = (x,\bar{s})\{f_{\bar{s}}\}\Lambda'\Lambda J_{\bar{s}}^{-1}$. □

Let us now recapitulate the results in Chapter 3.

Theorem 3.20. *The following categories are equivalent:* (i) $\mathfrak{A}\otimes(\mathcal{R}/Q)$; (ii) Ω; (iii) $\mathfrak{A}^{\vec{P}\,\mathrm{Cay}\,Q}$.

Proof. The equivalence of $\mathfrak{A}\otimes(\mathcal{R}/Q)$ and Ω is obtained from Proposition 3.18 and 3.19, while the equivalence of $\mathfrak{A}\otimes(\mathcal{R}/Q)$ and $\mathfrak{A}^{\vec{P}\,\mathrm{Cay}\,Q}$ has been proved in Theorem 3.10. □

Again if we start with a quasigroup module $\pi\colon A \to Q$, then by forgetting some structure we will have a right quasigroup module. Ignoring the coverings, Theorem 3.20 is a generalization of Theorem 2.23.

Theorem 3.17. *Let χ be a K-algeboid on $\overline{Q}$. Then a left χ-module M is a covering of χ (thought as a category) equivalent to a representation $\varphi\colon \chi \to \gamma$, where γ is the category of K-modules, such that*

$$\varphi(x+y) = \varphi(x) + \varphi(y)$$
$$\varphi(kx) = k\varphi(x)$$

for $x, y \in \overline{q'}\chi\bar{q}$ and $k \in k$.

Proof. Suppose we are given a representation $\varphi\colon \chi \to \gamma$ with the given properties, then let $\bar{q}M = \varphi(\bar{q})$; $\overline{q'}\chi\bar{q} \times \bar{q}M \to\!\!\to \overline{q'}M$ is given by $(x,m) \mapsto \varphi(x)(m)$ which is a K-bilinear left action satisfying $x'(xm) = (x'x)m$ for each $x' \in \bar{q}''\chi\bar{q}'$, $x \in \bar{q}'\chi\bar{q}$ and $m \in \bar{q}M$, for each $\bar{q}, \bar{q}', \bar{q}'' \in \overline{Q}$. If we have a morphism in γ, i.e., a natural transformation from φ_1 to φ_2, a family of γ-morphisms, $\tau(\bar{q})\colon \varphi_1(\bar{q}) \to \varphi_2(\bar{q})$ such that for every χ-morphism $x\colon \bar{q} \to \bar{r}$, the following diagram commutes

$$\begin{tikzcd}
\bar{q}M = \varphi_1(\bar{q}) \arrow[r, "\tau(\bar{q})"] \arrow[d, "\varphi_1(x)"'] & \varphi_2(\bar{q}) = \bar{q}M' \arrow[d, "\varphi_2(x)"] \\
\bar{r}M = \varphi_1(\bar{r}) \arrow[r, "\tau(\bar{r})"'] & \varphi_2(\bar{r}) = \bar{r}M'
\end{tikzcd} .$$

By defining $\bar{q}f\colon \bar{q}M \to \overline{q'}M$ as $\bar{q}f = \tau(\bar{q})$, which is a K-module homomorphism, we have for each $\bar{q}, \overline{q'} \in \overline{Q}$, $x \in \overline{q'}\chi\bar{q}$, and $m \in \bar{q}M$;

$$\begin{aligned}
\overline{q'}f(xm) &= \tau(\overline{q'})(xm) = \tau(\overline{q'})\,[\varphi_1(x)(m)] \\
&= \left[\tau(\overline{q'})\varphi_1(x)\right](m) = [\varphi_2(x)\tau(\bar{q})]\,(m) \\
&= \varphi(x)\,[\tau(\bar{q})(m)] = x\,[\bar{q}f(m)]\,.
\end{aligned}$$

Hence $\{\bar{q}f\}_{\bar{q}\in\overline{Q}}$ is a morphism in χ-mod. Also $1(\bar{q}) = \bar{q}1$ and we have a functor $\Lambda\colon \gamma^\chi \to \chi$-mod. since $\Lambda(\tau\tau') = ff' = \Lambda(\tau)\Lambda(\tau')$.

