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THE QUENCHING OF SOLUTIONS OF LINEAR PARABOLIC AND HYPERBOLIC EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS*

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Abstract. In this paper we examine the initial-boundary value problems (α): $u_t=u_{xx}$, $0<x<L$, $t>0$, $u(0,t)=u(x,0)=0$, $u_x(L,t)=\phi(u(L,t))$ and (β): $u_{tt}=u_{xx}$, $0<x<L$, $t>0$, $u(0,t)=u(x,0)=u_t(x,0)$, $u_x(L,t)=\phi(u(L,t))$ where $\phi(-\infty,1)\to(0,\infty)$ is continuously differentiable, monotone increasing and $\lim_{u\to 1^-}\phi(u)=+\infty$. For problem ($\alpha$) we show that there is a positive number L_0 such that if $L\leq L_0$, $u(x,t)\leq 1-\delta$ for some $\delta>0$ for all $t>0$, while if $L>L_0$, $u(L,t)$ reaches one in finite time while $u_t(L,t)$ becomes unbounded in that time. For problem (β) it is shown that if L is sufficiently small, then $u(L,t)\leq 1-\delta$ for all $t>0$ while if L is sufficiently large and $\int_0^1\phi(\eta)\,d\eta<\infty$, $u(L,t)$ reaches one in finite time whereas if $\int_0^1\phi(\eta)\,d\eta=\infty$, $u(L,t)$ reaches one in finite or infinite time.

In either of the last two situations $u_t(L,t)$ becomes unbounded if the time interval is finite. If u reaches one in infinite time, then $\int_0^1 u_x^2(x,t)\,dx$ and $u(x,t)$ are unbounded on the half line and half strip respectively.

1. Introduction. In his paper [5], Kawarada studied the behavior of solutions of

$$\text{(A)}\qquad \begin{aligned} &u_t=u_{xx}+1/(1-u(x,t)), && t>0,\quad 0<x<L,\\ &u(0,t)=u(L,t)=0, && t>0,\\ &u(x,0)=0, && 0\leq x\leq L. \end{aligned}$$

He showed that if $u(L/2,t)$ reached one in finite time, T, then $u_t(L/2,t)$ was unbounded on $(0,T)$, in fact $\lim_{t\to T^-}u_t(L/2,t)=+\infty$. He called this type of regularity loss quenching. In the same paper, he showed that if $L>2\sqrt{2}$, then quenching must occur as $u(L/2,t)$ does then reach one in finite time. In [1,2] and independently in [6] it was shown that there is a number $L_0<2\sqrt{2}$ ($L_0\cong 1.5307$) such that if $L<L_0$ then u cannot quench, even in infinite time whereas if $L>L_0$ u must quench in finite time. In [6] it was also shown that if $L=L_0$ the former situation holds. In [1], [2], [6] more general nonlinearities were also studied.

Let us make the following operational definition, which is weaker than Kawarada's. We will say that a solution of an evolutionary equation quenches in some seminorm (in x) depending on t if (i) the solution remains bounded in this norm while (ii) some derivative in some seminorm of the solution becomes unbounded *in finite time*. We shall sometimes say that a solution quenches in infinite time if (i) and (ii) occur but the solution exists on $[0,L]\times[0,\infty)$.

In [3] the following nonlinear initial boundary value problem for the wave equation was studied.

$$\text{(B)}\qquad \begin{aligned} &u_{tt}=u_{xx}+1/(1-u), && t>0,\quad 0<x<L,\\ &u(0,t)=u(L,t)=0, && t>0,\\ &u(x,0)=u_t(x,0)=0, && 0<x<L. \end{aligned}$$

The interest in (B) was theoretical. Whereas for (A) heavy use of the maximum principle was made, for (B) it was necessary to employ other arguments, specifically

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energy arguments, to establish global existence and no quenching, even in infinite time, for small L and a differential inequality argument to establish quenching for large L.

Although our interest in (A) and (B) was theoretical, both problems have their origin in physics. Problem (A) arises in the study of electric current transients in polarized ionic conductors [5]. Problem (B) can be viewed as the initial-boundary value problem describing the motion of a wire composed of a magnetic material carrying an electric current in the presence of a second wire also carrying a current. Stoker and Minorsky [9], [10] give a phase plane analysis of the analogous ordinary differential equation which describes the motion of a current carrying conductor restrained by springs and subject to the force due to a magnetic field of an infinitely long parallel wire conducting a current I. The equation has the form

$$\ddot{x}(t)=-kx(t)+k\lambda/(a-x(t))$$

where $x(0)$, $\dot{x}(0)$ are prescribed.

We were aware of the physical motivation for (A) before we wrote [6]. However Arje Nachman kindly brought to our attention the references [9], [10] (unfortunately after [3] had appeared). In the same spirit and in the hope that the knowledgeable reader will have a ready application for them, we present our results for problems (α), (β) below.

It is the purpose of this paper to examine the corresponding problems when the solution is driven by the boundary conditions rather than by the forcing term. Specifically, we study

$$(\alpha)\qquad \begin{aligned} &u_t=u_{xx}, && t>0,\quad 0<x<L,\\ &u(0,t)=0, && t\ge 0,\\ &u_x(L,t)=\phi(u(L,t)), && t>0,\\ &u(x,0)=0, && 0<x\le L \end{aligned}$$

and

$$(\beta)\qquad \begin{aligned} &u_{tt}=u_{xx}, && t>0,\quad 0<x<L,\\ &u(0,t)=0, && t\ge 0,\\ &u_x(L,t)=\phi(u(L,t)), && t>0,\\ &u(x,0)=u_t(x,0)=0, && 0\le x\le L. \end{aligned}$$

