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Subgradients of algebraically convex functions: A Galois connection relating convex sets and subgradients of convex functions

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Subgradients of algebraically convex functions:
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LIST OF SYMBOLS

∂f	subgradient of function f	28
	-for distributive lattices and modals	43
dom$f$	domain of the function f	
E	function of sections of π	25
$e^{(i)}$	i'th standard basis vector in $\mathbb{R}^n$	23
epi$f$	epigraph of function f	2
$\mathrm{epi}_{\mathbb{R}}(\mathbf{a}, b)$	epigraph of $(\mathbf{a}, b)^\mu$ for ordered pair $(\mathbf{a}, b) \in \mathrm{L}_{-\infty}$	24
$\mathrm{epi}_{\mathbb{R}} f$	finitary epigraph of function f	2
f^∂	function of normal vectors	28
f_e	extension of the function f to $\mathbb{R}^n$	3
Γ	set of sections of π	20
gr$f$	graph of function f	2
$\mathrm{gr}_{\mathbb{R}}(\mathbf{a}, b)$	graph of $(\mathbf{a}, b)^\mu$ for ordered pair $(\mathbf{a}, b) \in \mathrm{L}_{-\infty}$	24
$\mathrm{gr}_{\mathbb{R}} f$	finitary graph of function f	2
H	set of I°-homomorphisms from $\mathbb{R}^n$ to $\mathbb{R}$	20
$\mathrm{H}_{\mathbf{a}}$	section of H at $\mathbf{a}$	23
$\overline{\mathrm{H}}$	set of I°-homomorphisms from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$	18
${}_f\mathrm{H}_c$	set of homomorphisms less than function f, equal to f at c	33
Hom(A, B)	set of homomorphisms from A to B	9
(I)	idempotence identity	10
I	the closed interval $[0, 1]$	2
I°	the open interval $(0, 1)$	2
im$f$	image of the function f	
$\underline{\lambda}$	operation on $\mathbb{R}^n$ and $\overline{\mathbb{R}}$: $xy\underline{\lambda} = x(1-\lambda) + y\lambda, \lambda \in \mathrm{I}^\circ$	3
L	the set $\mathbb{R}^n \times \mathbb{R}$	20
$\mathrm{L}_{\mathbf{a}}$	the set $\{\mathbf{a}\} \times \overline{\mathbb{R}}$ for $\mathbf{a} \in \mathbb{R}^n$	20
$\overline{\mathrm{L}}$	the set $\mathbb{R}^n \times \overline{\mathbb{R}}$	20
M_2	the square distributive lattice	

$\max W$	largest element of chain W, in meet semilattice	32
μ	isomorphism from L to H	22
$\mathbb{N}$	the set of natural numbers, $\{0,1,2,\dots\}$	2
$\mathbb{N}^+$	the set of positive integers, $\{1,2,3,\dots\}$	2
Ω	set of basic operations on an algebra	9
(Ω)	category of Ω-algebras and Ω-homomorphisms between them	9
$\underline{p}$	$xy\underline{p} = x(1-p) + yp,\ p \in \mathrm{I}^\circ$	3
P	set of nonvertical hyperplanes in $\mathbb{R}^{n+1}$	20
$\mathrm{P}_{\mathbf{a}}$	section of P at a	23
$\overline{\mathrm{P}}$	the set $\mathrm{P} \cup [\mathbb{R}^n \times \{-\infty\}] \cup [\mathbb{R}^n \times \{+\infty\}]$	20
π	projection of $\overline{\mathrm{L}}$ onto $\overline{\mathbb{R}}$	20
$\mathbb{Q}$	the set of rational numbers	2
$\mathbb{R}$	the set of all reals, $(-\infty, +\infty)$	2
$\mathbb{R}^{+\infty}$	the interval $(-\infty, +\infty]$	2
$\mathbb{R}_{-\infty}$	the interval $[-\infty, +\infty)$	2
$\overline{\mathbb{R}}$	the interval $[-\infty, +\infty]$	2
ρ	isomorphism from L to P	22
$\overline{\rho}$	extension of function ρ	24
(S)	category with elements of semilattice S as objects and morphisms $a \to b \iff a \leq b$	14
$A\mathbf{S}$	set of nonempty subalgebras of A	9
$A\mathbf{S}_{\varnothing}$	set of all subalgebras of A	9
W_b	set of elements in chain W below element b	32
χ_{A}	characteristic function of the set A	5

INTRODUCTION

Various forms of abstract differentiation have been developed as a generalization of differential calculus. They use such algebraic structures as rings of polynomials or differential groupoids (see [RS1]). The structures used in this paper are modes and modals, which are relatively new. These are convenient media for study because Euclidean spaces naturally possess a mode structure, and the various extensions of the reals form modals.

The well-developed theory of real subgradients is the groundwork for our study. We interpret supporting hyperplanes as mode homomorphisms and formulate results from convex analysis in this algebraic setting.

The first two chapters are a review of classical convexity and universal algebra. Most proofs are omitted, but can be found in the supporting texts. The equivalence of lower-semicontinuity and closedness for convex functions is particularly noteworthy. In Chapter 3 we exhibit a Galois connection between convex subsets and the functions that choose a hyperplane for each normal vector. The fourth chapter puts the results of Chapter 3 into modal-theoretic terms. Semilattices and distributive lattices are examined in Chapter 5 as an example to determine a course of action and deal with possible complications in defining subgradients for algebraically convex functions. In Chapter 6 we present the major result of defining a subgradient for convex functions from modes into completely distributive modals. The final chapter describes the present situation in research.

CHAPTER 1: TRADITIONAL CONVEXITY

This chapter reviews the theory of subgradients for convex functions of Euclidean spaces and clarifies the definitions to be used in the rest of the paper.

Notation 1.1. Let $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}^+ = \{1, 2, 3, \dots\}$, $\mathrm{I} = [0, 1]$, $\mathrm{I}^\circ = (0, 1)$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^{+\infty} = (-\infty, +\infty]$, $\mathbb{R}_{-\infty} = [-\infty, +\infty)$, $\overline{\mathbb{R}} = [-\infty, +\infty]$, $\mathbb{Q}$ be the set of rational numbers, and for $n \in \mathbb{N}$, let $\mathbb{R}^n$ be the Euclidean space of dimension n with the standard inner product and topology.

Definition 1.1. For sets A and B, let B^A be the set of functions from A to B. The *graph* of $f \in \mathrm{B}^\mathrm{A}$ is the set

$$\mathrm{gr} f = \{(x, xf) \mid x \in \mathrm{A}\},$$

and when $(\mathrm{B}, \leq)$ is a poset, the *epigraph* of f is

$$\mathrm{epi} f = \{(x, y) \mid x \in \mathrm{A},\ xf \leq y \in \mathrm{B}\}.$$

Definition 1.2. Let A be a set and $(\mathrm{B}, \leq)$ a partially ordered set. The *pointwise order* on B^A is the partial order defined by:

$$f_1 \leq f_2 \text{ in } \mathrm{B}^\mathrm{A} \iff \forall x \in \mathrm{A},\ xf_1 \leq_{\mathrm{B}} xf_2.$$

We can partially order the product $\mathrm{A} \times \mathrm{B}$ also, by:

$$(a, b) \leq_{\mathrm{A} \times \mathrm{B}} (a', b') \iff a = a' \text{ and } b \leq_{\mathrm{B}} b'.$$

Fact 1.1. Let $f_1, f_2 \in \mathrm{B}^\mathrm{A}$, where B is a poset. Then

$$f_1 \leq f_2 \iff \mathrm{epi} f_1 \supseteq \mathrm{epi} f_2.$$

Proof: Let $f_1, f_2 \in \mathrm{B}^\mathrm{A}$. Then

$$f_1 \leq f_2 \iff \forall x \in \mathrm{A},\ xf_1 \leq xf_2 \iff$$
$$\forall x \in \mathrm{A},\ [y \leq xf_1 \implies y \leq xf_2] \iff \mathrm{epi} f_1 \supseteq \mathrm{epi} f_2. \quad \square$$

Definition 1.3. Given a set X, for a function $f : \mathrm{X} \to \overline{\mathbb{R}}$, the *finitary graph* and *finitary epigraph* of f are defined to be

$$\mathrm{gr}_{\mathbb{R}} f = \{(x, xf) \mid x \in \mathrm{X}, xf \in \mathbb{R}\} \qquad \text{and}$$
$$\mathrm{epi}_{\mathbb{R}} f = \{(x, y) \mid x \in \mathrm{X},\ xf \leq y \in \mathbb{R}\}, \text{ respectively.}$$

Thus $\mathrm{gr}_{\mathbb{R}} f = \mathrm{gr} f \cap (\mathbb{R}^n \times \mathbb{R})$ and $\mathrm{epi}_{\mathbb{R}} f = \mathrm{epi} f \cap (\mathbb{R}^n \times \mathbb{R})$. If $\mathrm{im} f \subseteq \mathbb{R}$, then $\mathrm{gr} f = \mathrm{gr}_{\mathbb{R}} f$.

Definition 1.4. For a convex set $\mathrm{X} \subseteq \mathbb{R}^n$, a function $f : \mathrm{X} \to \mathbb{R}$ is *convex* (called "real-convex" when needed for clarity) iff it satisfies the property
($\mathrm{CX}_{\mathbb{R}}$) $\forall x, y \in \mathrm{X}, \forall \lambda \in \mathrm{I}^{\circ}, [x(1-\lambda) + y\lambda]f \leq xf(1-\lambda) + yf\lambda.$

A function $f : \mathrm{X} \to \mathbb{R}^{+\infty}$ is convex iff it satisfies the same inequality, where any expression involving infinity equals infinity. (See Example 2.10 for a precise definition.) We can write $\lambda \in \mathrm{I}^{\circ}$ as a function of two variables by

$$\underline{\lambda} : (\mathbb{R}^{+\infty})^2 \to \mathbb{R}^{+\infty};\ (x, y) \mapsto (1-\lambda)x + \lambda y.$$

If X is not the whole space $\mathbb{R}^n$, then f can be extended to a new function

$$f_e : \mathbb{R}^n \to \mathbb{R}^{+\infty};\ x \mapsto \begin{cases} xf, & x \in \mathrm{X} \\ +\infty, & x \notin \mathrm{X}. \end{cases}$$

Fact 1.2. If f is convex, then f_e is convex:

$$x,\ y \in \mathrm{dom} f \implies xf_e yf_e \underline{\lambda} = xfyf\underline{\lambda} \geq xy\underline{\lambda}f = xy\underline{\lambda}f_e, \text{ and}$$
$$\{x, y\} \not\subseteq \mathrm{dom} f \implies \{+\infty\} \in \{xf_e, yf_e\} \implies xf_e yf_e \underline{\lambda} = +\infty \geq xy\underline{\lambda}f_e.$$

Thus we can consider convex functions to be defined everywhere. Note that $\mathrm{gr}_{\mathbb{R}} f_e = \mathrm{gr}_{\mathbb{R}} f$, and $\mathrm{epi}_{\mathbb{R}} f_e = \mathrm{epi}_{\mathbb{R}} f$.

Example 1.1. Both $|x|$ and e^x are finite convex functions.

The extensions of two convex functions are shown in the following examples. These functions, along with those in Example 1.6, are continuous, but each is different in an important way. In Example 1.2, the epigraph is closed, but not in Example 1.3. See Example 1.7 for more on this.

Example 1.2. For $xf = -\sqrt{x}$, $xf_e = \begin{cases} -\sqrt{x}, & x \geq 0 \\ +\infty, & x < 0. \end{cases}$ See Figure 1.1.

Example 1.3. Define $f : (0, +\infty) \to \mathbb{R}$ by $xf := -\sqrt{x}$. Then its extension is given by $xf_e = \begin{cases} -\sqrt{x}, & x > 0 \\ +\infty, & x \leq 0. \end{cases}$ See Figure 1.1.

Figure 1.1. Graphs of functions in Examples 1.2 and 1.3.

Fact 1.3. Convexity of a function is equivalent to convexity (as a set) of its finitary epigraph. ([BP], Prop. 1.2, p.85.) First we would like to restrict our attention to "proper" convex functions.

Definition 1.5. A function $f : \mathrm{X} \to \mathrm{Y}$ is *proper* iff $f(\mathrm{X}) \subseteq \mathbb{R}^{+\infty}$ and $f \not\equiv +\infty$. (cf. [BP], p.84.)

Definition 1.6. For a set $\mathrm{C} \subseteq \mathrm{X} \times \mathrm{Y}$ and a point $x \in \mathrm{X}$, define the set $\mathrm{C}_x := \{y \in \mathrm{Y} \mid (x, y) \in \mathrm{C}\}$, the *cylinder of* C *at* x.

Fact 1.4. A nonempty convex set $\mathrm{C} \subseteq \mathbb{R}^{n+1}$ is the finitary epigraph of a proper convex function if and only if

i. C is an *upset* ($(\mathbf{x}, y) \in \mathrm{C}$, $y \leq y' \in \mathbb{R} \implies (\mathbf{x}, y') \in \mathrm{C}$),

ii. C is *cylindrically bounded below* (for $\mathbf{x} \in \mathbb{R}^n$, $\mathrm{C}_{\mathbf{x}}$ is bounded below in $\mathbb{R}$), and

iii. C is *cylindrically closed* (for $\mathbf{x} \in \mathbb{R}^n$, $\mathrm{C}_{\mathbf{x}}$ is closed in the standard topology on $\mathbb{R}$).

Proof: Note that the finitary epigraph of every proper convex function satisfies all three conditions. For a set C satisfying conditions (i), (ii), and (iii), define a function $f : \mathbb{R}^n \to \mathbb{R}^{+\infty}$; $\mathbf{x}f := \inf \mathrm{C}_{\mathbf{x}}$. By properties (ii) and (iii), f is well-defined and the infimum is actually a minimum. If $\mathrm{C}_{\mathbf{x}}$ is nonempty, $(\mathbf{x}, \mathbf{x}f) \in \mathrm{C}$. So $\mathrm{C} = \mathrm{epi}_{\mathbb{R}} f$ by the upset property. By Fact 1.3, f is convex. $\square$

Definition 1.7. Given a topological space $(\mathrm{X}, \mathcal{T})$, a function $f : \mathrm{X} \to \mathbb{R}^{+\infty}$ is *lower-semicontinuous* iff for every $t \in \mathbb{R}$, the set $\{x \in \mathrm{X} \mid xf > t\}$ is open in X. ([BP], Prop. 1.3 i. and ii., p.87.)

Fact 1.5. Let $\mathcal{S}$ be the standard topology on $\mathbb{R}$, and $\mathcal{S}'$ be the topology on $\mathbb{R}$ with basis consisting of the set of intervals $\{(t, +\infty) \mid t \in \mathbb{R}\}$. Let $(\mathrm{X}, \mathcal{T})$ be a topological space and $f : \mathrm{X} \to \mathbb{R}$. Then $f : (\mathrm{X}, \mathcal{T}) \to (\mathbb{R}, \mathcal{S})$ is lower-semicontinuous if and only if $f : (\mathrm{X}, \mathcal{T}) \to (\mathbb{R}, \mathcal{S}')$ is continuous.

Fact 1.6. For a topological space X, a function $f : \mathrm{X} \to \mathbb{R}^{+\infty}$ is lower-semicontinuous if and only if

$$\forall x_0 \in \mathrm{X},\ x_0 f = \liminf_{x \to x_0} xf.$$

This is often used as the definition of lower-semicontinuity. ([BP], Def. 1.2, p.86.)

Example 1.4. We could define the characteristic function of a subset A of a set X as

$$f_{\mathrm{A}} : \mathrm{X} \to \mathbb{R}; xf_{\mathrm{A}} = \begin{cases} 1, & x \in \mathrm{A} \\ 0, & x \notin \mathrm{A}. \end{cases}$$

For such a definition, characteristic functions of open sets in a subspace X of $\mathbb{R}^n$ are lower-semicontinuous, although they are not convex.

Example 1.5. Let us, however, define the characteristic function of $\mathrm{A} \subseteq \mathrm{X}$ by

$$\chi_{\mathrm{A}} : \mathrm{X} \to \mathbb{R}^{+\infty};\ x\chi_{\mathrm{A}} = \begin{cases} 0, & x \in \mathrm{A} \\ +\infty, & x \notin \mathrm{A}. \end{cases}$$

Then for $\mathrm{X} = \mathbb{R}^n$ characteristic functions of convex sets are convex, and characteristic functions of closed sets are lower-semicontinuous. Therefore the characteristic function of a closed convex set is "closed" by Fact 1.7 below.

Definition 1.8. For a convex function $f : \mathbb{R}^n \to \mathbb{R}^{+\infty}$, the *closure* of f is the function

$$\mathrm{cl}f : \mathbb{R}^n \to \mathbb{R}^{+\infty};\ \mathbf{x} \mapsto \liminf_{\mathbf{y}\to\mathbf{x}} \mathbf{y}f \text{ ([BP] p. 89, eq. 1.8).}$$

A convex function f is *closed* iff $\mathrm{cl}f = f$.

Example 1.6. The extensions of the functions $-\sqrt{x}$ and $-\log x$ are closed. In spite of this, $-\sqrt{x}$ is not subdifferentiable at 0, and the domain of $-\log x$ is not a closed interval. See Figures 4.1 and 1.2. Note how the supporting lines of $-\log x$ approach the asymptote $x = 0$.

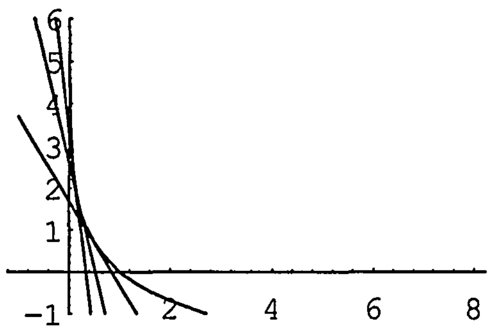

Figure 1.2. Graph and some tangent lines of $-\log x$.

Example 1.7. The function f defined in Example 1.3 is convex and continuous, but its extension is not lower-semicontinuous (or closed), since $\liminf_{x\to 0^+} xf_e = 0 \neq +\infty = 0f_e$.