Notice that the K-module bilinear action structure of $\bar{\bar{q}}\chi\bar{q}$ comes from the equations $\varphi(x+y) = \varphi(x)+\varphi(y)$ and $\varphi(kx) = xy(x)$ by definint x to be $\varphi(x)$. Conversely, given a left χ-module M, it is easy to get the obvious representation $\varphi\colon \chi \to \gamma$, where $\varphi(x)(m) = xm$ for $x \in \overline{q'}\chi\bar{q}$, $m \in \bar{q}M$, and

$$\begin{aligned}
\varphi(x+y)(m) &= (x+y)m = xm + ym \\
&= \varphi(x)(m) + \varphi(y)(m) \\
\varphi(kx)(m) &= (kx)m = k(xm) = k\,(\varphi(x)(m))\,.
\end{aligned}$$

So that we have $\Lambda'\colon \chi$-mod $\to \gamma^\chi$ and Λ, Λ' will give the equivalence. $\square$

CHAPTER 4

MODULES IN A VARIETY OVER A RING

If we take a look at Q-modules in a variety $\mathfrak{V}$, then we have replaced $\mathfrak{Q}$ with $\mathfrak{V}$ when Q is a quasigroup, and $\mathcal{R}$ with $\mathfrak{V}$ when Q is a right quasigroup. Secondly, one may consider a unital K-module in $\mathfrak{V}/Q$ for some commutative ring K, i.e. an abelian group A in $\mathfrak{V}/Q$ equipped with mappings $s\colon A \to A$; $a \mapsto as$ for each s in K such that $s\colon A \to A$ is an endomorphism of the abelian group A, and such that for the unit element, $1_K\colon A \to A$ is the identity mapping $1_A\colon A \to A$ of A. Expressing this as comuting diagrams in the category $\underline{\underline{\mathrm{Set}}}$, and interpreting as diagrams in $\mathfrak{V}/Q$ will give us a unital K-module A in $\mathfrak{V}/Q$.

Definition 4.1. A *unital K-module* A in $\mathfrak{V}/Q$ is an abelian group in $\mathfrak{V}/Q$ equipped with an additional $\mathfrak{V}/Q$-morphism $s\colon A \to A$ for each s in K such that the diagrams representing unital K-module identities, now interpreted as diagrams in $\mathfrak{V}/Q$, commute. The category $K \otimes (\mathfrak{V}/Q)$ of Q-modules in $\mathfrak{V}$ over K is the subcategory of $\mathfrak{V}/Q$ whose objects are unital K-modules in $\mathfrak{V}/Q$ and whose morphisms are Q-morphisms $f\colon A \to A'$ such that $sf = fs$ for all s in K. In particular, if $K = \mathbb{Z}$, then $K \otimes (\mathfrak{V}/Q) = \mathfrak{A} \otimes (\mathfrak{V}/Q)$.

The theory of Q-modules in a variety over a ring has been developed by [S2, §3] for a quasigroup Q in a variety $\mathfrak{V}$ of quasigroups. In the next few theorems, we will generalize to the right quasigroup case.

Let Q be a right quasigroup. For $r, q \in Q$, let $\bar{r}S\bar{q}$ be a set consisting of symbols

$$\tilde{\rho}(\bar{q}, q)\widetilde{L}(t)\tilde{\rho}(\overline{tq}, tq)^{-1} \qquad \text{where } \overline{tq} = \bar{r},$$

$$\tilde{\rho}(\bar{q}, q)\widetilde{R}(t')\tilde{\rho}(\overline{qt'}, qt')^{-1} \qquad \text{where } \overline{qt'} = \bar{r}$$

and

$$\left[\tilde{\rho}(\bar{r}, r)\widetilde{R}(t'')\tilde{\rho}(\overline{rt''}, rt'')^{-1}\right]^{-1} \qquad \text{where } \overline{rt''} = \bar{q}.$$

If $\bar{q}, \overline{q'} \in \overline{Q}$, we define a composable sequence from $\bar{q}$ to $\overline{q'}$ to be a finite sequence $x = \langle \bar{q}_0, s_1, \bar{q}_1, s_2, \bar{q}_2, \ldots, s_n, \bar{q}_n \rangle$ for some natural number n s.t. $\forall i$, $1 \leq i \leq n$, $s_i \in \bar{q}_{i-1} S \bar{q}_i$, $\bar{q}_0 = \bar{q}$, and $\bar{q}_n = \overline{q'}$. If $s_i = s_{i+1}^{-1}$, then

$$\langle \bar{q}_0, s_1, \bar{q}_1, \ldots, s_{i-1}, \bar{q}_{i-1}, s_{i+1} \bar{q}_{i+1}, \ldots, s_n, \bar{q}_n \rangle$$

will also be a composable sequence from $\bar{q}$ to $\overline{q'}$ and is called a *simple reduction* of x. Define $x \sim x'$ if x' is a finite reduction of x or x is a finite reduction of x'. Now $\sim$ will generate an equivalence relation on composable sequences from $\bar{q}$ to $\overline{q'}$. Let $[x]$ be the equivalence class of the composable sequence x. Let $\overline{q'}\chi\bar{q}$ be the free K-module on the equivalence classes of composable sequences (from $\bar{q}$ to $\overline{q'}$).

