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A NORMAL MODE SOLUTION TO RADIO WAVE PROPAGATION
IN A TERRESTRIAL ENVIRONMENT

by

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I. INTRODUCTION

The study of the propagation of electromagnetic waves over the surface of the earth has long commanded a great deal of interest because of the mathematical techniques involved and because of the obvious practical significance of the problem. The history of this study can be traced to the modal representation presented by Lord Rayleigh (29) for sound waves emanating from a point source beside a large spherical boundary. In this representation, each mode took the form of a Bessel function of large order.

G. N. Watson (36) adapted the Rayleigh solution for electromagnetic propagation, and transformed the resulting series to an alternate form which converged rapidly enough to permit numerical evaluation. van der Pol and Bremmer (33) extended the Watson residue series to accommodate an arbitrary conductivity of the earth and an arbitrary dielectric constant. The model here involved a homogeneous atmosphere.

Schelleng, Burrows, and Ferrell (31) then proposed a model in which tropospheric refraction would be taken into account by using an effective radius of the earth equal to about $\frac{4}{3}$ the actual value, thus extending the effective radio horizon. The validity of this general approach and the "classical $\frac{4}{3}$ earth radius" theory was assumed for some time.

A more critical study of the entire problem was stimulated by the publication of reliable experimental field strength measurements, such as those by Megaw (22), and Gerks (15), which indicated that fields beyond the radio horizon were commonly much stronger than the classical values. Three separate hypotheses were advanced to explain this discrepancy. The "turbulent scatterer" theory was advanced by Booker and Gordon (2), who

argued that local, time-varying anomalies in the refractive index effectively scatter energy beyond the horizon. Bullington (8) formulated the scattering of radiation due to the roughness of the surface of the earth, and obtained field strengths greater than the classical values.

A third group of investigators made a more critical study of the original mode theory in an attempt to bring together experimental and theoretical results. A brief history of this effort is desirable, since the classical mode theory forms the basis for this report.

Pekeris and Ament (27) completed the normal mode solution in cylindrical coordinates for several profiles of refractive index. Kerr (19) has collected several analyses, also concerning the flat earth, which were stimulated primarily by military work with radar. Bremmer (5) presents an exhaustive review of the classical results in spherical coordinates, including a physical interpretation of ray tracing and the Wentzel-Brillouin-Kramers method (10).

Wait (34) carefully analyzed the hypothesis of the homogeneous earth and concluded that such a model was justified. Ghose and Albright (16) studied the normal modes for a choice of smooth profile which yielded a height gain equation that could be solved exactly. Experimental data quoted by Ghose and Albright has been compared with the results of the present study, with good agreement.

Carroll and Ring (9) showed, in an exhaustive treatment, that for several particular profiles, higher order modes could make a significant contribution to the field strength. Bremmer (3) has studied the mode expansion for smooth profiles and obtained field strengths smaller than those of Carroll and Ring.

Budden (6,7) has reviewed the standard mode theory with particular emphasis on ionospheric propagation. Included is a treatment of Stoke's equation and the Airy integral solution. Post (28) has used earth-flattened coordinates and a Green's function technique to allow the normal mode solution to accomodate an arbitrary profile. Some numerical data from Post has been included in this report to allow comparison of results. Gerks (14) has reviewed the entire mode solution as it applies to a spherical earth and a stratified atmosphere. Wait (35) has discussed the most recent work with the normal mode theory and low frequency applications.

The attempt to find solutions for a general class of height gain differential equations was begun by Furry (13), who applied the well-known W.K.B. solution to a bilinear profile. Pekeris (26) derived an asymptotic solution by means of a power series expansion. Langer (20), motivated by Pekeris' work, showed that the results of one of his earlier papers (21) could be applied to the problem of microwave propagation, but did not complete any quantitative check on the solution.

Friedman (12) has made extensive use of the Langer solution in formulating the mode solution for a general stratified atmosphere. He obtains, for any smooth, monotonically decreasing profile, an expression for the eigenvalues which leaves them essentially unchanged from the homogeneous atmosphere values. Northover (23) has obtained the same qualitative result by a different method. Bremmer (4) has studied the dependence of the Langer solution on the complex root h_1 .

This analysis draws freely on the classical mode solution as written by Friedman. A particular smooth profile, approximating an exponential profile, is specified by a three term power series, or quadratic. The

proper integration is performed to generate the Langer solution for this profile, and the transcendental equations defining the eigenvalues are obtained. Numerical examples are presented to compare the results with those of Friedman and Post. The Friedman expression of the free space eigenvalues is obtained as a special case of the present solution.

II. FORMULATION OF THE PROBLEM AND THE GENERALIZED SOLUTION

This analysis deals with the electromagnetic fields produced by an elemental monochromatic magnetic dipole vertically oriented above a homogeneous conducting sphere. Of particular interest is the case in which the radius of the sphere is very large compared to the free space wavelength, and in which the dielectric medium surrounding the sphere is described by a refractive index that is given a specified functional dependence on the radial coordinate.

For the purposes of this analysis, a point magnetic dipole of strength m is defined as a planar loop of radius R , carrying current I , in the limiting case as R becomes arbitrarily small such that $m = \lim_{\substack{R \rightarrow 0 \\ I \rightarrow \infty}} RI$. This source is somewhat analogous to a horizontally oriented antenna in that the polarization of the resulting electric field is the same.

The problem is presented in ordinary spherical coordinates r , θ , and ϕ , as depicted in Figure 1. The dipole is radially oriented and located at $r = b$ on the axis defined by $\theta = 0$. The radius of the sphere will be given by a . This geometric configuration is obviously an appropriate idealized model with which to study the propagation of electromagnetic energy around the earth at high frequencies.

The choice of the magnetic dipole source as opposed to the electric dipole is made because the resulting boundary conditions are considerably simpler. This choice is further justified by the fact that experimental evidence indicates the field strengths in the diffraction region, or beyond the radio horizon, to be relatively independent of the polarization of the source (9). The frequency range for which the model is

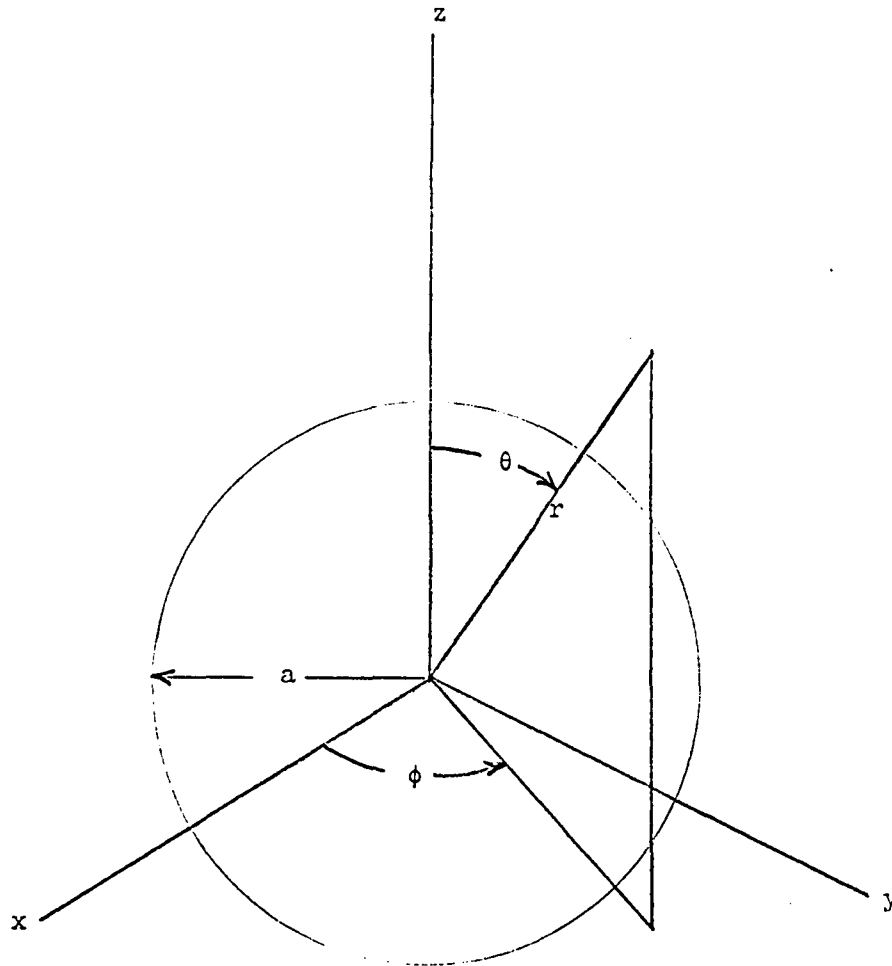


Figure 1. The spherical coordinates of the problem

accurate will have a lower bound due to the lack of consideration of the ionosphere. Field strengths calculated on the basis of a perfectly conducting earth will be valid for those frequencies at which the earth appears to be a very good conductor.

It will be instructive to review the classical construction of solutions to Maxwell's equations in terms of a magnetic vector potential \vec{A} and an electric potential \vec{F} . The notation here will be that due to Harrington (17). The electric field intensity \vec{E} and the magnetic field intensity \vec{H} must satisfy Maxwell's equations for a source-free region, except at the source point.

$$\begin{aligned} \nabla \times \vec{E} &= j\omega\mu\vec{H} & \nabla \cdot \vec{H} &= 0 \\ \nabla \times \vec{H} &= -j\omega\epsilon\vec{E} & \nabla \cdot \vec{E} &= 0 \end{aligned} \quad 1$$

The time dependence of $e^{-j\omega t}$ has been removed, so that real time expressions can be obtained for these field intensities by taking the real part of the products $\vec{E}e^{-j\omega t}$ and $\vec{H}e^{-j\omega t}$.

It is customary to relate potentials to intensities by requiring that

$$\vec{E} = -\nabla \times \vec{F} \quad 2$$

when only magnetic sources are present, and

$$\vec{H} = \nabla \times \vec{A} \quad 3$$

when only electric sources are present. In the case of both sources, the fields due to each source are superimposed to yield the relations

$$\begin{aligned} \vec{E} &= -\nabla \times \vec{F} + \frac{j}{\omega\epsilon} \nabla \times \nabla \times \vec{A} \\ \vec{H} &= \nabla \times \vec{A} + \frac{j}{\omega\mu} \nabla \times \nabla \times \vec{F} \end{aligned} \quad 4$$

The potentials \vec{A} and \vec{F} are still arbitrary to the extent that a particular

"gauge" transformation can be made. For one such choice, the potentials satisfy the wave equations

$$\begin{aligned}\nabla^2 \vec{A} + k^2 \vec{A} &= 0 \\ \nabla^2 \vec{F} + k^2 \vec{F} &= 0\end{aligned}\quad , \quad 5$$

where k is the wave number defined by

$$k^2 = \omega^2 \mu \epsilon . \quad 6$$

It is possible to represent, by means of the potentials \vec{A} and \vec{F} , an arbitrary electromagnetic field as the superposition of a magnetic field transverse to the radius vector, (TM), and an electric field transverse to \vec{r} , (TE). For this purpose it will be required that

$$\begin{aligned}\vec{A} &= A_r \vec{e}_r \\ \vec{F} &= F_r \vec{e}_r\end{aligned}\quad 7$$

where \vec{e}_r is a unit vector in the radial direction. Equations 5 require that the rectangular components of \vec{A} and \vec{F} satisfy the scalar Helmholtz wave equation. By substitution into these same equations, it can be determined that the functions $\frac{A_r}{r}$ and $\frac{F_r}{r}$ also satisfy the Helmholtz equation,

$$[\nabla^2 + k^2] \frac{A_r}{r} = 0 \quad 8$$

$$[\nabla^2 + k^2] \frac{F_r}{r} = 0 . \quad 9$$

At this point it is appropriate to recognize that the potential vector \vec{F} is essentially identical to the radial "Hertz" vector used extensively by Friedman (12), and Bremmer (5), and it is also equivalent to the "magnetic potential" \vec{Z}_m discussed by Panofsky and Phillips (25). It is known that an arbitrary field can be constructed from the TE and TM modes which are generated from the radial vectors \vec{F} and \vec{A} , since these

modes form a complete set (17). It is well established from the work of Bremmer (5) and Friedman (12), however, that for the particular problem outlined here, involving a radial dipole and radially stratified index of refraction, the fields can be generated from a single radial potential F . This assertion need not be proven before-hand, since the uniqueness theorem, as given by Harrington (17), provides assurance that a solution obtained from a single potential, radially oriented, will be the only one possible.