Example 1.8. The function $g : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ defined by

$$(r,s)g = \begin{cases} \frac{r^2+s^2}{s}, & s > 0 \\ 0, & (r,s) = (0,0) \end{cases}$$

is lower-semicontinuous since

1. g is continuous when $s > 0$, and
2. for s approaching 0, we have

$$\forall r \neq 0, \lim_{s\to 0} \tfrac{r^2+s^2}{s} = +\infty = (r,0)g_e, \text{ and}$$
$$\liminf_{(r,s)\to(0,0)} \tfrac{r^2+s^2}{s} = 0 = (0,0)g.$$

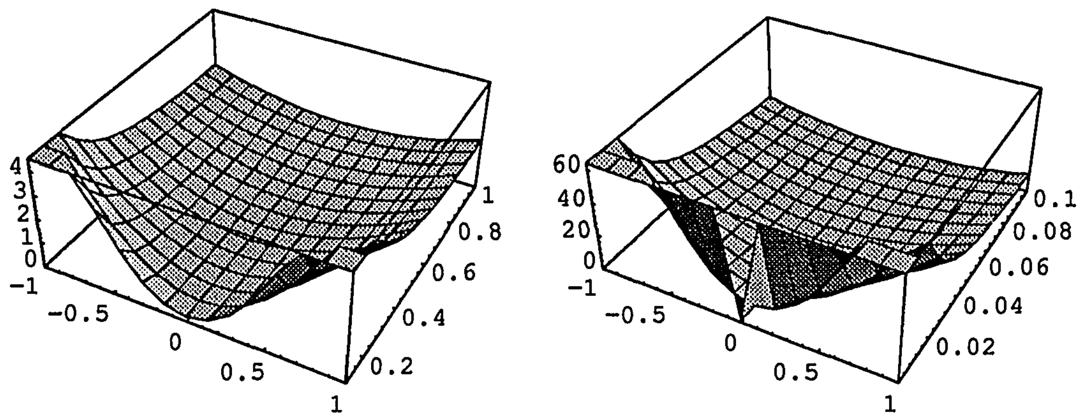

Figure 1.3. Two views of the function g in Example 1.8.

The behavior of g near the origin can be seen in Figure 1.3. Note how the graph itself becomes closed if we add the ray above $(0, 0, 0)$. The epigraph is closed and thus g_e is also closed, by Fact 1.7 below.

Fact 1.7. A proper convex function $f : \mathbb{R}^n \to \mathbb{R}^{+\infty}$ is closed $\iff$ f is lower-semicontinuous $\iff$ $\text{epi}_{\mathbb{R}} f$ is closed. ([BP], Prop 1.3, p.87 & §1.3, p.88.) Thus for a proper convex function f, $\text{cl} f$ is lower-semicontinuous.

Example 1.9. Although the convex function g in Example 1.8 is not continuous at $(0, 0)$, it is lower-semicontinuous there, as is its extension g_e. Thus, g_e is closed.

Fact 1.8. A proper function is convex and lower-semicontinuous if and only if it is the supremum of a family of affine continuous functions. ([BP], Cor. 1.6, p.99.)

Definition 1.9. A *hyperplane* of $\mathbb{R}^{n+1}$ is an affine subspace p of codimension one. We call p *vertical* iff it satisfies:

$$\exists \mathbf{x} \in \mathbb{R}^n \bullet \{\mathbf{x}\} \times \mathbb{R} \subset p,$$

and *nonvertical* iff it contains exactly one point $(\mathbf{x}, y)$ for each $\mathbf{x} \in \mathbb{R}^n$.

Definition 1.10. A *normal vector* to a hyperplane p of $\mathbb{R}^{n+1}$ is a vector $\mathbf{u} \in \mathbb{R}^{n+1}$ lying in $\ell = p^{\perp}$, the line in $\mathbb{R}^{n+1}$ which is perpendicular to every line in p. (This line $p^{\perp}$ is known in linear algebra as the orthogonal complement to p in $\mathbb{R}^{n+1}$.) A *unit* normal vector $\mathbf{u}$ has norm 1. The *regular* normal vector to a nonvertical hyperplane $y = \mathbf{a} \cdot \mathbf{x} + b$ in $\mathbb{R}^{n+1}$ is $(\mathbf{a}, -1)$. The n-vector $\mathbf{a}$ will be called the *projected normal vector* of the hyperplane $y = \mathbf{a} \cdot \mathbf{x} + b$.

Example 1.10. The hyperplane $2x_1 + 3x_2 - 6y = 12$ in $\mathbb{R}^3$ has unit normal vectors $\pm(\frac{2}{7}, \frac{3}{7}, -\frac{6}{7})$, regular normal vector $(\frac{1}{3}, \frac{1}{2}, -1)$, and projected normal vector $(\frac{1}{3}, \frac{1}{2})$. Note the uniqueness of the regular and projected normal vectors.

CHAPTER 2: CONVEXITY IN UNIVERSAL ALGEBRA: MODES AND MODALS

In this chapter we introduce the terms from universal algebra needed to generalize from classical convexity to modal theory and show how the Euclidean spaces and their extensions fit into the theory of modes and modals.

Definition 2.1. An *operation* on a set A is a function

$$\omega : \mathrm{A}^n \to \mathrm{A};\ (a_1, \ldots, a_n) \mapsto a_1 \ldots a_n\omega$$

for some nonnegative integer $n = \omega\tau$, called the *arity* of ω. An *algebra* $\mathbf{A} = (\mathrm{A}, \Omega)$ is a set A along with a set Ω of *basic* operations on A. Those operations that can be built up from the basic operations are called *derived* operations. The map $\tau : \Omega \to \mathbb{N}$ is the *type* of $\mathbf{A}$. An algebra of the form (A, Ω) is also called an Ω-*algebra.* An algebra (A, Ω') is a *reduct* of (A, Ω) iff $\Omega' \subseteq \Omega$, and a subset A' of A is a *subalgebra* of A (written $(\mathrm{A}', \Omega) \leq (\mathrm{A}, \Omega)$, or briefly, $\mathrm{A}' \leq \mathrm{A}$) iff

$$\forall\omega \in \Omega,\ \forall a_1, \ldots, a_{\omega\tau} \in \mathrm{A}',\ a_1 \ldots a_{\omega\tau}\omega \in \mathrm{A}'.$$

Definition 2.2. For an algebra (A, Ω), let $\mathbf{AS}$ denote the set of nonempty subalgebras of A, and $\mathbf{AS}_{\varnothing}$ denote the set of all subalgebras of A. For algebras (A, Ω) and (B, Ω) of the same type, define

$$\mathrm{Hom(A,B)} := \{h : \mathrm{A} \to \mathrm{B} \mid \forall\omega \in \Omega, \forall x_1, \ldots, x_{\omega\tau} \in \mathrm{A},\ x_1 \ldots x_{\omega\tau}\omega h = x_1 h \ldots x_{\omega\tau} h\omega\},$$

the set of Ω-*homomorphisms* (or just *homomorphisms*) from A to B.

Definition 2.3. For a given type $\tau : \Omega \to \mathbb{N}$, define (Ω) to be the category whose objects are all of the Ω-algebras of type τ and whose morphisms are all Ω-homomorphisms between these algebras. That this forms a category is easy to check.

Definition 2.4. A category of Ω-algebras is called *plural* iff $\Omega\tau \subseteq \{n \in \mathbb{N} \mid n > 1\}$. (cf.[RS2], Prop. 235.)

Definition 2.5. An operation $\omega \in \Omega$ is *idempotent* iff $\forall x \in \mathrm{A}$,

(I) $x \dots x\omega = x$,

commutative iff $\forall x_1, \dots, x_{\omega\tau} \in \mathrm{A}$, $\forall 1 \leq i < j \leq \omega\tau$,

(C) $x_1 \dots x_i \dots x_j \dots x_{\omega\tau}\omega = x_1 \dots x_{i-1}x_jx_{i+1} \dots x_{j-1}x_ix_{j+1} \dots x_{\omega\tau}\omega$,

and *associative* iff $\omega\tau = 2$ and $\forall x, y, z \in \mathrm{A}$,

(A) $(xy\omega)z\omega = x(yz\omega)\omega$.

The equations (I), (C), and (A) are *identities* because they hold for all possible choices of arguments. (cf.[RS2], p.13.) An identity is *regular* iff the sets of arguments on both sides of the equality are identical.

Example 2.1. In the ring $(\mathbb{Z}_2 \times \mathbb{Z}_2, +, \cdot)$, both operations $+$ and $\cdot$ are commutative and associative, and $\cdot$ is idempotent, since in $\mathbb{Z}_2$, $1 \cdot 1 = 1$ and $0 \cdot 0 = 0$.

Definition 2.6. An algebra (A, Ω) is *idempotent* or *commutative* iff every $\omega \in \Omega$ is, and *entropic* iff $\forall \omega, \omega' \in \Omega$, $\forall 1 \leq i \leq \omega\tau$, $\forall 1 \leq j \leq \omega'\tau$, $\forall x_{ij} \in \mathrm{A}$,

(E) $x_{11} \dots x_{\omega\tau 1}\omega \dots x_{1\omega'\tau} \dots x_{\omega\tau\omega'\tau}\omega\omega' = x_{11} \dots x_{1\omega'\tau}\omega' \dots x_{\omega\tau 1} \dots x_{\omega\tau\omega'\tau}\omega'\omega$.

Definition 2.7. A *mode* is an idempotent, entropic algebra. (cf. [RS2], p.14.)

Example 2.2. The group $(\mathbb{Z}_2 \times \mathbb{Z}_2, \cdot)$ is a mode, since $(xy)(zw) = (xz)(yw)$ holds by commutativity and associativity.

Example 2.3. *Differential groupoids* are defined to be modes $(\mathrm{G}, *)$ of type $\{(*, 2)\}$ satisfying the additional identity $x * (y * z) = x * y$. (cf. [RS1], p.284.)

Example 2.4. Notice that the functions $\underline{\lambda}$, defined in Definition 1.4, are operations on $\mathbb{R}$. Thus $(\mathbb{R}, \mathrm{I}^\circ)$ is an algebra, one of our most important examples of modes.

Proof: Clearly, $\forall p, q \in \mathrm{I}^\circ$, $\forall w,\ x,\ y,\ z \in \mathbb{R}$,

$$xx\underline{p} = x \text{ and}$$

$$\begin{aligned} xy\underline{p}wz\underline{p}\,\underline{q} &= [x(1-p) + py](1-q) + [w(1-p) + pz]q \\ &= [x(1-q) + qw](1-p) + [y(1-q) + qz]p \end{aligned}$$

$$= xw\underline{q}yz\underline{q}\,\underline{p}.$$

So $(\mathbb{R}, \mathrm{I}^\circ)$ is a mode. $\square$

Actually, we have by extension, applying $\underline{p}$ componentwise in the product,

Lemma 2.1. For every $n \in \mathbb{N}$, $(\mathbb{R}^n, \mathrm{I}^\circ)$ is a mode.

Definition 2.8. A subalgebra (S, Ω) of a mode (M, Ω) is called a *submode.*

Note 2.1. The idempotence condition gives us that one-element subsets of modes are submodes. See Example 2.5.

Example 2.5. Let Ω be a set of basic operations for some mode M. Let $\mathrm{X} = \{x\}$ be any one-element set. If we define for each $\omega \in \Omega$,

$$\omega : \mathrm{X}^{\omega\tau} \to \mathrm{X};\ (x, \ldots, x) \mapsto x,$$

then (X, Ω) is a mode.

Example 2.6. The submodes of $(\mathbb{R}^n, \mathrm{I}^\circ)$ are exactly the convex subsets of $\mathbb{R}^n$. ([RS5], Ex. 5.1.)

Proposition 2.1. The set MS of nonempty submodes of a mode (M, Ω) itself forms a mode (MS, Ω) where the operation $\omega \in \Omega$ acts on MS by:

$$\forall \mathrm{S}_1, \ldots, \mathrm{S}_{\omega\tau} \in \mathrm{MS},\ \mathrm{S}_1 \ldots \mathrm{S}_{\omega\tau}\omega := \{s_1 \ldots s_{\omega\tau}\omega \mid s_i \in \mathrm{S}_i\}.\ \text{([RS2], pp.13-14.)}$$

Proof: We must show that Ω is a set of operations on MS, and that (MS, Ω) is idempotent and entropic.

1. Let $\omega \in \Omega$ and $\mathrm{S}_1, \ldots, \mathrm{S}_{\omega\tau} \in \mathrm{MS}$. Then

$$\forall \omega' \in \Omega,\ \forall 1 \leq j \leq \omega'\tau,\ \forall s_{1j} \ldots s_{\omega\tau j}\omega \in \mathrm{S}_1 \ldots \mathrm{S}_{\omega\tau}\omega, \text{ with } s_{ij} \in \mathrm{S}_i,$$

$$s_{11} \ldots s_{\omega\tau 1}\omega \ldots s_{1\omega'\tau} \ldots s_{\omega\tau\omega'\tau}\omega\omega' = s_{11} \ldots s_{1\omega'\tau}\omega' \ldots s_{\omega\tau 1} \ldots s_{\omega\tau\omega'\tau}\omega'\omega,$$

which is back in $\mathrm{S}_1 \ldots \mathrm{S}_{\omega\tau}\omega$. Thus $\mathrm{S}_1 \ldots \mathrm{S}_{\omega\tau}\omega \in \mathrm{MS}$, and $\omega : (\mathrm{MS})^{\omega\tau} \to \mathrm{MS}$.

2. Let $\mathrm{S} \in \mathrm{MS}$ and $\omega \in \Omega$. Then

$$\forall s \in \mathrm{S},\ s = s \ldots s\omega \implies \mathrm{S} \subseteq \mathrm{S} \ldots \mathrm{S}\omega, \text{ and}$$
$$\mathrm{S} \leq \mathrm{M} \implies \mathrm{S} \ldots \mathrm{S}\omega \subseteq \mathrm{S}.$$

So $\mathrm{S} \ldots \mathrm{S}\omega = \mathrm{S}$, and idempotence is shown.

3. Finally, let $\omega, \omega' \in \Omega$, $\mathrm{S}_{11}, \ldots, \mathrm{S}_{\omega\tau 1}, \ldots, \mathrm{S}_{\omega'\tau 1}, \ldots, \mathrm{S}_{\omega\tau\omega'\tau} \in$ **MS**. Then

$$s \in \mathrm{S}_{11} \ldots \mathrm{S}_{\omega\tau 1}\omega \ldots \mathrm{S}_{1\omega'\tau} \ldots \mathrm{S}_{\omega\tau\omega'\tau}\omega\omega' \implies \forall 1 \le i \le \omega\tau,\ \forall 1 \le j \le \omega'\tau,$$
$$\exists s_{ij} \in \mathrm{S}_{ij} \bullet s = s_{11} \ldots s_{\omega\tau 1}\omega \ldots s_{1\omega'\tau} \ldots s_{\omega\tau\omega'\tau}\omega\omega',$$

which by entropicity of M is

$s = s_{11} \ldots s_{1\omega'\tau}\omega' \ldots s_{\omega\tau 1} \ldots s_{\omega\tau\omega'\tau}\omega'\omega \in \mathrm{S}_{11} \ldots \mathrm{S}_{1\omega'\tau}\omega' \ldots \mathrm{S}_{\omega\tau 1} \ldots \mathrm{S}_{\omega\tau\omega'\tau}\omega'\omega$,

so $\mathrm{S}_{11} \ldots \mathrm{S}_{\omega\tau 1}\omega \ldots \mathrm{S}_{1\omega'\tau} \ldots \mathrm{S}_{\omega\tau\omega'\tau}\omega\omega' \subseteq \mathrm{S}_{11} \ldots \mathrm{S}_{1\omega'\tau}\omega' \ldots \mathrm{S}_{\omega\tau 1} \ldots \mathrm{S}_{\omega\tau\omega'\tau}\omega'\omega$,

and similarly for the reverse containment. So **MS** is entropic. □

Definition 2.9. A *variety* is a class of algebras that contains all subalgebras, products, and homomorphic images of its members.

Lemma 2.2. (Birkhoff's Theorem) A class K of algebras is a variety $\iff$ there is a set of identities such that K is the class of all algebras satisfying those identities. ([BS], Def. 11.7, Thm. 11.9, p.75.)

Lemma 2.3. A product of modes with the same type is again a mode of the same type.

Proof: Modes are characterized as satisfying the identities (I) and (E), so they form a variety. Therefore, products of modes are modes.

Note 2.2. For modes (A, Ω) and (B, Ω), an operation $\omega \in \Omega$ acts on $\mathrm{A} \times \mathrm{B}$ by

$$\omega : (\mathrm{A} \times \mathrm{B})^{\omega\tau} \to \mathrm{A} \times \mathrm{B};\ (a_1, b_1) \ldots (a_{\omega\tau}, b_{\omega\tau}) \mapsto (a_1 \ldots a_{\omega\tau}\omega, b_1 \ldots b_{\omega\tau}\omega).$$

([RS2], p.6.) □

Example 2.7. We get the mode $(\mathbb{R}^n \times \mathbb{R}, \mathrm{I}^\circ)$ as a product of the modes $(\mathbb{R}^n, \mathrm{I}^\circ)$ and $(\mathbb{R}, \mathrm{I}^\circ)$ by Lemma 2.3.

Definition 2.10. Recall that a *semilattice* is an algebra $(\mathrm{S}, *)$ with a single binary operation that is idempotent, commutative, and associative. A *join* semilattice has the order defined by $x \le_* y \iff x * y = y$, and a *meet* semilattice has the order $x \le_* y \iff x * y = x$. An algebra $(\mathrm{S}, \vee, \wedge)$ is a *lattice* iff $(\mathrm{S}, \vee)$ and $(\mathrm{S}, \wedge)$ are semilattices and the partial orders $\le_\vee$ and $\le_\wedge$ coincide.

Definition 2.11. A *chain* in a semilattice $(S, *)$ is a set $W \subseteq S$ such that for any two elements $x, y \in W$, either $x \leq y$ or $y \leq x$.

Fact 2.1. Chains are lattices, where for $x \leq y$, $x \wedge y := x$ and $x \vee y := y$.

Example 2.8. Semilattices, and in particular chains, such as $(\mathbb{R}, \max)$, are modes.

Proof: A semilattice $(S, *)$ is clearly idempotent, and entropicity follows from the identity $(x * y) * (z * w) = (x * z) * (y * w)$, which uses commutativity and associativity. $\square$

Definition 2.12. Let X be a subset of a poset $(S, \leq)$. A *lower bound* (resp. *upper bound*) of X is an element $y \in S$ satisfying $x \in X \implies y \leq x$ (resp. $y \geq x$). Let Y be the set of lower bounds (resp. upper bounds) of X. The *greatest lower bound* (resp. *least upper bound*) of X, if it exists, is that $y' \in Y$ satisfying

$$y \in Y \implies y \leq y' \text{ (resp. } y \geq y').$$

Example 2.9. In a meet semilattice $(S, \wedge)$, the set $X = \{x_1, x_2\}$ has greatest lower bound $x_1 \wedge x_2$, and the set $X = \{x_1, \ldots, x_n\}$ has greatest lower bound $y = ((\ldots((x_1 \wedge x_2) \wedge x_3) \wedge \ldots) \wedge x_n)$. Since the meet operation is associative, we can leave off the parentheses, and write $y = \bigwedge X$. Similarly, the least upper bound of a finite subset X of a join semilattice S is written $\bigvee X$.

Definition 2.13. Given an ordered set S, if for every $X \subseteq S$, the greatest lower bound (resp. least upper bound) of X exists, we again write $\bigwedge X$ (resp. $\bigvee X$), and call S a *complete* meet (resp. join) semilattice.

