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THEORETICAL AND NUMERICAL STUDIES OF SOME ILL-POSED PROBLEMS IN PARTIAL DIFFERENTIAL EQUATIONS

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Theoretical and numerical studies of some ill-posed problems in partial differential equations

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TABLE OF CONTENTS

CHAPTER 1. INTRODUCTION

Consider the problem of obtaining a solution $u = u(x_1,\ldots,x_m)$ of a partial differential equation

$$P(u,u_{x_1},\ldots,u_{x_m},u_{x_1x_1},\ldots) = 0$$

given data f. Special cases include the *Cauchy* (or *initial-value*) *problem*, where the data are values of u prescribed on a hypersurface in $\mathbf{R}^m$ (or $\mathbf{C}^m$); and the *initial-boundary value* (or *mixed*) *problem*, where u is prescribed for $x_m = 0$ on a domain D in $\mathbf{R}^{m-1}$ (or $\mathbf{C}^{m-1}$), and for $x_m > 0$ on the boundary D. Such problems are said to be *well-posed* in the sense of Hadamard provided

(a) a solution u exists for given data f;

(b) u is determined uniquely by f; and

(c) u depends continuously on f.

Prototypical well-posed problems include the Cauchy problem

$$\frac{dy}{dt} = y, \quad t > 0; \quad y(0) = a;$$

and the Dirichlet problem for Laplace's equation (John [10, p. 155]).

A problem is *ill-posed* if solutions fail to satisfy one or more of (a), (b), and (c). Examples for which (a) fails include the Cauchy problem

$$\frac{dy}{dt} = y^2, \quad t > 0; \quad y(0) = 1,$$

whose solution $y(t) = (1-t)^{-1}$ cannot be continued past $t = 1$; and the general Cauchy problem for the Laplace equation ([10, p. 98]). An extreme example is furnished by Hans Lewy's linear partial differential equation with no solutions, regardless of the type or form of data prescribed ([24]). Examples for which (b) or (c) fails abound (see [10, pp. 155-156], and Smoller [34]).

One of the major advances in the study of ill-posed problems has been the development of a concept of weak or distribution solution. The notions of uniqueness, continuity, differentiability, and satisfaction of differential equation and side conditions are sufficiently generalized, so that in the new context a problem may become well-posed. Functions with weak derivatives were first used by Friedrichs [7,8] and spaces $W^{k,p}$ of such functions by Sobolev [35].

Weak solutions are especially useful for studying nonlinear problems, which can model physical phenomena in such fields as hydrodynamics, chemical kinetics, and biophysics ([34]). Solutions of such problems often exhibit behavior, such as the development of gradient catastrophes or shock waves, unknown to solutions of linear problems.

In this work three nonlinear initial-boundary value problems are considered. Each is already known to be ill-posed under certain conditions on initial data. Two basic questions confront the would-be solver of these problems. In what space of functions should one look for a solution? Under what conditions on the data can solutions be continued for all time, and under what conditions is this impossible? These questions are considered in the next three chapters.

CHAPTER 2. A POTENTIAL WELL THEORY FOR THE WAVE EQUATION WITH A NONLINEAR BOUNDARY CONDITION

Introduction

Let D be an open, bounded, connected subset of $\mathbb{R}^n$ with a Lipschitz boundary ∂D. Let ∂D be the union of two disjoint $(n-1)$-dimensional submanifolds σ, Σ (each of positive Lebesgue measure) and their Lipschitz confluence. Consider the initial-boundary value problem

$$
(\mathrm{W}) \qquad
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \Delta u && \text{in } D \times (0,T) \\
&u(x,0) = U(x), \quad \frac{\partial u}{\partial t}(x,0) = V(x) && \text{in } D \\
&u(x,t) = 0 && \text{on } \sigma \times (0,T) \\
&\frac{\partial u}{\partial n} = f(u(x,t)) && \text{on } \Sigma \times (0,T),
\end{aligned}
$$

where Δ denotes the n-dimensional Laplacian, and $\frac{\partial}{\partial n}$ the outward normal derivative. The kinetic and potential energy functionals associated with (W) are given by

$$K(u) = \frac{1}{2} \int_D \left|\frac{\partial u}{\partial t}\right|^2 dx \,, \tag{2.1}$$

$$J(u) = \frac{1}{2} \sum_{i=1}^{n} \int_D \left|\frac{\partial u}{\partial x_i}\right|^2 dx - \int_\Sigma F(u)\, dS \,, \tag{2.2}$$

respectively, where $F(u) = \int_0^u f(s)ds$. The total energy $E(t)$ of a solution of (W) at time t is defined to be the sum

$$E(t) \equiv K\big(u(\cdot,t)\big) + J\big(u(\cdot,t)\big) . \tag{2.3}$$

For a class of nonlinearities f (which includes the example $f(u) = |u|^p$ with $p > 1$), it has been shown [21] that solutions of (W) cannot exist for all time whenever $E(0) < 0$. Generalizations of this nonexistence result to a wider class of problems are given in [22], [12]. In this chapter, the existence of global solutions of (W) by means of potential well arguments, and the nonexistence of global solutions for some data with $E(0) > 0$, is established.

The arguments herein are adapted from those in [28] and [32], with the following important exceptions:

(a) the presence of a nonlinearity in a boundary condition necessitates certain trace and imbedding results, to be presented in the following section;

(b) global solutions in [32] are approximated by expansions in the Dirichlet eigenfunctions of the Laplacian on the domain of interest. In this work, global solutions are approximated by a double expansion in these eigenfunctions, and the eigenfunctions for the modified Steklov problem

$$\Delta\psi = 0 \qquad \text{in } D$$

$$\psi = 0 \qquad \text{on } \sigma$$

$$\frac{\partial\psi}{\partial n} = \mu\psi \qquad \text{on } \Sigma ,$$

again because of the nonlinear boundary condition;

(c) global existence of solutions is proved in [32] for $n = 1,2,$ and 3 space variables, and for higher dimensions when the potential well is infinitely deep. The proof of global existence in Lemma 2.13 of this chapter is valid for all dimensions n and all potential wells of positive depth, and can be modified to prove global existence for all dimensions and potential wells considered in [32].

(d) the authors of [28] have noted (see [18]) that the proof of Lemma 2.7 in [28] is in error for certain nonlinearities f. The lemma is a key step in establishing nonexistence. In this chapter a corresponding result, Lemma 2.15, is proved under slightly stronger hypotheses on f (see (2.71)). The proof of Lemma 2.15 can supplant that of Lemma 2.7 in [28] when the hypotheses there on f are correspondingly strengthened.

Preliminaries

This section presents the notation, Sobolev spaces, and compact imbedding results required in the sequel.

For a domain G in $\mathbb{R}^m$ and a (possibly empty) subset Γ of ∂G denote by $C^k(G \cup \Gamma)$ the set of real-valued functions g such that g and all its partial derivatives of orders $\leq k$ are continuous on G, and can be extended to be continuous on $G \cup \Gamma$. The set $C_0^k(G \cup \Gamma)$ consists of those $g \in C^k(G \cup \Gamma)$ with compact support, supp g, such that $(\text{supp } g) \cap (G \cup \Gamma)$ is compact. Set $C^\infty(G \cup \Gamma) = \bigcap_{k=0}^{\infty} C^k(G \cup \Gamma)$, with a similar definition for $C_0^\infty(G \cup \Gamma)$.

The volume element of integration in R^m is denoted by $dx = dx_1 \ldots dx_m$. The symbol ∇u denotes the gradient, grad $u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_m})$, of u. Define

$$\|u\|_{q,G} = [\int_G |u|^q \, dx]^{\frac{1}{q}}, \quad 1 \leq q < \infty \tag{2.4}$$

$$\|u\|_G = [\int_G |\nabla u|^2 \, dx]^{1/2} \tag{2.5}$$

$$\|u\|_{1,2,G} = [\|u\|_{2,G}^2 + \|u\|_G^2]^{1/2} \tag{2.6}$$

where u is any function on G for which the right-hand side makes sense. $L_q(G)$, $1 \leq q < \infty$, denotes the Banach space of all measurable functions u on G for which the norm (2.4) is finite.

Let $u,_i$ or $\frac{\partial u}{\partial x_i}$ denote the weak first partial derivative of u with respect to x_i, satisfying

$$\int_G u \frac{\partial \eta}{\partial x_i} dx = - \int_G u,_i \, \eta \, dx \tag{2.7}$$

for all $\eta \in C_0^1(G)$. The Sobolev space $W^{1,2}(G)$ consists of those $u \in L_2(G)$ having weak first partial derivatives $u,_1, \ldots, u,_m$ which are also in $L_2(G)$. Alternatively, $W^{1,2}(G)$ can be viewed as the completion of $\{u \in C^1(G) : \|u\|_{1,2,G} < \infty\}$ with respect to the norm $\|\cdot\|_{1,2,G}$ ([2]).

The following compact imbedding theorem of Sobolev will be required.

<u>Lemma 2.1</u>: If G is a bounded region in $\mathbb{R}^m$ possessing the cone property [2], then bounded sets in $W^{1,2}(G)$ are precompact as subsets of $L_q(G)$ for $1 \leq q < \frac{2m}{m-2}$ if $m > 2$, and $1 \leq q < \infty$ if $m = 2$.

A proof of Lemma 2.1 may be found in [2].

A one-to-one transformation Φ of a domain $G_1 \subseteq \mathbb{R}^m$ onto a domain $G_2 \subseteq \mathbb{R}^m$ is called <u>Lipschitz</u> if each component of both Φ and Φ^{-1} is Lipschitz continuous. An (m-1)-dimensional submanifold Γ of G (with boundary $\partial_{m-1}\Gamma$ in the manifold ∂G) is said to be <u>Lipschitz</u> if there exist an open cover $\{\mathcal{O}_1,\ldots,\mathcal{O}_J\}$ of Γ and corresponding Lipschitz transformations $\Phi_1, \ldots, \Phi_J$ satisfying

(i) Φ_j maps $\mathcal{O}_j$ onto the unit ball B in $\mathbb{R}^m$;

(ii) If $\mathcal{O}_j$ contains no points of $\partial_{m-1}\Gamma$, then

$$\Phi_j(G \cap \mathcal{O}_j) = B^+ \equiv \{y \in B : y_m > 0\} \quad \text{and}$$
$$\Phi_j(\Gamma \cap \mathcal{O}_j) = B_0 \equiv \{y \in B : y_m = 0\} \; ; \tag{2.8}$$

(iii) If $\mathcal{O}_j$ contains points of $\partial_{m-1}\Gamma$ then $\Phi_j(G \cap \mathcal{O}_j)$ (respectively $\Phi_j(\Gamma \cap \mathcal{O}_j)$) is a connected subset of B^+ (resp. B_0).

If a function g has support in $\Gamma \cap \mathcal{O}_j$ with Γ Lipschitz one may define the surface integral of g over Γ as

$$\int_\Gamma g(x)dS = \int_{\Phi_j(\Gamma \cap \mathcal{O}_j)} g\circ\Phi_j^{-1}(y',0)J_j(y')dy' \quad .$$

where $y' = (y_1,\ldots,y_{m-1})$, $x = \Phi_j^{-1}(y)$, and

$$J_j(y') = \left[\sum_{i=1}^{m}\left(\frac{\partial(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_m)}{\partial(y_1,\ldots,y_{m-1})}\right)^2\right]^{1/2}\Bigg|_{y_n=0} \; .$$

For arbitrary functions g defined on Γ one may select a C^∞ partition of unity $\{w_j\}$ subordinate to $\{\mathcal{O}_j\}$ and set

$$\int_\Gamma g(x)dS = \sum_{j=1}^{J}\int_\Gamma g(x)w_j(x)dS \; . \tag{2.9}$$

$L_q(\Gamma)$, $1 \leq q < \infty$, denotes the Banach space of functions g for which the norm

$$\|g\|_{q,\Gamma} = \left[\int_\Gamma |g|^q \, dS\right]^{1/q} \tag{2.10}$$

is defined and finite.

The following result makes precise what is meant by the values of a function $u \in W^{1,2}(G)$ on Γ.

Lemma 2.2: Let G be bounded and have a Lipschitz boundary. Each $u \in W^{1,2}(G)$ is the limit in $W^{1,2}(G)$ of a sequence $\{u_n\} \subseteq C^\infty(G)$. For $2 \leq q < \frac{2(n-1)}{n-2}$ if $n > 2$, and for $2 \leq q < \infty$ if $n = 2$, the sequence $\{u_n|_\Gamma\}$ of restrictions of $u_1,u_2,\ldots$ to Γ converges in $L_q(\Gamma)$ to a function $\underset{\sim}{B}_\Gamma u$ satisfying

$$\|\underset{\sim}{B}_\Gamma u\|_{q,\Gamma} \leq K_q \|u\|_{1,2,G} , \qquad (2.11)$$

where $K_q > 0$ is independent of u. Moreover, $\underset{\sim}{B}_\Gamma : W^{1,2}(G) \to L_q(\Gamma)$ is a well-defined, compact linear operator.

$\underset{\sim}{B}_\Gamma u$ is called the trace of u on Γ. Lemma 2.2 is a consequence of the fact that for bounded domains, having a Lipschitz boundary is equivalent to possessing the uniform cone property (see [9]); Calderón's Extension Theorem; the hypothesis that Γ can be flattened locally by a finite number of Lipschitz transformations; and the Rellich-Kondrachov Theorem. See [2, pp. 91, 144].

$H_0^1(G)$ denotes the completion of $C_0^1(G)$ under the norm (2.6). For $\gamma = \partial G \cap \Gamma^c$ (where Γ^c denotes the complement of Γ in $\mathbb{R}^m$), let $H_{0\gamma}^1(G)$ denote the completion of $C_0^1(G \cup \Gamma)$ under the norm (2.6). Functions in the closed subspace $H_{0\gamma}^1(G)$ of $W^{1,2}(G)$ vanish on γ in the weak sense, and their images under $\underset{\sim}{B}_\Gamma$ are in $L_2(\Gamma)$ by (2.11).

Lemma 2.3: If G is bounded, open, connected, and has a Lipschitz boundary, and if γ is an (m-1)-dimensional submanifold of ∂G of positive measure with a Lipschitz boundary in the manifold ∂G, then Poincaré's inequality

$$\|u\|_{2,G} \leq C\|u\|_G \qquad (2.12)$$

is valid for all $u \in H_{0\gamma}^1(G)$, where $C > 0$ is independent of u.

Thus $H^1_{0\gamma}(G)$ becomes a Hilbert space when endowed with inner product

$$(u,v)_{G,\Gamma} = \int_G \nabla u \cdot \nabla v \, dx + \int_\Gamma (\underset{\sim}{B}u)(\underset{\sim}{B}v) dS, \tag{2.13}$$

since $\|\cdot\|_G$ and the norm

$$\|u\|_{H^1_{0\gamma}(G)} = \sqrt{(u,u)} \tag{2.14}$$

induced by (2.13) are each equivalent to $\|\cdot\|_{1,2,G}$ by Lemmas 2.3 and 2.2.

Note that the conclusions of Lemmas 2.1, 2.2, and 2.3 hold with $G = D$, $\gamma = \sigma$, $\Gamma = \Sigma$, and $m = n$.

For any set A and any $T > 0$, let $A_T = A \times (0,T)$, $A_\infty = A \times (0,\infty)$. Then Lemmas 2.1, 2.2, 2.3 also hold with $G = D_T$, $\gamma = \sigma_T$, $\Gamma = \Sigma_T$, and $m = n+1$. In particular, Lemma 2.2 can be applied to yield compact operators $\underset{\sim}{B}_\Sigma : W^{1,2}(D) \to L_q(\Sigma)$ and $\underset{\sim}{B}_{\Sigma_T} : W^{1,2}(D_T) \to L_q(\Sigma_T)$. These will both be denoted $\underset{\sim}{B}$ when no confusion can result.

The Potential Well

In this section, the functional $J(u) = \frac{1}{2} \|u\|^2_D - \int_\Sigma F(\underset{\sim}{B}u) dS$ of (2.2) is shown to determine a potential well with local minimum at the origin in the function space $H^1_{0\sigma}(D)$.

In this and the following section, the assumptions on the functions, f, F are similar to those in [28]:

(i) $f \in C^1(\mathbb{R})$ and $f(0) = f'(0) = 0$; f does not vanish identically in a neighborhood of the origin;

(ii) Either (a) $f(s)$ is monotone increasing, and convex for $s > 0$, concave for $s < 0$; or

(b) f is convex; and

(iii)
$$(p+1)F(s) < sf(s), \text{ and} \tag{2.15}$$
$$|sf(s)| < \gamma|F(s)|, \tag{2.16}$$
for all $s \in \mathbb{R}$, where $2 < p+1 < \gamma < \frac{2(n-1)}{n-2}$.

Lemma 2.4: Let f, F satisfy (i)-(iii). Then as $s \to +\infty$

$$|F(s)| = \mathcal{O}(|s|^{\gamma}), \tag{2.17}$$

$$|f(s)| = \mathcal{O}(|s|^{\gamma-1}), \text{ and} \tag{2.18}$$

$$|f'(s)| = \mathcal{O}(|s|^{\gamma-2}) . \tag{2.19}$$

Moreover

$$F(s) = \mathcal{O}(|s|^{p+1}) \text{ as } s \to 0^+; \tag{2.20}$$

and in case f satisfies (ii)(a), then (2.20) also holds as $s \to 0^-$.

Proof: Multiplication of both sides of (2.15) by $s^{-(p+2)}$ for $s > 0$ yields $\frac{d}{ds}(s^{-(p+1)}F(s)) > 0$. Hence $F(s) = s^{p+1}I_1(s)$ for $s > 0$, where $I_1 \in C^2((0,+\infty))$ is positive and monotone increasing. This establishes (2.20).

For $s > 0$ (2.16) yields $\frac{d}{ds}(s^{-\gamma}F(s)) \leq 0$, so $F(s) = s^{\gamma}D_1(s)$ where $D_1 \in C^2((0,+\infty))$ is positive and monotone decreasing. Thus $F(s) = \mathcal{O}(s^{\gamma})$ as $s \to +\infty$.

Suppose f satisfies (ii)(a). Then (2.16) yields $F(s) = |s|^{\gamma}I_2(s)$ for $s < 0$, where $I_2 \in C^2((-\infty,0))$ is positive and monotone increasing; so $F(s) = \mathcal{O}(|s|^{\gamma})$ as $s \to -\infty$ and (2.17) is verified in this case. Inequality (2.15) can be used to show (2.20) holds as $s \to 0^-$.

One may also verify that if f satisfies (ii)(b) then (2.16) also implies (2.17).

The growth restriction (2.18) follows from (2.17) and (2.16).

To obtain (2.19) note that for $0 \leq s \leq s_0$, $f(s_0) \geq \int_s^{s_0} f'(\eta)d\eta \geq f'(s)(s_0-s)$. In particular, when $s_0 = 2s \geq 0$, $0 \leq sf'(s) \leq f(2s)$. Similar considerations when $s_0 \leq s \leq 0$ in either of the cases (ii)(a), (ii)(b) show that $|sf'(s)| \leq |f(2s)|$. The growth restriction (2.19) then follows from (2.18). ∎

The order conditions (2.17) and (2.20) are the best obtainable when $\gamma > p+1$, as may be seen by taking

$$F(s) = \begin{cases} s^{p+1} + s^{\gamma}, & s \geq 0 \\ |s|^{p+1}, & s < 0 . \end{cases}$$

The behavior of the functional J along rays emanating from the origin in $H^1_{0\sigma}(D)$ may now be considered.

<u>**Lemma 2.5**</u>: For f, F satisfying (i)-(iii) and for fixed $u \in H^1_{0\sigma}(D)$, the function $j_u(\lambda) = J(\lambda u)$ is in $C^2(\mathbb{R})$ and satisfies

$$j_u'(\lambda) = \lambda \|u\|_D^2 - \int_\Sigma (\underset{\sim}{B}u) f(\lambda \underset{\sim}{B}u) dS, \tag{2.21}$$

$$j_u''(\lambda) = \|u\|_D^2 - \int_\Sigma (\underset{\sim}{B}u)^2 f'(\lambda \underset{\sim}{B}u) dS. \tag{2.22}$$

<u>Proof</u>: If suffices to show $g(\lambda) = \int_\Sigma F(\lambda \underset{\sim}{B}u) dS$ is in $C^2(\mathbb{R})$ and possesses the appropriate expressions for its derivatives.

For $h_m \neq 0$ one may write

$$\frac{g(\lambda+h_m)-g(\lambda)}{h_m} = \int_\Sigma \int_0^1 f\big((\lambda+\eta h_m)\underset{\sim}{B}u\big)(\underset{\sim}{B}u) d\eta dS. \tag{2.23}$$

By (2.18) and Lemma 2.2, for λ fixed, $0 < \eta < 1$, and a sequence $\{h_m\}$ tending to zero, the integrand in (3.11) is dominated in absolute value by a fixed integrable function $(A_1+A_2C|\underset{\sim}{B}u|^{\gamma-1})|\underset{\sim}{B}u|$, where A_1, A_2, C are positive constants and $C > |\lambda+\eta h_m|^{\gamma-1}$ for all m. By the Lebesgue dominated convergence theorem the integral in (2.23) approaches

$$\int_\Sigma \int_0^1 f(\lambda \underset{\sim}{B}u)(\underset{\sim}{B}u) d\eta dS = \int_\Sigma f(\lambda \underset{\sim}{B}u)(\underset{\sim}{B}u) dS \tag{2.24}$$

as $m \to +\infty$. Therefore $g'(\lambda)$ exists and is given by (2.24) for each λ.

Similar proofs employing (2.19) and Lemma 2.2 establish the existence and continuity of, and the integral expression for, g''. ∎

Now $j_u(0) = j_u'(0) = 0$ and $j_u''(0) = \|u\|_D^2 > 0$ for u nonzero, so $j_u(\lambda)$ is a convex function of λ for small λ.

<u>Lemma 2.6</u>: Let f, F satisfy conditions (i)-(iii), and let u be a function in $H^1_{0\sigma}(D)$ with $\underset{\sim}{B}u$ nonzero.

When f satisfies (ii)(a),

(1) $\lim\limits_{\lambda \to +\infty} j_u(\lambda) = -\infty$;

(2) there exists a unique positive root $\lambda^* = \lambda^*(u)$ of $j_u'(\lambda) = 0$; and

(3) $j_u''(\lambda^*) < 0$.

In case f satisfies (ii)(b), then

(4) $\lim\limits_{\lambda \to +\infty} j_u(\lambda) = \pm \infty$;

(5) if $\underset{\sim}{B}u > 0$ a.e., then (1), (2), and (3) hold.

<u>Proof:</u> Let $g(\lambda) = \int_\Sigma F(\lambda \underset{\sim}{B}u)dS$. Then from (2.15), $(p+1)g(\lambda) \leq \lambda \int_\Sigma (\underset{\sim}{B}u)f(\lambda \underset{\sim}{B}u)dS = \lambda g'(\lambda)$. Hence $g(\lambda) = \lambda^{p+1}I(\lambda)$ where $I \in C^2(0,+\infty)$ is monotone increasing, and $j_u(\lambda) = \frac{\|u\|_D^2}{2}\lambda^2 + \lambda^{p+1}(-I(\lambda))$. Let $L = \lim\limits_{\lambda \to +\infty} (-I(\lambda))$. Then $L > 0$ implies $j_u(\lambda) \to +\infty$, and $L < 0$ or $L = -\infty$ implies $j_u(\lambda) \to -\infty$, as $\lambda \to +\infty$. In particular, (4) is verified.

Suppose now that either f satisfies (ii)(a), or f satisfies (ii)(b) and $\underset{\sim}{B}u > 0$ is nonzero. Then L is negative or $-\infty$, so (1) holds. The existence of a positive λ^* satisfying $j_u'(\lambda^*) = 0$ is then guaranteed by the convexity of $j_u(\lambda)$ in a neighborhood of the origin. One has

$$\begin{aligned} j_u''(\lambda^*) &= \|u\|_D^2 - \int_\Sigma (\underset{\sim}{B}u)^2 f'(\lambda^* \underset{\sim}{B}u)dS \\ &= (\lambda^*)^{-2} \int_\Sigma (\lambda^* \underset{\sim}{B}u)[f(\lambda^* \underset{\sim}{B}u) - (\lambda^* \underset{\sim}{B}u) f'(\lambda^* \underset{\sim}{B}u)]dS. \end{aligned} \tag{2.25}$$

Note that $f(s) - sf'(s)$ is the y-intercept of the tangent line to the graph of f at the point $(s,f(s))$, so that

$$s(f(s)-sf'(s)) < 0 \tag{2.26}$$

for $s \neq 0$ in case (ii)(a), and for $s > 0$ in case (ii)(b). Thus the integrand in (2.25) is negative and $j_u''(\lambda^*) < 0$. By (1), λ^* must be the only positive critical value of j_u, so (2), (3), and (5) are verified. ∎

When f satisfies (ii)(a) let $\underset{\sim}{C}$ consist of all $u \in H^1_{0\sigma}(D)$ with nonzero trace $\underset{\sim}{B}u$ on Σ. Lemma 2.6 shows that for $u \in \underset{\sim}{C}$, $j_u(\lambda^*)$ is the maximum value of j_u achieved when leaving the origin in $H^1_{0\sigma}(D)$ along a ray in the direction u. Let

$$d = \inf_{u \in \underset{\sim}{C}} j_u(\lambda^*) \, . \tag{2.27}$$

Clearly $0 \leq d < \infty$, and in Lemma 2.7 below it is shown that $d > 0$. A potential well W in $H^1_{0\sigma}(D)$ of depth d may then be defined by

$$W = \{u \in H^1_{0\sigma}(D) : 0 \leq j_u(\lambda) < d \quad \text{for} \quad 0 \leq \lambda \leq 1\} . \qquad (2.28)$$

Note that if $u \in \underset{\sim}{C}$ is always scaled so that $\lambda^* = 1$, then the variational problem (2.27) is equivalent to the problem

$$d = \inf_{u \in \underset{\sim}{C}} J(u) \qquad (2.29)$$

subject to the constraint

$$Q(u) \equiv \|u\|_D^2 - \int_\Sigma (\underset{\sim}{B}u)f(\underset{\sim}{B}u)dS = 0 . \qquad (2.30)$$

When nonzero $u \in H^1_{0\sigma}(D)$ has zero trace on Σ, $\lim_{\lambda \to +\infty} j_u(\lambda) = +\infty$ and the well is "infinitely deep" in the direction u.

One must proceed more carefully when f satisfies (ii)(b), <u>since critical values of</u> j_u, <u>even when they exist, may not be unique.</u> Let $\underset{\sim}{C}$ consist of all $u \in H^1_{0\sigma}(D)$ for which positive roots of $j_u'(\lambda) = 0$ exist, and denote by $\lambda^* = \lambda^*(u)$ the smallest such positive root. The well W and depth d are then defined as in (2.28), (2.27). (If nonzero $u \in H^1_{0\sigma}(D)$ is not in $\underset{\sim}{C}$, then by Lemma 2.6 $\lim_{\lambda \to +\infty} j_u(\lambda) = +\infty$). Note that for $u \in \underset{\sim}{C}$ one has $J(\lambda u) \leq J(\lambda^* u)$ for $0 \leq \lambda \leq \lambda^*$. Also $J(\lambda|u|) \leq J(\lambda u)$ for all $\lambda \geq 0$, and

$Q(\lambda^*|u|) < Q(\lambda^* u) = 0$. By Lemma 2.6 (5) there exists a unique $\bar{\lambda}$, $0 < \bar{\lambda} < \lambda^*$, satisfying $Q(\bar{\lambda}|u|) = 0$, and so $J(\lambda^* u) > J(\bar{\lambda}u) > J(\bar{\lambda}|u|)$. Hence it suffices in (2.27) to minimize only over $u \in \underset{\sim}{C}^+ \equiv \{u \in H^1_{0\sigma}(D) : \underset{\sim}{B}u$ is nonzero and $\underset{\sim}{B}u > 0$ a.e.$\}$. The variational problem (2.27) is therefore equivalent to the problem

$$d = \inf_{u \in \underset{\sim}{C}^+} J(u) \tag{2.31}$$

subject to the constraint (2.30).