(By simultaneous scaling in x,t we can take $L=1$ in Problems (α), (β) provided the boundary condition at the right endpoint takes the form

$$u_x(1,t)=L\phi(u(1,t)),\qquad t>0.$$

We shall therefore take $L=1$ and use the boundary condition above without further mention of this reduction.) Here ϕ: $(-\infty,1)\to(0,\infty)$ is continuously differentiable, monotone increasing and $\lim_{u\to 1^-}\phi(u)=+\infty$. The boundary and initial data are taken to be zero, not only for convenience, but also so that one can isolate the effects of the nonlinearity on the solution. While the results for (α), (β) are similar to those obtained for (A), (B), there are several differences worthy of mention. In the first place we show here that not only does, for large L, $u(1,t)$ (problem (α)) become one in finite time but also $u_t(1,t)$ becomes infinite in finite time. The same is true for problem (β) if $\int_0^1\phi(\eta)\,d\eta<\infty$. If, for (β), $\int_0^1\phi(\eta)\,d\eta=+\infty$, then u must quench in finite or infinite time. If this time is finite, $u_t(1,t)$ becomes unbounded in finite time also. If this time is infinite then u is unbounded on $[0,1]\times[0,\infty)$ and $\int_0^1 u_x^2(x,t)\,dx$ is also unbounded on $[0,\infty)$. This is a somewhat weaker result when $\phi(u)=1/(1-u)$ than for (B). On the other hand, it is shown for both Problems (α), (β) that if L is small then quenching cannot occur for either problem, even in infinite time. The results for problem (α) are sharp while there is a gap for problem (β). That is, for problem (α), there is $L_0>0$ such that if $L\le L_0$ no quenching at all is possible while if $L>L_0$ u quenches. For problem (β) there are L_1, L_2 with $0<L_1<L_2$ such that if $L<L_1$ no quenching at all can occur whereas if $L>L_2$ some kind of quenching must occur. These results are in accord with the general principle that small domains are more stable than large domains.

It is perhaps worth mentioning that, via the change of variable

$$v(x,t)=\int_1^{u(x,t)} d\eta/\phi(\eta)\equiv\Psi(u(x,t)),$$

we may reduce (α) to

$$(\alpha')\qquad \begin{array}{ll} v_t=v_{xx}+\phi'(\Psi^{-1}(v))v_x^2, & 0<x<1,\quad t>0,\\ v(x,0)=\Psi(0), & 0\le x\le 1,\\ v(0,t)=\Psi(0), & t\ge 0,\\ v_x(1,t)=L, & t>0. \end{array}$$

Using the techniques of [1], [2], [6], it is possible to study the problem

$$(\alpha'')\qquad \begin{array}{ll} u_t=u_{xx}+L^2\phi(u), & 0<x<1,\quad t>0,\\ u(x,0)=0, & 0<x<1,\\ u(0,t)=0, & t\ge 0,\\ u_x(1,t)=0, & t\ge 0. \end{array}$$

(In [1], [2], [6] the condition at $x=1$ was $u(1,t)=0$.) The same substitution reduces this problem to

$$(\alpha''')\qquad \begin{array}{ll} v_t=v_{xx}+\phi'(\Psi^{-1}(v))v_x^2+L^2, & 0<x<1,\quad t>0,\\ v(x,0)=\Psi(0), & 0<x<1,\\ v(0,t)=\Psi(0), & \\ v_x(0,t)=0. & \end{array}$$

Certainly (α') and (α''') are similar looking problems and we might therefore expect (indeed it is our goal to show) that the results obtained for (α) are similar to those obtained for (α''). However, there is no obvious correspondence between the solutions of (α) and (α'') or between (α') and (α'''). For example, the stationary solutions of (α) (when they exist) are linear in x so that (for $\phi(u)=1/(1-u)$) the stationary solutions of (α') are quadratic polynomials in x since

$$\int_1^u (1-\eta)\,d\eta=-\frac{1}{2}(1-u)^2.$$

On the other hand, (again for $\phi(u)=1/(1-u)$) the stationary solutions of (α''') (when they exist) are transcendental functions of x (see [1], [2], [6]). Therefore a separate treatment is needed for (α). Similar remarks apply to problem (β).

The plan of the paper is as follows. In §2, we treat problem (α) first establishing global existence for small L and then quenching for large L. In §3, problem (β) is analyzed in the same manner. In §4 we discuss local existence. The local existence result for (β) is of special interest. This ordering of topics introduces a slight nonlinearity in the development of our results. We apologize to the reader for this. However, the more interesting results, namely Theorem 2.5, Corollary 2.7, Theorem 3.1 and Corollary 3.4, come first and provide the main thrust of the paper.

A word about notation. We let $D_T=(0,1)\times(0,T)$ if $T<\infty$ and $D=(0,1)\times(0,\infty)$. Likewise, if $T<\infty$, $\Gamma_T=(0,1)\times\{0\}\cup\{0,1\}\times[0,T)$ and $\Gamma=(0,1)\times\{0\}\cup\{0,1\}\times[0,\infty)$ denote the parabolic boundary of D_T and D respectively.

The results of this paper, as well as those of [1]–[6], have some higher dimensional analogues. For example, in [1] and [2], problem (A) was studied in several dimensions. However, the blowup of u_t has yet to be shown for such problems. Likewise, Lieberman and the author have obtained some extensions of the results for problem (α) in several variables, but again with less sharp results than in one dimension. (However, we have an example of infinite time quenching in two dimensions, which cannot occur in one dimension.)

The hyperbolic problems (B), (β) present a more difficult challenge in several space dimensions. For both problems, it is fairly easy to obtain quenching if the space domain is large enough. However, the question of global existence for small domains is open when d^2/dx^2 is replaced by a second order elliptic operator. Smiley and the author have extended the global existence result when d^2/dx^2 is replaced by an elliptic operator of sufficiently high order. The essential ingredient in the global existence argument is the existence of a continuous imbedding of $W_0^{1,p}(\Omega)$ into $L^\infty(\Omega)$ for sufficiently large p, i.e., an inequality of the form

$$|u(x,t)|^2\leq\text{const.}\times\left(\int_\Omega|u(y,t)|^{2p}dy\right)^{1/p}$$

for $\Omega\subset R^n$, Ω bounded, $\partial\Omega$ smooth and $u=0$ on a portion of $\partial\Omega$ of dimension $n-1$. Finally, nothing has been established about the behavior of u_{tt}, even in one dimension.