Definition 2.14. For sets A and B, and an operation ω on B, we can define

$$\omega : (B^A)^{\omega\tau} \to B^A;\ (f_1, \ldots, f_{\omega\tau}) \mapsto (x \mapsto x f_1 \ldots x f_{\omega\tau} \omega).$$

Proposition 2.2. For modes (A, Ω) and (B, Ω) of the same type, the set of homomorphisms $H = (\mathrm{Hom}(A, B), \Omega)$ is a mode.

Proof: Idempotence and entropicity are inherited from the codomain:
$\forall\omega,\ \omega' \in \Omega,\ \forall 1 \le i \le \omega\tau,\ \forall 1 \le j \le \omega'\tau,\ \forall h, h_{ij} \in \mathrm{H},\ \forall x \in \mathrm{A},$

$$x(hh\omega) = xhxh\omega = xh, \text{ and}$$

$$\begin{aligned} x(h_{11}\dots h_{\omega\tau 1}\omega\dots h_{1\omega'\tau}\dots h_{\omega\tau\omega'\tau}\omega\omega') \\ &= xh_{11}\dots xh_{\omega\tau 1}\omega\dots xh_{1\omega'\tau}\dots xh_{\omega\tau\omega'\tau}\omega\omega' \\ &= xh_{11}\dots xh_{1\omega'\tau}\omega'\dots xh_{\omega\tau 1}\dots xh_{\omega\tau\omega'\tau}\omega'\omega \\ &= x(h_{11}\dots h_{1\omega'\tau}\omega'\dots h_{\omega\tau 1}\dots h_{\omega\tau\omega'\tau}\omega'\omega). \end{aligned}$$

So all we need to show is that every $\omega \in \Omega$ is an operation on H. We have

$\forall\omega, \omega' \in \Omega,\ \forall x_1, \dots, x_{\omega\tau} \in \mathrm{A},\ \forall h_1, \dots, h_{\omega'\tau} \in \mathrm{H},$

$$\begin{aligned} (x_1 \dots x_{\omega\tau}\omega)(h_1\dots h_{\omega'\tau}\omega') \\ &= (x_1\dots x_{\omega\tau}\omega)h_1\dots(x_1\dots x_{\omega\tau}\omega)h_{\omega'\tau}\omega' \\ &= (x_1h_1\dots x_{\omega\tau}h_1\omega)\dots(x_1h_{\omega'\tau}\dots x_{\omega\tau}h_{\omega'\tau}\omega)\omega' && [\Omega\text{-homomorphisms}] \\ &= (x_1h_1\dots x_1h_{\omega'\tau}\omega')\dots(x_{\omega\tau}h_1\dots x_{\omega\tau}h_{\omega'\tau}\omega')\omega && [\text{entropicity in B}] \\ &= x_1(h_1\dots h_{\omega'\tau}\omega')\dots x_{\omega\tau}(h_1\dots h_{\omega'\tau}\omega')\omega. \end{aligned}$$

Thus $h_1\dots h_{\omega'\tau}\omega' \in \mathrm{H}$. □

There is a particular construction, that of Płonka sums, that conveniently gives us the modes $(\mathbb{R}^{+\infty}, \mathrm{I}^\circ)$ and $(\mathbb{R}_{-\infty}, \mathrm{I}^\circ)$, and a new mode $(\overline{\mathbb{R}}, \mathrm{I}^\circ)$.

Definition 2.15. We can think of a meet semilattice $(\mathrm{S}, \wedge)$ as a category (S) where the objects are the elements of S and the set of morphisms is $\{(t \to s) \mid t \le s\}$. If (Ω) is plural, S can be made into an Ω-algebra by defining, for each $\omega \in \Omega$,

$$\omega : \mathrm{S}^{\omega\tau} \to \mathrm{S};\ (s_1, \dots, s_{\omega\tau}) \mapsto s_1 \wedge \dots \wedge s_{\omega\tau}.$$

A similar construction works for a join semilattice. Let $G : (\mathrm{S}) \to (\Omega)$ be a contravariant functor. Define $\mathrm{S}G := \coprod_{s\in\mathrm{S}} sG$ (which is the disjoint union $\dot{\bigcup}_{s\in\mathrm{S}} sG$) and make it into an Ω-algebra by defining:

$$\forall\omega \in \Omega,\ \forall s_1, \dots, s_{\omega\tau} \in \mathrm{S}, \text{ and } t = s_1 \wedge \dots \wedge s_{\omega\tau} \in \mathrm{S},$$
$$\omega : s_1G \times \dots \times s_{\omega\tau}G \to tG;\ (x_1, \dots, x_{\omega\tau}) \mapsto x_1(t \to s_1)G \dots x_{\omega\tau}(t \to s_{\omega\tau})G\omega.$$

Then $(\mathrm{S}G, \Omega)$ is the *Płonka sum* of the Ω-algebras sG over the semilattice $(\mathrm{S}, \wedge)$ by the functor G. The algebras sG for $s \in \mathrm{S}$ are called the *fibers* of $\mathrm{S}G$.

Lemma 2.4. A Płonka sum satifies the regular identities satisfied by each of its fibers. ([RS2], Prop. 238, p.34.)

Example 2.10. Let S={0,1} and $(\mathrm{S}, \wedge)$ be the meet semilattice with $0 \wedge 1 = 0$. Consider $\mathbb{R}, \{-\infty\} \in (\mathrm{I}^\circ)$. (Each $\lambda \in \mathrm{I}^\circ$ is the identity on $\{-\infty\}$. See Example 2.5.) Define the functor

$$G : (\mathrm{S}) \to (\mathrm{I}^\circ);\ 0 \mapsto \mathbb{R},\ 1 \mapsto \{-\infty\},\ (0 \to 1) \mapsto (\mathbb{R} \to \{-\infty\}).$$

Then the resulting Płonka sum $\mathbb{R}_{-\infty} = \mathbb{R}\,\dot{\cup}\{-\infty\}$ is again a mode. Similarly, we get the mode $\mathbb{R}^{+\infty} = \mathbb{R}\,\dot{\cup}\{+\infty\}$, where $(+\infty)x\underline{\lambda} = +\infty$ for any $\lambda \in \mathrm{I}^\circ$, $x \in \mathbb{R}^{+\infty}$.

Example 2.11. Now take $\mathrm{S} = \{0 < 1 < 2\}$. For the modes $\mathbb{R}, \{-\infty\}, \{+\infty\} \in (\mathrm{I}^\circ)$, define a functor $\mathrm{G} : (\mathrm{S}) \to (\mathrm{I}^\circ)$ by

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & 1 \arrow[d] & & \mathbb{R} \arrow[r] \arrow[d, equal] & \{-\infty\} \arrow[d] \\ 0 \arrow[r] & 2 & \mapsto & \mathbb{R} \arrow[r] & \{+\infty\}. \end{tikzcd}$$

Then $(\overline{\mathbb{R}}, \mathrm{I}^\circ) \cong \mathbb{R}_{-\infty} \coprod \{+\infty\}$ is a mode. Anytime $+\infty$ is an argument of $\underline{\lambda}$, the image is $+\infty$, and whenever $-\infty$ is an argument and $+\infty$ is not, the image is $-\infty$.

Example 2.12. For a plural mode (M, Ω), $\mathrm{MS}_\varnothing = \mathrm{MS}\,\dot{\cup}\{\varnothing\}$. Note that $\varnothing$ is an Ω-algebra (and a mode) where any $\omega \in \Omega$ is the empty function $\omega : \varnothing^{\omega\tau} \to \varnothing$. Let $\mathrm{S} = \{0 < 1\}$ as above. The functor

$$G : (\mathrm{S}) \to (\Omega);\ 0 \mapsto \mathrm{MS},\ 1 \mapsto \{\varnothing\},\ (0 \to 1) \mapsto (\mathrm{MS} \to \{\varnothing\})$$

gives us the mode $(\mathrm{MS}_\varnothing, \Omega)$ by the Płonka sum construction.

Definition 2.16. In an algebra $(\mathrm{A}, \vee, \Omega)$ with join-semilattice reduct $(\mathrm{A}, \vee)$, an operation $\omega \in \Omega$ *distributes* over the operation $\vee$ iff

$\forall 1 \leq j \leq \omega\tau,\ \forall x_1, \ldots, x_{\omega\tau}, x'_j \in \mathrm{A},$

(D) $x_1 \ldots (x_j \vee x'_j) \ldots x_{\omega\tau}\omega = (x_1 \ldots x_j \ldots x_{\omega\tau}\omega) \vee (x_1 \ldots x'_j \ldots x_{\omega\tau}\omega).$

In the case that $(\mathrm{A}, \bigvee)$ is a complete semilattice, we say $\omega \in \Omega$ *distributes completely* over $\bigvee$ iff $\forall 1 \leq j \leq \omega\tau$, $\forall x_1, \ldots, x_{\omega\tau} \in \mathrm{A}$, $\forall \mathrm{X} \subseteq \mathrm{A}$,
(CD) $x_1 \ldots x_{j-1}(\bigvee \mathrm{X})x_{j+1} \ldots x_{\omega\tau}\omega = \bigvee\{x_1 \ldots x_{j-1}xx_{j+1} \ldots x_{\omega\tau}\omega \mid x \in \mathrm{X}\}$.

Definition 2.17. A *modal* is an algebra $(\mathrm{D}, \vee, \Omega)$ with join-semilattice reduct $(\mathrm{D}, \vee)$ and mode reduct (D, Ω) such that the operations in Ω distribute over $\vee$.

Definition 2.18. A modal $(\mathrm{D}, \vee, \Omega)$ is *completely distributive* iff $(\mathrm{D}, \vee)$ is complete and every operation ω in Ω satisfies property (CD).

Example 2.13. A distributive lattice $(\mathrm{D}, \vee, \wedge) = (\mathrm{D}, \vee, \{\wedge\})$ is a modal, with $\Omega = \{\wedge\}$, since meet distributes over join.

Proposition 2.4. The algebra $(\mathbb{R}, \max, \mathrm{I}^\circ)$ is a modal and $(\overline{\mathbb{R}}, \sup, \mathrm{I}^\circ)$ is a completely distributive modal.
Proof: First note that $(\mathbb{R}, \max)$ and $(\overline{\mathbb{R}}, \sup)$ are join semilattices, and $\overline{\mathbb{R}}$ is complete. Next, we have

$$\begin{aligned}
&\forall x \in \mathbb{R},\ \forall \mathrm{Y} \subseteq \mathbb{R} \bullet\ 1 \leq |\mathrm{Y}| < +\infty,\ \forall p \in \mathrm{I}^\circ, \\
x(\max \mathrm{Y})\underline{p} &= x(1-p) + (\max \mathrm{Y})p \\
&= \max\{x(1-p) + yp \mid y \in \mathrm{Y}\} \\
&= \max\{xy\underline{p} \mid y \in \mathrm{Y}\},
\end{aligned}$$

and similarly for $(\max \mathrm{Y})x\underline{p}$, so $\underline{p}$ distributes over max. Also, supremum is the same for $|\mathrm{Y}| < +\infty$, even if Y contains an infinite element. Finally, consider $x \in \overline{\mathbb{R}}$, $|\mathrm{Y}| = +\infty$, sup in place of max above.
1. If $x = +\infty$ then both sides are $+\infty$.
2. If $x < +\infty$ and one side is $+\infty$, there must be a sequence $\{y_n\} \subseteq \mathrm{Y}$ with $\lim_{n \to +\infty} y_n = +\infty$. Continuity of $y \mapsto xy\underline{p}$ has both sides being $+\infty$.
3. If $x < +\infty$ and $\sup \mathrm{Y} < +\infty$, then both sides are finite when both x and $\sup \mathrm{Y}$ are finite, and $-\infty$ when either x or $\sup \mathrm{Y}$ is $-\infty$. $\square$

Definition 2.19. A map between partially ordered sets $(\mathrm{S}, \leq_{\mathrm{S}})$ and $(\mathrm{D}, \leq_{\mathrm{D}})$ is *monotone* iff $\forall x, y \in \mathrm{S}$, $x \leq_{\mathrm{S}} y \implies xf \leq_{\mathrm{D}} yf$.

Lemma 2.5. (Monotonicity Lemma) For a modal $(\mathrm{D}, \vee, \Omega)$, every $\omega \in \Omega$ is monotone as a map, $\omega : (\mathrm{D}^{\omega\tau}, \leq) \to (\mathrm{D}, \leq)$. This means

$$\forall x_1, \ldots, x_{\omega\tau}, y_1, \ldots, y_{\omega\tau} \in \mathrm{D},\ (\forall i \in \{1, \ldots, \omega\tau\},\ x_i \leq y_i) \implies$$
$$x_1 \ldots x_{\omega\tau}\omega \leq y_1 \ldots y_{\omega\tau}\omega.$$ ([RS2], Prop. 315, p.58.) □

Recall how in the Euclidean space case a function $f : (\mathbb{R}^n, \mathrm{I}^\circ) \to (\mathbb{R}^{+\infty}, \sup, \mathrm{I}^\circ)$ is convex iff it satisfies property $(\mathrm{CX}_{\mathbb{R}})$. (See p.2.) Note that the domain is a mode and the codomain is a modal. The generalization of this property is the following:

Definition 2.20. A mode (M, Ω) and a modal $(\mathrm{D}, \vee, \Omega)$ are called *compatible* iff the modes (M, Ω) and (D, Ω) have the same type. Let (M, Ω) be a mode and $(\mathrm{D}, \vee, \Omega)$ be a compatible modal. Let $f : \mathrm{M} \to \mathrm{D}$. We say f is *Ω-convex*, or just *convex*, iff for every $\omega \in \Omega$, and every $x_1, \ldots, x_n \in \mathrm{M}$, we have (CX) $x_1 \ldots x_n \omega^{\mathrm{M}} f \leq_{\vee} x_1 f \ldots x_n f \omega^{\mathrm{D}}$.

We call f *concave* iff it satisfies the reverse inequality. Let $\mathrm{Conv}(\mathrm{M}, \mathrm{D})$ be the set of all convex functions from M to D.

Example 2.14. A function whose logarithm is real-convex is called "log-convex" in the literature ([RV], p.18). Precisely, a function $f : \mathbb{R} \to \mathbb{R}^+$ is *log-convex* iff it satisfies

$$\forall p \in \mathrm{I}^\circ,\ (x(1-p) + yp)f \leq xf^{1-p} yf^p.$$

If we define, for $p \in \mathrm{I}^\circ$, the maps $\underline{\underline{p}} : (\mathbb{R}^+)^2 \to \mathbb{R}^+;\ (x, y) \mapsto x^{1-p}y^p$, then $(\mathbb{R}^+, \max, \underline{\underline{p}})_{p \in \mathrm{I}^\circ}$ is a modal, and the log-convex functions are exactly the elements of $\mathrm{Conv}((\mathbb{R}, \underline{p})_{p \in \mathrm{I}^\circ}, (\mathbb{R}^+, \vee, \underline{\underline{p}})_{p \in \mathrm{I}^\circ})$. The test functions (i.e., mode-reduct homomorphisms) are the exponential functions ke^{cx}.

Example 2.15. Call a function $f : \mathbb{R}^+ \to \mathbb{R}$ *exponentially-convex* iff it satisfies

$$\forall p \in \mathrm{I}^\circ,\ (x^{1-p}y^p)f \leq xf(1-p) + yfp.$$

This is equivalent to $tg = e^t f$ being real-convex. (Use $x = e^t$, $y = e^s$.) Now f is exponentially-convex iff $f \in \mathrm{Conv}((\mathbb{R}^+, \underline{\underline{p}})_{p \in \mathrm{I}^\circ}, (\mathbb{R}, \vee, \underline{p})_{p \in \mathrm{I}^\circ})$. The mode-reduct homomorphisms here are the log functions $c \log x + d$.

Proof: A typical homomorphism h must satisfy $(x^{1-p}y^p)h = xh(1-p) + yhp$ for every $p \in \mathrm{I}^\circ$. Let $x = 1$. We then have: $y^p h = 1h(1-p) + yhp$. Differentiating, this becomes: $(y^p)h'py^{p-1} = yh'p$, i.e., $(y^p)h'y^{p-1} = yh'$. This is $(y^p h')y^p = (yh')y$. This is solved by $zh = c\log z + d$, $d = 1h, c = 1h'$. $\square$

Example 2.16. Yet another kind of convexity is sometimes called "quasi-convexity" ([BP], p.84). Let $\mathrm{X} \subseteq \mathbb{R}^n$ be convex. A function $f : \mathrm{X} \to \overline{\mathbb{R}}$ is *quasi-convex* iff for every $\alpha \in \overline{\mathbb{R}}$, the set $\mathrm{C}_\alpha := \{x \in \mathrm{X} \mid xf \leq \alpha\}$ is convex. By [RV], p.230, this condition is equivalent to the condition:

$$\forall x, y \in \mathrm{X},\ \forall p \in \mathrm{I}^\circ,\ (x(1-p) + yp)f \leq xf \vee yf.$$

So quasiconvex functions are the elements of $\mathrm{Conv}((\mathbb{R}, \underline{p})_{p\in\mathrm{I}^\circ}, (\mathbb{R}, \vee, \vee_p)_{p\in\mathrm{I}^\circ})$. The codomain has the I°−join semilattice structure analogous to Definition 2.15 ($x \vee_p y = x \vee y = \max\{x, y\}$).

Proposition 2.5. For a mode (M, Ω) and a compatible modal $(\mathrm{D}, \vee, \Omega)$, $(\mathrm{Conv}((\mathrm{M}, \Omega), (\mathrm{D}, \vee, \Omega)), \vee, \Omega)$ is a modal.