To proceed with the classical solution, one writes the operator $[\nabla^2 + k^2]$ in spherical coordinates and uses the method of séparation of variables to solve Equation 9. Since by symmetry there is no ϕ dependence, let

$$\frac{F}{r} = \bar{\Theta}(\theta) R(r) . \quad 10$$

The separated equations which result are

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \left[k^2 - \frac{v(v+1)}{r^2} \right] R = 0 \quad 11$$

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\bar{\Theta}}{d\theta} \right) + [v(v+1)] \bar{\Theta} = 0 . \quad 12$$

The separation constant is given by $v(v+1)$. The requirement that $\bar{\Theta}$ be defined for all θ on the closed interval $[0, \pi]$ requires that v be a positive integer. The solution for $\bar{\Theta}$ is then

$$\bar{\Theta}_v(\theta) = P_v(\cos\theta) \quad 13$$

where $P(x)$ is the well known Legendre function. In the case of a homogeneous medium, the solutions for R are the well known spherical Bessel functions

$$R_v = \sqrt{\frac{\pi}{2kr}} B_{v+1/2}(kr) , \quad 14$$

where $B(kr)$ is an ordinary Bessel function.

The solution for F_r then becomes

$$F_r = \sum_v C_v r R_v P_v(\cos\theta) \quad 15$$

where C_v is a constant.

It is helpful to define the function \hat{B}_v by

$$\hat{B}_v = r R_v \quad 16$$

The differential equation which \hat{B}_v must satisfy is

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{v(v+1)}{r^2} \right] \hat{B}_v = 0 \quad 17$$

In the special case of the homogeneous medium, the functions \hat{B}_v are, apart from a multiplicative constant, the spherical Bessel functions due to Schelkunoff (30).

Inside the homogeneous earth, the choice for \hat{B}_v is that Schelkunoff function which is regular at $r = 0$, so that

$$\hat{B}_v = r^{1/2} J_{v+1/2}(k_1 r) . \quad 18$$

The complex wave number k_1 characterizes the permittivity and conductivity of the earth. Thus, for $r < a$, F_r is given by

$$F_r = \sum_v C_v r^{1/2} J_{v+1/2}(k_1 r) P_v(\cos\theta) \quad 19$$

From Equation 2, the electric field intensity can be obtained from F by the relations

$$E_r = E_\theta = 0 \quad 20$$

$$E_\phi = \frac{1}{r} \frac{\partial F_r}{\partial \theta} \quad 21$$

To form the series for F_r valid for $r > a$, one must take into account the dipole source and the dependence of k on r . Friedman's form of the solution will be used because the series for F_r can be completed in terms of generalized functions without specifying the exact "profile", or dependence of ϵ on r . Panofsky and Phillips (25) have shown that the potential F satisfies a wave equation similar to 5, but with the magnetization vector $\omega\mu\mathbf{m}$ as a source, where μ_0 is the permeability of the medium. The scalar wave equation for F_r then becomes

$$[\nabla^2 + k^2] \frac{F_r}{r} = -j\omega\mu_0 \frac{\delta(r-b)\delta(\theta)}{2\pi r^2 \sin\theta} \quad 22$$

where δ designates the Dirac delta function and the factor $2\pi r^2 \sin\theta$ normalizes the dipole moment to unit strength.

Next, a series for F_r is assumed in the form

$$F_r = \sum_v \left(\frac{2v+1}{2} \right) \hat{B}_v(r) P_v(\cos\theta) \quad 23$$

Using this representation in Equation 22, one can multiply both sides of the equation by $P_v(\cos\theta) \sin\theta d\theta$ and integrate from 0 to π . The result is the following equation:

$$\frac{d^2}{dr^2} (\hat{B}) + [k^2(r) - \frac{v(v+1)}{r^2}] \hat{B} = -j\omega\mu_0 \frac{\delta(r-b)}{2\pi r} \quad 24$$

Friedman has constructed, by the method of variation of parameters, the solution to this equation in terms of solutions to the homogeneous Equation 17. The solutions to the "height gain" Equation 17 are known to behave approximately like Bessel functions, but the exact behavior depends on the profile. In any case, one of the linearly independent solutions, to be designated as f_1 , can be expected to be asymptotic to $e^{+ik_0 r}$ for

large r . The second, f_2 , will be asymptotic to $e^{-ik_0 r}$ for large r . Here, k_0 is the free space wave number. Let g designate some linear combination of the solutions f_1 and f_2 . Then the solution to Equation 24 has been shown by Friedman to take the forms

$$\hat{B} = \frac{j\omega\mu_0 g(r)f_1(b)}{2\pi r W}, \quad a < r < b \quad 25$$

$$\hat{B} = \frac{j\omega\mu_0 g(b)f_1(r)}{2\pi r W}, \quad r > b \quad 26$$

where W is the Wronskian of the two solutions.

$$W = f_1 \frac{dg}{dr} - g \frac{df_1}{dr} \quad 27$$

Friedman has shown that the Wronskian is independent of r for the magnetic dipole problem. W will be written as $W(v)$ to emphasize the dependence on the parameter v . The series for F_r then become

$$F_r = \frac{j\omega\mu_0}{2\pi} \sum_v \left(\frac{2v+1}{2}\right) \frac{g(r)f_1(b)}{r W(v)} P_v(\cos\theta) \quad 28$$

for $a < r < b$, and

$$F_r = \frac{j\omega\mu_0}{2\pi} \sum_v \left(\frac{2v+1}{2}\right) \frac{g(b)f_1(r)}{r W(v)} P_v(\cos\theta) \quad 29$$

for $r > b$.

The above series converge so slowly that numerical evaluation, even by means of a digital computer, is not feasible. The Watson transformation, reviewed in Appendix A, is commonly applied to these series. The well known result is

$$F_r = \sum_s \frac{\left(v_s + \frac{1}{2}\right) g(b)f_1(r) P_{v_s}(-\cos\theta) j\omega\mu_0}{r W'(v_s) (\sin\pi v_s)} \quad 30$$

for $r > b$. Here $W'(v_s)$ is defined by

$$W'(v_s) = \left[\frac{d}{dt} W(t) \right]_{t=v_s} \quad 31$$

and the numbers v_s are defined by

$$W(v_s) = 0. \quad 32$$

Note that v_s is now complex, since the above series results from integration on the complex v plane. The approximation is commonly made that

$$\frac{P_{v_s}(-\cos\theta)}{\sin \pi v_s} = - \left[\frac{2j}{\pi(v_s+1) \sin\theta} \right]^{1/2} \exp[j(v_s + \frac{1}{2})\theta] \quad 33$$

From physical considerations, v_s must have a positive imaginary part to give modes which decay with increasing θ . Only those eigenvalues, v_s , with small imaginary part need be considered for the diffraction, or beyond-the-horizon zone, since other v_s 's correspond to highly attenuated modes. The functions f and g are both dependent on the eigenvalue v_s .

The series representation for E_ϕ can now be formed using Equations 30, 33, and 21. The exponential dependence on θ dominates the derivative, so that

$$E_\phi = \sum_s \frac{\omega \mu (v_{s+1/2})^2 g(b) f_1(r)}{r^2 W'(v_s)} \left[\frac{2j}{\pi(v_s+1) \sin\theta} \right]^{1/2} \exp[j(v_{s+1/2})\theta]$$

for $r > b$. (34)

It should be noted that multiplication of the functions g and f by non zero constants will still allow them to satisfy Equation 17. In addition, the above series will be unchanged by such scaling factors, due to the function $W'(v_s)$.

To insure the continuity of tangential components of \vec{E} and \vec{H} at the surface $r=a$, one must require the continuity of F_r and $\frac{dF}{dr}$ at $r=a$. The eigenvalues in Equation 34 must be identical with those in Equation 19, since the θ dependence of F_r must be the same above and below the surface $r=a$. Equation 30, with the arguments of f and g interchanged, forms the function F_r for $a < r < b$. The boundary condition on $g(r)$ then becomes

$$\left. \frac{\frac{d}{dr} g}{g} \right|_{r=a} = \frac{\frac{d}{dr} [r^{1/2} J_{v+1/2}(k_1 r)]}{r^{1/2} J_{v+1/2}(k_1 r)} \bigg|_{r=a} \quad 35$$

The series for F_r involves values of v for which the Wronskian vanishes. When this occurs, the functions f_1 and g are identical, except possibly for some multiplicative constant. The function g can then be replaced by f in Equation 35. It is also customary to use the asymptotic or large argument approximation for $J_{v+1/2}(k_1 r)$, to give the final form for the boundary or eigenvalue equation as

$$\left. \frac{df_1}{f_1} \right|_{r=a} = -i \left(k_1^2 - \frac{v(v+1)}{a^2} \right)^{1/2} \quad 36$$

For the special case in which the conductivity of the earth tends to infinity, the wave number k_1 becomes unbounded, and the eigenvalues are those which allow f to satisfy

$$f_1(a) = 0. \quad 37$$

III. SPECIFICATION OF THE PROFILE

The dependence of ϵ on r will now be specified. An exponential dependence of refractive index on height above the surface of the earth has been frequently suggested as characterizing a stable, uniform atmosphere (1,14). This can be written as

$$n = 1 + (n_0 - 1)e^{-ch} \quad 38$$

where n is the index of refraction, n_0 is the surface index, c is a constant parameter, and h is the height defined by

$$r = h + a. \quad 39$$

It is then true that

$$n^2 = 1 + 2(n_0 - 1)e^{-ch} + (n_0 - 1)^2 e^{-2ch} \quad 40$$

Since $(n_0 - 1)$ takes on values on the order of 10^{-4} , the second exponential can be neglected. Equation 17 then becomes, in terms of the independent variable h ,

$$\frac{d^2 \hat{B}}{dh^2} + [k_0^2 + 2k^2(n_0 - 1)e^{-ch} - \frac{v(v+1)}{(h+a)^2}] \hat{B} = 0 \quad 41$$

Now define $Y(h)$ by writing the above equation as

$$\frac{d^2 \hat{B}}{dh^2} + Y(h) \hat{B} = 0 \quad 42$$

No rigorous, exact solution has been found for $Y(h)$ as written above (14). The Langer asymptotic solution, whose derivation is outlined in Appendix B, is felt to be the best possible approximate solution (23) and has been used in attempts to draw general conclusions about wide classes of index profiles (12,23).

To complete the Langer solution, $Y(h)$ must be a function of h whose square root can be integrated without undue difficulty. A quadratic function of h meets this requirement, and can be obtained by generating a three term power series for e^{-ch} and for $\frac{1}{(h+a)^2}$. The quadratic form of $Y(h)$ which results can be used for h on the interval $[0, \frac{3}{c}]$, while $Y(h)$ assumes its free space form for $h > 3/c$. The surface $h = 3/c$ is then a boundary between the troposphere and free space. If $Y(h)$ and $\frac{dY(h)}{dh}$ are made continuous at $h = \frac{3}{c}$, the boundary there will be a fictitious one and no reflected wave from this boundary need be considered. The procedure outlined here will now be developed in detail.

