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The monotonicity of component importance measures in linear consective-k-out-of-n systems

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The monotonicity of component importance measures in linear consecutive-k-out-of-n systems

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1 INTRODUCTION

1.1 Motivation

A system is an orderly arrangement of a number of components that performs a certain function. The performance of a system may depend entirely on the simultaneous performance of its components. Each component may have a different contribution to the functioning of the system, where some play more significant roles than others. For example, consider a system consisting of two serially connected subsystems S_1 and S_2, where S_1 is a series system of k components, and S_2 is a parallel system of $n-k$ components. Intuitively, each component in S_1 is considered to be more important than components in S_2 because, given that the other components are functioning, the failure of a component in S_1 will cause the system to fail; whereas the failure of a component in S_2 will not necessarily do so.

For a system whose components are arranged in series, or in parallel, one can consider each component equally important to system performance. However, if component reliability is available, the importance of all components in the system may be uneven. Furthermore, if other characteristics are incorporated into all components, the importance of one component may change with respect to the importance of the others.

Based on these facts, it is, practically, important for system analysts to have a measure of importance for system components that is based on the particular characteristics available for each component. Such a measure will permit an analyst to determine which component merits the most additional research and development in order to improve overall system performance.

For this purpose, several measures of importance have been suggested. The first measure of importance in reliability theory was introduced by Birnbaum (1969). This measure is based on the critical sets and on the reliabilities of all components in the system. A number of importance measures have been proposed since then. These measures are based essentially on the combination of system structure and reliability performance. In terms of system structure, these measures are based on critical sets, minimal-path sets, and minimal-cut sets of a system. In terms of reliability performance, they are based on reliability and expected life of a system.

In general, each importance measure gives different weight to a component, with respect to other components in the system. The importance of a component depends on the type of measure being used and the type of system to which the component pertains. It is interesting, in this case, to see how an importance measure behaves in various types of systems. It is, however, more interesting to investigate the behavior of various measures in a particular type of system.

So far, there has been no study focused on this issue. On that account, we propose to examine the behavior of component importance measures on a specific type of system: the linear consecutive-k-out-of-n system. This system is a restricted type of k-out-of-n system, where all n components are arranged linearly, and where the k components insuring system performance must be successive.

1.2 Objectives

One interesting behavior of a component importance measure for linear consecutive-k-out-of-n systems is the monotonicity of this measure for the system having independently and equally reliable components. An importance measure of a component in this type of system is said to be monotone if the value of this measure is nondecreasing as the component approaches to the center of the system.

The objective of this study is to investigate the monotonicity behavior of a number of importance measures in linear consecutive-k-out-of-n systems. For the purpose of this study, we will develop specific formulations of these measures in corresponding systems. These formulations will be obtained through the study of structure functions and reliability functions of linear consecutive-k-out-of-n systems. It is assumed throughout the study that all components in these systems are statistically independent.

1.3 Thesis Overview

As mentioned above, the objective of this study is to investigate the monotonicity behavior of a number of component importance measures for linear consecutive-k-out-of-n systems. During the process of this study, some results insight into this issue are made. Results on structure and reliability functions for this type of system are also discussed.

The study begins, in Chapter 2, with a review of some basic concepts of system structure. It is assumed that a system, as well as each component in the system, can be in one and only one of two states: functioning and failed. It is also assumed that the performance of a system is determined completely by the performance of its components. The state of a system can then be indicated by a function ϕ, called a structure function. The notion of monotone structure and component relevance forming the definition of coherent structure are presented.

The concepts of minimal-path and minimal-cut vectors of Barlow and Proschan (1981) as well as the concepts of generating and veto vectors of Park (1985) are considered to be very useful in formulating the representation of system structure. In addition, the concepts of pivotal decomposition and Shapley value (Shapley, 1953) contribute other types of structure representations. The concepts of critical-path and critical-cut vectors are derived from the notion of component relevance. The concept of the dual of a structure function is also reviewed. Finally, the extension of a deterministic aspect to the probabilistic aspect of system structure is discussed in terms of reliability function.

Using these concepts, in Chapter 3, we study the structure of linear consecutive-k-out-of-n systems. Three types of structure functions for linear consecutive-k-out-of-n:G systems are derived. The first one is an explicit formula of structure function as a function of component states $\mathbf{x}$, n, and k. The second is a structure function of order n, in terms of a recursion relation with a structure of order $n-1$. The last, called a reduced form of structure function, is a recursion relation of the structure of order n with the structure functions of order $n-1$ and $n-k-1$. This function is derived from the second function, by taking into account the fact that x_j^m, where $m > 1$, can be replaced by x_j.

The procedure employed to derive these structure functions is based on minimal-path representation of a structure. The Shapley representation is also used to derive a structure function for the special case $k \leq n \leq 2k$. The extension of these results to the linear consecutive-k-out-of-n:F system is made with the help of the concept of dual structure function. The structure functions in the reduced form are ready to be transformed into reliability functions by the replacement of x_j with p_j.

In Chapter 4, we survey the existing measures of component importance. Examples, as well as the application of these measures to k-out-of-n systems, are presented in order to clarify the idea. As mentioned in the previous section, in terms of system structure, there are three types of measures. First, the importance measure based on critical-path and critical-cut vectors associated with component c_i. This includes the importance measures of Birnbaum (1969), Barlow and Proschan (1975), Lambert (1975), Shapley and Shubik (1954), Natvig (1979), and Bergman (1985). The more general form of importance measures of this type was given by Xie (1987).

The second type is importance measures based on the minimal-path vector associated with component c_i. This measure was suggested by Deegan and Packel (1978) in the context of an n-person game. Park (1985) reviewed this measure in the context of both game theory and reliability theory. He also extended this measure to the probabilistic interpretation. The other type of importance is the one based on minimal-cut vectors associated with component c_i. This includes the importance measure of Vesely (1972) and Fussel (1975).

Chapter 5 is devoted entirely to the main objective of this thesis. Here, we developed the formulation of several importance measures in linear consecutive-k-out-of-n systems. We consider the Birnbaum measure, Barlow-Proschan measure, Deegan-Packel measure, and Vesely-Fussel measure. In particular, we study the monotonicity behavior of the Birnbaum measure in both the "G" and "F" systems; the Barlow-Proschan measure in the "G" systems; the Deegan-Packel measure in the "G" systems; and the Vesely-Fussel in the "F" systems.

In general, the study of the behavior of these measures is divided into study in the system where $k \leq n \leq 2k$ and study in the system where $n > 2k$. When $k \leq n \leq 2k$, all importance measures being studied are monotone with respect to components in the corresponding systems. However, for $n > 2k$, only the Deegan-Packel and Vesely-Fussel measures are monotone. The Birnbaum and Barlow-Proschan measures show their nonmonotonicity in the range $2k+1 \leq n \leq 3k+1$. For $n > 3k+1$, the Birnbaum measure is nonmonotone in the "G" systems when $(1-p^k)^k \geq q$, and nonmonotone in the "F" systems when $(1-q^k)^k \geq p$, where p is the common reliability of all components and q is the common unreliability of all components in the systems. The Barlow-Proschan measure also shows its nonmonotonicity in the "G" systems when n is sufficiently large.

2 PRELIMINARIES ON SYSTEM STRUCTURE

2.1 Introduction

Modern system reliability theory is based on the theory of coherent structure. Birnbaum, Esary, and Saunders (1961), inspired by a paper of Moore and Shannon (1956) on relay networks, published a paper giving a systematic treatment to this theory. The main idea of the paper was to show that practically all engineering systems could be treated in a simple and unified fashion when determining its reliability in terms of its components. Esary and Proschan (1963) developed some general aspects of the reliability of coherent systems having independent components, but not necessarily having the same reliability. They also established lower and upper bounds of system reliability. Esary and Proschan (1962) presented an expository description of some potential results on the reliability of coherent system and discussed a generalization of some results given by Moore and Shannon (1956).

There has been considerable works in this area since then. A comprehensive discussion of reliability theory can be found in Barlow and Proschan (1981) and in Kaufmann, Grouchko, and Cruon (1977). Recently, Park (1985) studied reliability theory in connection with game theory and revealed some results applicable to both areas. A detailed discussion of the applications of reliability ideas in economics and social sciences is given by Bhattacharjee (1988).

Based on these literature, this chapter will present several terminologies, definitions, and results regarding the theory of system structure, in support of our discussion throughout this study.

2.2 Structure Functions

Consider a system whose performance is completely determined by the performance of its components. We assume that the system can be in one and only one of two states, that it is either functioning or failed. Each component of the system is also assumed to have two states, either functioning or failed. This is usually known as a binary system.

Let $C = \{c_1, c_2, \ldots, c_n\}$ be the set of all components, where c_i is the i-th component, and n is the number of components in the system. Let x_i be the state of component c_i. The joint performance of all components in the system can be indicated by vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, called a state vector, where

$$x_i = \begin{cases} 1 & \text{if component } c_i \text{ is functioning;} \\ 0 & \text{if component } c_i \text{ is failed.} \end{cases}$$

The number of functioning components when the states of all components are represented by vector $\mathbf{x}$ is called the size of $\mathbf{x}$ and is denoted by $s(\mathbf{x})$, where $s(\mathbf{x}) = \sum_{i=1}^{n} x_i$. The state of the system can be indicated by a function $\phi(\mathbf{x})$, where

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if the system is functioning;} \\ 0 & \text{if the system is failed.} \end{cases}$$

This function is called the structure function, or the structure of a system. The structure function $\phi(\mathbf{x})$ of a system with n components is attributed either the value 0 or the value 1 by each of the 2^n vertices of the n-dimensional unit cube. In other words, some of the 2^n possibilities of component states will insure the system's functioning, and others will cause the system to fail. For example, consider a k-out-of-n success structure. This system is functioning when at least k of its n components are functioning and is failed otherwise, i.e., the value of $\phi(\mathbf{x})$ is 1 if $s(\mathbf{x}) \geq k$ and 0 if $s(\mathbf{x}) < k$.

The above structure function ϕ and component state x_i are defined in the functioning or success mode. Sometimes, as in fault-tree analysis, failure is more emphasized, rather than success. Both structure function and component state, in this case, should be defined in the failure mode. This means that the structure function ϕ will have the value 1 when the system is failed and 0 when the system is functioning. Similarly, the component state x_i is 1 when the component c_i has failed and 0 when it is functioning.

When a system is defined in the failure mode, the structure function of this system can be obtained by using the structure function of the corresponding system, which is defined in the success mode. This structure function, denoted by ϕ^D, is called a dual of structure function ϕ and is defined in Barlow and Proschan (1981) as follows:

Definition 2.1 Given a structure function ϕ, its dual ϕ^D, is defined by

$$\phi^D(\mathbf{x}) = 1 - \phi(\mathbf{1} - \mathbf{x}), \tag{2.1}$$

where $(\mathbf{1} - \mathbf{x}) = ((1 - x_1), (1 - x_2), \ldots, (1 - x_n))$.

A system with a set of components $C = \{c_1, c_2, \ldots, c_n\}$ and structure function

ϕ from now on will be described by the notation (C, ϕ), and a structure with n components will be called a structure of order n.

2.3 Coherent Structures

A physical system would be quite unusual if improving the performance of a component, that is, replacing a failed component by a functioning component, caused the system to deteriorate. Likewise, the system is unusual if changing the state of a certain component doesn't change the state of the system for all possible states of all components in the system. The component described in this latter situation would be called irrelevant and the system noncoherent. The definition of relevant a component is stated as follows:

Definition 2.2 A component c_i in system (C, ϕ) is called irrelevant to the structure function ϕ if ϕ is constant in x_i, that is

$$\phi(1_i, \mathbf{x}) = \phi(0_i, \mathbf{x}) \text{ for all } (\bullet_i, \mathbf{x}), \tag{2.2}$$

where

$$(1_i, \mathbf{x}) \equiv (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$$
$$(0_i, \mathbf{x}) \equiv (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$
$$(\bullet_i, \mathbf{x}) \equiv (x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n);$$

otherwise the component is relevant to the structure function ϕ.

When a structure has irrelevant components, it may be possible to simplify it by omitting those components irrelevant to its performance. This structure is called reducible. However, if all components are relevant, the structure is called irreducible. Coherent structure is an irreducible structure with monotone structure function. Next, we state the definition of monotone structure.

Definition 2.3 A structure function ϕ, is called monotone if $\phi(\mathbf{y}) \geq \phi(\mathbf{x})$ for all $\mathbf{y} \supseteq \mathbf{x}$, where $\mathbf{y} \supseteq \mathbf{x} \Leftrightarrow y_i \geq x_i$ for all i.

The system with a monotone structure function is called semi-coherent. A semi-coherent system having relevant components is then called a coherent system, which is formally stated as

Definition 2.4 A structure (C, ϕ) is called coherent if

(1) it is semi-coherent, and

(2) each component is relevant.

The first requirement in this definition essentially states that replacing a failed component with a functioning component will not cause a functioning system to fail; and the second requirement rules out trivial systems not encountered in engineering practice. Given the structure function ϕ, there are only two semi-coherent structures that are not coherent. These structures are $\phi(\mathbf{x}) = 0$, which fails for every state of its components; and the structure $\phi(\mathbf{x}) = 1$, which performs for every state of its components. In other words, the class of structure functions of coherent systems is simply the class of nondecreasing binary functions, excluding these two functions.

Examples of typical coherent systems are: (a) A series system of order n with structure function $\phi(\mathbf{x}) = \prod_{i=1}^{n} x_i$, (b) A parallel system of order n with structure function $\phi(\mathbf{x}) = \coprod_{i=1}^{n} x_i = 1 - \prod_{i=1}^{n}(1 - x_i)$, (c) A k-out-of-n system with structure function $\phi(\mathbf{x}) = 1$ if $s(\mathbf{x}) \geq k$ and 0 otherwise, and (d) A parallel-series (series-parallel) system: the system consisting of a parallel (series) arrangement of series (parallel) subsystems. Consecutive-k-out-of-n systems, which will be discussed in this thesis, is a restricted k-out-of-n system.

The structure function of every coherent system is bound below by the structure function of a series system and bound above by the structure function of a parallel system formed by its components. Formally, as in Barlow and Proschan (1981), it is stated as follows:

Theorem 2.1 Let ϕ be a coherent structure of order n. Then

$$\prod_{i=1}^{n} x_i \leq \phi(\mathbf{x}) \leq \coprod_{i=1}^{n} x_i, \tag{2.3}$$

where $\coprod_{i=1}^{n} x_i = 1 - \prod_{i=1}^{n}(1 - x_i)$.

The following theorem (Barlow and Proschan, 1981) shows that, in a coherent structure, parallel redundancy at the component level is better than parallel redundancy at the system level. On the contrary, serial redundancy at the system level is better than serial redundancy at the component level.

Theorem 2.2 Let ϕ be a structure function of the coherent structure of order n. Denote $\mathbf{x} \coprod \mathbf{y} \equiv (x_1 \coprod y_1, \ldots, x_n \coprod y_n)$, where $x_i \coprod y_i = 1 - (1 - x_i)(1 - y_i)$, for $i = 1, 2, \ldots, n$, and $\mathbf{x}.\mathbf{y} = (x_1 y_1, \ldots, x_n y_n)$. Then

(a) $\phi(\mathbf{x} \coprod \mathbf{y}) \geq \phi(\mathbf{x}) \coprod \phi(\mathbf{y})$, and

(b) $\phi(\mathbf{x}.\mathbf{y}) \leq \phi(\mathbf{x})\phi(\mathbf{y})$

where equality holds in (a) for all $\mathbf{x}$ and $\mathbf{y}$ if and only if the structure is parallel, and equality holds in (b) for all $\mathbf{x}$ and $\mathbf{y}$ if and only if the structure is series.

A structure function $\phi(\mathbf{x})$ of a system of order n has a property that it can be expressed in terms of the structure functions of systems of order $n-1$, which is called a pivotal decomposition and is written as

$$\phi(\mathbf{x}) = x_i \phi(1_i, \mathbf{x}) + (1 - x_i)\phi(0_i, \mathbf{x}) \tag{2.4}$$

for all $(\bullet_i, \mathbf{x})$ and $i = 1, 2, \ldots, n$. The structure function ϕ of a coherent system can therefore be written in the form

$$\phi(\mathbf{x}) = x_i[\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x})] + \phi(0_i, \mathbf{x}) = x_i \delta_i(\mathbf{x}) + \mu_i(\mathbf{x}) \tag{2.5}$$

where $\delta_i(\mathbf{x}) = \phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x}) = \partial\phi(\mathbf{x})/\partial x_i$ and $\delta_i(\mathbf{x})$ as well as $\mu_i(\mathbf{x})$ do not depend on the state x_i of component c_i.

2.4 Representation of System Structure

2.4.1 Path vector and cut vector

Several combinations of component states can insure the functioning of a system, several others will insure the failure of the system. The state vector that insures the functioning of a system is called a path vector and the state vector that insures the failure of a system is called a cut vector. Before we state the formal definitions of path and cut vectors, we need to introduce some useful operators for state vectors.

A state vector $\mathbf{x}$ is said to be dominated by another state vector $\mathbf{y}$, denoted by $\mathbf{x} \subseteq \mathbf{y}$ or $\mathbf{y} \supseteq \mathbf{x}$, if $x_i \leq y_i$ for all i; and it is said to be strictly dominated by $\mathbf{y}$, denoted by $\mathbf{x} \subset \mathbf{y}$ or $\mathbf{y} \supset \mathbf{x}$, if $\mathbf{x} \subseteq \mathbf{y}$ but $\mathbf{x} \neq \mathbf{y}$.

Definition 2.5 A state vector $\mathbf{x}$ is called a path vector for the structure function ϕ if $\phi(\mathbf{x}) = 1$, and it is called a cut vector for ϕ if $\phi(\mathbf{x}) = 0$.

Definition 2.6 A path vector $\mathbf{x}$ is called a minimal-path vector if for any vector $\mathbf{y}$ such that $\mathbf{y} \subset \mathbf{x}$, then $\phi(\mathbf{y}) = 0$. A cut vector $\mathbf{x}$ is called a minimal-cut vector if for any vector $\mathbf{y}$ such that $\mathbf{y} \supset \mathbf{x}$, then $\phi(\mathbf{y}) = 1$.

The set of components associated with path vectors is called a path set and is denoted by $C_1(\mathbf{x}) = \{i \mid x_i = 1\}$. The set of components associated with cut vectors is called a cut set and is denoted by $C_0(\mathbf{x}) = \{i \mid x_i = 0\}$. In fact, the cut sets of a coherent system (C, ϕ) are precisely the path sets of the coherent system (C, ϕ^D), where $\phi^D(\mathbf{x})$ is the dual of $\phi(\mathbf{x})$. Path sets corresponding to the minimal path vectors are called minimal-path sets, and cuts set corresponding to minimal-cut vectors are called minimal-cut sets.

A minimal-path set can be interpreted as a minimal set of components whose functioning insures the functioning of the system. A minimal-cut set is a minimal set of components whose failure would insure the failure of the system.

A given structure ϕ may have paths and cuts of sizes ranging from 0 to n. The number of paths for ϕ of size j is called the path number, denoted by P_j, for $j = 0, 1, \ldots, n$; and the number of cuts for ϕ of size j is called the cut number, denoted by K_j, for $j = 0, 1, \ldots, n$. Let S_j be the set of all vectors of size j, that is, $S_j = \{\mathbf{x} : s(\mathbf{x}) = j\}$, then clearly the path number $P_j = \sum_{\mathbf{x} \in S_j} \phi(\mathbf{x})$ and the cut number $K_j = \sum_{\mathbf{x} \in S_j} [1 - \phi(\mathbf{x})]$ so that $P_j + K_j = \sum_{\mathbf{x} \in S_j} 1 = \binom{n}{j}$, which is the number of possible state vectors of size j.

Using the concept of path and cut, the relationship between structure (C, ϕ) and its dual (C, ϕ^D), as in Barlow and Proschan (1981) and Kaufmann et al. (1977), can be expressed as follows:

(1) a path of structure (C, ϕ) is a cut of structure (C, ϕ^D), and conversely;

(2) a cut of structure (C, ϕ) is a path of structure (C, ϕ^D), and conversely;

(3) a minimal path of structure (C, ϕ) is a minimal cut of structure (C, ϕ^D), and conversely;

(4) a minimal cut of structure (C, ϕ) is a minimal path of structure (C, ϕ^D), and conversely.

From the point of view of components, we now consider the role of a state vector in insuring the performance of a system. If a state vector causes the performance of a system to be the same as the performance of component c_i, we say that this vector is a critical vector for component c_i. A critical vector can be a critical-path vector, or a critical-cut vector, as given in the following definition:

Definition 2.7 State vectors $(1_i, \mathbf{x})$ and $(0_i, \mathbf{x})$ are respectively called as critical-path vector and critical-cut vector for component c_i if

$$\delta_i(\mathbf{x}) = \phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x}) = 1. \tag{2.6}$$

This means that, given the performance of the other components indicated by vector $\mathbf{x}$, if component c_i is functioning, the system is functioning; and if component c_i has failed, the system has also failed.

The corresponding set $C_1(1_i, \mathbf{x})$ of critical-path vectors for component c_i is called the critical-path set of component c_i; and the corresponding set $C_0(0_i, \mathbf{x})$ of critical-cut vectors is called the critical-cut set of component c_i. The number of critical-path vectors for c_i is

$$n_\phi(i) = \sum_{\{\mathbf{x} | x_i = 1\}} \delta_i(\mathbf{x}) = \sum_{\{\mathbf{x} | x_i = 1\}} [\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x})], \tag{2.7}$$

which is equal to the number of critical-cut vectors for component c_i.

2.4.2 Generating vector and veto vector

One requirement a system must fulfill in order to be coherent is that its structure function be monotone or nondecreasing. As stated in Definition 2.3, a structure function ϕ is monotone if $\phi(\mathbf{x}) \geq \phi(\mathbf{y})$ for any state vector $\mathbf{x}$ and $\mathbf{y}$, such that $\mathbf{x} \supseteq \mathbf{y}$. Of special interest to the monotone structure is structure function $\phi_{\mathbf{z}}$; constructed by using generating vector $\mathbf{z}$ and structure function $\psi_{\mathbf{y}}$, constructed by using veto vector $\mathbf{y}$. This concept was introduced by Park (1985). Before we present the definitions concerning generating and veto vectors of a structure function, we need to introduce additional operators for state vectors.

The union of state vectors $\mathbf{x}$ and $\mathbf{y}$, denoted by $\mathbf{x} \cup \mathbf{y}$, is defined by $\mathbf{x} \cup \mathbf{y} = (z_1, z_2, \ldots, z_n)$, where $z_i = x_i \bigoplus y_i$, for $i = 1, 2, \ldots, n$; and $\bigoplus$ is a Boolean disjunction operator. The intersection of state vectors $\mathbf{x}$ and $\mathbf{y}$, denoted by $\mathbf{x} \cap \mathbf{y}$, is defined by $\mathbf{x} \cap \mathbf{y} = (w_1, w_2, \ldots, w_n)$, where $w_i = x_i \bigodot y_i$, for $i = 1, 2, \ldots, n$; and $\bigodot$ is a Boolean conjunction operator.