<u>**Lemma 2.7**</u>: If f, F satisfy (i)-(iii), then $d > 0$.

<u>Proof:</u> Choose $u \in \underset{\sim}{C}$ when f satisfies (ii)(a), $u \in \underset{\sim}{C}^+$ when f satisfies (ii)(b), and let u satisfy the constraint (2.30). (2.15) implies

$$\begin{aligned} J(u) &> \frac{1}{2}\|u\|_D^2 - \frac{1}{p+1}\int_\Sigma (\underset{\sim}{B}u)f(\underset{\sim}{B}u)dS \\ &= \frac{p-1}{2(p+1)}\|u\|_D^2 . \end{aligned} \tag{2.32}$$

If one can show that in addition $\|u\|_D$ is bounded below by some positive constant K, then it follows that $d > \frac{p-1}{2(p+1)} K^2 > 0$.

By Lemma 2.4 there exist constants $A_1, A_2 > 0$ such that

$$0 < F(s) < A_1|s|^{p+1} + A_2|s|^\gamma$$

for all s when f satisfies (ii)(a), and for $s > 0$ when f satisfies (ii)(b). By Lemma 2.2, the constraint (2.30), and the inequality (2.16),

$$\begin{aligned}\|\underset{\sim}{B}u\|^2_{\gamma,\Sigma} &< C^2\|u\|^2_D = C^2 \int_\Sigma (\underset{\sim}{B}u)f(\underset{\sim}{B}u)dS \\ &< C^2\gamma \int_\Sigma F(\underset{\sim}{B}u)dS \\ &< C^2\gamma[A_1\|\underset{\sim}{B}u\|^{p+1}_{p+1,\Sigma} + A_2\|\underset{\sim}{B}u\|^{\gamma}_{\gamma,\Sigma}] \qquad (2.33) \\ &< A_3\|\underset{\sim}{B}u\|^{p+1}_{\gamma,\Sigma}\,[1 + \|\underset{\sim}{B}u\|^{\gamma-p-1}_{\gamma,\Sigma}]\ ,\end{aligned}$$

where C, A_3 are positive constants. Now $\|\underset{\sim}{B}u\|_{\gamma,\Sigma} \neq 0$, so (2.33) implies

$$\|\underset{\sim}{B}u\|_{\gamma,\Sigma} > \min\{1, (\frac{1}{2A_3})^{\frac{1}{p-1}}\} \overset{def}{=} A_4 .$$

Therefore, $\|u\|_D > \frac{A_4}{C}$. ∎

The following two lemmas will be used to construct a weak solution of problem (W).

Lemma 2.8: The potential well W is precompact as a subset of $L_2(D)$, and $\{\underset{\sim}{B}u : u \in W\}$ is precompact as a subset of $L_2(\Sigma)$.

Proof: For $u \in W$, $0 < J(u) < d$ and $Q(u) > 0$, so (2.32) holds and

$$\|u\|_D^2 < \frac{2(p+1)}{p-1} d \; . \tag{2.34}$$

Thus W is a bounded set in $H^1_{0\sigma}(D)$ by Lemma 2.3. The conclusions then follow from Lemmas 2.1 and 2.2. ∎

Lemma 2.9: If f, F satisfy (i)-(iii), then the functionals J and Q are continuous on $H^1_{0\sigma}(D)$.

Proof: One need only show $\int_\Sigma F(\underset{\sim}{B}u)dS$, $\int_\Sigma (\underset{\sim}{B}u)f(\underset{\sim}{B}u)dS$ are continuous for $u \in H^1_{0\sigma}(D)$. A proof for the first is given; the proof for the second is similar.

For $u,v \in H^1_{0\sigma}(D)$ write $u_b = \underset{\sim}{B}u$, $v_b = \underset{\sim}{B}v$; then

$$F(u_b) - F(v_b) = (u_b - v_b) \int_0^1 f\big((1-\tau)u_b + \tau v_b\big)d\tau \; .$$

Then

$$\begin{aligned} \Big|\int_\Sigma [F(u_b)-F(v_b)]dS\Big| &< \int_0^1 \Big\{\int_\Sigma |u_b - v_b|\,\big|f\big((1-\tau)u_b+\tau v_b\big)\big|dS\Big\}d\tau \\ &< \|u_b - v_b\|_{\gamma,\Sigma} \int_0^1 \big\|f\big((1-\tau)u_b+\tau v_b\big)\big\|_{\frac{\gamma}{\gamma-1},\Sigma}\, d\tau \end{aligned} \tag{2.35}$$

by Hölder's inequality.

Now by (2.18) there exist constants $C_1, C_2 > 0$ such that $|f(s)| < C_1 + C_2|s|^{\gamma-1}$ for all $s \in \mathbb{R}$. Thus

$$\|f((1-\tau)u_b+\tau v_b)\|_{\frac{\gamma}{\gamma-1},\Sigma}$$

$$\leq \|C_1+C_2|(1-\tau)u_b+\tau v_b|^{\gamma-1}\|_{\frac{\gamma}{\gamma-1},\Sigma} \tag{2.36}$$

$$\leq C_3 + C_2\|(1-\tau)u_b+\tau v_b\|_{\gamma,\Sigma}^{\gamma-1}$$

by the triangle inequality, where $C_3 = C_1[\int_\Sigma dS]^{\frac{\gamma-1}{\gamma}}$. Recall from Lemma 2.2 that

$$\|u_b-v_b\|_{\gamma,\Sigma} \leq K_\gamma\|u-v\|_D . \tag{2.37}$$

Combining (2.35)-(2.37), one obtains

$$|\int_\Sigma[F(u_b)-F(v_b)]dS|$$

$$\leq K_\gamma\|u-v\|_D \int_0^1 [C_3+C_2\|(1-\tau)u_b+\tau v_b\|_{\gamma,\Sigma}^{\gamma-1}]d\tau \tag{2.38}$$

$$\leq K(\|u_b\|_{\gamma,\Sigma},\|v_b\|_{\gamma,\Sigma})\|u-v\|_D ,$$

where K depends only on γ, $\|u_b\|_{\gamma,\Sigma}$, $\|v_b\|_{\gamma,\Sigma}$, and is bounded for bounded $\|u\|_D$, $\|v\|_D$. Hence J is in fact Lipschitz continuous on $H^1_{0\sigma}(D)$. ∎

Global Solution

One says that u is a weak solution of (W) on the interval $[0,T)$ provided

(1) $u(t) : [0,T) \to H^1_{0\sigma}(D)$, $u,_t(t) : [0,T) \to L_2(D)$; $\|u(t)\|_D$ and $\|u,_t(t)\|_{2,D}$ are uniformly bounded on compact subsets of $[0,T)$;

(2) for each t, $0 \leq t < T$, and every $\nu \in L_2(D)$,

$$\int_D [u(x,t) - U(x)]\nu(x)dx = \int_0^t \int_D u,_\tau(x,\tau)\nu(x)dxd\tau; \qquad (2.39)$$

(3) for each t, $0 \leq t < T$, and every $\eta(t) : [0,T) \to H^1_{0\sigma}(D)$ with the properties of u in (1) and (2) (with $\eta(x,0)$ replacing $U(x)$ in (2.39)),

$$\begin{aligned}
&\int_D [u,_t(x,t)\eta(x,t) - V(x)\eta(x,0)]dx \\
&\quad + \int_0^t \int_D [\nabla u\cdot\nabla\eta - u,_\tau\eta,_\tau]dxd\tau \qquad (2.40)\\
&\qquad - \int_0^t \int_\Sigma f(\underset{\sim}{B}u)(\underset{\sim}{B}\eta)dSd\tau = 0; \quad \text{and}
\end{aligned}$$

(4) for each t, $0 \leq t < T$,

$$E(t) \leq E(0), \qquad (2.41)$$

where $E(0) \equiv \frac{1}{2}\int_D V^2dx + \frac{1}{2}\|U\|^2_D - \int_\Sigma F(\underset{\sim}{B}U)dS$, and $E(t)$ is given by (2.3).

The function u is called a global weak solution provided it satisfies (1)-(4) with $T = +\infty$. In this section, the following result is established.

Theorem 2.1: Let W, d be as defined in (2.28-30), and let f, F satisfy (i)-(iii). Then provided $U \in W$, $V \in H^1_{0\sigma}(D)$, and $E(0) < d$, problem (W) has a global weak solution.

The weak solution in Theorem 2.1 is approximated by functions of the form

$$u_{MN}(x,t) = \sum_{i=1}^{M} q_i(t)\phi_i(x) + \sum_{k=1}^{N} p_k(t)\psi_k(x) , \tag{2.42}$$

where the functions q_i, p_k are solutions of a nonlinear initial value problem, to be detailed below. The ϕ_i, ψ_k are generalized eigenfunctions for a Dirichlet problem and a modified Steklov problem:

$$\begin{aligned} &\Delta\phi + \lambda\phi = 0 \quad \text{in } D , &&\text{and} \quad &&\Delta\psi = 0 \quad \text{in } D \\ &\phi = 0 \quad \text{on } \partial D && &&\psi = 0 \quad \text{on } \sigma \\ & && &&\frac{\partial\psi}{\partial n} = \mu\psi \quad \text{on } \Sigma . \end{aligned} \tag{2.43}$$

Lemma 2.10: There exist generalized eigenfunctions $\phi_i \in H^1_0(D)$, $\psi_k \in H^1_{0\sigma}(D)$, and corresponding eigenvalues $\lambda_i, \mu_k > 0$, which solve the weak formulations of (2.43); i.e.

$$\int_D \nabla\phi_i \cdot \nabla v \, dx = \lambda_i \int_D \phi_i v \, dx \ , \tag{2.44}$$

and

$$\int_D \nabla\psi_k \cdot \nabla w \, dx = \mu_k \int_\Sigma (\underset{\sim}{B}\psi_k)(\underset{\sim}{B}w) dS \tag{2.45}$$

for all $v \in H_0^1(D)$, $w \in H_{0\sigma}^1(D)$, and $i,k = 1,2,\ldots$. The eigenvalues λ_i, μ_k each have finite multiplicity and satisfy $\lambda_i \to +\infty$, $\mu_k \to +\infty$ as $i,k \to +\infty$. The collection $\{\phi_1,\phi_2,\ldots,\psi_1,\psi_2,\ldots\}$ (assumed normalized with respect to the norm $\|\cdot\|_{H_{0\sigma}^1(D)}$ defined in (2.14)) forms a complete orthonormal set in the Hilbert space $H_{0\sigma}^1(D)$ endowed with the inner product $(\cdot,\cdot)_{D,\Sigma}$ of (2.13).

<u>Proof</u>: The proofs of the existence of the λ_i and ϕ_i, and the fact that $\{\phi_i\}$ forms a complete orthonormal set in $H_0^1(D)$ endowed with the Dirichlet inner product, are standard and are omitted.

For any $h \in L_2(\Sigma)$, the linear functional ϕ given by $\phi(v) = \int_\Sigma h(\underset{\sim}{B}v)dS$ is bounded as a map $H_{0\sigma}^1(D) \to R$ by the Cauchy-Schwarz inequality and the imbedding (2.11). By Riesz's representation theorem there exists a unique $u \in H_{0\sigma}^1(D)$ such that

$$\phi(v) = \int_D \nabla u \cdot \nabla v \, dx \tag{2.46}$$

for all $v \in H_{0\sigma}^1(D)$; in other words, u solves the weak formulation (2.46) of the problem

$$\Delta u = 0 \qquad \text{in } D$$

$$\begin{aligned} u &= 0 && \text{on } \sigma \\ \frac{\partial u}{\partial n} &= h && \text{on } \Sigma \,. \end{aligned}$$

Therefore there is a well-defined linear Green's transformation $\underset{\sim}{G} : L_2(\Sigma) \to H^1_{0\sigma}(D)$ given by

$$\underset{\sim}{G}h = u \,,$$

where u solves (2.46). By Lemma 2.2, the linear map $\underset{\sim\sim}{BG} : L_2(\Sigma) \to L_2(\Sigma)$ is compact; and for any $h \in L_2(\Sigma)$,

$$\int_\Sigma h(\underset{\sim\sim}{BG}h)dS = \int_D |\nabla(\underset{\sim}{G}h)|^2 dx \,,$$

which is nonnegative and vanishes only when h is the zero function in $L_2(\Sigma)$, so that $\underset{\sim\sim}{BG}$ is a strictly positive operator. Hence $\underset{\sim\sim}{BG}$ possesses a countable spectrum of positive eigenvalues, each of finite multiplicity, which are written as $\frac{1}{\mu_1} > \frac{1}{\mu_2} > \dots$, satisfying $\mu_k \to +\infty$ as $k \to +\infty$; and corresponding eigenfunctions $\eta_1, \eta_2, \dots$ in $L_2(\Sigma)$. Moreover, the eigenfunctions $\{\eta_k\}$ can be chosen to form a complete orthonormal set in $L_2(\Sigma)$. For each k, $\tilde{\psi}_k \equiv \underset{\sim}{G}\eta_k$ is in $H^1_{0\sigma}(D)$ and satisfies $\eta_k = \mu_k \underset{\sim}{B} \tilde{\psi}_k$; hence $\mu_k, \tilde{\psi}_k$ satisfy (2.45). Let ψ_k be $\tilde{\psi}_k$ normalized with respect to the norm (2.14). The pairwise orthogonality of ϕ_i, ψ_k in $H^1_{0\sigma}(D)$ is a direct consequence of (2.44) and (2.45).

To show the completeness of the generalized eigenfunctions, choose any $u \in H^1_{0\sigma}(D)$, and set

$$\alpha_k = (u, \psi_k)_{D,\Sigma} \ , \qquad \beta_k = \int_\Sigma \underset{\sim}{B}u \ \underset{\sim}{B}\psi_k \ dS \ .$$

Then $\alpha_k = (\mu_k + 1)\beta_k$ by (2.45). Since the μ_k are positive, one has

$$\sum_{k=1}^{\infty} \beta_k^2 = \sum_{k=1}^{\infty} \frac{\alpha_k^2}{(1+\mu_k)^2} < \sum_{k=1}^{\infty} \alpha_k^2 < C\|u\|_D^2 < \infty$$

by Bessel's inequality and Lemma 2.3. Therefore

$$v \equiv \sum_{k=1}^{\infty} \beta_k \ \psi_k \in H^1_{0\sigma}(D) \ ;$$

and

$$w \equiv u - v \in H^1_0(D)$$

since $\underset{\sim}{B}u = \underset{\sim}{B}v$. ∎

The existence proof for $\{\eta_k\}$ above is adapted from Fichera ([5, pp. 108-110]). Regularity of the generalized eigenfunctions $\{\psi_k\}$ is difficult to prove and requires conditions on the confluence and smoothness of the boundary submanifolds σ, Σ (see, e.g., [25, pp. 233-236], [27], and the references cited therein), but is not required for the purposes of this chapter.

To obtain the nonlinear equations satisfied by q_i, p_k, substitute (2.42) into the kinetic and potential energy functionals K, J to yield

$$K(\dot{y}) \equiv K(u_{MN}) = \frac{1}{2} \dot{y}^T A_{MN} \dot{y} , \tag{2.47}$$

$$J(y) \equiv J(u_{MN}) = \frac{1}{2} y^T B_{MN} y - \int_{\Sigma} F(u_N) dS , \tag{2.48}$$

where $\quad y = (q_1,\ldots,q_M,p_1,\ldots,p_N)^T,$

$$A_{MN} = \int_D \zeta\zeta^T \, dx \text{ where } \zeta = (\phi_1,\ldots,\phi_M,\psi_1,\ldots,\psi_N)^T,$$

$$B_{MN} = \operatorname{diag}(\lambda_1,\ldots,\lambda_M,\mu_1 \int_{\Sigma} \psi_1^2 \, dS,\ldots,\mu_N \int_{\Sigma} \psi_N^2 \, dS), \quad \text{and}$$

$$u_N(x,t) = \sum_{k=1}^{N} p_k(t)\psi_k(x).$$

The Lagrangian for (2.47), (2.48) is the functional $L(\dot{y},y) = K(\dot{y}) - J(y)$. From Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right) - \frac{\partial L}{\partial y_i} = 0 \quad \text{for} \quad i = 1,\ldots,M+N,$$

one obtains the system of ordinary differential equations

$$A_{MN}\ddot{y} + B_{MN}y = H_N(y) \tag{2.49}$$

where

$$H_N(y) \equiv [0,\ldots,0,\int_{\Sigma} f(u_N)\psi_1 dS,\ldots,\int_{\Sigma} f(u_N)\psi_N \, dS]^T .$$

By Lemma 2.10 the eigenfunctions $\phi_1,\ldots,\phi_M,\psi_1,\ldots,\psi_N$ are linearly independent on $D \cup \Sigma$, so $\xi^T A_{MN} \xi = \int_D (\zeta^T \xi)^2 \, dx > 0$ for all nonzero $\xi \in \mathbb{R}^{M+N}$. Therefore A_{MN} is positive definite and (2.49) is equivalent to the system

$$\ddot{y} + \bar{B}_{MN} y = \bar{H}_N(y) \,, \tag{2.50}$$

where $\bar{B}_{MN} = A_{MN}^{-1} B_{MN}$, $\bar{H}_N(y) = A_{MN}^{-1} H_N(y)$.

Let U_{MN}, V_{MN} be the partial Fourier series

$$U_{MN}(x) = \sum_{i=1}^{M} \alpha_{1i}\phi_i(x) + \sum_{k=1}^{N} \alpha_{2k}\psi_k(x) \,, \tag{2.51}$$

$$V_{MN}(x) = \sum_{i=1}^{M} \beta_{1i}\phi_i(x) + \sum_{k=1}^{N} \beta_{2k}\psi_k(x) \,, \tag{2.52}$$

where $\alpha_{1i} = (U,\phi_i)_{D,\Sigma}$, $\alpha_{2k} = (U,\psi_k)_{D,\Sigma}$, $\beta_{1i} = (V,\phi_i)_{D,\Sigma}$, $\beta_{2k} = (V,\psi_k)_{D,\Sigma}$. By Lemmas 2.2, 2.3, and 2.10, for $U \in W$ and $V \in H^1_{0\sigma}(D)$ the norms $\|U-U_{MN}\|_{1,2,D}$, $\|\underset{\sim}{B}U-\underset{\sim}{B}U_{MN}\|_{2,\Sigma}$, $\|V-V_{MN}\|_{1,2,D}$, $\|\underset{\sim}{B}V-\underset{\sim}{B}V_{MN}\|_{2,\Sigma}$ all approach zero as $M,N \to +\infty$. The appropriate initial conditions for (2.50) are therefore

$$y(0) = \alpha, \quad \dot{y}(0) = \beta, \tag{2.53}$$

where $\alpha = (\alpha_{11},\ldots,\alpha_{1M},\alpha_{21},\ldots,\alpha_{2N})^T$, $\beta = (\beta_{11},\ldots,\beta_{1M},\beta_{21},\ldots,\beta_{2N})^T$.

Let $\underset{\sim}{M}_{MN}$ be the subspace of $H^1_{0\sigma}(D)$ spanned by the functions $\phi_1,\ldots,\phi_M,\psi_1,\ldots,\psi_N$, and set $\underset{\sim}{M}^+_{MN} = \{u \in \underset{\sim}{M}_{MN} : u \not\equiv 0 \text{ and } u \geq 0$ on $D\}$. For $u \in \underset{\sim}{M}_{MN}$ one may write $u = y^T\zeta$ for some $y \in \mathbb{R}^{M+N}$, and as in (2.48) define $J(y) = J(y^T\zeta)$. Let $\lambda^* = \lambda^*(y)$ denote the smallest positive root of $\frac{d}{d\lambda} J(\lambda y) = 0$, when it exists, and set

$$d_{MN} = \inf J(\lambda^* y) \tag{2.54}$$

where the infimum is taken over all $y \in \mathbb{R}^{M+N}$ such that $y^T\zeta \not\equiv 0$ in case f satisfies (ii)(a) (or such that $y^T\zeta \in M^+_{MN}$ in case f satisfies (ii)(b)). Since the minimization in (2.54) is over only a subclass of the set $\underset{\sim}{C}$ (or $\underset{\sim}{C}^+$) of the previous section, it follows that $d_{MN} \geq d$. A potential well W_{MN} in $\mathbb{R}^{M+N}$ of depth d_{MN} may then be defined by

$$W_{MN} = \{y \in \mathbb{R}^{M+N} : 0 \leq J(\lambda y) < d_{MN} \text{ for } 0 \leq \lambda \leq 1\}. \tag{2.55}$$

Note $W \subseteq W_{MN}$ in the sense that $W \subseteq \{y^T\zeta : y \in W_{MN}\}$.

<u>**Lemma 2.11**</u>: Let f, F satisfy (i)-(iii) of the previous section. If $\alpha \in W$ and

$$K(\beta) + J(\alpha) < d \leq d_{MN} , \tag{2.56}$$

for each M, N, then the initial value problem (2.50), (2.53) has a unique global solution $y = y(t)$ satisfying $(y(t))^T\zeta \in W$ for all $t \geq 0$.

Proof: The energy relation

$$K(\dot{y}(t)) + J(y(t)) = K(\beta) + J(\alpha) < d \; , \tag{2.57}$$

valid for all t in the existence interval for y, follows from the differential equations (2.50) and initial conditions (2.53).

For any $\alpha_0, \beta_0 \in \mathbb{R}^{M+N}$ satisfying $\alpha_0 \in W$ and $K(\beta_0) + J(\alpha_0) < d$, there are positive a, b depending only on M, N, and d such that $|\alpha_0| < a$, $|\beta_0| < b$ (here $|\cdot|$ denotes Euclidean norm). Indeed, from Lemma 2.8 it is clear that W is a bounded set in $\mathbb{R}^{M+N}$, so the bound for $|\alpha_0|$ follows. Also $\alpha_0 \in W$ implies $J(\alpha_0) > 0$, so that $0 < K(\beta_0) = \frac{1}{2} \beta_0^T A_{MN} \beta_0 < d$. Since A_{MN} is positive definite, a uniform upper bound for $|\beta_0|$ is obtainable.

The proof proceeds as in [32, p. 165]. Since $f \in C^1(\mathbb{R})$, an argument similar to the proof of Lemma 2.9 establishes that $\bar{H}_N(y)$ is a Lipschitz continuous function of y on compact subsets of $\mathbb{R}^{M+N}$. Therefore (2.50), (2.53) possess a unique local solution $y(t)$ on some interval $0 < t < \delta$, where δ depends on a, b, and the modulus of Lipschitz continuity of $\bar{H}_N(y)$. If y leaves the potential well W at some time t_0, then $J(y(t_0)) > d$ and (2.57) is violated. Therefore $y(t) \in W$ for $0 < t < \delta$, and (2.57) holds when $t = \delta$.

One has $|y(\delta)| < a$, $|\dot{y}(\delta)| < b$, and the argument of the preceding paragraph can be employed to extend the solution $y(t)$ uniquely to the interval $0 < t < 2\delta$. Continuing in this way one obtains a unique global solution $y(t)$ which remains in W for all $t > 0$. ∎

One may now obtain a candidate $u = u(x,t)$ for a global weak solution to problem (W). Choose $U \in W$, $V \in H^1_{0\sigma}(D)$ such that

$$E(U,V) \equiv \frac{1}{2}\int_D V^2 dx + \frac{1}{2}\|U\|_D^2 - \int_\Sigma F(\underset{\sim}{B}U)dS < d \ .$$

For simplicity in notation and argument relabel the orthogonal projections U_{MN}, V_{MN} of U, V given in (2.51), (2.52) as U_k, V_k, where $k \to +\infty$ if and only if both $M \to +\infty$ and $N \to +\infty$ (using, e.g., Cantor's ordering for the rationals in [0,1]). Clearly $\lim_{k \to +\infty} K(V_k) = K(V)$, and by Lemma 2.9 $\lim_{k \to +\infty} J(U_k) = J(U)$; so for sufficiently large k, $E(U_k,V_k) < d$.

Suppose that for all sufficiently large k, $U_k \notin W$. Then there is a subsequence $\{\lambda_k\}$ satisfying $0 < \lambda_k < 1$ and $J(\lambda_k U_k) > d$. Some subsequence $\{\lambda_{k'}\}$ of $\{\lambda_k\}$ converges to a limit $\lambda_0 \in [0,1]$. Then $\|\lambda_{k'}U_{k'} - \lambda_0 U\|_{1,2,D} \to 0$ as $k' \to +\infty$, so $\lim_{k' \to +\infty} J(\lambda_{k'}U_{k'}) = J(\lambda_0 U) < d$, a contradiction. Therefore there is an infinite subsequence $\{U_{k_1}\}$ such that $U_{k_1} \in W$. One may assume without loss of generality that $E(U_{k_1},V_{k_1}) < d$ for all k_1.

Using Lemma 2.11 and (2.42) one may obtain corresponding approximating functions $\{u_{k_1}(x,t)\}$ defined for all $t > 0$ and satisfying

$$u_{k_1} = U_{k_1}, \quad u_{k_1,t} = V_{k_1} \quad \text{at} \quad t = 0 \ , \tag{2.58}$$

$$u_{k_1} \in W, \quad E(u_{k_1},u_{k_1,t}) < d \quad \text{for all} \quad t > 0. \tag{2.59}$$

Thus for every $T > 0$, the sequence $\{u_{k_1}\}$ is bounded in norm in $W^{1,2}(D_T)$. Applying Lemma 2.1, one obtains a subsequence $\{u_{k_2}\}$ of the u_{k_1}'s such that u_{k_2} converges strongly in $L_2(D_T)$ to a limit $u \in L_2(D_T)$ for every $T > 0$. Bounded sets in $W^{1,2}(D_T)$ are also conditionally compact with respect to weak convergence in $W^{1,2}(D_T)$ (see, e.g., [26, p. 70]). Thus one may arrange (by taking subsequences if necessary) that the derivatives $u_{k_2,1},\ldots,u_{k_2,n},u_{k_2,t}$ converge weakly in $L_2(D_T)$ to limits $u_1,\ldots,u_n,u_{n+1} \in L_2(D_T)$, respectively, for all $T > 0$. It is easily verified that $u \in W^{1,2}(D_T)$ and that $u_{,t} = u_{n+1}$, $u_{,i} = u_i$ for $i = 1,\ldots,n$. Applying Lemma 2.2 and an elementary property of compact operators (see [37]), $\underset{\sim}{B}u_{k_2}$ converges strongly in $L_2(\Sigma_T)$ to $\underset{\sim}{B}u$ for every $T > 0$. By taking subsequences again if necessary, we may arrange that $u_{k_2} \to u$ a.e. on D_T, and $\underset{\sim}{B}u_{k_2} \to \underset{\sim}{B}u$ a.e. on Σ_T, for all $T > 0$.

<u>Lemma 2.12</u>: $\lim\limits_{k_2 \to +\infty} \int_0^T \int_\Sigma f(\underset{\sim}{B}u_{k_2})\eta \, dS\, dt = \int_0^T \int_\Sigma f(\underset{\sim}{B}u)\eta \, dS\, dt$ for every bounded measurable function η and each $T > 0$.

The proof is omitted since it is entirely analogous to that of Lemma 4.2 in [32].

<u>Lemma 2.13</u>: The limit function u satisfies (1)-(4) for every $T > 0$.

<u>Proof:</u> The weak convergence of $u_{k_2,i}$ to $u_{,i}$ and of $u_{k_2,t}$ to $u_{,t}$

implies

$$\int_A \|u_{,i}\|_{2,D}^2 \, dt \leq \int_A \|u_{k_2,i}\|_{2,D}^2 \, dt \tag{2.60}$$

for each k_2, each $i = 1,\ldots,n$, and every measurable subset A of $[0,\infty)$ of finite measure, and a similar inequality involving $u_{,t}$, $u_{k_2,t}$. Therefore

$$\|u\|_D^2 \leq \|u_{k_2}\|_D^2, \quad \|u_{,t}\|_{2,D}^2 \leq \|u_{k_2,t}\|_{2,D}^2 \tag{2.61}$$

almost everywhere on $[0,\infty)$ for each k_2. If necessary, u and its first partial derivatives may be modified on a set of measure zero in $[0,\infty)$ so that (2.61) holds everywhere, and $u(t) \in H^1_{0\sigma}(D)$, $u_{,t}(t) \in L_2(D)$ for all $t \geq 0$. This establishes (1) for each $T > 0$.