The function e^{-ch} is approximated by $1 + \bar{A}h + \bar{B}h^2$. When one requires this quadratic to have a zero value and a zero derivative at $h = 3/c$, \bar{A} and \bar{B} become

$$\bar{A} = -\frac{2c}{3} \quad 43$$

$$\bar{B} = c^2/9. \quad 44$$

The function $\frac{1}{(h+a)^2}$ can be expanded in a power series as

$$\frac{1}{(h+a)^2} = \frac{1}{a^2} \left(1 - \frac{2h}{a} + \frac{3h^2}{a^2} \right) \quad 45$$

Therefore, for $0 < h < 3/c$,

$$Y(h) = \overline{Y(h)} \quad 46$$

where

$$\begin{aligned}
\overline{Y(h)} &= k_o^2 + 2k_o^2(n_o-1) - \frac{v(v+1)}{a^2} \\
&+ h[-\frac{4c k_o^2(n_o-1)}{3} + \frac{2v(v+1)}{a^3}] \\
&+ h^2[\frac{2}{9} c^2 k_o^2(n_o-1) - \frac{3v(v+1)}{a^4}] .
\end{aligned} \tag{47}$$

So one can write

$$\overline{Y(h)} = k_o^2 [Ah^2 + Bh + C] \tag{48}$$

where

$$A = \frac{2}{9} c^2(n_o-1) - \frac{3v(v+1)}{k_o^2 a^4} \tag{49}$$

$$B = \frac{2v(v+1)}{k_o^2 a^3} - \frac{4}{3} c(n_o-1) \tag{50}$$

$$C = 1 + 2(n_o-1) - \frac{v(v+1)}{k_o^2 a^2} . \tag{51}$$

$Y(h)$, for $h > 3/c$, is given by

$$Y(h) = k_o^2 - \frac{v(v+1)}{(a+h)^2} = Y_o(h) \tag{52}$$

$Y(h)$ is thus discontinuous at $h = \frac{3}{c}$ by a small amount equal to the error in the three term expansion for $\frac{v(v+1)}{(a+h)^2}$ at $h = 3/c$. Since this error is more than three orders of magnitude smaller than the constant C in $\overline{Y(h)}$, it will be neglected, and these functions $Y_o(h)$ and $\overline{Y(h)}$ will be assumed continuous. The first derivatives are likewise essentially continuous.

It will then be true that the Taylor Series expansion for $Y_o(h)$ about $h = 3/c$ and the series for $\overline{Y(h)}$ about $h = 3/c$ will have identical first and second terms. The solutions to

$$\frac{d^2 B}{dh^2} + Y_0(h) B = 0 \quad 53$$

and

$$\frac{d^2 B}{dh^2} + \overline{Y(h)} B = 0 \quad 54$$

are therefore identical, except perhaps for a constant multiplier, for h on some interval including $h = 3/c$.

IV. THE HEIGHT GAIN FUNCTION

The Langer solutions for the differential equation

$$\frac{d^2 \hat{B}}{dh^2} + k_o^2(q(h)) \hat{B} = 0 \quad 55$$

take the form

$$f_1 = \left[\frac{u(h)}{Q(h)} \right]^{1/2} H_{1/3}^{(1)}[u(h)] \quad 56$$

$$f_2 = \left[\frac{u(h)}{Q(h)} \right]^{1/2} H_{1/3}^{(2)}[u(h)] . \quad 57$$

$Q(h)$ and h_1 are defined by

$$Q(h) = k_o(q(h))^{1/2} \quad 58$$

and

$$q(h_1) = 0 , \quad 59$$

and $u(h)$ is given by

$$u(h) = \int_{h_1}^h k_o q(h)^{1/2} dh \quad 60$$

In using the Langer solution, one adjusts the parameter v after the solution is formulated. The dependence of q on v can be seen by comparing Equations 55 and 17. The particular values of v which satisfy the boundary conditions imposed are the values v_s . For the Langer solution to be valid, it must be true that for each v_s , $k^2 |y'(h_1)|$ exceeds some large constant. In addition, it must be possible to extend the definition of $q(h)$ to complex values of h on some region including the points h_1 mentioned above. Such an extension is not difficult for the case studied here, and the condition on $y'(h)$ is satisfied for smooth, monotonically

decreasing profiles.

Under the above hypotheses, $q(h)$ has a root whose real part is positive and bounded from zero when h is real and positive. It is understood that this root is used in Equations 58 and 60. The path of integration extends from h_1 to some point on the real axis and then along the real axis to h . The real part of u is then an increasing function of h .

The Langer solutions for Equation 54 will be developed. The function f_1 appearing in Equations 30 and 34 will be given by

$$f_1 = \left[\frac{u(h)}{q(h)} \right]^{1/2} H_{1/3}^{(1)}[u(h)] \quad 61$$

for $0 < h < 3/c$, and by

$$f_1 = \frac{\left[\frac{u(3/c)}{q(3/c)} \right]^{1/2} H_{1/3}^{(1)}[u(3/c)]}{H_{v+1/2}^{(1)}(k_o 3/c)} H_{v+1/2}^{(1)}(k_o r) \quad 62$$

for $h > 3/c$.

The constants A , B , and C will now be evaluated for complex values of v . It is known from previous work (14) that the real part of v will be approximately equal to $k_o a$. Since $|v|$ is large compared to unity,

$$v(v+1) \approx v^2 \quad 63$$

The real variables α and β will be defined by

$$v = k_o a + \alpha + j\beta \quad 64$$

Then the following relations can be written, keeping only the dominant terms:

$$v^2 = (k_o a)^2 + 2k_o a\alpha + 2j\beta k_o a \quad 65$$

$$C = 2(n_o - 1) - \frac{2\alpha}{k_o a} - j \frac{2\beta}{k_o a} \quad 66$$

$$B = \frac{2}{a} - \frac{4c(n_o - 1)}{3} + j \frac{4\beta}{k_o a^2} \quad 67$$

$$A = \frac{2}{9} c^2 (n_o - 1) - \frac{3}{a^2} - j \frac{6\beta}{k_o a^3} \quad 68$$

By consideration of the typical values of c , n_o and k_o , and anticipated values of β , it is found that

$$\frac{\text{Re}A}{\text{Im}A} \approx 10^6 \quad 69$$

and

$$\frac{\text{Re}B}{\text{Im}B} \approx 10^4 \quad 70$$

A and B will therefore be taken as real parameters describing the troposphere. The relations

$$C_o = 2(n_o - 1) - \frac{2\alpha}{k_o a} \quad 71$$

and

$$C_1 = \frac{2\beta}{k_o a} \quad 72$$

will arbitrarily define the real variables C_o and C_1 . This will require that

$$C = C_o - jC_1 \quad 73$$

The root of $Ah^2 + Bh + C$ closest to the origin will now be found.

$$\begin{aligned} h_1 &= \frac{-B + (B^2 - 4AC)^{1/2}}{2A} \\ &= \left(\frac{B}{2A}\right) \left(-1 + \left[1 - \frac{4AC}{B^2}\right]^{1/2}\right) \end{aligned} \quad 74$$

Substituting for C under the radical yields

$$\left[1 - \frac{4AC_0}{B^2} + j\frac{4AC_1}{B^2}\right]^{1/2} \approx \left[\left(1 - \frac{4AC_0}{B^2}\right)^2 + \left(\frac{4AC_1}{B^2}\right)^2\right]^{1/4} \left(1 + j\frac{4AC_1}{2B^2}\right) \quad 75$$

Now make use of the relation that, for small ϵ ,

$$[1 + \epsilon]^{1/4} = 1 + \frac{\epsilon}{4} \quad 76$$

$$\left[\left(1 - \frac{4AC_0}{B^2}\right)^2 + \left(\frac{4AC_1}{B^2}\right)^2\right]^{1/4} \approx 1 - \frac{2AC_0}{B^2} + 4\left(\frac{AC_0}{B^2}\right)^2 + 4\left(\frac{AC_1}{B^2}\right)^2 \quad 77$$

After some algebraic manipulation the result is

$$h_1 \approx -\frac{C_0}{B} + \frac{2A}{B^3}(C_0^2 + C_1^2) + j\frac{C_1}{B}\left(1 - \frac{2AC_0}{B^2}\right) \quad 78$$

The root h_1 is thus seen to lie close to the root $-\frac{C}{B}$ which would result if $A = 0$. In the following integration, h_1 will be taken as

$$h_1 = -\frac{C_0}{B} + j\frac{C_1}{B} \quad 79$$

Of concern here is the integration designated in Equation 60, from the root h_1 to some point h on the real axis. This integration on the complex h plane will be carried out, for simplicity, along two rectilinear paths. Path one extends from the root h_1 to the real axis, while path two leads along the real axis to h . This integration path is shown in Figure 2.

Of particular interest is the integration to the origin, since the resulting function $u(0)$ will be involved in the eigenvalue equation for the perfectly conducting earth. It is easily seen that C_1 is a positive number, since the attenuation of each mode with θ , or distance, goes as $e^{-\theta\beta_s}$, where s is the mode integer. It is of interest to inquire into the sign of C_0 .

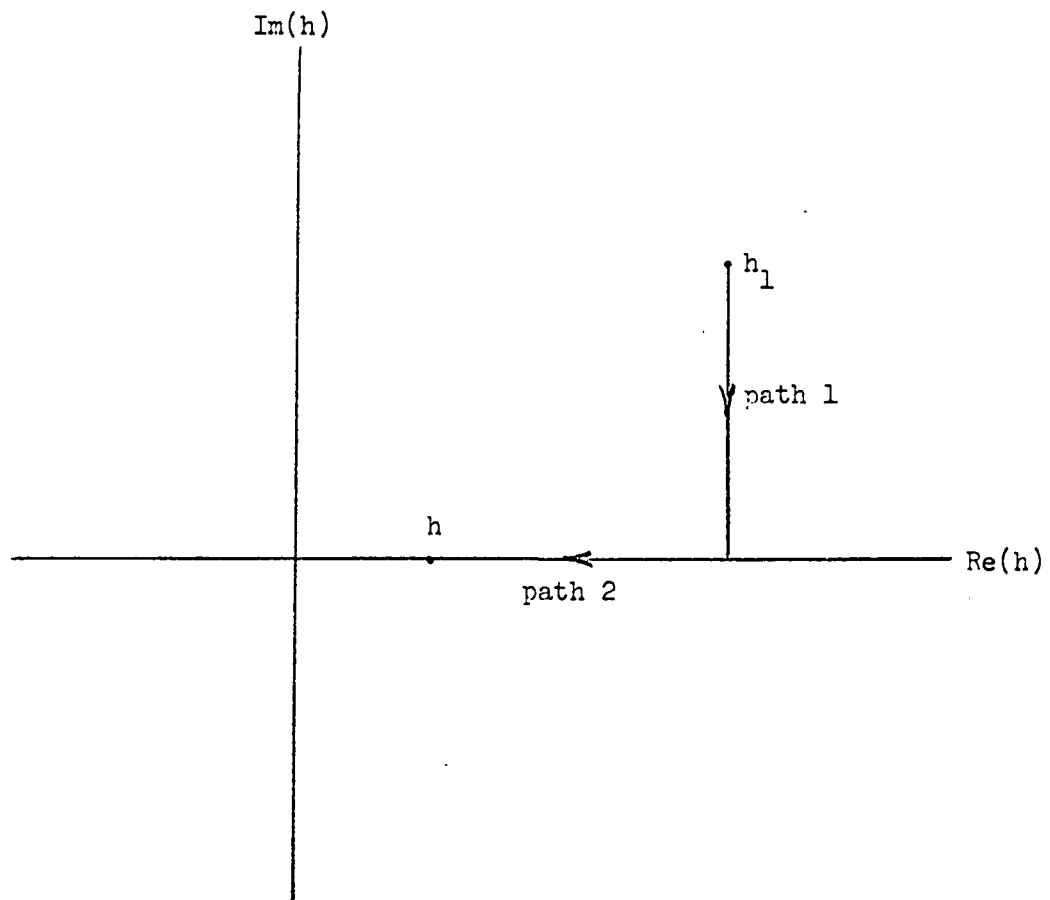


Figure 2. The path of integration for $u(h)$

If C_0 is positive, the root h_1 lies in the second quadrant of the h plane. As h takes on values along path one, $[Ah^2 + Bh + C]$ has a negative imaginary part. Then $[Ah^2 + Bh + C]^{1/2}$ moves along some contour in the second or fourth quadrant, depending on which square root is chosen. For example, suppose the root with the positive real part is taken. Then write

$$\text{path one} \int [Ah^2 + Bh + C]^{1/2} dh = \int_{C_1/B}^0 w_1(x) j dx - j \int_{C_1/B}^0 v_1(x) j dx \quad 80$$

where $w_1(x)$ and $v_1(x)$ are non-negative functions of the real variable x on the interval $[0, \frac{C_1}{B}]$. Then τ_1 and τ_2 , defined by

$$\int_0^{C_1/B} w_1(x) dx = \tau_1 \quad 81$$

and

$$\int_0^{C_1/B} v_1(x) dx = \tau_2 \quad 82$$

are non-negative numbers.

$$\text{path one} \int [Ah^2 + Bh + C]^{1/2} dh = -j\tau_1 - \tau_2 \quad 83$$

The choice of the square root with a negative real part along this path gives a similar result; with opposite sign. So one can write

$$\text{path one} \int [Ah^2 + Bh + C]^{1/2} dh = \pm (\tau_2 + j\tau_1) \quad 84$$

Along path two, h is real and $[Ah^2 + Bh + C]$ again has a negative imaginary part. Here the square root with a positive real part must be taken, since it is necessary that $u(h)$ be an increasing function of h for

large h . The root $[Ah^2 + Bh + C]^{1/2}$ therefore lies in the fourth quadrant.

$$\int_{\frac{-C_0}{B}}^0 [Ah^2 + Bh + C]^{1/2} dh = \int_{\frac{-C_0}{B}}^0 [w_2(x) - jv_2(x)] dx \quad 85$$

where $w_2(x)$ and $v_2(x)$ are non negative functions of x . Define τ_3 and τ_4 by

$$\int_{\frac{-C_0}{B}}^0 w_2(x) dx = \tau_3 \quad 86$$

$$\int_{\frac{-C_0}{B}}^0 v_2(x) dx = \tau_4, \quad 87$$

where τ_3 and τ_4 are non negative numbers if C_0 is positive as assumed. It is then true that the integration over path two gives

$$\int [Ah^2 + Bh + C]^{1/2} dh = \tau_3 - j\tau_4. \quad 88$$

The boundary condition at the surface of the perfectly conducting earth requires that

$$\pm (\tau_2 + j\tau_1) + \tau_3 - j\tau_4 = -\frac{\tau_s}{k}, \quad 89$$

as will be shown later, where τ_s is a positive real number. This equation cannot be satisfied for either choice of sign.