Definition 2.8 A vector $\mathbf{z}$ is called a generating vector of structure function $\phi_{\mathbf{z}}$ if it satisfies

$$\phi_{\mathbf{z}}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \supseteq \mathbf{z} \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

Consider a structure function $\phi_1(\mathbf{x})$ that equals to 1 if $\mathbf{x} \supseteq \mathbf{z}_1$ and 0 otherwise. Here, ϕ_1 corresponds to $\mathbf{z}_1$, and vice versa. Vector $\mathbf{z}_1$ is the generating vector of function ϕ_1. Consider another generating vector $\mathbf{z}_2$ and its corresponding function

$\phi_2(\mathbf{x})$. From ϕ_1 and ϕ_2 we can define a function ϕ_{12} that corresponds to $\mathbf{z}_1$ and $\mathbf{z}_2$, as follows:

$$\begin{aligned}
\phi_{12}(\mathbf{x}) &= \phi_1(\mathbf{x}) \bigoplus \phi_2(\mathbf{x}) \\
&= \max_{1 \le i \le 2} \phi_i(\mathbf{x}) \\
&= \begin{cases} 1 & \text{if } \mathbf{x} \supseteq \mathbf{z}_1 \text{ or } \mathbf{x} \supseteq \mathbf{z}_2 \\ 0 & \text{otherwise,} \end{cases}
\end{aligned} \tag{2.9}$$

where $\bigoplus$ is the Boolean disjunction operator for elements of state vectors. In general, for generating vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_t$, in a basis $\mathbf{B}$, with their corresponding $\phi_1, \phi_2, \ldots, \phi_t$, respectively, we can define a function

$$\begin{aligned}
\phi_{\mathbf{B}}(\mathbf{x}) &= \bigoplus_{i=1}^{t} \phi_i(\mathbf{x}) \\
&= \max_{1 \le i \le t} \phi_i(\mathbf{x}) \\
&= \begin{cases} 1 & \text{if } \mathbf{x} \supseteq \mathbf{z}_1, \text{ or } \mathbf{x} \supseteq \mathbf{z}_2, \ldots, \text{ or } \mathbf{x} \supseteq \mathbf{z}_t \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{2.10}$$

In the system context, generating vectors are equivalent to minimal-path vectors. The basis of structure function $\phi_{\mathbf{B}}$ is defined:

Definition 2.9 A set of generating vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_t$, denoted by $\mathbf{B} = \{\mathbf{z}_i\}_{i=1}^{t}$, is called a basis for $\phi_{\mathbf{B}}$ if none of these vectors dominates any other vectors, i.e., if there is no $\mathbf{z}_i$ such that $\mathbf{z}_i \supseteq \mathbf{z}_j$ for $i \neq j$.

Next, we consider the definition of a veto vector for a structure function.

Definition 2.10 A vector $\mathbf{y}$ is called a veto vector of structure function $\psi_{\mathbf{y}}$ if it satisfies

$$\psi_{\mathbf{y}}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \cap \mathbf{y} \neq \mathbf{0} \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

Now consider a veto vector $\mathbf{y}$ and structure function $\psi_1(\mathbf{x})$, where $\psi_1(\mathbf{x})$ equals 1 when $\mathbf{x} \cap \mathbf{y}_1 \neq \mathbf{0}$ and 0 otherwise. Considering another veto vector $\mathbf{y}_2$ and its corresponding function ψ_2, structure ψ_{12} is defined as

$$\begin{aligned} \psi_{12}(\mathbf{x}) &= \bigodot_{i=1}^{2} \psi_i(\mathbf{x}) \\ &= \min_{1 \le i \le 2} \psi_i(\mathbf{x}) \\ &= \begin{cases} 1 & \text{if } \mathbf{x} \cap \mathbf{y}_1 \neq \mathbf{0} \text{ and } \mathbf{x} \cap \mathbf{y}_2 \neq \mathbf{0} \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{2.12}$$

where $\bigodot$ is the Boolean conjunction operator for the elements of state vectors. In general, given veto vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r$ in a clique $\mathbf{Q}$ and given their corresponding functions $\psi_1, \psi_2, \ldots, \psi_r$, the structure $\psi_{\mathbf{Q}}$ is defined as

$$\begin{aligned} \psi_{\mathbf{Q}}(\mathbf{x}) &= \bigodot_{i=1}^{r} \psi_i(\mathbf{x}) \\ &= \min_{1 \le i \le r} \psi_i(\mathbf{x}) \\ &= \begin{cases} 1 & \text{if } \mathbf{x} \cap \mathbf{y}_i \neq \mathbf{0} \text{ for all } i \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.13}$$

In the system context, veto vectors correspond to minimal-cut vectors in the form of $\mathbf{w} = \mathbf{1} - \mathbf{y}$, where $\mathbf{w}$ is a minimal-cut vector. The so called clique of structure $\psi_{\mathbf{Q}}$ is defined:

Definition 2.11 The set of veto vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r$, denoted by $\mathbf{Q} = \{\mathbf{y}\}_{i=1}^r$, is called the clique of $\psi_{\mathbf{Q}}$ if none of these vectors dominates any other vectors, i.e., there is no $\mathbf{y}_i$ such that $\mathbf{y}_i \supseteq \mathbf{y}_j$ for $i \neq j$.

Having stated the above definitions, we are now in the position to discuss the representations of structure functions. Four types of structure functions will be discussed in this section. These are the Barlow-Proschan representation, the minimal-path vector representation, the minimal-cut vector representation, and the Shapley representation.

2.4.3 Barlow-Proschan representation

Any structure function ϕ of system of order n can be written in terms of structure functions of order $n-1$, as given in equation (2.4). Repeated application of this method will permit us to explicitly express the structure function $\phi(\mathbf{x})$ in terms of the state of its components. This application can be written as follows

$$\begin{aligned}
\phi(\mathbf{x}) &= x_i\phi(1_i, \mathbf{x}) + (1-x_i)\phi(0_i, \mathbf{x}) \\
&= x_1\phi(1, x_2, x_3, \ldots, x_n) + (1-x_1)\phi(0, x_2, x_3, \ldots, x_n) \\
&= x_1[x_2\phi(1, 1, x_3, \ldots, x_n) + (1-x_2)\phi(1, 0, x_3, \ldots, x_n)] + \\
&\quad (1-x_1)[x_2\phi(0, 1, x_3, \ldots, x_n) + (1-x_2)\phi(0, 0, x_3, \ldots, x_n)] \\
&\vdots \\
&= \sum_{\mathbf{y}} \prod_{i=1}^{n} x_i^{y_i}(1-x_i)^{1-y_i}\phi(\mathbf{y}), \qquad (2.14)
\end{aligned}$$

where the summation is taken over some possible values of vector $\mathbf{y}$ such that $\phi(\mathbf{y})$ is 1. Equation (2.14) is called Barlow-Proschan representation of the structures.

The following example illustrates the procedure for using this formula to obtain the structure function of a system.

Example 2.1 For the purpose of illustration, we will consider a linear consecutive-2-out-of-4:G system. The characteristic of this system is that it is functioning if at least two consecutive components are functioning, otherwise it is failed. The set of all possible state vectors, which insures the functioning of this system, is

$$\{(1,1,1,1),(1,1,1,0),(1,1,0,1),(1,1,0,0),$$
$$(1,0,1,1),(0,1,1,1),(0,1,1,0),(0,0,1,1)\}.$$

Using (2.14) we can represent this structure as follows:

$$\begin{aligned}
\phi(\mathbf{x}) &= \sum_{\mathbf{y}} \prod_{i=1}^{4} x_i{}^{y_i}(1-x_i)^{1-y_i}\phi(\mathbf{y}) \\
&= x_1x_2x_3x_4 + x_1x_2x_3(1-x_4) + x_1x_2(1-x_3)x_4 + \\
&\quad x_1x_2(1-x_3)(1-x_4) + x_1(1-x_2)x_3x_4 + (1-x_1)x_2x_3x_4 + \\
&\quad (1-x_1)x_2x_3(1-x_4) + (1-x_1)(1-x_2)x_3x_4 \\
&= x_1x_2 + x_2x_3 + x_3x_4 - x_1x_2x_3 - x_2x_3x_4. \qquad (2.15)
\end{aligned}$$

The representation of a structure function using this formula seems to be a very basic way of expressing the structure in terms of the states of system components. It is useful only for systems without the property of coherence. For coherent systems, however, this formula gives too many terms that in fact can be cancelled out in the process of simplication. The next method allows us to express a coherent system with a much smaller number of terms.

2.4.4 Minimal-path vector representation

When minimal-path vectors of a system are available, the structure function of a system can be represented using these vectors. Park (1985) derived this representation using the idea of generating vectors, which in fact, is equivalent to minimal-path vector. Let $\mathbf{z}_i = (z_{i1}, z_{i2}, \ldots, z_{in})$ be a minimal-path vector of ϕ. Then a structure function $\phi_{\mathbf{z}_i}$ can be defined as

$$\begin{aligned}\phi_{\mathbf{z}_i}(\mathbf{x}) &= \begin{cases} 1 & \text{if } \mathbf{x} \supseteq \mathbf{z}_i \\ 0 & \text{otherwise} \end{cases} \\ &= \min_{\{j:z_{ij}=1\}} x_j \\ &= \prod_{\{j:z_{ij}=1\}} x_j. \end{aligned} \tag{2.16}$$

For t minimal-path vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_t$ in the basis $\mathbf{B}$, a structure function ϕ can be defined as

$$\begin{aligned}\phi(\mathbf{x}) &= \bigoplus_{i=1}^{t} \phi_{\mathbf{z}_i}(\mathbf{x}) \\ &= \max_{1\le i\le t} \phi_{\mathbf{z}_i}(\mathbf{x}) \\ &= 1 - \min_{1\le i\le t} (1 - \phi_{\mathbf{z}_i}(\mathbf{x})) \\ &= 1 - \prod_{i=1}^{t} (1 - \prod_{\{j:z_{ij}=1\}} x_j) \\ &= \coprod_{i=1}^{t} \prod_{\{j:z_{ij}=1\}} x_j, \end{aligned} \tag{2.17}$$

which is called a minimal-path vector representation of system structure.

Example 2.2 Consider the linear consecutive-2-out-of-4:G system from the previous example. The basis, or the set of minimal-path vectors of this system, is

$$\mathbf{B} = \{(1,1,0,0),(0,1,1,0),(0,0,1,1)\}.$$

Using $\mathbf{B}$ and equation (2.17) we can write the structure function ϕ as

$$\begin{aligned}
\phi(\mathbf{x}) &= \coprod_{i=1}^{3} \prod_{\{j:z_{ij}=1\}} x_j \\
&= 1 - \prod_{i=1}^{2} (1 - \prod_{\{j:z_{ij}=1\}} x_j) \\
&= 1 - (1 - x_1x_2)(1 - x_2x_3)(1 - x_3x_4) \\
&= x_1x_2 + x_2x_3 + x_3x_4 - x_1x_2x_3 - x_2x_3x_4, \qquad (2.18)
\end{aligned}$$

which is equal to (2.15). In comparison to the Barlow-Proschan representation, this procedure is much simpler. Next, we will derive the formula for the minimal-cut vector representation of system structure.

2.4.5 Minimal-cut vector representation

When minimal-cut vectors of a system are available, we are able to represent the structure function of a system by using these vectors. We will use the notion of veto vectors to obtain this representation. The result will be expressed in a minimal-cut vector representation. Let $\mathbf{y}_i$ be a veto vector of ψ. Then the function $\psi_{\mathbf{y}_i}$ is defined as

$$\begin{aligned}
\psi_{\mathbf{y}_i}(\mathbf{x}) &= \begin{cases} 1 & \text{if } \mathbf{x} \cap \mathbf{y}_i \neq \mathbf{0} \\ 0 & \text{otherwise} \end{cases} \\
&= \max_{\{j:\mathbf{y}_{ij}=1\}} x_j
\end{aligned}$$

$$
\begin{aligned}
&= 1 - \min_{\{j:y_{ij}=1\}} (1 - x_j) \\
&= 1 - \prod_{\{j:y_{ij}=1\}} (1 - x_j). \qquad (2.19)
\end{aligned}
$$

For r veto vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r$ in the clique $\mathbf{Q}$, a structure function $\psi(\mathbf{x})$ can be defined as

$$
\begin{aligned}
\psi(\mathbf{x}) &= \bigodot_{i=1}^{r} \psi_{\mathbf{y}_i}(\mathbf{x}) \\
&= \min_{1 \le i \le r} \psi_{\mathbf{y}_i}(\mathbf{x}) \\
&= \prod_{i=1}^{r} \psi_{\mathbf{y}_i}(\mathbf{x}) \\
&= \prod_{i=1}^{r} (1 - \prod_{\{j:y_{ij}=1\}} (1 - x_j)) \\
&= \prod_{i=1}^{r} \coprod_{\{j:y_{ij}=1\}} x_j. \qquad (2.20)
\end{aligned}
$$

Using minimal-cut vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$, where $\mathbf{w}_i = \mathbf{1} - \mathbf{y}_i$ for $i = 1, 2, \ldots, r$, and replacing notation ψ by ϕ, we can write equation (2.20) as

$$
\phi(\mathbf{x}) = \prod_{i=1}^{r} \coprod_{\{j:w_{ij}=0\}} x_j, \qquad (2.21)
$$

which is called a minimal-cut vector representation of system structure.

Example 2.3 Again, for the purpose of illustration and comparison, we will consider the previous consecutive-2-out-of-4:G system. The set of minimal-cut vectors of this system is $\{(0,1,0,1), (1,0,0,1), (1,0,1,0)\}$. Using these vectors and employing equation (2.21), we can express structure function ϕ as follows:

$$
\phi(\mathbf{x}) = \prod_{i=1}^{3} \coprod_{\{j:w_{ij}=0\}} x_j
$$

$$
\begin{aligned}
&= \prod_{i=1}^{3}\Big(1 - \prod_{\{j:w_{ij}=0\}}(1-x_j)\Big) \\
&= [1-(1-x_1)(1-x_3)][1-(1-x_2)(1-x_3)] \\
&\quad [1-(1-x_2)(1-x_4)] \\
&= x_1x_2 + x_2x_3 + x_3x_4 - x_1x_2x_3 - x_2x_3x_4, \qquad (2.22)
\end{aligned}
$$

which agrees with (2.15) and (2.18). In comparison, the minimal-cut vector representation is not a better procedure for formulating the "G" system.

2.4.6 Shapley representation

This method originates in game theory. Shapley (1953) has provided a formula to express a super-additive characteristic function as a linear combination of characteristic functions $\phi_{\mathbf{y}}$ of symmetric games. Park (1985) modified this expression to represent the structure of coherent systems. His expression turns out to be the function of minimal-path vectors. The representation can be written as

$$
\phi(\mathbf{x}) = \sum_{i=1}^{t} \phi_{\mathbf{z}_i}(\mathbf{x}) - \sum_{1\le i<j\le t} \phi_{\mathbf{z}_i \cup \mathbf{z}_j}(\mathbf{x}) + \ldots (-1)^{t+1}\phi_{\bigcup_{i=1}^{t} \mathbf{z}_i}(\mathbf{x}), \qquad (2.23)
$$

where $\{\mathbf{z}_i\}_{i=1}^{t}$ is the minimal-path vectors of the coherent structure. The first term of (2.23) is the summation of structure functions, based on individual minimal-path vector $\mathbf{z}_i$, for $i = 1, 2, \ldots, t$, where $\phi_{\mathbf{z}_i}(\mathbf{x}) = \prod_{\{j:z_{ij}=1\}} x_j$. The second term is the summation of structure function based on the union of two minimal-path vectors. The last term is the structure function based on the union of all minimal-path vectors of the system. We present the illustration for this representation in the next example.

Example 2.4 Consider again the linear consecutive-2-out-of-4:G system. As explained in previous examples, the set of minimal-path vectors for this system is $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\} = \{(1,1,0,0),(0,1,1,0),(0,0,1,1)\}$. Thus the possible unions of these vectors are

$$\begin{aligned}
\mathbf{z}_1 \cup \mathbf{z}_2 &= (1,1,1,0), \\
\mathbf{z}_2 \cup \mathbf{z}_3 &= (0,1,1,1), \\
\mathbf{z}_1 \cup \mathbf{z}_3 &= (1,1,1,1), \text{ and} \\
\mathbf{z}_1 \cup \mathbf{z}_2 \cup \mathbf{z}_3 &= (1,1,1,1).
\end{aligned}$$

The representation of structure function using equation (2.23) is as follows:

$$\begin{aligned}
\phi(\mathbf{x}) &= \sum_{i=1}^{3} \phi_{\mathbf{z}_i}(\mathbf{x}) - \sum_{1 \le i < j \le 3} \phi_{\mathbf{z}_i \cup \mathbf{z}_j}(\mathbf{x}) + \phi_{\mathbf{z}_1 \cup \mathbf{z}_2 \cup \mathbf{z}_3}(\mathbf{x}) \\
&= x_1x_2 + x_2x_3 + x_3x_4 - x_1x_2x_3 - x_2x_3x_4 - \\
&\quad x_1x_2x_3x_4 + x_1x_2x_3x_4 \\
&= x_1x_2 + x_2x_3 + x_3x_4 - x_1x_2x_3 - x_2x_3x_4, \qquad (2.24)
\end{aligned}$$

which agrees with (2.15), (2.18), and (2.22).

2.5 Reliability Functions

So far, we have discussed the deterministic aspect of coherent structures. In this section, we will extend the discussion to the include the probabilistic aspect of coherent structure. It concerns about the reliability of a system based on the reliability of its components.

Assume that all components are statistically independent. Let the component state X_i be a Bernoulli random variable. That is, X_i is equal to 1 with probability

p_i and equal to 0 with probability q_i, where $q_i = 1 - p_i$. Then $P(X_i = 1) = p_i$ is called the reliability of component c_i, $i = 1, 2, \ldots, n$; and $P(X_i = 0) = q_i$ is the unreliability of component c_i, $i = 1, 2, \ldots, n$. The corresponding system reliability is given by

$$R_\phi(\mathbf{p}) = P\{\phi(\mathbf{x}) = 1 \mid \mathbf{p}\} = E[\phi(\mathbf{x}) \mid \mathbf{p}], \tag{2.25}$$

where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$.

Function R_ϕ is called the reliability function based on the structure function ϕ. By using the assumption of component independence, the reliability function R_ϕ is wholly determined by component reliabilities $(p_1, p_2, \ldots, p_n)$. So, R_ϕ can be written as $R_\phi(\mathbf{p})$; and in the case of $p_1 = p_2 = \ldots = p_n = p$, the reliability function can be written as $R_\phi(p)$, a function of common component reliability p.

When component c_i is functioning, the reliability function of a system is

$$R_\phi(1_i, \mathbf{p}) = P\{\phi(1_i, \mathbf{x}) = 1 \mid \mathbf{p}\}; \tag{2.26}$$

and the reliability function of a system given component c_i fails is

$$R_\phi(0_i, \mathbf{p}) = P\{\phi(0_i, \mathbf{x}) = 1 \mid \mathbf{p}\}, \tag{2.27}$$

where $\mathbf{p} = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$.

As structure function $\phi(\mathbf{x})$, $R_\phi(\mathbf{p})$ can also be expressed in the pivotal decomposition, where the reliability function of the structure of order n is expressed the reliability functions of the structure of order $n - 1$. This is written as

$$R_\phi(\mathbf{p}) = p_i R_\phi(1_i, \mathbf{p}) + (1 - p_i) R_\phi(0_i, \mathbf{p}). \tag{2.28}$$

One property of structure function $R_\phi(\mathbf{p})$ is that it strictly increases in each p_i on the domain $0 < p_i < 1$, for $i = 1, 2, \ldots, n$. As in Barlow and Proschan (1981), this property is formally stated as

Theorem 2.3 Let $R_\phi(\mathbf{p})$ be the reliability function of a coherent structure. Then $R_\phi(\mathbf{p})$ is strictly increasing in each p_i for $\mathbf{0} < \mathbf{p} < \mathbf{1}$, where $\mathbf{a} < \mathbf{b} \Leftrightarrow a_i < b_i$ for all i.

Another property of structure function $R_\phi(\mathbf{p})$ is the correponding statement of Theorem 2.2, which basically states that redundancy at the component level is more effective than redundancy at the system level.

Theorem 2.4 Let R_ϕ be the reliability function of a coherent system. Then, for all $\mathbf{0} \le \mathbf{p} \le \mathbf{1}$ and $\mathbf{0} \le \mathbf{p}' \le \mathbf{1}$,
(a) $R_\phi(\mathbf{p} \amalg \mathbf{p}') \ge R_\phi(\mathbf{p}) \amalg R_\phi(\mathbf{p}')$
(b) $R_\phi(\mathbf{p}.\mathbf{p}') \le R_\phi(\mathbf{p}) R_\phi(\mathbf{p}')$,
where equality holds in (a) for all $\mathbf{p}, \mathbf{p}'$ if and only if the system is parallel, in (b) for all $\mathbf{p}, \mathbf{p}'$ if and only if the system is series.

The shape of the reliability function $R_\phi(\mathbf{p})$ has been discussed by Moore and Shannon (1956) through an inequality that establishes the S-shapeness property of this function for the system with equally reliable components.

Theorem 2.5 The following inequality

$$p(1-p)R_\phi'(p) > R_\phi(p)[1 - R_\phi(p)] \tag{2.29}$$

is true for $0 < p < 1$, provided that $R_\phi(p)$ is neither identically zero, identically one, nor identically equal to p.

For the structure with no path set or cut set of size 1, there exists p_0, where $0 < p_0 < 1$, such that $R_\phi(p_0) = p_0$ (Esary and Proschan, 1962). If $R_\phi(p_0) = p_0$ for some p_0 in $(0,1)$, then $R_\phi(p) < p$ for $0 < p < p_0$; whereas $R_\phi(p) > p$ for $p_0 < p < 1$, that establishes the S-shapeness property of $R_\phi(p)$ for a system having equally reliable components.

The extension of inequality (2.5) to the reliability function of the system having unequally reliable components was suggested by Esary and Proschan (1962). Their inequality is written as

$$\sum_{i=1}^{n} p_i(1-p_i)\frac{\partial R_\phi(\mathbf{p})}{\partial p_i} > R_\phi(\mathbf{p})[1-R_\phi(\mathbf{p})]. \tag{2.30}$$

Exact system reliability for systems with independent components can be computed using the structure function $\phi(\mathbf{x})$. By minimal-path and minimal-cut representations, $R_\phi(\mathbf{p})$ can be calculated from

$$R_\phi(\mathbf{p}) = E\Big(\coprod_{i=1}^{t} \prod_{\{j:z_{ij}=1\}} X_j\Big) = E\Big(\prod_{i=1}^{r} \coprod_{\{j:w_{ij}=0\}} X_j\Big), \tag{2.31}$$

where z_{ij} is the j-th element of minimal-path vector $\mathbf{z}_i$; and where w_{ij} is the j-th element of minimal-cut vector $\mathbf{w}_i$.

For a structure function in "reduced" form, that is, a structure function having no power of x_i greater than 1, the reliability function is simply the structure function ϕ evaluated on $\mathbf{p}$. Using the Barlow-Proschan representation of the structure, $R_\phi(\mathbf{p})$ can be computed

$$R_\phi(\mathbf{p}) = \sum_{\mathbf{x}} \prod_{i=1}^{n} p_i^{x_i} q_i^{1-x_i} \phi(\mathbf{x}), \tag{2.32}$$

where x_i is the value of the i-th elements in path vectors $\mathbf{x}$. Typical examples of reliability functions are: (1) for series systems, $R_\phi(\mathbf{p}) = \prod_{i=1}^{n} p_i$ and (2) for parallel systems, $R_\phi(\mathbf{p}) = \coprod_{i=1}^{n} p_i$. If a system has a common component reliability p, $R_\phi(\mathbf{p})$ can be computed by

$$R_\phi(p) = \sum_{i=1}^{n} P_i p^i q^{n-i}, \tag{2.33}$$

where P_i denotes the number of path vectors $\mathbf{x}$ of size i. For example, the reliability function of k-out-of-n systems having a common component reliability p is

$$R(p,n,k) = \sum_{i=k}^{n} \binom{n}{i} p^i q^{n-i}. \tag{2.34}$$

For the system with a large number of components, the calculation of system reliability, sometimes, becomes intractable. In other cases, the assumption of independent used in computing exact system reliability may not be reasonable. For these reasons, bounds of system reliability are needed. Barlow and Proschan (1981) provide a detailed discussion of these bounds. For a system with independent components, the reliability function, $R_\phi(p)$, has upper and lower bounds which can be expressed as

$$\coprod_{i=1}^{r} \prod_{\{j:w_{ij}=0\}} p_j \leq R_\phi(p) \leq \prod_{i=1}^{s} \coprod_{\{j:z_{ij}=1\}} p_j, \tag{2.35}$$

where r is the number of minimal-cut vectors $\mathbf{w}_j$ and s is the number of minimal-path vectors $\mathbf{z}_j$.

