Fourier bases on the skewed Sierpinski gasket

by

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DEDICATION

For Dr. Lynne Butler, who invited me back, and Drs. Ruth Haas and Jim Henle, who brought me here.

And for Dr. Charles V Stephenson, in whose footsteps I belatedly follow.

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ABSTRACT

First, we demonstrate the construction of Fourier bases for fractal approximations via a construction analogous to the Fast Fourier Transformation. Then, we construct several full Fourier bases for the skewed Sierpinski gasket and related fractals.

CHAPTER 1. OVERVIEW

1.1 Introduction

We begin our investigations with the fractal $\mathcal{S}$, the invariant set of the iterated function system $\psi_0(x, y) = \frac{1}{3}(x, y)$, $\psi_1(x, y) = \frac{1}{3}(x + 2, y)$, $\psi_2(x, y) = \frac{1}{3}(x, y + 2)$. By Hutchinson's theorem, there exists an invariant measure ν_3 supported on the invariant set $\mathcal{S}$.

The set $\mathcal{S}$ takes the form of a skewed Sierpinski gasket; it is the set of points $(x, y) \in \mathbb{R}^2$ for which x, y, and $x + y$ are all in the standard middle-thirds Cantor set. This was the reason for the choice of fractal; while it has been shown that the middle-thirds Cantor set does not have an orthonormal Fourier basis, the question of whether it has a Fourier frame is still open, and we hope that investigations on related sets such as $\mathcal{S}$ will lead to such a frame. It is a difficult open question, but also one that has a close connection to undergraduate analysis topics via the Cantor set.

For $\mathcal{S}$, we first choose a dual iterated function system $\rho_0(x, y) = 3(x, y)$, $\rho_1(x, y) = 3(x, y) + (1, 2)$, $\rho_2(x, y) = 3(x, y) + (2, 1)$. This iterated function system generates the frequencies $\mathcal{R}_n$ of an orthogonal basis of exponentials on finite approximations $\mathcal{S}_n$ of $\mathcal{S}$.

We proved that the matrix M_n, where $(M_n)_{j,k} = \exp(2\pi i R_j \cdot S_k)$ with $(R_j)_{j=0}^{n-1}$ and $(S_k)_{k=0}^{n-1}$ orderings of $\mathcal{R}_n$ and $\mathcal{S}_n$ respectively, is a block matrix in the form of Diţă's construction for generating large Hadamard matrices. In particular, it is an orthogonal basis matrix of exponentials for S_n.

The blocks of M_n can be expressed in terms of M_1, M_{n-1} and diagonal matrices. The rows and columns of M_n^{-1} can be permuted into a block matrix of the same form. These block form allow a Fast Fourier Transform-type algorithm to quickly transform a function on $\mathcal{S}_n$, written as a column vector $\vec{v}$, into its representation in terms of the orthogonal basis of exponentials that are the rows of M_n. The block form reduces the matrix computation complexity to $O(N \log N)$, as in the case of the classical Fast Fourier Transformation.

This construction extends to a Fractal Fast Fourier Transformation in the general case of finite approximations of a fractal generated by an iterated function system of affine transformations on $\mathbb{R}^d$. With certain restrictions on the choice of affine transformations for the fractal and signal approximations, the construction generalizes fully with the same reduction in complexity.

After this success, I returned to $\mathcal{S}$, aiming to demonstrate that the infinite set $\mathcal{R}$ generated by $\{\rho_j\}$ was the frequency set for a Fourier basis on $L^2(\nu_3)$ of the form $\{e_r(x) = e^{2\pi i r \cdot x} | r \in \mathcal{R}\}$. I determined that the exponentials were orthogonal; however, the completeness failed. In fact, I could show that the set $\mathcal{R}$ was *not* a complete Fourier basis.

As it turned out there was a simple fix: following the work of Dorin Ervin Dutkay and Palle E. T. Jorgensen in (7), we took two of the "bad points" and added them into the set $\mathcal{R}$. Our original set of frequencies was the simple application of $\{\rho_j\}$ to the origin; to construct a set of Fourier frequencies, it was enough to apply them to two more points: in this case $(-1, -1/2)$ and $(-1/2, -1)$. In that way we constructed our first full Fourier basis for $\mathcal{S}$.

Once we had that construction, I was able to quickly construct two more Fourier bases for $\mathcal{S}$, based on different choices of dual iterated function systems for $\{\psi_j\}$. The most interesting such basis was the one constructed using $\rho_0 = 3(x, y)$, $\rho_1 = 3(x, y) + (2, 1)$ and $\rho_2(x, y) = 3(x, y) + (4, 2)$. The vectors $(2, 1)$ and $(4, 2)$ are linearly dependent, and in fact, the Fourier basis generated by these vectors has the form $\{e_{(u,u/2)}(x) | u \in \mathbb{Z}\}$.

We then extended our results to fractals related to $\mathcal{S}$ by rotations and reflections. It turns out that $\mathcal{S}$ shares some of its spectra with its with its reflection across the line $y = x$, suggesting possible future work on the union of these two fractals. The question also remains of how many possible Fourier bases there are for $\mathcal{S}$; so far, there have been the three we constructed with our Hadamard duals, as well as one constructed by Dutkay and Jorgensen in (9).

CHAPTER 2. REVIEW OF LITERATURE

2.1 The Fast Fourier Transform

The standard Fast Fourier Transform is a method of reducing the complexity of the Discrete Fourier Transform on 2^N equally-spaced data points on $[0,1)$. A function f on this discrete space can be viewed as a vector of length 2^N, and the Discrete Fourier Transform multiplies this vector by the change of basis matrix:

$$\mathcal{F}_N = (e^{-2\pi i \frac{jk}{2^N}})_{jk}, \quad 0 \le j, k < 2^N.$$

In general, multiplying a vector of length 2^N by a $2^N \times 2^N$ matrix requires $O(2^{2N})$ operations. The Fast Fourier Transform was developed first by Gauss in 1805 and then generalized by Cooley and Tukey in 1965, in order to reduce the complexity of this multiplication by permuting the Discrete Fourier Transform matrix into blocks and making use of symmetries in the complex matrix entries (3), (16).