The above paragraph corrects one of the concluding remarks of [3]. We note one other correction (typographical) for [3]. On p. 395, we should have

$$F(x,t,u)=-\varphi(-u)\quad\text{if } x\in[2n-1,2n).$$

2. The parabolic problem (α). By a solution of (α) in D_T we mean a function $u(x,t)$ continuous in $D_T\cup\Gamma_T$, $u<1$ on $D_T\cup\Gamma_T$, and twice continuously differentiable in x and once in t in D_T. Known regularity results permit differentiation of the equation in D_T. The following lemmas are easy consequences of the maximum principle and the boundary point lemma for parabolic equations (These are sometimes referred to as the first and second maximum principles for parabolic inequalities. See [7, pp. 164, ff.]).

LEMMA 2.1. *If u solves (α) in D_T, then $u>0$ there.*

Proof. For any $\varepsilon>0$, if u had a nonpositive minimum in $\bar{D}_{T-\varepsilon}$, by the maximum principle it would have to occur on $\bar{\Gamma}_{T-\varepsilon}$. Since $u_x(1,t)\geq 0$ if $0<t\leq T-\varepsilon$ it cannot occur on line $x=1$. Since u is not identically zero, $u>0$ in $\bar{D}_{T-\varepsilon}$ for all $\varepsilon>0$.

LEMMA 2.2. *If u solves (α) in D_T, then $u_x(x,t)>0$ in D_T.*

Proof. Put $\pi=u_x(x,t)$.Then $\pi_{xx}=\pi_t$ in D_T, $\pi(0,t)\geq 0$ by Lemma 2.1, $\pi(x,0)=0$ and $\pi(1,t)>0$ so $\pi>0$ in D_T again by the maximum principle and the boundary point lemma.

LEMMA 2.3. *If u solves (α) in D_T, then $u_t(x,t)>0$ in $D_T\cup\{1\}\times(0,T)$.*

Proof. We work in $D_{T-\varepsilon}$ with $0<h\le\varepsilon/2$ and $0\le t\le T-\varepsilon$. Define $v(x,t)=u(x,t+h)-u(x,t)$. Then v solves

$$\begin{aligned}
&v_t=v_{xx}, && 0<x<1,\quad 0\le t<T-\varepsilon,\\
&v(x,0)>0, && 0<x<1,\\
&v(0,t)=0, && 0\le t\le T-\varepsilon,\\
&v_x(1,t)=L[\phi(u(1,t+h))-\phi(u(1,t))], && 0<t\le T-\varepsilon\\
&\qquad =L\phi'(\xi)v, &&
\end{aligned}$$

where ξ is between $u(1,t)$ and $u(1,t+h)$. Since $t+h\le T-\varepsilon/2$, $u(1,t)$ and $u(1,t+h)$ are bounded above by $1-\delta'$ for some $\delta'>0$. Therefore there is a number $\lambda<0$ such that $\lambda+L\phi'(\xi)<0$ if $t\le T-\varepsilon$ and $h\le\varepsilon/2$. Now set $w=v\exp(\lambda x-\lambda^2t)$. Then

$$\begin{aligned}
&w_t-w_{xx}+2\lambda w_x=0 && \text{in } D_{T-\varepsilon},\\
&w(0,t)=0, && 0\le t\le T-\varepsilon,\\
&w(x,0)>0 \text{ (by Lemma 2.1)}, && 0\le x\le L,\\
&w_x(1,t)=(\lambda+L\phi(\xi))w, && 0<t\le T-\varepsilon.
\end{aligned}$$

By the maximum principle, w cannot have a nonpositive minimum in $D_{T-\varepsilon}\cap(0,1)\times\{T-\varepsilon\}$. Moreover w cannot have a negative minimum at a point $(1,t_0)$ $(0<t_0<T-\varepsilon)$, otherwise (because of the choice of λ) $w_x(1,t_0)>0$ at such a point whereas it must be nonpositive at a negative minimum. Therefore $w\ge0$ in $\bar{D}_{T-\varepsilon}$ and hence $v\ge0$ in D.

It follows that, wherever it exists, $u_t(x,t)\ge0$. Now $v=u_t$ satisfies

$$\begin{aligned}
&v_t=v_{xx}, && 0<x<1,\quad 0<t<T-\varepsilon,\\
&v(x,0)\ge0, && 0\le t\le T-\varepsilon,\\
&v_x(1,t)=L\phi'(u(1,t))v, && 0<T\le T-\varepsilon.
\end{aligned}$$

Since $v\ge0$, v cannot vanish at any point in the set $\{(x,t)|0\le x<1, 0<t\le T-\varepsilon\}$ unless $v\equiv0$ by the strong maximum principle. However if $v\equiv0$, then $u(x,t)=f(x)$ for some f and consequently $f(x)=0$ since $u(x,0)=0$. But then, for $0<t<T-\varepsilon$, $u_x(1,t)=0=L\phi(u(1,t))>0$, a contradiction. If $v(1,t_0)=0$ for some t_0, $0<t_0\le T-\varepsilon$, $v_x(1,t_0)=0$ also. Therefore, by the second form of the strong maximum principle, [7, p. 190], $v\equiv0$ in $\{(x,t)|0\le x<1, 0\le t\le t_0$ or $x=1$ and $0<t\le t_0\}$. Thus, as before, $u(x,t)=f(x)\equiv0$ in this latter point set and hence $u_x(1,t_0)=0=L\phi(u(1,t_0))>0$. Therefore $v=u_t>0$ whenever it is defined except along $x=0$.

Remark 2.1. The content of Lemmas 2.2 and 2.3 is that the maximum of u in any closed domain $\bar{D}_T$ must occur at the point $(1,T)$.

COROLLARY 2.4. *The solution of problem* (α) *is unique.*

Proof. One lets u_1, u_2 be two solutions. If $w=e^{\lambda x-\lambda^2t}(u_1-u_2)$, then w satisfies the same initial boundary value problem as the w of the preceding lemma except that $w_x=(\lambda+L\phi'(\xi))w$ where ξ is between $u_1(1,t)$ and $u_2(1,t)$. Since $w\ge0$ and $-w\ge0$ by the first part of the proof, we have $w\equiv0$.

LEMMA 2.5. *Let* $f(x)=ax$ *where* $a<1$ *is a root of* $a=L\phi(a)$ *and let* u *solve problem* (α). *Then* $u(x,t)<f(x)$ *for all* $(x,t)\in D_T$.