If C_0 is chosen to be negative, integration along path one is essentially unchanged. For h real and on path two, integration is in the negative h direction. The numbers τ_3 and τ_4 , still defined by Equations 86 and 87, are then both negative, and the above equation can be satisfied by the choice of the lower sign.

The integration necessary to form $u(h)$ is presented in Appendix C.

The result is

$$\begin{aligned} \frac{u(h)}{k_o} = & \left[\frac{2Ah+B}{4A} \right] [Ah^2 + Bh + C]^{1/2} \\ & + \frac{4AC-B^2}{8A^{3/2}} \text{Log} \frac{2A^{1/2} [Ah^2 + Bh + C]^{1/2} + 2Ah + B}{2A^{1/2} [-jC_1]^{1/2} - \frac{2AC_o}{B} + B} \\ & - \frac{B(1 - \frac{2AC_o}{B^2}) (-jC_1)^{1/2}}{4A} - j \frac{C_1^{3/2}}{2B} - \frac{5C_1^{3/2}}{8 \cdot 2B} \end{aligned} \quad 90$$

The evaluation of this function for $h = 0$ is also accomplished in Appendix C.

V. THE EIGENVALUES FOR THE PERFECTLY CONDUCTING EARTH

For the perfectly conducting earth, the eigenvalue equation is given by Equation 37 . Written in terms of the Langer solution, this becomes

$$\left[\frac{u(o)}{Q(o)} \right]^{1/2} H_{1/3}^{(1)}[u(o)] = 0 \quad 91$$

The eigenvalues v_s are those values of v for which

$$u(o) = -\tau_s \quad 92$$

where τ_s are defined by

$$H_{1/3}^{(1)}(-\tau_s) = 0 \quad 93$$

The two transcendental equations in C_0 and C_1 thus are written

$$\text{Re } u(o) = -\tau_s \quad 94$$

and

$$\text{Im } u(o) = 0 . \quad 95$$

From the results of Appendix C, the above relations become

$$\begin{aligned} & \frac{-D^2}{4A^{1/2}} + \frac{D^2 \cos^2 \frac{\Psi}{2}}{2A^{1/2}} + \frac{D^3 \cos \frac{\Psi}{2}}{B} - \frac{4D^3 \cos^3 \frac{\Psi}{2}}{3B} \\ & \frac{-C_0}{4A^{1/2}} + \frac{C_0 D \cos \frac{\Psi}{2}}{B} - \frac{C_1 D \sin \frac{\Psi}{2}}{B} = \frac{-\tau_s}{k_0} \end{aligned} \quad 96$$

and

$$\begin{aligned} & \frac{C_1}{4A^{1/2}} - \frac{C_1^{3/2}}{2\sqrt{2}B} - \frac{D^2 \sin \frac{\Psi}{2} \cos \frac{\Psi}{2}}{2A^{1/2}} - \frac{C_1 D \cos \frac{\Psi}{2}}{B} \\ & - \frac{C_0 D \sin \frac{\Psi}{2}}{B} = 0 . \end{aligned} \quad 97$$

D and Ψ are defined by the relations

$$D = (C_0^2 + C_1^2)^{1/4} = |C^{1/2}| \quad 98$$

and

$$\psi = \arctan \frac{-C_1}{C_0} . \quad 99$$

The technique for finding eigenvalues v_s for Equation 34 is to choose values for A and B which describe the profile to be analyzed, and solve Equations 96 and 97 simultaneously for C_0 and C_1 for each value τ_s . The values of C_0 and C_1 then determine α and β by Equations 71 and 72. The first four values of τ_s , to three places, have been given by Bremmer as 2.38, 5.50, 8.60, and 11.73.

Figure 3 presents the locus of points (C_0, C_1) which satisfy Equation 97, and the loci of solutions to Equation 96 for two typical values of $\frac{\tau_s}{k}$. As s takes on the values 1, 2, 3, ..., for a given frequency, the curves intersect at increasing values of C_1 . The points of intersection give the eigenvalues α_s and β_s . For a given integer s , the intersection occurs at smaller values of C_1 , as k_0 increases. The value of β increases with increasing frequency, however, due to Equation 72.

The attenuation parameter β_1 for the first and strongest mode has been plotted versus frequency for several choices of $(n_0 - 1)$ and c in Figure 4. The work of Post (28) indicates that the initial value of the gradient of refractive index is substantially more important as a parameter than the scale height. Therefore c has been given values for Figure 4 which match the initial gradient of the present model to those which are typical in the atmosphere. The upper curve gives Friedman's result for the corresponding variable. This curve is nearly independent of the parameters c and $(n_0 - 1)$, and indicates that Friedman's first mode is much

$$n_0 - 1 = 2.48 \times 10^{-4}$$

$$1/c = 5.15 \times 10^3 \text{ meters}$$

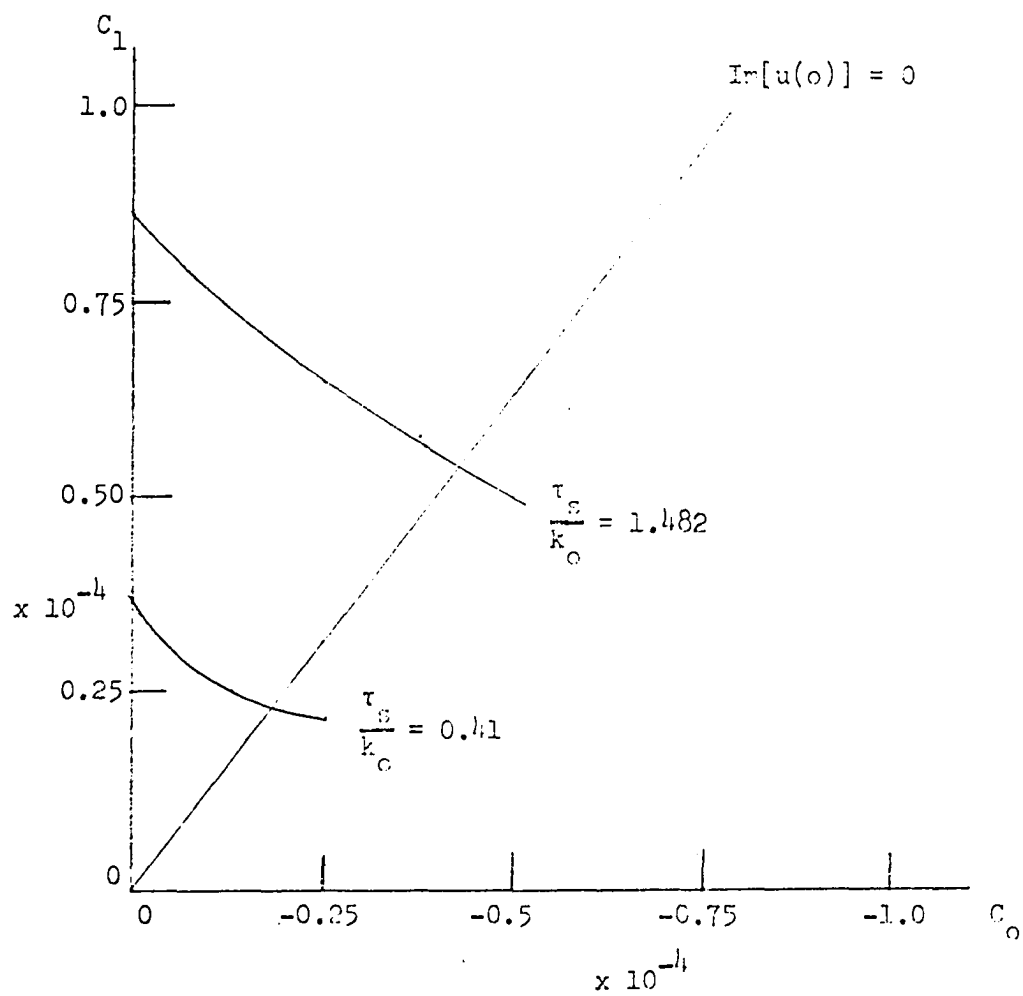


Figure 3. The loci of solutions to the transcendental equations

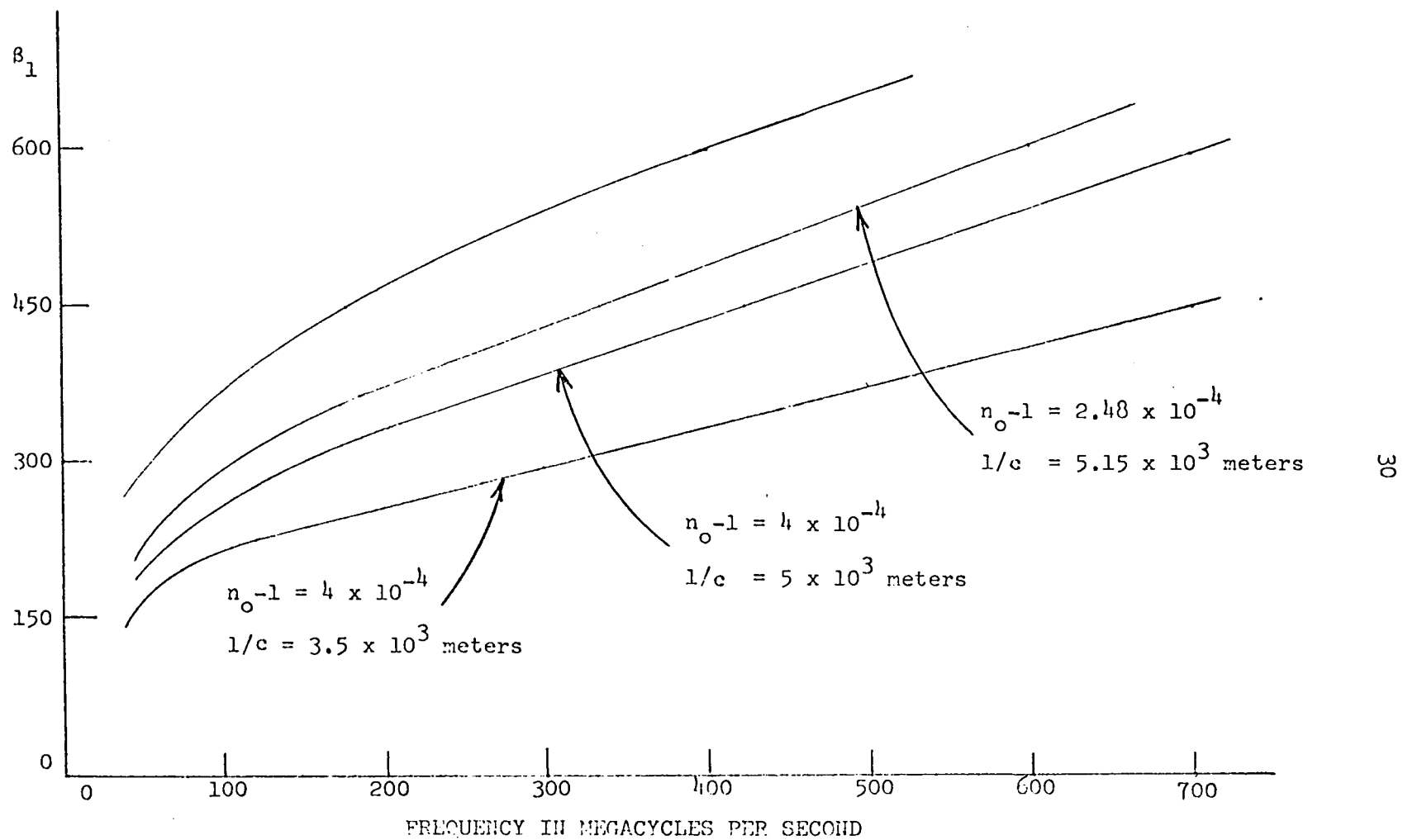


Figure 4. Attenuation of first mode versus frequency

more highly attenuated throughout the frequency range of interest.