Finally, let us consider the expression of reliability function when a system, as well as its components, has life distribution. Let $F_i(t)$ be a life distribution of component c_i, and let

$$X_i(t) = \begin{cases} 1 & \text{if } c_i \text{ is functioning until time } t \\ 0 & \text{if } c_i \text{ is failed before time } t. \end{cases} \tag{2.36}$$

Then the reliability of component c_i at time t is

$$P[X_i(t) = 1] = E[X_i(t)] = S_i(t) = 1 - F_i(t), \tag{2.37}$$

and the system reliability at time t is

$$R_\phi[S(t)] = P\{\phi[X(t)] = 1\} = E\phi[X(t)]. \tag{2.38}$$

3 LINEAR CONSECUTIVE-K-OUT-OF-N SYSTEMS

3.1 Introduction

Consecutive-k-out-of-n systems can be categorized into consecutive-k-out-of-n:G and consecutive-k-out-of-n:F systems. A consecutive-k-out-of-n:G system is an n component system that functions whenever at least k consecutive components are functioning. Such a system can either be a linear system, where all components are arranged linearly, or be a circular system, where all components are arranged circularly. A consecutive-k-out-of-n:F system is an n component system that fails whenever at least k consecutive components are failed. Similarly, such a system can be either a linear or a circular system.

The consecutive-k-out-of-n:F system was introduced by Kontoleon (1980). Chiang and Niu (1981) indicated the relevance of such a system to telecommunication and oil pipeline systems. Bollinger and Salvia (1982) remarked that such systems also arise in the design of integrated circuits. The applications to street light systems and microwave tower systems are discussed by Chao and Lin (1984). Reliability evaluations of these systems are also discussed by Bolinger (1982), Derman, Lieberman, and Ross (1982), Hwang (1982), Shantikumar (1982), Lambiris and Papastavridis (1985), Griffith and Govindarajulu (1985), Fu (1985), Papastavridis and Chrysaphinou (1988), and others. The designs of these systems were discussed by

Wei, Hwang, and Sös (1983), Chao and Lin (1984), Malon (1984, 1985), and others.

Regarding the consecutive-k-out-of-n:G systems, Zhang (1988), and Kuo, Zhang, and Zuo (1988) studied this system and indicated its application to certain traffic problems.

In this chapter, we derive the structure and reliability functions for both linear consecutive-k-out-of-n:G and linear consecutive-k-out-of-n:F systems through the minimal-path vector representation and through the notion of duality given in Chapter 2. Pertinent works in the literature include that of Shantikumar (1982), in which reliability functions are derived by using a probabilistic argument for the "F" systems, and the argument of Zhang (1988) for the "G" systems.

3.2 Structure and Reliability Functions of Linear Consecutive-k-out-of-n:G Systems

We will use the minimal-path representation to find the structure function of the linear consecutive-k-out-of-n:G system, where $2 \le k \le n$. The number of minimal-path vectors for this system is equal to $n-k+1$ because there are $n-k+1$ possibilities for placing the k consecutive 1's on n possible consecutive locations. So, the set of minimal-path vectors for linear consecutive k-out-of-n:G systems is

$$\mathbf{Z} = \{\mathbf{z}_i\}_{i=1}^{n-k+1} = \{(\mathbf{1}_k, \mathbf{0}_{n-k}), (0, \mathbf{1}_k, \mathbf{0}_{n-k-1}), \ldots, (\mathbf{0}_{n-k}, \mathbf{1}_k)\}, \tag{3.1}$$

where $\mathbf{0}_j$ is the j-dimensional zero vector $(0, 0, \ldots, 0)$ and $\mathbf{1}_j$ is the j-dimensional unit vector $(1, 1, \ldots, 1)$. Based on the minimal-path vector representation (2.17), the structure function for the linear consecutive-k-out-of-n:G system is then

$$\begin{aligned}
\phi(\mathbf{x}, n, k) &= 1 - \prod_{i=1}^{n-k+1} (1 - \prod_{\{j: z_{ij}=1\}} x_j) \\
&= 1 - \prod_{i=1}^{n-k+1} (1 - \prod_{j=i}^{i+k-1} x_j). \qquad (3.2)
\end{aligned}$$

The structure function $\phi(\mathbf{x}, n, k)$, for $n = k$, is simply $\prod_{j=1}^{k} x_j$, which is the structure function of the well-known series system. For n greater than k, the structure function $\phi(\mathbf{x}, n, k)$ satisfies a recursion relationship, as follows:

Lemma 3.1 The structure function $\phi(\mathbf{x}, n, k)$ of linear consecutive-k-out-of-n:G systems, for $n > k$, satisfies

$$\phi(\mathbf{x}, n, k) = \phi(\mathbf{x}, n-1, k) + (1 - \phi(\mathbf{x}, n-1, k)) \prod_{j=n-k+1}^{n} x_j. \qquad (3.3)$$

Proof:

The minimal-path vector representation (3.2) can be written as

$$\begin{aligned}
\phi(\mathbf{x}, n, k) &= 1 - \prod_{i=1}^{n-k} (1 - \prod_{j=i}^{i+k-1} x_j)(1 - \prod_{n-k+1}^{n} x_j) \qquad (3.4) \\
&= 1 - (1 - (1 - \prod_{i=1}^{n-k} (1 - \prod_{j=i}^{i+k-1} x_j))(1 - \prod_{n-k+1}^{n} x_j) \qquad (3.5) \\
&= 1 - (1 - \phi(\mathbf{x}, n-1, k))(1 - \prod_{n-k+1}^{n} x_j) \qquad (3.6) \\
&= \phi(\mathbf{x}, n-1, k) + (1 - \phi(\mathbf{x}, n-1, k)) \prod_{j=n-k+1}^{n} x_j, \qquad (3.7)
\end{aligned}$$

where equation (3.6) is due to the fact that, according to (3.2),

$$1 - \prod_{i=1}^{n-k} (1 - \prod_{j=i}^{i+k-1} x_j),$$

is the structure function of a linear consecutive-k-out-of-(n-1):G system. □

The reliability of a system is obtainable as its structure function ϕ with $\mathbf{x}$ replaced by $\mathbf{p}$, under the restriction that ϕ is in "reduced" form containing no power of any x_i greater than the first. Such a "reduced" structure function is obtainable through

Theorem 3.1 The structure function of a linear consecutive-k-out-of-n:G system, for $n > k$, can be obtained recursively by using

$$\begin{aligned}\phi(\mathbf{x}, n, k) &= \phi(\mathbf{x}, n-1, k) + \\ &\quad (1 - \phi(\mathbf{x}, n-k-1, k))(1 - x_{n-k}) \prod_{j=n-k+1}^{n} x_j.\end{aligned} \tag{3.8}$$

Proof:

Equation (3.3) can also be written as

$$\phi(\mathbf{x}, r, k) = \phi(\mathbf{x}, r-1, k)(1 - \prod_{j=r-k+1}^{r} x_j) + \prod_{j=r-k+1}^{r} x_j. \tag{3.9}$$

Equation (3.8) will be proved by repeated application of equation (3.9) to the second term of the recursion formula (3.3).

Substituting (3.9), for $r = n-1$, into (3.3), we obtain the result of the first application as follows:

$$\begin{aligned}\phi(\mathbf{x}, n, k) &= \phi(\mathbf{x}, n-1, k) + \\ &\quad (1 - \phi(\mathbf{x}, n-2, k)(1 - \prod_{j=n-k}^{n-1} x_j) - \prod_{j=n-k}^{n-1}) \prod_{j=n-k+1}^{n} x_j \\ &= \phi(\mathbf{x}, n-1, k) + \\ &\quad (1 - \phi(\mathbf{x}, n-2, k)(1 - x_{n-k}) - x_{n-k}) \prod_{j=n-k+1}^{n} x_j\end{aligned} \tag{3.10}$$

$$= \phi(\mathbf{x}, n-1, k) + (1 - \phi(\mathbf{x}, n-2, k))(1 - x_{n-k}) \prod_{j=n-k+1}^{n} x_j, \tag{3.11}$$

where the second equation is due to the fact that

$$\prod_{j=n-k}^{n-1} x_j \prod_{j=n-k+1}^{n} x_j = \prod_{j=n-k}^{n} x_j = x_{n-k} \prod_{j=n-k+1}^{n} x_j.$$

Substituting (3.9), for $r = n-2$, into the second term of equation (3.11), we obtain the result of the second application as follows:

$$\begin{aligned}
\phi(\mathbf{x}, n, k) &= \phi(\mathbf{x}, n-1, k) + \\
&\quad (1 - \phi(\mathbf{x}, n-3, k)(1 - \prod_{j=n-k-1}^{n-2} x_j) - \prod_{j=n-k-1}^{n-2}) \\
&\quad (1 - x_{n-k}) \prod_{j=n-k+1}^{n} x_j & (3.12) \\
&= \phi(\mathbf{x}, n-1, k) + \\
&\quad (1 - \phi(\mathbf{x}, n-3, k)(1 - \prod_{j=n-k-1}^{n-k} x_j) - \prod_{j=n-k-1}^{n-k} x_j) \\
&\quad (1 - x_{n-k}) \prod_{j=n-k+1}^{n} x_j & (3.13) \\
&= \phi(\mathbf{x}, n-1, k) + \\
&\quad (1 - \phi(\mathbf{x}, n-3, k))(1 - \prod_{j=n-k-1}^{n-k} x_j) \\
&\quad (1 - x_{n-k}) \prod_{j=n-k+1}^{n} x_j & (3.14) \\
&= \phi(\mathbf{x}, n-1, k) + \\
&\quad (1 - \phi(\mathbf{x}, n-3, k))(1 - x_{n-k}) \prod_{j=n-k+1}^{n} x_j, & (3.15)
\end{aligned}$$

where the second equation is due the fact that

$$\prod_{j=n-k-1}^{n-2} x_j \prod_{j=n-k+1}^{n} x_j = \prod_{j=n-k-1}^{n} = \prod_{j=n-k-1}^{n-k} x_j \prod_{j=n-k+1}^{n} x_j,$$

and the last equation is due to the fact that

$$(1 - \prod_{j=n-k-1}^{n-k} x_j)(1 - x_{n-k}) = (1 - x_{n-k}).$$

From equation (3.11) and (3.15), we can obtain the result of the $(k-1)$-th application as

$$\begin{aligned} \phi(\mathbf{x}, n, k) &= \phi(\mathbf{x}, n-1, k) + \\ &\quad (1 - \phi(\mathbf{x}, n-k, k))(1 - x_{n-k}) \prod_{j=n-k+1}^{n} x_j. \end{aligned} \tag{3.16}$$

Finally, substituting (3.9), for $r = n - k$, into the second term of (3.16), we obtain the result of the k-th application as follows:

$$\begin{aligned} \phi(\mathbf{x}, n, k) &= \phi(\mathbf{x}, n-1, k) + \\ &\quad (1 - \phi(\mathbf{x}, n-k-1, k)(1 - \prod_{j=n-2k+1}^{n-k} x_j) - \prod_{j=n-2k+1}^{n-k}) \\ &\quad (1 - x_{n-k}) \prod_{j=n-k+1}^{n} x_j & (3.17) \\ &= \phi(\mathbf{x}, n-1, k) + \\ &\quad (1 - \phi(\mathbf{x}, n-k-1, k))(1 - \prod_{j=n-2k+1}^{n-k} x_j) \\ &\quad (1 - x_{n-k}) \prod_{j=n-k+1}^{n} x_j & (3.18) \\ &= \phi(\mathbf{x}, n-1, k) + \\ &\quad (1 - \phi(\mathbf{x}, n-k-1, k))(1 - x_{n-k}) \prod_{j=n-k+1}^{n} x_j, & (3.19) \end{aligned}$$

where the last equation, which is due to the fact that

$$(1 - \prod_{j=n-2k+1}^{n-k})(1 - x_{n-k}) = (1 - n_{n-k}),$$

is the "reduced" structure function as stated in this theorem. □

Corollary 3.1 For $k \leq n \leq 2k$, the structure function of a linear consecutive-k-out-of-n:G system can be written as

$$\phi(\mathbf{x}, n, k) = \prod_{j=1}^{k} x_j + \sum_{i=1}^{n-k} (1 - x_i) \prod_{j=i+1}^{i+k} x_j. \tag{3.20}$$

Proof:

When $k < n \leq 2k$, the value of the structure function $\phi(\mathbf{x}, n - k - 1, k)$ is equal to 0, so that, by using (3.8),

$$\phi(\mathbf{x}, n, k) = \phi(\mathbf{x}, n - 1, k) + (1 - x_{n-k}) \prod_{j=n-k+1}^{n} x_j. \tag{3.21}$$

Equation (3.20) is then verified recursively as follows:

$$\begin{aligned}
\phi(\mathbf{x}, k, k) &= \prod_{j=1}^{k} x_j \\
\phi(\mathbf{x}, k+1, k) &= \prod_{j=1}^{k} x_j + (1 - x_1) \prod_{j=2}^{k+1} x_j \\
\phi(\mathbf{x}, k+2, k) &= \prod_{j=1}^{k} x_j + (1 - x_1) \prod_{j=2}^{k+1} x_j + (1 - x_2) \prod_{j=3}^{k+2} x_j.
\end{aligned}$$

Finally,

$$\phi(\mathbf{x}, n, k) = \prod_{j=1}^{k} x_j + (1 - x_1) \prod_{j=2}^{k+1} x_j + (1 - x_2) \prod_{j=3}^{k+2} x_j +$$

$$\cdots + (1 - x_{n-k}) \prod_{j=n-k+1}^{n} x_j$$

$$= \prod_{j=1}^{k} x_j + \sum_{i=1}^{n-k} (1 - x_i) \prod_{j=i+1}^{i+k} x_j. \ \square \tag{3.22}$$

The following argument is another procedure for proving equation (3.20). This procedure is based on the Shapley representation as discussed in Chapter 2.

For $k \leq n \leq 2k$, any two nonadjacent minimal-path vectors $\mathbf{z}_i$ and $\mathbf{z}_j$ have the property that

$$\begin{aligned} \mathbf{z}_i \cup \mathbf{z}_j &= \mathbf{z}_i \cup \mathbf{z}_l \cup \mathbf{z}_j, \quad l = i+1, \ldots, j-1 & (3.23) \\ &= \mathbf{z}_i \cup \mathbf{z}_{l_1} \cup \mathbf{z}_{l_2} \cup \mathbf{z}_j, \quad i < l_1 < l_2 = i+2, \ldots, j-1 & (3.24) \\ &\vdots \\ &= \mathbf{z}_i \cup \{\cup_{l=i+1}^{j-1} \mathbf{z}_l\} \cup \mathbf{z}_j & (3.25) \end{aligned}$$

because the elements of any vectors in between vectors $\mathbf{z}_i$ and $\mathbf{z}_j$ with a nonzero value are also the elements of the nonzero value of the union of vectors $\mathbf{z}_i$ and $\mathbf{z}_j$. The possibility that the number of vectors in between $\mathbf{z}_i$ and $\mathbf{z}_j$ is odd is given by

$$n_o(i,j) = \sum_{r \text{ odd}} \binom{j-i-1}{r}, \tag{3.26}$$

and the possibility that the number of vectors in between $\mathbf{z}_i$ and $\mathbf{z}_j$ is even is

$$n_e(i,j) = \sum_{r \text{ even}} \binom{j-i-1}{r}, \tag{3.27}$$

where the even number including 0. By putting $a = 1$ and $b = -1$ into the Binomial series

$$(a+b)^{j-i-1} = \sum_r \binom{j-i-1}{r} a^r b^{j-i-1-r}, \tag{3.28}$$

we can show that

$$n_o(i,j) = n_e(i,j). \tag{3.29}$$

The Shapley representation of the structure function (2.23) is

$$\phi(\mathbf{x}) = \sum_{i=1}^{t} \phi_{\mathbf{z}_i}(\mathbf{x}) - \sum_{1 \le i < j \le t} \phi_{\mathbf{z}_i \cup \mathbf{z}_j}(\mathbf{x}) + \ldots (-1)^{t+1} \phi_{\bigcup_{i=1}^{t} \mathbf{z}_i}(\mathbf{x}), \tag{3.30}$$

where $\phi_{\mathbf{z}}(\mathbf{x})$ is the product of x_j for any j such that the j-th element of $\mathbf{z}$ is 1. Structure function ϕ in the first term of the right-hand-side of equation (3.30) is based on a single minimal-path vector $\mathbf{z}_i$; in the second term it is based on the union of two minimal-path vectors $\mathbf{z}_i$ and $\mathbf{z}_j$; and in the last term it is based on the union of t minimal-path vectors $\mathbf{z}_i$, $i = 1, 2, \ldots, t$. This can be written as

$$\begin{aligned} \phi(\mathbf{x}) \;=\; & \sum_{i=1}^{t} \phi_{\mathbf{z}_i}(\mathbf{x}) - \sum_{i=1}^{t-1} \phi_{\mathbf{z}_i \cup \mathbf{z}_{i+1}}(\mathbf{x}) - \sum_{2 \le i+1 < j \le t} \phi_{\mathbf{z}_i \cup \mathbf{z}_j}(\mathbf{x}) + \\ & \sum_{1 \le i < l < j \le t} \phi_{\mathbf{z}_i \cup \mathbf{z}_l \cup \mathbf{z}_j}(\mathbf{x}) - \sum_{1 \le i < l_1 < l_2 < j \le t} \phi_{\mathbf{z}_i \cup \mathbf{z}_{l_1} \cup \mathbf{z}_{l_2} \cup \mathbf{z}_j}(\mathbf{x}) + \\ & \vdots \\ & +(-1)^{t+1} \phi_{\mathbf{z}_i \cup \{\bigcup_{l=i+1}^{j-1} \mathbf{z}_l\} \cup \mathbf{z}_j}(\mathbf{x}). \end{aligned} \tag{3.31}$$

Structure function ϕ in the second term of the right-hand-side of (3.31) is based on the union of the two adjacent minimal-path vectors $\mathbf{z}_i$ and $\mathbf{z}_{i+1}$, and ϕ's in the remaining terms are based on the union of more than two minimal-path vectors. The following argument shows that for a linear consecutive-k-out-of-n:G system, all terms, except the first and second terms in the right-hand-side of (3.31), add up to zero, which implies that $\phi(\mathbf{x})$ can be written as

$$\phi(\mathbf{x}) = \sum_{i=1}^{t} \phi_{\mathbf{z}_i}(\mathbf{x}) - \sum_{i=1}^{t-1} \phi_{\mathbf{z}_i \cup \mathbf{z}_{i+1}}(\mathbf{x}). \tag{3.32}$$

From (3.23)–(3.25), we know that any ϕ that is based on the union of more than two minimal-path vectors can be written as function ϕ, which is based on the union of two nonadjacent minimal-path vectors. Thus, by (3.26) and (3.27), all terms, except the first and the second terms of (3.31), can be written as

$$\sum_{2\le i+1<j\le t} \{n_O(i,j) - n_E(i,j)\}\phi_{\mathbf{z}_i\cup\mathbf{z}_j}(\mathbf{x}). \tag{3.33}$$

This quantity is equal to zero since $n_O(i,j) = n_E(i,j)$. In linear consecutive-k-out-of-n:G system, $t = n - k + 1$. So, the the structure function for this system, when $k \le n \le 2k$, is written as

$$\begin{aligned}
\phi(\mathbf{x}, n, k) &= \sum_{i=1}^{n-k+1} \phi_{\mathbf{z}_i}(\mathbf{x}) - \sum_{i=1}^{n-k} \phi_{\mathbf{z}_i\cup\mathbf{z}_{i+1}}(\mathbf{x}) &(3.34)\\
&= \sum_{i=1}^{n-k+1} \prod_{j=i}^{i+k-1} x_j - \sum_{i=1}^{n-k} \prod_{j=i}^{i+k} x_j. &(3.35)
\end{aligned}$$

By changing the range of index i, we are able to write this function as follows:

$$\begin{aligned}
\phi(\mathbf{x}, n, k) &= \sum_{i=0}^{n-k} \prod_{j=i+1}^{i+k} x_j - \sum_{i=1}^{n-k} x_i \prod_{j=i}^{i+k} x_j &(3.36)\\
&= \prod_{j=1}^{k} + \sum_{i=1}^{n-k} \prod_{j=i+1}^{i+k} x_j - \sum_{i=1}^{n-k} x_i \prod_{j=i+1}^{i+k} x_j &(3.37)\\
&= \prod_{j=1}^{k} x_j + \sum_{i=1}^{n-k} (1 - x_i) \prod_{j=i+1}^{i+k} x_j. \ \square &(3.38)
\end{aligned}$$

Example 3.1 Consider the linear consecutive-2-out-of-5:G system. This system is functioning when at least two consecutive components are functioning. Here $k = 2$ and $n = 5$. Using the equation (3.2) we can write the structure function of this system as

$$\begin{aligned}\phi_l(\mathbf{x},5,2) &= 1-\prod_{i=1}^{4}(1-\prod_{j=i}^{i+1}x_j)\\ &= 1-(1-x_1x_2)(1-x_2x_3)(1-x_3x_4)(1-x_4x_5) \qquad (3.39)\\ &= x_1x_2+x_2x_3+x_3x_4+x_4x_5-x_1x_2x_3-x_2x_3x_4\\ &\quad -x_3x_4x_5-x_1x_2x_4x_5+x_1x_2x_3x_4x_5. \qquad (3.40)\end{aligned}$$

The simplification of this function from equation (3.39) to (3.40) becomes very tedious when n is large. Using the recursion relationship (3.3), we can write the structure function of this system as

$$\begin{aligned}\phi(\mathbf{x},2,2) &= x_1x_2\\ \phi(\mathbf{x},3,2) &= \phi(\mathbf{x},2,2)+(1-\phi(\mathbf{x},2,2))\prod_{j=2}^{3}x_j\\ &= x_1x_2+(1-x_1x_2)x_2x_3\\ &= x_1x_2+x_2x_3-x_1x_2x_3.\\ \phi(\mathbf{x},4,2) &= \phi(\mathbf{x},3,2)+(1-\phi(\mathbf{x},3,2))\prod_{j=3}^{4}x_j\\ &= (x_1x_2+x_2x_3-x_1x_2x_3)+(1-(x_1x_2+x_2x_3-x_1x_2x_3))x_3x_4\\ &= x_1x_2+x_2x_3+x_3x_4-x_1x_2x_3-x_2x_3x_4.\end{aligned}$$

Therefore, the structure function of a linear consecutive-2-out-of-5:G system is

$$\begin{aligned}\phi(\mathbf{x},5,2) &= \phi(\mathbf{x},4,2)+(1-\phi(\mathbf{x},4,2))\prod_{j=4}^{5}x_j\\ &= x_1x_2+x_2x_3+x_3x_4-x_1x_2x_3-x_2x_3x_4+\\ &\quad (1-(x_1x_2+x_2x_3+x_3x_4-x_1x_2x_3-x_2x_3x_4))x_4x_5\\ &= x_1x_2+x_2x_3+x_3x_4+x_4x_5-x_1x_2x_3-x_2x_3x_4\end{aligned}$$

$$-x_3x_4x_5 - x_1x_2x_4x_5 + x_1x_2x_3x_4x_5.$$

Using (3.8) and (3.20), we obtain the structure function of this system as follows:

$$\begin{aligned}
\phi(\mathbf{x},2,2) &= x_1x_2 \\
\phi(\mathbf{x},3,2) &= \phi(\mathbf{x},2,2) + (1-x_1)\prod_{j=2}^{3} x_j \\
&= x_1x_2 + (1-x_1)x_2x_3 \\
\phi(\mathbf{x},4,2) &= \phi(\mathbf{x},3,2) + (1-x_2)\prod_{j=3}^{4} x_j \\
&= x_1x_2 + (1-x_1)x_2x_3 + (1-x_2)x_3x_4 \\
\phi(\mathbf{x},5,2) &= \phi(\mathbf{x},4,2) + (1-\phi(\mathbf{x},2,2))(1-x_3)\prod_{j=4}^{5} x_j \\
&= x_1x_2 + (1-x_1)x_2x_3 + (1-x_2)x_3x_4 + \\
&\quad (1-x_1x_2)(1-x_3)x_4x_5 \\
&= x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 - x_1x_2x_3 - x_2x_3x_4 \\
&\quad -x_3x_4x_5 - x_1x_2x_4x_5 + x_1x_2x_3x_4x_5.
\end{aligned}$$

The reliability of a system is the probability that its structure function $\phi(\mathbf{x}, n, k)$ equals 1, which, since ϕ is an indicator variable, equals its expectation:

$$R(\mathbf{p}, n, k) = P(\phi(\mathbf{x}, n, k) = 1) = E(\phi(\mathbf{x}, n, k)). \tag{3.41}$$

For a system with independent components, R may be found, simply, by replacing $\mathbf{x}$ by $\mathbf{p}$ in the "reduced" structure function ϕ. The following theorem states the reliability functions of linear consecutive-k-out-of-n:G system.