Proof. Put $v(x,t)=f(x)-u(x,t)$. Then $v(0,t)=0, v(x,0)=f(x)>0$, $v_t=v_{xx}$ and $v_x(1,t)=a-L\phi(u(1,t))=L(\phi(a)-\phi(u(1,t)))=L\phi'(\xi)v$ where ξ is between a and $u(1,t)$. By the same argument with $w=e^{\lambda x-\lambda^2t}v$ as in Lemma 2.3 we conclude that $v>0$ in D_T.

Remark 2.2. The equation $a=L\phi(a)$ need not have any solutions in $(0,1)$.

We shall assume for the purposes of this section that if $u\le 1-\delta$ on $\bar{D}_T$, then u may be extended to be a larger domain $\bar{D}_{T+\sigma}$ on which $u\le 1-\delta'$ ($\delta'<\delta$) for some $\sigma>0$ and sufficiently small. This will be established later.

THEOREM 2.5. *Either* (a) *u exists on D and* $\lim_{t\to+\infty}u(x,t)=ax$ *where* $a=L\phi(a)$ *and* $a<1$ *or* (b) *for some* $T<\infty$, $\lim_{t\to T^-}u(1,t)=1$ (*u quenches in finite time*).

Proof. Suppose (b) fails. Then since $u(1,T)\ge u(x,t)$ on D_T for all T, by the comment on continuation, we may assume u exists (and is less than one) for all $t\ge 0$. Let

$$G(x,y)=\begin{cases} x, & 0\le x\le y\le 1,\\ y, & 0\le y\le x\le 1\end{cases}$$

denote the Green's function for d^2/dx^2 with boundary conditions $G(0,y)=G_x(1,y)=0$. By Lemma 2.3,

$$\lim_{t\to+\infty} u(x,t)=h(x) \qquad (\le 1)$$

exists. Put

$$F(x,t)=\int_0^1 G(x,y)u(y,t)\,dy.$$

Then

$$F_t(x,t)=\int_0^1 G(x,y)u_t(y,t)\,dy>0$$

by Lemma 2.3. Using the differential equation and integrating by parts we find

$$F_t(x,t)=(yu_y-u)\big|_0^x+xu_y\big|_x^1=-u(x,t)+xL\phi(u(1,t)).$$

Clearly,

$$\lim_{t\to+\infty}F_t(x,t)=\begin{cases}-h(x)+xL\phi(h(1))\equiv M(x) & \text{if } h(1)<1,\\ +\infty & \text{if } h(1)=1.\end{cases} \tag{2.1}$$

where $M(x)\ge 0$. Now for any x,t we have

$$F(x,t)\le\int_0^1 G(x,y)\,dy\le\frac{1}{2}.$$

It is easy to see that if $f(t)$ is a bounded function such that $f'(t)\ge 0$ and $\lim_{t\to\infty}f'(t)=\alpha\ge 0$, then $\alpha=0$. Therefore $h(1)<1$ and $M(x)\equiv 0$ so that $h(x)=xL\phi(h(1))$ so that with $a=h(1)$ we have the theorem.

COROLLARY 2.6. *If $a=L\phi(a)$ has no solutions in* $(0,1)$ *then $u(1,t)$ reaches one in finite time.*

Example 2.1. $\phi(u)=(1-u)^{-\beta}$, $\beta>0$. Then it is easily checked that $a=L\phi(a)$ has no solutions if $L>L_0=\beta^\beta(1+\beta)^{-(1+\beta)}$, one solution smaller than one if $L=L_0$ and two solutions smaller than one if $L<L_0$. In particular, if $\beta=1$ and $0<L\le\frac{1}{4}$

$$\lim_{t\to+\infty}u(x,t)=a_-x,$$

where $a_-=\frac{1}{2}(1-\sqrt{1-4L})$.

COROLLARY 2.7. *Suppose L is so large that $u(1,t)$ reaches one in finite time. Then $u_t(1,t)$ becomes infinite in finite time.*

Proof. We invoke (4.3) of this paper which is used to establish local existence. With $f(x)\equiv 0$ there we find that

$$u_t(x,t)=LG(x,1;t)\phi(u(1,t))$$
$$+L\int_0^T G(x,1;t-\eta)\phi'(u(1,\eta))u_\eta(1,\eta))\,d\eta.$$

Since $\phi'>0$ and $u_\eta\geq 0$ it follows that

$$u_t(1,t)\geq LG(1,1;t)\phi(u(1,t)),$$

where G is the Green's function following (4.1). Since $G(1,1,t)>0$ on $[0,\infty)$ and $\phi(u(1,t))\to+\infty$ in finite time, the result follows.

3. The hyperbolic problem (β). Here we consider weak solutions of (β).

DEFINITION 3.1. A continuous function u on $D_T\cup\Gamma_T$ is a *weak solution* of problem (β) if

(i) $u(1,t)<1$ for $0\leq t<T$;

(ii) $u(0,t)=u(x,0)=u_t(x,0)=0$, $0\leq x\leq 1$, $0\leq t<T$;

(iii) u has weak derivatives u_x, u_t, which, as functions of x are in $L^2(0,1)$ for each $t\in(0,T)$;

(iv) for every $\psi\in C_p^1(\bar{D}_T)$ with $\psi(0,t)=0$

$$\int_0^1\psi(y,t)u_t(y,t)\,dy=L\int_0^t\psi(1,\eta)\phi(u(1,\eta))\,d\eta \tag{3.1}$$
$$+\int_0^t\int_0^1\left[\psi_\eta(y,\eta)u_\eta(y,\eta)-\psi_y(y,\eta)u_y(y,\eta)\right]dy\,d\eta$$

(C_p^1 denotes piecewise C^1 functions);

(v) The following conservation law holds:

$$E(t)\equiv\frac{1}{2}\int_0^1u_t^2(x,t)\,dx+\frac{1}{2}\int_0^1u_x^2(x,t)\,dx-L\int_0^{u(1,t)}\phi(\eta)\,d\eta=E(0)(=0). \tag{3.2}$$

Remark 3.1. Notice that the boundary condition at $x=1$ has been incorporated into (3.1) and (3.2) because $u_x(1,t)$ need not be defined for specific points. Notice also that (3.2) implies $0\leq u(1,t)$. Equation (3.2) can be obtained formally from (β) in the usual manner. See also Theorem 4.2.