Appendix D shows how Friedman's expression for β can be found as a special case of the present model.

It should be noted that the first mode is dependent on distance along the earth according to the function $e^{-\beta_1 \theta}$, where θ is the angle away from the source. A lower value of β_1 means a slower decay of the field strength versus distance. It can be seen from Figure 4 that an increase in $(n_o - 1)$, or in c , or both, results in a smaller β_1 and stronger diffraction fields.

VI. THE COMPLETED SERIES FOR E_ϕ

To complete the series for the electric field, the factor $W'(v_s)$ must be evaluated for this particular profile.

$$W' = \frac{\partial f_1}{\partial v} \frac{dg}{\partial r} + f_1 \frac{\partial^2 g}{\partial v \partial r} - \frac{\partial g}{\partial v} \frac{\partial f_1}{\partial r} - g \frac{\partial^2 f_1}{\partial v \partial r} \quad 100$$

If the wave number of the earth dominates the right hand side of Equation 36, it must be approximately true that

$$\frac{\partial}{\partial v} \left[\frac{\frac{dg}{dr}}{g} \right]_{r=a} = 0 \quad 101$$

Then the factor W' assumes the customary form (12)

$$W'(v_s) = \frac{\partial f_1}{\partial v} \frac{\partial f_1}{\partial r} \bigg|_{r=a} - f_1 \frac{\partial^2 f_1}{\partial v \partial r} \bigg|_{r=a} \quad 102$$

where f_1 is now treated as a function of the two independent variables v and r . The above relation is adequate unless the earth is taken to be poorly conducting.

For the case of the perfectly conducting earth, Equation 102 simplifies to

$$W'(v_s) = \frac{\partial f_1}{\partial v} \frac{\partial f_1}{\partial r} \bigg|_{\substack{v=v_s \\ r=a}} \quad 103$$

The expression 103 can be evaluated using the form for $f_1(r)$ already developed. However, the resulting series for E_ϕ does not lend itself to numerical evaluation as well as an alternate method due to Friedman (12).

Suppose that the earth is a good, but not perfect conductor. Then Equation 102 can be written as

$$W' = (f_1)^2 \frac{\partial}{\partial v} \left(\frac{\partial/\partial r f_1}{f_1} \right) \bigg|_{r=a} \quad 104$$

$$= [f_1(a)]^2 M \quad 105$$

where M is defined by the last equation. Friedman has found an approximate value for M by an involved procedure that will only be outlined here. To proceed, it must be recognized that the dominant term in the expression

$\frac{\partial f_1}{\partial r/f_1}$ involves the function

$$Z(x) = \frac{\frac{d}{dx} H_{1/3}^{(1)}(x)}{H_{1/3}^{(1)}(x)} \quad 106$$

Only this dominant term is saved. Then, by examining the equation satisfied by $H_{1/3}^{(1)}(x)$, it is found that $Z(x)$ satisfies the Ricatti equation

$$Z' + Z^2 + \frac{Z}{x} + 1 - \frac{1}{9x^3} = 0 \quad 107$$

Therefore, for large x ,

$$Z' \approx 1 - Z^2 - \frac{Z}{x} \quad 108$$

When the differentiation with respect to v is performed in Equation 104, Z' is eliminated with the above equation, and Z is eliminated by Equation 36. The result, when the earth is a good but finite conductor, is

$$M = \frac{k_1^2}{k_0} \left(1 - \frac{k^2(a)}{k_1^2} \right) \quad 109$$

where $k(a)$ is the surface wave number. The use of this approximate formula is probably justified if, as in the present study, the variation of field strength with distance is of prime importance.

The series for E_ϕ thus takes the form

$$E_\phi = \sum_s \frac{\omega \mu (\nu_s + 1/2)^2}{k_1^2 \left(1 - \frac{k^2(a)}{k_1^2}\right) r^2} \begin{bmatrix} \frac{f_1(b)}{f_1(a)} & \frac{f_1(r)}{f_1(a)} \end{bmatrix} \times \left[\frac{2j}{\pi (\nu_s + 1) \sin \theta} \right]^{1/2} \exp[j(\nu_s + \frac{1}{2})\theta] \quad 110$$

The results of the present analysis have been compared with other results by computing the ratio of field strength E_ϕ to the free space field strength which would result in the absence of the earth. This ratio has been expressed in decibels and plotted in Figures 5 and 6 for two choices of frequency. Curve 1 indicates the result computed by Post (28) for a linear atmosphere, while curve 3 indicates the first normal mode as given by Friedman. Curve 2 represents the field as written in Equation 110. Friedman's formula gives a highest attenuation throughout this range of frequencies. Above about 412 megacycles per second the field strength of the present method suffers somewhat less attenuation than that of Post (28).

Experimental field strength measurements at 190 megacycles per second are presented in Ghose and Albright (16) which were taken during periods when meteorological stations indicated a uniform linear gradient of refractive index along the 127 mile path. An attenuation of about 3/4 decibel per mile was measured. This agrees well with curve 2 in Figure 5.

In Figure 7, attenuation in decibels of the field strength is plotted versus the initial or surface value of the gradient of refractive index.