Proof: Again we only need $\mathrm{C} = \mathrm{Conv}(\mathrm{M}, \mathrm{D})$ to be closed under the operations. We have

$\forall \omega, \omega' \in \Omega$, $\forall x_1, \ldots, x_{\omega\tau} \in \mathrm{A}$, $\forall f_1, \ldots, f_{\omega'\tau} \in \mathrm{C}$,

$$\begin{aligned}
&(x_1 \ldots x_{\omega\tau}\omega)(f_1 \vee f_2) \\
&\qquad = (x_1 \ldots x_{\omega\tau}\omega)f_1 \vee (x_1 \ldots x_{\omega\tau}\omega)f_2 \\
&\qquad \leq (x_1 f_1 \ldots x_{\omega\tau} f_1 \omega) \vee (x_1 f_2 \ldots x_{\omega\tau} f_2 \omega) && [f_i \text{ convex}] \\
&\qquad = (x_1 f_1 \vee x_1 f_2) \ldots (x_{\omega\tau} f_1 \vee x_{\omega\tau} f_2)\omega && [(\mathrm{D}) \text{ in } \mathrm{D}] \\
&\qquad = x_1(f_1 \vee f_2) \ldots x_{\omega\tau}(f_1 \vee f_2)\omega,
\end{aligned}$$

so $f_1 \vee f_2 \in \mathrm{C}$, and

$$\begin{aligned}
&(x_1 \ldots x_{\omega\tau}\omega)(f_1 \ldots f_{\omega'\tau}\omega') \\
&\qquad = (x_1 \ldots x_{\omega\tau}\omega)f_1 \ldots (x_1 \ldots x_{\omega\tau}\omega)f_{\omega'\tau}\omega' \\
&\qquad \leq (x_1 f_1 \ldots x_{\omega\tau} f_1 \omega) \ldots (x_1 f_{\omega'\tau} \ldots x_{\omega\tau} f_{\omega'\tau}\omega)\omega' && [f_i \text{ convex}] \\
&\qquad = (x_1 f_1 \ldots x_1 f_{\omega'\tau}\omega') \ldots (x_{\omega\tau} f_1 \ldots x_{\omega\tau} f_{\omega'\tau}\omega')\omega && [(\mathrm{E}) \text{ in } \mathrm{D}] \\
&\qquad = x_1(f_1 \ldots f_{\omega'\tau}\omega') \ldots x_{\omega\tau}(f_1 \ldots f_{\omega'\tau}\omega')\omega,
\end{aligned}$$

so $f_1 \ldots f_{\omega'\tau}\omega' \in \mathrm{C}$ also. $\square$

Corollary 2.1. The algebra $(\mathrm{Conv}((\mathbb{R}^n, \mathrm{I}^\circ), (\mathbb{R}, \max, \mathrm{I}^\circ)), \max, \mathrm{I}^\circ)$ is a modal.

There is a more general algebraic structure that will be useful in the next chapter.

Definition 2.21. An algebra $(\mathrm{A}, \leq, \Omega)$ is an *ordered mode* iff (A, Ω) is a mode, $(\mathrm{A}, \leq)$ is a poset, and every $\omega \in \Omega$ is monotone.

Example 2.16. All modals are ordered modes, with the join semilattice order, and all modes can be considered as ordered modes with the trivial order $x \leq y \iff x = y$.

Example 2.17. For a compatible mode M and modal D, since the mode $\mathrm{Hom}((\mathrm{M}, \Omega), (\mathrm{D}, \Omega))$ is a subset of the ordered mode $\mathrm{Conv}(\mathrm{M}, \mathrm{D})$ it is also a (non-trivial) ordered mode.

CHAPTER 3: A GALOIS CORRESPONDENCE IN EUCLIDEAN SPACES

In this chapter an order-theoretic approach to the concept of subgradient lines of convex functions is presented. We will exhibit a Galois correspondence between the finitary epigraphs of convex functions and those sections of an order-theoretic "bundle" that (conveniently) give subgradient hyperplanes.

The functions considered here have a Euclidean space for a domain and some extension of the reals as codomain. We are concentrating on the mode and poset structures of the Euclidean spaces, to be able to generalize the present results to arbitrary modes and modals.

Notation 3.1. For discussion purposes, let $n \in \mathbb{N}$. The algebras $(\mathbb{R}, \max, \mathrm{I}^\circ)$, $(\overline{\mathbb{R}}, \sup, \mathrm{I}^\circ)$, $(\mathbb{R}^{+\infty}, \sup, \mathrm{I}^\circ)$, and $(\mathbb{R}^n, \mathrm{I}^\circ)$ were introduced in Chapter 2. Here we define some more modes and ordered modes. Let

$$\mathrm{L} := \mathbb{R}^n \times \mathbb{R} \qquad \text{and} \qquad \overline{\mathrm{L}} := \mathbb{R}^n \times \overline{\mathbb{R}}.$$

These can be ordered in the last component as in Definition 1.2, and form I°-modes as products of I°-modes. The function

$$\pi : \overline{\mathrm{L}} \to \mathbb{R}^n;\ (\mathbf{a}, b) \mapsto \mathbf{a}$$

makes $\overline{\mathrm{L}}$ into a "bundle" of the fibers $\mathrm{L}_{\mathbf{a}} := \pi^{-1}\{\mathbf{a}\}$. Define

$$\Gamma := \{\sigma : \mathbb{R}^n \to \overline{\mathrm{L}} \mid \sigma\pi = \mathrm{id}_{\mathbb{R}^n}\}$$

to be the set of "sections" of π, ordered pointwise as a subset of $(\overline{\mathrm{L}})^{\mathbb{R}^n}$. Let

$$\mathrm{H} := \mathrm{Hom}((\mathbb{R}^n, \mathrm{I}^\circ), (\mathbb{R}, \mathrm{I}^\circ)),$$

the set of I°-homomorphisms (or affine functions) from $\mathbb{R}^n$ to $\mathbb{R}$ and

$$\overline{\mathrm{H}} := \mathrm{H}\dot{\cup}(\mathbb{R}\times\{-\infty\})\dot{\cup}(\mathbb{R}\times\{+\infty\}) = \mathrm{Hom}((\mathbb{R}^n, \mathrm{I}^\circ), (\overline{\mathbb{R}}, \mathrm{I}^\circ)).$$

Both H and $\overline{\mathrm{H}}$ have the ordered mode structure inherited from the modal $(\mathrm{Conv}((\mathbb{R}^n, \mathrm{I}^\circ), (\overline{\mathbb{R}}, \vee, \mathrm{I}^\circ)), \vee, \mathrm{I}^\circ)$. The sets

$$\begin{aligned} \mathrm{P} &:= \big\{\{(\mathbf{x}, y) \mid y = \mathbf{a}\cdot\mathbf{x} + b\} \mid \mathbf{a} \in \mathbb{R}^n, b \in \mathbb{R}\big\} \\ &= \{\text{non-vertical hyperplanes in } \mathbb{R}^{n+1}\}, \text{ and} \\ \overline{\mathrm{P}} &:= \mathrm{P}\dot{\cup}\{\{(\mathbf{x}, -\infty) \mid \mathbf{x} \in \mathbb{R}^n\}\}\dot{\cup}\{\{(\mathbf{x}, +\infty) \mid \mathbf{x} \in \mathbb{R}^n\}\} \end{aligned}$$

form ordered modes, as follows. Identify an element of $\overline{\mathrm{P}}$ by its equation. Order

P and $\overline{\mathrm{P}}$ by $\ell_1 \leq \ell_2$ iff no point in ℓ_1 lies above any point in ℓ_2 (in $\mathbb{R}^n \times \overline{\mathbb{R}}$), so for every $\ell \in \mathrm{P}$, $(y = -\infty) < \ell < (y = +\infty)$; and for $\lambda \in \mathrm{I}^\circ$ define

$$\underline{\lambda} : (\overline{\mathrm{P}})^2 \to \overline{\mathrm{P}};\ ((y = \mathbf{a}\cdot\mathbf{x} + b), (y = \mathbf{a}'\cdot\mathbf{x} + b')) \mapsto (y = \mathbf{a}\mathbf{a}'\underline{\lambda}\cdot\mathbf{x} + bb'\underline{\lambda}).$$

We will first show that $(\mathrm{L}, \leq, \mathrm{I}^\circ)$, $(\mathrm{H}, \leq, \mathrm{I}^\circ)$, and $(\mathrm{P}, \leq, \mathrm{I}^\circ)$ are isomorphic. This is not coincidental. We can think of an element $(\mathbf{a}, b) \in \mathrm{L}$ as the projected normal vector $\mathbf{a}$ and intercept b of the hyperplane $(y = \mathbf{a}\cdot\mathbf{x} + b) \in \mathrm{P}$, which is the graph of the I°-homomorphism $(\mathbf{x} \mapsto \mathbf{a}\cdot\mathbf{x} + b) \in \mathrm{H}$.

Lemma 3.1. For $\ell, \ell' \in \mathrm{P}$, $\ell \leq \ell'$ if and only if ℓ and ℓ' are parallel and ℓ has smaller intercept. In other words,

$$(y = \mathbf{a}\cdot\mathbf{x} + b) \leq (y = \mathbf{a}'\cdot\mathbf{x} + b') \iff \mathbf{a} = \mathbf{a}' \text{ and } b \leq b'.$$

Proof: Note that for $\mathbf{c} \neq \mathbf{0}$, $\mathbf{c}\cdot\mathbf{x}$ takes on every real value. Thus if $(\mathbf{a}' - \mathbf{a})\cdot\mathbf{x}$ is bounded from below, $\mathbf{a}' - \mathbf{a}$ must be $\mathbf{0}$. Now

$$\begin{aligned}(y = \mathbf{a}\cdot\mathbf{x} + b) \leq (y = \mathbf{a}'\cdot\mathbf{x} + b') &\iff \mathbf{a}\cdot\mathbf{x} + b \leq \mathbf{a}'\cdot\mathbf{x} + b' \\ &\iff b - b' \leq (\mathbf{a}' - \mathbf{a})\cdot\mathbf{x} \\ &\iff \mathbf{a}' = \mathbf{a} \text{ and } b' \geq b. \quad \square\end{aligned}$$

Definition 3.1. For $q \in \mathbb{N}^+$, a *convex combination* in $\mathbb{R}^q$ is a vector $(c_1, \ldots, c_q)$ satisfying $\sum_{i=1}^{q} c_i = 1$ and $\forall 1 \leq i \leq q$, $c_i > 0$.

The following lemma shows that all convex combinations are derivable from the basic operations (I°), and thus are also preserved by I°-homomorphisms.

Lemma 3.2. Let $h \in \overline{\mathrm{H}}$ and $q \in \mathbb{N}^+$. Then

$$\forall c_1, \ldots, c_q \in \mathbb{R}^+ \bullet \sum_{i=1}^{q} c_i = 1,\ \forall \mathbf{x}^{(i)} \in \mathbb{R}^n,\ \left(\sum_{i=1}^{q} c_i\mathbf{x}^{(i)}\right)h = \sum_{i=1}^{q} c_i(\mathbf{x}^{(i)}h).$$

If we let $\mathbf{c} = (c_1, \ldots, c_q)$, this can be written

$$\mathbf{x}^{(1)} \ldots \mathbf{x}^{(q)}\underline{\mathbf{c}}h = \mathbf{x}^{(1)}h \ldots \mathbf{x}^{(q)}h\underline{\mathbf{c}}.$$

Proof: Let $q \in \mathbb{N}^+$. For $q = 1$, the lemma is trivial. So assume $q \geq 2$. Let $\mathbf{c} = (c_1, \ldots, c_q)$ be a convex combination in $\mathbb{R}^q$. Define $\mathbf{c}' = (\frac{c_1}{1-c_q}, \ldots, \frac{c_{q-1}}{1-c_q})$, which is again a convex combination, but in $\mathbb{R}^{q-1}$. For $q = 2$, defining $\lambda = c_2 \in \mathrm{I}^\circ$ gives the operation $\underline{\mathbf{c}} = \underline{\lambda}$. For $q > 2$, we can define $\underline{\mathbf{c}} : (\mathbb{R}^n)^q \to \mathbb{R}^n$

recursively by

$$x^{(1)}\dots x^{(q)}\underline{\mathbf{c}} := c_q x^{(q)} + (1-c_q)(x^{(1)}\dots x^{(q-1)}\underline{\mathbf{c}'}). \quad \square$$

Proposition 3.1. The functions

$$\mu : (\mathrm{L}, \leq, \mathrm{I}^\circ) \to (\mathrm{H}, \leq, \mathrm{I}^\circ);\ (\mathbf{a}, b) \mapsto (\mathbf{x} \mapsto \mathbf{a}\cdot\mathbf{x} + b) \quad \text{and}$$
$$\rho : (\mathrm{L}, \leq, \mathrm{I}^\circ) \to (\mathrm{P}, \leq, \mathrm{I}^\circ);\ (\mathbf{a}, b) \mapsto \{(\mathbf{x}, y) \mid y = \mathbf{a}\cdot\mathbf{x} + b\}$$

are isomorphisms of ordered modes. Thus $(\mathrm{L}, \leq, \mathrm{I}^\circ) \cong (\mathrm{P}, \leq, \mathrm{I}^\circ) \cong (\mathrm{H}, \leq, \mathrm{I}^\circ)$, and also $(\overline{\mathrm{P}}, \leq, \mathrm{I}^\circ) \cong (\overline{\mathrm{H}}, \leq, \mathrm{I}^\circ)$.

Proof: Clearly μ and ρ are 1-1, and ρ is onto. What must be shown is that μ maps into H and is onto. Preservation of the order and the mode operations will then follow. For $(\mathbf{a}, b) \in \mathrm{L}$, let $k = (\mathbf{a}, b)^\mu$, and let $\lambda \in \mathrm{I}^\circ$. Then

$$\begin{aligned} \mathbf{x}k\mathbf{z}k\underline{\lambda} &= (\mathbf{a}\cdot\mathbf{x} + b)(1-\lambda) + (\mathbf{a}\cdot\mathbf{z} + b)\lambda \\ &= \mathbf{a}\cdot\mathbf{x}(1-\lambda) + b(1-\lambda) + \mathbf{a}\cdot\mathbf{z}\lambda + b\lambda \\ &= \mathbf{a}\cdot(\mathbf{x}(1-\lambda) + \mathbf{z}\lambda) + b(1-\lambda+\lambda) \\ &= \mathbf{a}\cdot(\mathbf{x}\mathbf{z}\underline{\lambda}) + b \\ &= \mathbf{x}\mathbf{z}\underline{\lambda}k, \end{aligned}$$

so $k \in \mathrm{H}$. Therefore $\mu : \mathrm{L} \to \mathrm{H}$.

We show μ is onto in three steps. Let $k \in \mathrm{H}$.

Case 1. When $n = 0$, the proposition reduces to the obvious statement

$$(\{0\} \times \mathbb{R}, \leq, \mathrm{I}^\circ) \cong (\{(0 \mapsto b) \mid b \in \mathbb{R}\}, \leq, \mathrm{I}^\circ) \cong (\{(y = b) \mid b \in \mathbb{R}\}, \leq, \mathrm{I}^\circ). \quad \square_1$$

Case 2. Consider the case $n = 1$. Let $a + b = 1k$ and $-a + b = (-1)k$. Then $0k = (1k(-1)k)\underline{\tfrac{1}{2}} = b$, the y-intercept, and a is the slope, i.e.,

$$\forall x \in \mathbb{R},\ xk = ax + b.$$

Proof: For $x \in (0,1)$, $xk = 01\underline{x}k = 0k1k\underline{x} = b(1-x) + (a+b)x = ax + b$. For $x > 1$, $a + b = 1k = 0x\underline{\tfrac{1}{x}}k = 0kxk\underline{\tfrac{1}{x}} = b(1 - \tfrac{1}{x}) + xk(\tfrac{1}{x}) = b + (xk - b)(\tfrac{1}{x})$. Solving for a gives $a = (xk - b)(\tfrac{1}{x})$, i.e., $ax = xk - b$, or $xk = ax + b$. For $x < 0$, we can use $\lambda = -x$ or $-\frac{1}{x}$, and -1 in place of 1 above, to get $xk = ax + b$ for all x. $\square_2$

Case 3. Let $n \in \mathbb{N}^+$ be arbitrary. For $1 \leq i \leq n$, let $\mathbf{e}^{(i)}$ be the standard basis vector $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$ where the "1" is in the i'th position. Let $a_i + b = e^{(i)}k$, with $b = \mathbf{0}k$, the y-intercept. Then $\mathbf{x}k = \mathbf{a}\cdot\mathbf{x} + b$.

Proof: Now

$$b = \mathbf{0}k = e^{(i)}(-e^{(i)})\underline{\frac{1}{2}}k = e^{(i)}k(-e^{(i)})k\underline{\frac{1}{2}} = \frac{1}{2}\left[e^{(i)}k + (-e^{(i)})k\right]$$
$$= \frac{1}{2}\left[a_i + b + (-e^{(i)})k\right],$$

which upon solving yields $(-e^{(i)})k = -a_i + b$. Thus by the one-dimensional case, we have:

$$\forall c \in \mathbb{R},\ \forall 1 \leq i \leq n,\ (ce^{(i)})k = ca_i + b.$$

Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{x} = (x_1, \ldots, x_n) = \sum\limits_{i=1}^{n} x_i e^{(i)}$. Then

$$\mathbf{x}k = \left(\sum_{i=1}^{n} x_i e^{(i)}\right)k = \left(\sum_{i=1}^{n} \frac{1}{n}(nx_i e^{(i)})\right)k$$
$$= \frac{1}{n}\sum_{i=1}^{n}(nx_i e^{(i)})k = \frac{1}{n}\sum_{i=1}^{n}(nx_i a_i + b)$$
$$= \sum_{i=1}^{n}(x_i a_i) + b = \mathbf{a}\cdot\mathbf{x} + b,$$

where we use $q = n$, and for each i, $c_i = \frac{1}{n}$, in Lemma 3.2. $\Box_3$

Note that $\mathbf{a}$ is the projected normal vector to the hyperplane $y = \mathbf{x}k$. Thus $(\mathbf{a}, b)^\mu = k$ so μ is onto. Since the sets

$$\mathrm{L}_{\mathbf{a}} \setminus (\{(\mathrm{a}, -\infty)\} \cup \{(\mathrm{a}, +\infty)\}) = \{(\mathrm{a}, b) \mid b \in \mathbb{R}\},$$
$$\mathrm{P}_{\mathbf{a}} = \{(y = \mathbf{a}\cdot\mathbf{x} + b) \mid b \in \mathbb{R}\}, \text{ and}$$
$$\mathrm{H}_{\mathbf{a}} = \{\mathbf{x} \mapsto \mathbf{a}\mathbf{x} + b \mid b \in \mathbb{R}\}$$

are the .5order components of L, P, and H respectively, μ and ρ clearly preserve the order. For $\lambda \in \mathrm{I}^\circ$ and $h_1, h_2 \in \mathrm{H}$ with $\mathbf{x}h_1 = \mathbf{a}_1\cdot\mathbf{x} + b_1$ and $\mathbf{x}h_2 = \mathbf{a}_2\cdot\mathbf{x} + b_2$,

$$\mathbf{x}h_1h_2\underline{\lambda} = \mathbf{x}h_1\mathbf{x}h_2\underline{\lambda} = (\mathbf{a}_1\cdot\mathbf{x} + b_1)(\mathbf{a}_2\cdot\mathbf{x} + b_2)\underline{\lambda} = (\mathbf{a}_1\mathbf{a}_2\underline{\lambda})\cdot\mathbf{x} + b_1b_2\underline{\lambda}.$$

Thus $\underline{\lambda}$ acts on H as it acts on L, and clearly $\underline{\lambda}$ acts the same on L and P. Therefore μ and ρ are mode isomorphisms also. Thus,

$$\rho^{-1}\mu : \mathrm{P} \to \mathrm{H};\ (y = \mathbf{a}\cdot\mathbf{x} + b) \mapsto (\mathbf{x}k = \mathbf{a}\cdot\mathbf{x} + b)$$

is an ordered mode isomorphism and can be extended to the isomorphism

$\overline{\rho^{-1}\mu} : \overline{\mathrm{P}} \to \overline{\mathrm{H}}$ by mapping $y = -\infty$ to $k \equiv -\infty$ and $y = +\infty$ to $k \equiv +\infty$. Also, μ and ρ can be extended to the surjections $\overline{\mu} : \overline{\mathrm{L}} \to \overline{\mathrm{H}}$ and $\overline{\rho} : \overline{\mathrm{L}} \to \overline{\mathrm{P}}$ by mapping any infinite element in $\overline{\mathrm{L}}$ to the corresponding infinite element in the codomain. $\square$

Definition 3.2. For $k = (\mathbf{a}, b) \in \overline{\mathrm{L}}$, let

$$\mathrm{gr}_{\mathbb{R}}k := \{(\mathbf{x}, y) \in \mathbb{R}^n \times \mathbb{R} \mid y = \mathbf{a}\cdot\mathbf{x} + b\} \text{ and}$$
$$\mathrm{epi}_{\mathbb{R}}k := \{(\mathbf{x}, y) \in \mathbb{R}^n \times \mathbb{R} \mid y \geq \mathbf{a}\cdot\mathbf{x} + b\}.$$

Note 3.1. We have $\mathrm{gr}_{\mathbb{R}}k = \mathrm{gr}_{\mathbb{R}}(k^{\overline{\mu}})$. Also for any $\mathbf{a} \in \mathbb{R}^n$, if $k = (\mathbf{a}, -\infty) \in \overline{\mathrm{L}}$ then $\mathrm{epi}_{\mathbb{R}}k = \mathbb{R}^{n+1}$ and $\mathrm{gr}_{\mathbb{R}}k = \varnothing$, and if $k = (\mathbf{a}, +\infty) \in \overline{\mathrm{L}}$ then $\mathrm{epi}_{\mathbb{R}}k = \varnothing$ and $\mathrm{gr}_{\mathbb{R}}k = \varnothing$.