For each $\nu \in L_2(D)$ and each t in $[0,\infty)$,

$$\int_D [u_{k_2}(x,t) - U_{k_2}(x)]\nu(x)dx = \int_0^t \int_D u_{k_2,\tau}(x,\tau)\nu(x)dx \, d\tau.$$

Letting $k_2 \to \infty$ and using the facts that u_{k_2} tends strongly to u in $L_2(D)$ for almost all $t \geq 0$, U_{k_2} tends strongly to U in $L_2(D)$, and $u_{k_2,t}$ tends weakly to $u_{,t}$ in $L_2(D_T)$ for each $T > 0$, one obtains (2.39) for almost all $t \geq 0$. Again, if necessary, one may redefine u on a subset of $[0,\infty)$ of measure zero, so that (2.39) holds for every $t \geq 0$.

To verify (3), let $\rho_\ell(x,t) = C_\ell(t)\psi_\ell(x)$ where $C_\ell \in C^1([0,\infty))$. Choose k_2 so large that $u_{k_2} = u_{MN}$ where $N \geq \ell$, multiply the

$(M+\ell)$-th equation of (2.49) by $C_\ell(t)$ and integrate over $[0,t)$ to obtain

$$0 = \int_0^t \Big[\sum_{i=1}^{M} \Big(\int_D \phi_i \psi_\ell dx\Big) \ddot{q}_i + \sum_{k=1}^{N} \Big(\int_D \psi_k \psi_\ell dx\Big) \ddot{p}_k$$

$$+ \mu_\ell \Big(\int_\Sigma \psi_\ell^2 \, dS\Big) p_\ell - \int_\Sigma f(B\underset{\sim}{u}_N) \psi_\ell dS \Big] C_\ell \, d\tau$$

$$= \int_0^t \int_D \Big[\sum_{i=1}^{M} \ddot{q}_i \phi_i + \sum_{k=1}^{N} \ddot{p}_k \psi_k \Big] \rho_\ell \, dx \, d\tau$$

$$+ \int_0^t \int_D \Big[\sum_{k=1}^{N} (\nabla\psi_k \cdot \nabla\psi_\ell) p_k \Big] C_\ell \, dx \, d\tau$$

$$- \int_0^t \int_\Sigma f(B\underset{\sim}{u}_N) \rho_\ell \, dS \, d\tau$$

$$= \int_0^t \int_D \Big[\frac{\partial^2 u_{MN}}{\partial t^2} \rho_\ell + \nabla u_{MN} \cdot \nabla \rho_\ell \Big] dx \, d\tau$$

$$- \int_0^t \int_\Sigma f(B\underset{\sim}{u}_N) \rho_\ell \, dS \, d\tau \, .$$

Integration by parts yields

$$0 = \Phi(u_{k_2}, V_{k_2}, \rho_\ell) \, , \tag{2.62}$$

where $\Phi(u,V,\eta)$ denotes the left-hand side of (2.40). Letting $k_2 \to +\infty$ in (2.62), and using $\|V_{k_2} - V\|_{2,D} \to 0$, $u_{k_2,t} \to u_{,t}$ and $\nabla u_{k_2} \to \nabla u$ weakly in $L_2(D_T)$, $u_{k_2,t} \to u_{,t}$ strongly in $L_2(D)$ and almost everywhere on $[0,\infty)$, and Lemma 2.12, one obtains $\Phi(u,V,\rho_\ell) = 0$ for almost all $t > 0$.

Similar considerations show $\Phi(u,V,\nu_j) = 0$ for almost all $t > 0$ and all $\nu_j(x,t) = B_j(t)\phi_j(x)$, where $B_j \in C^1([0,\infty))$. From the linearity of Φ in its third argument it is evident that $\Phi(u,V,\eta_{MN}) = 0$ for almost all $t > 0$ and all functions η_{MN} of the form

$$\eta_{MN} = \sum_{j=1}^{M} B_j\phi_j + \sum_{\ell=1}^{N} C_\ell\psi_\ell \ . \qquad (2.63)$$

Now, for any $\eta = \eta(x,t)$ which is C^1 in t on $[0,\infty)$, and in $H^1_{0\sigma}(D)$ for each $t > 0$,

$$\begin{aligned} |\Phi(u,V,\eta)| &= |\Phi(u,V,\eta-\eta_{MN})| \\ &\leq \|u_{,t}\|_{2,D}\|\eta-\eta_{MN}\|_{2,D} + \|V\|_{2,D}\|\eta(x,0)-\eta_{MN}(x,0)\|_{2,D} \qquad (2.64) \\ &\quad + \|u\|_{D_t}\|\eta-\eta_{MN}\|_{D_t} + \|u_{,\tau}\|_{2,D_t}\|\eta_{,\tau}-\eta_{MN,\tau}\|_{2,D_t} \\ &\quad + \|f(\underset{\sim}{B}u)\|_{s,\Sigma_t}\|\underset{\sim}{B}\eta-\underset{\sim}{B}\eta_{MN}\|_{\gamma,\Sigma_t} \end{aligned}$$

by Hölder's inequality, where $s = \frac{\gamma}{\gamma-1}$. (Note that by (2.18) and Lemma 2.2, $f(\underset{\sim}{B}u) \in L_s(\Sigma_t)$). Using Lemma 2.10 one may choose a sequence $\{\eta_{MN}\}$ of functions of the form (2.63) (where B_j, C_ℓ are the Fourier coefficients of η) such that for each $t > 0$,

$$\|\eta(x,t)-\eta_{MN}(x,t)\|_D \to 0 \qquad (2.65)$$

as $M,N \to +\infty$. Now $\{\|\eta_{MN}\|_D\}$ is a monotone increasing sequence of functions continuous on $[0,\infty)$, which converge pointwise on $[0,\infty)$ to $\|\eta\|_D$. By the monotone convergence theorem,

$$\int_0^t \|\eta_{MN}\|_D \, d\tau \to \int_0^t \|\eta\|_D \, d\tau \,, \tag{2.66}$$

and

$$\|\eta-\eta_{MN}\|_{D_t} = \int_0^t \|\eta-\eta_{MN}\|_D \, d\tau \to 0 \tag{2.67}$$

as $M,N \to +\infty$, for each $t > 0$.

Since $\|\eta,_t\|_D$ is uniformly bounded on compact subsets of $[0,\infty)$, one may obtain from (2.65) and Bessel's inequality that $\|\eta,_t(x,t)-\eta_{MN,t}(x,t)\|_D \to 0$ for $t > 0$. By the argument of the preceding paragraph one may conclude that $\|\eta,_\tau - \eta_{MN,\tau}\|_{D_t} \to 0$ for each $t > 0$. Then, by Lemma 2.3, for each $t > 0$

$$\|\eta,_\tau-\eta_{MN,\tau}\|_{2,D_t} \to 0 \tag{2.68}$$

as $M,N \to +\infty$. By (2.65) and Lemma 2.2,

$$\|\eta(x,0)-\eta_{MN}(x,0)\|_{2,D} \to 0 \;; \tag{2.69}$$

and by (2.67) and Lemma 2.2, for each $t > 0$

$$\|\eta-\eta_{MN}\|_{\gamma,\Sigma_t} \to 0 \;. \tag{2.70}$$

Let $M,N \to +\infty$ in (2.64) and use (2.65–70) to obtain $\Phi(u,V,\eta) = 0$ for all $t > 0$. If $\eta : [0,\infty) \to H^1_{0\sigma}(D)$ has the properties of u in (1) and (2), simply choose $\{\eta_k\}$ which are each C^1 in t on $[0,\infty)$ and in $H^1_{0\sigma}(D)$ for each $t > 0$, such that $\eta_k \to \eta$ strongly in $H^1_{0\sigma_T}(D_T)$ for each $T > 0$, and $\eta_k \to \eta$ strongly in $L_2(D)$ for almost all $t > 0$, to obtain $\Phi(u,V,\eta) = 0$ almost everywhere. By redefining u on a set of measure zero one may arrange that (2) holds for all $T > 0$.

For all measurable subsets A of $[0,\infty)$ of finite measure one may establish

$$\lim_{k_2 \to \infty} \int_A \int_\Sigma F(\underset{\sim}{B} u_{k_2}) dS\, dt = \int_A \int_\Sigma F(\underset{\sim}{B} u) dS\, dt$$

by a proof similar to that of Lemma 4.2 of [32]. Thus by (2.61)

$$\begin{aligned} \int_A E(t)dt &\le \lim_{k_2 \to +\infty} \int_A E(u_{k_2}, u_{k_2,t})dt \\ &= \lim_{k_2 \to +\infty} \int_A E(U_{k_2}, V_{k_2})dt \\ &= \int_A E(0)dt \ . \end{aligned}$$

Consequently

$$\int_A [E(0) - E(t)]dt \ge 0$$

for all such A, so that $E(t) \leq E(0)$ for almost all $t > 0$. The function u and its weak derivatives may be altered on a set of measure zero in $[0,\infty)$ to make (2.41) hold everywhere. ∎

Thus (1)-(4) hold with $T = +\infty$, and u is the global weak solution of Theorem 2.1.

Nonexistence

In this section, the following result is established.

Theorem 2.2: Let f satisfy the conditions below, and let u be a weak solution to (W) in the sense of (1)-(4) of the previous section. There is a region $\underset{\sim}{E}$, exterior to the potential well W and characterized by

$$\underset{\sim}{E} = \{v \in H^1_{0\sigma}(D) : J(v) < d \text{ and } Q(v) < 0\},$$

such that if $U \in \underset{\sim}{E}$ and $E(0) = E(U,V) < d$, then u can only exist on a set $D \times [0,T_0)$ with $T_0 < \infty$.

In this section, it is assumed that either f satisfies (i), (ii)(a), and (iii); or else f satisfies (i), (ii)(b), (2.16), and the condition

$$psf(s) \leq s^2 f'(s) \tag{2.71}$$

for all $s \in \mathbf{R}$. Note that (2.15) can be derived from (2.71) by integration, but the two conditions are not equivalent.

Lemma 2.14: Let f satisfy the above conditions, and let u be any nonzero function in $H^1_{0\sigma}(D)$. Then either

(a) $j_u'(\lambda) = 0$ has a unique positive root $\lambda^* = \lambda^*(u)$, $j_u'(\lambda) < 0$ and $j_u''(\lambda) < 0$ for $\lambda > \lambda^*$, and $\lim_{\lambda \to +\infty} j_u(\lambda) = -\infty$; or else

(b) $j_u(\lambda)$ is strictly increasing for $\lambda > 0$, and $\lim_{\lambda \to +\infty} j_u(\lambda) = +\infty$.

Proof: Suppose first that f satisfies (i), (ii)(a), and (iii). By Lemma 2.6 (1)-(3), whenever $\underset{\sim}{B}u \geq 0$ is nonzero one has $j_u'(\lambda) < 0$ for $\lambda > \lambda^*$, so

$$j_u''(\lambda) \leq \frac{1}{\lambda^2} \int_\Sigma (\lambda \underset{\sim}{B}u)[f(\lambda \underset{\sim}{B}u)-(\lambda \underset{\sim}{B}u)f'(\lambda \underset{\sim}{B}u)]dS < 0$$

for $\lambda > \lambda^*$, by (2.26).

Suppose now that f satisfies (i), (ii)(b), (2.71), and (2.16). By (2.71),

$$\begin{aligned} j_u''(\lambda) &\leq \|u\|_D^2 - \frac{p}{\lambda} \int_\Sigma (\underset{\sim}{B}u)f(\lambda \underset{\sim}{B}u)dS \\ &= (1-p)\|u\|_D^2 + \frac{p}{\lambda} j_u'(\lambda) \end{aligned} \tag{2.72}$$

for all $\lambda > 0$. If $j_u'(\lambda) = 0$ has a smallest positive root λ^*, multiply both sides of (2.72) by $\lambda^{-p} > 0$ and integrate from

λ^* to $\lambda > \lambda^*$ to obtain

$$j_u'(\lambda) < \|u\|_D^2 \, \lambda^p[\lambda^{1-p}-(\lambda^*)^{1-p}] \; .$$

Since $p > 1$, $j_u'(\lambda) < 0$ for $\lambda > \lambda^*$; which in turn implies $j_u''(\lambda) < 0$ for $\lambda > \lambda^*$ by (2.72). In particular, λ^* must be the only positive root of $j_u'(\lambda) = 0$, and $\lim\limits_{\lambda \to +\infty} j_u(\lambda) = -\infty$ by Lemma 2.6 (4).

Therefore (a) is established. The result in (b) follows from the convexity of j_u in a neighborhood of the origin, and Lemma 2.6. ∎

The following lemma will play a central role in the proof of Theorem 2.2.

Lemma 2.15: Suppose $\{u_n\}$ is a sequence in $H^1_{0\sigma}(D)$ satisfying $\|u_n\|_D \neq 0$ and $Q(u_n) < 0$ for all n, and $Q(u_n) \to 0$ as $n \to +\infty$, where Q is as defined in (2.30). Then

$$\varliminf_{n \to +\infty} J(u_n) \geq d \; .$$

Proof: Suppose to the contrary that there exists an $\varepsilon > 0$ and a subsequence of $\{u_n\}$ (still denoted $\{u_n\}$) such that $J(u_n) \leq d - \varepsilon$ for all n.

Define $j_n(\lambda) = J(\lambda u_n)$. Since $Q(u_n) < 0$, by Lemma 2.14 for each n there exists a unique positive root λ_n of $j_n'(\lambda) = 0$, and $0 < \lambda_n < 1$. Also by Lemma 2.14, $j_n''(\lambda) < 0$ for $\lambda_n \leq \lambda \leq 1$, so

$$0 > j_n(1) - j_n(\lambda_n) = \int_{\lambda_n}^{1} j_n'(s)ds$$

$$> j_n'(1)(1-\lambda_n) > j_n'(1) \to 0^- \qquad (n \to \infty) .$$

This is a contradiction, since $j_n(1) < d - \varepsilon$ for all n, and $j_n(\lambda_n) > d$ for all n by the variational definition of d. ∎

The proof of Theorem 2.2 uses a well-known concavity technique ([15],[19],[21],[22]), and is very similar to that of [28]. It is included here for the convenience of the reader. Suppose u is a global weak solution of (W) in the sense of the previous section, and suppose $U \in \underset{\sim}{E}$ and $E(0) < d$. Define

$$M(t) = \int_D u^2(x,t)dx .$$

It will be shown that $M \to +\infty$ in finite time, contradicting the assumption that u is a global solution of ($\bar{W}$).

Define $P(t,s) = \int_D u(x,t)u(x,s)dx$. By (2.39) with $\nu(x) = u(x,s)$,

$$P(t,s) = P(0,s) + \int_0^t \int_D u,_\tau(x,\tau)u(x,s)dx\, d\tau$$

for $t,s > 0$. Since $\int_D u,_t(x,t)u(x,s)dx$ is an integrable function of t on compact subsets of $[0,\infty)$ for each $s > 0$,

$$\frac{\partial}{\partial t} P(t,s) = \int_D u,_t(x,t)u(x,s)dx$$

for $s \geq 0$ and almost all t in $[0,\infty)$. Since P is symmetric in its arguments,

$$\dot{M}(t) = \left[\frac{\partial}{\partial t} P(t,s) + \frac{\partial}{\partial s} P(t,s)\right]\Big|_{s=t}$$

$$= 2 \int_D u_{,t}(x,t)u(x,t)dx \qquad (2.73)$$

almost everywhere on $[0,\infty)$. If necessary, u and $u_{,t}$ may be modified on a set of measure zero in D_∞ so that (2.73) holds for all $t > 0$.

Now, taking $\eta = u$ in (2.40), one obtains

$$\dot{M}(t_2) - \dot{M}(t_1) = 2 \int_{t_1}^{t_2} \int_D [|u_{,t}|^2 - |\nabla u|^2]dx\, dt$$
$$+ 2 \int_{t_1}^{t_2} \int_\Sigma (\underset{\sim}{B}u)f(\underset{\sim}{B}u)dS\, dt$$

for $t_2 > t_1 \geq 0$. Since $\|u_{,t}\|_{2,D}$, $\|u\|_D$, and $\int_\Sigma (\underset{\sim}{B}u)f(\underset{\sim}{B}u)dS$ are uniformly bounded on compact subsets of $[0,\infty)$, $\dot{M}$ is Lipschitz continuous on such sets. Hence $\ddot{M}$ exists a.e. in $[0,\infty)$ and is given by

$$\ddot{M}(t) = 2[\|u_{,t}\|^2_{2,D} - \|u\|^2_D + \int_\Sigma (\underset{\sim}{B}u)f(\underset{\sim}{B}u)dS] \,. \qquad (2.74)$$

One may show that $u(t) \in \underset{\sim}{E}$ for all $t \geq 0$. Indeed, if u leaves $\underset{\sim}{E}$ at some smallest time $t = t_0 > 0$, then

$$Q(u(t_0)) \leq \varliminf_{n \to \infty} Q(u(t_n)) \leq 0$$

where $t_n \to t_0^-$. If $Q(u(t_0)) < 0$, then $u(t_0) \in \underset{\sim}{E}$. If $Q(u(t_0)) = 0$, then $J(u(t_0)) \geq d$ by the variational definition of d; but this contradicts the energy inequality (2.41), which requires that $J(u(t)) \leq E(0) < d$ for all $t \geq 0$.

Therefore

$$Q(u) = \|u\|_D^2 - \int_\Sigma (\underset{\sim}{B}u)f(\underset{\sim}{B}u)dS \leq 0 \quad \text{on} \quad [0,\infty),$$

and

$$\ddot{M}(t) \geq 0 \qquad \text{for all} \quad t \geq 0.$$

One may show $\dot{M}(t_1) > 0$ for some $t_1 \geq 0$. Suppose to the contrary that $\dot{M}(t) \leq 0$ for all $t \geq 0$. Then, since $M \geq 0$ and is convex, $L \equiv \lim_{t \to +\infty} M(t)$ exists and is finite. Furthermore, $L > 0$ since u remains in the region $\underset{\sim}{E}$ exterior to the potential well W for all positive time. Choose $t_n \to \infty$ such that $M(t_n) \to L$, $\dot{M}(t_n) \to 0$, and $\ddot{M}(t_n) \to 0$. Since $Q(u(t_n)) \leq 0$ for all n, one sees from (2.74) that

$$\|u_{,t}(x,t_n)\|_{2,D}^2 \to 0 \tag{2.75}$$

and

$$Q(u(t_n)) \to 0$$

as $n \to \infty$. Applying Lemma 2.15,

$$\varliminf_{n \to \infty} J(u(t_n)) \geq d > E(0) .$$

But from (2.75) and the energy inequality (2.41), $\underline{\lim}_{n \to \infty} J(u(t_n)) < E(0)$, a contradiction.

Using (2.15), (2.41), and Lemma 2.3,

$$\ddot{M}(t) > 2[\|u,_t\|^2_{2,D} - \|u\|^2_D + (p+1)\int_\Sigma F(\underset{\sim}{B}u)dS]$$

$$> 2\,\|u,_t\|^2_{2,D} - 2\|u\|^2_D$$

$$+ 2(p+1)[\frac{1}{2}\,\|u,_t\|^2_{2,D} + \frac{1}{2}\,\|u\|^2_D - E(0)] \tag{2.76}$$

$$= (p+3)\|u,_t\|^2_{2,D} + (p-1)\|u\|^2_D - 2(p+1)E(0)$$

$$> (p+3)\|u,_t\|^2_{2,D} + C(p-1)M(t) - 2(p+1)E(0)$$

for $t > 0$, where $C > 0$ is a constant. Since M is convex and $\dot{M}(t_1) > 0$ by the argument of the previous paragraph, one may conclude that $M(t)$ is increasing for all $t > t_1$, and that the quantity

$$C(p-1)M(t) - 2(p+1)E(0)$$

must eventually become positive, and remain positive thereafter. Therefore

$$\ddot{M}(t) > (p+3)\|u,_t\|^2_{2,D}$$

for all sufficiently large t. Let $\alpha = \frac{p-1}{4} > 0$. Then for t sufficiently large,

$$(M^{-\alpha})'' = \frac{\alpha}{M^{\alpha+2}} [(\alpha+1)\dot{M}^2 - M\ddot{M}]$$

$$< \frac{\alpha}{M^{\alpha+2}} [(\alpha+1)(\int_D u_{,t}\, u\, dx)^2 - (p+3)\|u_{,t}\|^2_{2,D}\|u\|^2_{2,D}]$$

$$= \frac{\alpha(p+3)}{M^{\alpha+2}} [\frac{1}{4} (\int_D u_{,t}\, u\, dx)^2 - \|u_{,t}\|^2_{2,D}\|u\|^2_{2,D}] < 0 .$$

Therefore there exists a finite time $T_0 > 0$ for which $\lim_{t \to T_0^-} M^{-\alpha}(t) = 0$; and consequently

$$\lim_{t \to T_0^-} M(t) = +\infty .$$

This completes the proof of Theorem 2.2.

CHAPTER 3. THE POTENTIAL WELL THEORY APPLIED TO THE HEAT EQUATION WITH A NONLINEAR BOUNDARY CONDITION

Introduction

In this chapter, the following problem for the linear heat equation is considered:

$$
(\mathrm{H}) \qquad
\begin{aligned}
u_t &= \Delta u && \text{in } D \times (0,T) \\
u &= 0 && \text{on } \sigma \times (0,T) \\
\frac{\partial u}{\partial n} &= f(u) && \text{on } \Sigma \times (0,T) \\
u(x,0) &= U(x) && \text{in } D\,.
\end{aligned}
$$

In general, methods employed to study solutions of hyperbolic problems cannot be employed to study solutions of parabolic problems and conversely. Nevertheless the arguments in Chapter 2 used to study problem (W) can be modified to obtain corresponding results for problem (H) and can be used to prove Sattinger's assertions about the parabolic problems he mentions in [32]. The key results of this chapter, being of a function analytic nature, are similar to those in Chapter 2. However, there are several important differences in their proofs which are emphasized here.

Note that the potential energy functional associated with problem (H) is the same as that associated with problem (W):

$$J(u) = \frac{1}{2}\int_D |\nabla u|^2 dx - \int_\Sigma F(\underset{\sim}{B}u)dS. \tag{3.1}$$

Suppose that f, F satisfy the following conditions (i)-(iii) of Chapter 2. Then as in Chapter 2, there is a potential well W of positive depth d in the function space $H^1_{0\sigma}(D)$ characterized by

$$W = \{u \in H^1_{0\sigma}(D) : 0 < J(\lambda u) < d \quad \text{for} \quad 0 < \lambda < 1\}, \tag{3.2}$$

and

$$d = \inf J(u) \tag{3.3}$$

where u is subject to the constraint

$$Q(u) \equiv \|u\|_D^2 - \int_\Sigma (\underset{\sim}{B}u)f(\underset{\sim}{B}u)\, dS = 0. \tag{3.4}$$

The infimum in (3.3) is taken over all $u \in H^1_{0\sigma}(D)$ with nonzero trace $\underset{\sim}{B}u$ on Σ in case f satisfies (ii)(a); and may be taken over all $u \in H^1_{0\sigma}(D)$ with $\underset{\sim}{B}u > 0$ a.e. in case f satisfies (ii)(b).

Under conditions on the nonlinearity f weaker than (i)-(iii) it has been shown ([21]) that solutions of problem (H) cannot exist for all time whenever $J(U) < 0$. Generalizations of this result to a wider class of initial-boundary value problems are given in [12], [22]. In this chapter, the following theorem is proved, which is the analogue of the main results, Theorems 2.1 and 2.2, of Chapter 2:

Theorem 3.1: Let W, d be as defined in (3.2)-(3.4).

(a) Suppose f satisfies (i)-(iii). Then if $U \in W$, problem (H) has a global weak solution in the sense of (1)-(4) of the following section.

(b) Suppose that either f satisfies (i), (ii)(a), and (iii), or else f satisfies (i), (ii)(b), (2.16), and the condition (2.71). Let u be a weak solution of problem (H). Then there is a region $\underset{\sim}{E}$, exterior to W and characterized by

$$\underset{\sim}{E} = \{v \in H^1_{0\sigma}(D) : J(v) < d \text{ and } Q(v) < 0\},$$

such that if $U \in \underset{\sim}{E}$, then u can only exist on a set $D \times [0,T_0)$ with $T_0 < \infty$.

The proof of Theorem 3.1 will be outlined giving particular attention to how it differs from the proofs of Theorems 2.1 and 2.2.

Existence

In this section, part (a) of Theorem 3.1 is established. Recall the following results:

(A) the potential well W is bounded as a subset of $W^{1,2}(D)$;

(B) bounded sets in $W^{1,2}(D)$ (respectively, $W^{1,2}(D_T)$) are pre-compact as subsets of $L_2(D)$ $(L_2(D_T))$, and are conditionally compact with respect to weak convergence in $W^{1,2}(D)$ $(W^{1,2}(D_T))$;

(C) the trace operator $\underset{\sim}{B}$ is compact as a linear mapping from $W^{1,2}(D)$ into $L_q(\Sigma)$ for $2 \leq q < \frac{2(n-1)}{n-2}$ if $n > 2$, and for $2 \leq q < \infty$ if $n = 2$; and

(D) Poincaré's inequality

$$\|u\|_{2,D} \leq C\|u\|_D$$

holds for all $u \in H^1_{0\sigma}(D)$, where $C > 0$ is independent of u.