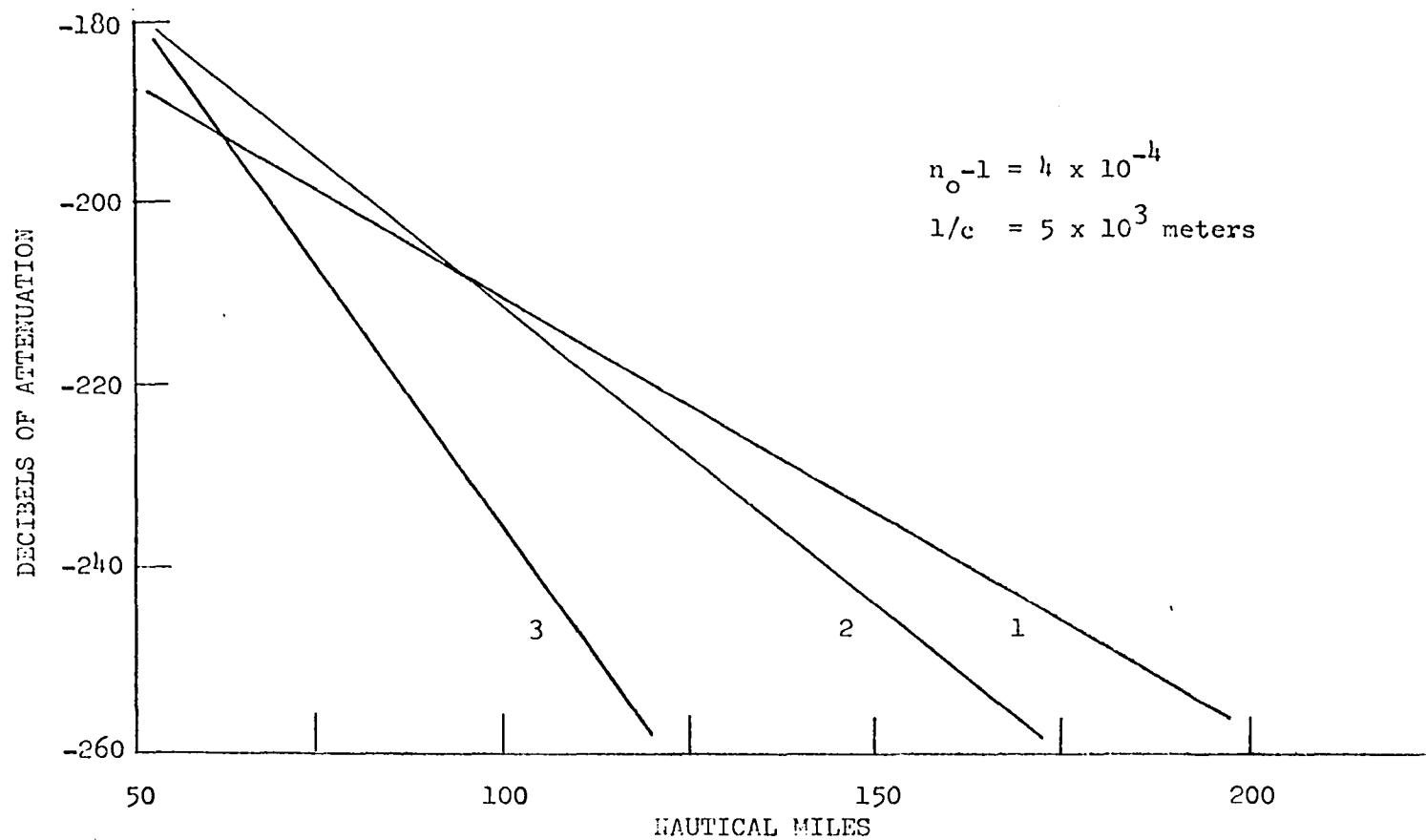


Figure 5. Decibels of attenuation versus distance at 190 megacycles per second

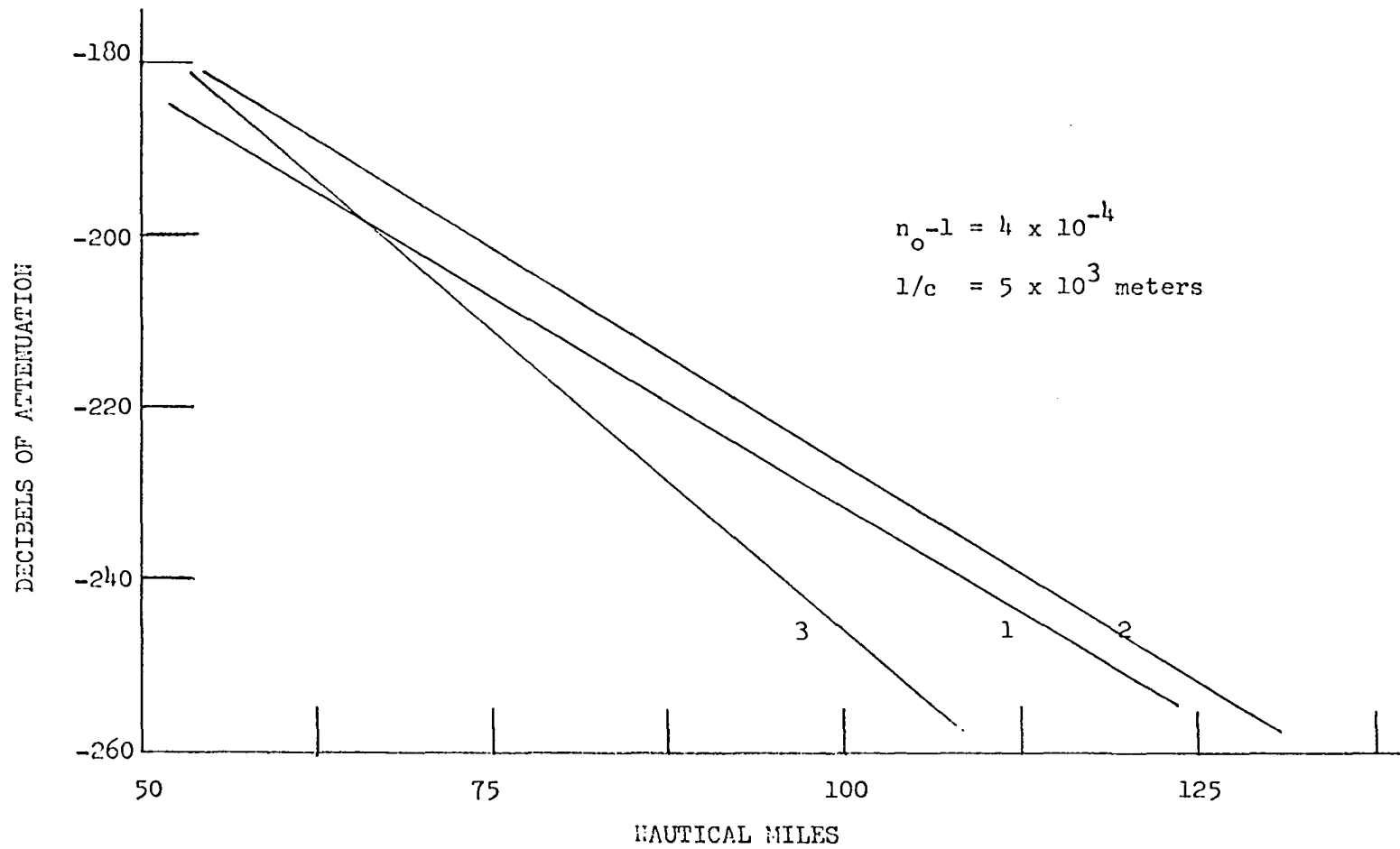


Figure 6. Decibels of attenuation versus distance at 412 megacycles per second

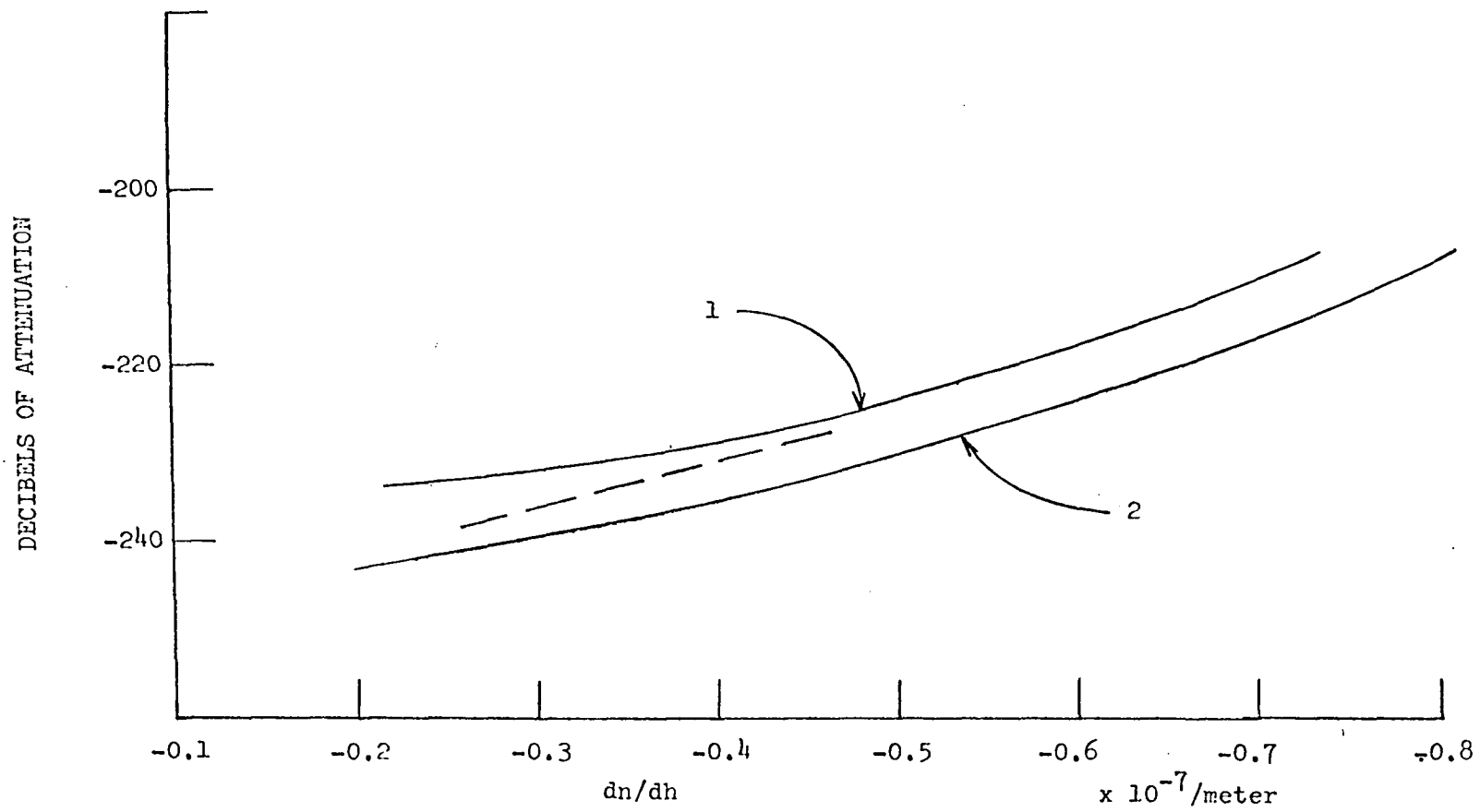


Figure 7. Decibels of attenuation at 127 miles and 190 megacycles per second versus initial gradient of refractive index

Curve 2 represents the results of the present study, while curve 1 gives the attenuation predicted by Post (28). Curve 2 is plotted for $n_o - 1 = 4 \times 10^{-4}$ and c changes as

$$\frac{dn(o)}{dh} = - \frac{2}{3}(n_o - 1)c . \quad 111$$

The selection of frequency and distance given in Figure 7 was made so that these curves could be compared with similar experimental measurements reported by Ghose and Albright (16). The dotted line in Figure 7 indicates the experimental data. Only the slope of this curve is significant since absolute field strengths were not reported.

The agreement with experiment that has been discussed is thought to be particularly significant in view of the fact that the data was taken when meteorological instruments indicated that the actual profile was comparable to this model.

VII. CONCLUSIONS

The classical or normal mode solution for electromagnetic wave propagation around a spherical earth can be found for the particular profile of refractive index which is described by a quadratic function. The inclusion of the quadratic term in the profile description alters the dependence of the Langer solution on the parameter v . The careful solution of the boundary value problem for this profile results in eigenvalues which differ significantly from the values obtained for the homogeneous atmosphere.

The dependence of the eigenvalues on frequency is essentially unchanged from the homogeneous case. The eigenvalues, and hence the diffraction field strength, show a marked dependence on the surface value of the refractive index and its gradient. An increase in the surface refractive index or in its initial gradient effectively increases the diffraction field strength. The dependence of field strength on the initial gradient of refractive index, predicted by this analysis, agrees with the results of other investigations and with corresponding experimental data. Attenuation of field strength with distance, computed by the present method, is in good agreement with experimental measurements taken under meteorological conditions which suggest a meaningful comparison.

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IX. ACKNOWLEDGEMENTS

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X. APPENDIX A

Watson (36) considered a series of the form

$$S = \sum_{n=0}^{\infty} \left(\frac{2n+1}{2} \right) a_n P_n(\cos\theta) \quad 112$$

and showed that it could be represented as the integral

$$I = \frac{1}{2\pi j} \int_C \frac{(t + 1/2) a_t P_t(-\cos\theta)}{\sin\pi t} dt \quad 113$$

where C is a contour which starts at $\infty - j\delta$ on the t plane, goes below the real axis to $t = -1/2$, and returns above the real axis to $\infty + j\delta$, as in Figure 8.

If a_t and $P_t(-\cos\theta)$ are analytic functions of t , the singularities of the integrand occur for those real values of t inside C for which $\sin\pi t$ vanishes. Then I can be written as a sum of residues

$$I = - \sum_{n=0}^{\infty} \frac{(n + 1/2) a_n P_n(-\cos\theta)}{\pi \cos n\pi} = - \frac{1}{\pi} \sum_{n=0}^{\infty} (n + 1/2) a_n P_n(\cos\theta) \quad 114$$

When this procedure is applied to Equation 29 ,

$$F_r = - \frac{\omega\mu}{4\pi} \int_C \frac{(v + 1/2)g(b)f_1(r)}{rW_v} \frac{P_v(-\cos\theta)}{\sin\pi v} dv \quad 115$$

On the part of C below the real axis, designated by C_1 in Figure 4, v is replaced by $-v-1$, so that C_1 is transformed into C_3 and

$$\begin{aligned} F_r = & - \frac{\omega\mu}{4\pi} \int_{C_3} \frac{(v + 1/2)g(b)f_1(r)}{W_{-v-1}} \frac{P_{-v-1}(-\cos\theta)}{\sin\pi v} dv \\ & - \frac{\omega\mu}{4\pi} \int_{C_2} \frac{(v + 1/2)g(b)f_1(r)}{W_v} \frac{P_v(-\cos\theta)}{\sin\pi v} dv \end{aligned} \quad 116$$

The contours C_2 and C_3 are now rotated upward until they enclose the positive half of the line given by

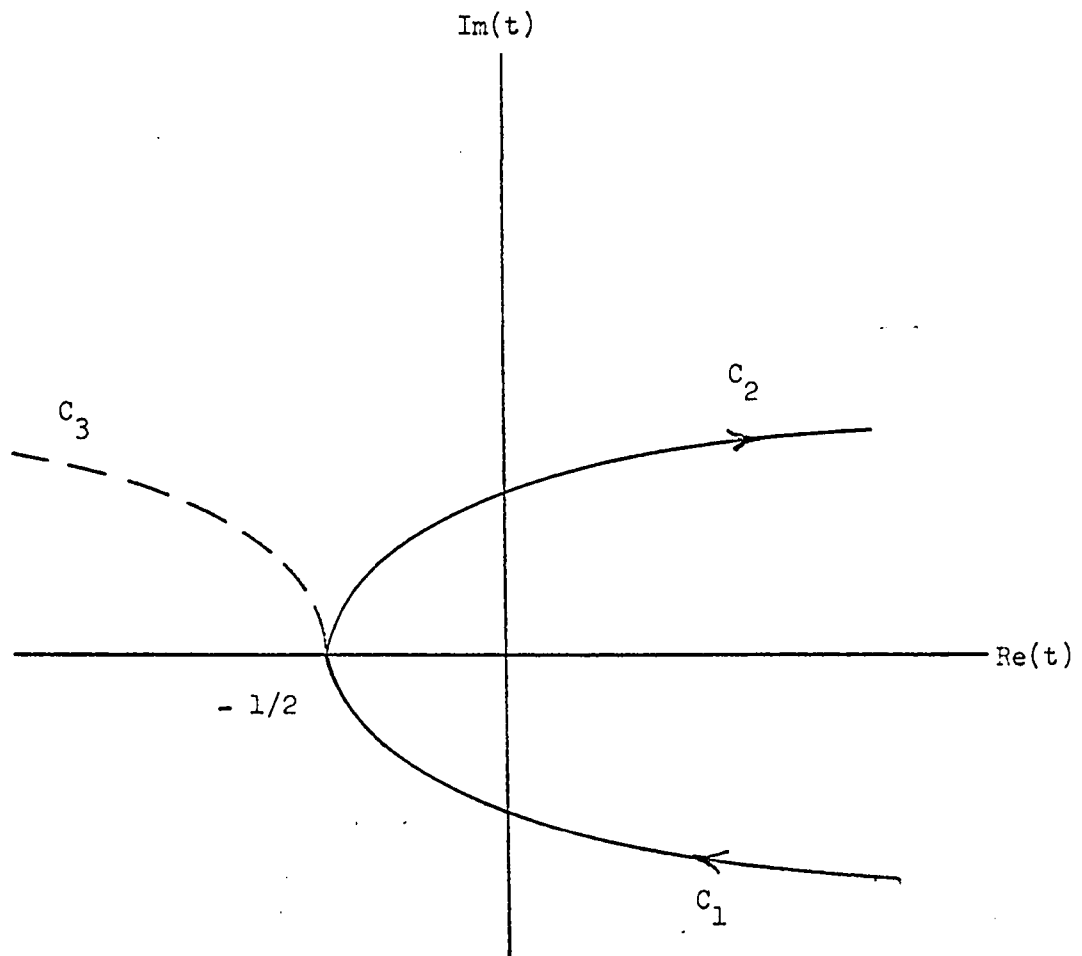


Figure 8. Contours of integration for the Watson transformation

$$\operatorname{Re} v = -\frac{1}{2} \quad 117$$

As the contour is moved in this manner, it may pass over some poles of the integrand. Then F_r can be expressed as the sum of the residues at these poles, plus two integrals over the line described above. Put $v = -\frac{1}{2} = j\tau$, so that

$$F_r = \Sigma \text{ residues} + \frac{\omega\mu}{4\pi} \int_0^\infty \frac{\tau P_{j\tau-1/2}(-\cos\theta)}{-\cosh\pi\tau} x$$

$$\left[\frac{g(b)f_1(r)}{r W_{j\tau-1/2}} - \frac{g(b)f_1(r)}{r W_{-j\tau-1/2}} \right] d\tau \quad 118$$

Friedman has shown this integral to be zero when the earth is perfectly conducting, and small enough to be dropped when the dielectric constant of the earth is large compared to the surrounding medium.