Lemma 3.3. Let $\mathbf{a} \in \mathbb{R}^n$. For $\mathrm{S} \subseteq \mathrm{L}_{\mathbf{a}}$, let $\mathrm{B} = \{b \in \overline{\mathbb{R}} \mid (\mathbf{a}, b) \in \mathrm{S}\}$. Let $b' = \sup\limits_{\overline{\mathbb{R}}} \mathrm{B}$, and $k' = (\mathbf{a}, b')$. Then

(a) $k' = \sup\limits_{\mathrm{L}_{\mathbf{a}}} \mathrm{S}$, and

(b) $\mathrm{epi}_{\mathbb{R}}k' = \bigcap\{\mathrm{epi}_{\mathbb{R}}k \mid k \in \mathrm{S}\}$.

Proof: For (a), clearly, $\forall k \in \mathrm{S},\ k' \geq k$. Let $k'' = (\mathbf{a}, b'')$. Then

$$\forall k \in \mathrm{S},\ k'' \geq k \implies \forall b \in \mathrm{B},\ b'' \geq b \implies b'' \geq b' \implies k'' \geq k'.$$

Thus $k' = \sup \mathrm{S}$. For (b)

$$(\mathbf{x}, y) \in \mathrm{epi}_{\mathbb{R}}k' \iff y \geq \mathbf{x}k'^{\overline{\mu}} \iff \forall k \in \mathrm{S},\ y \geq \mathbf{x}k^{\overline{\mu}}$$
$$\iff \forall k \in \mathrm{S},\ (\mathbf{x}, y) \in \mathrm{epi}_{\mathbb{R}}k \iff (\mathbf{x}, y) \in \bigcap\{\mathrm{epi}_{\mathbb{R}}k \mid k \in \mathrm{S}\}. \quad \square$$

Note 3.2. The suprema of the empty subsets of various ordered sets are $\sup\limits_{\overline{\mathbb{R}}} \varnothing = -\infty$, $\sup\limits_{\mathrm{L}_{\mathbf{a}}} \varnothing = (\mathbf{a}, -\infty)$, $\sup\limits_{\overline{\mathrm{P}}} \varnothing = (y = -\infty)$, and $\sup\limits_{\overline{\mathrm{H}}} \varnothing = (\mathbf{x} \mapsto -\infty)$.

Definition 3.3. Let $(\mathcal{C}, \subseteq)$ be the set of all convex subsets of $\mathbb{R}^{n+1}$ ordered by inclusion, and define the function

$$\alpha : \mathcal{C} \to \mathcal{P}(\overline{\mathrm{L}})^{\mathbb{R}^n};\ \mathrm{C} \mapsto (\alpha_{\mathrm{C}} : \mathbf{a} \mapsto \{k \in \pi^{-1}\{\mathbf{a}\} \mid \mathrm{C} \subseteq \mathrm{epi}_{\mathbb{R}}k\}).$$

The image $(\mathbf{a}\alpha_{\mathrm{C}})^{\overline{\rho}}$ contains all hyperplanes lying below the set C having projected normal vector $\mathbf{a}$. We want to identify the largest such hyperplane (if it exists), via the corresponding element of $\mathrm{L}_{\mathbf{a}}$.

Lemma 3.4. For every $\mathrm{C} \in \mathcal{C}$, for each $\mathbf{a} \in \mathbb{R}^n$, $\sup_{\mathrm{L_a}} \mathbf{a}\alpha_{\mathrm{C}}$ exists.

Proof: Since $\mathbf{a}\alpha_{\mathrm{C}} \subset \mathrm{L_a}$, Lemma 3.3 applies. □

Example 3.1. For the convex set $\mathrm{B} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, the closed unit ball, we get the line with slope $m \in \mathbb{R}$ tangent to the function $y = -\sqrt{1-x^2}$ as $\sup(m\alpha_{\mathrm{B}})$. Figure 3.1 shows some of these lines.

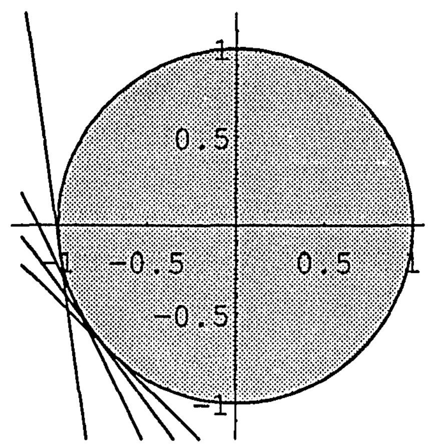

Figure 3.1. Some supporting lines of the unit ball in Example 3.1.

Definition 3.4. Define the functions

$\delta : \mathcal{C} \to \Gamma; \mathrm{C} \mapsto (\delta_{\mathrm{C}} : \mathbf{a} \mapsto \sup[\mathbf{a}\alpha_{\mathrm{C}}])$, and

$E : \Gamma \to \mathcal{C}; \sigma \mapsto \bigcap\{\mathrm{epi}_{\mathbb{R}}(\mathbf{a}\sigma) \mid \mathbf{a} \in \mathbb{R}^n\}$.

Note 3.3. We can also write $\sigma E = \mathrm{epi}_{\mathbb{R}}[\bigvee\{(\mathbf{a}\sigma)^{\overline{\mu}} \mid \mathbf{a} \in \mathbb{R}^n\}]$.

Example 3.2. Let n=1. Define the set $\mathrm{C} = \{(x,y) \in \mathbb{R}^2 \mid y \geq |x|, y > 0\}$, and the function

$$\sigma : \mathbb{R} \to \overline{\mathrm{L}}; \quad m\sigma = \begin{cases} (-1,0), & m = -1 \\ (1,0), & m = 1 \\ (m,-1), & m \in (-1,1) \\ (m,-\infty), & \text{otherwise.} \end{cases}$$

Then σE is the epigraph of the function $xf = |x|$, which is the closure of the set C, and δ_{C} is the function

$$m\delta_{\mathrm{C}} = \begin{cases} (m, 0), & m \in [-1, 1] \\ (m, -\infty), & \text{otherwise.} \end{cases}$$

We thus get $\delta_{\sigma E} = \delta_{\mathrm{C}}$ and $\delta_{\mathrm{C}} E = \sigma E$. See Figure 3.2.

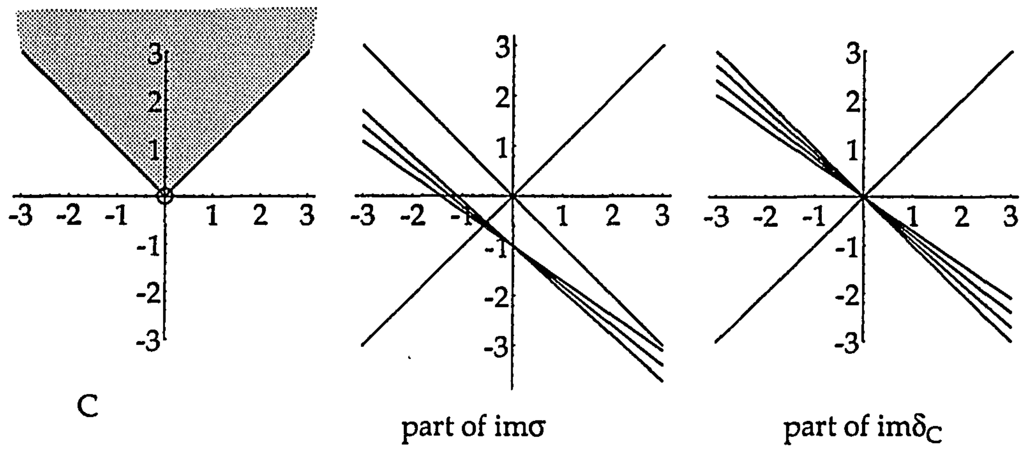

Figure 3.2. The set C, and some values of σ and δ_{C} from Ex. 3.2.

Lemma 3.6. For $\mathrm{C} \in \mathcal{C}$, for every $\mathbf{a} \in \mathbb{R}^n$, $\mathrm{C} \subseteq \mathrm{epi}_{\mathbb{R}}(\mathbf{a}\delta_{\mathrm{C}})$. Thus if for some $\mathbf{a}$, $\mathbf{a}\delta_{\mathrm{C}} = (\mathbf{a}, +\infty)$, then $\mathrm{C} = \varnothing$, so that $(\mathbb{R}^n)\delta_{\mathrm{C}} = \mathbb{R}^n \times \{+\infty\}$.

Proof: Let $\mathbf{a} \in \mathbb{R}^n$, $\mathrm{C} \in \mathcal{C}$. Lemma 3.3(b) gives

$$\mathrm{C} \subseteq \bigcap \{\mathrm{epi}_{\mathbb{R}} k \mid k \in \mathbf{a}\alpha_{\mathrm{C}}\} = \mathrm{epi}_{\mathbb{R}}(\sup \mathbf{a}\alpha_{\mathrm{C}}) = \mathrm{epi}_{\mathbb{R}}(\mathbf{a}\delta_{\mathrm{C}}). \quad \square$$

Now for the main result of this chapter.

Theorem 3.1. The pair (δ, E) is a Galois connection from $(\mathcal{C}, \subseteq)$ to $(\Gamma, \leq)$.

Proof: We need to prove the extensivity of δE and $E\delta$ and that E and δ are antitone.

1. To prove E is antitone, use Lemma 3.3 with $\mathrm{S} = \{\sigma_1, \sigma_2\} \subset \Gamma$. Then $\sigma_1 \leq \sigma_2 \Rightarrow \forall \mathrm{a} \in \mathbb{R}^n$, $\mathrm{a}\sigma_1 \leq \mathrm{a}\sigma_2 \Rightarrow \forall \mathrm{a}$, $\mathrm{epi}_{\mathbb{R}}(\mathrm{a}\sigma_2) \subseteq \mathrm{epi}_{\mathbb{R}}(\mathrm{a}\sigma_1) \Rightarrow \sigma_2 E \subseteq \sigma_1 E$.

2. To prove δ is antitone, let $\mathbf{a} \in \mathbb{R}^n$, $\mathrm{C}_1, \mathrm{C}_2 \in \mathcal{C}$ with $\mathrm{C}_1 \subseteq \mathrm{C}_2$. Then

$$k \in \mathbf{a}\alpha_{\mathrm{C}_2} \implies \mathrm{C}_2 \subseteq \mathrm{epi}_{\mathbb{R}} k \implies \mathrm{C}_1 \subseteq \mathrm{epi}_{\mathbb{R}} k \implies k \in \mathbf{a}\alpha_{\mathrm{C}_1}.$$

So $\mathrm{a}\alpha_{\mathrm{C}_2} \subseteq \mathrm{a}\alpha_{\mathrm{C}_1}$, and thus $\mathrm{a}\delta_{\mathrm{C}_2} = \sup(\mathrm{a}\alpha_{\mathrm{C}_2}) \leq \sup(\mathrm{a}\alpha_{\mathrm{C}_1}) = \mathrm{a}\delta_{\mathrm{C}_1}$. Therefore, $\delta_{\mathrm{C}_2} \leq \delta_{\mathrm{C}_1}$.

3. The functions $\delta E : \mathcal{C} \rightarrow \mathcal{C}$ and $E\delta : \Gamma \rightarrow \Gamma$ are extensive:

a. Let $\mathrm{C} \in \mathcal{C}$. By Lemma 3.6, for each $\mathrm{a} \in \mathbb{R}^n$, $\mathrm{C} \subseteq \mathrm{epi}_{\mathbb{R}}(\mathrm{a}\delta_{\mathrm{C}})$. Therefore

$$\mathrm{C} \subseteq \bigcap\{\mathrm{epi}_{\mathbb{R}}(\mathrm{a}\delta_{\mathrm{C}}) \mid \mathrm{a} \in \mathbb{R}^n\} = \delta_{\mathrm{C}}E.$$

b. Let $\sigma \in \Gamma$. For $\mathrm{a} \in \mathbb{R}^n$,

$$\sigma E \subseteq \mathrm{epi}_{\mathbb{R}}(\mathrm{a}\sigma) \implies \mathrm{a}\sigma \in \mathrm{a}\alpha_{\sigma E} \implies \mathrm{a}\sigma \leq \sup(\mathrm{a}\alpha_{\sigma E}) = \mathrm{a}\delta_{\sigma E}.$$

Thus $\delta_{\sigma E} \geq \sigma$, as required. $\square$

The resulting Galois correspondence is between maximal sections of the projection π and maximal convex sets. These will be subgradient sets and finitary epigraphs of closed convex functions.

CHAPTER 4: SUBGRADIENTS IN EUCLIDEAN SPACES

At this point we present traditional subgradients in an algebraic context and relate them to the functions E and δ from the last chapter.

Definition 4.1. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be real-convex. The *effective domain* of f is the set $f^{-1}(\mathbb{R})$. Recall $\mathrm{epi}_{\mathbb{R}} f \in \mathcal{C}$, and $(\mathbb{R}^n)\delta_{\mathrm{epi}_{\mathbb{R}} f} \subseteq \overline{\mathrm{L}}$. Define the functions
$f^{\partial} : \mathbb{R}^n \to \overline{\mathrm{H}};\ \mathrm{a} \mapsto (\mathrm{a}\delta_{\mathrm{epi}_{\mathbb{R}} f})^{\overline{\mu}}$, and
$\tilde{\partial} f : \mathbb{R}^n \to \mathcal{P}(\mathrm{H});\ \mathbf{x} \mapsto (\mathrm{im} f^{\partial}) \cap \{k \in \mathrm{H} \mid \mathbf{x}k = \mathbf{x}(\mathrm{cl} f)\}$.

Claim 4.1. For a convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, an image $\tilde{\partial} f(\mathbf{x})$ is an element of $(\mathrm{H},\mathrm{I}^\circ)\mathbf{S}_{\varnothing}$, the set of (possibly empty) submodes of the mode $(\mathrm{H},\mathrm{I}^\circ)$.
Proof: Let $\mathbf{x}_0 \in \mathbb{R}^n$. If $\tilde{\partial} f(\mathbf{x}_0)$ is empty, it is vacuously a submode. So assume it is nonempty. Let $p \in \mathrm{I}^\circ$ and $h, k \in \tilde{\partial} f(\mathbf{x}_0)$, not necessarily different. Now

$$\mathbf{x}_0 h = \mathbf{x}_0(\mathrm{cl} f) = \mathbf{x}_0 k \implies \mathbf{x}_0(hk\underline{p}) = \mathbf{x}_0 h \mathbf{x}_0 k \underline{p} = \mathbf{x}_0(\mathrm{cl} f)\mathbf{x}_0(\mathrm{cl} f)\underline{p} = \mathbf{x}_0(\mathrm{cl} f),$$

and

$$h, k \le f \implies \forall \mathbf{x} \in f^{-1}(\mathbb{R}),\ \mathbf{x}(hk\underline{p}) = \mathbf{x}h\mathbf{x}k\underline{p} \le \mathbf{x}f\mathbf{x}f\underline{p} = \mathbf{x}f \implies hk\underline{p} \in \tilde{\partial} f(\mathbf{x}_0).$$

Therefore, $\tilde{\partial} f(\mathbf{x}_0) \in (\mathrm{H},\mathrm{I}^\circ)\mathbf{S}_{\varnothing}$. □

Definition 4.2. For a convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, define the *subgradient* of f to be the function

$$\partial f : \mathbb{R}^n \to (\mathrm{H},\mathrm{I}^\circ)\mathbf{S}_{\varnothing};\ \mathbf{x} \mapsto \tilde{\partial} f(\mathbf{x}).$$

We call $\partial f(\mathbf{x})$ the *subgradient of f at* $\mathbf{x}$. We will also refer to $\delta_{\mathrm{epi}_{\mathbb{R}} f}$ as the subgradient of f (when speaking of the Galois connection from the last chapter). Note that the arguments of $\delta_{\mathrm{epi}_{\mathbb{R}} f}$ and f^{∂} are projected normal vectors, while the arguments of ∂f are the same vectors as the arguments of f. See Example 4.1.

Example 4.1. Let $f = \chi_{(0,1)} : \mathbb{R} \to \mathbb{R}^{+\infty}$. Then
$\partial f(0) = \{(x \mapsto mx) \mid m \in (-\infty, 0]\}$, $\partial f(-1) = \varnothing$, and
$\forall x_0 \in (0,1)$, $\partial f(x_0) = \{x \mapsto 0\}$, while
$0 f^{\partial} = (x \mapsto 0)$, $1 f^{\partial} = (x \mapsto x - 1)$, and $\forall a \le 0$, $a f^{\partial} = (x \mapsto ax)$.

Lemma 4.1. Let $f : \mathbb{R}^n \to \mathbb{R}^{+\infty}$ be a proper convex function. For every $\mathbf{x_0}$ in the interior of $f^{-1}(\mathbb{R})$, $\partial f(\mathbf{x_0}) \neq \varnothing$. ([BP], Cor. 2.1, p.105.)