The kinetic and total energy functionals associated with problem (H) are given by

$$K(u) = \int_0^t \int_D u^2_{,\tau}(x,\tau)\, dx\, d\tau, \tag{3.5}$$

and

$$E(t) = K(u(\cdot,t)) + J(u(\cdot,t)),$$

respectively. A function $u = u(x,t)$ is called a weak solution of problem (H) on D_T provided

(1) $u(t) : [0,T) \to H^1_{0\sigma}(D)$, $u_{,t}(t) : [0,T) \to L_2(D)$; $\|u(t)\|_D$ and $\|u_{,t}(t)\|_{2,D}$ are uniformly bounded on compact subsets of $[0,T)$;

(2) for each t, $0 \leq t < T$, and every $\eta \in L_2(D)$,

$$\int_D [u(x,t) - U(x)]\eta(x)dx = \int_0^t \int_D u,_\tau(x,\tau)\eta(x)dx\, d\tau; \qquad (3.6)$$

(3) for each t, $0 \leq t < T$, and every $\eta(t) : [0,T) \to H^1_{0\sigma}(D)$ with the properties of u in (1) and (2),

$$\int_D u,_t \eta\, dx + \int_D \nabla u\cdot\nabla\eta\, dx - \int_\Sigma f(\underset{\sim}{B}u)(\underset{\sim}{B}\eta)dS = 0; \qquad (3.7)$$

and

(4) for each t, $0 \leq t < T$,

$$E(t) \leq E(0) = J(U). \qquad (3.8)$$

The function u is a global weak solution of problem (H) provided u satisfies (1)-(4) with $T = +\infty$. The global weak solution of Theorem 3.1(a) is approximated by functions of the form

$$u_{MN}(x,t) = \sum_{i=1}^{M} q_i(t)\phi_i(x) + \sum_{k=1}^{N} p_k(t)\, \psi_k(x), \qquad (3.9)$$

where $\phi_1,\phi_2,\ldots \in H^1_0(D)$, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots$, and $\psi_1, \psi_2, \ldots \in H^1_{0\sigma}(D)$, with corresponding eigenvalues $\mu_1, \mu_2, \ldots$, are as in Chapter 2. The q_i's and p_k's are chosen to satisfy a nonlinear system of ordinary differential equations. Let

$$y = (q_1,\ldots,q_M,p_1,\ldots,p_N)^T,$$

$$\zeta = (\phi_1,\ldots,\phi_M,\psi_1,\ldots,\psi_N)^T,$$

and substitute (3.9) into the expressions (3.5), (3.1) to obtain

$$K(\dot{y}) \equiv K(u_{MN}) = \int_0^t (\dot{y}(\tau))^T A_{MN} \dot{y}(\tau)d\tau,$$

$$J(y) \equiv J(u_{MN}) = \frac{1}{2} y^T B_{MN} y - \int_\Sigma F(u_N)dS,$$

where

$$A_{MN} = \int_D \zeta\zeta^T dx,$$

$$B_{MN} = \mathrm{diag}\left(\lambda_1,\ldots,\lambda_M, \mu_1 \int_\Sigma \psi_1^2 dS, \ldots, \mu_N \int_\Sigma \psi_N^2 dS\right),$$

$$u_N = \sum_{k=1}^N p_k \psi_k .$$

Form the Lagrangian $L(\dot{y},y) = K(\dot{y}) - J(y)$. From Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_i}\right) - \frac{\partial L}{\partial y_i} = 0, \quad i = 1,\ldots,M+N,$$

one obtains the system of $M+N$ ordinary differential equations

$$A_{MN} \dot{y} + B_{MN} y = H_N(y), \tag{3.10}$$

where

$$H_N(y) = [0,\ldots,0,\int_\Sigma f(u_N)\psi_1 dS, \ldots, \int_\Sigma f(u_N)\psi_N dS]^T.$$

In Chapter 2, it was shown that A_{MN} is a real symmetric, positive definite matrix. Thus (3.10) is equivalent to the system

$$\dot{y} + \bar{B}_{MN}\, y = \bar{H}_N(y), \tag{3.11}$$

where $\bar{B}_{MN} = A_{MN}^{-1}B_{MN}$, $\bar{H}_N = A_{MN}^{-1}H_N$. Expand $U \in W$ in the partial Fourier series

$$U_{MN}(x) = \sum_{i=1}^{M} \alpha_{1i}\phi_i(x) + \sum_{k=1}^{N} \alpha_{2k}\, \psi_k(x), \tag{3.12}$$

where $\alpha_{1i} = (U,\phi_i)_{D,\Sigma}$, $\alpha_{2k} = (U,\psi_k)_{D,\Sigma}$, to obtain the appropriate initial conditions for (3.11):

$$y(0) = \alpha, \tag{3.13}$$

where $\alpha = (\alpha_{11}, \ldots, \alpha_{1M}, \alpha_{21}, \ldots, \alpha_{2N})^T$. The energy equality

$$K(\dot{y}(t)) + J(y(t)) = J(\alpha), \tag{3.14}$$

valid for all t in the existence interval for y, follows directly from (3.11) and (3.13).

One may then prove the following analogue of Lemma 2.11.

Lemma 3.1: Provided $U_{MN} = \alpha^T\zeta \in W$, the initial-value problem (3.11), (3.13) has a unique global solution y satisfying $(y(t))^T\zeta \in W$ for all $t > 0$.

The proof uses the boundedness of $\{\alpha_0 \in \mathbf{R}^{M+N} : \alpha_0^T\zeta \in W\}$ in $\mathbf{R}^{M+N}$, and the Lipschitz continuity of $\bar{H}_N(y)$ on compact subsets of $\mathbf{R}^{M+N}$, both established in Chapter 2.

Just as in Chapter 2, one may choose a doubly infinite subsequence $\{U_{M_1N_1}\}$ of functions of the form (3.12) satisfying $U_{M_1N_1} \in W$ for all M_1, N_1. Using Lemma 3.1 and (3.9), one obtains corresponding approximating functions $\{u_{M_1N_1}(x,t)\}$ defined for all $t > 0$ and satisfying

$$u_{M_1N_1}(x,0) = U_{M_1N_1}(x),$$

$$u_{M_1N_1} \in W \quad \text{and} \quad K(u_{M_1N_1}) + J(u_{M_1N_1}) = J(U_{M_1N_1}) \quad \text{for all} \quad t > 0.$$

Using (A), (B), and (C), choose a doubly infinite subsequence $\{u_{M_2N_2}\}$ of approximating functions, and a function $u \in W^{1,2}(D_T)$, such that for all $T > 0$, as $M_2, N_2 \to \infty$ the sequence $u_{M_2N_2}$ converges to u strongly in $L_2(D_T)$ and a.e. on D_T; $u_{M_2N_2,t}, u_{M_2N_2,1}, \ldots, u_{M_2N_2,n}$ converges to $u_{,t}, u_{,1}, \ldots, u_{,n}$ weakly in $L_2(D_T)$; and $\underset{\sim}{B}u_{M_2N_2}$ converges to $\underset{\sim}{B}u$ strongly in $L_2(\Sigma_T)$ and a.e. on Σ_T. By Lemma 2.12,

$$\begin{aligned} &\lim_{M_2,N_2 \to \infty} \int_0^T \int_\Sigma f(\underset{\sim}{B}u_{M_2N_2})\eta \, dS \, dt \\ &= \int_0^T \int_\Sigma f(\underset{\sim}{B}u)\eta \, dS \, dt \end{aligned} \tag{3.15}$$

for every bounded measurable function η on Σ_T and each $T > 0$; and

$$\lim_{M_2,N_2 \to \infty} \int_G F(\underset{\sim}{B}u_{M_2N_2}) dS \, dt = \int_G F(\underset{\sim}{B}u) dS \, dt \tag{3.16}$$

for each measurable subset G of $[0,T)$ and each $T > 0$.

Conditions (1) and (2) on a weak solution of (H) are identical to conditions in Chapter 2 on a weak solution of problem (W). Consequently the limit function u of the preceding paragraph satisfies (1) and (2) for each $T > 0$ and all $0 < t < T$ by the same arguments as in Chapter 2.

To establish (3), let $\Psi(u,\eta)$ denote the left-hand side of (3.7). Choose any $T > 0$, and for each integer $\ell \geq 1$ let $C_\ell = C_\ell(t)$ be any continuously differentiable function on $[0,T]$. For fixed ℓ choose $N_2 \geq \ell$, multiply both sides of the $(M_2+\ell)$-th equation in (3.10) by C_ℓ, integrate both sides with respect to t over any measurable subset G of $[0,T)$, and use the orthogonality properties of the ψ_k's to obtain

$$0 = \int_G \Big[\sum_{i=1}^{M_2} \Big(\int_D \phi_i\psi_\ell \, dx\Big)\dot{q}_i + \sum_{k=1}^{N_2} \Big(\int_D \psi_k\psi_\ell \, dx\Big)\dot{p}_k$$

$$+ \mu_\ell\Big(\int_\Sigma \psi_\ell^2 \, dS\Big)p_\ell - \int_\Sigma f(\underset{\sim}{B}u_{N_2})\psi_\ell dS\Big]C_\ell \, dt \qquad (3.17)$$

$$= \int_G \Psi(u_{M_2N_2}, C_\ell\psi_\ell)dt \; .$$

Let $M_2, N_2 \to \infty$ in (3.17) and use (3.15) and the facts that $u_{M_2N_2,t} \to u_{,t}$ and $\nabla u_{M_2N_2} \to \nabla u$ weakly in $L_2(D_T)$ to obtain $\int_G \Psi(u, C_\ell\psi_\ell)dt = 0$ for each $\ell \geq 1$. In a similar manner one may show that $\int_G \Psi(u, B_j\phi_j)dt = 0$ for each $j \geq 1$, where $B_j = B_j(t)$ is any continuously differentiable function on $[0,T]$. Since $\Psi(u,\eta)$ is linear in η, it follows that

$$\int_G \Psi(u,\eta_{MN})dt = 0 \tag{3.18}$$

for all functions η_{MN} of the form

$$\eta_{MN} = \sum_{j=1}^{M} B_j\phi_j + \sum_{\ell=1}^{N} C_\ell\psi_\ell \ . \tag{3.19}$$

Since (3.18) holds for all measurable subsets G of $[0,T)$, it follows that $\Psi(u,\eta_{MN}) = 0$ a.e. on $[0,T)$. If necessary, u may be modified on a set of measure zero in $[0,T)$ so that $\Psi(u,\eta_{MN}) = 0$ everywhere.

For any $\eta(t) : [0,T) \to H^1_{0\sigma}(D)$ satisfying the same conditions as u in (1) and (2), we may choose a sequence $\{\eta_{MN}\}$ of functions of the form (3.19) such that for each t, $0 \le t < T$,

$$\|\eta(t) - \eta_{MN}(t)\|_D \to 0 \tag{3.20}$$

as $M,N \to \infty$. By Hölder's inequality,

$$\begin{aligned} |\Psi(u,\eta)| &= |\Psi(u,\eta-\eta_{MN})| \\ &\le \|u,_t\|_{2,D}\|\eta-\eta_{MN}\|_{2,D} + \|u\|_D\|\eta-\eta_{MN}\|_D \\ &\quad + \|f(\underset{\sim}{B}u)\|_{s,\Sigma}\|\underset{\sim}{B}\eta-\underset{\sim}{B}\eta_{MN}\|_{\gamma,\Sigma} \ , \end{aligned} \tag{3.21}$$

where $s = \frac{\gamma}{\gamma-1}$ and $\|v\|_{q,\Sigma} \equiv [\int_\Sigma |v|^q \, dS]^{1/q}$. (Using the uniform boundedness of $\|u(t)\|_D$ on compact subsets of $[0,T)$ from (1), the order condition $|f(u)| = (\mathcal{O}|u|^{\gamma-1})$ as $|u| \to +\infty$ proved in Chapter 2, and results (C) and (D), $f(\underset{\sim}{B}u) \in L_s(\Sigma)$ for each t, $0 \leq t < T$). By (3.20), (C), and (D), the right-hand side of (3.21) approaches zero as $M,N \to \infty$. Therefore $\Psi(u,\eta) = 0$ for each t, $0 \leq t < T$, and (3) is established for each $T > 0$.

To establish (4), note that since $u_{M_2N_2,t} \to u_{,t}$ and $\nabla u_{M_2N_2} \to \nabla u$ weakly in $L_2(D_T)$, one has

$$K(u(t)) \leq K(u_{M_2N_2}(t))$$

for each $0 \leq t < T$ and each M_2,N_2, and

$$\int_G \int_D |\nabla u|^2 \, dx \, d\tau \leq \int_G \int_D |\nabla u_{M_2N_2}|^2 \, dx \, d\tau$$

for each measurable subset G of $[0,T)$ and each M_2,N_2. Thus

$$\int_G E(\tau) d\tau = \int_G [K(u) + J(u)] d\tau$$

$$\leq \lim_{M_2N_2 \to \infty} \int_G [K(u_{M_2N_2}) + J(u_{M_2N_2})] d\tau$$

$$= \lim_{M_2N_2 \to \infty} \int_G J(U_{M_2N_2}) d\tau = \int_G J(U) d\tau$$

using the energy equality for the approximating solutions $u_{M_2N_2}$, the continuity of J on $H^1_{0\sigma}(D)$ (proved in Lemma 2.9), and the fact that $U_{M_2N_2} \to U$ strongly in $H^1_{0\sigma}(D)$. Therefore $E(t) \leq J(U)$ for almost all $t \in [0,T)$. If necessary, u may be redefined on a set of measure zero so that (4) holds everywhere on $[0,T)$ for each $T > 0$.

Nonexistence

Suppose f satisfies the conditions of part (b) of Theorem 3.1, suppose $U \in \underset{\sim}{E}$, and suppose u is a global weak solution to problem (H). Then one can show that

$$M(t) \equiv \int_0^t \int_D u^2(x,\tau)dxd\tau$$

approaches infinity in finite time, which contradicts (1) of the previous section and establishes Theorem 3.1(b). The proof is similar to that in Section 5 of [28] and uses a well-known concavity technique ([15], [19]); an outline is given here.

Since $\|u(t)\|^2_{2,D}$ is integrable on compact subsets of $[0,\infty)$,

$$\dot{M}(t) = \int_D u^2(x,t)dx = \|U\|^2_{2,D} + 2\int_0^t \int_D u(x,\tau)u,_\tau(x,\tau)dxd\tau \qquad (3.22)$$

a.e. on $[0,\infty)$. Since $\|u(t)\|_{2,D}$, $\|u,_t(t)\|_{2,D}$ are uniformly bounded on compact subsets of $[0,\infty)$, one may use (3.22) and (possibly) redefine u on a set of measure zero to show that $\dot{M}$ is Lipschitz continuous on compact subsets of $[0,\infty)$. Therefore $\ddot{M}$ exists a.e. on $[0,\infty)$, and using (3.22) and (3.6),

$$\ddot{M}(t) = 2\int_D u,_t(x,t)u(x,t)dx$$

$$= 2[\int_\Sigma (\underset{\sim}{B}u)f(\underset{\sim}{B}u)dS - \|u\|_D^2] = -2Q(u) \tag{3.23}$$

a.e. in $[0,\infty)$, where Q is as defined in (3.4).

Suppose u leaves $\underset{\sim}{E}$ at some smallest time $t = t_0 > 0$. Then

$$Q(u(t_0)) \leq \underline{\lim}_{n\to\infty} Q(u(t_n)) \leq 0,$$

where $t_n \to t_0^-$. If $Q(u(t_0)) < 0$, then $u(t_0) \in \underset{\sim}{E}$, a contradiction. If $Q(u(t_0)) \geq d$, then $J(u(t_0)) \geq d$ by the variational definition (3.3), (3.4) of d; but this contradicts (3.8), which requires that $J(u(t)) \leq J(U) < d$ for all $t \geq 0$. Therefore $u(t) \in \underset{\sim}{E}$, and $Q(u(t)) < 0$, for all $t \geq 0$.

If there exists a sequence $t_n \to \infty$ such that $Q(u(t_n)) \to 0^-$, then one may obtain the contradiction

$$\underline{\lim}_{n\to\infty} J(u(t_n)) \geq d > J(U)$$

of (3.8), as a consequence of Lemma 2.15.

Therefore one has in fact that $Q(u(t))$ is negative and bounded away from zero for $0 \leq t < \infty$. Using (3.23) one may easily show that both $M(t)$ and $\dot{M}(t)$ approach $+\infty$ as $t \to \infty$.

By (3.23) and the energy inequality (3.8),

$$\ddot{M}(t) \geq 2(p+1)K(u) + (p-1)C^2\dot{M} - 2(p+1)J(U) \tag{3.24}$$

for all $t > 0$, where C is as in (D) of the previous section. If one sets $\alpha = \frac{p-1}{2} > 0$, then using (3.24) and the expression (3.22) for $\dot{M}$, one obtains

$$
\begin{aligned}
(M^{-\alpha})^{\cdot\cdot} &= -\frac{\alpha}{M^{\alpha+2}}\left(M\ddot{M} - \frac{p+1}{2}\dot{M}^2\right) \\
&\leq -\frac{\alpha}{M^{\alpha+2}}\Big\{2(p+1)\Big[K(u)M - \Big(\int_0^t\int_D uu,_{\tau}\,dx\,d\tau\Big)^2\Big] \\
&\qquad + (p-1)C^2 M\dot{M} - (p+1)\|U\|_{2,D}^2\,\dot{M} \\
&\qquad - 2(p+1)J(U)M + \frac{(p+1)}{2}\|U\|_{2,D}^4\Big\}.
\end{aligned}
\tag{3.25}
$$

The term in square brackets in (3.25) is nonnegative by Schwarz's inequality, and $(p-1)C^2M\dot{M}$ will eventually become and stay larger than the remaining three terms in (3.25). Therefore $(M^{-\alpha})^{\cdot\cdot} < 0$ for all sufficiently large t, so that $M^{-\alpha} \to 0^+$, and $M \to +\infty$, in finite time. This is the desired contradiction.

CHAPTER 4. A HYPERBOLIC QUENCHING PROBLEM IN SEVERAL DIMENSIONS

Introduction

Let D be an open, bounded subset of $\mathbb{R}^n$ with boundary ∂D. Let $\phi : (-\infty, M) \to (0,\infty)$, $M > 0$, be continuously differentiable, monotone increasing, convex, and satisfy $\lim_{u \to M^-} \phi(u) = \infty$; and let $\varepsilon > 0$. In this chapter the following initial-boundary value problem is considered:

$$(\mathrm{An}) \qquad \begin{aligned} u_{tt} &= \Delta_n u + \varepsilon\phi(u) && \text{in} \quad D \times (0,T) \\ u &= 0 && \text{on} \quad \partial D \times (0,T) \\ u(x,0) &= u_0(x), \quad u_t(x,0) = v_0(x) && \text{in} \quad D, \end{aligned}$$

where Δ_n denotes the n-dimensional Laplacian.

Let (A1) denote problem (An) with $D = (0,1)$. When $\phi(u) = \frac{1}{M-u}$, a solution of (A1) has a physical interpretation as describing the motion of a wire composed of a magnetic material and carrying an electric current, in the presence of another current-carrying wire ([17]). Chang and Levine [4] showed that for suitably regular initial data, problem (A1) has a unique local piecewise C^2 solution u which can be continued as long as $u < M$. They also established the existence of numbers $\varepsilon_1 > \varepsilon_0 > 0$ such that

(a) if $\varepsilon > \varepsilon_1$, then for some finite $T > 0$,

$$\lim_{t \to T^-} \Big(\sup_{0 < x < 1} u(x,t) \Big) = M.$$

Hence one of u_{tt}, u_{xx} becomes infinite on $[0,1] \times [0,T)$;

(b) if $0 < \varepsilon < \varepsilon_0$, and the initial data u_0, v_0 are appropriately restricted, there is a $\delta = \delta(\varepsilon) > 0$ such that

$$|u(x,t)| < M - \delta \quad \text{on} \quad [0,1] \times [0,\infty).$$

Note that by applying to (A1) the change of scale $x' = Lx$, $t' = Lt$, $\varepsilon = L^2$, one obtains (A1) with ε replaced by 1 and x' varying between 0 and L. Results (a) and (b) assert, therefore, that global solutions do not exist for long wires, but do exist for short wires.

If u behaves as in (a), it is said to <u>quench</u> in finite time. Speaking loosely, one says that a solution of some evolutionary problem quenches in finite (or infinite) time T if some norm of the solution remains bounded, while some norm of one of its derivatives becomes unbounded, on the interval $[0,T)$ ([16]).

For space dimensions $n > 2$ and ε sufficiently large, solutions of problem (An) also quench in finite time ([4]). However, the proof of (b) in [4] relies strongly on the inequality

$$4\, u^2(x,t) < \int_0^1 |u_x(x,t)|^2 \, dx, \tag{4.1}$$

which guarantees the imbedding of $H_0^1(0,1)$ into $C([0,1])$. In general, no such imbedding $H_0^1(D) \to C(\bar{D})$, or even $H_0^1(D) \to L_\infty(D)$, is possible in higher dimensions, and the question of existence of global solutions of (An) for $n > 2$ remains open.

If instead of (An) one considers the abstract problem

$$\text{(B)} \qquad \begin{aligned} &\frac{d^2u}{dt^2} + Au = \varepsilon\phi(u) \qquad && 0 < t < \infty \\ &u(0) = u_0 \in V, \quad u'(0) = v_0 \in L_2(D), && \end{aligned}$$

where $V \subseteq L_2(D)$ imbeds in $L_\infty(D)$ and A is an operator of elliptic type mapping V into its dual, then a global existence theorem for ε sufficiently small may be proved; see Levine and Smiley [23]. Their results apply, for example, to solutions of (B) when D is the interior of a rectangle in $\mathbb{R}^2$, $u(x,t) = \Delta_2 u(x,t) = 0$ on $\partial D \times [0,\infty)$, A is the biharmonic operator Δ_2^2, and $V = H^2(D) \cap H_0^1(D)$.

Acker and Walter [1] have proved a higher-dimensional global existence theorem for small ε for solutions of

$$\text{(C)} \qquad \begin{aligned} u_t &= \Delta_n u + \varepsilon\phi(u) \qquad && \text{in} \quad D \times (0,T) \\ u &= 0 && \text{on} \quad \partial D \times (0,T) \\ u(x,0) &= u_0(x) && \text{in} \quad D\,. \end{aligned}$$

Their proof relies on consequences of the maximum principle for parabolic problems, which are available only in much weaker forms for hyperbolic problems. Hyperbolic problems in which the driving term

$\varepsilon\phi(u)$ appears in a boundary condition instead of in the differential equation have also been studied ([17]), but the question of global existence of solutions in space dimensions higher than one also remains unanswered. For a comprehensive survey of the literature on quenching see Levine [16].

In this chapter, problem (An) is shown to have a unique local continuous solution u in low dimensions for small ε under appropriate assumptions on ϕ, u_0, v_0, and ∂D, which can be continued as long as $u < M$. It is also shown that for any $\varepsilon > 0$, there exists no potential well about any equilibrium solution of (An), so that a proof of global existence along the lines of Sattinger [32] is not possible. Nevertheless, an a priori inequality for solutions of (An) similar to (4.1) is shown, via energy considerations, to guarantee global existence. Numerical evidence is given which suggests that such an a priori inequality is sometimes satisfied by solutions of (An).

Theoretical Considerations

Local continuous solutions of (An) for $n = 2,3$ are obtained by applying the abstract theory of Reed [30] to an appropriately modified problem.

In this section ϕ' will be assumed to be bounded and uniformly Lipschitz continuous on intervals of the form $(-\infty, M-d]$, $\delta > 0$. For $0 < \delta < M$ define

$$\phi_\delta(u) = \begin{cases} \phi(u), & u < M - \delta \\ \phi\left(M - \frac{\delta}{2}\right), & u > M - \frac{\delta}{2} . \end{cases}$$

Then by suitably defining ϕ_δ on the interval $\left(M - \delta, M - \frac{\delta}{2}\right)$, one may arrange that $\phi_\delta \in C^1(\mathbb{R})$ and ϕ' is bounded and uniformly Lipschitz continuous on $\mathbb{R}$. Let (An,δ) represent problem (An) with ϕ replaced by ϕ_δ.

It is assumed that problem (An) has a stationary solution $f \in C^2(\bar{D})$, which is analytic in D and satisfies

$$\begin{aligned} \Delta_n f + \varepsilon\phi(f) &= 0 && \text{in} \quad D . \\ f &= 0 && \text{on} \quad \partial D . \end{aligned}$$

Such is indeed the case when, e.g., D is a ball in $\mathbb{R}^n$ and $\phi(u) = (M+\alpha u)^\beta$, $\alpha,\beta < 0$; see Joseph and Lundgren [11].

Applying the transformation $\bar{u} = u - f$ to problem (An,δ), one obtains the problem

$$\begin{aligned} \bar{u}_{tt} &= \Delta_n \bar{u} + \varepsilon \psi_\delta(x,\bar{u}) && \text{in} \quad D \times (0,T) \\ \bar{u} &= 0 && \text{on} \quad \partial D \times (0,T) \\ \bar{u}(x,0) &= u_0(x) - f(x), \quad \bar{u}_t(x,0) = v_0(x) && \text{in} \quad D, \end{aligned} \tag{4.2}$$

where $\psi_\delta(x,u) \equiv \phi_\delta\big(u + f(x)\big) - \phi_\delta\big(f(x)\big)$. By setting

$$\bar{v} = \bar{u}_t, \quad \eta = \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}, \quad \eta_0 = \begin{pmatrix} u_0 - f \\ v_0 \end{pmatrix},$$

$$F(\eta) = \begin{pmatrix} 0 \\ \varepsilon\psi_\delta(x,\bar{u}) \end{pmatrix}, \quad \text{and} \quad A = -\begin{pmatrix} 0 & I \\ \Delta_n & 0 \end{pmatrix},$$

one may write (4.2) as the equivalent system

$$\begin{aligned} &\eta'(t) = -A\eta(t) + F\big(\eta(t)\big), \quad 0 < t < T \\ &\eta(0) = \eta_0 \, . \end{aligned} \tag{4.3}$$

Let H denote the Hilbert space of real-valued functions $H_0^1(D) \oplus L_2(D)$, with inner product

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} w \\ z \end{pmatrix} \right\rangle = \int_D \nabla u \cdot \nabla w \, dx + \int_D vz \, dx \, . \tag{4.4}$$

Provided ∂D is of class C^2, A is a closed skew-adjoint operator on H with domain $\mathrm{Dom}(A) \equiv [H^2(D) \cap H_0^1(D)] \oplus H^1(D)$, and generates on H the continuous one-parameter group $W(t) = e^{-tA}$. Therefore (4.3) can be reformulated as the integral equation problem

$$\eta(t) = e^{-tA}\eta_0 + \int_0^t e^{-(t-s)A} F\big(\eta(s)\big) ds, \tag{4.5}$$

which may then be solved by the contraction mapping principle.

The following theorem summarizes results proved in [30].

Theorem 4.1: Let ∂D be of class C^2, and for a fixed $m \geq 1$ let η_0 be in $\mathrm{Dom}(A^m)$. Let $\|\cdot\|$ denote the norm on H induced by (4.4). Suppose that for all $1 \leq j \leq m$,

$$\|A^j F(\eta)\| \leq C(\|\eta\|, \ldots, \|A^{j-1}\eta\|)\|A^j\eta\|, \tag{4.6}$$

and

$$\|A^j(F(\eta) - F(\nu))\| \leq C(\|\eta\|,\|\nu\|,\ldots,\|A^j\eta\|,\|A^j\nu\|)\|A^j(\eta-\nu)\| \tag{4.7}$$

for all η, ν in $\mathrm{Dom}(A^j)$, where the constants C are nondecreasing, everywhere finite functions of all their variables. Then there is a $T > 0$ such that (4.3) has a unique continuously differentiable solution $\eta(t)$ for $0 \leq t < T$, with $\phi(t)$ in $\mathrm{Dom}(A^m)$ for all $0 \leq t < T$.

If in addition $\|\eta(t)\|$ is bounded on any finite interval of existence of $\eta(t)$, then $\eta(t)$ exists globally in time.

Let $\|\cdot\|_p$ denote the norm in $L_p(D)$, and let K_1, K_1, ... denote positive constants. If $\eta \in \mathrm{Dom}(A)$ has first component u, then

$$\begin{aligned}\|AF(\eta)\|^2 &= \varepsilon^2\int_D |\nabla\psi_\delta(x,u)|^2 dx \\ &\leq K_1\left(\|\phi_\delta'(u+f)|\nabla u|\|_2^2 + \||\phi_\delta'(u+f) - \phi_\delta'(f)||\nabla f|\|_2^2\right] \\ &\leq K_2\left[\|\nabla u\|_2^2 + \|u\|_2^2\right] \leq K_3\|\nabla u\|_2^2 \leq K_4\|A\eta\|^2 ,\end{aligned}$$

where use was made of the boundedness and uniform Lipschitz continuity of ϕ_δ' on $\mathbb{R}$, the boundedness of $|\nabla f|$ on $\bar{D}$, and the Poincaré inequalities

$$\|u\|_2 \leq K\|\nabla u\|_2 \leq K^2\|\Delta_n u\|_2 , \tag{4.8}$$

valid for $u \in H^2(D) \cap H_0^1(D)$. This establishes (4.6) with $j = 1$.

If $\eta,\nu \in \mathrm{Dom}(A)$ have respective first components u, w, then by applying Hölder's inequality, the Sobolev inequality

$$\|u\|_p \leq C_p\|\nabla u\|_2 , \quad 1 \leq p \leq \frac{2n}{n-2} \tag{4.9}$$

valid for $u \in H^1(D)$, and (4.8), one obtains

$$\begin{aligned}
&\|A(F(\eta) - F(\nu))\| \\
&\quad \leq K_1\left[\|u-w\|_2^2 + \|\nabla(u-w)\|_2^2 + \|\nabla u\|_4^2\|u-w\|_4^2\right] \\
&\quad \leq K_2\|\Delta_n(u-w)\|_2^2 + K_3\|\Delta_n u\|_2^2 \;\|\nabla(u-w)\|_2^2 \\
&\quad \leq K_4(1 + \|A\eta\|^2)\|A(\eta-\nu)\|^2 .
\end{aligned}$$

This establishes (4.7) with $j = 1$ for $1 \leq n \leq 4$.