The poles of the original integrand are at those values of v for which $W = 0$. Then the residues at these points v_s can be written

$$F_r = \sum_s \omega\mu \frac{(v_s + 1/2)g(b)f_1(r) P_{v_s}(-\cos\theta)}{r W'(v_s) (\sin\pi v_s)} \quad 119$$

This brief explanation of the Watson transformation is due mainly to Bremner (5) and Friedman (12).

XI. APPENDIX B

It is desirable to present a brief discussion of the Langer asymptotic solution for a particular ordinary differential equation. It is important that the symbols used to define functions and parameters in this section should not be confused with those in the remainder of this thesis. The notation follows that of Langer (21).

The differential equation to be solved has the form

$$u''(z) + [\rho^2 \phi^2(z) - X(z)]u(z) = 0, \quad 120$$

where $X(z)$ is assumed to be an analytic function in the region of interest R . The coefficient $\phi^2(z)$ is of the form

$$\phi^2(z) = z^v \phi_1^2(z) \quad 121$$

where v is a real non negative constant and $\phi_1^2(z)$ is a single-valued analytic function bounded from zero.

Now define

$$\phi = \int_0^z \phi(z) dz \quad 122$$

where the integration is performed on a Riemann surface which is appropriate to a single-valued representation of $\phi(z)$. The integral is then independent of path and has the form

$$\phi = z^{v/2+1} \phi_1(z), \quad 123$$

with $\phi_1(z)$ single-valued and analytic in R and $\phi_1(0) \neq 0$.

Define μ by

$$\mu = \frac{1}{v+2} \quad 124$$

and $\Psi(z)$ by

$$\Psi(z) = \frac{\{\phi(z)\}^{\frac{1}{2-\mu}}}{\phi(z)^{1/2}} \quad 125$$

If

$$\xi = \rho\phi(z) \quad 126$$

and

$$y(z) = \Psi(z)\xi^{\mu}C_{\pm\mu}(\xi), \quad 127$$

where $C_{\pm\mu}$ is a cylindrical function, $y(z)$ satisfies the related equation

$$y''(z) + \{\rho^2\phi^2(z) - \omega(z)\}y(z) = 0 \quad 128$$

where $\omega(z)$ is analytic and single-valued. The technique now is to show that the solutions to 120 are expressible in terms of $y(z)$. The cylindrical functions to be used are Hankel functions, so

$$y_{k,j}(z) = \frac{\Psi(z)}{j^k A_i} \xi^{\mu} H_{\mu}^{(i)}(\xi e^{-k\pi j}), \quad k \text{ even} \quad 129$$

$$\frac{\Psi(z)}{j^k A_{3-i}} \xi^{\mu} H_{\mu}^{(3-i)}(\xi e^{-k\pi j}), \quad k \text{ odd},$$

where

$$A_i = \left(\frac{2}{\pi}\right)^{1/2} e^{\pm(\mu + \frac{1}{2})\pi i/2}. \quad 130$$

By defining

$$\theta(z) = X(z) - \omega(z), \quad 131$$

Equation 120 becomes

$$u''(z) + \{\rho^2\phi^2(z) - \omega(z)\}u(z) = \theta(z)u(z). \quad 132$$

The solutions for $u(z)$ then are

$$u_{k,i}(z) = y_{k,i}(z) + \frac{1}{2j\rho^{2\mu}} \int^z \left[y_{k,1}(z)y_{k,2}(z_1) - y_{k,2}(z)y_{k,1}(z_1) \right] \\ \times \theta(z_1)u_{k,i}(z_1)dz_1 \quad 133$$

Langer has evaluated the integral above as a series of terms involving ρ^{-n} , where n is the number of the series term. An examination of the convergence properties of the series shows that, for large ρ , the sum goes to zero. The solutions $u_{k,i}(z)$ go asymptotically to $y_{k,i}(z)$ as ρ becomes large.

XII. APPENDIX C

To complete the integration for $u(h)$ along path one, make the change of variable

$$h = \frac{-C}{B} + jx . \quad 134$$

Then one can write

$$Ah^2 + Bh + C = x^2(-A) + x(jB - j\frac{2AC}{B}) + C - C_o + \frac{AC_o^2}{B^2} . \quad 135$$

It is approximately true that

$$C - C_o + \frac{AC_o^2}{B^2} \approx -jC_1 , \quad 136$$

since C_1 is much larger than the anticipated values of $\frac{AC_o^2}{B^2}$. Now define the parameters M, N, P by the relations

$$M = A$$

$$N = -jB(1 - \frac{2AC_o}{B^2}) \quad 137$$

$$P = +jC_1$$

and note that

$$dh/dx = j . \quad 138$$

Then the integrand can be expressed as

$$[Ah^2+Bh+C]^{1/2} = j[Mx^2+Nx+P]^{1/2} . \quad 139$$

The root on the left is to have a positive real part, so that the root on the right, above, must have a negative imaginary part. The integral then becomes

$$\text{path} \int_{\text{one}} [Ah^2+Bh+C]^{1/2} dh = \int_{C_{1/B}}^0 j[Mx^2+Nx+P]^{1/2} jdx \quad 140$$

$$= \int_0^{C_{1/B}} [Mx^2+Nx+P]^{1/2} dx \quad 141$$

It can be determined by differentiation that

$$\begin{aligned} \int [Mx^2+Nx+P]^{1/2} dx = \\ \left[\frac{2Mx+N}{4M} \right] [Mx^2+Nx+P]^{1/2} \\ + \frac{4MP-N^2}{8M^{3/2}} \text{Log}[2M^{1/2}(Mx^2+Nx+P)^{1/2} + 2Mx + N] . \end{aligned} \quad 142$$

In this analysis, Log will designate the logarithm to the base e of a complex number, while log will indicate the logarithm to the base e of a real number.

At the upper limit of expression 141, Mx^2+Nx+P vanishes, by definition. The integral in Equation 141 then becomes

$$\begin{aligned} \int_0^{C_{1/B}} [Mx^2+Nx+P]^{1/2} dx = \left(\frac{4MP-N^2}{8M^{3/2}} \right) \text{Log}\left[\frac{2MC_1}{B} + N \right] \\ - \frac{NP^{1/2}}{4M} - \left(\frac{4MP-N^2}{8M^{3/2}} \right) \text{Log}[2M^{1/2}P^{1/2}+N] , \end{aligned} \quad 143$$

where $P^{1/2}$ must have a positive real part.

$$[jC_1]^{1/2} = C_1^{1/2} \left(\frac{-1}{\sqrt{2}} - j \frac{1}{\sqrt{2}} \right) \quad 144$$

If the logarithm functions are combined as the logarithm of a quotient, the numerator and denominator of the argument can be divided by N. The

integral can then be written as

$$\begin{aligned} \int_0^{C_1/B} [Mx^2 + Nx + P]^{1/2} dx &= \left[\frac{4MP - N^2}{8M^{3/2}} \right] \left\{ \text{Log} \left[1 + \frac{2MC_1}{NB} \right] \right. \\ &\quad \left. - \text{Log} \left[\frac{2M^{1/2} P^{1/2}}{N} + 1 \right] \right\} - \frac{NP^{1/2}}{4M} \end{aligned} \quad 145$$

$$- \frac{NP^{1/2}}{4M} = -j \frac{BC_1^{1/2}}{4 \cdot 2A} + \frac{BC_1^{1/2}}{4 \cdot 2A} + j \frac{C_0 C_1^{1/2}}{2 \cdot 2B} - \frac{C_0 C_1^{1/2}}{2 \cdot 2B} \quad 146$$

Using Equations 137 ,

$$\begin{aligned} \frac{4MP - N^2}{8M^{3/2}} &= \frac{j4AC_1 + B^2(1 - \frac{4AC_0}{B^2})}{8A^{3/2}} \\ &= j \frac{C_1}{2A^{1/2}} + \frac{B^2}{8A^{3/2}} - \frac{C_0}{2A^{1/2}} \end{aligned} \quad 147$$

To evaluate the first logarithm in Equation 145 , write

$$\frac{2MC_1}{BN} = j \frac{2AC_1}{B^2} \quad 148$$

Now note that, if x and y take real values,

$$\begin{aligned} \text{Log}[1 + x + jy] &= \frac{1}{2} \log|1 + x + jy|^2 \\ &\quad + j \arctan \frac{y}{1+x} \end{aligned} \quad 149$$

$$= \frac{1}{2} \log[1 + 2x + x^2 + y^2] + j \arctan (y - yx), \quad 150$$

where only two terms have been included in the argument of the arc tan

function. Therefore, it is true that

$$\text{Log} \left[1 + \frac{2MC_1}{NB} \right] = \frac{1}{2} \log \left(1 + \frac{4A^2 C_1^2}{B^4} \right) + j \arctan \frac{2AC_1}{B^2} . \quad 151$$

Using the series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots \quad 152$$

for $x < 1$, it is found that

$$\log\left(1 + \frac{4A^2 C_1^2}{B^4}\right) = \frac{4A^2 C_1^2}{B^4} + \text{higher terms} \quad 153$$

It can be determined by investigating the orders of magnitude of the terms involved here that quantities involving B to the fourth power, or higher, in the denominator, can be neglected. In addition, the arguments of the arc tan functions in this appendix are generally less than 0.05, so from the series

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots \quad 154$$

it can be seen that $\arctan \theta$ can be approximated by θ , with negligible error. Therefore one can write

$$\text{Log}\left[1 + \frac{2MC_1}{NB}\right] = j \frac{2AC_1}{B^2} . \quad 155$$

To evaluate the second logarithm in Equation 145, use the relation

$$\frac{2M^{1/2}P^{1/2}}{N} = -j \frac{\sqrt{2AC_1}}{B} - j \frac{2AC_0 \sqrt{2AC_1}}{B^3} + \frac{\sqrt{2AC_1}}{B} + \frac{2AC_0 \sqrt{2AC_1}}{B^3} \quad 156$$

Using the relation 150 for the above expression, let

$$\begin{aligned} x &= \frac{\sqrt{2AC_1}}{B} + \frac{2AC_0 \sqrt{2AC_1}}{B^3} \\ x^2 &= \frac{2AC_1}{B^2} \\ y &= -\frac{\sqrt{2AC_1}}{B} - \frac{2AC_0 \sqrt{2AC_1}}{B^3} \\ y^2 &= x^2 = -yx . \end{aligned} \quad 157$$

The result is

$$\begin{aligned} \text{Log}\left[1 + \frac{2M^{1/2}P^{1/2}}{N}\right] &= \log\left(1 + \frac{2\sqrt{2AC_1}}{B} + \frac{4AC_0\sqrt{2AC_1}}{B^3} + \frac{4AC_1}{B^2}\right) \\ &+ j\left(-\frac{\sqrt{2AC_1}}{B} - \frac{2AC_0\sqrt{2AC_1}}{B^3} + \frac{2AC_1}{B^2}\right) \end{aligned} \quad 158$$