We shall assume for the purposes of this section that if u is a weak solution on $D_T\cup\Gamma_T$ and $u\leq 1-\delta$ on $\{1\}\times[0,T]$ then u may be continued as a weak solution on $D_{(T+\sigma)}\cup\Gamma_{(T+\sigma)}$ for σ sufficiently small and positive while $u\leq 1-\delta'$ ($\delta'<\delta$) on $x=1$. This will be established in the next section.

THEOREM 3.1. *Let u be a weak solution on (β) on $D_T\cup\Gamma_T$. Let*

$$L_1=\sup_{0\leq\delta\leq 1}\Psi(\delta),$$

where

$$\Psi(\delta)=\frac{1}{2}(1-\delta)^2\left(\int_0^{1-\delta}\phi(\sigma)\,d\sigma\right)^{-1},$$

while

$$L_2=\sup_{0\leq\delta\leq 1}(1-\delta)/\phi(1-\delta).$$

(a) *If* $L<L_1$ *then* $T=+\infty$ *and u cannot quench, even in infinite time, i.e.*, $|u(x,t)|\leq 1-\delta$ *on* $x=1$ *for some* $\delta>0$.

(b) *If* $L>L_2$ *and* $\int_0^1\phi(\eta)\,d\eta<+\infty$, *then* $T<\infty$ *and* $\lim_{t\to T^-}u(1,t)=1$.

(c) *If* $L>L_2$ *and* $\int_0^1\phi(\eta)\,d\eta=+\infty$, *then* $T\leq\infty$ *and* $\lim_{t\to T^-}u(1,t)=1$.

Proof. We first note that since $\Psi(1)=0$ and $\Psi(0)\geq 0$, $L_1=\Psi(\delta_0)$ for some $\delta_0\in[0,1)$ which solves $\Psi'(\delta_0)=0$ or

$$\int_0^{1-\delta_0}\phi(\sigma)\,d\sigma=\frac{1}{2}(1-\delta_0)\phi(1-\delta_0)$$

($\Psi(1)=0$ by l'Hôpital's rule, $\Psi(0)>0$ if $\int_0^1\phi(\sigma)\,d\sigma<+\infty$, otherwise $\Psi(0)=0$). Therefore

$$L_1=\frac{(1-\delta_0)}{\phi(1-\delta_0)}\leq L_2,$$

as should be the case.

(a). For any $L<L_1$ there is a $\delta_1\in(0,1)$ such that $L<\Psi(\delta_1)$. Let T be the largest time such that $u(x,t)\leq 1-\delta_1$ on $x=1$. From (3.2) and the (sharp) inequality

$$u^2(x,t)\leq\int_0^1 u_x^2(x,t)\,dx,$$

we have, for $0<x<1$,

$$u^2(x,t)\leq 2L\int_0^{u(1,t)}\phi(\eta)\,d\eta. \tag{3.3}$$

If we take note of the monotonicity of the integral with respect to the upper limit, and take the supremum over $\{1\}\times[0,T]$ on the left of (3.3), we find that

$$L\geq\Psi(\delta_1),$$

which contradicts the choice of δ_1. This, together with the remarks on continuation, proves that $u(1,t)<1-\delta_1$. Using (3.3) and the definition of δ_1, (a) follows.

(b). Suppose (b) fails, i.e., $T=+\infty$ and $u<1$ on $x=1$. Define

$$F(x,t)=\int_0^1 G(x,y)u(y,t)\,dy,$$

where G is the Green's function given in Theorem 2.5. There results

$$F_t(x,t)=\int_0^1 G(x,y)u_t(y,t)\,dy.$$

Since for each x, $G(x,\cdot)$ is admissible in (3.1), we find

$$F_t(x,t)=L\int_0^t G(x,1)\phi(u(1,\eta))\,d\eta-\int_0^t\int_0^1 G_y(x,y)u_y(y,\eta)\,dy\,d\eta.$$

Therefore F_{tt} exists and

$$\begin{aligned}F_{tt}(x,t)&=Lx\phi(u(1,t))-\int_0^1 G_y(x,y)u_y(y,t)\,dy\\&=Lx\phi(u(1,t))-\int_0^x u_y(y,t)\,dy\\&=Lx\phi(u(1,t))-u(x,t).\end{aligned}$$

Since $0\leq u(1,t)<1$ and $L>L_2$, we have $L\geq u(1,t)/\phi(u(1,t))+\varepsilon$ for some $\varepsilon>0$. Thus $F_{tt}(1,t)\geq\varepsilon\phi(u(1,t))\geq\varepsilon\phi(0)$ so that $F(1,t)\geq\frac{1}{2}\varepsilon\phi(0)t^2$. On the other hand, if we take square roots of both sides of (3.3) multiply through the resulting inequality by $G(1,y)$ and integrate over $[0,1]$, we find

$$\frac{1}{2}\varepsilon\phi(0)t^2 \leq F(1,t) \leq \sqrt{2L}\left(\int_0^1 G(1,y)\,dy\right)\left(\int_0^{u(1,t)}\phi(\eta)\,d\eta\right)^{1/2}. \tag{3.4}$$

Since we are assuming $\int_0^1\phi(\sigma)\,d\sigma<\infty$, (3.4) is not possible for all $t>0$. Hence $u(1,t)$ reaches one in finite time.

(c) This also follows from (3.4). If $u(1,t)\leq 1-\delta$ on $[0,\infty)$, then (3.4) will again be contradicted. Hence $\lim_{t\to T^-}u(1,t)=1$ where $T\leq+\infty$.

Before stating an important corollary, we look at an example.