If $\overline{q''} \in \overline{Q}$ also, and $[x]$, $[x']$ are elements of $\overline{q'}\chi\bar{q}$, $\overline{q''}\chi\overline{q'}$ respectively, then $[x][x']\colon = [xx']$ is an element of $\overline{q''}\chi\bar{q}$. This composition has $[1] = \left[\tilde{\rho}(\bar{q}, q)\widetilde{R}(t)\tilde{\rho}(\overline{qt}, qt)^{-1}\right]\left[\tilde{\rho}(\bar{q}, q)\widetilde{R}(t)\tilde{\rho}(\overline{qt}, qt)^{-1}\right]^{-1}$ as an identity from $\bar{q}$ to $\bar{q}$, and can be extended K-bilinearly to make χ a K-algebroid.

It is now possible to state the first theorem in this section.

Theorem 4.2. *Let Q be a right quasigroup, and let $\widetilde{G} = RU(Q, \mathcal{R})$. Then for each right quasigroup word $x_1 x_2 \ldots x_n w$ with n arguments, there is an n-vector $\tilde{w}_i\colon Q^n \to \chi$; $\tilde{w}_i(q_1, \ldots, q_n) \in {}_{\overline{q_1 \ldots q_n w}}\chi_{\bar{q}_i}$ such that for a given Q-module in $\mathcal{R}/Q$ [or an element of Ω exactly], we have*

$$(m_1, q_1)\ldots(m_n, q_n)w = (m_1\tilde{w}_1(q_1, \ldots, q_n) + \ldots$$
$$+ m_n\tilde{w}_n(q_1, \ldots, q_n), q_1 \ldots q_n w)\,.$$

Proof. By induction on the length of the word w. If w is of length 1, i.e., is just x, then $n = 1$, and $\tilde{w}_1(q) = 1$. Suppose the theorem is true for all words of length λ or less.

Let $x_1 \ldots x_n w$ be a word of length $\lambda + 1$. Then $x_1 \ldots x_n w$ may be written as $u \cdot v$ or u/v, with words $u = y_1 \ldots y_k u$ and $v = z_1 \ldots z_\ell v$ of length less than λ, and where

$\{y_1, \ldots, y_k\} \cup \{z_1, \ldots, z_\ell\} = \{x_1, \ldots, x_n\}$. Suppose for example that $w = u/v$ (the other case is treated similarly), with

$$q_1 \ldots q_n w = r_1 \ldots r_k u / s_1 \ldots s_\ell v.$$

Then by the induction hypothesis:

$$(n_1, r_1) \ldots (n_k, r_k) u = \left(\sum_{i=1}^{k} n_i \tilde{u}_i(r_1, \ldots, r_k), r_1 \ldots r_k u \right) \quad \text{and}$$
$$(p_1, s_1) \ldots (p_\ell, s_\ell) v = \left(\sum_{j=1}^{\ell} p_j \tilde{v}_j(s_1, \ldots, s_\ell), s_1 \ldots s_\ell v \right).$$

We will have

$$\begin{aligned}
&(m_1, q_1) \ldots (m_n, q_n) w \\
&\quad = (n_1, r_1) \ldots (n_k, r_k) u / (p_1, s_1) \ldots (p_\ell, s_\ell) v \\
&\quad = \left(\sum_{i=1}^{k} n_i \tilde{u}_i(r_1, \ldots, r_k), r_1 \ldots r_k u \right) / \left(\sum_{j=1}^{\ell} p_j \tilde{v}_j(s_1, \ldots, s_\ell), s_1 \ldots s_\ell v \right) \\
&\quad = \Bigg(\sum_{i=1}^{k} n_i \tilde{u}_i \tilde{\rho}(\overline{r_1 \ldots r_k u}, r_1 \ldots r_k u) \widetilde{R}^{-1}(s_1 \ldots s_\ell v) \\
&\qquad \tilde{\rho}(\overline{r_1 \ldots r_k u / s_1 \ldots s_\ell v}, r_1 \ldots r_k u / s_1 \ldots s_\ell v)^{-1} \\
&\qquad - \sum_{j=1}^{\ell} p_j \tilde{v}_j \tilde{\rho}(\overline{s_1 \ldots s_\ell v}, s_1 \ldots s_\ell v) \widetilde{L}(r_1 \ldots r_k u / s_1 \ldots s_\ell v) \\
&\qquad \cdot \widetilde{R}(s_1 \ldots s_\ell v)^{-1} \tilde{\rho}(\overline{r_1 \ldots r_k u / s_1 \ldots s_\ell v}, r_1 \ldots r_k u / s_1 \ldots s_\ell v)^{-1}, \\
&\qquad r_1 \ldots r_k u / s_1 \ldots s_\ell v \Bigg) \\
&\quad = \sum n_i \tilde{u}_{i'} + \sum p_j \tilde{v}_{j'},
\end{aligned}$$