Theorem 3.2 The reliability function of linear consecutive-k-out-of-n:G systems when all components are independent and $n \geq k$ is obtainable recursively by

$$R_G(\mathbf{p}, n, k) = R_G(\mathbf{p}, n-1, k) +$$

$$(1 - R_G(\mathbf{p}, n-k-1, k))q_{n-k} \prod_{j=n-k+1}^{n} p_j. \quad (3.42)$$

If $p_1 = p_2 = \ldots = p_n = p$, the reliability function becomes

$$R_G(p, n, k) = R_G(p, n-1, k) + (1 - R_G(p, n-k-1, k))qp^k. \quad (3.43)$$

If $k \leq n \leq 2k$, the reliability function is obtainable by

$$R_G(\mathbf{p}, n, k) = \prod_{j=1}^{k} p_j + \sum_{i=1}^{n-k} q_i \prod_{j=i+1}^{i+k} p_j. \quad (3.44)$$

If $k \leq n \leq 2k$ and $p_1 = p_2 = \ldots = p_n = p$, the reliability function becomes

$$R_G(p, n, k) = [(n-k)(1-p) + 1]p^k. \quad (3.45)$$

Proof:

Equations (3.42) and (3.44) are obtained by replacing x_i by p_i in (3.8) and (3.20), and equations (3.43) and (3.45) follow by replacing p_i by p. □

To compute system reliability, especially for a large system, equation (3.42) can be used directly to produce an algorithm. The algorithm should begin with reading and checking the input n, k, and p_j such that $1 \leq k \leq n$ and $0 \leq p_j \leq 1$. The next step is to compute $R_G(\mathbf{p}, k, k) = \prod_{j=1}^{k} p_j$, and the last step is to compute $R_G(\mathbf{p}), n, k)$ using equation (3.42) or using (3.44) for a system with $k \leq n \leq 2k$.

3.3 Structure and Reliability Functions of Linear Consecutive-k-out-of-n:F Systems

The set of minimal-cut vectors of linear consecutive-k-out-of-n:F systems is

$$\mathbf{W} = \{\mathbf{w}_i\}_{i=1}^{n-k+1} = \{(\mathbf{0}_k, \mathbf{1}_{n-k}), (1, \mathbf{0}_k, \mathbf{1}_{n-k-1}), \ldots, (\mathbf{1}_{n-k}, \mathbf{0}_k)\}, \quad (3.46)$$

where $\mathbf{0}_j$ is the j-dimensional zero vector $(0,0,\ldots,0)$ and $\mathbf{1}_j$ is the j-dimensional unit vector $(1,1,\ldots,1)$. There are $n-k+1$ minimal-cut vectors since there are $n-k+1$ possibilities of placing the k consecutive zeroes out of n locations. Using the minimal-cut vector representation (2.21), the structure function of the linear consecutive-k-out-of-n:F system can be written as

$$\begin{aligned}\psi(\mathbf{x},n,k) &= \prod_{i=1}^{n-k+1}(1-\prod_{\{j:w_{ij}=0\}}(1-x_j)) \\ &= \prod_{i=1}^{n-k+1}(1-\prod_{j=i}^{i+k-1}(1-x_j)). \qquad (3.47)\end{aligned}$$

The structure function of this system can also be represented in a recursion relationship, as shown in the next lemma.

Lemma 3.2 The structure function of linear consecutive-k-out-of-n:F systems for $n > k$ can be expressed as

$$\psi(\mathbf{x},n,k) = \psi(\mathbf{x},n-1,k) - \psi(\mathbf{x},n-1,k)\prod_{j=n-k+1}^{n}(1-x_j). \qquad (3.48)$$

Proof:

Minimal-cut vector representation of this system is

$$\begin{aligned}\psi(\mathbf{x},n,k) &= \prod_{i=1}^{n-k}(1-\prod_{j=i}^{i+k-1}(1-x_j))(1-\prod_{j=n-k+1}^{n}(1-x_j)) \qquad (3.49)\\ &= \psi(\mathbf{x},n-1,k)(1-\prod_{j=n-k+1}^{n}(1-x_j)) \qquad (3.50)\\ &= \psi(\mathbf{x},n-1,k) - \psi(\mathbf{x},n-1,k)\prod_{j=n-k+1}^{n}(1-x_j). \ \square \qquad (3.51)\end{aligned}$$

As in the "G" systems, this recursion doesn't directly produce a "reduced" form of structure function. The following theorem gives a recursion relationship that directly produces a structure function of linear consecutive-k-out-of-n:F systems in "reduced" form.

Theorem 3.3 The structure function of a linear consecutive-k-out-of-n:F system, for $n > k$, can be obtained recursively by using

$$\begin{aligned} \psi(\mathbf{x}, n, k) &= \psi(\mathbf{x}, n-1, k) - \\ &\quad \psi(\mathbf{x}, n-k-1, k) x_{n-k} \prod_{j=n-k+1}^{n} (1 - x_j). \end{aligned} \tag{3.52}$$

Proof:

A linear consecutive-k-out-of-n:F system can be regarded as the dual of linear consecutive-k-out-of-n:G systems since $\mathbf{w}_i = \mathbf{1} - \mathbf{z}_i$. Using the duality definition of Barlow and Proschan (1981), we can write the structure function of a linear consecutive-k-out-of-n:F system as

$$\psi(\mathbf{x}, n, k) = 1 - \phi(\mathbf{1} - \mathbf{x}, n, k), \tag{3.53}$$

where $\phi(\mathbf{1} - \mathbf{x}, n, k)$ is the structure function of a linear consecutive-k-out-of-n:F in the context of failure. Replacing $\mathbf{x}$ by $\mathbf{1} - \mathbf{x}$ in (3.8), and using (3.53), we obtain

$$\begin{aligned} \psi(\mathbf{x}, n, k) &= [1 - \phi(\mathbf{1} - \mathbf{x}, n-1, k)] - \\ &\quad [1 - \phi(\mathbf{1} - \mathbf{x}, n-k-1, k)] x_{n-k} \prod_{j=n-k+1}^{n} (1 - x_j) \\ &= \psi(\mathbf{x}, n-1, k) - \\ &\quad \psi(\mathbf{x}, n-k-1, k) x_{n-k} \prod_{j=n-k+1}^{n} (1 - x_j), \end{aligned} \tag{3.54}$$

which verifies equation (3.52). □

Corollary 3.2 For $k \leq n \leq 2k$, the structure function of a linear consecutive-k-out-of-n:F system can be written as

$$\psi(\mathbf{x}, n, k) = 1 - \prod_{j=1}^{k}(1 - x_j) - \sum_{i=1}^{n-k} x_i \prod_{j=i+1}^{i+k}(1 - x_j). \tag{3.55}$$

Proof:

When $k < n \leq 2k$, the value of the structure function $\psi(\mathbf{x}, n-k-1, k)$ is equal to 1, so that, applying (3.52),

$$\psi(\mathbf{x}, n, k) \;=\; \psi(\mathbf{x}, n-1, k) - x_{n-k} \prod_{j=n-k+1}^{n}(1 - x_j). \tag{3.56}$$

Equation (3.55) is then obtainable, and hence verified, using the similar procedure employed in Corollary 3.1. □

Example 3.2 Consider linear consecutive-2-out-of-5:F systems. This system is failed when at least two consecutive components are failed. Here $k = 2$ and $n = 5$. Using equation (3.47), we can write the structure function of this system as

$$\begin{aligned}
\psi(\mathbf{x}, 5, 2) &= \prod_{i=1}^{4}(1 - \prod_{j=i}^{i+1}(1 - x_j)) \\
&= (1-(1-x_1)(1-x_2))(1-(1-x_2)(1-x_3) \\
&\quad (1-(1-x_3)(1-x_4))(1-(1-x_4)(1-x_5) \\
&= x_2x_4 + x_1x_3x_4 + x_1x_3x_5 + x_2x_3x_5 - x_1x_2x_3x_4 - \\
&\quad x_1x_3x_4x_5 - x_1x_2x_3x_5 - x_2x_3x_4x_5 - x_1x_2x_3x_4x_5.
\end{aligned}$$

Using the recursion relationship (3.48), we obtain

$$\psi(\mathbf{x}, 2, 2) \;=\; x_1 + x_2 - x_1x_2$$

$$\begin{aligned}
\psi(\mathbf{x},3,2) &= \psi(\mathbf{x},2,2) - \psi(\mathbf{x},2,2)(1-x_2)(1-x_3)\\
&= x_2 + x_1x_3 - x_1x_2x_3\\
\psi(\mathbf{x},4,2) &= \psi(\mathbf{x},3,2) - \psi(\mathbf{x},3,2)(1-x_3)(1-x_4)\\
&= x_1x_3 + x_2x_3 + x_2x_4 - x_1x_2x_3 - x_2x_3x_4\\
\psi(\mathbf{x},5,2) &= \psi(\mathbf{x},4,2) - \psi(\mathbf{x},4,2)(1-x_4)(1-x_5)\\
&= x_2x_4 + x_1x_3x_4 + x_1x_3x_5 + x_2x_3x_5 - x_1x_2x_3x_4 -\\
&\quad x_1x_3x_4x_5 - x_1x_2x_3x_5 - x_2x_3x_4x_5 - x_1x_2x_3x_4x_5.
\end{aligned}$$

Using equation (3.52), we obtain

$$\begin{aligned}
\psi(\mathbf{x},2,2) &= x_1 + x_2 - x_1x_2\\
\psi(\mathbf{x},3,2) &= \psi(\mathbf{x},2,2) - x_1(1-x_2)(1-x_3)\\
&= x_2 + x_1x_3 - x_1x_2x_3\\
\psi(\mathbf{x},4,2) &= \psi(\mathbf{x},3,2) - x_2(1-x_3)(1-x_4)\\
&= x_1x_3 + x_2x_3 + x_2x_4 - x_1x_2x_3 - x_2x_3x_4\\
\psi(\mathbf{x},5,2) &= \psi(\mathbf{x},4,2) - \psi(\mathbf{x},2,2)x_3(1-x_4)(1-x_5)\\
&= x_2x_4 + x_1x_3x_4 + x_1x_3x_5 + x_2x_3x_5 - x_1x_2x_3x_4 -\\
&\quad x_1x_3x_4x_5 - x_1x_2x_3x_5 - x_2x_3x_4x_5 - x_1x_2x_3x_4x_5.
\end{aligned}$$

As in linear consecutive-k-out-of-n:G systems, the reliability functions of linear consecutive-k-out-of-n:F systems are stated as follows:

Theorem 3.4 The reliability function of a linear consecutive-k-out-of-n:F system, when all components are independent and $n \geq k$, is obtainable through

$$R_F(\mathbf{p},n,k) = R_F(\mathbf{p},n-1,k) - R_F(\mathbf{p},n-k-1,k))p_{n-k} \prod_{j=n-k+1}^{n} q_j. \quad (3.57)$$

If $p_1 = p_2 = \ldots = p_n = p$, the reliability function becomes

$$R_F(p,n,k) = R_F(p,n-1,k) - R_F(p,n-k-1,k)pq^k. \tag{3.58}$$

If $k \leq n \leq 2k$, the reliability function is

$$R_F(\mathbf{p},n,k) = 1 - \prod_{j=1}^{k} q_j - \sum_{i=1}^{n-k} p_i \prod_{j=i+1}^{i+k} q_j. \tag{3.59}$$

If $k \leq n \leq 2k$ and $p_1 = p_2 = \ldots = p_n = p$,

$$R_F(p,n,k) = 1 - [(n-k)p+1]q^k. \tag{3.60}$$

Proof:

Equations (3.57) and (3.59) are obtained by replacing x_i by p_i in (3.52) and (3.55). Equations (3.58) and (3.60) are obtained by the replacing p_i by p. □

By using (3.57) and (3.59), an algorithm can be produced to compute the reliability of this system This can easily be done by modifying the algorithm for the "G" systems. In this case, use input q_j instead of p_j to obtain system unreliability. Then, substract this quantity from 1 to obtain system reliability.

4 IMPORTANCE MEASURES OF SYSTEM COMPONENTS

4.1 Introduction

Several measures of importance have been proposed. Some of them originate in reliability theory, and others in game theory. Some measures are merely based on the structure of the system; and some others are based on the reliability and life of each component, in addition to the system structure. In terms of the structures of the system, all measures can be categorized as measures based on either critical vectors, minimal-path vectors, or minimal-cut vectors.

Birnbaum (1969) introduced a concept of component importance for coherent systems. He defined two types of component importance measures: structural importance and reliability importance. Barlow and Proschan (1975) proposed another measure of importance. Their definition of component importance is essentially the conditional probability of system failure caused by the failure of a given component. As in Birnbaum's definition of importance, in the absence of information about component reliability, Barlow and Proschan also defined the structural importance of a specific component, given that all components have the same reliability.

Vesely (1972) and Fussel (1975) introduced another concept of importance. Their importance measure is based on the number of cut sets that have failed and cause the failure of the system. This definition takes into account the fact that a failure of a component can be contributing to the failure of the system, without being critical. Lambert (1975) reviewed this measure and suggested another type of measure called criticality importance.

In the spirit of the Vesely and Fussel importance measure, Natvig (1979) defined a concept of importance measure based on the reduction in the remaining system life-time due to the failure of a specific component. Bergman (1985) proposed another measure related to this measure based on the expected life-time of a system. A general expression of some importance measures is given by Xie (1987).

The above importance measures were introduced originally in reliability theory. There are other importance measures that can be applied to reliability theory, which originate in game theory. Park (1985) studied those importance measures in the context of both areas. One importance measure, introduced as an importance measure of a player in an n-person game, was suggested by Shapley and Shubik (1954). This measure is based on the critical vectors of component c_i. In fact, this importance is equivalent to Barlow and Proschan's structural importance.

Another measure of importance originally introduced in game theory is a measure suggested by Deegan and Packel (1978). Different from previously discussed measures, this importance measure is based on minimal-path vectors associated with component c_i. The extension to the probabilistic interpretation of this measure was done by Park (1985), and we call it Park importance.

The objective of this chapter is to survey these measures by introducing the basic concept and properties of each measure. Some examples will be presented to clarify the ideas. Interrelationships among some measures, as well as the application of some measures to k-out-of-n systems, will be investigated.

4.2 Birnbaum Importance

The Birnbaum importance measure is based on the critical vectors of component c_i, as defined in Definition 2.7. There are two situations considered in Birnbaum importance measure: first, a situation where only the design of the system is known, but no information is available about the component reliabilities of the system; and second, a situation where both the structure and the component reliabilities are known. In the first situation, the Birnbaum's measure of importance is called the structural importance of component c_i; and for the second situation, the measure is defined as the reliability importance of component c_i.

4.2.1 Birnbaum structural importance

As stated in Definition 2.2, a component c_i is called relevant to the structure ϕ when $\delta_i(\mathbf{x}) = \phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x}) = 1$. Further, c_i is relevant to the functioning of ϕ when $(1 - x_i)\delta_i(\mathbf{x}) = 1$; that is, when $x_i = 0$, and $\delta_i(\mathbf{x}) = 1$. It is relevant for the failure of ϕ when $x_i\delta_i(\mathbf{x}) = 1$; that is, when $x_i = 1$, and $\delta_i(\mathbf{x}) = 1$.

The structural importance of component c_i for the functioning of structure ϕ is defined as

$$I_B^i(\phi, 1) = 2^{-n} \sum_{\mathbf{x}} (1 - x_i)\delta_i(\mathbf{x}), \tag{4.1}$$

and the structural importance of c_i for the failure of ϕ is defined as

$$I_B^i(\phi, 0) = 2^{-n} \sum_{\mathbf{x}} x_i\delta_i(\mathbf{x}). \tag{4.2}$$

The summation of both structural importances is the definition of Birnbaum structural importance. Formally, it is stated as follows:

Definition 4.1 The Birnbaum structural importance for component c_i in a system with structure ϕ is

$$I_B^i(\phi) = 2^{-(n-1)} n_\phi(i), \tag{4.3}$$

where

$$n_\phi(i) = \sum_{\{\mathbf{x}|x_i=1\}} \delta_i(\mathbf{x}) = \sum_{\{\mathbf{x}|x_i=1\}} [\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x})]. \tag{4.4}$$

In fact, the definition of Birnbaum structural importance is the number of critical-path vectors (or critical-cut vectors) associated with component c_i, divided by the number of all possible state vectors when $x_i = 1$ (or when $x_i = 0$).

Example 4.1 Consider a 4 component system with minimal-path vectors $(1, 1, 1, 0)$ and $(1, 0, 0, 1)$. The structure function of this system is then

$$\phi(\mathbf{x}) = x_1 x_2 x_3 + x_1 x_4 - x_1 x_2 x_3 x_4.$$

The derivatives of this function with respect to x_i, $i = 1, 2, 3, 4$, are

$$\begin{aligned}
\delta_1(\mathbf{x}) &= x_2 x_3 + x_4 - x_2 x_3 x_4, \\
\delta_2(\mathbf{x}) &= x_1 x_3 - x_1 x_3 x_4, \\
\delta_3(\mathbf{x}) &= x_1 x_2 - x_1 x_2 x_4, \\
\delta_4(\mathbf{x}) &= x_1 - x_1 x_2 x_3.
\end{aligned}$$

Using all possible values of x_i, for $i = 1, 2, 3, 4$, we obtain the Birnbaum importance for each component as follows:

$$I_B^1(\phi) = 2^{-(n-1)} \sum_{\{\mathbf{x}:x_1=1\}} \delta_1(\mathbf{x}) = 5/8,$$

$$I_B^2(\phi) = 2^{-(n-1)} \sum_{\{\mathbf{x}:x_2=1\}} \delta_2(\mathbf{x}) = 1/8,$$
$$I_B^3(\phi) = 2^{-(n-1)} \sum_{\{\mathbf{x}:x_3=1\}} \delta_3(\mathbf{x}) = 1/8,$$
$$I_B^4(\phi) = 2^{-(n-1)} \sum_{\{\mathbf{x}:x_4=1\}} \delta_4(\mathbf{x}) = 3/8.$$

Applying this measure to k-out-of-n systems, we find that all components in this system are equally important, as stated in the following lemma.

Lemma 4.1 All components in k-out-of-n systems have the same Birnbaum structural importance.

Proof:

For a k-out-of-n system, $\delta_i(\mathbf{x}) = 1$ if and only if exactly $k-1$ out of $n-1$ components other than c_i are functioning. Therefore, the Birnbaum structural importance of c_i is

$$I_B^i(\phi) = 2^{-(n-1)} \binom{k-1}{n-1}, \tag{4.5}$$

which is the same for all i. □

This lemma implies that the Birnbaum structural importance of all components in series and parallel systems are also equal since series systems are n-out-of-n systems and parallel system are 1-out-of-n systems.

4.2.2 Birnbaum reliability importance

Based on the structure function ϕ, and applying equation (2.5), the system reliability $R_\phi(\mathbf{p})$ can be written as

$$R_\phi(\mathbf{p}) \quad = \quad P[\phi(\mathbf{x}) = 1] = E[\phi(\mathbf{x})]$$

$$\begin{aligned} &= E[x_i\delta_i(\mathbf{x}) + \mu_i(\mathbf{x})] \\ &= p_i E[\delta_i(\mathbf{x})] + E[\mu_i(\mathbf{x})], \end{aligned}$$

where $\mu_i(\mathbf{x})$ is independent of p_i. Then the derivative of system reliability with respect to the reliability of component c_i is

$$\frac{\partial R_\phi(\mathbf{p})}{\partial p_i} = E[\delta_i(\mathbf{x})] = E[\frac{\partial \phi(\mathbf{x})}{\partial x_i}].$$

The reliability importance of c_i for the functioning of ϕ is defined as

$$I_B^i(R_\phi, 1) = E[(1 - x_i)\delta(\mathbf{x})] = (1 - p_i)\frac{\partial R_\phi(\mathbf{p})}{\partial p_i}, \tag{4.6}$$

and the reliability importance for the failure of ϕ is defined as

$$I_B^i(R_\phi, 0) = E[x_i\delta(\mathbf{x})] = p_i\frac{\partial R_\phi(\mathbf{p})}{\partial p_i}. \tag{4.7}$$

As with structural importance, the reliability importance of component c_i is obtained by summing $I_B^i(R_\phi, 1)$ and $I_B^i(R_\phi, 0)$, as stated in the following definition:

Definition 4.2 The Birnbaum reliability importance of component c_i in the structure ϕ is

$$I_B^i(R_\phi) = E(\delta_i(\mathbf{x})) = \frac{\partial R_\phi(\mathbf{p})}{\partial p_i} = R_\phi(1_i, \mathbf{p}) - R_\phi(0_i, \mathbf{p}). \tag{4.8}$$

One meaning of the Birnbaum reliability importance is the rate at which system reliability improves when the reliability of component c_i improves. That is, the difference of system reliability when component c_i is functioning and when component c_i is failed. Another meaning is the functioning probability of the system with component c_i is critical, given that the component is functioning.

Example 4.2 Consider the structure described in Example 4.1. Let the vector $\mathbf{p} = (p_1, p_2, p_3, p_4)$ be the reliabilities of c_i, for $i = 1, 2, 3, 4$. The reliability function of this structure is $R_\phi(\mathbf{p}) = p_1p_2p_3 + p_1p_4 - p_1p_2p_3p_4$. The Birnbaum reliability importance of each component is as follows:

$$\begin{aligned}
I_B^1(R_\phi) &= p_2p_3 + p_4 - p_2p_3p_4, \\
I_B^2(R_\phi) &= p_1p_3 - p_1p_3p_4, \\
I_B^3(R_\phi) &= p_1p_2 - p_1p_2p_4, \\
I_B^4(R_\phi) &= p_1 - p_1p_2p_3.
\end{aligned}$$

The application of this measure to series and parallel systems is stated in the following lemma.

Lemma 4.2 The most important component in series systems is the component with the lowest reliability and the most important component in parallel system is the component with the highest reliability.

Proof:

The reliability function of a series system is $R_\phi(\mathbf{p}) = \prod_{j=1}^n p_j$. Therefore, the Birnbaum importance of component c_i is

$$I_B^i(R_\phi) = \prod_{j \neq i}^n p_j = \frac{1}{p_i} \prod_{j=1}^n p_j, \tag{4.9}$$

which obtains its largest value at the smallest p_i. Similarly, the reliability function of a parallel system is $R_\phi(\mathbf{p}) = 1 - \prod_{j=1}^n (1 - p_j)$. Hence the Birnbaum importance of c_i is

$$I_B^i(R_\phi) = \prod_{j \neq i}^n (1 - p_j) = \frac{1}{1 - p_i} \prod_{j=1}^n (1 - p_j), \tag{4.10}$$

which obtains its largest value at the highest p_i. □

4.3 Shapley-Shubik Importance

Shapley and Shubik (1954) introduced an index for measuring the voting power of individuals, which is a special application of a general value concept introduced by Shapley (1953) in n-person game. It is assumed that a voter's value is the a priori chance that he will be the last member added to turn a losing coalition into a winning one. Taking this idea into systems context, we define this measure as an importance measure of components in an n-component system as follows:

Definition 4.3 Shapley-Shubik importance of component c_i in structure ϕ is

$$I^i_{SS}(\phi) = \sum_{\{\mathbf{x}:x_i=1\}} \frac{(x-1)!(n-x)!}{n!}\delta_i(\mathbf{x}), \tag{4.11}$$

where x is the number of nonzero elements in vector $\mathbf{x}$, and n is the number of components in the system.