Note 4.1. For any convex function $f : \mathbb{R}^n \to \mathbb{R}^{+\infty}$, if $\mathbf{x_0}$ is not in the closure of $f^{-1}(\mathbb{R})$, then $\partial f(\mathbf{x_0}) = \varnothing$, since $\mathbf{x_0}(\mathrm{cl} f) = +\infty$ and no elements of H take on this value.

Fact 4.1. A proper convex function $f : \mathbb{R}^n \to \mathbb{R}^{+\infty}$ has minimum at $\mathbf{x}$ if and only if there exists a constant function $k \equiv c$ such that $k \in \partial f(\mathbf{x})$ and $xk = xf$. A constant function is an affine function with projected normal vector $\mathbf{a} = 0$. Thus the existence of a constant function $k \in \partial f(\mathbf{x})$ corresponds to Rockafellar's notation "$0 \in \partial f(\mathbf{x})$." (See [R2], Prop. 5A.)

Example 4.2. For $xf = -\sqrt{x}$, $x \geq 0$, $\partial f(0) = \varnothing$, since the supporting line at $(0,0)$ is vertical. (See Figure 4.1.) In spite of this, $f_e : \mathbb{R} \to \mathbb{R}^{+\infty}$ is closed. See Example 1.2.

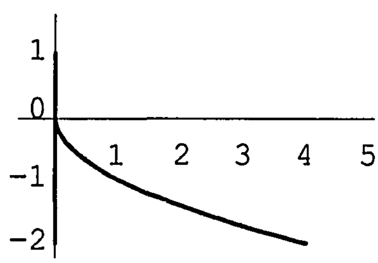

Figure 4.1. The function $-\sqrt{x}$ with vertical supporting line.

Lemma 4.2. If a convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper and closed, then $f = \bigvee[f^{\partial}(\mathbb{R}^n)]$.

Proof: By Facts 1.6 and 1.8, f is the supremum of a set $\mathrm{K}' \subseteq \mathrm{H}$ of affine functions. Now, let $k \in \mathrm{K}'$, and $\mathbf{a} = (k^{\mu^{-1}})\pi$. Then $\mathbf{a}\delta_{\mathrm{epi}_{\mathbb{R}} f} \in \mathrm{L}$ and

$$f \geq (\mathbf{a}\delta_{\mathrm{epi}_{\mathbb{R}} f})^{\mu} \geq k \implies f \geq \bigvee f^{\partial}(\mathbb{R}^n) \geq \bigvee \mathrm{K}' = f \implies f = \bigvee f^{\partial}(\mathbb{R}^n). \quad \square$$

Theorem 4.1. The closed elements of the Galois connection $\delta : \mathcal{C} \rightleftarrows \Gamma : E$ are exactly the finitary epigraphs and subgradients of closed convex functions

$f : \mathbb{R}^n \to \overline{\mathbb{R}}$. In other words,

$\delta_{\sigma E} = \sigma \iff$ there is a closed convex $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ with $\sigma = \delta_{\mathrm{epi}_{\mathbb{R}} f}$, &

$\delta_{\mathrm{C}} E = \mathrm{C} \iff$ there is a closed convex $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ with $\mathrm{C} = \mathrm{epi}_{\mathbb{R}} f$.

Proof: Let $\sigma \in \Gamma$, and $\mathrm{C} \in \mathcal{C}$. Now, if $\sigma \equiv -\infty$, then $f_\sigma \equiv -\infty$ gives $\sigma E = \mathbb{R}^{n+1} = \mathrm{epi}_{\mathbb{R}}(f_\sigma)$. If $\sigma \equiv +\infty$, then $f_\sigma \equiv +\infty$ gives $\sigma E = \varnothing = \mathrm{epi}_{\mathbb{R}}(f_\sigma)$. So assume there is some $\mathbf{a} \in \mathbb{R}^n$ with $\mathbf{a}\sigma \in \mathbb{R}$. Then σE is an intersection of at least one closed, cylindrically-closed, cylindrically-bounded-below, convex upset, so by Facts 1.4 and 1.7, it is the finitary epigraph of a closed convex function, $f = f_\sigma$. Thus we always have $\sigma E = \mathrm{epi}_{\mathbb{R}} f_\sigma$, and since $\delta_{\mathrm{C}} \in \Gamma$ and $\sigma E \in \mathcal{C}$,

$$\delta_{\mathrm{C}} E = \mathrm{C} \implies \mathrm{C} = \mathrm{epi}_{\mathbb{R}} f_{\delta_{\mathrm{C}}}, \text{ and}$$
$$\delta_{\sigma E} = \sigma \implies \sigma = \delta_{\mathrm{epi}_{\mathbb{R}} f_\sigma}.$$

For the converse, we must show $\delta_{\mathrm{epi}_{\mathbb{R}} f} E \subseteq \mathrm{epi}_{\mathbb{R}} f$ and $\delta_{\delta_{\mathrm{epi}_{\mathbb{R}} f} E} \leq \delta_{\mathrm{epi}_{\mathbb{R}} f}$, the reverse inclusion and inequality holding by Theorem 3.1. Lemma 4.2 gives

$$\mathrm{epi}_{\mathbb{R}} f = \mathrm{epi}_{\mathbb{R}}\left(\bigvee f^{\partial}(\mathbb{R}^n)\right) = \bigcap\{\mathrm{epi}_{\mathbb{R}} k \mid k \in f^{\partial}(\mathbb{R}^n)\} = \delta_{\mathrm{epi}_{\mathbb{R}} f} E,$$

and for $\mathbf{a} \in \mathbb{R}^n$,

$$\begin{aligned}
\mathbf{a}\delta_{\delta_{\mathrm{epi}_{\mathbb{R}} f} E} &= \sup\left\{k \in \pi^{-1}\{\mathbf{a}\} \mid \mathrm{epi}_{\mathbb{R}} k \supseteq \delta_{\mathrm{epi}_{\mathbb{R}} f} E\right\} \\
&= \sup\left\{k \in \pi^{-1}\{\mathbf{a}\} \mid \mathrm{epi}_{\mathbb{R}} k \supseteq \left(\cap\{\mathrm{epi}_{\mathbb{R}}(\mathbf{c}\delta_{\mathrm{epi}_{\mathbb{R}} f}) \mid \mathbf{c} \in \mathbb{R}^n\}\right)\right\} \\
&= \sup\left\{k \in \pi^{-1}\{\mathbf{a}\} \mid \mathrm{epi}_{\mathbb{R}} k \supseteq \mathrm{epi}_{\mathbb{R}}(\mathbf{a}\delta_{\mathrm{epi}_{\mathbb{R}} f})\right\} \\
&= \mathbf{a}\delta_{\mathrm{epi}_{\mathbb{R}} f}. \quad \square
\end{aligned}$$

It seems that the identification of "closed" convex functions is the key to getting subgradients defined at "almost every" point in the domain.

CHAPTER 5: SUBGRADIENTS IN SEMILATTICE THEORY

In this chapter we present a basic example to motivate and clarify the definitions to be used in Chapter 6. Since semilattices and distributive lattices are about the simplest nontrivial examples of modes and modals, with only one mode operation, we consider functions from $\wedge$-semilattices into distributive lattices. What makes this example especially simple is that there is an additional order $\leq_\wedge$ on both the domain and codomain, and the two orders $\leq_\wedge$, and $\leq_\vee$ in the codomain coincide.

Lemma 5.1. In a meet semilattice $(\mathrm{S}, \wedge)$, if $\mathrm{W} \subseteq \mathrm{S}$ is a chain and $a, b \in \mathrm{S}$, then the sets $\{t \in \mathrm{W} \mid t = t \wedge a \wedge b\}$ and $\{x \wedge y \mid x = x \wedge a \in \mathrm{W},\ y = y \wedge b \in \mathrm{W}\}$ are equal.

Proof: If $t = t \wedge a \wedge b \in \mathrm{W}$ then $t = t \wedge a = t \wedge b$, and we know $t = t \wedge t$, so let $x = y = t \in \mathrm{W}$. Then t is of the form $x \wedge y$ with $x = x \wedge a$ and $y = y \wedge b$ both in W. Conversely,

$$
\begin{aligned}
&x = x \wedge a \in \mathrm{W},\ y = y \wedge b \in \mathrm{W} \implies \\
&\qquad x \wedge y = (x \wedge a) \wedge (y \wedge b) = (x \wedge y) \wedge (a \wedge b) \in \mathrm{W},
\end{aligned}
$$

so let $t = x \wedge y$. Then $t = t \wedge a \wedge b$ and is in W. $\square$

Lemma 5.2. A function from a meet semilattice to a distributive lattice is convex if and only if it is monotone.

Proof: First recall that the orders $\leq_\wedge$ and $\leq_\vee$ coincide in a distributive lattice. Let S be a meet semilattice and D a distributive lattice. Let $g : \mathrm{S} \to \mathrm{D}$ be $\wedge$-convex, and $x, y \in \mathrm{S}$. Then we have

$$x \leq y \iff x = x \wedge y \implies xg = (x \wedge y)g \leq xg \wedge yg \leq yg,$$

so g is monotone. Conversely, let g be monotone, and $x, y \in \mathrm{S}$. Since $x \wedge y \leq x$ and $x \wedge y \leq y$, we have $(x \wedge y)g \leq xg$ and $(x \wedge y)g \leq yg$. Thus $(x \wedge y)g \leq xg \wedge yg$. $\square$

Now fix a semilattice $(\mathrm{S}, \wedge)$ and a completely distributive lattice $(\mathrm{D}, \vee, \wedge)$. Let $f : \mathrm{S} \to \mathrm{D}$ be a fixed convex function. We want to define subgradients of f as sets of homomorphisms from S to D. We first define a function h as a candidate for a subgradient homomorphism of f. For a fixed $c \in \mathrm{S}$, let W be a maximal chain in S containing c. For $b \in \mathrm{S}$, let $\mathrm{W}_b = \{x \mid x \wedge b = x \in \mathrm{W}\}$ and define

$$h : \mathrm{S} \to \mathrm{D};\ b \mapsto \bigvee\{xf \mid x \in \mathrm{W}_b\}.$$

Definition 5.1. Suppose S has no infinite chains. Let V be a chain in S. Define $\max \mathrm{V}$ to be the element $x \in \mathrm{V}$ such that for every $y \in \mathrm{V}$, $y \leq x$. (Since V is finite, max V exists.)

Fact 5.1. If S has no infinite chains, we can define a function g to agree with f on the maximal chain W, and then for every $b \notin \mathrm{W}$, find the largest element x of W less than b. Defining $bg = xg$ will give a $\wedge$-homomorphism

$$g : \mathrm{S} \to \mathrm{D};\ b \mapsto (\max\{x \mid x \in \mathrm{W}, x \leq b\})f.$$

Actually, g is the function h defined above, since

$$\begin{aligned} bg &= (\max\{x \mid x \in \mathrm{W}, x \leq b\})f \\ &= \max\{xf \mid x \in \mathrm{W}, x \leq b\} && [f \text{ monotone}] \\ &= \max\{xf \mid x = x \wedge b \in \mathrm{W}\} \\ &= \bigvee\{xf \mid x \in \mathrm{W}_b\} \\ &= bh. \end{aligned}$$

Lemma 5.3. The function h defined above is a semilattice homomorphism.
Proof: Let $a, b \in \mathrm{S}$. Then

$$\begin{aligned} ah \wedge bh &= (\bigvee\{xf \mid x = x \wedge a \in \mathrm{W}\}) \wedge (\bigvee\{yf \mid y = y \wedge b \in \mathrm{W}\}) \\ &= (\bigvee\{xf \wedge yf \mid x = x \wedge a,\ y = y \wedge b \in \mathrm{W}\}) && [\text{(CD) in D}] \\ &= (\bigvee\{(x \wedge y)f \mid x = x \wedge a,\ y = y \wedge b \in \mathrm{W}\}) && [f \text{ monotone}] \\ &= (\bigvee\{tf \mid t \wedge a \wedge b = t \in \mathrm{W}\}) && [\text{Lemma 5.1}] \\ &= (a \wedge b)h. \quad \square \end{aligned}$$

Lemma 5.4. The functions h and f and the element $c \in \mathrm{S}$ above satisfy the conditions $h \leq f$ and $ch = cf$.

Proof: Let $b \in \mathrm{S}$. Now

$$x \in \mathrm{W}_b \implies x \wedge b = x \implies x \leq b \implies xf \leq bf.$$

So $bh \leq bf$. Also,

$$b \in \mathrm{W} \implies b \wedge b = b \in \mathrm{W}_b \implies bh \geq bf.$$

So, for $b \in \mathrm{W}$, $bh = bf$. In particular, since $c \in \mathrm{W}$, $ch = cf$. □

Definition 5.2. Define the set ${}_f\mathrm{H}_c := \{k \in \mathrm{Hom}(\mathrm{S}, \mathrm{D}) \mid k \leq f,\ ck = cf\}$.

Proposition 5.1. For every $c \in \mathrm{S}$, ${}_f\mathrm{H}_c$ has a maximal element.

Proof: We want to use Zorn's Lemma to conclude ${}_f\mathrm{H}_c$ has a maximal element. By the above construction, ${}_f\mathrm{H}_c$ is nonempty. Now suppose we have a chain $\Theta = \{h_\alpha \mid \alpha \in \mathcal{A}\}$ of homomorphisms in ${}_f\mathrm{H}_c$. Then

$$\alpha, \beta \in \mathcal{A} \implies (h_\alpha \vee h_\beta = h_\alpha \text{ or } h_\alpha \vee h_\beta = h_\beta) \implies h_\alpha \vee h_\beta \in \Theta.$$

Let $h = \bigvee\{h_\alpha \mid \alpha \in \mathcal{A}\}$. We need h to be a homomorphism. Then we will have $h \in {}_f\mathrm{H}_c$, since

$$(\forall k \in \Theta,\ ck = cf \text{ and } k \leq f) \implies (ch = cf, \text{ and } h \leq f) \implies h \in {}_f\mathrm{H}_c.$$

Let $x, y \in \mathrm{S}$. Note first that since every $k \in \Theta$ is convex, h is convex (Prop. 2.5). So $(x \wedge y)h \leq xh \wedge yh$, and also,

$$\begin{aligned}
xh \wedge yh &= \left(\bigvee\{xh_\alpha \mid \alpha \in \mathcal{A}\}\right) \wedge \left(\bigvee\{yh_\beta \mid \beta \in \mathcal{A}\}\right) \\
&= \bigvee\{xh_\alpha \wedge yh_\beta \mid \alpha,\ \beta \in \mathcal{A}\} && \text{[(CD) in D]} \\
&\leq \bigvee\{xk \wedge yk \mid k = h_\alpha \vee h_\beta, \alpha, \beta \in \mathcal{A}\} && [\wedge \text{ monotone}] \\
&= \bigvee\{xk \wedge yk \mid k \in \Theta\} \\
&= \bigvee\{(x \wedge y)k \mid k \in \Theta\} && \text{[homomorphism]} \\
&= (x \wedge y)h.
\end{aligned}$$

Therefore $h \in \Theta$, showing chains in ${}_f\mathrm{H}_c$ have upper bounds in ${}_f\mathrm{H}_c$. Thus by Zorn's Lemma, ${}_f\mathrm{H}_c$ has maximal elements. □

Note 5.3. If we let ${}_f\mathrm{H}_c^*$ be the set of all maximal elements of ${}_f\mathrm{H}_c$, we could define ${}_f\mathrm{H}_c^*$ to be the subgradient of f at c, but this does not always give a submode of $\mathrm{Hom}(\mathrm{S}, \mathrm{D})$, as we have in the Euclidean case. So, we use a different notion of maximal, that at only one point.

Definition 5.3. Define $\partial f(c) := {}_f\mathrm{H}_c$, the set of those homomorphisms $h \leq f$ which take the maximal value $ch = cf$ at c, to be the *subgradient of f at c.*

Proposition 5.2. The set ${}_f\mathrm{H}_c \subseteq \mathrm{Hom(S,D)}$ is convex, so $\partial f : \mathrm{S} \rightarrow (\mathrm{Hom(S,D)})\mathbf{S}$.
Proof: Let $h, k \in {}_f\mathrm{H}_c$. Now,

$$h \leq f \implies h \wedge k \leq f, \text{ and}$$

$ch = ck = cf \implies c(h \wedge k) = ch \wedge ck = cf \wedge cf = cf \implies c(h \wedge k) = cf.$
So $h \wedge k \in {}_f\mathrm{H}_c$. Therefore ${}_f\mathrm{H}_c = \partial f(c)$ is a submode of $\mathrm{Hom(S, D)}$. □

Theorem 5.1. A convex function from a meet semilattice to a completely distributive lattice is a join of semilattice homomorphisms.
Proof: Let $f : (\mathrm{S}, \wedge) \rightarrow (\mathrm{D}, \vee, \wedge)$ be convex. We have shown

$$\forall c \in \mathrm{S}, \exists h \in \partial f(c) \bullet ch = cf.$$

Thus $\bigvee \partial f(\mathrm{S}) \geq f$, and since $\bigvee \partial f(\mathrm{S}) \leq f$, the theorem is proved. □

Example 5.1. A meet semilattice S and a completely distributive lattice D are shown by their Hasse diagrams in Figure 5.1. Also shown is the position

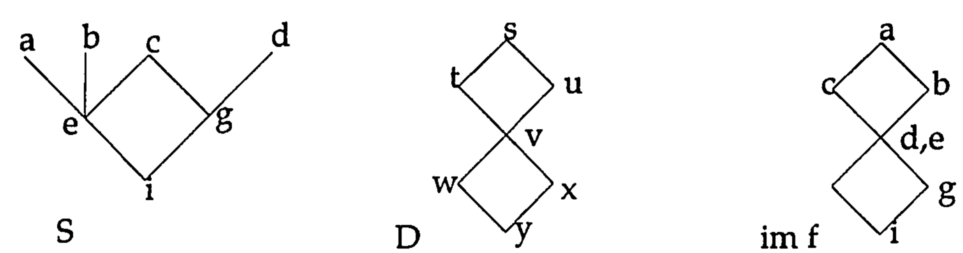

Figure 5.1. Hasse diagrams of $(\mathrm{S}, \wedge)$, $(\mathrm{D}, \vee, \wedge)$, and im$f$.

in D of the images of the elements of S under the function $f : \mathrm{S} \rightarrow \mathrm{D}$ defined in Table 5.1 below. Note that f is convex and that there is one subsemilattice that is not a chain, the one consisting of $\{b, c, e, i\}$, on which f is a homomorphism. We will find that there is a maximal homomorphism that agrees with f only on a proper subset of this set. (Compare $xf = |x|$ in the reals.) The process of finding the subgradient at a point is shown is the steps below. There are surprisingly many homomorphisms that are equal to f at at least one point. In fact, in this example, all homomorphisms below f satisfy $ih = if$, so we get that all homomorphisms below f are subgradients. Now because every homomorphism listed is a subgradient of f at i, the set ${}_f\mathrm{H}_i$ of all these homomorphisms forms a semilattice. In Table 5.1, some homomorphisms have been partially defined on choice subsets of S. The subscripts identify where we originally decide to make them agree with f. Tables 5.2 and 5.3 show the

Table 5.1. Some homomorphisms partially defined to agree with f.