where

$$\begin{aligned}
\tilde{u}_{i'} &= \tilde{u}_i \tilde{\rho}(\overline{r_1 \dots r_k u}, r_1 \dots r_k u) \widetilde{R}^{-1}(s_1 \dots s_\ell v) \\
&\qquad \tilde{\rho}(\overline{q_1 \dots q_n w}, q_1 \dots q_n w)^{-1} \\
&= \tilde{u}_i \Big[\tilde{\rho}(\overline{q_1 \dots q_n w}, q_1 \dots q_n w) \widetilde{R}(s_1 \dots s_\ell v) \\
&\qquad \tilde{\rho}(r_1 \dots r_k u, r_1 \dots r_k u)^{-1} \Big]^{-1} \\
\tilde{v}_{j'} &= -\tilde{v}_j \tilde{\rho}(\overline{s_1 \dots s_\ell v}, s_1 \dots s_\ell v) \widetilde{L}(r_1 \dots r_k u / s_1 \dots s_\ell v) \\
&\qquad \tilde{\rho}(\overline{r_1 \dots r_k u}, r_1 \dots r_k u)^{-1} \big[\tilde{\rho}(\overline{q_1 \dots q_n w}, q_1 \dots q_n w) \\
&\qquad \widetilde{R}(s_1 \dots s_\ell v) \tilde{\rho}(\overline{r_1 \dots r_k u}, r_1 \dots r_k u)^{-1} \Big]^{-1} .
\end{aligned}$$

If

$$\begin{aligned}
&q_i = r_j \notin \{s_1, \dots, s_\ell\}, && \text{set } \tilde{w}_i = \tilde{u}_{j'} \ ; \\
&q_i = s_j \notin \{r_1, \dots, r_k\}, && \text{set } \tilde{w}_i = \tilde{v}_{j'} \ ; \\
&q_i = r_n = s_j, && \text{set } \tilde{w}_i = \tilde{u}'_n + \tilde{v}'_j \ .
\end{aligned}$$

Then

$$(m_1, q_1) \dots (m_n, q_n) w = \left(\sum_{i=1}^{n} m_i \tilde{w}_i(q_1, \dots, q_n), q_1 \dots q_n w \right) ,$$

as required to complete the induction step. □

Given a unital K-module in $\mathfrak{V}/Q$, we can construct a K-algebroid χ and a left χ-module as described below. Take K as the field in the construction of the K-algebroid χ, let $\bar{q}M = \pi^{-1}(\bar{q})$ and define $\bar{r}\chi\bar{q} \times \bar{q}M \to \bar{r}M$; $(w, m) \mapsto w(m)$. Then clearly $\bar{q}M$ is an K-module and the action $\bar{r}\chi\bar{q} \times \bar{q}M \to \bar{r}M$ is K-bilinear, i.e.,

$$(k_1 w_1 + k_2 w_2)(m) = k_1 (w_1(m)) + k_2 (w_2(m)) \quad \text{and}$$

$$w' (w(m)) = (w'w)(m).$$

Consider the words

$$w(x_1, x_2) = (x_1 \cdot x_2)/x_2 \ ,$$
$$v(x_1, x_2) = x_1 \ ,$$
$$u(x_1, x_2) = (x_1/x_2) \cdot x_2 \ .$$

Then $w = v = u$. Using Theorem 4.2, by considering (m_i, q_i), we will have (for $m_2 = \bar{q}_2$)

$$m_1\left(\tilde{w}_1(q_1, q_2) - \tilde{v}_1(q_1, q_2)\right) = \bar{q}_1 \quad \text{and}$$
$$m_1\left(\tilde{u}_1(q_1, q_2) - \tilde{v}_1(q_1, q_2)\right) = \bar{q}_1 \ ,$$

i.e., $\tilde{w}_1 - \tilde{v}_1$ and $\tilde{u}_1 - \tilde{v}_1$ are in the annihilator of M. [Analogously for $\tilde{w}_2 - \tilde{v}_2$ and $\tilde{u}_2 - \tilde{v}_2$.]

Suppose further that $\mathfrak{V}$ is defined by the set of identities

$$\{J^k(x_1, \ldots, x_{n_{j_k}}) = J'^k(x_1, \ldots, x_{njk}) \mid k \in I\}.$$

Again, using Theorem 4.2, we have that the

$$\widetilde{J}^k_{\ell_k}(q_1, \ldots, q_{njk}) - \widetilde{J}^{k'}_{\ell_k}(q_1, \ldots, q_{njk})$$

are in the annihilator for $k \in I$, $\ell_k = 1, \ldots, n_{j_k}$.

Definition 4.3. Let $\Omega_{\mathfrak{V}}$ be the category of left χ-modules M such that $\tilde{w}_1 - \tilde{v}_1$, $\tilde{u}_1 - \tilde{v}_1$, $\tilde{w}_2 - \tilde{v}_2$, $\tilde{u}_2 - \tilde{v}_2$ and $\widetilde{J}^k_{\ell_k} - \widetilde{J}^{k'}_{\ell_k}$ for $k \in I$, $\ell_k = 1, \ldots, n_{j_k}$ are in the annihilator of M, where $\mathfrak{V} \models \{J^k = J^{k'} \mid k \in I\}$. A morphism in $\Omega_{\mathfrak{V}}$ from M to M' is a K-module homomorphism $f \colon M \to M'$ such that $\overline{q'} f(w(m)) = w_{\bar{q}} f(m)$ for $m \in \bar{q}M$, and $w \in \overline{q'}\chi\bar{q}$.