Equation 152 can be used to evaluate the log function, to yield

$$\begin{aligned} \text{Log}\left[1 + \frac{2M^{1/2}P^{1/2}}{N}\right] &= \frac{\sqrt{2AC_1}}{B} + \frac{2AC_0\sqrt{2AC_1}}{B^3} - \frac{3AC_1\sqrt{2AC_1}}{2B^3} \\ &+ j\left(-\frac{\sqrt{2AC_1}}{B} - \frac{2AC_0\sqrt{2AC_1}}{B^3} + \frac{2AC_1}{B^2}\right) \end{aligned} \quad 159$$

Now define Δ by

$$\Delta = \text{Log}\left[1 + \frac{2MC_1}{BN}\right] - \text{Log}\left[1 + \frac{2M^{1/2}P^{1/2}}{N}\right] \quad 160$$

Combining Equations 159 and 155 gives

$$\begin{aligned} \Delta &= -\frac{\sqrt{2AC_1}}{B} - \frac{2AC_0\sqrt{2AC_1}}{B^3} + \frac{3AC_1\sqrt{2AC_1}}{2B^3} \\ &+ j\frac{\sqrt{2AC_1}}{B} + j\frac{2AC_0\sqrt{2AC_1}}{B^3} \end{aligned} \quad 161$$

$$\begin{aligned} \left[\frac{4MP-N^2}{8M^{3/2}}\right] \Delta &= \left(j\frac{C_1}{2A^{1/2}} - \frac{C_0}{2A^{1/2}}\right)\left(-\frac{\sqrt{2AC_1}}{B} + j\frac{\sqrt{2AC_1}}{B}\right) \\ &+ \frac{B^2}{8A^{3/2}}\left(-\frac{\sqrt{2AC_1}}{B} - \frac{2AC_0\sqrt{2AC_1}}{B^3} + \frac{3AC_1\sqrt{2AC_1}}{2B^3} + j\frac{\sqrt{2AC_1}}{B} \right. \\ &\quad \left. + j\frac{2AC_0\sqrt{2AC_1}}{B^3}\right) \end{aligned} \quad 162$$

In the above relation, the smaller terms have been deleted from Δ in the first product. Combining Equations 162 and 146 in 145, all but two terms cancel to give

$$\text{path one} \int [Ah^2+Bh+C]^{1/2} dh = -j \frac{C_1^{3/2}}{\sqrt{2B}} - \frac{5C_1^{3/2}}{8\sqrt{2B}} \quad 163$$

The integration along the real axis (path two) can be accomplished in the same manner. Using expression 142, one obtains

$$\begin{aligned} \int_{\frac{-C_0}{B}}^h [Ah^2+Bh+C]^{1/2} dh &= \left[\frac{2Ah+B}{4A} \right] [Ah^2+Bh+C]^{1/2} \\ &+ \left(\frac{4AC-B^2}{8A^{3/2}} \right) \text{Log} \left[\frac{2A^{1/2}(Ah^2+Bh+C)^{1/2} + 2Ah+B}{2A^{1/2}(-jC_1)^{1/2} - \frac{2AC_0}{B} + B} \right] \\ &- \frac{B(1 - \frac{2AC_0}{B^2})(-jC_1)^{1/2}}{4A} \quad 164 \end{aligned}$$

This function will now be evaluated for $h = 0$.

$$\begin{aligned} \int_{\frac{-C_0}{B}}^0 [Ah^2+Bh+C]^{1/2} dh &= \frac{BC_1^{1/2}}{4A} - \frac{B(1 - \frac{2AC_0}{B^2})(-jC_1)^{1/2}}{4A} \\ &+ \left(\frac{4AC-B^2}{8A^{3/2}} \right) \text{Log} \left[\frac{2A^{1/2}C_1^{1/2}}{B} + 1 \right] - \left(\frac{4AC-B^2}{8A^{3/2}} \right) \text{Log} \left[1 - \frac{2AC_0}{B^2} + \frac{2A^{1/2}}{B} \right. \\ &\quad \left. (-jC_1)^{1/2} \right] \quad 165 \end{aligned}$$

Since $[Ah^2+Bh+C]^{1/2}$ is to have a positive real part, $C^{1/2}$ should also have a positive real part wherever it appears in Equation 165. This root of C can be expressed as

$$C^{1/2} = D(\cos \frac{\psi}{2} - j \sin \frac{\psi}{2}) \quad 166$$

where D is defined by Equation 98 and ψ is defined by

$$\psi = \arctan \frac{-C_1}{C_0} . \quad 167$$

Therefore

$$\frac{BC}{4A}^{1/2} = \frac{BD \cos \frac{\psi}{2}}{4A} - \frac{BD \sin \frac{\psi}{2}}{4A} \quad 168$$

$$\begin{aligned} -\frac{B}{4A} \left(1 - \frac{2AC_0}{B^2}\right) (-jC_1)^{1/2} &= \frac{-B(1 - \frac{2AC_0}{B^2})C_1^{1/2}(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}})}{4A} \\ &= \frac{-BC_1^{1/2}}{4\sqrt{2}A} + j \frac{BC_1^{1/2}}{4\sqrt{2}A} + \frac{C_0C_1^{1/2}}{2\sqrt{2}B} - j \frac{C_0C_1^{1/2}}{2\sqrt{2}B} \end{aligned} \quad 169$$

To evaluate the first logarithm, write

$$\frac{2A^{1/2}C^{1/2}}{B} = \frac{2A^{1/2}D \cos \frac{\psi}{2}}{B} - j \frac{2A^{1/2}D \sin \frac{\psi}{2}}{B} \quad 170$$

Substitute this into the form 150 to get

$$\begin{aligned} \log\left[1 + \frac{2A^{1/2}C^{1/2}}{B}\right] &= \frac{1}{2} \log\left(1 + \frac{4A^{1/2}D \cos \frac{\psi}{2}}{B} + \frac{4AD^2}{B^2}\right) \\ &+ j \left(-\frac{2A^{1/2}D \sin \frac{\psi}{2}}{B} + \frac{4AD^2}{B^2} \sin \frac{\psi}{2} \cos \frac{\psi}{2}\right), \end{aligned} \quad 171$$

where

$$\begin{aligned} \log\left(1 + \frac{4A^{1/2}D \cos \frac{\psi}{2}}{B} + \frac{4AD^2}{B^2}\right) &= \frac{4A^{1/2}D \cos \frac{\psi}{2}}{B} + \frac{4AD^2}{B^2} \\ &- \frac{8AD^2 \cos^2 \frac{\psi}{2}}{B^2} - \frac{16A^{3/2}D^3 \cos \frac{\psi}{2}}{B^3} + \frac{64}{3} \frac{A^{3/2}D^3 \cos^3 \frac{\psi}{2}}{B^3}. \end{aligned} \quad 172$$

Second logarithm becomes

$$\begin{aligned}
 \text{Log}\left[1 - \frac{2AC_0}{B^2} + \frac{\sqrt{2AC_1}}{B} - j \frac{\sqrt{2AC_1}}{B}\right] &= \frac{1}{2} \log\left(1 - \frac{4AC_0}{B^2}\right. \\
 &+ \frac{2\sqrt{2AC_1}}{B} + \frac{4AC_1}{B^2} - \frac{4AC_0\sqrt{2AC_1}}{B^3}\Big) \\
 &+ j\left(-\frac{\sqrt{2AC_1}}{B} - \frac{2AC_0\sqrt{2AC_1}}{B^3} + \frac{2AC_1}{B^2} - \frac{2AC_1\sqrt{2AC_1}}{B^3}\right)
 \end{aligned} \tag{173}$$

$$\begin{aligned}
 &= \frac{1}{2}\left(\frac{-4AC_0}{B^2} + \frac{2\sqrt{2AC_1}}{B} + \frac{4AC_0\sqrt{2AC_1}}{B^3} - \frac{3AC_1\sqrt{2AC_1}}{B^3}\right) \\
 &+ j\left(-\frac{\sqrt{2AC_1}}{B} - \frac{2AC_0\sqrt{2AC_1}}{B^3} + \frac{2AC_1}{B^2} - \frac{2AC_1\sqrt{2AC_1}}{B^3}\right)
 \end{aligned} \tag{174}$$

Now note that

$$\frac{4AC-B^2}{8A^{3/2}} = \frac{C_0}{2A^{1/2}} - j \frac{C_1}{2A^{1/2}} - \frac{B^2}{8A^{3/2}} \tag{175}$$

Substituting Equations 168 , 169 , 171 , 174 , and 175 in the Equation 165 yields

$$\begin{aligned}
 \text{path} \int_{\text{two}} [Ah^2+Bh+C]^{1/2} dh &= j \left[\frac{C_1}{4A^{1/2}} + \frac{C_1^{3/2}}{2\sqrt{2}B} - \frac{D^2 \sin \frac{\psi}{2} \cos \frac{\psi}{2}}{2A^{1/2}} \right. \\
 &- \frac{C_1 D \cos \frac{\psi}{2}}{B} - \frac{C_0 D \sin \frac{\psi}{2}}{B} \Big] + \left[\frac{D^2 \cos^2 \frac{\psi}{2}}{2A^{1/2}} - \frac{D^2}{4A^{1/2}} + \frac{D^3 \cos \frac{\psi}{2}}{B} \right. \\
 &- \frac{4D^3 \cos^3 \frac{\psi}{2}}{3B} - \frac{C_0}{4A^{1/2}} + \frac{5C_1^{3/2}}{8\sqrt{2}B} + \frac{C_0 D \cos \frac{\psi}{2}}{B} - \frac{C_1 D \sin \frac{\psi}{2}}{B} \Big].
 \end{aligned} \tag{176}$$

XIII. APPENDIX D

Friedman (12) has obtained, by a series of approximations and a single real integration, the relation

$$v_s = k_o a + \frac{1}{2}(k_o a)^{1/3}(3\tau_s)^{2/3} e^{j\pi/3} \quad 177$$

The corresponding relation for β then becomes

$$\beta_s = \frac{\sqrt{3}}{4} (k_o a)^{1/3} (3\tau_s)^{2/3} \quad 178$$

This compares with the results of the present work if A is set equal to zero before the integration is performed.

$$\int [Bh+C]^{1/2} dh = \frac{2}{3B} [Bh+C]^{3/2} \quad 179$$

Along path one, let

$$h = \frac{-C}{B} + jx \quad 180$$

so that

$$\frac{dh}{dx} = j \quad 181$$

Then

$$\begin{aligned} \text{path one } \int [Bh+C]^{1/2} dh &= j \int_{C_{1/B}}^0 [jBx-jC_1]^{1/2} dx \\ &= \frac{2}{3B} [jBx-jC_1]^{3/2} \bigg|_{C_{1/B}}^0 \\ &= \frac{2}{3B} [-jC_1]^{3/2} \end{aligned} \quad 182$$

Integration along the real axis gives the result

$$\frac{-C_o}{B} \int_0^{\infty} [Bh+C]^{1/2} dh = \frac{2}{3B} [C^{3/2} - (-jC_1)^{3/2}] \quad 183$$

Combining Equations 182 and 183 yields the eigenvalue equation

$$\frac{2}{3B} C^{3/2} = \frac{-\tau_s}{k} \quad 184$$

In order that $C^{3/2}$ be real, one must require that

$$\psi = \frac{2\pi}{3} \quad 185$$

and

$$-C_o = \frac{C_1}{2} . \quad 186$$

Then

$$|C| = \frac{5}{2} C_1 . \quad 187$$

From Equation 184

$$|C| = \left(\frac{3B\tau_s}{2k_o} \right)^{2/3} \quad 188$$

so

$$C_1 = \frac{2}{5} \left(\frac{3B\tau_s}{2k_o} \right)^{2/3} \quad 189$$

For the free space atmosphere,

$$B = \frac{2}{a} \quad 190$$

so

$$C_1 = \frac{2}{5} \left(\frac{3\tau_s}{k_o a} \right)^{2/3} = \frac{2B}{k_o a} \quad 191$$

The final result is

$$\beta = \frac{1}{5} (k_o a)^{1/3} (3\tau_s)^{2/3} \quad 192$$

In comparing Equations 192 and 178 , it seems that Friedman's result is equivalent to using a two term expansion for $Y(h)$ in the height gain differential equation, instead of the three term expansion used in this analysis. The solution for the height gain function in the case of a linear function $Y(h)$ can be obtained rigorously, without recourse to the Langer asymptotic method, by making a change of variable to put the differential equation into the form of Stokes equation. The solution then takes the form of the Airy integral, as discussed by Budden (7) and others.