From the integral equation (4.5), from the fact that $|\psi_\delta(x,u)| \leq C|u|$ for some constant $C > 0$ for all $x \in \bar{D}$ and $u \in \mathbb{R}$, and from (4.8), one may obtain

$$\|\eta(t)\| \leq \|e^{-tA}\eta_0\| + \|\int_0^t e^{-(t-s)A}F(\eta(s))ds\|$$

$$\leq \|\eta_0\| + \int_0^t \|F(\eta(s))\|ds$$

$$\leq K_1[\|\eta_0\| + \int_0^t \|\eta(s)\|ds] \; ;$$

hence by quadrature

$$\|\eta(t)\| \leq K_1\|\eta_0\|e^{K_1 t}$$

for all t in the existence interval for $\eta(t)$.

The above arguments and Theorem 4.1 together yield the following

Corollary 4.1: Let $D \subseteq \mathbb{R}^n$ with $1 \leq n \leq 4$ have a C^2 boundary, and suppose that $\eta_0 \in \text{Dom}(A) = [H^2(D) \cap H_0^1(D)] \oplus H^1(D)$. Then for all $0 < \delta < M$ and for all sufficiently small $\varepsilon > 0$, problem (4.3) has a unique continuously differentiable solution $\eta(t)$ which is global in time and remains in $\text{Dom}(A)$ for all $t \geq 0$.

Four remarks are in order. Note that the proof of Corollary 4.1 does not require ϕ to be convex. Theorem 4.1 cannot be used to obtain greater regularity of solutions of (4.3) due to the lack of a Poincaré inequality of the form

$$\|\nabla^j u\|_2 \leq C\|\nabla^{j+1}u\|_2, \quad u \in H^{j+1}(D) \cap H_0^1(D)$$

for $j \geq 2$. The first component $\bar{u}$ of the solution $\eta = \begin{pmatrix} \bar{u} \\ \bar{u}_t \end{pmatrix}$ in

Corollary 4.1 with $n = 2$ or 3 is continuous on $\bar{D} \times [0,\infty)$ in view of the imbedding inequality

$$\max_{x \in \bar{D}} |\bar{u}(x,t)| < C\|\Delta_n \bar{u}(\cdot,t)\|_2$$

valid for $\bar{u}(\cdot,t) \in H^2(D) \cap H_0^1(D)$, $1 < n < 3$, and the continuity in time of $\bar{u}$ in the norm on $H^2(D) \cap H_0^1(D)$. Since $H^2(D)$ imbeds in $C(\bar{D})$ only for $n = 1,2,3$ ([6, p. 30]), any extension of Corollary 4.1 to cases $n > 5$ would not be useful for the purposes of this paper.

Now suppose that the solution η in Corollary 4.1 with $n = 2$ or 3 begins with $\max_{x \in \bar{D}} [u_0(x)] < M - \delta$. If $\max_{x \in \bar{D}} [\bar{u}(x,t) + f(x)] < M - \delta$ for all $t > 0$, then $u = \bar{u} + f$ is a global solution of problem (An). Otherwise there is a first time $T_0 > 0$ at which $\max_{x \in \bar{D}} [\bar{u}(x,T_0) + f(x)] = M - \delta$; by choosing $0 < \delta_1 < \delta$ and applying Corollary 4.1 to problem (4.3) with δ replaced by δ_1, one may extend $\bar{u} + f$ uniquely to an interval $[0,T_1)$ with $T_1 > T_0$ such that $\max_{x \in \bar{D}} [\bar{u}(x,t) + f(x)] < M - \delta_1$ for $0 < t < T_1$, and $u = \bar{u} + f$ solves (An) on $\bar{D} \times [0,T_1)$. This argument establishes the following

<u>**Corollary 4.2**</u>: Let $D \subseteq \mathbb{R}^n$ with $n = 2$ or 3 have a C^2 boundary, and suppose that $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \mathrm{Dom}(A)$ with $\max_{x \in \bar{D}} u_0(x) < M$. Then for all sufficiently small $\varepsilon > 0$, the system form of problem (An)

$$\begin{pmatrix} u \\ u_t \end{pmatrix}' = -A\begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon\phi(u) \end{pmatrix}, \quad 0 < t < T$$

$$\begin{pmatrix} u \\ u_t \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

has a unique solution $\begin{pmatrix} u \\ u_t \end{pmatrix} \in \mathrm{Dom}(A)$ on some time interval $[0,T)$ which is continuously differentiable in time in the norm on H. The solution can be continued as long as $\max_{x \in \bar{D}} u(x,t) < M$.

Define $\Phi(u) = \int_0^u \phi(s)\, ds$. A solution u of (An) with the regularity properties given in Corollary 4.2 satisfies the energy equality

$$E(t) \equiv \frac{1}{2}\int_D |u_t|^2\, dx + J(u) = E(0), \tag{4.10}$$

where

$$J(u) = \frac{1}{2}\int_D |\nabla u|^2\, dx - \varepsilon \int_D \Phi(u)dx$$

represents the potential energy of u at time t.

By defining $j(\lambda) = J(f + \lambda u)$ for $\lambda > 0$, one may examine the behavior of J along rays emanating from the stationary solution f of problem (An) in the function space $H_0^1(D)$. Note

$$j'(0) = \int_D \nabla f \cdot \nabla u \, dx - \varepsilon \int_D \phi(f)\, u \, dx$$

$$= \int_D \nabla f \cdot \nabla u \, dx + \int_D (\Delta f) u \, dx = 0.$$

Let $f_\infty = \max_{x \in \bar{D}} f(x)$; then by (4.8),

$$j''(0) = \int_D |\nabla u|^2 \, dx - \varepsilon \int_D \phi'(f)\, u^2 \, dx$$

$$\geq \int_D |\nabla u|^2 \, dx - \varepsilon \phi'(f_\infty) \int_D u^2 \, dx$$

$$\geq \left(1 - \varepsilon K^2 \phi'(f_\infty)\right) \int_D |\nabla u|^2 \, dx .$$

Suppose f satisfies $f_\infty \to 0$ as $\varepsilon \to 0^+$. (Equilibrium solutions with this property do exist; see [11]). Then for all sufficiently small ε one has $1 - \varepsilon K^2 \phi'(f_\infty) > 0$, and hence $j''(0) > 0$; i.e., J is convex along rays emanating from f. This is a necessary condition that f be a local minimum of J in $H_0^1(D)$.

Nevertheless, as the following results show, there exists no potential well in the function space $H_0^1(D)$ about any stationary solution of (An) for any $\varepsilon > 0$ and any $n \geq 2$.

<u>Lemma 4.1</u>: Let $\varepsilon > 0$ be fixed, and let f be an equilibrium solution of (An), $n \geq 2$. Choose $x_0 \in D$ such that $f(x_0) = f_\infty$. Then for any $\delta > 0$ one may find a ball $B_M \subseteq D$ with center x_0, and functions $\{w_\lambda : f_\infty < \lambda \leq M\} \subseteq C_0^\infty(D)$, such that $w_\lambda = \lambda - f$ on B_M and $\int_D |\nabla w_\lambda|^2 \, dx < \delta$ for all $f_\infty < \lambda \leq M$.

Proof: Let $B(x_0,a)$, $B(x_0,b)$ denote concentric open balls in $\mathbb{R}^n$ with center x_0 and respective radii $0 < a < b$. By Friedman ([6, p. 9]) there exists a $\zeta \in C^\infty(\mathbb{R}^n)$ such that $\zeta = 1$ in $B(x_0,a)$, $0 \leq \zeta \leq 1$ in $B(x_0,b) - B(x_0,a)$, and $\zeta = 0$ outside $B(x_0,b)$. The function ζ will be called a C^∞ cutoff from $B(x_0,a)$ to $B(x_0,b)$. Note that for $0 < \rho < 1$, $\zeta_\rho(x) \equiv \zeta(\frac{x}{\rho})$ is a C^∞ cutoff from $B(x_0,a\rho)$ to $B(x_0,b\rho)$ which satisfies $|\nabla\zeta_\rho(x)| \leq \frac{K_0}{\rho}$ for all $x \in \mathbb{R}^n$, where $K_0 > 0$ depends on a, b, but is independent of ρ.

The proof for $n \geq 3$ proceeds as follows. Choose $0 < a < b$, and for each $f_\infty \leq \lambda \leq M$ and each $\rho > 0$ define

$$w_{\lambda,\rho} = \zeta_\rho(\lambda - f) .$$

Then for all ρ sufficiently small $w_{\lambda,\rho} \in C_0^\infty(D)$, and $w_{\lambda,\rho} = \lambda - f$ on $B(x_0,a\rho)$. Now f is nonnegative on $\bar{D}$ by the maximum principle, so for $f_\infty \leq \lambda \leq M$

$$\int_D |\nabla w_{\lambda,\rho}|^2\, dx = 2 \int_{B(x_0,b\rho)} [|\nabla\zeta_\rho|^2|\lambda-f|^2 + |\zeta_\rho|^2|\nabla f|^2]dx$$

$$\leq 2V_n(K_0^2M^2 + \rho^2 \max_{\bar{D}}|\nabla f|^2)\rho^{n-2} ,$$

where V_n denotes the volume of the unit ball in $\mathbb{R}^n$. By taking $\rho = \rho_0 > 0$ sufficiently small, one may arrange that

$$\int_D |\nabla w_{\lambda,\rho_0}|^2\, dx < \delta \quad \text{for all } f_\infty \leq \lambda \leq M .$$

One may then take B_M to be $B(x_0,a\rho_0)$, w_λ to be w_{λ,ρ_0} for $f_\infty < \lambda < M$.

When $n = 2$, choose fixed $b > a > 0$ so small that $B(x_0,b) \subseteq D$. Let $r = |x - x_0|$ denote the distance from x_0 to x in $\mathbb{R}^2$. For $f_\infty < \lambda < M$ choose $B \equiv \lambda - f_\infty > A > 0$, and choose $0 < 2\rho < a$. For $\alpha < 0$ let

$$C = \frac{B - A}{\rho^\alpha - b^\alpha}, \qquad D = \frac{A\rho^\alpha - Bb^\alpha}{\rho^\alpha - b^\alpha},$$

so that $C\rho^\alpha + D = B$ and $Cb^\alpha + D = A$. Let ζ denote a C^∞ cutoff from $B(x_0,1)$ to $B(x_0,2)$, and define $\zeta_\rho(r) = \zeta(\frac{r}{\rho})$. Let $\zeta_{a,b}$ be a C^∞ cutoff from $B(x_0,a)$ to $B(x_0,b)$. For $f_\infty < \lambda < M$ define

$$w_{\lambda,\rho}(x) = \zeta_{a,b}(r)[\zeta_\rho(r)(\lambda - f(x) - Cr^\alpha - D) + Cr^\alpha + D] .$$

Then setting

$$I_1 = \int_{B(x_0,2\rho)} |\nabla f|^2 \, dx ,$$

$$I_2 = \int_{B(x_0,2\rho)-B(x_0,\rho)} |\nabla\zeta_\rho|^2 |\lambda-f-Cr^\alpha-D|^2 \, dx$$

$$I_3 = \int_\rho^b |\nabla(Cr^\alpha)|^2 \, r \, dr$$

$$I_4 = \int_a^b |\nabla\zeta_{a,b}|^2 |Cr^\alpha+D|^2 \, r \, dr ,$$

one may obtain that

$$\int_D |\nabla w_{\lambda,\rho}|^2 \, dx \le K_1 \sum_{j=1}^{4} I_j$$

for an absolute constant $K_1 > 0$.

Since $|\nabla f|$ is bounded on $B(x_0,b)$, one may choose a $\rho_0 > 0$ independent of A, α, and λ such that $I_4 < \delta/(4K_1)$ for all $0 < \rho \le \rho_0$.

Using the facts that $Cr^\alpha + D$ is positive and decreasing for $0 < r \le b$, and that $|\nabla\zeta_\rho| \le \frac{K_0}{\rho}$ for some absolute constant $K_0 > 0$, one may show that

$$I_2 \le 6\pi K_0^2 [C^2(1-2^\alpha)^2 \rho^{2\alpha} + \max_{x \in B(x_0,2\rho)} |f(x) - f_\infty|^2] \, ,$$

$$I_3 = \frac{\alpha\, C^2}{2} \left(b^{2\alpha} - \rho^{2\alpha}\right) \, ,$$

$$I_4 \le K_2 [A^2 + C^2(a^\alpha - b^\alpha)^2] \, ,$$

for some positive absolute constant K_2. It is a simple matter to show that the expressions $C^2(1 - 2^\alpha)^2 \rho^{2\alpha}$, $\frac{\alpha C^2}{2}(b^{2\alpha} - \rho^{2\alpha})$, $C^2(a^\alpha - b^\alpha)^2$ can be made arbitrarily small independent of λ, $f_\infty < \lambda \le M$, by taking $|\alpha| > 0$, $\rho > 0$ sufficiently small. Hence by choosing $A = A_0$, $\rho = \rho_1 < \rho_0$, $\alpha = \alpha_0$ all sufficiently close to zero, one may arrange that $I_j < \frac{\delta}{4K_1}$ for $1 \le j \le 4$ for all $f_\infty < \lambda \le M$. Set $B_M = B(x_0,\rho_1)$, and $w_\lambda = w_{\lambda,\rho_1}$ with $A = A_0$, $\alpha = \alpha_0$. Then $\int_D |\nabla w_\lambda|^2 \, dx < \delta$ for all $f_\infty < \lambda \le M$. ∎

In particular, for any $\varepsilon > 0$, $n > 2$, there are functions $v_M = w_M + f$ with essential supremum equal to M in any neighborhood of f in $H_0^1(D)$.

Lemma 4.2: Let $y(t)$ denote the solution of the ordinary initial-value problem

$$\begin{aligned} &\ddot{y} = \varepsilon\phi(y) \qquad t > 0 \\ &y(0) = y_0, \quad \dot{y}(0) = 0, \end{aligned} \tag{4.11}$$

where $y_0 < M$. Then there is a finite $T_q > 0$ such that $\lim_{t \to T_q^-} y(t) = M$; i.e., y quenches in finite time. As $y_0 \to M^-$, $T_q \to 0^+$.

Proof: The uniform Lipschitz continuity of ϕ on intervals of the form $(-\infty, M-\delta]$, $\delta > 0$, guarantees that (4.11) has a unique local C^2 solution $y(t)$ which can be continued as long as $y(t) < M$. On the existence interval $[0,T_q)$ for y, $\ddot{y} > 0$ and hence $\dot{y}(t)$ is strictly increasing in t. Since $\dot{y}(0) = 0$, $\dot{y}(t) > 0$ and hence $y(t) > y_0$ for $0 < t < T_q$. Since ϕ is strictly increasing,

$$\ddot{y}(t) > \varepsilon\phi(y_0) ,$$

so that

$$y(t) - y_0 > \varepsilon\phi(y_0)t^2 ,$$

for $0 < t < T_q$. Hence

$$\sqrt{\frac{M - y_0}{\varepsilon\phi(y_0)}} > T_q .$$

Clearly $T_q \to 0^+$ as $y_0 \to M^-$. ■

Theorem 4.2: Let ε, δ, T_0 be any fixed positive numbers, and let k be any nonnegative integer. Let $f \in C^k(\bar{D})$ be an equilibrium solution of (An) with $n > 2$. Then there exists $u_0 \in C^k(\bar{D})$ with $u_0 = 0$ on ∂D, $\max_{\bar{D}} u_0 < M$, and $\int_D |\nabla(u_0 - f)|^2 \, dx < \delta$, such that the solution u of problem (An) with $v_0 = 0$ quenches in finite time $T < T_0$.

Proof: Let $w_\lambda \in C_0^\infty(\bar{D})$ be the functions satisfying $w_\lambda = \lambda - f$ on B_M, $\int_D |\nabla w_\lambda|^2 \, dx < \delta$ for all $f_\infty < \lambda < M$, whose existence is guaranteed by Lemma 4.1. Let $\rho > 0$ denote the radius of B_M. By Lemma 4.2, one may choose $y_0 < M$ so close to M that the solution $y = y(t,y_0)$ of (4.11) quenches in time $T_q < \min\{T_0,\rho\}$.

Define $u_0 = w_{y_0} + f$; then $u_0 \in C^k(\bar{D})$, $u_0 = 0$ on ∂D, $u_0 = y_0$ on B_M, and $\int_D |\nabla(u_0 - f)|^2 \, dx < \delta$. If u denotes the solution of problem (An) with this u_0 and with $v_0 = 0$, one has

$$u(x,t) = y(t,y_0)$$

for all (x,t) in the retrograde characteristic cone with vertex (x_0,ρ) and base $B_M \times \{0\}$. Hence u must quench in time $T < T_q$. ■

The idea of comparing solutions inside retrograde characteristic cones in the half-space $t > 0$ was first used by Keller [13] to show pointwise blow-up in finite time of solutions of $u_{tt} = c^2 \Delta_n u + f(u)$ for certain $f \in C^2(\mathbb{R})$.

Suppose u is a continuous solution of (An) with sufficient regularity to satisfy energy equality (4.10) (or the inequality $E(t) \leq E(0)$) for all t in its existence interval. Define $u_\infty^+ = u_\infty^+(t) \equiv \max_{x \in \bar{D}} \left(u(x,t),0\right)$, $\gamma = \gamma(t) \equiv \left(u_\infty^+\right)^{-2} \int_D |\nabla u|^2 \, dx$. Then

$$\gamma \leq 2(u_\infty^+)^{-2}[E(0) + \varepsilon\mu(D)\Phi(u_\infty)] \equiv g(u_\infty^+) , \qquad (4.12)$$

where $\mu(D)$ denotes the n-dimensional Lebesgue measure of D.

When $E(0) > 0$, $g(s)$ achieves a positive absolute minimum $g_m = g(s_0)$ on the interval $(0,M]$. Note that $g(\frac{M}{2})$, and hence g_m itself, can be made arbitrarily small by taking both $\varepsilon > 0$ and $E(0) > 0$ sufficiently small.

If u satisfies an a priori inequality of the form

$$\gamma(t) \geq g_m , \qquad t \geq 0 \qquad (4.13)$$

and begins in the region R depicted in Figure 1 (i.e. with $u_{0,\infty}^+ < s_0$), then u_∞^+ remains bounded away from M for all time, and u will be a global solution of (An).

When $n = 1$ one has $\gamma(t) \geq 4$ for all $t \geq 0$ by (4.1), and these observations underlie the proof of the result (b) of the previous section in [4].

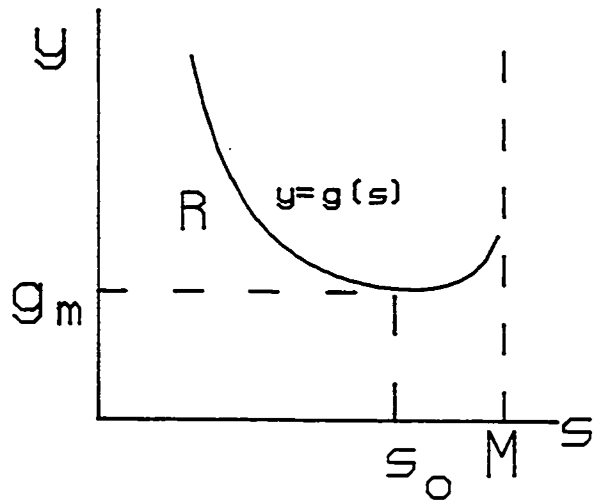

Figure 1. An invariant region for solutions of (An) satisfying (4.13).

Numerical Results

An explicit finite-difference scheme was used to approximate solutions $u = u(r,t)$ of

$$u_{tt} = u_{rr} + \frac{n-1}{r} u_r + \varepsilon\phi(u), \qquad 0 < r < 1, \quad 0 < t < T \quad (4.14)$$

$$\text{(Rn)} \qquad u_r(0,t) = u(1,t) = 0, \qquad 0 < t < T \quad (4.15)$$

$$u(t,0) = u_0(r), \quad u_t(r,0) = v_0(r), \quad 0 < r < 1, \quad (4.16)$$

which are radial solutions of (An) when $r = |x|$ and D is the unit ball centered at the origin in R^n. (Note that to ensure compatibility, the initial data must satisfy $u_{0,r}(0) = v_{0,r}(0) = u_0(1) = v_0(1) = 0$.)

The difference scheme used is adapted from John [10, pp. 172-174]. Divide the interval $[0,1]$ into N subintervals of equal length $h = \frac{1}{N}$, and let k denote the stepsize in time, with

$$\lambda \equiv \frac{k}{h} < 1. \tag{4.17}$$

For $1 < i < N+1$, $j > 0$, $w = w(r,t)$ define

$$w_{ij} = w\big((i-1)h,\ jk\big).$$

Let δ_t denote the forward divided difference operator

$$\delta_t w_{ij} = \frac{1}{k}\left[w_{i,j+1} - \frac{1}{2}\left(w_{i+1,j} + w_{i-1,j}\right)\right],$$

with space averaging in the lower step; and let δ_r denote the central divided difference operator

$$\delta_r\, w_{ij} = \frac{1}{2h}\left[w_{i+1,j} - w_{i-1,j}\right].$$

For $2 < i < N-1$ and $j > 0$, (4.14) was replaced by the difference equation

$$\delta_t^2\, w_{ij} = \delta_r^2\, w_{ij} + \frac{n-1}{(i-1)h}\,\delta_r\, w_{ij} + \varepsilon\phi(w_{ij})\ . \tag{4.18}$$

(Values $w_{0,j}$, $w_{-1,j}$, ... are interpreted by extending w as an even function of r through $r = 0$). When $i = 1$, (4.14) was replaced by

$$\delta_t^2 w_{1j} = n\,\delta_r^2 w_{1j} + \varepsilon\phi(w_{1j}) \;. \tag{4.19}$$

When $i = N$, backward differences in r must be used, and space averaging abandoned, wherever necessary to avoid going past $r = 1$; consequently (4.14) was replaced by

$$\begin{aligned}\frac{1}{k}\,\delta_t\left(w_{N,j+1} - w_{Nj}\right) &= \frac{1}{h}\,\delta_r\left(w_{Nj} - w_{N-1,j}\right) \\ &\quad + \frac{n-1}{(i-1)h}\,\delta_r\, w_{Nj} + \varepsilon\phi(w_{Nj}) \;.\end{aligned} \tag{4.20}$$

Of course, when $i = N + 1$ the boundary condition $u(1,t) = 0$ of (4.15) translates to

$$w_{N+1,j+2} = 0. \tag{4.21}$$

Since each of (4.18-21) can be solved for $w_{i,j+2}$ in terms of w_{i_0,j_0} with $j_0 < j+2$, the scheme is explicit. The Taylor series approximation

$$\begin{aligned}w_{i1} &= (u_0)_i + k(v_0)_i \\ &\quad + \frac{1}{2}\,k^2[\delta_r^2(u_0)_i + \frac{n-1}{(i-1)h}\,\delta_r(u_0)_i + \varepsilon\phi(u_{0i})] + \ldots\end{aligned}$$

was used to obtain the values w_{i1} (with the obvious modification when $i = 1$).

The difference scheme used is stable, consistent, and convergent when applied to the pure initial-value problem obtained by linearizing (4.14) about a stationary solution f. Grave difficulties are encountered, however, in attempts to prove consistency and converence for (4.18-21), due to the boundary conditions and the presence of the nonlinearity ϕ. In particular, one is unable to derive useful upper bounds for higher difference quotients of w. This is analogous to the difficulties encountered with the abstract approach to the differential problem in the previous section. Therefore for the numerical tests the following checks and safeguards were implemented:

(a) the Courant-Friedrichs-Lewy condition (4.17), which is a necessary condition for stability, was ensured to be satisfied by taking $\lambda = \frac{1}{4}$ in all tests;

(b) stationary solutions f of (Rn) were approximated by a procedure described below. The difference scheme (4.18-21) was then applied with $u_0 = f$, $v_0 = 0$ as a check of the computer code. Since the approximate stationary solutions are not exact, these checks (as well as checks with $v_0 = 0$, u_0 = small perturbation of f) served as empirical evidence of the scheme's stability;

(c) the convergence of the scheme (4.18-21) was checked empirically for several examples by letting $h,k \to 0$ while keeping $\lambda = \frac{1}{4}$; and

(d) the energy equality (4.10) was checked for the difference scheme at selected time steps using Simpson's rule to approximate the integrals involved.

Double-precision arithmetic was used for all computations. The experiments were performed on a National Advanced Systems AS/9160 computer with MVS/SP operating system.

To isolate the effects of the term $\varepsilon\phi(u)$, solutions of (4.18-21) with $u_0 = v_0 = 0$ were computed in dimensions $n = 2,3,$ and 7. The behavior of such solutions agrees qualitatively with behavior reported in [4] for solutions in the case $n = 1$. In particular, in each dimension n considered there appears to be an $\varepsilon_n > 0$ such that solutions quench in finite time when $\varepsilon > \varepsilon_n$, and do not quench (even in infinite time) when $\varepsilon < \varepsilon_n$. For $\varepsilon < \varepsilon_n$ the solution displays a sequence of relative maxima which occur along the line $r = 0$, $t > 0$; the first such relative maximum appears to be an absolute maximum, which approaches 1 from below as ε approaches ε_n from below.

The following table lists values of ε_n obtained when $\phi(u) = (1-u)^{-1}$; the value of ε_1 is taken from [4]. Figures 2 and 3 contrast the behavior of a solution w of (4.18-21) for values of ε respectively greater than or less than ε_n. Both figures were generated using $\phi(u) = (1-u)^{-1}$, $n=2$, and $h = 1/200$. In Figure 2, $\varepsilon = 1.5 > \varepsilon_2$, and the solution quences in time $T = 1.01$; while in Figure 3, $\varepsilon = 0.9 < \varepsilon_2$, and the solution is displayed for $0 < t < 8$.

n	ε_n	ε_*
1	0.341	0.383
2	1.017	1.309
3	1.520	2.139
7	2.563	6.000

Table. Values of ε_n, ε_* for $\phi(u) = (1-u)^{-1}$.

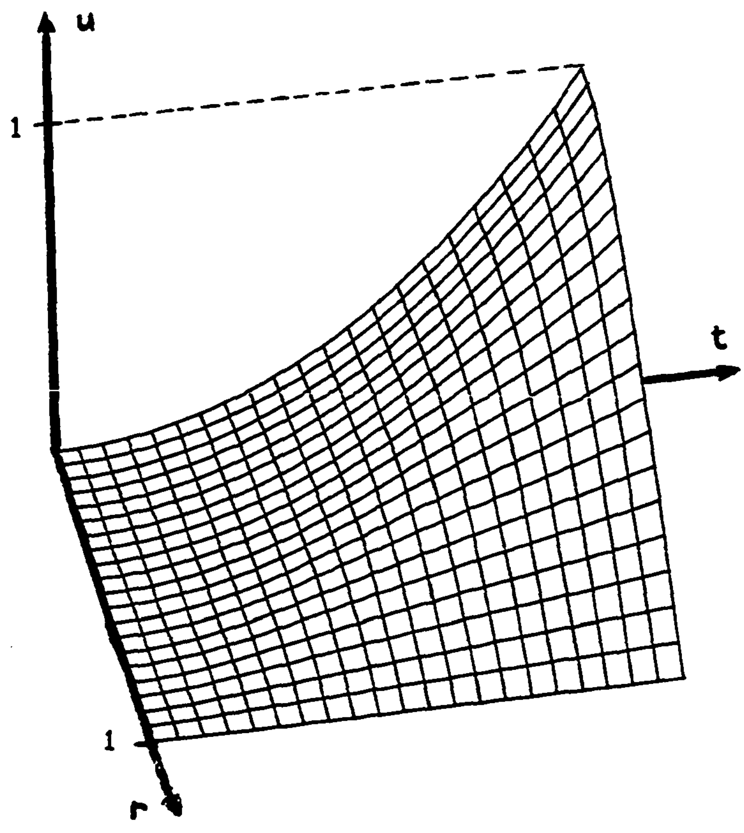

Figure 2. A solution of (4.18-21) with $\varepsilon > \varepsilon_n$.