$\setminus$ *fcn* / *arg* $\setminus$	f	h_a	h_b	h_{bc}	h_c	h_{cg}	h_d
a	s	s					
b	u		u	u			
c	t			t	t	t	
d	v						v
e	v	v	v	v	v		
g	x					x	x
i	y	y	y	y	y	y	y

progression from Table 5.1 of finding maximal values for each homomorphism. Note that every homomorphism started in Table 5.1 is to be a subgradient of f at either e or g. The images of the elements in S lying above e or g are then listed in Table 5.2. Table 5.3 is the completion of this process, and also includes an additional homomorphism h_i which agrees with f only at i but is a maximal element of the set of all subgradients in Hom(S, D). Figure 5.2 shows the images of each homomorphism in Table 5.3 while Figure 5.3 shows the other generators of the semilattice of subgradients of f at i. The simplest subgradients are shown in Figure 5.4. Their proper place in the semilattice order is easy to see.

Table 5.2. Homomorphisms from Table 5.1 defined above e or g.

$arg \diagdown fcn$	f	h_a	h_b	h_{bc}	h_c	h_{cg}	h_d
a	s	s	t	v	u		
b	u	v	u	u	v		
c	t	v	v	t	t	t	x
d	v					x	v
e	v	v	v	v	v		
g	x					x	x
i	y	y	y	y	y	y	y

Table 5.3. Completion of Table 5.1, and one more homomorphism.

$arg \diagdown fcn$	f	h_a	h_b	h_{bc}	h_c	h_{cg}	h_d	h_i
a	s	s	t	v	u	w	y	x
b	u	v	u	u	v	w	y	x
c	t	v	v	t	t	t	x	x
d	v	y	y	y	y	x	v	w
e	v	v	v	v	v	w	y	x
g	x	y	y	y	y	x	x	y
i	y	y	y	y	y	y	y	y

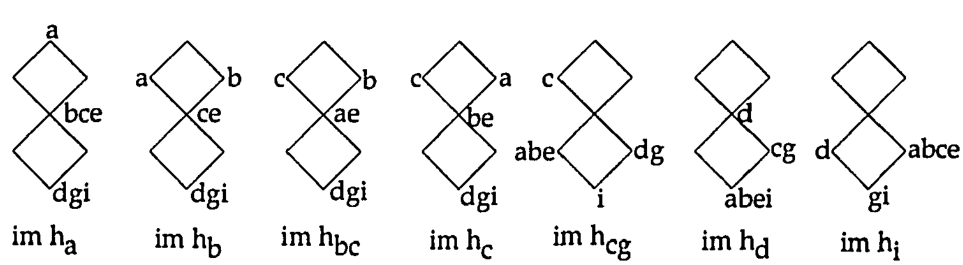

Figure 5.2. Images of the maximal homomorphisms.

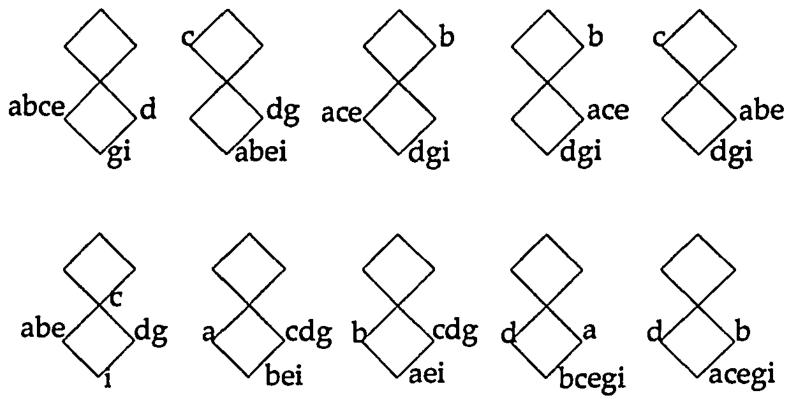

Figure 5.3. The other generating homomorphisms.

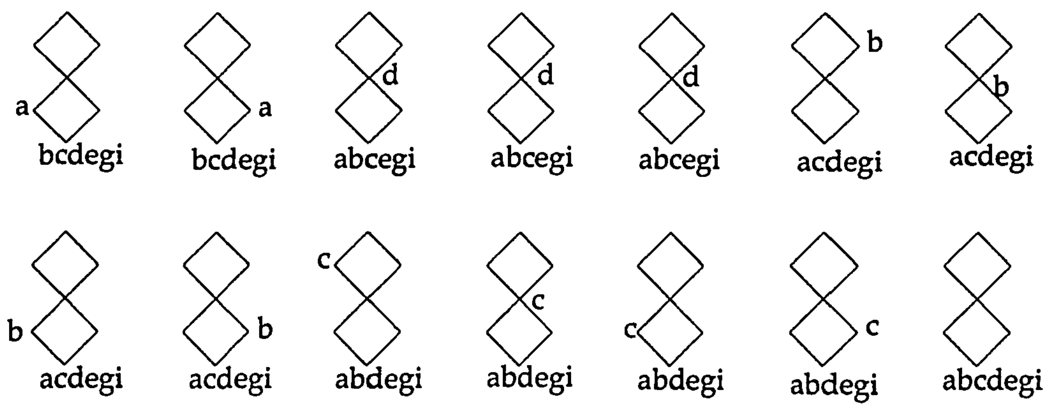

Figure 5.4. The simple homomorphisms.

There are sixteen homomorphisms that together form a subsemilattice isomorphic to the semilattice reduct of the lattice $M_2 \times M_2$. These satisfy the conditions $ah \in \{s, t, u, v\}$, $bh = ch = eh \in \{v, w, x, y\}$, and $dh = gh = ih = y$. This subsemilattice is shown in Figure 5.5. Figure 5.6 is the full semilattice ${}_f\mathrm{H}_i$ of subgradients at i except that, to avoid confusion, the homomorphisms in Figures 5.4 and 5.5 are eliminated. Their position relative to any given homomorphism in the figure is included when deemed enlightening.

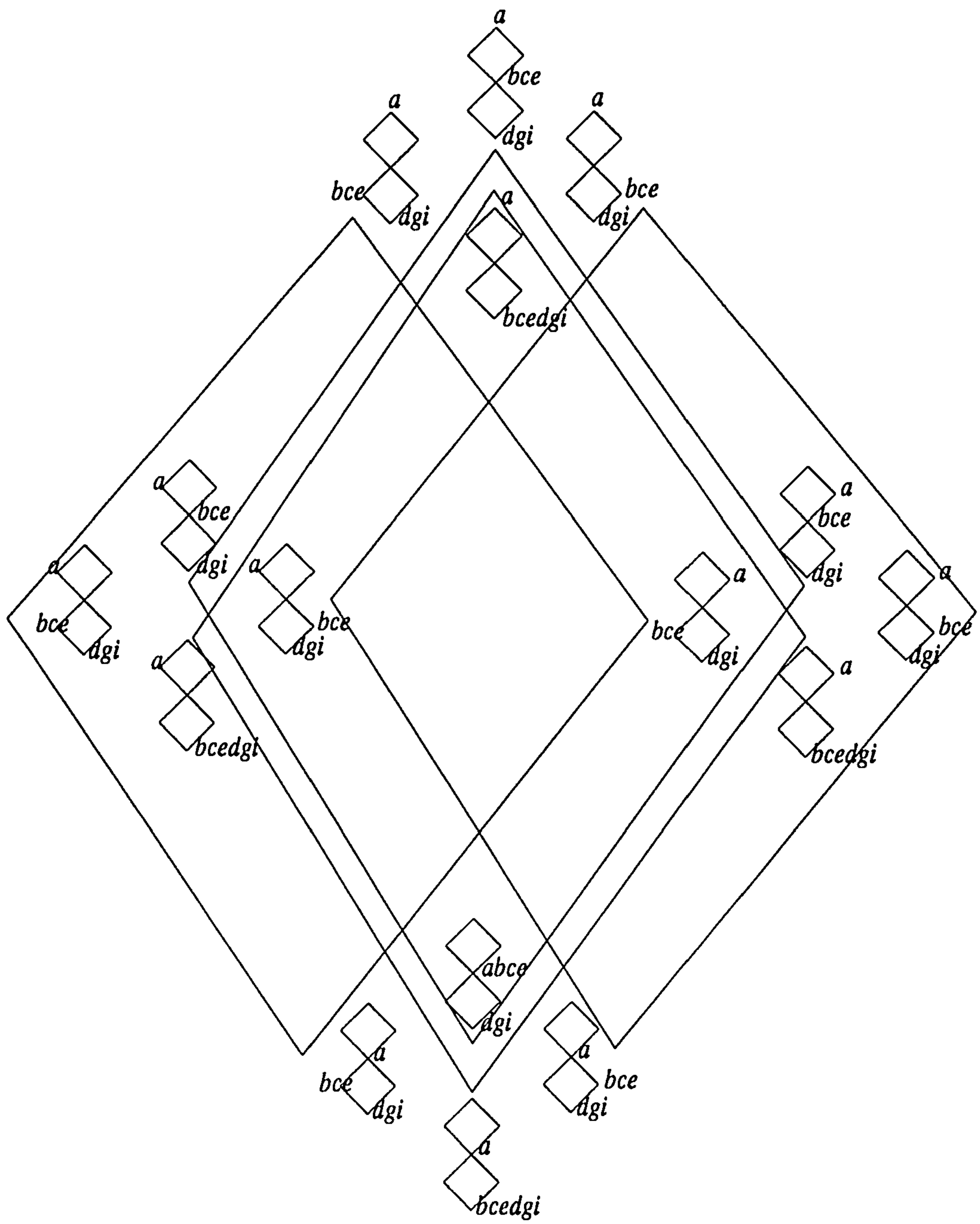

Figure 5.5. The $M_2 \times M_2$ semilattice.

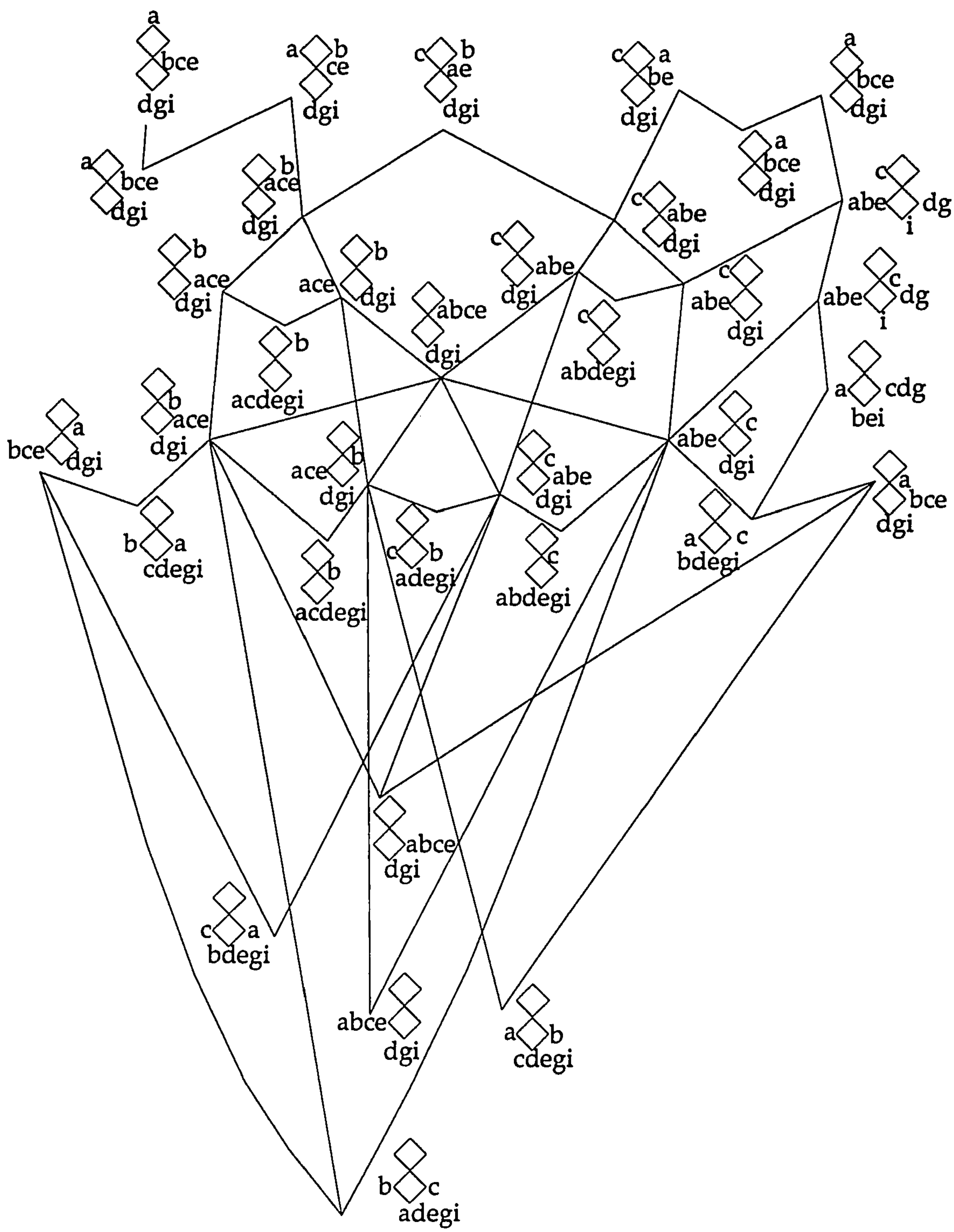

Figure 5.6. The subgradient semilattice (H_i).

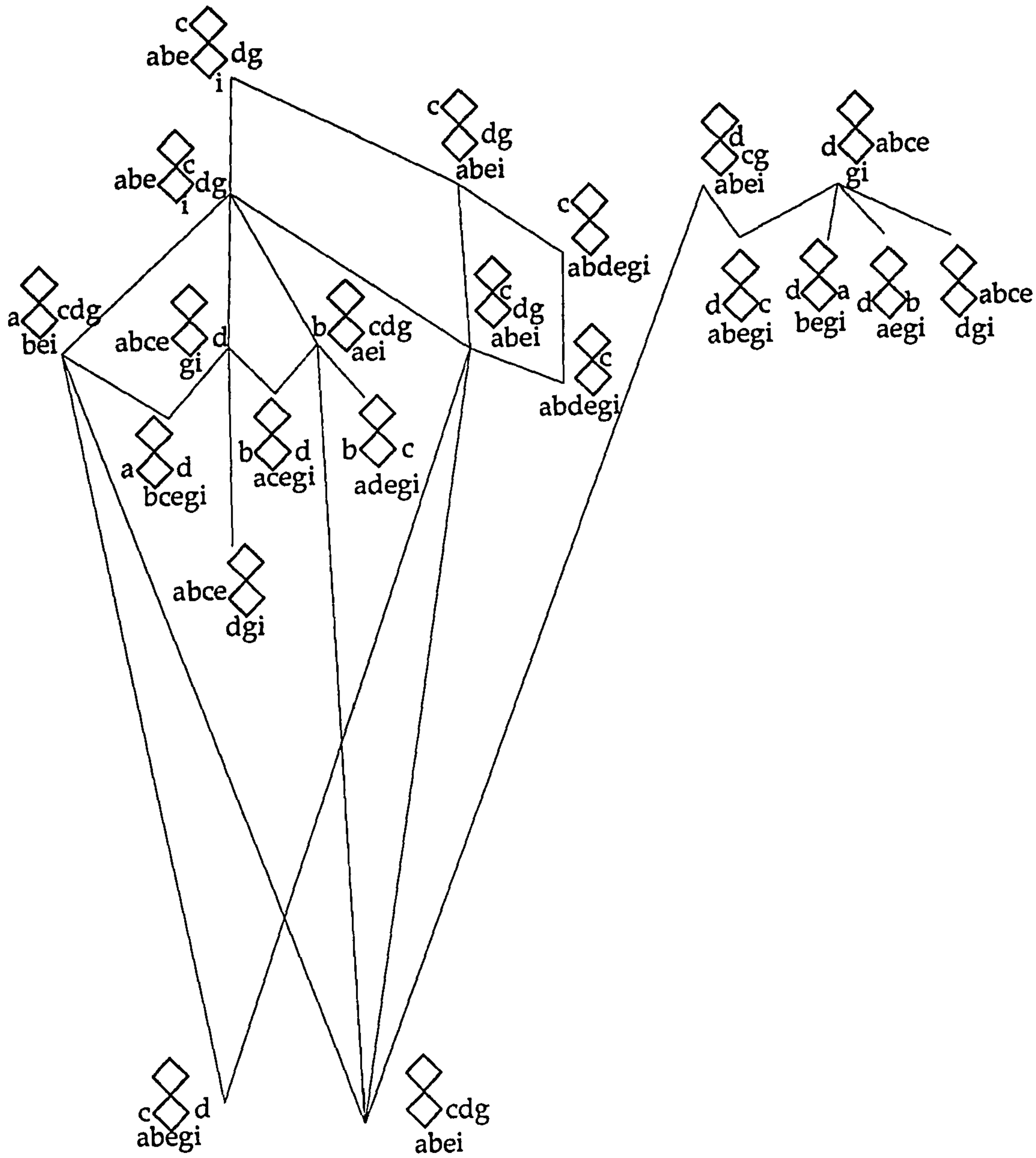

Figure 5.6. (Continued.)

CHAPTER 6: SUBGRADIENTS IN MODAL THEORY

We finally have the foundations set to define subgradients of modal-theoretically convex functions.

Proposition 6.1. Let (M,Ω) be a mode and $(\mathrm{D},\vee,\Omega)$ a compatible completely distributive modal. Then the modal $(\mathrm{Conv}(\mathrm{M},\mathrm{D}),\vee,\Omega)$ of convex functions is completely distributive also.