Different from the Birnbaum importance, where the same weight is assigned to all possible values of $\delta_i(\mathbf{x})$, the Shapley-Shubik importance weighs each value of $\delta_i(\mathbf{x})$ differently. The computation of this measure, for a relatively small system, involves more terms than does the Birnbaum importance. However, using the relationship between multilinear extension and the Shapley value, this computation becomes much simpler. This relationship, as pointed out by Owen (1975), is

$$\int_0^1 g_i(t,t,\ldots,t)dt = \alpha_i(\phi),$$

where $g_i(\mathbf{x})$ is the partial derivative of multilinear extension $g_\phi(\mathbf{x})$ with respect to x_i. The form of $g_\phi(\mathbf{x})$ is the same as the Barlow-Proschan form of structure function. So we can treat $\phi(\mathbf{x})$ as $g_\phi(\mathbf{x})$, and hence $\delta_i(\mathbf{x})$ as $g_i(\mathbf{x})$. By integrating

$\delta_i(\mathbf{x})$ on the main diagonal $x_1 = x_2 = \ldots = x_n = x$ of unit cube $[0,1]^n$, we can obtain the Shapley-Shubik importance. Further, for coherent structure, we can replace the multilinear extension $g_\phi(\mathbf{x})$ by the structure function $\phi(\mathbf{x})$ of coherent systems. Then the Shapley-Shubik importance can be written as

$$I_{SS}^i(\phi) = \int_0^1 \delta_i(x, x, \ldots, x)dx. \tag{4.12}$$

Example 4.3 Consider a system with structure function

$$\phi(\mathbf{x}) = x_1x_2x_3 + x_1x_4 - x_1x_2x_3x_4.$$

Evaluating $\delta_i(\mathbf{x})$ at $x_i = x$, for $i = 1, 2, 3, 4$, we have

$$\begin{aligned}
\delta_1(x,x,x,x) &= x^2 + x - x^3, \\
\delta_2(x,x,x,x) &= x^2 - x^3, \\
\delta_3(x,x,x,x) &= x^2 - x^3, \\
\delta_4(x,x,x,x) &= x - x^3.
\end{aligned}$$

Using (4.12), we obtain

$$I_{SS}^1(\phi) = 7/12, \;\; I_{SS}^2(\phi) = 1/12, \;\; I_{SS}^3(\phi) = 1/12, \;\; I_{SS}^4(\phi) = 3/12.$$

As with the Birnbaum structural importance, the Shapley-Shubik importance is also equal for all components in k-out-of-n system as stated in the following lemma.

Lemma 4.3 All components in the k-out-of-n system have the same Shapley-Shubik importance.

Proof:

For k-out-of-n systems, $\delta_i(\mathbf{x}) = 1$ if and only if exactly $k-1$ out of $n-1$ components other than c_i are functioning. In this case, x is the cardinal of vector $\mathbf{x}$ such that $\delta_i(\mathbf{x}) = 1$ is equal to k for all i. Therefore, the Shapley-Shubik importance for component c_i is

$$I_{SS}^i(\phi) = \frac{(k-1)!(n-k)!}{n!}\binom{n-1}{k-1} = \frac{1}{n}, \tag{4.13}$$

which is constant for all i. □

Obviously, this lemma is also applicable to series and parallel systems.

4.4 Barlow-Proschan Importance

Barlow and Proschan (1975) consider a time-dependent approach when defining a component importance. They assume that if components fail sequentially in time and that if two or more components have a vanishingly small probability of occurring at the same instant, then one component must have caused the system to fail. Their measure is essentially the conditional probability of system failure caused by the failure of a given component. This measure reveals the relative extent to which each component is contributing to system failure. The derivation of this measure is based on the assumption of independent components.

4.4.1 Barlow-Proschan reliability importance

Let $F_i(t)$ be the continuous life distribution of component c_i; and, as defined in equation (2.36), let $X_i(t) = 1$ if c_i is functioning until time t, and 0 if c_i is failed before time t. The probability of c_i functioning until time t can be expressed by

$$P[X_i(t) = 1] \quad = \quad E[X_i(t)] = 1 - F_i(t) = S_i(t). \tag{4.14}$$

Denoting the vectors of component reliabilities by $S(t) = (S_1(t), S_2(t), \ldots, S_n(t))$, we can describe the system reliability as

$$R_\phi[S(t)] = P\{\phi[X(t)] = 1\} = E\{\phi[X(t)]\}. \tag{4.15}$$

The probability that at time t, the system is functioning if c_i is functioning; but it is failed otherwise, is given by

$$\begin{aligned} P\{[\phi(1_i, X(t)) - \phi(0_i, X(t))] = 1\} &= E\{[\phi(1_i, X(t)) - \phi(0_i, X(t))]\} \\ &= R_\phi[1_i, S(t)] - R_\phi[0_i, S(t)]. \end{aligned} \tag{4.16}$$

Thus, the probability that component c_i causes the system to fail in interval $[t, t+dt]$ is given by

$$\{R_\phi[1_i, S(t)] - R_\phi[0_i, S(t)]\}f_i(t)dt. \tag{4.17}$$

Integrating this between 0 and t, we obtain the probability that component c_i causes the system to fail:

$$\int_0^1 \{R_\phi[1_i, S(t)] - R_\phi[0_i, S(t)]\}f_i(t)dt. \tag{4.18}$$

The probability that the system has failed before time t is then given by

$$\sum_{j=1}^{n} \int_0^1 \{R_\phi[1_j, S(t)] - R_\phi[0_j, S(t)]\}f_j(t)dt. \tag{4.19}$$

So the probability that component c_i causes the system to fail in interval $[0, t]$, given the system failure in $[0, t]$, is

$$\frac{\int_0^t \{R_\phi[1_i, S(u)] - R_\phi[0_i, S(u)]\} f_i(u)du.}{\sum_{j=1}^n \int_0^t \{R_\phi[1_j, S(u)] - R_\phi[0_j, S(u)]\} f_j(u)du}. \tag{4.20}$$

Letting $t \to \infty$, and hence the denumerator becomes unity, then the numerator of equation (4.20) is defined to be the Barlow-Proschan importance measure.

Definition 4.4 The Barlow-Proschan importance of component c_i in a structure ϕ is defined as

$$I_{BP}^i(R_\phi) = \int_0^\infty \{R_\phi[1_i, S(t)] - R_\phi[0_i, S(t)]\} f_i(t)dt, \tag{4.21}$$

where $S(t) = [S_1(t), \ldots, S_{i-1}(t), S_{i+1}(t), \ldots, S_n(t)]$.

Basically, the Barlow-Proschan importance of component c_i is the probability of system failure because of a critical cut set containing component c_i failing, with c_i failing last. This measure has the following properties:

(1) $0 \leq I_{BP}^i \leq 1$

(2) $\sum_{i=1}^n I_{BP}^i = 1$

(3) If $n \geq 2$ and the intersection of supports of F_j $(j = 1, 2, \ldots, n)$ has positive probability with respect to the product $\prod_{j=1}^n F_j(t)$, then $0 < I_{BP}^i < 1$.

4.4.2 Proportional hazard case

It is difficult to compute I_{BP}^i for an arbitrary F_i. However, if we assume $S_i(t) = \exp[-\lambda_i R(t)]$, for $i = 1, 2, \ldots, n$, i.e., proportional hazard distribution, where $R(t)$

is the common hazard, then the calculation becomes more tractable. By a change of variable, we may assume that $R(t) = t$. So the Barlow-Proschan reliability importance for c_i in this case becomes

Definition 4.5 The Barlow-Proschan reliability importance for component c_i in a proportional hazard distribution, where $S_i(t)$=exp$(-\lambda_i t)$, is defined as

$$\begin{aligned} I^i_{BP}(R_\phi) &= \int_0^1 [R_\phi(p^{\lambda_1},\ldots,p^{\lambda_{i-1}},1,p^{\lambda_{i+1}},\ldots,p^{\lambda_n})] - \\ &\quad [R_\phi(p^{\lambda_1},\ldots,p^{\lambda_{i-1}},0,p^{\lambda_{i+1}},\ldots,p^{\lambda_n})]\lambda_i p^{\lambda_i - 1}dp, \quad (4.22) \end{aligned}$$

where $p = \exp(-t)$.

Example 4.4 Consider a system with structure function

$$\phi(\mathbf{x}) = x_1x_2x_3 + x_1x_4 - x_1x_2x_3x_4.$$

For $i = 1$ we obtain

$$\begin{aligned} R_\phi(1_1, S(t)) &= S_2(t)S_3(t) + S_4(t) - S_2(t)S_3(t)S_4(t) \\ &= p^{\lambda_2+\lambda_3} + p^{\lambda_4} - p^{\lambda_2+\lambda_3+\lambda_4}, \end{aligned}$$

and $R_\phi(0_1, S(t)) = 0$. By (4.22), the reliability importance of component c_1 is

$$\begin{aligned} I^1_{BP}(R_\phi) &= \int_0^1 (p^{\lambda_2+\lambda_3} + p^{\lambda_4} - p^{\lambda_2+\lambda_3+\lambda_4})\lambda_1 p^{\lambda_1-1}dp \\ &= \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} + \frac{\lambda_1}{\lambda_1+\lambda_4} - \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3+\lambda_4}. \end{aligned}$$

Similarly, for components c_2, c_3, c_4, we obtain

$$\begin{aligned} I^2_{BP}(R_\phi) &= \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3} - \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3+\lambda_4} \\ I^3_{BP}(R_\phi) &= \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3} - \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3+\lambda_4} \\ I^4_{BP}(R_\phi) &= \frac{\lambda_4}{\lambda_1+\lambda_4} - \frac{\lambda_4}{\lambda_1+\lambda_2+\lambda_3+\lambda_4}. \end{aligned}$$

For series and parallel systems, Barlow and Proschan (1975) obtained the following formulations:

Lemma 4.4 Assuming component c_i follows a proportional hazard distribution, where $S_i(t) = p^{\lambda_i}$, then the Barlow-Proschan importance of c_i for series systems is

$$I^i_{BP} = \lambda_i / \sum_{j=1}^{n} \lambda_j; \tag{4.23}$$

and for parallel systems, it is

$$I^i_{BP} = \lambda_i[\lambda_i^{-1} - \sum_{j \neq i}(\lambda_i + \lambda_j)^{-1} + \ldots + (-1)^{n-1}(\lambda_1 + \ldots + \lambda_n)^{-1}]. \tag{4.24}$$

4.4.3 Barlow-Proschan structural importance

In the absence of information concerning component reliabilities, it may be reasonable to assume that component life distribution for all components in the system are equal, that is, $S_1(t) = S_2(t) = \ldots = S_n(t) = p$. Assuming $S_i(t) = p$ for all i, Barlow-Proschan reliability importance is defined to be the Barlow-Proschan structural importance, or

Definition 4.6 Barlow-Proschan structural importance for component c_i in the structure ϕ is defined as

$$I^i_{BP}(\phi) = \int_0^1 [R_\phi(1_i, \mathbf{p}) - R_\phi(0_i, \mathbf{p})]dp, \tag{4.25}$$

where $\mathbf{p} = (p, p, \ldots, p)$.

Example 4.5 Consider Example 4.4. By setting $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$, we obtain the Barlow-Proschan structural importances for c_1, c_2, c_3, c_4 as

$$I^1_{BP}(\phi) = 7/12, \;\; I^2_{BP}(\phi) = 1/12, \;\; I^3_{BP}(\phi) = 1/12, \;\; I^4_{BP}(\phi) = 3/12,$$

where the order of these values agrees with the order of the Birnbaum structural importance described in Example 4.1.

In fact, Barlow-Proschan structural importance coincides with the Shapley-Shubik importance. The following lemma was given by Park (1985), which is equivalent to Theorem 4.1 of Barlow and Proschan (1975).

Lemma 4.5 When $S_i(t) = p_i = p$ for all $i = 1, 2, \ldots, n$, the Barlow-Proschan measure is the Shapley-Shubik measure.

Proof:

From (2.32) we know that

$$R_\phi(p_1, \ldots, p_n) = \sum_{\mathbf{x}} \prod_{i=1}^{n} p_i^{x_i}(1-p_i)^{(1-x_i)}\phi(\mathbf{x}). \tag{4.26}$$

When $p_i = p$, for $i = 1, 2, \ldots, n$,

$$R_\phi(p, \ldots, p) = \sum_{\mathbf{x}} p^x(1-p)^{n-x}\phi(\mathbf{x}), \tag{4.27}$$

where x is the number of nonzero elements in path vector $\mathbf{x}$. The reliability function when component c_i is functioning is

$$R_\phi(1_i, \mathbf{p}) = \sum_{\mathbf{x}} p^{x-1}(1-p)^{n-x}\phi(1_i, \mathbf{x}); \tag{4.28}$$

and the reliability function when component c_i is failed is

$$R_\phi(0_i, \mathbf{p}) = \sum_{\mathbf{x}} p^{x-1}(1-p)^{n-x}\phi(0_i, \mathbf{x}). \tag{4.29}$$

The Barlow-Proschan structural importance is then

$$I_{BP}^i(\phi) = \int_0^1 \sum_{\mathbf{x}} p^{x-1}(1-p)^{n-x}[\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x})]dp \tag{4.30}$$

$$= \sum_{\mathbf{x}} [\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x})] \int_0^1 p^{x-1}(1-p)^{n-x} dp \tag{4.31}$$

$$= \sum_{\mathbf{x}} \frac{(x-1)!(n-x)!}{n!} [\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x})] \tag{4.32}$$

$$= \sum_{\mathbf{x}} \frac{(x-1)!(n-x)!}{n!} \delta_i(\mathbf{x}), \tag{4.33}$$

where (4.32) is due to the solution of Euler's first integral. □

In fact, equation (4.33) can be written as

$$I^i_{BP}(\phi) = \sum_{r=1}^{n} n^{-1} \binom{n-1}{r-1} n_r(i), \tag{4.34}$$

where $n_r(i)$ is the number of critical-path vectors of size r associated with component c_i. This representation leads us to a comparison between the Barlow-Proschan measure $I^i_{BP}(\phi)$ and the Birnbaum measure $I^i_B(\phi)$.

From Definition 4.1, the Birnbaum measure is

$$I^i_B(\phi) = 2^{-(n-1)} \sum_{\{\mathbf{x}|x_i=1\}} \delta_i(\mathbf{x}) = 2^{-(n-1)} \sum_{r=1}^{n} n_r(i). \tag{4.35}$$

Clearly, the Barlow-Proschan structural importance attaches a weight $n^{-1}\binom{n-1}{k-1}^{-1}$ to $n_r(i)$ that depends on the size of critical path-vectors, whereas the Birnbaum measure attaches a common weight of $2^{-(n-1)}$ to $n_r(i)$.

4.5 Lambert Importance

Lambert (1975) suggested an importance measure which is called criticality importance. With this measure, the importance of component c_i is defined to be the conditional probability that a system is in a state at time t in which component c_i is critical and has failed given the failure of the system to which the component pertains. In terms of reliability functions and the failure distribution of component c_i, this measure is formally stated as follows:

Definition 4.7 The Lambert importance of component c_i in the structure ϕ is defined as

$$I_L^i = \frac{[R_\phi(1_i, S(t)) - R_\phi(0_i, S(t))]F_i(t),}{1 - R_\phi(S(t))} \tag{4.36}$$

where $F_i(t)$ is the failure probability of c_i at time t.

Suppose that at time t, the functioning probability of component c_i is $S_i(t) = p_i$. Then we can write this measure as

$$I_L^i = \frac{[R_\phi(1_i, \mathbf{p}) - R_\phi(0_i, \mathbf{p})](1 - p_i)}{1 - R_\phi(\mathbf{p})} = I_B^i \frac{q_i}{Q_\phi(\mathbf{p})}, \tag{4.37}$$

where I_B^i is the Birnbaum reliability importance for component c_i; q_i is the unreliability of component c_i; and $Q_\phi(\mathbf{p})$ is the unreliability of the system ϕ.

Example 4.6 Consider the system with the following reliability function:

$$R_\phi(\mathbf{p}) = p_1p_2p_3 + p_1p_4 - p_1p_2p_3p_4.$$

The Lambert importance for each component is as follows:

$$\begin{aligned}
I_L^1(R_\phi) &= q_1(p_2p_3 + p_4 - p_2p_3p_4)/(1 - p_1p_2p_3 - p_1p_4 + p_1p_2p_3p_4) \\
I_L^2(R_\phi) &= q_2(p_1p_3 - p_1p_3p_4)/(1 - p_1p_2p_3 - p_1p_4 + p_1p_2p_3p_4) \\
I_L^3(R_\phi) &= q_3(p_1p_2 - p_1p_2p_4)/(1 - p_1p_2p_3 - p_1p_4 + p_1p_2p_3p_4) \\
I_L^4(R_\phi) &= q_4(p_1 - p_1p_2p_3)/(1 - p_1p_2p_3 - p_1p_4 + p_1p_2p_3p_4).
\end{aligned}$$

The application of this measure to series and parallel systems is stated as

Lemma 4.6 **In series systems, components with the lowest reliability have the highest Lambert importance; whereas in parallel systems, all components have the same Lambert importance.**

Proof:

The reliability function of series systems is $R(\mathbf{p}) = \prod_{j=1}^{n} p_j$, so that $Q(\mathbf{p}) = 1 - \prod_{j=1}^{n} p_j$. The Lambert importance of component c_i is then

$$\begin{aligned} I_L^i &= q_i I_B^i / Q(\mathbf{p}) \\ &= q_i \prod_{j \neq i} p_j / (1 - \prod_{j=1}^{n} p_j) \\ &= (p_i^{-1} - 1) \prod_{j=1}^{n} p_j / (1 - \prod_{j=1}^{n} p_j), \end{aligned}$$

which obtains its largest value at the smallest p_i. For parallel system, the reliability of this system is $R(\mathbf{p}) = 1 - \prod_{j=1}^{n}(1 - p_j)$, so that $Q(\mathbf{p}) = \prod_{j=1}^{n}(1 - p_j)$. The Lambert importance of component c_i is then

$$\begin{aligned} I_L^i &= q_i I_B^i / Q(\mathbf{p}) \\ &= (1 - p_i) \prod_{j \neq i} (1 - p_j) / \prod_{j=1}^{n} (1 - p_j) \\ &= \prod_{j=1}^{n} (1 - p_j) / \prod_{j=1}^{n} (1 - p_j) = 1, \end{aligned}$$

which are the same for all components in the system. □

4.6 Natvig Importance

Natvig (1979) proposed another kind of importance measure. He suggested that the importance of component c_i is proportional to the expected reduction in the remaining life-time of a system due to the failure of this component. This measure is stated as follows:

Definition 4.8 Let Z_i be the reduction in the remaining system life-time due to the failure of component c_i. Then the Natvig importance for component c_i is defined as

$$I_N^i = \frac{E(Z_i)}{\sum_{j=1}^n E(Z_j)}, \tag{4.38}$$

with tacitly assuming that $E(Z_i) < \infty$ for $i = 1, 2, \ldots, n$.

It is obvious that this measure has the following properties:

(1) $0 \leq I_N^i \leq 1$ and

(2) $\sum_{i=1}^n I_N^i = 1$,

which are also true for the Barlow-Proschan measure. Natvig (1979) provided the formulation of the expected reduction in the remaining life-time of component c_i as

$$E(Z_i) = \int_0^\infty \sum_{(\bullet_i, \mathbf{x})} \prod_{j \neq i} F_j(t)^{1-x_j} S_j(t)^{x_j} \tag{4.39}$$

$$\cdot \int_0^\infty [R(H_t^{(1_i, \mathbf{x})}(u)) - R(H_t^{(0_i, \mathbf{x})}(u))] du f_i(t) dt, \tag{4.40}$$

where

$$R(H_t^{\mathbf{x}}(u)) = P[\phi(\mathbf{X}(t+u)) = 1 | \mathbf{X}(t) = \mathbf{x}] \tag{4.41}$$

$$H_t^{\mathbf{x}}(u) = (H_{1,t}^{x_1}(u), \ldots, (H_{n,t}^{x_n}(u)) \tag{4.42}$$

$$H_{i,t}^1 = S_i(t+u)/S_i(t), \quad H_{i,t}^0(u) = 0. \tag{4.43}$$

The application of this measure to series and parallel systems is given by Natvig (1979) in the following lemma.

Lemma 4.7 The Natvig importance for series systems is defined as

$$I_N^i = \frac{\int_0^\infty S_i(v)ln(S_i(v))\prod_{j\neq i} S_j(v)dv}{\sum_{i=1}^n \int_0^\infty S_i(v)ln(S_i(v))\prod_{j\neq i} S_j(v)dv}, \tag{4.44}$$

whereas for parallel systems it is

$$I_N^i = \frac{\int_0^\infty S_i(v)ln(S_i(v))\prod_{j\neq i} F_j(v)dv}{\sum_{i=1}^n \int_0^\infty S_i(v)ln(S_i(v))\prod_{j\neq i} F_j(v)dv}. \tag{4.45}$$

The computation of this measure for arbitrary distribution is very complicated, except for a special type such as proportional hazard distribution. The Natvig measure was furher studied by Natvig (1982, 1985) and Aven (1986). A result from Natvig (1985) is that I_N^i can be expressed as

$$I_N^i = \frac{\int_0^\infty S_i(v)\{-ln(S_i(v))\}I_B^i dv}{\sum_{i=1}^n \int_0^\infty S_i(v)\{-ln(S_i(v))\}I_B^j dv}, \tag{4.46}$$

which is a function of Birnbaum measure I_B^i.

4.7 Bergman Importance

Bergman (1985) proposed an importance measure based on system life-time. He assumes that a component is the most important in a system if a small improvement in its reliability performance, gives the best improvement of system reliability performance.

Let $\tau_1, \ldots, \tau_n$ be the random life of the n components in the system and let $F_1, \ldots, F_n$ be the corresponding life distributions of these components, which are assumed to be continuous. Then the expected system life is written as

$$\begin{aligned} E(\tau_\phi) &= \int_0^\infty P(\tau_\phi > t)dt = \int_0^\infty E(\phi(X(t)))dt & (4.47)\\ &= \int_0^\infty S_i(t)I_B^i dt + \int_0^\infty E(0_i, X(t))dt. & (4.48) \end{aligned}$$

Let $\tau_{\phi,i}$ be the life of the system when component c_i is replaced by a component with life distribution G_i. The difference in expected life length is then

$$\Delta_i = E(\tau_{\phi,i}) - E(\tau_{\phi}). \tag{4.49}$$

Representing the small component reliability improvement with the replacing life distribution G_i, a natural importance measure is the ratio of Δ_i to the summation of Δ_i, for $i = 1, 2, \ldots, n$. By studying the reduction of expected system life-time due to the infinitessimal scale change of the life distribution of c_i, Bergman (1985) defines an importance measure which can be stated as follows:

Definition 4.9 The Bergman importance of component c_i is defined as

$$I_{BE}^i = \frac{\int_0^\infty t I_B^i(t) dF_i(t)}{\sum_{j=1}^n \int_0^\infty t I_B^j(t) dF_j(t)}. \tag{4.50}$$

This measures coincides with Natvig's measure I_N^i when life distributions are Weibull with the same shape parameters. An application of this measure to series and parallel systems under exponential distribution is made by Bergman (1985) in the following lemma.