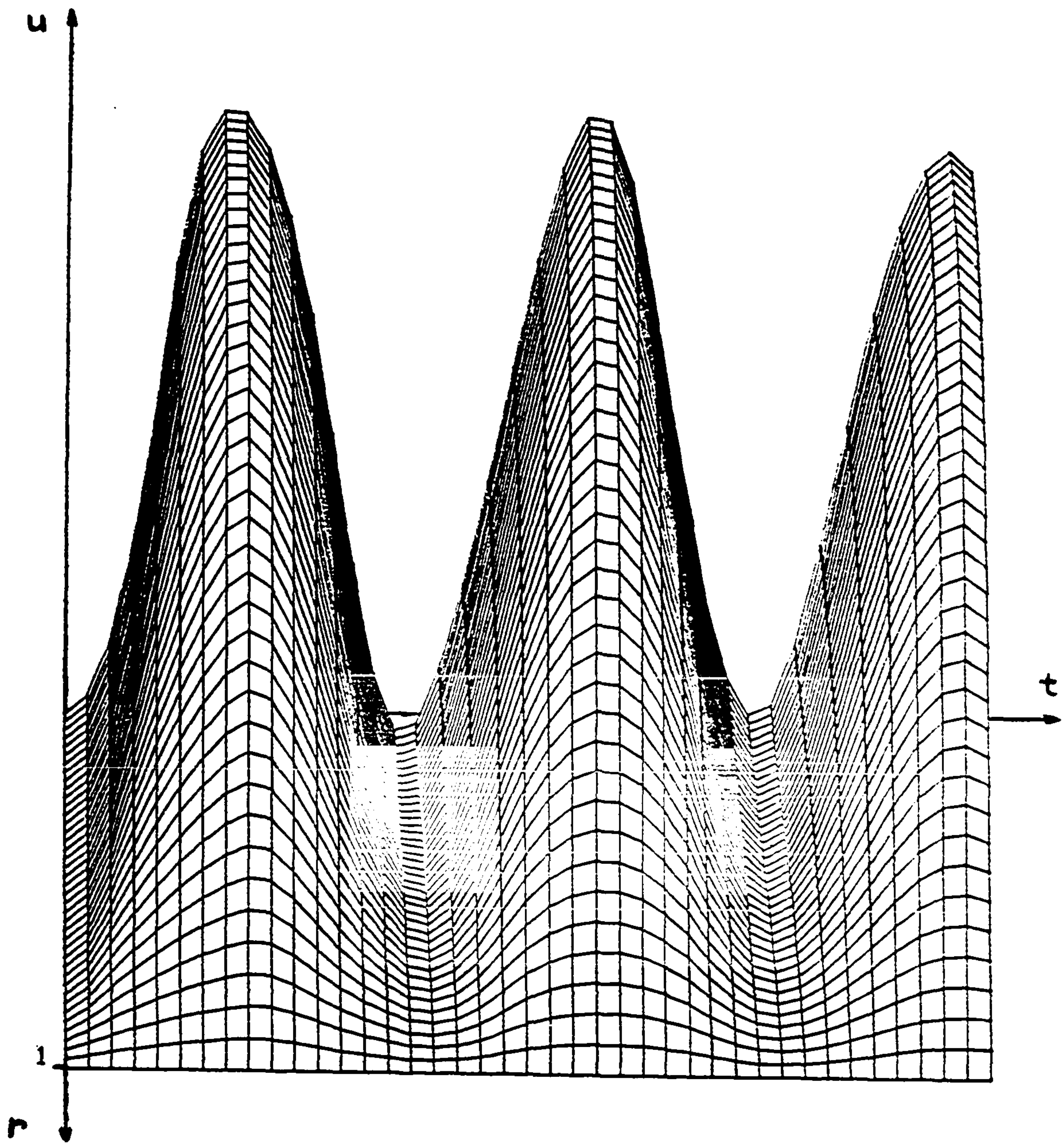

Figure 3. A solution of (4.18-21) with $\varepsilon < \varepsilon_n$.

The stability of the solution f of the stationary problem

$$f''(r) + \frac{n-1}{r} f'(r) + \varepsilon\phi\big(f(r)\big) = 0, \quad 0 < r < 1 \tag{4.22}$$

$$f'(0) = f(1) = 0 \tag{4.23}$$

satisfying $f_\infty \to 0^+$ as $\varepsilon \to 0^+$, was also investigated. The solutions f were obtained via the shooting method; i.e., the boundary conditions (4.23) are replaced with "initial" conditions

$$f'(0) = 0, \qquad f(0) = A, \tag{4.24}$$

and A is chosen so that the solution $f(r,A)$ of (4.22), (4.24) (obtained via the classical fourth-order Runge-Kutta method) satisfies $f(1,A) = 0$. Since A is a root of the nonlinear equation $f(1,A) = 0$, it may be found using an iterative procedure such as Newton's method. The secant method was used for the numerical experiments since it does not require the extra calculation of $\frac{\partial f}{\partial A}$.

Care must be taken in the choice of $A = A_0$ to begin the shooting procedure, since solutions of (4.22),(4.24) are not always unique. Indeed, Joseph and Lundgren showed in [11] that for $\phi(u) = (1+\alpha u)^\beta$ with $\alpha,\beta < 0$, $\tau = \frac{2}{\beta-1}$, $\bar{\varepsilon} = \frac{\tau}{\alpha}(n-2-\tau)$, and $f(\beta) = 2\beta\tau + 2(2\beta\tau)^{1/2}$,

(a) there is an $\varepsilon_* > 0$ such that positive solutions of (4.22), (4.23) do not exist when $\varepsilon > \varepsilon_*$;

(b) for $3 < n < 2 + f(\beta)$ and $\varepsilon_* > \bar{\varepsilon}$, there are a large but finite number of positive solutions when $\varepsilon < \bar{\varepsilon}$ is close to $\bar{\varepsilon}$, and a countably infinite number of solutions when $\varepsilon = \bar{\varepsilon}$; and

(c) for $n > 2 + f(\beta)$ and $\bar{\varepsilon} = \varepsilon_*$, there is exactly one positive solution when $\varepsilon < \varepsilon_*$.

See the previous table for values of ε_* when $\phi(u) = (1-u)^{-1}$ and $n = 1,2,3,7$.

Bifurcation diagrams plotting ε as a function of f_∞ were generated using the procedure described in [11]. It could then be checked that the shooting method above converged to the stationary solution with smallest maximum, f_∞. Figure 4 contains bifurcation diagrams for space dimensions $n = 2,3,9$ when $\alpha = -1$, $\beta = -3$.

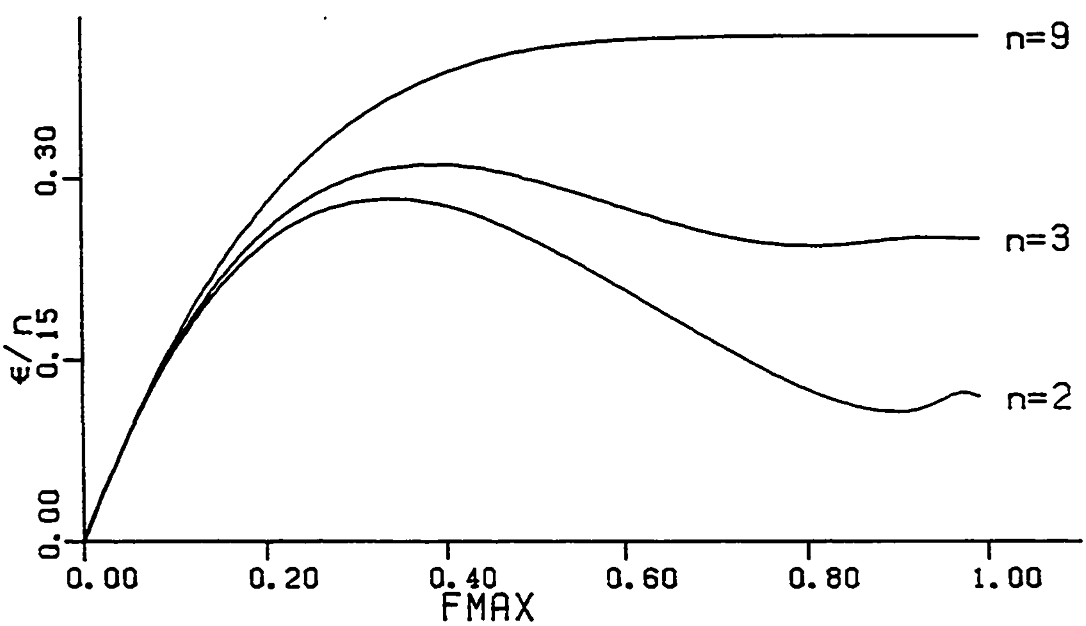

Figure 4. Bifurcation diagram for positive solutions of (4.22), (4.23) when $\phi(u) = (1-u)^{-3}$.

Perturbations p of f used as initial data u_0 were of the form $p = \nu f$ for $0 < \nu < \frac{1}{f_\infty}$; or of the form

$$p(r) = \begin{cases} \nu & 0 \le r \le r_0, \\ f(r) & r_1 \le r \le 1 \end{cases}$$

where $0 < r_0 < r_1 < 1$, $f_\infty < \nu < 1$, and p is defined on $r_0 \le r \le r_1$ to be strictly decreasing and C^2 on $[0,1]$. Numerical experiments with initial data $u_0 = p$, v_0 chosen so that $E(0) > 0$, indicate that whenever p is sufficiently close to f in sup norm and $E(0)$ is small, the a priori inequality (4.13) is satisfied for all time by solutions of (4.18-21). The following diagrams, Figures 5 and 6, contrast the trajectories of $\gamma(t)$ in quenching and nonquenching cases. Each figure was generated with $n = 3$, $\phi(u) = (1-u)^{-1}$, $u_0 = \nu f$, $v_0(r) = \mu(1-r^2)$ where μ is chosen so that $E(0) > 0$, $\varepsilon = 0.5$, $h = 1/100$.

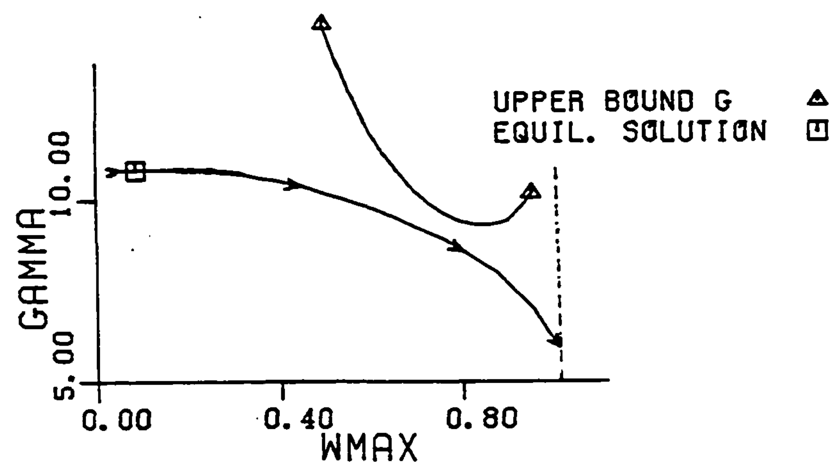

Figure 5. Quenching trajectory for γ when $\nu = 0.5$, $E(0) = 2.0$.

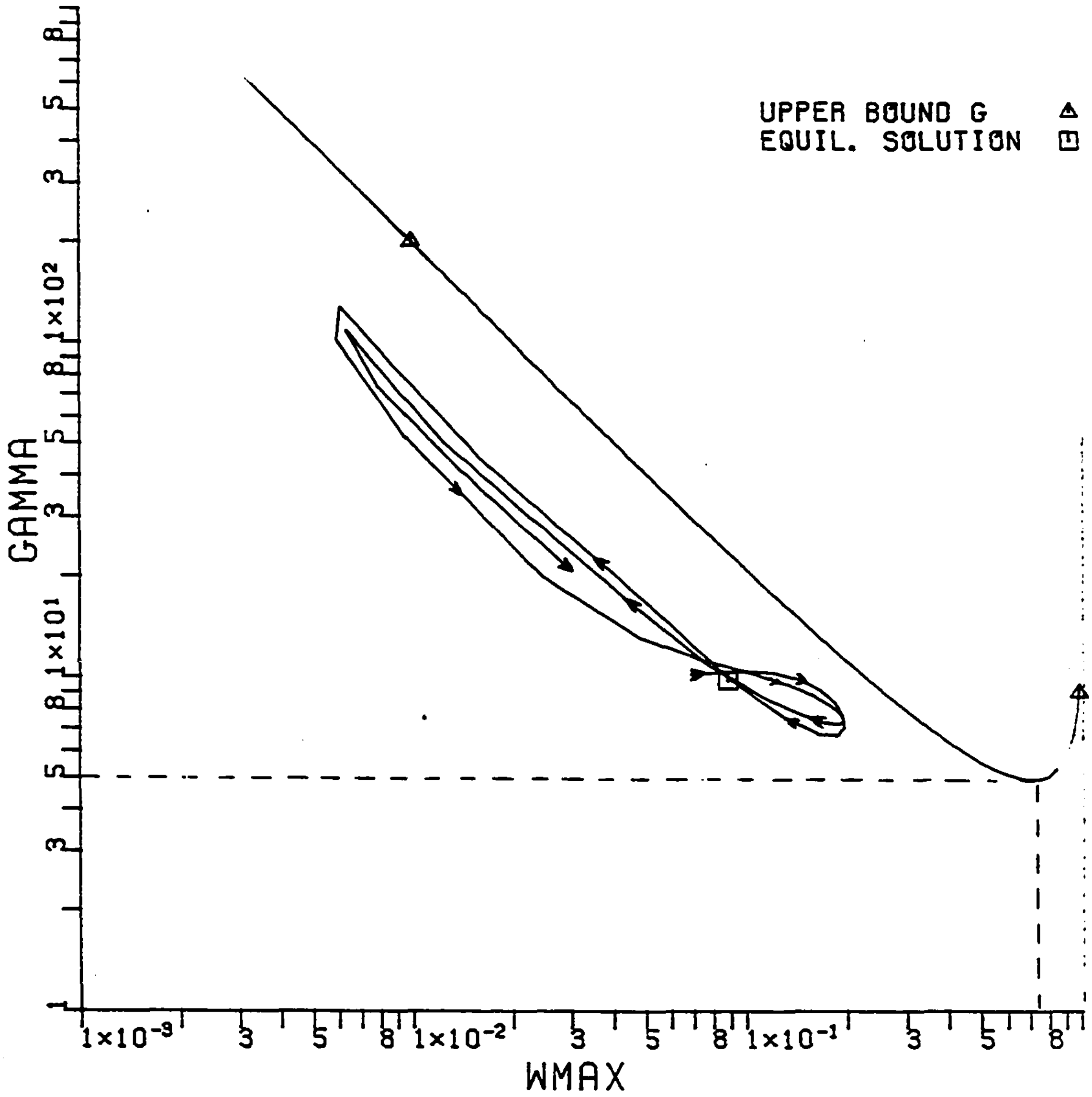

Figure 6. Nonquenching trajectory for γ when $\nu = 0.5$, $E(0) = 0$.

CHAPTER 5. DISCUSSION

One can more generally require that the initial velocity V of problem (W) belong to $L_2(D)$, at the expense of losing uniqueness of the Fourier expansion of V in terms of the eigenfunctions ϕ, ψ.

The global solutions obtained in [32] and in Chapters 2 and 3 are not proved to be unique. In addition, it remains to be shown whether global solutions have greater regularity under more restrictive assumptions on initial data and geometry; and whether global solutions depend continuously, in some sense, on initial data.

Ball [3] was the first to note that concavity arguments like those of Chapters 2 and 3 do not prove that nonexistence actually occurs by "blow-up" of some norm of the solution. It is an open question whether nonexistence of global solutions of problems (W), (H), and of problems considered in [28], is indeed caused by such blow-up.

The definition of a weak solution of problem (H) given in (1)-(4) of Chapter 3 is nonstandard. One might ask whether the global solution obtained in Chapter 3 satisfies the conventional definition of weak solution (see, e.g., [14, pp. 418-419]).

Potential well arguments similar to those of Chapters 2 and 3 are also applicable to problems of the form

$$
\begin{aligned}
u_{tt} &= \Delta u + g(u) && \text{in } D_T \\
u &= 0 && \text{on } \sigma_T \\
\frac{\partial u}{\partial n} &= f(u) && \text{on } \Sigma_T \\
u(x,0) = U(x), &\quad u_t(x,0) = V(x) && \text{in } D,
\end{aligned}
$$

where g satisfies (i)-(iii) of Chapter 2 with $1 < p'+1 < \gamma' < \frac{2n}{n-2}$. Here four cases need be considered, depending on whether each of f,g satisfies (ii)(a) or (ii)(b). When f satisfies (ii)(a), and g (ii)(b), one must require that $f(-s) > -f(s)$ for all $s < 0$, in order to ensure that the depth of the potential well is determined by nonzero functions $u \in H^1_{0\sigma}(D)$ with $u > 0$ a.e. (This condition on f holds when, for example, f is symmetric through the origin). Similarly, when f satisfies (ii)(b), and g (ii)(a), one must require $g(-s) > -g(s)$ for all $s < 0$. When f,g both satisfy (ii)(a), or both satisfy (ii)(b), no additional restrictions on f or g are necessary. The positive depth of the potential well is established by showing that

$$
\alpha \equiv \max\{\|u\|_{\gamma',D}, \ \|\underset{\sim}{B}u\|_{\gamma,\Sigma}\}
$$

is uniformly bounded below for all appropriately restricted $u \in H^1_{0\sigma}(D)$ satisfying

$$
Q_0(u) \equiv \|u\|_D^2 - \int_D u\, g(u)\, dx - \int_\Sigma (\underset{\sim}{B}u)\, f(\underset{\sim}{B}u)\, dS = 0.
$$

The extraction of a suitable convergent subsequence in the proof of global existence requires the compact imbedding of $H^1_{0\sigma}(D)$ into $L_q(D)$ for $2 \le q < \frac{2n}{n-2}$.

In [28] Payne and Sattinger considered the stability properties of the positive solution (ground state) of $\Delta u + g(u) = 0$ with Dirichlet boundary conditions. In a pending work Sternberg [36] demonstrates the instability of higher modes of $u_{tt} = \Delta u + g(u)$. It appears that problems (H),(W) do not possess higher modes, at least in one dimension. The question of existence and stability properties of higher modes of problems (H),(W) in several dimensions is being studied.

Sacks [31] derived decay estimates and described the asymptotic behavior of solutions of the problem

$$\begin{aligned} u_t &= \Delta(|u|^{m-1} u) + \lambda |u|^{p-1} u && \text{in } D_T \\ u &= 0 && \text{on } (\partial D)_T \\ u(x,0) &= u_0(x) && \text{in } D . \end{aligned}$$

A similar analysis may be possible when the nonlinearity appears in a boundary condition:

$$\begin{aligned} u_t &= \Delta(|u|^{m-1} u) && \text{in } D_T \\ u &= 0 && \text{on } \sigma_T \\ \frac{\partial u}{\partial n} &= \lambda |u|^{p-1} u && \text{on } \Sigma_T \\ u(x,0) &= u_0(x) && \text{in } D , \end{aligned}$$

and when nonlinearities appear in both equation of motion and boundary condition.

The phenomenon of barrier penetration, i.e., of a quantum-mechanical particle's "tunneling" to a region which is classically inaccessible, is well known (see, e.g., [29]). One might consider an initial-boundary value problem for Schrödinger's equation with a nonlinear boundary condition of the form

$$
\text{(S)}\qquad
\begin{aligned}
\frac{1}{i}\frac{\partial u}{\partial t} &= c^2 \Delta u && \text{in } D_T \\
u &= 0 && \text{on } \sigma_T \\
\frac{\partial u}{\partial n} &= f(u) && \text{on } \Sigma_T \\
u(x,0) = U(x), \quad u_t(x,0) &= V(x) && \text{in } D\ ,
\end{aligned}
$$

where the complex-valued function u of space and time represents the wave function of a particle. It may be possible for a solution of (S) which begins in a potential well of finite positive depth d, and with total initial energy less than d, to tunnel out of the well in finite time.

Much more research needs to be done on the difficult problem (An) of Chapter 4, particularly on the existence of local continuous solutions in dimensions $n > 4$, and on the obtainment of greater regularity of solutions for $n > 2$. Due to the limited success of the abstract theory, progress on these fronts must rely heavily on the particular form of the differential equation and side conditions. Future numerical research might be directed toward determining whether there are initial data u_0 in each neighborhood in $H_0^1(D)$ of each equilibrium solution for which global solutions of (An) appear to exist.