Example 3.2. If $\phi(u)=(1-u)^{-\beta}$, $\beta>0$, then u reaches one along $x=1$ in finite or infinite time provided

$$L>L_2(\beta)=\beta^\beta(1+\beta)^{-(1+\beta)}.$$

If $0<\beta<1$, part (b) of Theorem 3.1 applies. On the other hand, if $L<L_1(\beta)$, where

$$L_1(\beta)=\begin{cases}\max\limits_{0\leq\delta\leq 1}\dfrac{1}{2}(1-\delta)^2/\ln(1/\delta)=\delta_0(1-\delta_0), & \beta=1,\\[2ex] \max\limits_{0\leq\delta\leq 1}\dfrac{1}{2}(1-\beta)(1-\delta)^2/(1-\delta^{1-\beta})=\delta_0(1-\delta_0)^\beta, & \beta\neq 1,\end{cases}$$

where $2\ln\delta_0=1-1/\delta_0$ if $\beta=1$ and $2\delta_0^\beta-(1+\beta)\delta_0-(1-\beta)=0$ if $\beta\neq 1$, then u cannot quench, even in infinite time. To see this, one simply calculates $\Psi'(\delta)$ and shows that it has a unique root $\delta_0\in(0,1)$ while $\Psi'(\delta)$ changes from being positive for $\delta<\delta_0$ to being negative for $\delta>\delta_0$. We notice that for $\beta=1$, $L_2(\beta)=0.25$ while $L_1(\beta)\cong 0.20365$ and $\delta_0\simeq 0.2847$.

Remark 3.3. Since $\chi(\delta)\equiv(1-\delta)/\phi(1-\delta)$ vanishes at $\delta=0$ and $\delta=1$, $L_1=L_2$ provided there is $\delta_0\in(0,1)$ such that δ_0 maximizes both χ and Ψ. Then we have $\chi(\delta_0)=\Psi(\delta_0)$. This reduces to the requirement that the equations

$$\int_0^{1-\delta}\phi(\sigma)\,d\sigma=\frac{1}{2}(1-\delta)\phi(1-\delta), \qquad \phi(1-\delta)=(1-\delta)\phi'(1-\delta)$$

have a common solution $\delta=\delta_0$. This cannot happen if ϕ' is strictly increasing since then

$$\frac{(1-\delta_0)^2}{2}\phi'(1-\delta_0)=\int_0^{1-\delta_0}\eta\phi'(\eta)\,d\eta<\frac{(1-\delta_0)^2}{2}\phi'(1-\delta_0)$$

as an integration by parts shows. Thus these techniques are unlikely to yield optimal results.

COROLLARY 3.4. *If u solves (β) and*

(a) $u(1,t)$ *reaches one in finite time T, then*

$$\lim_{t\to T^-}u_x(1,t)=\lim_{t\to T^-}u_t(1,t)=+\infty, \quad \textit{or}$$

(b) $u(1,t)$ *reaches one in infinite time, then*

$$\limsup_{t\to+\infty}\max_{0\leq x\leq 1}u(x,t)=\limsup_{t\to+\infty}\int_0^1 u_x^2(1,t)\,dx=+\infty.$$

The proof of this corollary is postponed to the end of §4 because it depends upon a certain auxiliary function introduced in the next section.

COROLLARY 3.5. *If* $u(1,t)\le 1-\delta$ *for all* t, *then for all* $t>0$ *and all* x, $0<x<1$,

$$|u(x,t)|\le\sqrt{2L}\left(\int_0^{1-\delta}\varphi(\eta)\,d\eta\right).$$

Proof. This is an obvious consequence of (3.3).

4. Local existence. In this section we examine the questions of local existence and continuation of local solutions of problems (α), (β). Since these questions reduce to the study of nonlinear Volterra integral equations, copious detail will not be needed.

We begin with problem (α). The solution of

$$(\alpha_1)\qquad \begin{aligned} &w_t=w_{xx}+F(x,t), \qquad 0<x<1, \quad t>0,\\ &w(0,t)=0,\\ &w_x(1,t)=0\\ &w(x,0)=f(x)\end{aligned}$$

can be found by elementary means. It is

$$(4.1)\qquad w(x,t)=\int_0^1 G(x,y;t)f(y)\,dy+\int_0^t\int_0^1 G(x,y;t-\eta)F(y,\eta)\,dy\,d\eta,$$

where $G(x,y;t)$ is the heat kernel for the homogeneous problem. In fact, with $\lambda_n=\frac{1}{2}(2n+1)\pi$,

$$G(x,y;t)=2\sum_{n=1}^{\infty}\sin\lambda_n x\sin\lambda_n y\exp(-\lambda_n^2 t).$$

It is well known that $G>0$ on the half strip and $G_{xx}=G_{yy}=G_t$, $G_y(x,1;t)=G_x(1,y;t)=G(0,y;t)=G(x,0,t)=0$, and

$$\int_0^1 G(x,y;t)\,dy\le 1.$$

Consider next problem (α) *with inhomogeneous nonnegative* initial data $u(x,0)=f(x)$. If we set

$$w(x,t)=u(x,t)-xL\phi(u(1,t)),$$

then w solves problem (α_1) with $w(x,0)=f(x)-L\phi(u(1,0))$, $F(x,t)=-xL\phi'(u(1,t))u_t(1,t)$. Therefore u must solve, on $D_T\cup\Gamma_T$, the nonlinear Volterra equation

(4.2)

$$\begin{aligned}u(x,t)=xL\phi(u(1,t))&+\int_0^1 G(x,y;t)f(y)\,dy\\ &-L\phi(u(1,0))\int_0^1 yG(x,y;t)\,dy-L\int_0^t\int_0^1 G(x,y;t-\eta)\frac{d}{d\eta}\phi(u(1,\eta))\,dy\,d\eta.\end{aligned}$$

If one integrates by parts in this last integral, uses $G_t=G_{yy}$ and again integrates by parts, one sees that (4.2) takes the more pleasant form

$$(4.3)\qquad u(x,t)=\int_0^1 G(x,y;t)f(y)\,dy+L\int_0^t G(x,1;t-\eta)\phi(u(1,\eta))\,d\eta.$$

It is now a straightforward matter to prove the convergence (pointwise) of the following iteration scheme on $\bar{D}_T$ provided T is sufficiently small:

$$u_0(x,t)\equiv 0,$$
$$u_{n+1}(x,t)=\int_0^1 G(x,y;t)f(y)\,dy+L\int_0^t G(x,1;t-\eta)\phi(u_n(1,\eta))\,d\eta.$$