Lemma 4.8 If the life distributions are exponential with failure rates $\lambda_1, \ldots, \lambda_n$, and the system is a series, then

$$I_{BE}^i \propto \lambda_i / (\sum_{j=1}^n \lambda_j)^2; \tag{4.51}$$

and if the system is parallel, then

$$I_{BE}^i \propto \lambda_i [\lambda_i^{-2} - \sum_{j \neq i} (\lambda_i + \lambda_j)^{-2} + \ldots + (-1)^{n-1} (\lambda_1 + \ldots + \lambda_n)^{-2}]. \tag{4.52}$$

4.8 General Reliability Importance

In general, for nonstructural importance measures, a reasonable requirement of a measure is that it indicate how important the components are with respect to the chosen system reliability performance measure. Bergman (1985) listed some different reliability measures of systems, with respect to which it may be adequate to measure the reliability performance of a system in different situations:

1. Survival probability at time t_0, $P(\tau_\phi > t_0)$. This reliability performance measure is suitable if we want to assure that the system is functioning during a critical time interval $(0, t)$. This measure may be generalized to $P(\tau_\phi > t_2 | \tau_\phi > t_1)$, if the critical interval is (t_1, t_2).
2. Expected life, $E(\tau_\phi)$. This is the natural measure if we want a long life of the system rather than a high survival probability during a certain time interval.
3. Expected restricted life, $E(\min(\tau_\phi, T))$. This measure may be of interest in the same type of cases as the one above, but a finite time horizon T exists, after which the reliability of the system is of no interest, for instance because of obsolescence due to economic or technical reasons.
4. Discounted expected life, $E(1 - e^{-\alpha\tau_\phi})/\alpha$. Sometimes a failure late in the life of the system is not judged as critical as it would have been if the failure had occured early. This may be an effect of obsolescence.
5. Expected yield, $E(Y(\tau_\phi))$. Assuming $Y(.)$ to be an increasing random process, the expected yield during the sytem life-time is a natural performance measure in many situations.

He also pointed out that survival probability $P(\tau_\phi > t_0)$, expected life $E(\tau_\phi)$, and discounted expected life $E(1 - e^{-\alpha\tau_\phi})/\alpha$, are special cases of expected yield

$E(Y(\tau_\phi))$. Based on this fact, Xie (1987) developed another measure as a general form of importance measures of this type.

Definition 4.10 For any given yield function $Y(t)$, the general importance (Xie importance) of component c_i in a system is defined as

$$I_X^i = \int_0^\infty Y'(t).I_B^i(t)dF_i(t)/E(Y'), \tag{4.53}$$

where

$$E(Y') = \sum_{i=1}^n \int_0^\infty Y'(t).I_B^i(t)dF_i(t) = \int_0^\infty Y'(t)dF(t), \tag{4.54}$$

and $F(t)$ is the life-time distribution of the system.

As a special case, when $Y(t) = t^{p+1}$ for some constant $p \geq 0$, this measure then becomes

$$I_X^i \propto \int_0^\infty t^p.I_B^i(t)dF_i(t). \tag{4.55}$$

When $p = 0$, this measure turns out to be the Barlow-Proschan measure; and when $p = 1$, it coincides with the Bergman measure.

4.9 Deegan-Packel Importance

Deegan and Packel (1978) proposed an importance measure for a player in an n-person game. This measure is based on generating vectors. In the reliability context, we can express this measure in terms of minimal-path vectors.

Let $M(\phi) = \{\mathbf{z}_j\}_{j=1}^m$ be the set of minimal-path vectors of a structure function ϕ. Define the subset $M_i(\phi)$ of $M(\phi)$ as

$$M_i(\phi) = \{\mathbf{z}_j : z_{ij} = 1\},$$

where z_{ij} is the i-th coordinate of minimal-path vector $\mathbf{z}_j$. This set is interpreted as the set of minimal paths passing through component c_i.

Definition 4.11 The Deegan-Packel importance for component c_i is defined as

$$I_{DP}^{i}(\phi) = \frac{1}{m} \sum_{\mathbf{z}_j \in M_i} \frac{1}{z_j}, \tag{4.56}$$

where m is the number of minimal-path vectors of structure ϕ; and z_j is the number of nonzero elements in vector $\mathbf{z}_j$.

It is originally interpreted as the payoff expectation for player i for participating in forming the set of minimal winning coalitions in a simple n-person game. The following assumptions are used in defining this measure:

(1) Only minimal winning coalitions will emerge victorious;

(2) Each minimal coalition has equal probability of forming; and

(3) Players in a victorious minimal winning coalition divide the spoils equally.

Park (1985) interprets this measure as the probability of being selected when a minimal-path vector is selected at random from the set of all minimal-path vectors whose i-th coordinate is 1, followed by a random selection from among the nonzero coordinates of the selected minimal-path vector.

Suppose, in systems context, we can assume that: (1) only minimal-path vectors will emerge from the functioning of the system, (2) each minimal-path vector has equal probability of forming, and (3) components in a minimal-path vector have equal probability of functioning. Then we can use this measure to determine the importance of a component in a particular system.

Example 4.7 Consider the system described in Example 4.1. The set of minimal-path vectors for this structure is $P(\phi) = \{\mathbf{z}_1, \mathbf{z}_2\} = \{(1,1,1,0),(1,0,0,1)\}$, so that

$$P_1 = \{(1,1,1,0),(1,0,0,1)\} \quad P_2 = \{(1,1,1,0)\}$$
$$P_3 = \{(1,1,1,0)\} \quad P_4 = \{(1,0,0,1)\}.$$

Hence, the Deegan-Packel importance for c_1, c_2, c_3, c_4 are

$$I^1_{DP} = 5/12, \;\; I^2_{DP} = 1/6, \;\; I^3_{DP} = 1/6, \;\; I^4_{DP} = 3/12.$$

As the Birnbaum importance, Shapley importance, and Lambert importance, the application of this measure to k-out-of-n systems gives the result that the importance of all components in this system are the same.

Lemma 4.9 The Deegan-Packel importance of all components in k-out-of-n systems are equal.

Proof:

A minimal-path vector $\mathbf{z}_j$ in a k-out-of-n system has k nonzero elements and $n-k$ zero elements. There are $\binom{n}{k}$ minimal-path vectors in M and $\binom{n-1}{k-1}$ minimal-path vectors in M_i. Hence, $\mathbf{z}_j = k$, $m = \binom{n}{k}$, and $m_i = \binom{n-1}{k-1}$. The Deegan-Packel importance is then

$$I^i_{DP} = \frac{1}{\binom{n}{k}} \sum_{\mathbf{z}_j \in M_i} \frac{1}{k} = \frac{1}{\binom{n}{k}} \binom{n-1}{k-1} \frac{1}{k} = 1/n,$$

which is equal for all i. □

4.10 Park Importance

Park (1985) suggested an importance measure as an extension of the Deegan-Packel importance. His argument is that the probabilistic interpretation of the

Deegan-Packel importance is based on the assumption of uniformity. If we consider X_i as an independent binomial random variable, such that $P(X_i = 1) = p_i$ and $P(X_i = 0) = 1 - p_i$ for $i = 1, 2, \ldots, n$, then the uniformity assumption is no longer in effect.

Definition 4.12 Assume that the probability of forming a particular minimal-path vector $\mathbf{z}_j$ is

$$P_{\mathbf{z}_j} = \prod_{i=1}^{n} p_i^{z_{ji}} (1 - p_i)^{1 - z_{ji}}. \tag{4.57}$$

Then, the Park importance of component c_i in structure ϕ is defined as

$$I_P^i(\phi) = \sum_{\mathbf{z}_j \in M_i} \frac{P_{\mathbf{z}_j}}{\sum_{j=1}^{m} P_{\mathbf{z}_j}} \frac{1}{z_j}, \tag{4.58}$$

where $M_i = \{\mathbf{z}_j : z_{ij} = 1\}$; and m is the number of minimal-path vectors for structure ϕ.

Example 4.8 Consider the structure in previous examples. The minimal-path vectors of this structure are $\mathbf{z}_1 = (1, 1, 1, 0)$ and $\mathbf{z}_2 = (1, 0, 0, 1)$. Let $P = (p_1, p_2, p_3, p_4)$, so that $P_{\mathbf{z}_1} = p_1 p_2 p_3 (1 - p_4)$ and $P_{\mathbf{z}_2} = p_1 (1 - p_2)(1 - p_3) p_4$. The Park importance for component c_1, c_2, c_3, c_4 are then

$$\begin{aligned}
I_P^1 &= \frac{P_{\mathbf{z}_1}}{3(P_{\mathbf{z}_1} + P_{\mathbf{z}_2})} + \frac{P_{\mathbf{z}_2}}{2(P_{\mathbf{z}_1} + P_{\mathbf{z}_2})}, \\
I_P^2 &= \frac{P_{\mathbf{z}_1}}{3(P_{\mathbf{z}_1} + P_{\mathbf{z}_2})}, \\
I_P^3 &= \frac{P_{\mathbf{z}_1}}{3(P_{\mathbf{z}_1} + P_{\mathbf{z}_2})}, \\
I_P^4 &= \frac{P_{\mathbf{z}_1}}{3(P_{\mathbf{z}_1} + P_{\mathbf{z}_2})} + \frac{P_{\mathbf{z}_2}}{2(P_{\mathbf{z}_1} + P_{\mathbf{z}_2})}.
\end{aligned}$$

When $p_1 = p_2 = p_3 = p_4 = 1/2$, we obtain the Deegan-Packel importance, as shown in the above example.

The following lemma and corollaries show an application of this measure to k-out-of-n systems.

Lemma 4.10 Park importance of component c_l in k-out-of-n systems with unequally reliable components can be expressed as

$$I_P^l = \frac{1}{k} \frac{\sum_{j_l=1}^{m_l} \prod_{\{i:z_{j_l i}=1\}} p_i^*}{\sum_{j=1}^{m} \prod_{\{i:z_{ji}=1\}} p_i^*}, \tag{4.59}$$

where $p_i^* = p_i/(1-p_i)$, $m_l = \binom{n-1}{k-1}$, and $m = \binom{n}{k}$.

Proof:

A minimal-path vector $\mathbf{z}_j$ in k-out-of-n systems has exactly k nonzero elements and $n-k$ zero elements. There are $\binom{n}{k}$ minimal-path vectors for this system, where $\binom{n-1}{k-1}$ of them are minimal-path vectors with $z_{jl} = 1$. Hence $z_j = k$, $m = \binom{n}{k}$, and $m_l = \binom{n-1}{k-1}$. Note that j_l is the index of minimal-path vectors $\mathbf{z}_j$ in the set M_l, that is $M_l = \{\mathbf{z}_{j_l} : z_{ij_l} = 1\}$.

The probability $P_{\mathbf{z}_j}$ can be written as

$$\begin{aligned} P_{\mathbf{z}_j} &= \Big(\prod_{\{i:z_{ji}=1\}} (1-p_i)\Big)\Big(\prod_{\{i:z_{ji}=0\}} p_i\Big) \\ &= \Big(\prod_{\{i:z_{ji}=1\}} \frac{p_i}{1-p_i}\Big)\Big(\prod_{i=1}^{n} (1-p_i)\Big) \\ &= \Big(\prod_{\{i:z_{ji}=1\}} p_i^*\Big)\Big(\prod_{i=1}^{n} (1-p_i)\Big), \end{aligned}$$

where $p_i^* = p_i/(1-p_i)$. Therefore, from Definition 4.12, we obtain

$$\begin{aligned} I_P^l &= \sum_{z_j \in M_l} \frac{(\prod_{\{i:z_{ji}=1\}} p_i^*)(\prod_{i=1}^n (1-p_i))}{\sum_{j=1}^m (\prod_{\{i:z_{ji}=1\}} p_i^*)(\prod_{i=1}^n (1-p_i))} \frac{1}{z_j} \\ &= \frac{1}{k} \frac{\sum_{j_l=1}^{m_l} \prod_{\{i:z_{j_l i}=1\}} p_i^*}{\sum_{j=1}^m \prod_{\{i:z_{ji}=1\}} p_i^*}, \end{aligned}$$

which verifies (4.59). □

Corollary 4.1 The Park importance for component c_l in series systems is

$$I_P^l = \frac{1}{n} \tag{4.60}$$

and the Park importance for component c_l in parallel systems is

$$I_P^l = \frac{p_l^*}{\sum_{i=1}^n p_l^*}. \tag{4.61}$$

Proof:

For series systems, $k = n$ and $\prod_{\{i:z_{j_l i}=1\}} p_i^* = \prod_{\{i:z_{ji}=1\}} p_i^* = \prod_{i=1}^n p_i^*$. Substituting these quantities into (4.59) we obtain (4.60). For parallel system $k = 1$, $\prod_{\{i:z_{j_l i}=1\}} p_i^* = p_l^*$, and $\prod_{\{i:z_{ji}=1\}} p_i^* = p_i^*$. Substituting these quantities into (4.59) we obtain (4.61). □

Corollary 4.2 The Park importance of all components in k-out-of-n systems with equally reliable components are equal.

Proof:

When the system has equally reliable components, then $p_i^* = p/(1-p)$; and hence, $\prod_{\{i:z_{j_l i}=1\}} p_i^* = \prod_{\{i:z_{ji}=1\}} p_i^*$. Substituting these into (4.59), we obtain $I_P^l = 1/n$, which is equal for all l. □

These corollaries show that component reliability p_i in series systems, where all components in the system are equally important, has no effect to the Park importance. However, in parallel systems, component reliability p_i provides a contribution to the importance of that component, where a component with the highest reliability is the most important component according to this measure. In k-out-of-n systems, when all components have the same reliability, the importance of these components are also the same.

4.11 Vesely-Fussel Importance

Vesely (1972) introduced the concept of component importance later described by Fussel (1975). Lambert (1975) reviewed this measure and called it the Vesely-Fussel importance. The idea of this measure is that it is possible that two or more cut sets could have failed when a system has failed. In this case, restoring a failed component to a functioning state does not necessarily mean that the system is restored to a functioning state. Hence, it is possible that a failure of a component can be contributing to system failure without, its being critical.

Component c_i contributes to system failure if a cut set containing i has failed, that is, a structure function of a subsystem based on all cut sets containing component c_i has failed, i.e.,

$$\psi_i(\mathbf{x}(t)) = \prod_{j=1}^{r_i} \left(1 - \prod_{\{l:w_{jl}=0\}} (1 - x_l(t))\right) = 0, \tag{4.62}$$

where r_i is the number of minimal-cut sets consisting of component c_i. The probability that component c_i is contributing to the system is the probability that structure $\psi_i(\mathbf{x}(t))$ is equal to 0. Vesely-Fussel importance is defined to be the probability that component c_i is contributing to system failure, given that the sytem has failed by the time t, that is

Definition 4.13 The Vesely-Fussel importance of component c_i in structure ψ is defined as

$$I^i_{VF} = \frac{P(\psi_i(\mathbf{x}(t)) = 0)}{P(\psi(\mathbf{x}(t)) = 0)}, \tag{4.63}$$

where $\psi_i(\mathbf{x}(t))$ is given by (4.62).

Example 4.9 Consider a system with minimal-cut vectors $\{(0,1,1,1),(1,0,1,0),$ $(1,1,0,0)\}$. The structure function of this system is

$$\psi(\mathbf{x}) = x_1x_2x_3 + x_1x_4 - x_1x_2x_3x_4,$$

so the unreliability of the system is

$$P(\psi(\mathbf{x}) = 0) = 1 - p_1p_2p_3 - p_1p_4 + p_1p_2p_3p_4.$$

The structure functions $\psi_i(\mathbf{x})$, for $i = 1,2,3,4$, are then

$$\begin{aligned}
\psi_1(\mathbf{x}) &= x_1 \\
\psi_2(\mathbf{x}) &= x_2 + x_4 - x_2x_4 \\
\psi_3(\mathbf{x}) &= x_3 + x_4 - x_3x_4 \\
\psi_4(\mathbf{x}) &= x_2x_3 + x_4 - x_2x_3x_4;
\end{aligned}$$

and the unreliabilities of these subsystems are

$$\begin{aligned}
P(\psi_1(\mathbf{x}) = 0) &= 1 - p_1 \\
P(\psi_2(\mathbf{x}) = 0) &= 1 - p_2 - p_4 + p_2p_4 \\
P(\psi_3(\mathbf{x}) = 0) &= 1 - p_3 - p_4 + p_3p_4 \\
P(\psi_4(\mathbf{x}) = 0) &= 1 - p_2p_3 - p_4 + p_2p_3p_4.
\end{aligned}$$

Hence, the Vesely-Fussel importance of component c_i, $i = 1, 2, 3, 4$, is

$$
\begin{aligned}
I_{VF}^{1} &= (1 - p_1)/(1 - p_1p_2p_3 - p_1p_4 + p_1p_2p_3p_4) \\
I_{VF}^{2} &= (1 - p_2 - p_4 + p_2p_4)/(1 - p_1p_2p_3 - p_1p_4 + p_1p_2p_3p_4) \\
I_{VF}^{3} &= (1 - p_3 - p_4 + p_3p_4)/(1 - p_1p_2p_3 - p_1p_4 + p_1p_2p_3p_4) \\
I_{VF}^{4} &= (1 - p_2p_3 - p_4 + p_2p_3p_4)/(1 - p_1p_2p_3 - p_1p_4 + p_1p_2p_3p_4).
\end{aligned}
$$

5 THE MONOTONICITY OF COMPONENT IMPORTANCE MEASURES

5.1 Introduction

In Chapter 4, several measures of component importance have been surveyed, and the behavior of some of these measures in k-out-of-n systems have been investigated. Basically, in terms of system structure, these measures can be categorized as importance measures based on critical vectors, importance measures based on minimal-path vectors, or importance measures based on minimal-cut vectors. The first type of measure includes the Birnbaum measure and all measures involving the Birnbaum measure. The second type of measure includes the Deegan-Packel and Park measures, and the third type includes the Vesely-Fussel measure.

This chapter is devoted to studying the behavior of these measures in linear consecutive-k-out-of-n systems. For the first type, we include the Birnbaum and Barlow-Proschan measures, for the second type, we include the Deegan-Packel measure, and for the third type, we include the Vesely-Fussel measure. Specific formulations of these measures for linear consecutive-k-out-of-n systems, under the assumption of component independence, will be derived.

Before proceeding with this study, we need to clarify the term "monotonicity." As mentioned in Chapter 1, an importance measure of a component in linear consecutive-k-out-of-n systems is said to be monotone if the value of this measure is nondecreasing when the component gets closer to the center of the system. Formally, it can be stated as

Definition 5.1 An importance measure I^i for component c_i in linear consecutive-k-out-of-n system is said to be monotone if it satisfies

$$I^i \geq I^j \quad \text{when} \quad |\frac{1}{2}n - i| < |\frac{1}{2}n - j|. \tag{5.1}$$

Otherwise, it is called nonmonotone in this system.

5.2 The Monotonicity of Birnbaum Importance in Linear Consecutive-k-out-of-n:G Systems

As mentioned in Chapter 4, the Birnbaum importance of a component in a system depends on the availability of information about the reliability of that particular component in the system. If the reliability of the component is available, the measure is called reliability importance; otherwise it is called structural importance. In fact, the structural importance is a special case of the reliability importance where the common reliability of all components is a half. In this chapter, we will use the Birnbaum reliability importance to measure the importance of a component in linear consecutive-k-out-of-n systems.

Let $R_G(\mathbf{p}, i-1, k)$ be the reliability of a linear consecutive-k-out-of-(i-1):G system consisting of components 1 through component $i-1$, and let $R'_G(\mathbf{p}, n-i, k)$ be the reliability of a linear consecutive-k-out-of-(n-i):G system consisting of component $i+1$ through component n. The following lemma gives the expression of Birnbaum importance for linear consecutive-k-out-of-n:G systems:

Lemma 5.1 The Birnbaum importance of component c_i in a linear consecutive-k-out-of-n:G system with unequally reliable components is

$$I_B^i = \frac{1}{p_i}[R_G(\mathbf{p},n,k) - R_G(\mathbf{p},i-1,k) - R'_G(\mathbf{p},n-i,k) + R_G(\mathbf{p},i-1,k)R'_G(\mathbf{p},n-i,k)]. \qquad (5.2)$$

The proof of this lemma can be found in Zhang (1988) and is based on the pivotal decomposition of the reliability function $R_G(\mathbf{p},n,k)$ and the definition of linear consecutive-k-out-of-n:G systems. His work is parallel to the work of Papastavridis (1987) on the Birnbaum importance of linear consecutive-k-out-of-n:F systems. The following theorem shows that the Birnbaum importance is monotone in this system for $k \leq n \leq 2k$.

Theorem 5.1 In a linear consecutive-k-out-of-n:G system with equally reliable components, where $p_1 = p_2 = \ldots = p_n = p$, the Birnbaum importance is monotone when $k \leq n \leq 2k$.

Proof:

If all components are equally reliable, i.e., $p_1 = p_2 = \ldots = p_n = p$, then $R'_G(\mathbf{p},n-i,k) = R_G(\mathbf{p},n-i,k)$, so that (5.2) becomes

$$I_B^i = \frac{1}{p}[R_G(p,n,k) - R_G(p,i-1,k) - R_G(p,n-i,k) + R_G(p,i-1,k)R_G(p,n-i,k)]. \qquad (5.3)$$

Consider the case where $n = 2k$. For $i \leq k$, $R_G(p,i-1,k) = 0$, implying that

$$I_B^i = \frac{1}{p}[R_G(p,n,k) - R_G(p,n-i,k)], \qquad (5.4)$$

which increases from component 1 to component k since $R_G(p, n-i, k)$ decreases on i in this range. For $i \geq k+1$, $R_G(p, n-i, k) = 0$, implying that

$$I_B^i = \frac{1}{p}[R_G(p, n, k) - R_G(p, i-1, k)], \tag{5.5}$$

which decreases from component $k+1$ to component n since $R_G(p, i-1, k)$ increases on i in this range.

Now, consider the case when $n < 2k$. For $i \leq n-k+1$, $R_G(p, i-1, k) = 0$, implying that I_B^i is equal to (5.4). Thus, I_B^i is increasing on i in this range. When $i \geq k$, $R_G(p, n-i, k) = 0$, implying that I_B^i is equal to (5.5), which is decreasing on i in this range. It is obvious that for $n = 2k-1$, component k and component $n-k+1$ are the same component and that there is no component in between them. For $n < 2k-1$, however, there are components in between component $n-k+1$ and component k. The values of I_B^i for these components are equal since both $R_G(p, i-1, k)$ and $R_G(p, n-i, k)$ are vanished. Further, these values are equal to I_B^{n-k+1}, as well as to I_B^k. $\square$

The behavior of the Birnbaum importance in linear consecutive-k-out-of-n:G systems, especially for $n > 2k$, is interesting in the sense that even if the reliability of all components in the system are equal, numerical study shows that the importance of all components between component k and component $n-k+1$ are not equal.

Regarding the linear consecutive-k-out-of-n:G system, we would suppose, on heuristic ground, that the importance of components in this system is greater, the nearer the component is to the center of the system. However, using (5.3), it can easily be seen that for $n > 2k$, the Birnbaum importance increases steadily only from component 1 to component k and from component n to component $n-k+1$. A drop occurs from component k to component $k+1$, and also from component

$n - k + 1$ to component $n - k$. The nonmonotonicity of the Birnbaum importance will be studied throughout the investigation of the importance of these particular components.