Proposition 4.4. *The construction of an element of $\Omega_{\mathfrak{V}}$ from an element of $K \otimes \mathfrak{V}/Q$ as described before* Definition 4.3 *gives a functor Υ from $K \otimes \mathfrak{V}/Q$ to $\Omega_{\mathfrak{V}}$.*

Proof. If $f\colon A \to A'$ is a morphism in the category of unital K-module in $\mathfrak{V}/Q$, then $\bar{q}f\colon \bar{q}M \to \bar{q}M'$; $m \mapsto f(m)$ is a K-module homomorphism such that $\overline{q'}f\,(w(m)) = w_{\bar{q}}f(m)$ for $m \in \bar{q}M$ and $w \in \overline{q'}\chi\bar{q}$. $\square$

Given a left χ-module M in $\Omega_{\mathfrak{V}}$, let $A = \{(x,q) \mid x \in {}_{\bar{q}}M,\ q \in [\bar{q}]\}$. Define

$$\begin{aligned}(x,q)\cdot(y,r) = \Big(&x\tilde{\rho}(\bar{q},q)\widetilde{R}(r)\tilde{\rho}(\overline{qr},qr)^{-1}\\ &+ y\tilde{\rho}(\bar{r},r)\widetilde{L}(q)\tilde{\rho}(\overline{qr},qr)^{-1}, qr\Big) \quad \text{and}\\ (x,q)/(y,r) = \Big(&x\left[\tilde{\rho}(\overline{q/r},q/r)\widetilde{R}(r)\tilde{\rho}(\bar{q},q)^{-1}\right]^{-1}\\ &- y\left[\tilde{\rho}(\bar{r},r)\widetilde{L}(q/r)\tilde{\rho}(\bar{q},q)^{-1}\right]\left[\tilde{\rho}(\overline{q/r},q/r)\widetilde{R}(r)\tilde{\rho}(\bar{q},q)^{-1}\right]^{-1}, qr\Big).\end{aligned}$$

From the proof of Theorem 4.2, we have

$$\begin{aligned}\tilde{w}_1(q_1,q_2) &= \left(\tilde{\rho}(\overline{q_1},q_1)\widetilde{R}(q_2)\tilde{\rho}(\overline{q_1q_2},q_1q_2)^{-1}\right)\cdot\\ &\qquad\left(\tilde{\rho}(\overline{q_1},q_1)\widetilde{R}(q_2)\tilde{\rho}(\overline{q_1q_2},q_1q_2)^{-1}\right)^{-1}\end{aligned}$$

and

$$\begin{aligned}\tilde{w}_2(q_1,q_2) &= \left(\tilde{\rho}(\overline{q_2},q_2)\widetilde{L}(q_1)\tilde{\rho}(\overline{q_1q_2},q_1q_2)^{-1}\right)\cdot\\ &\qquad\left[\tilde{\rho}(\overline{q_1},q_1)\widetilde{R}(q_2)\tilde{\rho}(\overline{q_1q_2},q_1q_2)^{-1}\right]^{-1}\\ &\qquad- \tilde{\rho}(\overline{q_2},q_2)\widetilde{L}(q_1)\tilde{\rho}(\overline{q_1q_2},q_1q_2)^{-1}\cdot\\ &\qquad\quad\left[\tilde{\rho}(\overline{q_1},q_1)\widetilde{R}(q_2)\tilde{\rho}(\overline{q_1q_2},q_1q_2)^{-1}\right]^{-1},\end{aligned}$$

while
$$\tilde{v}_1(q_1,q_2) = 1, \qquad \tilde{v}_2(q_1,q_2) = 0,$$

$$\tilde{u}_1(q_1q_2) = \left[\tilde{\rho}(\overline{q_1}/q_2,q_1/q_2)\widetilde{R}(q_2)\tilde{\rho}(\overline{q_1},q_1)^{-1}\right]^{-1}\tilde{\rho}(\overline{q_1/q_2},q_1/q_2)\widetilde{R}(q_2)\tilde{\rho}(\overline{q_1},q_1)^{-1},$$

and

$$\tilde{u}_2(q_1,q_2) = -\tilde{\rho}(\overline{q_2},q_2)\widetilde{L}(q_1/q_2)\tilde{\rho}(\overline{q_1},q_1)^{-1}$$
$$\left[\tilde{\rho}(\overline{q_1/q_2},q_1/q_2)\widetilde{R}(q_2)\tilde{\rho}(\overline{q_1},q_1)^{-1}\right]^{-1}.$$

Thus from $\tilde{w}_1 - \tilde{v}_1, \tilde{w}_2 - \tilde{v}_2, \tilde{u}_1 - \tilde{v}_1, \tilde{u}_2 - \tilde{v}_2 \in AnnM$, it is easy to show that A is indeed a right quasigroup.