Proof: Let $\mathrm{C} = \mathrm{Conv}(\mathrm{M},\mathrm{D})$ and $\mathrm{F} \subseteq \mathrm{C}$. First we show that $(\mathrm{C},\vee)$ is a complete semilattice. Let $f = \bigvee \mathrm{F}$. This exists as a function since D is complete. We need $f \in \mathrm{C}$. Let $\omega \in \Omega$, $x_1,\ldots,x_{\omega\tau} \in \mathrm{M}$ be arbitrary. Then

$$\begin{aligned}
(x_1 \ldots x_{\omega\tau}\omega)f &= \bigvee\{(x_1 \ldots x_{\omega\tau}\omega)f' \mid f' \in \mathrm{F}\} \\
&\leq \bigvee\{(x_1 f' \ldots x_{\omega\tau} f')\omega\} && [\omega \text{ monotone}] \\
&= [\bigvee(x_1 f') \ldots \bigvee(x_{\omega\tau} f')]\omega && [\text{(CD) in D}] \\
&= [x_1(\bigvee f') \ldots x_{\omega\tau}(\bigvee f')]\omega \\
&= x_1 f \ldots x_{\omega\tau} f\omega.
\end{aligned}$$

So, $f \in \mathrm{C}$. To prove complete distributivity, let $\omega \in \Omega$, $f_1,\ldots,f_{\omega\tau} \in \mathrm{C}$, $\mathrm{F} \subseteq \mathrm{C}$, $1 \leq j \leq \omega\tau$, and $x \in \mathrm{M}$ be arbitrary. Then

$$\begin{aligned}
&x(f_1 \ldots f_{j-1}(\bigvee \mathrm{F})f_{j+1} \ldots f_{\omega\tau}\omega) \\
&\quad = (xf_1 \ldots [x(\bigvee \mathrm{F})] \ldots xf_{\omega\tau})\omega \\
&\quad = (xf_1 \ldots \bigvee\{xf_j \mid f_j \in \mathrm{F}\} \ldots xf_{\omega\tau})\omega \\
&\quad = \bigvee\{xf_1 \ldots xf_j \ldots xf_{\omega\tau})\omega \mid f_j \in \mathrm{F}\} \\
&\quad = \bigvee\{x(f_1 \ldots f_j \ldots f_{\omega\tau}\omega) \mid f_j \in \mathrm{F}\},
\end{aligned}$$

so $f_1 \ldots f_{j-1}(\bigvee \mathrm{F})f_{j+1} \ldots f_{\omega\tau}\omega = \bigvee\{f_1 \ldots f_j \ldots f_{\omega\tau}\omega \mid f_j \in \mathrm{F}\}$. □

Definition 6.1. Let $f : (\mathrm{M},\Omega) \to (\mathrm{D},\vee,\Omega)$ be a convex function from a mode to a compatible completely distributive modal. The *closure* of f is the function

$$\mathrm{cl}f = \bigvee\{h \in \mathrm{Hom}(\mathrm{M},\mathrm{D}) \mid h \leq f\}.$$

Note 6.1. By the above proposition, since $\mathrm{Hom}(\mathrm{M},\mathrm{D}) \subseteq \mathrm{Conv(M,D)}$, $\mathrm{cl}f$ is convex. Also, $\mathrm{cl}f \leq f$ clearly.

Definition 6.2. A convex function f from a mode to a compatible completely distributive modal is *closed* iff $f = \mathrm{cl}f$.

Thus the closed convex functions are precisely the functions that are joins of some set of homomorphisms. In particular, all convex functions from $\wedge$-semilattices to completely distributive lattices are closed, by Theorem 5.1. Closed functions will be useful in the study of possible duality results.

Definition 6.3. Let (M, Ω) be a mode and $(\mathrm{D}, \vee, \Omega)$ a compatible completely distributive modal. For a convex function $f : \mathrm{M} \to \mathrm{D}$, define for each $c \in \mathrm{M}$,

$$_f\mathrm{H}_c := \{h \in \mathrm{Hom}(\mathrm{M}, \mathrm{D}) \mid h \leq f,\, ch = c(\mathrm{cl}f)\}.$$

The set $_f\mathrm{H}_c$ has the pointwise order inherited from $(\mathrm{Conv}(\mathrm{M}, \mathrm{D}), \leq_\vee)$.

As in the last chapter, $_f\mathrm{H}_c$ will have maximal elements when it is nonempty.

Theorem 6.1. Let $f : (\mathrm{M}, \Omega) \to (\mathrm{D}, \vee, \Omega)$ be a convex function from a mode to a compatible completely distributive modal. Let $c \in \mathrm{M}$. If the set $_f\mathrm{H}_c$ is nonempty, then it has maximal elements.

Proof: Assume $_f\mathrm{H}_c$ is nonempty. Let $\Theta = \{h_\alpha \mid \alpha \in \mathcal{A}\}$ be a chain of homomorphisms in $_f\mathrm{H}_c$. Let

$$h = \bigvee\{h_\alpha \mid \alpha \in \mathcal{A}\}.$$

Clearly, $h \leq f$. We need h to be a homomorphism. Then we will have $h \in {_f\mathrm{H}_c}$, since

$$[\forall k \in \Theta,\, ck = c(\mathrm{cl}f) \text{ and } k \leq f] \implies [ch = c(\mathrm{cl}f) \text{ and } h \leq f],$$

and $_f\mathrm{H}_c$ will have maximal elements by Zorn's Lemma. Take arbitrary $\omega \in \Omega$ and $x_1, \ldots, x_n \in \mathrm{M}$. Then

$$\begin{aligned}
&x_1 h \ldots x_n h\omega \\
&= \left(\bigvee\{x_1 h_1 \mid h_1 \in \Theta\}\right) \ldots \left(\bigvee\{x_n h_n \mid h_n \in \Theta\}\right)\omega \\
&= \bigvee\{x_1 h_1 \ldots x_n h_n \omega \mid (h_1, \ldots, h_n) \in \Theta^n\} && \text{[complete distributivity]} \\
&\leq \bigvee\{x_1 h_\alpha \ldots x_n h_\alpha \omega \mid h_\alpha = \max\{h_1, \ldots, h_n\}, (h_1, \ldots, h_n) \in \Theta^n\} && [\omega \text{ monotone}] \\
&= \bigvee\{x_1 \ldots x_n \omega h_\alpha \mid (h_1, \ldots, h_n) \in \Theta^n, h_\alpha = \max\{h_1, \ldots, h_n\}\} && \text{[homomorphism]} \\
&= \bigvee\{x_1 \ldots x_n \omega h_\alpha \mid h_\alpha \in \Theta\}
\end{aligned}$$

$= x_1 \dots x_n \omega h$.
So h is concave. Since h is a join of homomorphisms, which are always convex, h is also convex. Thus h is a homomorphism, proving the theorem. $\square$

Definition 6.4. Let $f : (\mathrm{M}, \Omega) \to (\mathrm{D}, \vee, \Omega)$ be a convex function from a mode to a compatible completely distributive modal. The *subgradient* of f is the function

$$\partial f : \mathrm{M} \to [\mathrm{Hom}(\mathrm{M}, \mathrm{D})]\mathrm{S}_{\varnothing};\ c \mapsto {}_f\mathrm{H}_c.$$

For $c \in \mathrm{M}$, $\partial f(c) = {}_f\mathrm{H}_c$ is the *subgradient of f at c*. If ${}_f\mathrm{H}_c$ is nonempty, then f is said to be *subdifferentiable at c*. The elements of $\partial f(c)$ are called subgradient homomorphisms. A function is *subdifferentiable* iff it is subdifferentiable at every point in its domain.

Example 6.1. Let S be any set. Define the projection operators

$$p_1, p_2 : \mathrm{S}^2 \to \mathrm{S};\ (s,t)p_1 = s,\ (s,t)p_2 = t,$$

and let $\Omega = \{p_1, p_2\}$. Then every function $f : \mathrm{S} \to \mathrm{S}$ is an Ω-homomorphism. (For $i = 1, 2$, $x_1 x_2 p_i f = x_i f = x_1 f x_2 f p_i$.) Take the order $0 < 1$ on $\mathbb{Z}_2$. For $n \in \mathbb{N}^+$, define the product order $\leq$ on $(\mathbb{Z}_2^n, \vee)$. Then $(0, \dots, 0)$ is the smallest element, and $(1, \dots, 1)$ is the largest. It is easy to show $(\mathbb{Z}_2^n, \cdot, \Omega)$ is a modal. For the algebras $(\mathbb{R}, \Omega)$ and $(\mathbb{Z}_2, \Omega)$, define the function

$$f : \mathbb{R} \to \mathbb{Z}_2;\ x \mapsto \begin{cases} 0, & x \in \mathbb{R}\backslash\mathbb{Q} \\ 1, & x \in \mathbb{Q}. \end{cases}$$

Then

1. $\forall x \in \mathbb{R},\ f \in \partial f(x)$,
2. $\forall t \in \mathbb{R}\backslash\mathbb{Q},\ h \equiv 0 \in \partial f(t)$, and
3. $\forall g : \mathbb{R} \to \mathbb{Z}_2 \bullet h \leq g \leq f,\ \forall t \notin \mathbb{Q},\ g \in \partial f(t)$.

Actually, if $g \leq f$, then for any $x \in \mathbb{R}$ where $xg = xf$, $g \in \partial f(x)$. So f is subdifferentiable everywhere, f is a subgradient homomorphism, and yet there are many subgradient homomorphisms.

Now we take another look at Euclidean spaces for examples of algebraically convex functions.

Note 6.2. For a convex function $f : \mathbb{R}^n \to \mathbb{R}^{+\infty}$, the above definition of ∂f agrees with that given in Definition 4.2.

Recall: For convex $f : \mathrm{X} \to \mathbb{R}$, we have

$$f_e : \mathbb{R}^n \to \mathbb{R}^{+\infty};\ \mathbf{x} \mapsto \begin{cases} \mathbf{x}f, & \mathbf{x} \in \mathrm{X} \\ +\infty, & \mathbf{x} \notin \mathrm{X}. \end{cases}$$

Proposition 6.3. For $\mathrm{X} \in (\mathbb{R}^n, \mathrm{I}^\circ)\mathbf{S}$, if $f : \mathrm{X} \to \mathbb{R}$ is convex, then

$$\forall \mathbf{x} \in \mathrm{X},\ \partial f(\mathbf{x}) = \{k \restriction \mathrm{X} \mid k \in \partial f_e(\mathbf{x})\}.$$

Also, if the absolute closure of $\mathrm{epi}_{\mathbb{R}} f_e$ in $\mathbb{R}^n \times \mathbb{R}$ and the relative closure of $\mathrm{epi} f$ in $\mathrm{X} \times \mathbb{R}$ are the same, then

$$\{k \restriction \mathrm{X} \mid k \in \partial f_e(\mathbb{R}^n)\} = \{k \mid k \in \partial f(\mathrm{X})\}.$$

Proof: First note that $\mathrm{Hom}(\mathrm{X},\mathbb{R}) = \{k \restriction \mathrm{X} \mid k \in \mathrm{H} = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})\}$, and $\mathrm{epi}_{\mathbb{R}} f = \mathrm{epi}_{\mathbb{R}} f_e$ is contained in $\mathbb{R}^n \times \mathbb{R}$. Now the relative closure of $\mathrm{epi}_{\mathbb{R}} f$ in $\mathrm{X} \times \mathbb{R}$ is the intersection of the absolute closure of $\mathrm{epi}_{\mathbb{R}} f_e$ with $\mathrm{X} \times \mathbb{R}$. Therefore, $\mathrm{cl} f = (\mathrm{cl} f_e) \restriction \mathrm{X}$. This implies

$$\forall \mathbf{x} \in \mathrm{X},\ \forall k \in \mathrm{H},\ \mathbf{x}k = \mathbf{x}(\mathrm{cl} f) \iff \mathbf{x}k = \mathbf{x}(\mathrm{cl} f_e).$$

Therefore, for $\mathbf{x}$ in X,

$$\begin{aligned} k \in \partial f_e(\mathbf{x}) &\iff [\mathbf{x}k = \mathbf{x}f_e \text{ and } k \le f_e] \\ &\iff [\mathbf{x}k = \mathbf{x}f \text{ and } (k \restriction \mathrm{X}) \le f] \iff (k \restriction \mathrm{X}) \in \partial f(\mathbf{x}). \end{aligned}$$

Let F_e be the closure of $\mathrm{epi}_{\mathbb{R}} f_e$ and F be the relative closure of $\mathrm{epi}_{\mathbb{R}} f$ in $\mathrm{X} \times \mathbb{R}$. If $\mathrm{F}_e = \mathrm{F}$, then

$$\mathrm{epi}_{\mathbb{R}} \mathrm{cl} f_e = \mathrm{F}_e = \mathrm{F} = \mathrm{epi}_{\mathbb{R}} \mathrm{cl} f \implies \mathrm{gr}_{\mathbb{R}} \mathrm{cl} f_e = \mathrm{gr}_{\mathbb{R}} \mathrm{cl} f.$$

Thus

$$\forall \mathbf{x} \in \mathbb{R}^n,\ \mathbf{x}k = \mathbf{x}\mathrm{cl} f_e \iff \mathbf{x}k = \mathbf{x}(\mathrm{cl} f)_e,$$

so

$$k \in \partial f_e(\mathbb{R}^n) \iff (k \restriction \mathrm{X}) \in \partial f(\mathbf{x}). \quad \square$$

Thus the modal-theoretic Definition 6.4 of ∂f is a sensible generalization of the Euclidean Definition 4.2.

Example 6.2. Let $f : [0,1] \to \mathbb{R}$; $x \mapsto 0$. Then f is real-convex. Notice that $k : [0,1] \to \mathbb{R}$; $x \mapsto -x$ and $h : [0,1] \to \mathbb{R}$; $x \mapsto 0$ are both in $\partial f(0)$ even though $k < h$. Considering the fact that for proper convex functions of $\mathbb{R}^n$, all subgradient elements were maximal homomorphisms, it seemed, at the point of Chapter 4, that k should not be included in $\partial f(0)$. At the same time, though, for the extension of f, $f_e = \chi_{[0,1]}$, we had that the function $\tilde{k} : \mathbb{R} \to \mathbb{R}$; $x \mapsto -x$ was in $\partial f_e(0)$. This caused a dilemma that was solved by modal theory, which told us that $\partial f(0)$ should include k (because $0k = 0(\mathrm{cl} f)$) even though k is not maximal in the absolute sense.

Example 6.3. Recall, for $p \in \mathrm{I}^\circ$, $\underline{\underline{p}} : (\mathbb{R}^+)^2 \to \mathbb{R}^+$; $(x, y) \mapsto x^{1-p}y^p$. In Example 2.15 the homomorphisms are the log-linear functions. Thus subdifferentiating exponentially-convex functions is approximation by the test functions $c \log x + d$. Consider the function $xf = (\log x)^2$. The subgradient homomorphisms may be found by differentiating f and finding which log-linear function has the same value and slope as f at any particular point. Let $xh_{cd} = c \log x + d$. Now, $x_0 f' = \frac{2 \log x_0}{x_0}$ and $x_0 f = (\log x_0)^2 = \log x_0 \log x_0$ while $x_0 h'_{cd} = \frac{c}{x_0}$. Therefore $h_{cd} \in \partial f(x_0) \iff c = 2 \log x_0$ and $d = -\log x_0$.

Review and Extension

Let C be any nonempty convex subset of $\mathbb{R}^n$, and $f : \mathrm{C} \to \mathbb{R}$ be convex. We have shown the following:

1. $\forall x \in \mathrm{C}$, $\partial f(x) \in (\mathrm{Hom}(\mathrm{C}, \mathbb{R}), \mathrm{I}^\circ)\mathbf{S}_\varnothing$,
2. $k \in \partial f(x) \iff xk = x\mathrm{cl}f$ and $k \leq f$, and
3. $f = \mathrm{cl} f \iff f = \bigvee \partial f(\mathrm{C})$.

Note that

$$(\mathrm{Hom}(\mathrm{C}, \mathbb{R}))\mathbf{S} \cong (\mathrm{Hom}(\mathrm{C}, \mathbb{R}_{-\infty}))\mathbf{S} \setminus \{\{k \equiv -\infty\}\}.$$

Thus, by sending the empty subalgebra of $\mathrm{Hom}(\mathrm{C}, \mathbb{R})$ to the singleton subalgebra $\{k \equiv -\infty\}$ of $\mathrm{Hom}(\mathrm{C}, \mathbb{R}_{-\infty})$, we get an isomorphism

$$(\mathrm{Hom}(\mathrm{C}, \mathbb{R}))\mathbf{S}_\varnothing \cong (\mathrm{Hom}(\mathrm{C}, \mathbb{R}_{-\infty}))\mathbf{S},$$

and thus

$$\partial f_e \subseteq \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})\mathbf{S}_{\varnothing} \cong \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}_{-\infty})\mathbf{S}.$$

This is natural, in that $k \equiv -\infty \leq f$, always. We would like to generalize this last statement to modal theory. That will require a study of duality.

CHAPTER 7: SUMMARY AND TOPICS FOR FURTHER STUDY

Summary

The Galois connection $\delta : \mathcal{C} \rightleftarrows \Gamma : E$ of Theorem 3.1, connecting epigraphs of convex functions and subgradient functions of convex functions, gave an algebraic interpretation of the duality realized in traditional convex analysis. By the resulting Galois correspondence we obtained: "A proper convex function is closed if and only if its epigraph is maximal (or 'closed') with respect to the Galois connection."

By way of example we found that a $\wedge$-convex function from a semilattice into a completely distributive lattice is always closed and subdifferentiable. Then for compatible modes and modals, we defined $\partial f : \mathrm{M} \to [\mathrm{Hom}(\mathrm{M}, \mathrm{D})]\mathbf{S}_{\varnothing}$ in terms of the closure cl$f$ of a convex function $f : \mathrm{M} \to \mathrm{D}$, which turned out to be the proper generalization of the subgradients in Euclidean spaces, enabling us to define subgradients for functions of proper convex subsets of a Euclidean space. Thus, the closed Ω-convex functions of modes and modals are an appropriate generalization of closed real-convex functions of real spaces.

Future study topics

1. The results of Chapter 6 (Prop. 6.3 ff.) suggest that we study, for a mode (M, Ω) and a modal $(\mathrm{D}, \vee, \Omega)$, the possibility of finding a mode M_e containing M to extend a convex function $f : \mathrm{M} \to \mathrm{D}$ to some convex function

$$f_e : \mathrm{M}_e \to \mathrm{D}^{+\infty}$$

so that the subgradients of f are always maximal homomorphisms

$$k : \mathrm{M}_e \to \mathrm{D}_{-\infty}.$$

2. The question of whether a Galois connection exists in the more general setting of modes and modals also deserves further study. The dualities of $\mathbb{R}^{+\infty}$ with $\mathbb{R}_{-\infty}$, and of $\mathbb{R}^n$ with itself seem a good place to start. These have already been studied in convex analysis ([R1], p.79ff). Rockafellar has, for a proper, closed, convex function $f : \mathbb{R}^n \to \mathbb{R}^{+\infty}$, a function $f^* : \mathbb{R}^n \to \mathbb{R}^{+\infty}$, which is

convex, and actually, $(z, zf^*) = -z\delta_{\mathrm{epi}_{\mathbb{R}}f}$. As f^* is convex, $\delta_{\mathrm{epi}_{\mathbb{R}}f}$ is concave. The duality result is that $f^{**} = f$.

The definition of subgradient might be extended to non-convex functions, such as convex-concave functions. Also submode reducts such as $(\mathbb{Q}, \mathbb{Q} \cap \mathrm{I}^\circ)$ could use some more investigation.

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APPENDIX: INDEX OF CONCEPTS