The next theorem shows that, for $2k + 1 \leq n \leq 3k + 1$, or for $n > 3k + 1$ where $(1 - p^k)^k \geq 1 - p$, the importance of the k-th component is always greater than the importance of the $k + 1$-th component. Hence, by the symmetry of linear consecutive-k-out-of-n:G systems with equally reliable components, the importance of the $n - k + 1$-th component is always greater than the importance of the $n - k$-th component. Before we proceed with the theorem, we will prove the following lemmas:

Lemma 5.2 In the linear consecutive-k-out-of-r:G system with $k \geq 2$ and all components equally reliable with reliability p, if $k \leq r \leq 2k$ then

$$\frac{Q_G(p, r, k)}{Q_G(p, r - k, k)} > q. \tag{5.6}$$

Proof:

In proving this lemma, we use the abbreviated notation $Q(j)$ in place of $Q_G(p, j, k)$. First, look at the range $k \leq r \leq 2k - 1$. In this case $Q(r - k) = 1$ because in this range $(r - k)$ is less than k. This simplifies inequality (5.6) into $Q(r) > q$. Using equation (3.45), we can write $Q(r)$ as

$$Q(r) = 1 - [(r - k)(1 - p) + 1]p^k. \tag{5.7}$$

So, we need to show that $1 - [(r - k)(1 - p) + 1]p^k > q$, which is equivalent to showing

$$[(r - k)(1 - p) + 1]p^k < p. \tag{5.8}$$

The left-hand-side of (5.8) is maximized over the range $k \leq r \leq 2k-1$, at $r = 2k-1$. Denoting this maximum by $f(k)$, we find

$$f(k) = [(k-1)(1-p)+1]p^k = [k-(k-1)p]p^k. \tag{5.9}$$

This function decreases in k since

$$\begin{aligned} f(k+1) - f(k) &= [k+1-kp]p^{k+1} - [k-(k-1)p]p^k \\ &= -kp^k(1-p)^2 < 0, \end{aligned} \tag{5.10}$$

so that $f(k)$ is maximum at $k = 1$, where $f(1) = p$. So, for $k \geq 2$, the value of $f(k)$ will always be less than p. This shows that (5.8) holds, which was to be verified in order to show that (5.6) holds in the range $k \leq r \leq 2k-1$, or

$$\frac{Q(r)}{Q(r-k)} > q \quad \text{for } r = k, k+1, \ldots, 2k-1. \tag{5.11}$$

For $r = 2k$, we need to show that

$$Q(2k) > qQ(k). \tag{5.12}$$

From (3.43) and (3.45) we know that $Q(2k) = Q(2k-1) - qp^k$ and $Q(k) = 1 - p^k$. Substituting these into (5.12), we get $Q(2k-1) > q$, which is true by (5.11) for $r = 2k-1$. Consequently, we can conclude that

$$\frac{Q(r)}{Q(r-k)} > q \quad \text{holds for } r = k, k+1, \ldots, 2k, \tag{5.13}$$

which verifies (5.6). □

Lemma 5.3 In the linear consecutive-k-out-of-r:G system with $k \geq 2$ and all components equally reliable with reliability p, if $r > 2k$, the inequality

$$\frac{Q_G(p,r,k)}{Q_G(p,r-k,k)} > q \tag{5.14}$$

holds if $(1-p^k)^k \geq q$.

Proof:

The abbreviated notation $Q(j)$ will also be used in place of $Q_G(p,j,k)$. From (3.43), we can write $Q(r) = Q(r-1) - Q(r-k-1)qp^k$, so that the ratio of $Q(r)$ to $Q(r-1)$ is expressible as

$$\frac{Q(r)}{Q(r-1)} = 1 - \frac{Q(r-k-1)}{Q(r-1)}qp^k. \tag{5.15}$$

Furthermore, the ratio of $Q(r)$ to $Q(r-k)$ can be written as

$$\frac{Q(r)}{Q(r-k)} = \frac{Q(r-k+1)}{Q(r-k)} \cdot \frac{Q(r-k+2)}{Q(r-k+1)} \cdot \ldots \cdot \frac{Q(r-1)}{Q(r-2)} \cdot \frac{Q(r)}{Q(r-1)}; \tag{5.16}$$

and using (5.15), this ratio can be expressed as

$$\begin{aligned}
\frac{Q(r)}{Q(r-k)} &= (1 - \frac{Q(r-2k)}{Q(r-k)}qp^k)(1 - \frac{Q(r-2k+1)}{Q(r-k+1)}qp^k)\ldots \\
&\quad (1 - \frac{Q(r-k-2)}{Q(r-2)}qp^k)(1 - \frac{Q(r-k-1)}{Q(r-1)}qp^k) \qquad (5.17) \\
&= \prod_{i=1}^{k} (1 - \frac{Q(r-2k-1+i)}{Q(r-k-1+i)}qp^k). \qquad (5.18)
\end{aligned}$$

Equation (5.18) shows a recursion relationship between $Q(r)/Q(r-k)$ and k "previous" ratios $Q(r-2k-1+i)/Q(r-k-1+i)$, for $i = 1,2,\ldots,k$. This relationship can be used to obtain the ratio $Q(r)/Q(r-k)$ for $r \geq 2k+1$.

Starting with $r = 2k+1$, we write (5.18) as

$$\frac{Q(2k+1)}{Q(k+1)} = \prod_{i=1}^{k} (1 - \frac{Q(i)}{Q(k+i)}qp^k). \tag{5.19}$$

Note that (5.13) can be written as

$$\frac{Q(k+i)}{Q(i)} > q, \quad \text{for } i = 0,1,2,\ldots,k, \tag{5.20}$$

which is equivalent to

$$\frac{Q(i)}{Q(k+i)} < \frac{1}{q}, \quad \text{for } i = 0, 1, 2, \ldots, k. \tag{5.21}$$

Applying these inequalities, for $i = 1, 2, \ldots, k$, to (5.19), we obtain

$$\frac{Q(2k+1)}{Q(k+1)} > (1 - \frac{1}{q} qp^k)^k = (1 - p^k)^k. \tag{5.22}$$

Consequently, the inequality

$$\frac{Q(2k+1)}{Q(k+1)} > q \tag{5.23}$$

holds if $(1 - p^k)^k \geq q$. Hence, for p and k, such that $(1 - p^k)^k \geq q$, the inequality (5.6) holds for $r = k, k+1, \ldots, 2k+1$, which, in terms of i, can be written as

$$\frac{Q(k+i)}{Q(i)} > q, \quad \text{for } i = 0, 1, 2, \ldots, k+1. \tag{5.24}$$

This is equivalent to

$$\frac{Q(i)}{Q(k+i)} < \frac{1}{q}, \quad \text{for } i = 0, 1, 2, \ldots, k+1. \tag{5.25}$$

This completes the analysis for $r = k + 1$. For $r = 2k + 2$, we will write (5.18) as

$$\frac{Q(2k+2)}{Q(k+2)} = \prod_{i=2}^{k+1} (1 - \frac{Q(i)}{Q(k+i)} qp^k). \tag{5.26}$$

Applying (5.25), for $i = 2, 3, \ldots, k+1$, to (5.26), we obtain

$$\frac{Q(2k+2)}{Q(k+2)} > (1 - \frac{1}{q} qp^k)^k = (1 - p^k)^k, \tag{5.27}$$

and hence

$$\frac{Q(2k+2)}{Q(k+2)} > q \tag{5.28}$$

when $(1-p^k)^k \geq q$; in other words, we can conclude that for p and k, such that $(1-p^k)^k \geq q$, the inequality (5.6) holds for $r = k, k+1, \ldots, 2k+2$.

For $r > 2k+2$, the above procedure can be used inductively to show that $Q(r)/Q(r-k)$ is greater than q when $(1-p^k)^k \geq q$, by showing that the truth of (5.6) for $k \leq r \leq s$ implies the truth of (5.6) for $k \leq r \leq s+1$.

In particular, then, let us assume, for $r = s$, that

$$\frac{Q(s-2k+i)}{Q(s-k+i)} > q \quad \text{for } i = 1, 2, \ldots, k, \tag{5.29}$$

which is equivalent to

$$\frac{Q(s-k+i)}{Q(s-2k+i)} < \frac{1}{q} \quad \text{for } i = 1, 2, \ldots, k. \tag{5.30}$$

Using (5.18), for $r = s+1$, we obtain

$$\frac{Q(s+1)}{Q(s-k+1)} = \prod_{i=1}^{k} \left(1 - \frac{Q(s-2k+i)}{Q(s-k+i)} q p^k\right), \tag{5.31}$$

and applying (5.30) to (5.31), we obtain

$$\frac{Q(s+1)}{Q(s-k+1)} > \left(1 - \frac{1}{q} q p^k\right)^k = (1-p^k)^k; \tag{5.32}$$

hence,

$$\frac{Q(s+1)}{Q(s-k+1)} > q \tag{5.33}$$

for p and k such that $(1-p^k)^k > q$, which completes the proof that (5.6) holds under the given condition. □

Computer simulation study shows that the inequality $(1-p^k)^k \geq q$ for the condition of this lemma holds for $p \leq 0.61803$. However, for $p > 0.61803$, it requires that k be large. For example, if $p = 0.75$, then k should be greater than 4, and, if $p = 0.9$, then k should be at least 22. The required values of k increase rapidly for $p > 0.9$. Also, the study shows that the restriction on p and k in Lemma 5.3 can be relaxed, but no corresponding analytical proof has yet been developed.

Theorem 5.2 In linear consecutive-k-out-of-n:G systems with $k \geq 2$ and all components equally reliable with reliability p, the inequalities

$$I_B^k > I_B^{k+1} \text{ and } I_B^{n-k} < I_B^{n-k+1} \tag{5.34}$$

are true if one of the following conditions holds:

(1) $2k + 1 \leq n \leq 3k + 1$, or

(2) $n > 3k + 1$ and $(1 - p^k)^k \geq q$.

Proof:

Here, we use the abbreviated notation $R(j)$ in place of $R_G(\mathbf{p}, j, k)$. For $p_1 = p_2 = \ldots = p_n = p$, $R'(n-i)$ is equal to $R(n-i)$. Hence, the Birnbaum importance in (5.2) becomes $I_B^i = [R(n) - R(i-1) - R(n-i) + R(i-1)R(n-i)]/p$. The importance of the k-th component is then

$$\begin{aligned} I_B^k &= \frac{1}{p}[R(n) - R(k-1) - R(n-k) + R(k-1)R(n-k)] \\ &= \frac{1}{p}[R(n) - R(n-k)] \end{aligned} \tag{5.35}$$

because $R(k-1) = 0$, and the importance of the $k+1$-th component is

$$I_B^{k+1} = \frac{1}{p}[R(n) - R(k) - R(n-k-1) + R(k)R(n-k-1)]. \tag{5.36}$$

From equation (3.43) we know that

$$R(n-k) = R(n-k-1) + (1 - R(n-2k-1))qp^k; \tag{5.37}$$

and hence

$$R(n-k-1) = R(n-k) - (1 - R(n-2k-1))qp^k. \tag{5.38}$$

Substituting (5.38) into (5.36), we get

$$\begin{aligned} I_B^{k+1} = \frac{1}{p}[R(n) - R(k) - R(n-k) + (1 - R(n-2k-1))qp^k \\ +R(k)R(n-k-1)]. \end{aligned} \tag{5.39}$$

The difference between the importance of the $k+1$-th component and the importance of the k-th component becomes

$$\begin{aligned} I_B^{k+1} - I_B^k &= \frac{1}{p}[(1 - R(n-2k-1))qp^k - R(k)(1 - R(n-k-1))] \\ &= p^{k-1}[q(1 - R(n-2k-1)) - (1 - R(n-k-1))] \\ &= p^{k-1}[qQ(n-2k-1) - Q(n-k-1)]. \end{aligned} \tag{5.40}$$

This difference is negative when

$$qQ(n-2k-1) - Q(n-k-1) < 0; \tag{5.41}$$

that is, when

$$\frac{Q(n-k-1)}{Q(n-2k-1)} > q, \tag{5.42}$$

which is seen, in view of Lemma 5.2 and Lemma 5.3, to hold under the given conditions by the substitution of $r = n - k - 1$. By virtue of the symmetry of linear consecutive-k-out-of-n:G systems, the inequality $I_B^{n-k+1} > I_B^{n-k}$ is consequently true. □

This theorem establishes the nonmonotonicity of Birnbaum importance in "G" systems for n, p, and k on the range in question. Next, we investigate the monotonicity behavior of the Birnbaum measure in linear consecutive-k-out-of-n:F systems.

5.3 The Monotonicity of Birnbaum Importance in Linear Consecutive-k-out-of-n:F Systems

The analysis for linear consecutive-k-out-of-n:F systems is actually an extension of the analysis for linear consecutive-k-out-of-n:G systems. In this section, we will apply the duality property of these systems to obtain some results for the "F" systems from the "G" systems.

Let $R_F(\mathbf{p}, i-1, k)$ be the reliability function of a linear consecutive-k-out-of-(i-1):F system and $R'_F(\mathbf{p}, n-i, k)$ be the reliability function of a linear consecutive-k-out-of-(n-i):F system consisting of component $i+1$ through component n. The expression of Birnbaum importance for linear consecutive-k-out-of-n:F system, as given by Papastavridis (1987), is stated in the following lemma.

Lemma 5.4 The Birnbaum importance of component c_i in a linear consecutive-k-out-of-n:F system with unequally reliable components is

$$I_B^i = \frac{1}{q_i}[R_F(\mathbf{p}, i-1, k)R'_F(\mathbf{p}, n-i, k) - R_F(\mathbf{p}, n, k)]. \tag{5.43}$$

Note that this formulation can also be obtained by taking the dual expression of (5.2). The following theorem shows that for $k \leq n \leq 2k$, the Birnbaum importance is monotone in linear consecutive-k-out-of-n:F systems having equally reliable components.

Theorem 5.3 In linear consecutive-k-out-of-n:F systems with equally reliable components, the Birnbaum importance is monotone when $k \leq n \leq 2k$.

Proof:

When all components are equally reliable with common reliability p, the reliability functions $R'_F(\mathbf{p}, n-i, k)$ and $R_F(\mathbf{p}, n-i, k)$ are equal, so that, (5.43) becomes

$$I_B^i = \frac{1}{q}[R_F(p, i-1, k)R_F(p, n-i, k) - R_F(p, n, k)]. \tag{5.44}$$

Consider the case where $n = 2k$. When $i \leq k$, $R_F(p, i-1, k) = 1$, implying that

$$I_B^i = \frac{1}{q}[R_F(p, n-i, k) - R_F(p, n, k)], \tag{5.45}$$

which increases as i increases on this range. For $i \geq k+1$, $R_F(p, n-i, k) = 1$, implying that

$$I_B^i = \frac{1}{q}[R_F(p, i-1, k) - R_F(p, n, k)], \tag{5.46}$$

which decreases as i increases on this range.

Next, consider the case where $n < 2k$. For $i \leq n-k+1$, $R_F(p, i-1, k) = 1$, implying that I_B^i is equal to (5.45). So, I_B^i is increasing on i in this range. For $i \geq k$, $R_F(p, n-i, k) = 1$, implying I_B^i is equal (5.46), which is decreasing on i in this range. Note that for $n = 2k-1$, component $n-k+1$ and component k are overlapped so that there is no component between them. For $n < 2k-1$, however, there are components in between these two components. The value of I_B^i for these components are equal since both $R_F(p, i-1, k)$ and $R_F(p, n-i, k)$ are equal to 1. Further, these values are equal to I_B^{n-k+1}, as well as to I_B^k. □

For linear consecutive-k-out-of-n:F systems with $n > 2k$, the following theorem shows that for $2k+1 \leq n \leq 3k+1$, or for $n > 3k+1$ and $(1-q^k)^k \geq p$, the importance of the k-th component is always greater than the importance of the $k+1$-th component. Thus, by the symmetry of a linear consecutive-k-out-of-n:F system having equally reliable components, the importance of the $n - k + 1$-th component is always greater than the importance of the $n - k$-th component.

Theorem 5.4 In linear consecutive-k-out-of-n:F systems with $k \geq 2$ and all components equally reliable with reliability p, the inequalities

$$I_B^k > I_B^{k+1} \text{ and } I_B^{n-k} < I_B^{n-k+1} \tag{5.47}$$

are true if one of the following conditions holds:

(1) $2k + 1 \leq n \leq 3k + 1$, or

(2) $n > 3k + 1$ and $(1 - q^k)^k \geq p$.

Proof:

From the duality expression (3.53), we find that

$$P(\psi(\mathbf{x}, n, k) = 1) = P(\phi(\mathbf{1} - \mathbf{x}, n, k) = 0), \tag{5.48}$$

which is equivalent to

$$R_F(\mathbf{p}, n, k) = Q_G(\mathbf{q}, n, k). \tag{5.49}$$

From Lemma 5.2 and Lemma 5.3, we know that the inequality

$$\frac{Q_G(p, r, k)}{Q_G(p, r - k, k)} > q \tag{5.50}$$

holds under one of the conditions: (1) $k \leq r \leq 2k$ and (2) $r > 2k$ and $(1 - p^k)^k \geq q$. This implies that the inequality

$$\frac{Q_G(q, r, k)}{Q_G(q, r - k, k)} > p, \tag{5.51}$$

or by (5.49), the inequality

$$\frac{R_F(p,r,k)}{R_F(p,r-k,k)} > p, \tag{5.52}$$

holds under one of the following conditions:

(1) $k \le r \le 2k$ or

(2) $r > 2k$ and $(1-q^k)^k \ge p$.

The Birnbaum importance for component c_i in a linear consecutive-k-out-of-n:F system when the system has equally reliable components, is given by (5.44). Then the importance of the k-th component is

$$\begin{aligned} I_B^k &= \frac{1}{q}[R_F(p,k-1,k)R_F(p,n-k,k) - R_F(p,n,k)] && (5.53)\\ &= \frac{1}{q}[R_F(p,n-k,k) - R_F(p,n,k)] && (5.54) \end{aligned}$$

because $R_F(p,k-1,k) = 1$, and the importance of the $k+1$-th component is

$$I_B^{k+1} = \frac{1}{q}[R_F(p,k,k)R_F(p,n-k-1,k) - R_F(p,n,k)]. \tag{5.55}$$

The difference of the importance between the two components is

$$I_B^{k+1} - I_B^k = \frac{1}{q}[R_F(p,k,k)R_F(p,n-k-1,k) - R_F(p,n-k,k)]. \tag{5.56}$$

Using the fact that $R_F(p,k,k) = 1-q^k$, and by applying (3.58) to $R_F(p,n-k,k)$, we can write equation (5.56) as

$$I_B^{k+1} - I_B^k = q^{k-1}[-R_F(p,n-k-1,k) + R_F(p,n-2k-1,k)p]. \tag{5.57}$$

This difference is negative when

$$\frac{R_F(p,n-k-1,k)}{R_F(p,n-2k-1,k)} > p, \tag{5.58}$$

which is seen, in view of (5.52) and its conditions, to hold by the substitution of $r = n - k - 1$. By the symmetry of linear consecutive-k-out-of-n:F systems, the inequality $I_B^{n-k+1} > I_B^{n-k}$ is consequently true. □

This theorem establishes the nonmonotonicity of the Birnbaum importance in linear consecutive-k-out-of-n:F systems. In the following section we will investigate the monotonicity of the Barlow-Proschan importance in the "G" systems.

5.4 The Monotonicity of Barlow-Proschan Importance in Linear Consecutive-k-out-of-n:G Systems

As mentioned in Definition 4.4, the Barlow-Proschan importance measure for component c_i is in fact the integration of the Birnbaum measure with respect to the failure distribution of that component. In the proportional hazard case, where the reliability of c_i is assumed to be p^{λ_i}, this measure becomes simply the integration of the Birnbaum measure with respect to p. In this section, we investigate the importance of components in linear consecutive-k-out-of-n:G systems based on the proportional hazard assumption. The monotonicity of this measure will be investigated under the assumption of equal proportional hazard distribution, that is, $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \lambda$.

The Barlow-Proschan importance for this system cannot be evaluated for an arbitrary number of components in the system. This is because the explicit expression, in nonrecursion form, of reliability function R for this system is not available for arbitrary n except for $k \leq n \leq 2k$. The following lemma is the formulation of the Barlow-Proschan importance for $k \leq n \leq 2k$.

Lemma 5.5 The Barlow-Proschan importance of component c_i in linear consecutive-k-out-of-n:G systems with component reliability p^{λ_i}, for $i = 1, 2, \ldots, n$, and $k \le n \le 2k$, can be formulated as

$$I^i_{BP} = \lambda_i \Big[\sum_{l=0}^{n-k} \Big(\sum_{j=l+1}^{l+k} \lambda_j \Big)^{-1} \mathcal{I}_1(i) - \sum_{l=1}^{n-k} \Big(\lambda_l \sum_{j=l+1}^{l+k} \lambda_j \Big)^{-1} \mathcal{I}_2(i) \Big], \tag{5.59}$$

where

$$\mathcal{I}_1(i) = \begin{cases} 1 & \text{if } i \in \{l+1, \ldots, l+k\} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{I}_2(i) = \begin{cases} 1 & \text{if } i \in \{l, \ldots, l+k\} \\ 0 & \text{otherwise.} \end{cases}$$

Proof:

The structure function of linear consecutive-k-out-of-n:G systems for $k \le n \le 2k$ from Corollary 3.1 can be written as

$$\phi(\mathbf{x}, n, k) = \sum_{l=0}^{n-k} \prod_{j=l+1}^{l+k} x_j - \sum_{l=1}^{n-k} x_l \prod_{j=l+1}^{l+k} x_j, \tag{5.60}$$

and since all components are assumed to be independent, the reliability function of this system is

$$R(\mathbf{p}, n, k) = \sum_{l=0}^{n-k} \prod_{j=l+1}^{l+k} p_j - \sum_{l=1}^{n-k} p_l \prod_{j=l+1}^{l+k} p_j. \tag{5.61}$$

Making $p_j = p^{\lambda_j}$, we obtain

$$\begin{aligned} R(\mathbf{p}, n, k) &= \sum_{l=0}^{n-k} \prod_{j=l+1}^{l+k} p^{\lambda_j} - \sum_{l=1}^{n-k} p^{\lambda_l} \prod_{j=l+1}^{l+k} p^{\lambda_j} && (5.62) \\ &= \sum_{l=0}^{n-k} p^{\sum_{j=l+1}^{l+k} \lambda_j} - \sum_{l=1}^{n-k} p^{\lambda_l + \sum_{j=l+1}^{l+k} \lambda_j}. && (5.63) \end{aligned}$$

The Birnbaum importance of c_i is the derivative of $R(\mathbf{p}, n, k)$, with respect to component reliability p^{λ_i}. So, the Barlow-Proschan importance is

$$\begin{aligned}
I^i_{BP} &= \int_0^1 \frac{\partial}{\partial p^{\lambda_i}}[R(\mathbf{p}, n, k)]\lambda_i p^{\lambda_i - 1} dp \\
&= \lambda_i \int_0^1 \frac{\partial}{\partial p^{\lambda_i}}[\sum_{l=0}^{n-k} p^{\sum_{j=l+1}^{l+k} \lambda_j} - \sum_{l=1}^{n-k} p^{\lambda_l + \sum_{j=l+1}^{l+k} \lambda_j}] p^{\lambda_i - 1} dp \\
&= \lambda_i [\sum_{l=0}^{n-k} (\sum_{j=l+1}^{l+k} \lambda_j)^{-1} \mathcal{I}_1(i) - \sum_{l=1}^{n-k} (\lambda_l + \sum_{j=l+1}^{l+k} \lambda_j)^{-1} \mathcal{I}_2(i)], \quad (5.64)
\end{aligned}$$

where the last equation is the result of the derivation of the terms in the brackets of the previous equation multiplied by $p^{\lambda_i - 1}$ and followed by integration with respect to p on the domain [0,1]. The identity functions $\mathcal{I}_1(i)$ and $\mathcal{I}_2(i)$ are used to keep the terms $(\lambda_{l+1} + \ldots + \lambda_{l+k})^{-1}$ and $(\lambda_l + \ldots + \lambda_{l+k})^{-1}$ containing λ_i and to eliminate the ones containing no λ_i. □

Example 5.1 Consider a linear consecutive-3-out-of-5:G system with component reliability p^{λ_i}, for $i = 1, 2, \ldots, 5$. The Barlow-Proschan importance of the first component is

$$\begin{aligned}
I^1_{BP} &= \lambda_1[(\lambda_1 + \lambda_2 + \lambda_3)^{-1} + (\lambda_2 + \lambda_3 + \lambda_4)^{-1}.0 + (\lambda_3 + \lambda_4 + \lambda_5)^{-1}.0 \\
&\quad -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^{-1} - (\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)^{-1}.0] \\
&= \lambda_1[(\lambda_1 + \lambda_2 + \lambda_3)^{-1} - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^{-1}].
\end{aligned}$$

Similarly, the Barlow-Proschan importance of the remaining components are:

$$\begin{aligned}
I^2_{BP} &= \lambda_2[(\lambda_1 + \lambda_2 + \lambda_3)^{-1} - (\lambda_2 + \lambda_3 + \lambda_4)^{-1}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^{-1} \\
&\quad -(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)^{-1}]
\end{aligned}$$

$$\begin{aligned}
I^3_{BP} &= \lambda_3[(\lambda_1+\lambda_2+\lambda_3)^{-1}+(\lambda_2+\lambda_3+\lambda_4)^{-1}+(\lambda_3+\lambda_4+\lambda_5)^{-1}\\
&\quad -(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^{-1}-(\lambda_2+\lambda_3+\lambda_4+\lambda_5)^{-1}]\\
I^4_{BP} &= \lambda_4[(\lambda_2+\lambda_3+\lambda_4)^{-1}+(\lambda_3+\lambda_4+\lambda_5)^{-1}\\
&\quad -(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^{-1}-(\lambda_2+\lambda_3+\lambda_4+\lambda_5)^{-1}]\\
I^5_{BP} &= \lambda_5[(\lambda_3+\lambda_4+\lambda_5)^{-1}-(\lambda_2+\lambda_3+\lambda_4+\lambda_5)^{-1}].
\end{aligned}$$

For the system with $n > 2k$ and unequally reliable components, the explicit formulation of the Barlow-Proschan importance becomes very complex since there is no simple formula available for the reliability function, except in the recursion relation. However, if the system has equal component reliability, this measure becomes the Barlow-Proschan structural importance which turns out to be the Shapley-Shubik importance. This can be written as

$$I^i_{BP} = \sum_{\{\mathbf{x}:x_i=1\}} \frac{(x-1)!(n-x)!}{n!}[\phi(1_i,\mathbf{x},n,k)-\phi(0_i,\mathbf{x},n,k)], \qquad (5.65)$$

where x is the number of nonzero elements in vector $\mathbf{x}$; $\phi(1_i,\mathbf{x},n,k)$ is the structure function of linear consecutive-k-out-of-n:G systems when component c_i is functioning; and $\phi(0_i,\mathbf{x},n,k)$ is the structure function of this system when component c_i has failed.