Define $\pi\colon A \to Q$; $(x,q) \mapsto q$. Then π is an object in $\mathfrak{V}/Q$ since

$$m_{\ell_k}\left(\widetilde{J}^k_{\ell_k}(q_1,\ldots,q_{njk}) - \widetilde{J}^{k'}_{\ell_k}(q_1,\ldots,q_{njk})\right) = 0,$$

i.e., by Theorem 4.2,

$$\left((m_1,q_1)\ldots(m_{n_{jk}},q_{n_{jk}})\right) J^k = \left((m_1,q_1)\ldots(m_{n_{jk}},q_{n_{jk}})\right) J^{k'}$$

(by the fact that π is a Q-module in $\mathcal{R}/Q$). Now A satisfies the condition of an abelian group in $\mathcal{R}/Q$ since

$$(x,q) + (x',q) = (x+x',q)\ ,$$
$$-(x,q) = (-x,q)\ ,$$
$$O_Q(q) = (0,q)\ ,$$

and

$$[(x,q)+(x',q)]\cdot[(y,r)+(y',r)]$$
$$= (x,q)\cdot(y,r) + (x',q)\cdot(y',r)$$
$$[(x,q)+(x',q)]\,/\,[(y,r)+(y',r)]$$
$$= (x,q)/(y,r) + (x',q)/(y',r)$$

while
$$O_Q(q\cdot r) = O_Q(q)\cdot O_Q(r)\ ,$$
$$O_Q(q/r) = O_Q(q)/O_Q(r)\ ,$$

and
$$-\left[(x,q)\cdot(y,r)\right] = [-(x,q)]\cdot[-(y,r)]\ .$$

The abelian group identities diagrams commute.

Also A is equipped with an additional $\mathfrak{V}/Q$ morphism $k\colon A \to A$; $(x,q) \mapsto (kx,q)$ for each $k \in K$ such that the diagrams representing unital K-modules identities commute. For example

$$\begin{tikzcd} A \times_Q A \arrow[r, "k\times k"] \arrow[d, "+"'] & A \times_Q A \arrow[d, "+"] \\ A \arrow[r, "k"'] & A \end{tikzcd}$$

commutes since

$$\begin{aligned} ((m,q),(n,q))\,(k \times k)(+) &= ((km,q),(kn,q))+ \\ &= (km+kn,q) = (k(m+n),q) \\ &= (m+n,q)k = ((m,q)(n,q)) + (k). \end{aligned}$$

Proposition 4.5. *The construction of an element of $K \otimes \mathfrak{V}/Q$ from an element of $\Omega_{\mathfrak{V}}$ as described above gives a functor Υ' from $\Omega_{\mathfrak{V}}$ to $K \otimes \mathfrak{V}/Q$.*

Proof. If $f\colon M \to M'$ is a $\overline{Q}$-tuple of K-module homomorphisms $\bar{q}f\colon \bar{q}M \to \bar{q}M'$ such that for $\bar{q},\bar{r} \in \overline{Q}$, $w \in {}_{\bar{r}}\chi_{\bar{q}}$, and $m \in \bar{q}M$, we have ${}_{\bar{r}}f(wm) = w_{\bar{q}}f(m)$, then

$$f = \{\bar{q}f\}\colon A \to A' \;;\; (m,q) \mapsto (\bar{q}f(m),q)$$

is clearly a Q-morphism s.t. $kf = fk$ $\forall k \in K$, since

$$\begin{aligned} kf(m,q) &= k\,({}_{\bar{q}}f(m),q) = (k_{\bar{q}}f(m),q) \\ &= ({}_{\bar{q}}f(km),q) = f(km,q) = fk(m,q). \end{aligned}$$

Hence we conclude that the assignment

$$\Upsilon'\colon \Omega_{\mathfrak{V}} \to K \otimes \mathfrak{V}/Q \;;\; M \mapsto M\Upsilon' = A$$

is a functor. $\square$

Theorem 4.6. Υ *and* Υ' *give an equivalence between* $K \otimes \mathfrak{V}/Q$ *and* $\Omega_{\mathfrak{V}}$.