Since $G>0$ and ϕ is increasing, $u_n>0$ for all n. Moreover $u_1>u_0=0$ so $\phi(u_n(1,t))\geq\phi(u_{n-1}(1,t))$ if $u_n\geq u_{n-1}$. Thus by induction $u_{n+1}\geq u_n$ on $\bar{D}_T$. Now suppose $f(x)\leq 1-2\delta$ and $u_n\leq 1-\delta$, then $u_{n+1}\leq 1-\delta$ also provided T is so small that

$$(1-2\delta)+L\phi(1-\delta)\int_0^T G(x,1,T-\eta)\,d\eta\leq 1-\delta,$$

i.e., provided T is so small that

$$\sup_{0\leq x\leq 1}\int_0^T G(x,1,T-\eta)\,d\eta\leq\frac{\delta L^{-1}}{\phi(1-\delta)}.$$

Clearly this is always possible. Thus the sequence $\{u_n\}_{n=1}^{\infty}$ of iterates is an increasing sequence of continuous functions, bounded above by $1-\delta$ if T is sufficiently small. Now by the monotone convergence theorem, $\lim_{n\to\infty}u_n=u$ exists and satisfies (4.3) and hence (4.1). Thus we have established the following:

THEOREM 4.1. *If T is sufficiently small, then problem* (α) *possesses a unique solution, C^1 in t, C^2 in x in D_T and continuous in $\bar{D}_T$. Moreover, if $u\leq 1-\delta$ on $\bar{D}_T$ then u may be continued as a solution on $\underline{D_{T+T'}}\cup\Gamma_{T+T'}$ for T' sufficiently small and $u\leq 1-\delta'$ where $\delta'<\delta$ on $\bar{D}_{T+T'}$.*

We turn next to problem (β). We consider first the inhomogeneous problem

$$(\beta_1)\qquad\begin{aligned}&w_{tt}=w_{xx}+F(x,t), && 0<x<1,\quad t>0,\\ &w(x,0)=f(x), &&\\ &w_t(x,0)=g(x) &&\\ &w(0,t)=w_x(1,t)=0, && t>0.\end{aligned}$$

We let

$$B=\{f\colon R^1\to R^1 | f,f' \text{ are piecewise continuous},$$
$$f(x)=f(2-x)=-f(-x)=f(x+4)\}.$$

We extend f,g and $F(\cdot,t)$ for each t so that $f,g,F(\cdot,t)\in B$. This amounts to requiring that $f(0)=g(0)=F(0,t)=0$ and that f,f',g,g',F,F_x are continuous on [0, 1]. The solution of (β_1), which is then given by the d'Alembert formula

$$u(x,t)=\frac{1}{2}[f(x+t)+f(x-t)]$$
$$+\frac{1}{2}\int_{x-t}^{x+t}g(\sigma)\,d\sigma+\frac{1}{2}\int_0^t\int_{x-t+\eta}^{x+t-\eta}F(\xi,\eta)\,d\xi\,d\eta, \tag{4.4}$$

again has the property that $w(\cdot,t)\in B$ for each $t\geq 0$.

For the purpose of the argument that follows, we shall assume $\phi\in C^2(-\infty,1)$. Let $H(x)\in B$ be defined by

$$H(x)=\begin{cases}2-x, & 1<x<2,\\ x, & -1\leq x\leq 1,\\ -(2+x), & -2\leq x<-1\end{cases}$$

on $(-2,2)$.

Suppose we have a solution of (β) *with inhomogeneous initial data*, f, g. Define, for $0 \le x \le 1$,

$$w(x,t)=u(x,t)-LH(x)\phi(u(1,t)).$$

Then w solves (β_1) with

$$\begin{aligned}
&F(x,t)=-LH(x)\,d^2\phi(u(1,t))/dt^2,\\
&w(x,0)=f(x)-LH(x)\phi(f(1)),\\
&w_t(x,0)=g(x)-LH(x)\phi'(f(1))g(1),\\
&w(0,t)=w_x(1,t)=0.
\end{aligned}$$

Therefore, for w, the data are in B (for each t). Therefore u solves problem (β) weakly if and only if u solves

$$\begin{aligned}
(4.5)\qquad u(x,t)=u_0(x,t)&-\frac{1}{2}L\phi(f(1))[H(x+t)+H(x-t)]\\
&+LH(x)\phi(u(1,t))-\frac{1}{2}Lg(1)\phi'(f(1))\int_{x-t}^{x+t}H(\eta)\,d\eta\\
&-\frac{1}{2}L\int_0^t\frac{d^2}{d\eta^2}\phi(u(1,\eta))\left(\int_{x-t+\eta}^{x+t-\eta}H(\xi)\,d\xi\right)d\eta,
\end{aligned}$$

where $u_0(x,t)$, the so-called free solution, is given by

$$(4.6)\qquad u_0(x,t)=\frac{1}{2}[f(x+t)+f(x-t)]+\frac{1}{2}\int_{x-t}^{x+t}g(\eta)\,d\eta.$$

If one integration by parts is carried out in the last integral on the right of (4.5), we find

$$\begin{aligned}
(4.7)\quad u(x,t)=u_0(x,t)&+LH(x)\phi(u(1,t))-\frac{1}{2}L\phi(f(1))[H(x+t)+H(x-t)]\\
&-\frac{1}{2}L\int_0^t\frac{d}{d\eta}\phi(u(1,\eta))[H(x+t-\eta)+H(x-t+\eta)]\,d\eta.
\end{aligned}$$

The quantity in brackets in the integral on the right of (4.7) is piecewise continuously differentiable. Thus we may integrate by parts one more time and obtain

$$(4.8)\quad u(x,t)=u_0(x,t)+\frac{1}{2}L\int_0^t\phi(u(1,\eta))\frac{d}{d\eta}[H(x-t+\eta)+H(x+t-\eta)]\,d\eta$$

where now $-\infty<x<\infty$, $t>0$. The kernel in the integrand in (4.8) is piecewise constant and cannot exceed two in absolute value.