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APPENDIX. FORTRAN CODE FOR THE NUMERICAL EXPERIMENTS

```
C  PROGRAM TO SOLVE
C           DTTU = LAPLACIAN(U) + EPSL*PHI(U)
C              U = 0,  R = |X| = 1, T > 0
C         U(R,0) = U0(R),  0 < R < 1           (*)
C       DTU(R,0) = V0(R),  0 < R < 1
C  VIA FINITE DIFFERENCES, WHERE
C       U IS A RADIAL FUNCTION OF  DIM > 1  SPACE VARIABLES
C         X1,...,XDIM AND TIME T.  DIM NEED NOT BE AN INTEGER;
C       EPSL IS A POSITIVE GEOMETRIC FACTOR EQUAL TO THE SQUARE
C         OF THE RADIUS OF THE ORIGINAL DOMAIN OF SOLUTION;
C       PHI(S) IS CONTINUOUSLY DIFFERENTIABLE, STRICTLY
C         INCREASING, CONVEX, AND APPROACHES INFINITY AS S
C         APPROACHES  EM > 0  FROM THE LEFT.
C  INITIAL DATA  U0,V0  ARE DETERMINED BY SUBROUTINE INIDAT.
C
C  LATEST VERSION CODED BY R.A. SMITH, IOWA STATE UNIVERSITY,
C    NOVEMBER 25, 1985.
C
C  ALL COMPUTATIONS USE DOUBLE-PRECISION ARITHMETIC.
C
      IMPLICIT REAL*8 (A-H,O-Z)
C
C  W CONTAINS THE FINITE DIFFERENCE APPROXIMATION OF U.  W(.,1),
C    W(.,2),W(.,3) CONTAIN VALUES OF W AT THREE SUCCESSIVE
C    TIME STEPS.
C  R(.,1) CONTAINS VALUES OF THE RADIAL VARIABLE R.  R(.,2) CONTAINS
C    VALUES OF  R**(DIM-1), WHICH ARE USED REPEATEDLY IN ENERGY
C    COMPUTATIONS.
C
      DIMENSION  W(1000,3), R(1000,2)
      COMMON  EM, EPSL, DIM, DIMM1, H, N, N1, NMIN1, NRSKIP, R,
     1        OMEGA, FAC1, FAC2, FAC3
C
C  FUNCTION DEFINITIONS OF PHI,DPHI,CAPPHI FOLLOW.  DPHI IS THE
C    DERIVATIVE OF PHI AND IS USED IN THE TAYLOR SERIES
C    APPROXIMATION BELOW.  CAPPHI(S) IS THE INTEGRAL FROM 0 TO S
C    OF PHI AND IS USED IN ENERGY COMPUTATIONS.
C
      PHI(S) = 1.D0/(1.D0 - S)
      DPHI(S) = 1.D0/(1.D0 - S)**2
      CAPPHI(S) = -DLOG(1.D0 - S)
C
      EM = 1.D0
C
C  NOTE: CHANGING PHI REQUIRES CHANGING THE FUNCTION DEFINITIONS
C    OF DPHI,CAPPHI, AND THE VALUE OF EM (THE BLOW-UP POINT OF
C    PHI).  MUST ALSO CHANGE LITERAL OUTPUT DEFINING PHI IN
```

```
C    FORMAT STATEMENT 20.
C
C  READ IN PARAMETERS FOR (*) AND THE DIFFERENCE SCHEME.
C
C       XLAMDA = (DELTA T)/(DELTA R). MUST BE POSITIVE, AND LESS
C         THAN 1 TO SATISFY THE COURANT-FRIEDRICHS-LEWY CONDITION.
C       N = NUMBER OF SUBINTERVALS INTO WHICH THE INTERVAL (0,1) FOR
C         THE RADIAL VARIABLE R IS DIVIDED.  MUST BE EVEN,
C         GREATER THAN 3, AND LESS THAN 1000.
C       KEND = NUMBER OF TIME STEPS AT WHICH W IS COMPUTED.  TIME
C         INTERVAL OF SOLUTION IS  (0, KEND*XLAMDA/N).  MUST BE
C         POSITIVE.
C       NRSKIP: AT A GIVEN TIME STEP, VALUES OF W ARE OUTPUT EVERY
C         NRSKIP/N UNITS IN THE R VARIABLE.  MUST BE POSITIVE AND
C         CANNOT EXCEED N.
C       NTSKIP: CHOSEN VALUES OF W ARE OUTPUT EVERY  NTSKIP*XLAMDA/N
C         UNITS IN THE T VARIABLE.  MUST BE POSITIVE AND CANNOT
C         EXCEED KEND.
C
      READ(5,10)  EPSL, DIM, XLAMDA, N, KEND, NRSKIP, NTSKIP
 10   FORMAT(3(D23.16/),3(I4/),I4)
      WRITE(6,20)  EPSL, DIM
 20   FORMAT(' SOLUTION OF  DTTU = LAPLACIAN(U) + EPSL*',
     1  'PHI(U)',15X,'R.A.SMITH'/,' PHI(S) = 1/(1-S)'//,
     2  ' EPSILON = SQUARE OF RADIUS = ',D13.6/,
     3  ' DIM = NUMBER OF SPACE VARIABLES = ',D13.6)
      WRITE(6,30)  XLAMDA, N, KEND, NRSKIP, NTSKIP
 30   FORMAT(' XLAMDA = (DELTA T)/(DELTA R) = ',D13.6/,
     1  ' N = NUMBER OF R PARTITION POINTS = ',I4/,
     2  ' KEND = NUMBER OF TIME STEPS = ',I4/,
     3  ' NRSKIP = ',I4/,' NTSKIP = ',I4/)
C
C  CHECKS THAT INPUT PARAMETERS ARE ACCEPTABLE.
C
      IF (EPSL .LE. 0.D0)  GO TO 270
      IF (DIM .LE. 1.D0)  GO TO 270
      IF ( (XLAMDA .LE. 0.D0) .OR. (XLAMDA .GT. 1.D0) )  GO TO 270
      IF ( (N .LE. 3) .OR. (N .GE. 1000) )  GO TO 270
      IF (KEND .LE. 0)  GO TO 270
      IF ( (NRSKIP .LT. 1) .OR. (NRSKIP .GT. N) )  GO TO 270
      IF ( (NTSKIP .LT. 1) .OR. (NTSKIP .GT. KEND) )  GO TO 270
C
C  ASSIGN PRIMARY VARIABLES USED BY DIFFERENCE SCHEME.
C
C        H = DELTA R = STEPSIZE IN R VARIABLE.
C       AK = DELTA T = STEPSIZE IN T VARIABLE.
C
      DIMM1 = DIM - 1.D0
      XN = N
      H = 1.D0/XN
```

```
      AK = XLAMDA*H
      AKSQ = AK*AK
      N1 = N+1
C
C  ASSIGN VARIABLES USED IN ENERGY COMPUTATIONS.
C
C       OMEGA = SURFACE AREA OF UNIT SPHERE IN DIM DIMENSIONS.
C       FAC1,FAC2,FAC3 ARE FACTORS FOR ENERGY EXPRESSIONS.
C
      PI = 4.D0*DATAN(1.D0)
      OMEGA = 2.D0*PI**(DIM/2.D0)/DGAMMA(DIM/2.D0)
C
      FAC1 = OMEGA/(24.D0*H)
      FAC2 = OMEGA*H/(6.D0*AKSQ)
      FAC3 = EPSL*OMEGA*H/3.D0
      NMIN1 = N-1
C
C  ASSIGN VALUES OF R IN R(.,1), AND VALUES OF  R**DIMM1 IN R(.,2).
C
      R(1,1) = 0.D0
      R(1,2) = 0.D0
      DO  40  I = 2,N
        TEMP = I-1
        R(I,1) = TEMP*H
        R(I,2) = R(I,1)**DIMM1
 40   CONTINUE
      R(N1,1) = 1.D0
      R(N1,2) = 1.D0
C
C  COMPUTE INITIAL POSITION DATA U0 AND STORE IN W(.,1);
C    MAXIMUM VALUE U0MAX OF U0 AND VALUE R0MAX OF R AT WHICH
C    IT OCCURS; INITIAL VELOCITY DATA V0 AND TEMPORARILY STORE
C    IN W(.,3); INITIAL GAMMA=GAM0; AND INITIAL ENERGIES.  SEE
C    SUBROUTINE INIDAT FOR DETAILS.
C
      CALL INIDAT(W,U0MAX,R0MAX,GAM0,IERR)
      IF (IERR .NE. 0)  STOP
C
C  ASSIGN VALUES OF  W(I,2) = W(R,DELTA T) USING THE PDE AND THE
C    TAYLOR EXPANSION OF U IN DELTA T ABOUT T=0.
C  ALSO CHECK TO SEE IF W(.,2) EXCEEDS EM (I.E., IF QUENCHING
C    OCCURS).
C
      FC1 = 0.5D0*AKSQ
      FC2 = 1.D0/(H*H)
      FC3 = DIMM1/(2.D0*H)
      FC4 = AKSQ*AK/6.D0
C
      W(1,2) = W(1,1) + AK*W(1,3) +
     1         FC1*(2.D0*DIM*FC2*(W(2,1)-W(1,1))+EPSL*PHI(W(1,1))) +
```

```
     2          FC4*( 2.D0*DIM*FC2*( W(2,3)-W(1,3) ) +
     3          EPSL*DPHI(W(1,1))*W(1,3) )
      IF (W(1,2) .LT. EM)  GO TO 45
      I = 1
      TLAST = AK
      GO TO 260
C
 45   DO  50  I = 2,N
        W(I,2) = W(I,1) + AK*W(I,3) +
     1            FC1*( FC2*( W(I+1,1)-2.D0*W(I,1)+W(I-1,1) ) +
     2            FC3*( W(I+1,1)-W(I-1,1) )/R(I,1) +
     3            EPSL*PHI(W(I,1)) )
        W(I,2) = W(I,2) +
     1            FC4*( FC2*( W(I+1,3)-2.D0*W(I,3)+W(I-1,3) ) +
     2            FC3*( W(I+1,3)-W(I-1,3) )/R(I,1) +
     3            EPSL*DPHI(W(I,1))*W(I,3) )
        IF (W(I,2) .LT. EM)  GO TO 50
        TLAST = AK
        GO TO 260
 50   CONTINUE
C
      W(N1,2) = 0.D0
C
C  ASSIGN SECONDARY VARIABLES USED IN THE DIFFERENCE SCHEME.
C
      AKSQEP = AKSQ*EPSL
      XLAMSQ = XLAMDA*XLAMDA
      ALL1 = 0.25D0*(XLAMSQ - 1.D0)
      ALL2 = 0.5D0*(XLAMSQ + 1.D0)
      FACTO1 = 2.D0*XLAMSQ*DIMM1
      FACTO2 = AKSQ*DIMM1/(2.D0*H)
      FACTO3 = 0.25D0*(3.D0*XLAMSQ + 1.D0)
C
C  COMPUTE SOLUTION, ENERGIES AT LATER TIME STEPS.
C
C  THE FINITE DIFFERENCE SCHEME USES CENTRAL R DIFFERENCES, AND
C    BACKWARD TIME DIFFERENCES WITH SPACE AVERAGING IN THE LOWER
C    STEP.  SEE FRITZ JOHN, PARTIAL DIFFERENTIAL EQUATIONS
C    (REFERENCE 10).  THIS SCHEME IS STABLE.
C
C  FOR EACH FIXED T, U IS EXTENDED AS AN EVEN FUNCTION OF R THROUGH
C    R = 0.  FOR REGULARITY IT IS ASSUMED THAT DRU(0,T) = 0; HENCE
C    VALUES OF W(0,T) ARE OBTAINED BY REPLACING THE DRU(R,T)/R
C    EXPRESSION IN A TERM OF LAPLACIAN(U) BY A DIFFERENCE
C    APPROXIMATION TO DRRU(0,T).  VALUES OF W NEAR OR AT R = 1
C    ARE OBTAINED USING THE BOUNDARY CONDITIONS U(1,T) =
C    DTU(1,T) = 0, AND BACKWARD R DIFFERENCES WHERE NECESSARY TO
C    AVOID GOING PAST R = 1.
C
C  WMAX = GLOBAL MAXIMUM VALUE OF W ACHIEVED.
```

```
C  RMAX = VALUE OF R AT WHICH WMAX OCCURS.
C  TMAX = VALUE OF T AT WHICH WMAX OCCURS.
C  GAMOLD, GAMNEW ARE USED TO COMPUTE THE TIME DERIVATIVE OF GAMMA.
C
C  GAMMA(T) IS TWICE THE LINEAR POTENTIAL ENERGY OF W(T),
C    DIVIDED BY WMAX(T)**2.
C
      WMAX = UOMAX
      RMAX = ROMAX
      TMAX = 0.D0
      GAMOLD = GAM0
      TSKIP = NTSKIP
C
      DO  230  J = 1,KEND
C
C  W(.,3) CONTAINS THE VALUES OF W COMPUTED AT EACH NEW TIME STEP.
C    W(1,3),W(2,3),W(N,3),W(N1,3) MUST BE COMPUTED INDIVIDUALLY.
C    EACH W(I,3) IS CHECKED TO SEE IF IT EXCEEDS EM (I.E., TO SEE
C    IF QUENCHING HAS OCCURRED).  THE LOCAL MAXIMUM VALUE WMAXJ OF
C    W AT EACH TIME STEP IS DETERMINED, ALONG WITH THE VALUE RMAXJ
C    OF R AT WHICH WMAXJ OCCURS.
C
        W(1,3) = 2.D0*( W(2,2)+ALL1*W(3,1) ) - ALL2*W(1,1) +
     1           FACTO1*( W(2,1)-W(1,1) ) + AKSQEP*PHI(W(1,1))
        IF (W(1,3) .LT. EM)  GO TO 60
        I = 1
        GO TO 130
 60     WMAXJ = W(1,3)
        RMAXJ = 0.D0
C
        W(2,3) = W(3,2) + W(1,2) + ALL1*( W(4,1)+W(2,1) ) -
     1           ALL2*W(2,1) + FACTO2*( W(3,1)-W(1,1) )/H +
     2           AKSQEP*PHI(W(2,1))
        IF (W(2,3) .LT. EM)  GO TO 70
        I = 2
        GO TO 130
 70     IF (W(2,3) .LT. WMAXJ)  GO TO 80
        WMAXJ = W(2,3)
        RMAXJ = R(2,1)
C
 80     DO  90  I = 3,NMIN1
          W(I,3) = W(I+1,2) + W(I-1,2) + ALL1*( W(I+2,1)+W(I-2,1) ) -
     1             ALL2*W(I,1) + FACTO2*( W(I+1,1)-W(I-1,1))/R(I,1) +
     2             AKSQEP*PHI(W(I,1))
          IF (W(I,3) .GE. EM)  GO TO 130
          IF (W(I,3) .LE. WMAXJ)  GO TO 90
          WMAXJ = W(I,3)
          RMAXJ = R(I,1)
 90     CONTINUE
C
```

```
      W(N,3) = W(N-1,2) - FACTO3*W(N,1) + ALL1*W(N-2,1) -
     1         FACTO2*W(N-1,1)/R(N,1) + AKSQEP*PHI(W(N,1))
      IF (W(N,3) .LT. EM)  GO TO 100
      I = N
      GO TO 130
 100  IF (W(N,3) .LE. WMAXJ)  GO TO 110
      WMAXJ = W(N,1)
      RMAXJ = R(N,1)
C
 110  W(N1,3) = 0.D0
C
C  IF WMAXJ EXCEEDS THE GLOBAL MAXIMUM WMAX, REPLACE THE PREVIOUS
C    VALUE STORED IN WMAX WITH WMAXJ.
C
      IF (WMAXJ .LE. WMAX)  GO TO 120
      WMAX = WMAXJ
      RMAX = RMAXJ
      TMAX = J
C
C  IF QUENCHING HAS NOT OCCURRED (JFLAG=0), DETERMINE WHETHER W,
C    ENERGIES ARE TO BE OUTPUT AT THIS TIME STEP (I.E., WHETHER
C    J+1 IS AN INTEGRAL MULTIPLE OF NTSKIP).
C
 120  JFLAG = 0
      JLIM = J+1
      IF ((JLIM - (JLIM/NTSKIP)*NTSKIP) .NE. 0)  GO TO 210
      XJ = J
      T = (XJ + 1.D0)*AK
      GO TO 160
C
C  IF QUENCHING HAS OCCURRED (JFLAG=1), OUTPUT W,ENERGIES AT THE
C    TIME STEP JUST PRIOR TO QUENCHING.  MUST FIRST CHECK THAT THESE
C    VALUES HAVE NOT ALREADY BEEN OUTPUT.  MUST ALSO RESTORE W TO ITS
C    STATE BEFORE QUENCHING.
C
 130  JFLAG = 1
      IF ((J - (J/NTSKIP)*NTSKIP) .EQ. 0)  GO TO 250
      DO  140  K = 1,I
        W(I,3) = W(I,2)
 140  CONTINUE
      DO  150  K = 1,N1
        W(I,2) = W(I,1)
 150  CONTINUE
      XJ = J
      T = XJ*AK
C
 160  WRITE(6,165) T, (W(K,3), K=1,N1,NRSKIP)
 165  FORMAT(//' SOLUTION AND ENERGIES AT TIME LEVEL T = ',
     1  D13.6//,50(' ',7(D13.6,3X),/))
C
```

```
C  EVALUATE ENERGIES USING SIMPSON'S RULE.
C        S1 = LINEAR POTENTIAL ENERGY = 0.5D0*OMEGA*
C          (THE INTEGRAL FROM 0 TO 1 OF  (R**DIMM1)*(DRW**2)
C          WITH RESPECT TO R).
C        S2 = KINETIC ENERGY = 0.5D0*OMEGA*(THE INTEGRAL
C          FROM 0 TO 1 OF  (R**DIMM1)*(DTW**2)  WITH RESPECT
C          TO R).
C        S3 = MINUS NONLINEAR POTENTIAL ENERGY =
C          EPSL*OMEGA*(THE INTEGRAL FROM 0 TO 1 OF
C          (R**DIMM1)*CAPPHI(W)  WITH RESPECT TO R).
C  DERIVATIVES IN THE INTEGRANDS ARE APPROXIMATED BY BACKWARD
C    TIME DIFFERENCES AND CENTRAL R DIFFERENCES.
C
C  THE EXACT SOLUTION OF (*) SATISFIES THE ENERGY EQUALITY
C    S2 + S1 - S3 = E0, WHERE E0 IS THE TOTAL INITIAL ENERGY.
C
        S1 = 0.D0
        S2 = R(1,2)*( W(1,3)-W(2,2) )**2
        S3 = R(1,2)*CAPPHI(W(1,3))
        DO  170  K = 2,NMIN1,2
          TEMP1 = 4.D0*R(K,2)
          TEMP2 = 2.D0*R(K+1,2)
          S1 = S1 + TEMP1*( W(K+1,3)-W(K-1,3) )**2 +
     1         TEMP2*( W(K+2,3)-W(K,3) )**2
          S2 = S2 + TEMP1*( W(K,3)-0.5D0*( W(K+1,2)+W(K-1,2) ) )**2
     1         + TEMP2*( W(K+1,3)-0.5D0*( W(K+2,2)+W(K,2) ) )**2
          S3 = S3 + TEMP1*CAPPHI(W(K,3)) +
     1         TEMP2*CAPPHI(W(K+1,3))
 170    CONTINUE
        S1 = FAC1*(S1 + 4.D0*(R(N,2)*W(N-1,3)**2 + W(N,3)**2))
        S2 = FAC2*(S2 + 4.D0*R(N,2)*( W(N,3)-0.5D0*W(N-1,2) )**2)
        S3 = FAC3*(S3 + 4.D0*R(N,2)*CAPPHI(W(N,3)))
C
C  T3 = NONLINEAR POTENTIAL ENERGY
C  PE = TOTAL POTENTIAL ENERGY
C  TE = TOTAL ENERGY
C
        T3 = -S3
        PE = S1 - S3
        TE = PE + S2
        WRITE(6,180)  S1, T3, PE, S2, TE
 180    FORMAT(/' LPE=',D13.6,' ,  NPE=',D13.6,' ,  PE=',D13.6,
     1    ' ,  KE=',D13.6,5X,'E(T) = ',D15.8)
C
C  OUTPUT LOCAL MAXIMUM WMAXJ AND VALUE GAMNEW OF GAMMA.
C
        GAMNEW = 2.D0*S1/WMAXJ**2
        WRITE(6,190) WMAXJ, GAMNEW
 190    FORMAT(' MAXIMUM W(T) = ',D13.6/,' GAMMA(T) = ',D13.6)
C
```

```
      IF (JFLAG .EQ. 1)  GO TO 250
C
C  DTGAM = TIME DERIVATIVE OF GAMMA, COMPUTED USING A BACKWARD
C    DIFFERENCE BETWEEN TWO SUCCESSIVE OUTPUT VALUES OF GAMMA.
C
      DTGAM = (GAMNEW-GAMOLD)/(AK*TSKIP)
      WRITE(6,200) DTGAM
 200  FORMAT(' DTGAMMA(T) = ',D13.6)
      GAMOLD = GAMNEW
C
C  SHIFT VECTORS TO BEGIN COMPUTATIONS AT THE NEXT TIME STEP.
C
 210  DO  220  I = 1,N1
        W(I,1) = W(I,2)
        W(I,2) = W(I,3)
 220  CONTINUE
 230  CONTINUE
C
C  IF W NEVER QUENCHED, OUTPUT ITS GLOBAL MAXIMUM WMAX
C    AND THE R VALUE RMAX AND T VALUE TMAX AT WHICH IT OCCURRED.
C
      IF(TMAX .EQ. 0.D0)  GO TO 240
      TMAX = (TMAX + 1.D0)*AK
 240  WRITE(6,245) RMAX, TMAX, WMAX
 245  FORMAT(/' JOB COMPLETED WITH NO QUENCHING'/,
     1  ' MAXIMUM VALUE OF U ACHIEVED = U(',D13.6,' , ',
     2  D13.6,') = ',D13.6)
      STOP
C
C  IF W QUENCHED, OUTPUT WHERE AND WHEN.
C
 250  TLAST = (XJ + 2.D0)*AK
 260  WRITE(6,265) R(I,1),TLAST
 265  FORMAT(/' SOLUTION EQUALED OR EXCEEDED  A  AT  R = ',
     1  D13.6,4X,'AT TIME  TLAST = ',D13.6)
      STOP
C
C  IF AN INPUT VARIABLE EXCEEDS ACCEPTABLE PARAMETERS:
C
 270  WRITE(6,275)
 275  FORMAT(' **ERROR: INPUT VARIABLE IN MAIN PROGRAM EXCEEDS',
     1  ' ACCEPTABLE PARAMETERS'/,' COMPUTATION DISCONTINUED.')
      STOP
C
      END
C
C
C
C
      SUBROUTINE INIDAT(W,U0MAX,R0MAX,GAM0,IERR)
```

```
C
C  SUBROUTINE TO GENERATE INITIAL DATA FOR THE MAIN PROGRAM.
C    THIS VERSION:
C       A. DETERMINES U0 AS A PERTURBATION OF AN EQUILIBRIUM
C          SOLUTION F OF (*) AND STORES U0 IN W(.,1).  SEE
C          INIU0 FOR DETAILS.
C       B. COMPUTES THE MAXIMUM VALUE U0MAX OF U0, AND THE
C          VALUE R0MAX OF R AT WHICH U0MAX OCCURS.
C       C. DETERMINES V0(R)=XMU*(1-R**2), AND STORES V0
C          IN W(.,3).  XMU IS CHOSEN SO THAT THE TOTAL INITIAL
C          ENERGY OF THE SOLUTION OF (*) IS E0.
C          NOTE: E0 MUST EQUAL OR EXCEED THE TOTAL POTENTIAL
C          ENERGY OF U0.
C       D. COMPUTES THE INITIAL ENERGIES OF THE SOLUTION OF (*).
C       E. DETERMINES THE INITIAL VALUE GAM0 OF GAMMA.
C       F. DETERMINES WHETHER GAMMA BEGINS IN THE REGION R OF
C          FIGURE 1.  SEE DISSERTATION AND SUBROUTINE FINMIN
C          FOR DETAILS.
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION  W(1000,3), R(1000,2), F(1000,2)
      COMMON  EM, EPSL, DIM, DIMM1, H, N, N1, NMIN1, NRSKIP, R,
     1        OMEGA, FAC1, FAC2, FAC3
C
      CAPPHI(S) = -DLOG(1.D0 - S)
C
C  READ IN INITIAL TOTAL ENERGY E0, AND PARAMETER SEED FOR
C    SUBROUTINE EQSOLN. SEED IS CHOSEN USING THE BIFURCATION
C    DIAGRAM PLOTTING EPSL AS A FUNCTION OF FMAX.
C
      READ(5,400)  E0, SEED
 400  FORMAT(D23.16/D23.16)
      WRITE(6,405)  SEED
 405  FORMAT(/' EQUILIBRIUM SOLUTION SEED = ',D13.6/)
C
C  CHECK THAT SEED IS WITHIN ACCEPTABLE PARAMETERS.
C
      IF ( (SEED .LE. 0.D0) .OR. (SEED .GE. EM) )  GO TO 500
C
C  COMPUTE AN EQUILIBRIUM SOLUTION F OF (*) WITH FMAX CLOSEST
C    TO SEED.  F(.,1) CONTAINS VALUES OF F; F(.,2) CONTAINS
C    VALUES OF DRF.  SEE SUBROUTINE EQSOLN FOR DETAILS.
C
      CALL EQSOLN(F,SEED,NRITS,JERR)
      IF (JERR .NE. 0)  GO TO 510
C
C  OUTPUT F.
C
      WRITE(6,410) NRITS,(F(I,1), I=1,N1,NRSKIP)
 410  FORMAT(//' EQUILIBRIUM SOLUTION F  (',I3,
```

```
     1  ' ITERATIONS REQUIRED)'//,50(' ',7(D13.6,3X),/))
C
C  COMPUTE U0 AND STORE IN W(.,1).  SEE SUBROUTINE INIU0.
C
      CALL INIU0(W,F,U0MAX,R0MAX,JERR)
      IF (JERR .NE. 0)  GO TO 510
C
C  OUTPUT U0.
C
      WRITE(6,420) (W(I,1), I=1,N1,NRSKIP)
 420  FORMAT(//' INITIAL VALUES OF U'//,50(' ',7(D13.6,3X),/))
C
C  COMPUTE
C       S10 = INITIAL LINEAR POTENTIAL ENERGY;
C       S30 = INITIAL MINUS NONLINEAR POTENTIAL ENERGY.
C  SEE MAIN PROGRAM FOR DESCRIPTION OF HOW ENERGIES ARE COMPUTED.
C
      S10 = 0.D0
      S30 = R(1,2)*CAPPHI(W(1,1))
      DO  430  I = 2,NMIN1,2
        TEMP1 = 4.D0*R(I,2)
        TEMP2 = 2.D0*R(I+1,2)
        S10 = S10 + TEMP1*( W(I+1,1)-W(I-1,1) )**2 +
     1          TEMP2*( W(I+2,1)-W(I,1) )**2
        S30 = S30 + TEMP1*CAPPHI(W(I,1)) + TEMP2*CAPPHI(W(I+1,1))
 430  CONTINUE
      S10 = FAC1*(S10 + 4.D0*(R(N,2)*W(N-1,1)**2 + W(N,1)**2))
      S30 = FAC3*(S30 + 4.D0*R(N,2)*CAPPHI(W(N,1)))
C
C  COMPUTE V0(R) = XMU*(1-R**2), WHERE XMU IS CHOSEN SO THAT
C    THE TOTAL INITIAL ENERGY IS E0.  STORE V0 IN W(.,3).  S20=
C    INITIAL KINETIC ENERGY.
C
      S20 = E0 + S30 - S10
      IF (S20 .LT. 0.D0)  GO TO 520
      XMU = DSQRT(DIM*(DIM+2.D0)*(DIM+4.D0)*S20/(4.D0*OMEGA))
      DO  440  I = 1,N1
        W(I,3) = XMU*(1.D0-R(I,1)**2)
 440  CONTINUE
C
C  OUTPUT V0.
C
      WRITE(6,450) (W(I,3), I=1,N1,NRSKIP)
 450  FORMAT(//' INITIAL VALUES OF DTU'//,50(' ',7(D13.6,3X),/))
C
C  OUTPUT INITIAL ENERGIES.  SEE MAIN PROGRAM FOR EXPLANATION OF
C    T3,PE.
C
      T30 = -S30
      PE0 = S10 + T30
```

```
      WRITE(6,460)  S10, T30, PE0, S20, E0
 460  FORMAT(/' LPE0=',D13.6,' ,  NPE0=',D13.6,' ,  PE0=',D13.6,
     1  ' ,  KE0=',D13.6/,' INITIAL TOTAL ENERGY E0 = ',D13.6/)
C
C  COMPUTE INITIAL VALUE GAM0 OF GAMMA.
C
      GAM0 = 2.D0*S10/U0MAX**2
      WRITE(6,47C)  R0MAX, U0MAX, GAM0
 470  FORMAT(' MAXIMUM VALUE OF INITIAL U = U0MAX = ',
     1  'U0(',D13.6,') = ',D13.6/,' INITIAL GAMMA = GAM0 = ',D13.6/)
C
C  DETERMINE WHETHER INITIAL GAMMA LIES IN REGION R OF FIGURE 1.
C
      CALL FINMIN(E0,S0,GMIN,JERR)
      IF (JERR .NE. 0)  GO TO 510
      IERR = 0
      IF ( (U0MAX .GT. S0) .OR. (GAM0 .LT. GMIN) )  GO TO 490
      WRITE(6,480)
 480  FORMAT(/' INITIAL GAMMA LIES IN REGION R'/)
      RETURN
 490  WRITE(6,495)
 495  FORMAT(/' INITIAL GAMMA IS EXTERIOR TO REGION R'/)
      RETURN
C
C  ERROR-HANDLING SECTION.
C
C  IF SEED WAS OUTSIDE INTERVAL (0,EM):
C
 500  WRITE(6,505)
 505  FORMAT(' **ERROR: SEED EXCEEDS ACCEPTABLE PARAMETERS'/)
      IERR = 1
      RETURN
C
C  IF ONE OF SUBROUTINES EQSOLN,RUNKUT,INIU0,FINMIN FAILED
C     (ERROR MESSAGE WAS OUTPUT BY FAILED ROUTINE):
C
 510  IERR = 1
      RETURN
C
C  IF INITIAL KINETIC ENERGY WAS NEGATIVE:
C
 520  PE0 = S10 - S30
      WRITE(6,525)  PE0, E0, S20
 525  FORMAT(/' **ERROR: PE0 = ',D13.6,'  EXCEEDS E0 = ',D13.6/,
     1  ' YIELDING NEGATIVE INITIAL KINETIC ENERGY S20 = ',D13.6/)
      IERR = 1
      RETURN
      END
C
C
```

```
C
C
      SUBROUTINE EQSOLN(F,SEED,NRITS,JERR)
C
C  SUBROUTINE TO OBTAIN AN EQUILIBRIUM SOLUTION F OF THE MAIN
C    PROGRAM'S PROBLEM (*), VIA THE SHOOTING METHOD.
C
C  LET F(R,A) DENOTE THE SOLUTION OF
C       DRRF + (DIM-1)*DRF/R + EPSL*PHI(F) = 0
C                                      F(0) = A    (**)
C                                    DRF(0) = 0,
C    WHERE A IS A PARAMETER.  THE SECANT METHOD IS USED TO
C    FIND THE A FOR WHICH F(1,A)=0, SO THAT F(R,A) IS THE
C    DESIRED STATIONARY SOLUTION OF (*).
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION  F(1000,2), R(1000,2)
      COMMON  EM, EPSL, DIM, DIMM1, H, N, N1, NMIN1, NRSKIP, R,
     1        OMEGA, FAC1, FAC2, FAC3
C
C  A1,A2 ARE INITIAL ESTIMATES FOR A, AND ARE USED TO START