Proof. The natural isomorphism in $K \otimes \mathfrak{V}/Q$ is defined by

$$\tau_A \colon A\Upsilon\Upsilon' \mapsto A \ ; \ (x, q) \mapsto x\tilde{\rho}(\bar{q}, q)$$

and

$$\tau_A^{-1} \colon A \mapsto A\Upsilon\Upsilon' \ ; \ x \mapsto \left(x\tilde{\rho}(\bar{q}, q)^{-1}, q\right).$$

If $f \colon A \to A'$ is a $K \otimes \mathfrak{V}/Q$ morphism, then the required diagrams commute. The natural isomorphism in $\Omega_{\mathfrak{V}}$ is defined by

$$\sigma_M \colon M\Upsilon'\Upsilon \mapsto M \ ; \ (m, \bar{s}) \mapsto m$$

and

$$\sigma_M^{-1} \colon M \mapsto M\Upsilon'\Upsilon \ ; \ m \mapsto (m, \bar{s}).$$

Again, if $f \colon M \to M'$ is an $\Omega_{\mathfrak{V}}$-morphism then the required diagrams commute. $\square$

If we take $\mathfrak{V}$ to be the category of quandles, then $\mathfrak{V}$ is defined by the set of identities:

(I) $x \cdot x = x$;

(D) $(x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z)$.

Let $Q \in \mathfrak{V}$. Then by Definition 4.3 and Theorem 4.2, $\Omega_{\mathfrak{V}}$ is defined by the set of identities:

(i) $\left[\tilde{\rho}(\bar{q}, q)\widetilde{R}(r)\tilde{\rho}(\overline{qr}, qr)^{-1}\right]\left[\tilde{\rho}(\bar{q}, q)\widetilde{R}(r)\tilde{\rho}(\overline{qr}, qr)^{-1}\right]^{-1} = 1_{\bar{q}}$;

(ii) $\left[\tilde{\rho}(\bar{q}, q)\widetilde{R}(r)\tilde{\rho}(\overline{qr}, qr)^{-1}\right]^{-1}\left[\tilde{\rho}(\bar{q}, q)\widetilde{R}(r)\tilde{\rho}(\overline{qr}, qr)^{-1}\right] = 1_{\bar{q}}$;

(iii) $\left[\tilde{\rho}(\bar{q}, q)\widetilde{R}(q)\tilde{\rho}(\bar{q}, q)^{-1}\right] + \left[\tilde{\rho}(\bar{q}, q)\widetilde{L}(q)\tilde{\rho}(\bar{q}, q)^{-1}\right] = 1_{\bar{q}}$;

(iv) $\left[\tilde{\rho}(\bar{q}, q)\widetilde{R}(r)\tilde{\rho}(\overline{qr}, qr)^{-1}\right]\left[\tilde{\rho}(\overline{qr}qr)\widetilde{R}(s)\tilde{\rho}(\overline{qr \cdot s}, qr \cdot s)^{-1}\right]$
$= \left[\tilde{\rho}(\bar{q}, q)\widetilde{R}(s)\tilde{\rho}(\overline{qs}, qs)^{-1}\right]\left[\tilde{\rho}(\overline{qs}, qs)\widetilde{R}(rs)\tilde{\rho}(\overline{qr \cdot s}, qr \cdot s)^{-1}\right]$;

(v) $\left[\tilde{\rho}(\bar{r}, r)\widetilde{L}(q)\tilde{\rho}(\overline{qr}, qr)^{-1}\right]\left[\tilde{\rho}(\overline{qr}, qr)\widetilde{R}(s)\tilde{\rho}(\overline{qr \cdot s}, qr \cdot s)^{-1}\right]$
$= \left[\tilde{\rho}(\bar{r}, r)\widetilde{R}(s)\tilde{\rho}(\overline{rs}, rs)^{-1}\right]\left[\tilde{\rho}(\overline{rs}, rs)\widetilde{L}(qr)\tilde{\rho}(\overline{qr \cdot s}, qr \cdot s)^{-1}\right]$;

(vi) $\tilde{\rho}(\bar{s}, s)\widetilde{L}(qr)\tilde{\rho}(\overline{qr \cdot s}, qr \cdot s)^{-1}$

$$= \left[\tilde{\rho}(\bar{s}, s)\widetilde{L}(q)\tilde{\rho}(\overline{qs}, qs)^{-1}\right]\left[\tilde{\rho}(\overline{qs}, qs)\widetilde{R}(rs)\tilde{\rho}(\overline{qr \cdot s}, qr \cdot s)^{-1}\right]$$
$$+ \left[\tilde{\rho}(\bar{s}, s)\widetilde{L}(r)\tilde{\rho}(\overline{rs}, rs)^{-1}\right]\left[\tilde{\rho}(\overline{rs}, rs)\widetilde{L}(qs)\tilde{\rho}(\overline{qr \cdot s}, qr \cdot s)^{-1}\right].$$

Theorem 4.7. *Let χ be a K-algebroid on $\overline{Q}$. Then a left χ-module M is a covering of χ (thought of as a category) equivalent to a representation $\varphi\colon \chi \to \gamma$, where γ is the category of K-modules, such that*

$$\varphi(x + y) = \varphi(x) + \varphi(y)$$

and

$$\varphi(kx) = k\varphi(x)$$

for $x, y \in \overline{q'}\chi\bar{q}$ and $k \in K$.