The following theorem establishes the monotonicity of the Barlow-Proschan measure in linear consecutive-k-out-of-n:G systems when $k \le n \le 2k$ and equal proportional hazard distribution is assumed.

Theorem 5.5 In the linear consecutive-k-out-of-n:G system with all its components following equal proportional hazard distribution, the Barlow-Proschan importance is monotone when $k \le n \le 2k$.

Proof:

If $\lambda_i = \lambda$, then the Barlow-Proschan importance in (5.59) becomes

$$I^i_{BP} = \lambda[\sum_{l=0}^{n-k} \frac{1}{k\lambda}\mathcal{I}_1(i) - \sum_{l=1}^{n-k} \frac{1}{(k+1)\lambda}\mathcal{I}_2(i)] \tag{5.66}$$

$$= \frac{1}{k}\sum_{l=0}^{n-k} \mathcal{I}_1(i) - \frac{1}{(k+1)}\sum_{l=1}^{n-k} \mathcal{I}_2(i) \tag{5.67}$$

$$= \frac{1}{k(k+1)}[\sum_{l=0}^{n-k} \mathcal{I}_1(i) + k(\sum_{l=0}^{n-k} \mathcal{I}_1(i) - \sum_{l=1}^{n-k} \mathcal{I}_2(i))]. \tag{5.68}$$

If $n = 2k$, then $\sum_{l=0}^{n-k} \mathcal{I}_1(i) = \sum_{l=1}^{n-k} \mathcal{I}_2(i)$ for all i; and hence

$$I^i_{BP} = \frac{1}{k(k+1)} \sum_{l=0}^{n-k} \mathcal{I}_1(i),$$

which increases from component 1 through component k, remains constant from component k to component $k+1$, and decreases from component $k+1$ to component n.

When $n < 2k$,

$$\sum_{l=0}^{n-k} \mathcal{I}_1(i) = \sum_{l=1}^{n-k} \mathcal{I}_2(i) \text{ for } i = 1, 2, \ldots, n-k, k+1, \ldots, n;$$

and

$$\sum_{l=0}^{n-k} \mathcal{I}_1(i) = \sum_{l=1}^{n-k} \mathcal{I}_2(i) + 1 = n-k+1 \text{ for } i = n-k+1, \ldots, k.$$

In this case, the Barlow-Proschan importance becomes

$$I^i_{BP} = \begin{cases} \frac{1}{k(k+1)} \sum_{l=0}^{n-k} \mathcal{I}_1(i) & \text{for } i = 1, 2, \ldots, n-k, k+1, \ldots, n, \\ \frac{1}{k(k+1)}(n+1) & \text{for } i = n-k+1, \ldots, k \end{cases}$$

which increases from component 1 through component $n-k+1$, remains constant from component $n-k+1$ through component k, and decreases from component k through component n. $\square$

The nonmonotonicity of the Barlow-Proschan measure in linear consecutive-k-out-of-n:G systems when $n > 2k$ can be evaluated by checking the importance of component k and component $k+1$, and also component $n-k$ and $n-k+1$. The following theorem states that, for $0 < p < 1$, and with $2k+1 \leq n \leq 3k+1$, the importance of component k is greater than the importance of component $k+1$. Similarly, the importance of component $n-k+1$ is greater than the importance of component $n-k$, so that monotonicity does not pertain for any p in (0,1).

Theorem 5.6 In a linear consecutive-k-out-of-n:G system with all components following equal proportional hazard distribution, if $2k+1 \leq n \leq 3k+1$, then

$$I_{BP}^{k} > I_{BP}^{k+1} \text{ and } I_{BP}^{n-k} < I_{BP}^{n-k+1}. \tag{5.69}$$

Proof:

The Barlow-Proschan importance, in the case of equal proportional hazard, can be written in terms of the Birnbaum importance as

$$I_{BP}^{i} = \lambda \int_0^1 I_B^i p^{\lambda-1} dp. \tag{5.70}$$

Without loss of generality, we can assume $\lambda = 1$, so that

$$I_{BP}^{i} = \int_0^1 I_B^i dp. \tag{5.71}$$

The difference of the importance of component k and component $k+1$ for the Barlow-Proschan measure is then

$$I_{BP}^{k} - I_{BP}^{k+1} = \int_0^1 (I_B^k - I_B^{k+1}) dp. \tag{5.72}$$

This is greater than zero since, by Theorem 5.1, $I_B^k > I_B^{k+1}$ for k and n in the range in question. So, $I_{BP}^{k} > I_{BP}^{k+1}$. By the symmetry of this system under the assumption of independence and equal reliability, $I_{BP}^{n-k} < I_{BP}^{n-k+1}$. $\square$

The same conclusion can be derived for a given k and a sufficiently large n.

Theorem 5.7 In a linear consecutive-k-out-of-n:G system with equally reliable components,

$$I_{BP}^{k} > I_{BP}^{k+1} \text{ and } I_{BP}^{n-k} < I_{BP}^{n-k+1} \tag{5.73}$$

for a given k and a sufficiently large n.

Proof:

For $n > 3k+1$, the difference of the Birnbaum importance of component k and component $k+1$, in view of (5.40),

$$\begin{aligned} I_B^k - I_B^{k+1} &= p^{k-1}[Q(p,n-k-1,k) - q(Q(p,n-2k-1,k)] && (5.74) \\ &= p^{k-1}Q(p,n-2k-1,k)[\frac{Q(p,n-k-1,k)}{(Q(p,n-2k-1,k)} - q]. && (5.75) \end{aligned}$$

It has been shown, when proving Lemma 5.3, that for $r > 2k$,

$$\frac{Q(p,r,k)}{Q(p,r-k,k)} > (1-p^k)^k, \tag{5.76}$$

so that, for $r = n-k-1$, i.e. $n > 3k+1$

$$\frac{Q(p,n-k-1,k)}{Q(p,n-2k-1,k)} > (1-p^k)^k. \tag{5.77}$$

Equation (5.75) then leads to

$$I_B^k - I_B^{k+1} > p^{k-1}Q(p,n-2k-1,k)[(1-p^k)^k - q]. \tag{5.78}$$

A lower bound for $R(p,n,k)$ in a linear consecutive-k-out-of-n:F system, as in Chiang and Niu (1981), is $(1-(1-p)^k)^{n-k+1}$. A lower bound for $Q(p,n,k)$ in a linear consecutive-k-out-of-n:G system is thus $(1-p^k)^{n-k+1}$; and hence a lower bound for $Q(p, n-2k-1, k)$ in such a system is $(1-p^k)^{n-3k}$. Substituting this into (5.78), we obtain

$$I_B^k - I_B^{k+1} > p^{k-1}(1-p^k)^{n-3k}[(1-p^k)^k - q]. \tag{5.79}$$

In view of (5.71), the difference of the Barlow-Proschan importance of component k and component $k+1$ therefore satisfies

$$I_{BP}^k - I_{BP}^{k+1} > \int_0^1 p^{k-1}(1-p^k)^{n-3k}[(1-p^k)^k - q]dp. \tag{5.80}$$

It remains now to show that the right-hand-side of (5.80) is greater than 0.

Denote the integrand of the right-hand-side of (5.80) by $f(p,k,n)$. To show that the result of integration is positive means to show that the signed area of $f(p,k,n)$ on the interval $[0,1]$ exceeds zero. To this end, consider the three factors $f_1(p,k,n) = (1-p^k)^{n-3k}$, $f_2(p,k) = p^{k-1}$, and $f_3(p,k) = (1-p^k)^k - (1-p)$ of the integrand $f(p,k,n)$. Both f_1 and f_2 are positive on $(0,1)$, where f_1 is decreasing and f_2 is increasing on p. However, unlike f_1 and f_2, f_3 is not always positive on $(0,1)$. The following argument develops a lower bound for f_3 on $(0,1)$.

Consider the Binomial expansion of $(1-p^k)^k$:

$$\begin{aligned}(1-p^k)^k &= \sum_{i=0}^{k}(-1)^i k! p^{ik}/i!(k-i)! & (5.81)\\ &= 1 - kp^k + \frac{1}{2!}k(k-1)p^{2k} - \frac{1}{3!}k(k-1)(k-2)p^{3k} + \ldots, & (5.82)\end{aligned}$$

from which we obtain

$$\begin{aligned}(1-p^k)^k - (1-p) &= p - kp^k + \frac{1}{2!}k(k-1)p^{2k} - \frac{1}{3!}k(k-1)(k-2)p^{3k} + \ldots\\ &= p\sum_{i=0}^{k}(-1)^i k! p^{ik-1}/i!(k-i)!. & (5.83)\end{aligned}$$

Denote the i-th term of the series in the right-hand-side of (5.83) by a_i. The ratio of the magnitude of the term a_{i+1} to the magnitude of the term a_i, for $i = 0, 1, \ldots, k$, is then

$$\frac{|a_{i+1}|}{|a_i|} = \frac{(k-i)}{(i+1)} p^k. \tag{5.84}$$

This ratio decreases with i, so that all ratios are no greater than kp^{k-1}, which implies, in turn, that if $kp^{k-1} < 1$, then all ratios are less than 1. Hence, under this condition, using the first two terms of the right-hand-side of (5.83), we obtain

$$f_3(p,k,n) = (1-p^k)^k - (1-p) > p - kp^k, \tag{5.85}$$

which produces a lower bound for f_3. This lower bound holds for the condition kp^{k-1}, or, equivalently, holds for $p < p^* = k^{-(k-1)^{-1}}$. In fact, the right-hand-side of (5.85) is the bound of the positive part of f_3. The bound for the negative part of f_3 is obtained as follows:

Let p_0 be a value of p such that $p_0 > p^*$ and p_0 be the first root of f_3 to the right of p^*. If $p > p_0$, then $(1-p_0) - (1-p) > 0$, so that

$$(1-p^k)^k + (1-p_0) - (1-p) > 0 \tag{5.86}$$

or equivalent to

$$(1-p^k)^k - (1-p) > -(1-p_0), \tag{5.87}$$

which produces a lower bound for f_3 on $(p_0, 1)$.

Now, let p_1 be a value of p on $(0, p^*)$. Denote the integration in the right-hand-side of (5.80) by ΔI_{BP}. The following procedure produces the solution of this integration in terms of n, p_0, and p_1.

$$\begin{aligned}
\Delta I_{BP} &= \int_0^1 f(p,k,n)dp && (5.88)\\
&= \int_0^{p_1} f(p,k,n)dp + \int_{p_1}^1 f^+(p,k,n)dp - \int_{p_1}^1 f^-(p,k,n)dp && (5.89)\\
&> \int_0^{p_1} f(p,k,n)dp - \int_{p_1}^1 f^-(p,k,n)dp, && (5.90)
\end{aligned}$$

where $f^+(p,k,n)$ is the positive part of f and $f^-(p,k,n)$ is the negative part of f on $(p_1,1)$. Replacing f by f_1, f_2, and f_3, we have

$$\Delta I_{BP} > \int_0^{p_1} f_1(p,k,n)f_2(p,k)f_3(p,k) - \int_{p_0}^1 f_1(p,k,n)f_2(p,k)f_3(p,k). \quad (5.91)$$

The negative part of f is conservatively picked by assuming that f is negative on the domain $(p_0,1)$. Evaluating f_1 with p_1, and employing the lower bound (5.85) of f_3 on $(0,p^*)$ and the magnitude of the lower bound (5.87) of f_3 on $(p_0,1)$, we obtain

$$\begin{aligned}
\Delta I_{BP} &> f_1(p_1,k,n)\int_0^{p_1} p^{k-1}(p-kp^k)dp \\
&\quad -f_1(p_0,k,n)\int_{p_0}^1 p^{k-1}(1-p_0)dp && (5.92)\\
&> (1-p_1^k)^{n-3k}[\frac{1}{k+1}p_1^{k+1} - \frac{1}{2}p_1^{2k}] \\
&\quad -\frac{1}{k}(1-p_0)(1-p_0^k)^{n-3k+1}. && (5.93)
\end{aligned}$$

By the fact that f_1 is decreasing on n, for any $n > 3k+1$, there should be an n^* large enough so that, given p_0 on $(p^*,1)$, p_1 on $(0,p^*)$, and $n \geq n^*$, the right-hand-side of (5.93) will exceed zero. Therefore, the difference $I_{BP}^k - I_{BP}^{k+1}$ is greater then zero for large enough n. □

Numerical study shows that the value of n^* depends on the value of k. For $k=2$, where $p_0 = 0.61803$, by taking p_1 from the interval $(0.20, 0.45)$, we obtain

$n^* = 8k$. For $k = 3$, where $p_0 = 0.68233$, by taking p_1 from the interval $(0.40, 0.50)$, we obtain $n^* = 6k$. The value of n^* becomes smaller when k becomes larger. For $k = 10$, for example, where $p_0 = 0.83508$, we find $n^* = 5k$ for $p_1 = 0.70$. According to this study, $n^* = 8k$ is large enough for (5.73) to be true.

5.5 The Monotonicity of Deegan-Packel Importance in Linear Consecutive-k-out-of-n:G Systems

The Deegan-Packel importance measure for component c_i, as defined in Definition 4.11, is based on the number of minimal-path sets containing that component. The formulation of this measure in linear consecutive-k-out-of-n:G systems is given in the following lemma.

Lemma 5.6 The Deegan-Packel importance of component c_i in linear consecutive-k- out-of-n:G system is

$$I^i_{DP} = \frac{m_i}{k(n-k+1)}, \tag{5.94}$$

where

$$m_i = \begin{cases} i & \text{if } i \leq \min\,(k, n-k+1) \\ \min(k, n-k+1) & \text{if } \min\,(k, n-k+1) \leq i \leq \max\,(k, n-k+1) \\ n-i+1 & \text{if } \max\,(k, n-k+1) \leq i \leq n. \end{cases}$$

Proof:

There are $n - k + 1$ minimal-path vectors in linear consecutive k-out-of-n:G systems, so that $m = n-k+1$. Each minimal-path vector $\mathbf{z}_j$ has k nonzero elements and $n-k$ zero elements. Hence $z_j = k$, and the Deegan-Packel importance is then

$$I_{DP}^i(\phi) = \frac{1}{k(n-k+1)} \sum_{z_j \in M_i} 1 = \frac{m_i}{k(n-k+1)}, \tag{5.95}$$

where m_i depends on the following relationship between n and k.

For $n \leq 2k-1$, i.e., $n-k+1 \leq k$,

$$m_i = \begin{cases} i & \text{if } i \leq n-k+1 \\ n-k+1 & \text{if } n-k+1 \leq i \leq k \\ n-i+1 & \text{if } i \geq k. \end{cases} \tag{5.96}$$

For $n \geq 2k-1$, i.e., $n-k+1 \geq k$,

$$m_i = \begin{cases} i & \text{if } i \leq k \\ n-k+1 & \text{if } k \leq i \leq n-k+1 \\ n-i+1 & \text{if } i \geq n-k+1. \end{cases} \tag{5.97}$$

Expression (5.96) and (5.97) give expression (5.95). □

Example 5.2 Consider linear consecutive-3-out-of-6:G system. Here $k=3$, $n=6$, $n-k+1=4$, and $k(n-k+1)=12$. Using (5.94), we obtain $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, $m_4 = 3$, $m_5 = 2$, and $m_6 = 1$. Then the Deegan-Packel importance of component c_i, $i = 1, 2, \ldots, 6$, is:

$$I_{DP}^1 = 1/12, \; I_{DP}^2 = 2/12, \; I_{DP}^3 = 3/12,$$

$$I_{DP}^4 = 3/12, \; I_{DP}^5 = 2/12, \; I_{DP}^6 = 1/12.$$

The following theorem shows that this measure is always monotone in linear consecutive-k-out-of-n:G systems.

Theorem 5.8 The Deegan-Packel importance is monotone in linear consecutive-k-out-of-n:G systems.

Proof:

From (5.94) we know that the behavior of this measure depends on m_i. For $n < 2k$, I^i_{DP} increases from component 1 to component $n-k+1$, remains constant from component $n-k+1$ to component k, and decreases from component k to component n. For $n \geq 2k$, it increases from component 1 to component k, remains constant from component k to component $n-k+1$, and decreases from component $n-k+1$ to component n. □

5.6 The Monotonicity of Vesely-Fussel Importance in Linear Consecutive-k-out-of-n:F Systems

Contrary to the Deegan-Packel importance, the Vesely-Fussel importance is based on minimal-cut vectors associated with component c_i. One structure function of interest in evaluating the Vesely-Fussel importance is the structure function of a subsystem based on all minimal-cut sets that contain component c_i. This subsystem will always fail if c_i and other components within a minimal-cut containing c_i fail. This structure function is denoted as $\psi_i(\mathbf{x})$. The following lemma gives the formulation of $\psi_i(\mathbf{x})$ for linear consecutive-k-out-of-n:F systems.

Lemma 5.7 Let $\psi_i(\mathbf{x})$ be the structure function of a subsystem where every minimal-cut set contains component c_i. Then $\psi_i(\mathbf{x})$ can be formulated as

$$\psi_i(\mathbf{x}) = 1 - \prod_{j=1}^{k}(1-x_j) - \sum_{l=a}^{b-k} x_j \prod_{j=l+1}^{l+k}(1-x_j), \tag{5.98}$$

where, for $n < 2k$

$$\begin{cases} a = 1 \text{ and } b = i + k - 1 & \text{if } 1 \le i < n - k + 1 \\ a = 1 \text{ and } b = n & \text{if } n - k + 1 \le i \le k \\ a = i - k + 1 \text{ and } b = n & \text{if } k < i \le n. \end{cases} \tag{5.99}$$

and for $n \ge 2k$

$$\begin{cases} a = 1 \text{ and } b = i + k - 1 & \text{if } 1 \le i < k \\ a = i - k + 1 \text{ and } b = i + k - 1 & \text{if } k \le i \le n - k + 1 \\ a = i - k + 1 \text{ and } b = n & \text{if } n - k + 1 < i \le n. \end{cases} \tag{5.100}$$

Proof:

The set of minimal-cut vectors for structure $\psi_i(\mathbf{x})$ contains all minimal-cut vectors $\mathbf{w}_j$, such that $w_{ij} = 0$. For linear consecutive-k-out-of-n:F system, it is given in (3.46). From this set, we can see that the subystem with structure $\psi_i(\mathbf{x})$ has no more than k minimal-cut vectors. So the structure $\psi_i(\mathbf{x})$ involves no more than $2k - 1$ components. Therefore, by applying Corollary 3.2, this structure can be represented as

$$\psi_i(\mathbf{x}, n, k) = 1 - \prod_{j=1}^{k} (1 - x_j) - \sum_{l=a}^{b-k} x_l \prod_{j=l+1}^{l+k} (1 - x_j), \tag{5.101}$$

where a is the index of the first component and b is the index of the last component in the subsystem. These indices can be determined by indicating the components involved in constructing this function.

First consider, on one hand, the system with the number of components $n < 2k$. For $1 \le i < n-k+1$, the number of minimal-cut vectors contributing to the function ψ_i is i; and the components involved are components 1 to $i + k - 1$. So, $a = 1$ and

$b = i + k - 1$. For $n - k + 1 \leq i \leq k$, the number of minimal-cut vectors contributing to this function is $n - k + 1$, and the components involved are components 1 to n. So $a = 1$ and $b = n$. For $k < i \leq n$, the number of minimal-cut vectors contributing to this function is $n - i + 1$, and the components involved are components $i - k + 1$ to n. So $a = i - k + 1$ and $b = n$.

On the other hand, consider the system with $n \geq 2k$. By the same procedure, we obtain $a = 1$ and $b = i + k - 1$ when $1 \leq i < k$, $a = i - k + 1$ and $b = i + k - 1$ when $k \leq i \leq n - k + 1$, and $a = i - k + 1$ and $b = n$ when $n - k + 1 < i \leq n$. Using these facts, the formulation of the structure function $\psi_i(\mathbf{x}, n, k)$ in this lemma is then verified. □

Lemma 5.8 The Vesely-Fussel importance for component c_i in linear consecutive-k-out-of-n:F system can expressed as

$$I_{FV}^{i} = \frac{1}{Q_F(\mathbf{p}, n, k)}[\prod_{j=1}^{k} q_j + \sum_{l=a}^{b-k} p_j \prod_{j=l+1}^{l+k} q_j], \tag{5.102}$$

where $Q_F(\mathbf{p}, n, k)$ is the unreliability of linear consecutive-k-out-of-n:F systems, and the values of a and b are given in Lemma 5.7.

Proof:

By Lemma 5.7 and the fact that $P(\psi_i(\mathbf{x}, n, k) = 0) = 1 - P(\psi_i(\mathbf{x}, n, k) = 1)$, we obtain the expression

$$P(\psi_i(\mathbf{x}, n, k) = 0) = \prod_{j=1}^{k} q_j + \sum_{l=a}^{b-k} p_j \prod_{j=l+1}^{l+k} q_j, \tag{5.103}$$

along with the specifications of a and b given by (5.99) and (5.100). Noting that $P(\psi(\mathbf{x}, n, k) = 0)$ is in fact the unreliability $Q_F(\mathbf{p}, n, k)$, by the definition of the Vesely-Fussel importance, (5.102) is then verified. □

The following theorem shows the monotonicity behavior of this measure in a linear consecutive-k-out-of-n:F system when all components are equally reliable with common reliability p.

Theorem 5.9 The Vesely-Fussel importance is monotone in linear consecutive-k-out-of-n:F systems with equally reliable components.

Proof:

Replacing p_j by p in (5.102) we obtain the Vesely-Fussel importance as follows: For $n < 2k$,

$$I_{VF}^{i} = \begin{cases} \frac{q^k}{Q_F(p,n,k)}(q+pi) & \text{if } 1 \leq i < n-k+1 \\ \frac{q^k}{Q_F(p,n,k)}(q+np-kp) & \text{if } n-k+1 \leq i \leq k \\ \frac{q^k}{Q_F(p,n,k)}(1+np-pi) & \text{if } k < i \leq n \end{cases} \tag{5.104}$$

which increases from component 1 to component $n-k$, remains constant from component $n-k$ to component $k+1$, and decreases from component $k+1$ to component n.

For $n \geq 2k$,

$$I_{VF}^{i} = \begin{cases} \frac{q^k}{Q_F(p,n,k)}(q+pi) & \text{if } 1 \leq i < k \\ \frac{q^k}{Q_F(p,n,k)}(q+kp) & \text{if } k \leq i \leq n-k+1 \\ \frac{q^k}{Q_F(p,n,k)}(1+np-pi) & \text{if } n-k+1 < i \leq n \end{cases} \tag{5.105}$$

which increases from component 1 to component k, remains constant from component k to component $n-k+1$, and decreases from component $n-k+1$ to component n. □

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