C    THE SECANT METHOD.
C
      A1 = SEED
      A2 = 0.95D0*SEED
      NRITS = 0
C
C  COMPUTE F(R,A1)
C
      CALL RUNKUT(F,A1,KERR)
      IF (KERR .NE. 0) GO TO 720
      F0 = F(N1,1)
C
C  LOOP TO CALCULATE A.  SECANT METHOD IS ITERATED UNTIL THE
C    RELATIVE ERROR BETWEEN SUCCESSIVE A1,A2 IS LESS THAN
C    5.D-12.
C
  700 NRITS = NRITS + 1
      CALL RUNKUT(F,A2,KERR)
      IF (KERR .EQ. 1)  GO TO 720
      F1 = F(N1,1)
      TEMP = F1 - F0
      IF (TEMP .EQ. 0.D0)  GO TO 730
      TEMP = F1*(A2-A1)/TEMP
      IF (DABS(TEMP) .GE. (5.D-12*DABS(A2)))  GO TO 710
      JERR = 0
      RETURN
  710 A1 = A2
      A2 = A1 - TEMP
      F0 = F1
```

```
      IF (NRITS .GT. 10)  GO TO 740
      GO TO 700
C
C  ERROR-HANDLING SECTION.
C
C  IF A CALL TO RUNKUT FAILS:
C
 720  WRITE(6,725) NRITS
 725  FORMAT(/' DURING CALL ',I3,' TO SUBROUTINE RUNKUT')
      JERR = 1
      RETURN
C
C  IF SECANT METHOD FAILS DUE TO DIVISION BY ZERO:
C
 730  WRITE(6,735) NRITS
 735  FORMAT(/' **ERROR: DIVISION BY ZERO IN THE CALCULATION',
     1  ' OF A2 AT STEP ',I2,' OF EQSOLN'/)
C
C  IF SECANT METHOD FAILS TO CONVERGE:
C
 740  WRITE(6,745) F1
 745  FORMAT(/' **ERROR: SECANT METHOD DOES NOT CONVERGE',
     1  ' AFTER 10 ITERATIONS IN SUBROUTINE EQSOLN'/,
     2  ' LAST COMPUTED  F(1,A2) = ',D13.6/)
      JERR = 1
      RETURN
      END
C
C
C
C
      SUBROUTINE RUNKUT(F,ALPHA,KERR)
C
C  SUBROUTINE TO SOLVE (**) IN EQSOLN VIA THE CLASSICAL FOURTH-
C    ORDER RUNGE-KUTTA METHOD.
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION  F(1000,2), R(1000,2), TK(4,2)
      COMMON  EM, EPSL, DIM, DIMM1, H, N, N1, NMIN1, NRSKIP, R,
     1        OMEGA, FAC1, FAC2, FAC3
C
      PHI(S) = 1 DO/(1.D0 - S)
C
      F(1,1) = ALPHA
      F(1,2) = 0.D0
      J = 1
      TK(1,1) = F(1,2)
      IF (F(1,1) .GE. EM) GO TO 780
      TK(1,2) = -EPSL*PHI(F(1,1))/DIM
      COEF = 0.5D0*H
```

```
760  TEMP = F(J,1) + COEF*TK(1,1)
     IF (TEMP .GE. EM) GO TO 780
     TK(2,1) = F(J,2) + COEF*TK(1,2)
     QUANT = R(J,1) + COEF
     TK(2,2) = -(DIMM1*TK(2,1)/QUANT + EPSL*PHI(TEMP))
     TEMP = F(J,1) + COEF*TK(2,1)
     IF (TEMP .GE. EM) GO TO 780
     TK(3,1) = F(J,2) + COEF*TK(2,2)
     TK(3,2) = -(DIMM1*TK(3,1)/QUANT + EPSL*PHI(TEMP))
     TEMP = F(J,1) + H*TK(3,1)
     IF (TEMP .GE. EM) GO TO 780
     TK(4,1) = F(J,2) + H*TK(3,2)
     JTEMP = J
     J = J+1
     TK(4,2) = -(DIMM1*TK(4,1)/R(J,1) + EPSL*PHI(TEMP))
     F(J,1) = F(JTEMP,1) + H*(TK(1,1) + 2.DO*TK(2,1) +
    1          2.DO*TK(3,1) + TK(4,1))/6.DO
     F(J,2) = F(JTEMP,2) + H*(TK(1,2) + 2.DO*TK(2,2) +
    1          2.DO*TK(3,2) + TK(4,2))/6.DO
     IF (J .GE. N1) GO TO 770
     TEMP = F(J,1)
     IF (TEMP .GE. EM) GO TO 780
     TK(1,1) = F(J,2)
     TK(1,2) = -(DIMM1*TK(1,1)/R(J,1) + EPSL*PHI(TEMP))
     GO TO 760
C
C  IF QUENCHING OCCURS IN RUNKUT:
C
 770 KERR = 0
     RETURN
 780 WRITE(6,785) R(J,1)
 785 FORMAT(/' **ERROR: ARGUMENT OF PHI EQUALED OR',
    1        ' EXCEEDED EM AT  R = ',D13.6)
     KERR = 1
     RETURN
     END
C
C
C
C
     SUBROUTINE INIUO(W,F,UOMAX,ROMAX,JERR)
C
C  THIS VERSION CREATES INITIAL DATA U0 FOR THE MAIN
C    PROGRAM SUCH THAT:
C      A. U0 IS TWICE CONTINUOUSLY DIFFERENTIABLE ON (0,1);
C      B. U0(R) = U0 FOR 0 <= R <= R0;
C      C. U0(R) IS STRICTLY DECREASING ON R0 <= R <= R1;
C      D. U0(R) = F(R) FOR  R1<= R<= 1, WHERE F IS COMPUTED BY
C         SUBROUTINE EQSOLN.
C    THE VALUES OF U0 ON (R0,R1) ARE OBTAINED BY ADDING AN
```

```
C    APPROPRIATE FOURTH- OR FIFTH-ORDER POLYNOMIAL IN R TO F.
C    VALUES OF U0 ARE STORED IN W(.,1).
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION  W(1000,3), R(1000,2), F(1000,2)
      COMMON  EM, EPSL, DIM, DIMM1, H, N, N1, NMIN1, NRSKIP, R,
     1        OMEGA, FAC1, FAC2, FAC3
C
C  READ IN PARAMETERS FOR CREATING U0.
C
      READ(5,800) U0MAX,R0,R1
 800  FORMAT(2(D23.16/),D23.16)
      WRITE(6,805) U0MAX, R0, R1
 805  FORMAT(/' MAXIMUM VALUE OF INITIAL U = U0MAX = ',D13.6/,
     1  ' R0 = ',D13.6,5X,'R1 = ',D13.6/)
C
C  CHECK THAT PARAMETERS ARE ACCEPTABLE.
C
      IF ( (U0MAX .LT. F(1,1)) .OR. (U0MAX .GT. EM) )  GO TO 850
      IF ( (R0 .LT. 0.D0) .OR. (R0 .GE. R1) .OR.
     1  (R1 .GE. 1.D0) )  GO TO 850
C
C  CREATE U0.
C
      R0MAX = 0.D0
      W(1,1) = U0MAX
      NR0 = R0/H
      NR0 = NR0+1
      NR1 = R1/H
      NR1 = NR1+1
      IF (NR0 .EQ. NR1)  GO TO 860
      IF (NR0 .GT. 1) GO TO 815
      G4 = 3.D0*(F(1,1)-U0MAX)/R1**4
      G3 = 4.D0*(F(1,1)-U0MAX)/R1**3
      DO 810  I = 2,NR1
        W(I,1) = ((R(I,1) - R1)**3) *
     1            (G4*(R(I,1)-R1)+G3) + F(I,1)
 810  CONTINUE
      GO TO 830
 815  DO 820  I = 2,NR0
        W(I,1) = U0MAX
 820  CONTINUE
      NR0PL1 = NR0+1
      AL1 = R1-R0
      AL1SQ = AL1*AL1
      AL1CU = AL1*AL1SQ
      A1 = (F(NR0,1)-U0MAX)/AL1CU
      DRFR0 = (F(NR0PL1,1)-F(NR0-1,1))/(2.D0*H)
      A2 = (3.D0*AL1SQ*A1 + DRFR0)/AL1CU
      DRRFR0 = (F(NR0PL1,1)-2.D0*F(NR0,1)+F(NR0-1,1))/H**2
```

```
      A3 = (DRRFRO-6.DO*AL1*A1+6.DO*AL1SQ*A2)/(2.DO*AL1CU)
      DO 825  I = NROPL1,NR1
        W(I,1) = A3*(R(I,1)-RO) + A2
        W(I,1) = ((R(I,1)-RO)*W(I,1)+A1)*(R(I,1)-R1)**3 + F(I,1)
 825  CONTINUE
 830  IF (NR1 .GE. N1)  GO TO 840
      NR1PL1 = NR1+1
      DO 835  I = NR1PL1,N1
        W(I,1) = F(I,1)
 835  CONTINUE
 840  JERR = 0
      RETURN
C
C  ERROR-HANDLING SECTION.
C
C  IF INPUT VARIABLE UNACCEPTABLE:
C
 850  WRITE(6,855)
 855  FORMAT(/' **ERROR: INPUT VARIABLE IN SUBROUTINE INIUO',
     1  ' EXCEEDS ACCEPTABLE PARAMETERS'/)
      JERR = 1
      RETURN
C
C  IF RO,R1 TOO CLOSE FOR COMFORT:
C
 860  WRITE(6,865) NRO
 865  FORMAT(/' **ERROR: RO,R1 BOTH YIELD PARTITION',
     1        ' VALUE ',I5,' IN SUBROUTINE  INIDAT  AND ARE',
     2        ' INDISTINGUISHABLE'/)
      JERR = 1
      RETURN
      END
C
C
C
C
      SUBROUTINE FINMIN(E0,S0,GMIN,JERR)
C
C  SUBROUTINE TO DETERMINE THE ABSOLUTE MINIMUM POINT (S0,GMIN) OF
C    THE FUNCTION  G(S) = (E0 + EPSL*VOLUM*CAPPHI(S))/S**2, WHERE
C         VOLUM = THE LEBESGUE MEASURE OF THE DIM-DIMENSIONAL UNIT
C           SPHERE;
C         G(UMAX) IS AN UPPER BOUND FOR GAMMA AS DETAILED IN CHAPTER
C           FOUR OF DISSERTATION.
C
C  IMPORTANT: THIS SUBROUTINE ASSUMES  E0 >= 0.
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION  R(1000,2)
      COMMON  EM, EPSL, DIM, DIMM1, H, N, N1, NMIN1, NRSKIP, R,
```

```
     1        OMEGA, FAC1, FAC2, FAC3
C
      CAPPHI(S) = -DLOG(1.D0 - S)
      FZERO(S) = 1.D0 - S - DEXP(CONST - S/(2.D0*(1.D0-S)))
      DFZERO(S) = DEXP(CONST-S/(2.D0*(1.D0-S)))/(2.D0*(1.D0-S)**2)
     1            -1.D0
C
C  FZERO IS CHOSEN SO THAT SOLVING FZERO(S)=0 FOR S IS EQUIVALENT
C    TO SOLVING DG(S)=0.
C  SOLUTION OF FZERO(S)=0 IS ACCOMPLISHED BY APPLYING THE BISECTION
C    ALGORITHM TO LOCATE S0 TO WITHIN 2**(-10), AND THEN APPLYING
C    NEWTON'S METHOD.
C
C  NOTE: IF THE FUNCTION DEFINITION OF PHI AND VALUE OF EM ARE
C    CHANGED IN THE MAIN PROGRAM, ONE NEED ONLY CHANGE THE FUNCTION
C    DEFINITIONS OF CAPPHI,FZERO,DFZERO IN THIS SUBROUTINE.
C
      VOLUM = OMEGA/DIM
      CONST = E0/(EPSL*VOLUM)
C
C  APPLY THE BISECTION ALGORITHM.  BOUND1,BOUND2 ARE RESPECTIVELY
C    LOWER,UPPER BOUNDS FOR S0.  THE ALGORITHM IS WRITTEN USING
C    THE FACTS THAT S0 IS UNIQUE, S0 IS LOCATED BETWEEN 0 AND EM,
C    AND DFZERO(S0)>0.
C
      BOUND1 = 0.D0
      BOUND2 = EM
      NRITS = 0
 910  S0 = (BOUND1 + BOUND2)/2.D0
      NRITS = NRITS + 1
      IF (NRITS .GT. 11)  GO TO 940
      IF (FZERO(S0))  920, 960, 930
 920  BOUND1 = S0
      GO TO 910
 930  BOUND2 = S0
      GO TO 910
 940  NRITS = 0
C
C  APPLY NEWTON'S METHOD UNTIL THE RELATIVE ERROR BETWEEN TWO
C    SUCCESSIVE COMPUTED S0 IS LESS THAN 5.D-14.
C
C  THE METHOD IS DISCONTINUED, AND AN ERROR MESSAGE ISSUED, IF MORE
C    THAN 10 ITERATIONS ARE REQUIRED, IF DFZERO(S0) 'VANISHES' AT
C    SOME ITERATION, OR IF A COMPUTED S0 ESCAPES THE INTERVAL
C    (BOUND1,BOUND2) DETERMINED BY THE BISECTION ALGORITHM ABOVE.
C
 950  IF (NRITS .GT. 10)  GO TO 970
      NRITS = NRITS + 1
      TEMP = DFZERO(S0)
      IF (DABS(TEMP) .LT. 5.D-16)  GO TO 980
```

```
      TEMP = FZERO(S0)/TEMP
      S0 = S0 - TEMP
      IF ( (S0 .LT. BOUND1) .OR. (S0 .GT. BOUND2) )  GO TO 990
      IF (DABS(TEMP) .GE. (5.D-14*DABS(S0)))  GO TO 950
C
C  IF NEWTON'S METHOD WAS SUCCESSFUL, OUTPUT S0,GMIN,NRITS.
C
 960  GMIN = (E0 + EPSL*VOLUM*CAPPHI(S0))/S0**2
      WRITE(6,965) GMIN, S0, NRITS
 965  FORMAT(' G ASSUMES ABSOLUTE MINIMUM OF GMIN = ',D13.6,
     1  '  AT S0 = ',D13.6/,' ',I2,' ITERATIONS OF NEWTON''S',
     2  ' METHOD REQUIRED'/)
      JERR = 0
      RETURN
C
C  IF NEWTON'S METHOD WAS UNSUCCESSFUL, OUTPUT APPROPRIATE ERROR
C    MESSAGE.
C
 970  WRITE(6,975) S0
 975  FORMAT(' **ERROR: 11 ITERATIONS OF NEWTON''S METHOD',
     1  ' REQUIRED BY FINMIN'/,' LAST COMPUTED S0 = ',D13.6/)
      JERR = 1
      RETURN
C
 980  WRITE(6,985) S0, TEMP, NRITS
 985  FORMAT(' **ERROR: DFZERO(',D13.6,') = ',D13.6,
     1  '  IS CONSIDERED TO VANISH'/,' AFTER ',I2,
     2  ' ITERATIONS OF NEWTON''S METHOD IN FINMIN'/)
      JERR = 1
      RETURN
C
 990  WRITE(6,995) S0, NRITS, BOUND1, BOUND2
 995  FORMAT(' **ERROR: S0 = ',D13.6,' COMPUTED AT ITERATION ',I2,
     1  ' OF NEWTON''S METHOD IN FINMIN'/,' ESCAPED INTERVAL (',
     2  D13.6,' , ',D13.6,') DETERMINED BY BISECTION ALGORITHM'/)
      JERR = 1
      RETURN
      END
C
C  SAMPLE LIST OF INPUT VARIABLES.
C
1.0D0                         EPSL
2.D0                          DIM
0.25D0                        XLAMDA
 100                          N
 400                          KEND
  10                          NRSKIP
  40                          NTSKIP
1.D0                          E0
0.334589D0                    SEED
```

```
0.6D0                          U0MAX
0.2D0                          R0
0.4D0                          R1
```

```
C  PROGRAM TO OBTAIN THE BIFURCATION DIAGRAM OF EPSL VS. FMAX,
C    WHERE F=F(R) IS A POSITIVE SOLUTION OF THE STATIONARY PROBLEM
C        DRRF + (DIM-1)*DRF/R + EPSL*PHI(F) = 0        (*)
C                                F(1) = DRF(0) = 0
C    VIA EMDEN'S METHOD, WHERE PHI(S) = (1+ALPHA*S)**BETA.
C
C  FOR A DESCRIPTION OF EMDEN'S SOLUTION TECHNIQUE SEE
C    JOSEPH AND LUNDGREN, "QUASILINEAR DIRICHLET PROBLEMS
C    DRIVEN BY POSITIVE SOURCES" (REFERENCE 11).
C
C  LATEST VERSION CODED BY RICHARD A. SMITH, IOWA STATE UNIVERSITY,
C    NOVEMBER 27, 1985.
C
C  ALL COMPUTATIONS USE DOUBLE-PRECISION.
C
      IMPLICIT REAL*8 (A-H,O-Z)
C
C  V(.,1) CONTAINS VALUES V OF EMDEN'S SOLUTION.  V(.,2) CONTAINS
C    VALUES OF DXV.
C  X CONTAINS VALUES OF THE ARGUMENT OF V.
C  EPSL CONTAINS VALUES OF EPSL AS A FUNCTION OF FMAX.
C  VDIF: V(X) IS ASYMPTOTIC TO  X**(-TAU), WHERE  TAU = 2/(BETA-1).
C    VDIF CONTAINS VALUES OF  V(X) - X**(-TAU).
C  G: THE VECTOR (V(.,1),V(.,2)) SOLVES EMDEN'S ORDINARY
C    DIFFERENTIAL EQUATION  DX(V(.,1),V(.,2)) = (G(.,1),G(.,2)).
C
      DIMENSION  V(10001,2), X(10001), EPSL(1000), FMAX(1000),
     1     VDIF(10001),G(10001,2)
C
C  READ IN PARAMETERS.
C
C     W = THE HOMOLOGY CONSTANT OF THE SOLUTION OF THE
C       EMDEN PROBLEM. MUST BE POSITIVE;
C     N = THE NUMBER OF SUBINTERVALS INTO WHICH THE INTERVAL (0,1)
C       FOR X IS TO BE DIVIDED.  MUST BE POSITIVE, AND CANNOT
C       EXCEED 10000;
C     DIM = THE NUMBER OF SPACE VARIABLES IN (*). MUST BE
C       GREATER THAN OR EQUAL TO 1, BUT NEED NOT BE AN INTEGER.
C     NXSKIP SELECTS VALUES OF X AT WHICH TO OUTPUT V. MUST BE
C       POSITIVE, AND CANNOT EXCEED 10000.
C     FMAXIN = THE INTERVAL WIDTH FOR FMAX. MUST BE POSITIVE, AND
```

```
C       LESS THAN THE BLOW-UP POINT 1/DABS(ALPHA) OF PHI.
C     NFMAX = THE NUMBER OF SUBINTERVALS INTO WHICH THE INTERVAL
C       (0,FMAXIN) FOR FMAX IS TO BE DIVIDED. MUST BE POSITIVE.
C     ALPHA,BETA MUST BOTH BE NEGATIVE.
C     DELTA = ACCURACY WITHIN WHICH GPRED MUST PREDICT GCRC IN THE
C       PREDICTOR-CORRECTOR METHOD BELOW.
C
      READ (5,10) W,N,DIM,NXSKIP,FMAXIN,NFMAX,ALPHA,BETA,DELTA
 10   FORMAT(D23.16/I5/D23.16/I4/D23.16/I4/D23.16/D23.16/D23.16)
      WRITE(6,20) W,N,DIM,NXSKIP,FMAXIN,NFMAX,ALPHA,BETA,DELTA
 20   FORMAT(' EPSL VS. FMAX VIA EMDEN''S METHOD'//,' W = ',
     1 D13.6/,' N = ',I5/,' DIM = ',D13.6/,' NXSKIP = ',
     2 I4/,' FMAXIN = ',D13.6/,' NFMAX = ',I4/,
     3 ' ALPHA = ',D13.6/,' BETA = ',D13.6/,' DELTA = ',D13.6///,
     4 ' EMDEN''S SOLUTION WITH HOMOLOGY CONSTANT W'/)
C
C  CHECKS THAT INPUT VARIABLES ARE ACCEPTABLE.
C
      IF(W .LE. 0.D0)  GO TO 240
      IF ( (N .LE. 0) .OR. (N .GT. 10000) )  GO TO 240
      IF (DIM .LT. 1.D0)  GO TO 240
      IF ( (NXSKIP .LT. 1) .OR. (NXSKIP .GT. 10000) )  GO TO 240
      IF ( (FMAXIN .LE. 0.D0) .OR. (FMAXIN .GE.
     1 (1.D0/DABS(ALPHA))) )  GO TO 240
      IF (FMAXIN .LE. 0.D0)  GO TO 240
      IF ((ALPHA .GE. 0.D0) .OR. (BETA .GE. 0.D0)) GO TO 240
      IF (DELTA .LE. 0.D0)  GO TO 240
C
C  SET UP INTERNAL VARIABLES FOR DIFFERENCE SCHEME.
C      H IS THE STEPSIZE IN X VARIABLE.
C
      XN = N
      N1 = N+1
      H = 1.D0/XN
      DIMM1 = DIM - 1.D0
C
C  INITIALIZE X.
C
      DO 30  I = 1,N1
        TEMP = I-1
        X(I) = TEMP*H
 30   CONTINUE
C
C  COMPUTE EMDEN'S SOLUTION V USING SIXTH ORDER ADAMS-BASHFORTH
C    PREDICTOR WITH ADAMS-MOULTON CORRECTOR.
C
C  THE FIRST FIVE STARTING VALUES ARE GENERATED USING THE
C    CLASSICAL FOURTH-ORDER RUNGE-KUTTA METHOD.
C
      TAU = 2.D0/(BETA-1.D0)
```

```
      CONST = TAU*(DIM-2.D0-TAU)
      V(1,1) = 1.D0/W**(-TAU)
      V(1,2) = 0.D0
      TK11 = V(1,2)
      TK12 = -CONST/(DIM*V(1,1)**(-BETA))
      G(1,1) = TK11
      G(1,2) = TK12
      J = 1
      COEF = 0.5D0*H
 40   TEMP = V(J,1) + COEF*TK11
      IF (TEMP .LT. 1.D-15) GO TO 260
      TK21 = V(J,2) + COEF*TK12
      QUANT = X(J) + COEF
      TK22 = -DIMM1*TK21/QUANT - CONST/TEMP**(-BETA)
      TEMP = V(J,1) + COEF*TK21
      IF (TEMP .LT. 1.D-15) GO TO 260
      TK31 = V(J,2) + COEF*TK22
      TK32 = -DIMM1*TK31/QUANT - CONST/TEMP**(-BETA)
      TEMP = V(J,1) + H*TK31
      IF (TEMP .LT. 1.D-15) GO TO 260
      TK41 = V(J,2) + H*TK32
      JTEMP = J
      J = J+1
      TK42 = -DIMM1*TK41/X(J) - CONST/TEMP**(-BETA)
      V(J,1) = V(JTEMP,1) + H*(TK11 + 2.D0*TK21 +
     1         2.D0*TK31 + TK41)/6.D0
      V(J,2) = V(JTEMP,2) + H*(TK12 + 2.D0*TK22 +
     1         2.D0*TK32 + TK42)/6.D0
      G(J,1) = V(J,2)
      G(J,2) = -DIMM1*V(J,2)/X(J) - CONST/V(J,1)**(-BETA)
      IF (J .GE. 5) GO TO 50
      TEMP = V(J,1)
      IF (TEMP .LT. 1.D-15) GO TO 260
      TK11 = V(J,2)
      TK21 = -DIMM1*TK11/X(J) - CONST/TEMP**(-BETA)
      GO TO 40
C
C  INITIALIZE CONSTANTS FOR PREDICTOR-CORRECTOR SCHEME.
C
 50   D1 = 720.D0
      B0 = 1901.D0/D1
      B1 = -2774.D0/D1
      B2 = 2616.D0/D1
      B3 = -1274.D0/D1
      B4 = 251.D0/D1
      D2 = 1440.D0
      C0 = 475.D0/D2
      C1 = 1427.D0/D2
      C2 = -798.D0/D2
      C3 = 482.D0/D2
```

```
      C4 = -173.D0/D2
      C5 = 27.D0/D2
C
C  APPLY PREDICTOR-CORRECTOR METHOD.
C
C       VPRED1,VPRED2 CONTAIN PREDICTED VALUES OF V.
C       VCRC1,VCRC2 CONTAIN CORRECTED VALUES OF V.
C
C    A MAXIMUM OF 8 CORRECTIONS IS ALLOWED.  FOR DETAILS, SEE
C    HENRICI, DISCRETE VARIABLE METHODS IN ORDINARY DIFFERENTIAL
C    EQUATIONS, PAGE 200.
C
      DO 80  J = 6,N1
        VPRED1 = V(J-1,1) + H*(B4*G(J-5,1) + B3*G(J-4,1)
     1           + B2*G(J-3,1) + B1*G(J-2,1) + B0*G(J-1,1))
        VPRED2 = V(J-1,2) + H*(B4*G(J-5,2) + B3*G(J-4,2)
     1           + B2*G(J-3,2) + B1*G(J-2,2) + B0*G(J-1,2))
        GPRED1 = VPRED2
        GPRED2 = -DIMM1*VPRED2/X(J) - CONST/VPRED1**(-BETA)
        CON1 = V(J-1,1) + H*(C5*G(J-5,1) + C4*G(J-4,1) +
     1         C3*G(J-3,1) + C2*G(J-2,1) + C1*G(J-1,1))
        CON2 = V(J-1,2) + H*(C5*G(J-5,2) + C4*G(J-4,2) +
     1         C3*G(J-3,2) + C2*G(J-2,2) + C1*G(J-1,2))
        ITER = 1
 60     VCRC1 = CON1 + H*C0*GPRED1
        VCRC2 = CON2 + H*C0*GPRED2
        GCRC1 = VCRC2
        GCRC2 = -DIMM1*VCRC2/X(J) - CONST/VCRC1**(-BETA)
        TEMP = DABS(GCRC1-GPRED1) + DABS(GCRC2-GPRED2)
        IF (TEMP .LT. DELTA) GO TO 70
        IF (ITER .GT. 8) GO TO 320
        GPRED1 = GCRC1
        GPRED2 = GCRC2
        ITER = ITER + 1
        GO TO 60
 70     G(J,1) = GCRC1
        G(J,2) = GCRC2
        V(J,1) = VCRC1
        V(J,2) = VCRC2
 80   CONTINUE
C
C  OUTPUT V AND CHECK ITS ASYMPTOTIC BEHAVIOR USING VDIF.
C
      LINES = N1/(7*NXSKIP)
      IF ((LINES*7*NXSKIP) .NE. N1) LINES = LINES+1
      INCR = 7*NXSKIP
      JLOW = 1-INCR
      JHIGH = 1-NXSKIP
      DO 130  I = 1,LINES
        JLOW = JLOW + INCR
```

```
      JHIGH = JHIGH + INCR
      IF (JHIGH .GT. N1) JHIGH = N1
      WRITE(6,90) (X(J),J=JLOW,JHIGH,NXSKIP)
 90   FORMAT(' X:    ',7(D13.6,2X))
      WRITE(6,100) (V(J,1),J=JLOW,JHIGH,NXSKIP)
 100  FORMAT(' V:    ',7(D13.6,2X))
      DO 110  J = JLOW,JHIGH,NXSKIP
        VDIF(J) = V(J,1) - X(J)**(-TAU)
 110  CONTINUE
      WRITE(6,120)(VDIF(J),J=JLOW,JHIGH,NXSKIP)
 120  FORMAT(' VDIF: ',7(D13.6,2X)/)
 130 CONTINUE
C
C  CHECK FOR STRICTLY INCREASING V.
C
     DO 140  I = 1,N
       IF (V(I,1) .GE. V(I+1,1)) GO TO 300
 140 CONTINUE
C
C  USING V, DETERMINE THE UNIQUE EPSL CORRESPONDING TO EACH
C    FMAX. THIS IS ACCOMPLISHED USING THE OUTER BOUNDARY
C    CONDITION WHICH V SATISFIES; SEE JOSEPH AND LUNDGREN.
C
     XFMAX = NFMAX
     DELFMX = -FMAXIN/(XFMAX*ALPHA)
     JSTART = 1
     NUMP1 = NFMAX + 1
     DO 180  I = 2,NUMP1
       TEMP = I-1
       FMAX(I) = TEMP*DELFMX
       VVAL = V(1,1)/(1.D0 + ALPHA*FMAX(I))
       DO 150  J = JSTART,N1
         IF (V(J,1) .GE. VVAL)  GO TO 160
 150   CONTINUE
 160   IF (J .LT. N1) GO TO 170
       NFMAX = I-3
       GO TO 190
 170   JSTART = J-1
       XTEMP = X(JSTART) + H*(VVAL-V(JSTART,1))/
    1          (V(J,1)-V(JSTART,1))
       EPSL(I) = (CONST/ALPHA)*(XTEMP*W)**2*
    1             (1.D0+ALPHA*FMAX(I))**(1.D0-BETA)
 180 CONTINUE
C
C  OUTPUT EPSL VS. FMAX.
C
 190 IF (NFMAX .LE. 0) GO TO 280
     WRITE(6,200)
 200 FORMAT(//' EPSL VS. FMAX'/)
     LINES = NUMP1/7
```

```
      IF ((LINES*7) .NE. NUMP1) LINES = LINES+1
      JLOW = -5
      JHIGH = 1
      DO 230  I = 1,LINES
        JLOW = JLOW + 7
        JHIGH = JHIGH + 7
        IF (JHIGH .GT. NUMP1) JHIGH = NUMP1
        WRITE(6,210) (FMAX(J),J=JLOW,JHIGH)
 210    FORMAT(' FMAX:  ',7(D13.6,2X))
        WRITE(6,220) (EPSL(J),J=JLOW,JHIGH)
 220    FORMAT(' EPSL:  ',7(D13.6,2X)/)
 230  CONTINUE
      STOP
C
C  ERROR-HANDLING SECTION.
C
 240  WRITE(6,250)
 250  FORMAT(/' **ERROR: INPUT VARIABLE EXCEEDS ACCEPTABLE ',
     1         'PARAMETERS'/)
      STOP
C
 260  WRITE(6,270) J
 270  FORMAT(/' **ERROR:  R-K SOLUTION FOR V FAILS AT STEP ',
     1       I4/)
      STOP
C
 280  WRITE(6,290)
 290  FORMAT(/' NO VALUES OF FMAX, EPSL TO OUTPUT'/)
      STOP
C
 300  WRITE(6,310)
 310  FORMAT(/' **ERROR: V NOT STRICTLY INCREASING'/)
      STOP
C
 320  WRITE(6,330) X(J)
 330  FORMAT(' **ERROR: NINE ITERATIONS REQUIRED IN PREDICTOR-',
     1 'CORRECTOR AT  X = ',D13.6/,' COMPUTATION DISCONTINUED')
      STOP
      END
C
C  SAMPLE LIST OF INPUT VARIABLES.
C
1000.D0                     W
10000                       N
3.D0                        DIM
 100                        NXSKIP
0.99D0                      FMAXIN
  99                        NFMAX
-1.D0                       ALPHA
-1.D0                       BETA
0.00001D0                   DELTA
```