Proof. Suppose we are given a representation $\varphi\colon \chi \to \gamma$ with the given properties. Let $\bar{q}M = \varphi(\bar{q})$; $\overline{q'}\chi\bar{q} \times \bar{q}M \longrightarrow \overline{q'}M$ be given by $(x, m) \mapsto \varphi(x)(m)$, a K-bilinear left action satisfying $x'(xm) = (x'x)m$ for each $x' \in \bar{q}''\chi\bar{q}'$, $x \in \bar{q}'\chi\bar{q}$ and $m \in \bar{q}M$, for each $\bar{q}, \bar{q}', \bar{q}'' \in \overline{Q}$. Consider a morphism in the category of representations, i.e. a natural transformation from φ_1 to φ_2, consisting of a family of γ-morphisms, $\tau(\bar{q})\colon \varphi_1(\bar{q}) \to \varphi_2(\bar{q})$ such that for every χ-morphism $x\colon \bar{q} \to \bar{r}$, the following diagram commutes

$$\begin{array}{ccc}
\bar{q}M = \varphi_1(\bar{q}) & \xrightarrow{\tau(\bar{q})} & \varphi_2(\bar{q}) = \bar{q}M' \\
{\scriptstyle \varphi_1(x)}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_2(x)} \\
\bar{r}M = \varphi_1(\bar{r}) & \xrightarrow[\tau(\bar{r})]{} & \varphi_2(\bar{r}) = \bar{r}M'
\end{array} .$$

By defining $\bar{q}f\colon \bar{q}M \to \overline{q'}M$ as $\bar{q}f = \tau(\bar{q})$, which is a K-module homomorphism, we have for each $\bar{q}, \overline{q'} \in \overline{Q}$, $x \in \overline{q'}\chi\bar{q}$, and $m \in \bar{q}M$,

$$\begin{aligned}
\overline{q'}f(xm) &= \tau(\overline{q'})(xm) = \tau(\overline{q'})\left[\varphi_1(x)(m)\right] \\
&= \left[\tau(\overline{q'})\varphi_1(x)\right](m) = \left[\varphi_2(x)\tau(\bar{q})\right](m) \\
&= \varphi_2(x)\left[\tau(\bar{q})(m)\right] = x\left[\bar{q}f(m)\right].
\end{aligned}$$

Hence $\{\bar{q}f\}_{\bar{q}\in\overline{Q}}$ is a morphism in χ-mod. Also $1(\bar{q}) = \bar{q}1$ and we have a functor $\Lambda\colon \gamma^{\chi} \to \chi$-mod, since $\Lambda(\tau\tau') = ff' = \Lambda(\tau)\Lambda(\tau')$.

Notice that the K-module bilinear action structure of $q'\chi\bar{q}$ comes from the equations $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(kx) = k\varphi(x)$ by defining x to be $\varphi(x)$. Conversely, given a left χ-module M, it is easy to get the obvious representation $\varphi\colon \chi \to \gamma$, where $\varphi(x)(m) = xm$ for $x \in \overline{q'}\chi\bar{q}$, $m \in \bar{q}M$, and

$$\begin{aligned}\varphi(x+y)(m) &= (x+y)m = xm + ym \\ &= \varphi(x)(m) + \varphi(y)(m)\ , \\ \varphi(kx)(m) &= (kx)m = k(xm) = k\left(\varphi(x)(m)\right).\end{aligned}$$

Thus we have $\Lambda'\colon \chi$-mod $\to \gamma^{\chi}$ and Λ, Λ' will give the equivalence. $\square$

SUMMARY

Given a fixed quasigroup A, the following categories are equivalent:

(i) $\mathfrak{A} \otimes \mathfrak{Q}/Q$ (ii) $\mathfrak{A}^{\pi\,\mathrm{Cay}\,Q}$ (iii) $Ab\,\mathrm{Cov}\,Q$ (iv) $A^{\widetilde{G}_e}$.

Given a fixed right quasigroup Q, the following categories are equivalent:

(i) $\mathfrak{A} \otimes \mathcal{R}/Q$ (ii) $\mathfrak{A}^{\vec{P}\,\mathrm{Cay}\,Q}$ (iii) Ω.

Furthermore, for a right quasigroup Q, the category $K \otimes (\mathfrak{V}/Q)$ of Q-modules in $\mathfrak{V}$ over K is equivalent to the category of ordered modules $\Omega_{\mathfrak{V}}$. Also, we have the freeness of the right universal multiplication group $\mathcal{R}U(Q, \mathcal{R})$ of a fixed right quasigroup Q in the variety $\mathcal{R}$ of all right quasigroups.

The attempt to get an analogue of [S1, 244] fails.

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