It is clear from (4.8) that if u solves (4.8) in $R^1\times[0,\tau]$ and is continuous there, then u_x and u_t exist and are piecewise continuously differentiable except on the lines $x=n$, $x\pm t=n$ where n is an integer. In fact, the second derivatives exist except on that point set, so that the solution is classical except on that point set. Therefore (4.8) need only be solved in the larger space

$$\begin{aligned}
B_\tau=\{u\colon R^1\times[0,\tau]\to R^1 \mid u(x,t)&=u(1-x,t)\\
&=-u(-x,t)=u(x+4,t),\ u\text{ is continuous}\}.
\end{aligned}$$

Let

$$B(\delta)=\{u\in B_\tau \mid |u(1,t)|\le 1-\delta\}.$$

For $u \in B_\tau$, let

$$\|u\| = \sup\{|u(x,t)|, 0 \le t \le \tau, 0 \le x \le 1\}.$$

Let, for $f, g \in B$, τ be so small that

$$|u_0(1,t)| = \left| \frac{1}{2}[f(1+t)+f(1-t)] + \frac{1}{2}\int_{1-t}^{1+t} g(\eta)\,d\eta \right| \le 1-2\delta \tag{4.9}$$

for $0 \le t \le \tau_1$ say. This can be accomplished if $|f(1)| < 1-2\delta$. Let

$$B(\rho,\delta,u_0) = \{u \in B(\delta) \mid \|u-u_0\| \le \rho\}.$$

Define

$$\mathfrak{T}: B(\rho,\delta,u_0) \to B(\rho,\delta,u_0)$$

by

$$(\mathfrak{T}u)(x,t) = u_0(x,t) + \frac{1}{2}L\int_0^t \phi(u(1,\eta))K(x,t,\eta)\,d\eta,$$

where

$$K(x,t,\eta) = \frac{d}{d\eta}[H(x+t-\eta)+H(x-t+\eta)]$$

almost everywhere (except where $x \pm (t-\eta)$ is an integer).

We need to show that $\mathfrak{T}$ is well defined and that it is a contraction. We note that if $u \in B(\rho,\delta,u_0)$,

$$(\mathfrak{T}u)(1,t) = u_0(1,t) + \frac{1}{2}L\int_0^t [H'(1-(t-\eta)) - H'(1+(t-\eta))]\phi(u(1,\eta))\,d\eta.$$

Therefore, since $|K| \le 2$,

$$-1+2\delta - L\tau\phi(1-\delta) \le (\mathfrak{T}u)(1,t) \le 1-2\delta+L\tau\phi(1-\delta).$$

Thus if

$$\tau \le \tau_2 \equiv \delta/(L\phi(1-\delta)), \tag{4.10}$$

we have that $|\mathfrak{T}u(1,t)| \le 1-\delta$ also. Moreover

$$\|\mathfrak{T}u - u_0\| \le L\tau\phi(1-\delta) \le \rho$$

if

$$\tau \le \tau_3 \equiv \rho\tau_2/\delta. \tag{4.11}$$

Thus $\mathfrak{T}$ is well defined. Also one verifies readily that

$$\|\mathfrak{T}u - \mathfrak{T}v\| \le L\tau\left[\sup_{|\xi| \le 1-\delta} \phi'(\xi)\right]\|u-v\|$$

if $u, v \in \mathfrak{B}(\rho,\delta,u_0)$. Thus $\mathfrak{T}$ will be a contraction if, in addition to (4.9)–(4.11),

$$\tau < \tau_4 \equiv 1\Big/\left[L \sup_{|\xi| \le 1-\delta} \phi'(\xi)\right]. \tag{4.12}$$

(This assumes $\phi'\not\equiv 0$ in any neighborhood of zero.) Therefore for $T<\min(\tau_1,\tau_2,\tau_3,\tau_4)$ we have proved the following theorem:

THEOREM 4.2. *There exists, for any $L>0$ and $\delta\in(0,1)$ a unique weak solution of problem (β) on some domain $\bar{D}_T$ for $T=T(L,\delta)$ sufficiently small, which satisfies $|u(1,t)|\le 1-\delta$ for $0\le t\le T$. This solution is classical except on the characteristics, so that* (3.1) *and* (3.2) *hold and therefore $u(1,t)\ge 0$ on $[0,T]$. Moreover if $\delta'<\delta$, this solution may be continued to $D_{T+T'}$ with $0\le u(1,t)\le 1-\delta'$ on $[0,T+T']$ for sufficiently small $T'>0$. The extended (in x) solution satisfies* (4.8) *on $R^1\times[0,T]$.*

Proof of Corollary 3.4. (a) From (4.8) with $f=g=0$, we see that

$$u(1,t)=\frac{1}{2}L\int_0^t\phi(u(1,\eta))[H'(1-(t-\eta))-H'(1+(t-\eta))]\,dy. \tag{4.13}$$

A few moments reflection will convince the reader that

$$H'(1-x)-H'(1+x)=2H'(x-1).$$

Using this in (4.13) and taking the (distribution) derivative of the result yields (where n is the largest integer such that $2n\le t-1$),

$$u_t(1,t)=L\phi(u(1,t))+2L\sum_{p=0}^{n}(-1)^p\phi(u(1,t-2p-1)). \tag{4.14}$$

If u quenches in (finite) time T, and if N is the largest integer such that $2N\le T-1$, then as $t\to T^-$ the sum on the right of (4.14) approaches

$$\sum_{p=0}^{N}(-1)^p\phi(u(1,T-2p-1))$$

while the first term becomes unbounded. This proves part (a) of the corollary.

To prove part (b), we see from part (b) of the proof of Theorem 3.1 that if $u(1,t)$ reaches one in infinite time then

$$\lim_{t\to+\infty}\int_0^1 G(1,y)u(y,t)\,dy=+\infty.$$

Thus there is a sequence of points (x_n,t_n) with $0<x_n<1$ and $t_n\to+\infty$ such that

$$\lim_{n\to\infty}u(x_n,L_n)=+\infty.$$

Using these points in the inequality preceding (3.3) we find

$$\lim_{n\to\infty}\int_0^1 u_x^2(x,t_n)\,dx=+\infty.$$

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