In particular, the Fast Fourier Transform permutes the columns of $\mathcal{F}_N$ by the permutation:

$$\sigma(k) = \begin{cases} 2k & 0 \le k < 2^{N-1}, \\ 2k+1 & 2^{N-1} \le k < 2^N. \end{cases}$$

This gives $\mathcal{F}_N$ the block form:

$$\mathcal{F}_N P = \begin{pmatrix} \mathcal{F}_{N-1} & D\mathcal{F}_{N-1} \\ \mathcal{F}_{N-1} & -D\mathcal{F}_{N-1} \end{pmatrix} \tag{2.1}$$

where D is the $2^{N-1} \times 2^{N-1}$ diagonal matrix with kth diagonal entry $e^{2\pi i k/2^N}$ (21).

Multiplying our vector f by this block form is significantly easier than multiplying by the full matrix $\mathcal{F}_N$. And since $\mathcal{F}_{N-1}$ can be permuted in the same manner, and so forth down to F_1, the overall computational complexity is reduced down to $O(N \cdot 2^N)$.

2.1.1 Diţă's construction

A block matrix form strikingly similar to that of the Fast Fourier Transform shows up in the work of P. Diţă, who studies complex Hadamard matrices. (4; 22). A complex Hadamard matrix is an $N \times N$ matrix H, all of whose entries have norm one, and with the property that $H^*H = NI_N$. The Discrete Fourier Transform and Fast Fourier Transform Matrices are examples of Hadamard Matrices.

Diţă developed a way of constructing large Hadamard matrices from smaller ones that is, in a sense, the reverse of the Fast Fourier Transform. In particular, if A is a $K \times K$ Hadamard matrix, B is an $M \times M$ Hadamard matrix, and $E_1, \ldots, E_{K-1}$ are $M \times M$ unitary diagonal matrices, then the $KM \times KM$ block matrix H is a Hadamard matrix:

$$H = \begin{pmatrix} a_{00}B & a_{01}E_1B & \ldots & a_{0(K-1)}E_{K-1}B \\ a_{10}B & a_{11}E_1B & \ldots & a_{1(K-1)}E_{K-1}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{(K-1)0}B & a_{(K-1)1}E_1B & \ldots & a_{(K-1)(K-1)}E_{K-1}B \end{pmatrix}. \tag{2.2}$$

To show that H is Hadamard, first notice that, for $C = A^*$,

$$H^* = \begin{pmatrix} c_{00}B^* & c_{01}B^* & \ldots & c_{0(K-1)}B^* \\ c_{10}B^*E_1^* & c_{11}B^*E_1^* & \ldots & c_{1(K-1)}B^*E_1^* \\ \vdots & \vdots & \ddots & \vdots \\ c_{(K-1)0}B^*E_{K-1}^* & c_{1(K-1)}B^*E_{K-1}^* & \ldots & c_{(K-1)(K-1)}B^*E_{K-1}^* \end{pmatrix}. \tag{2.3}$$

Proposition 2.1.1. *$H^*H = (KM)I_{KM}$.*

Proof. Let G be the block matrix in Equation (2.3), and let $E_0 = I_M$. Note that the product of H and G will have a block form. Multiplying the j-th row of H with the ℓ-th column of G, we obtain that the j,ℓ block of HG is:

$$\sum_{k=0}^{K-1}(a_{jk}E_kB)(c_{k\ell}B^*E_k^*) = \sum_{k=0}^{K-1} a_{jk}c_{k\ell}(M)I_M.$$

Since $\sum_{k=0}^{K-1} a_{jk}c_{k\ell} = K\delta_{j,\ell}$, we obtain $HG = (KM)I_{KM}$. □

Let $\vec{v}$ be a vector of length KM. Consider $H\vec{v}$ where H is the block matrix as in Equation (2.2). We divide the vector $\vec{v}$ into K vectors of length M as follows:

$$\vec{v} = \begin{pmatrix} \vec{v}_0 \\ \vec{v}_1 \\ \vdots \\ \vec{v}_{K-1} \end{pmatrix}.$$

Then the matrix multiplication $H\vec{v}$ can be reduced in complexity, since

$$H\vec{v} = \begin{pmatrix} \sum_{j=0}^{K-1} a_{0j}E_jB\vec{v}_0 \\ \sum_{j=0}^{K-1} a_{1j}E_jB\vec{v}_1 \\ \vdots \\ \sum_{j=0}^{K-1} a_{(K-1)j}E_jB\vec{v}_{K-1} \end{pmatrix}.$$

Let $\mathcal{O}_M$ be the number of operations required to multiply a vector $\vec{w}$ of length M by the matrix B. The total number of operations required for each component of $H\vec{v}$ is $\mathcal{O}_M + M(K-1) + MK$ multiplications and $M(K-1)$ additions. The total number of operations for $H\vec{v}$ is then $K\mathcal{O}_M + 3MK^2 - 2MK$. We have just established the following proposition.

Proposition 2.1.2. *The product $H\vec{v}$ requires at most $K\mathcal{O}_M + 3MK^2 - 2MK$ operations.*

Since $\mathcal{O}_M = O(M^2)$, we obtain that the computational complexity of H is $O(M^2K+MK^2)$, whereas for a generic $KM \times KM$ matrix, the computational complexity is $O(K^2M^2)$. Thus, the block form of H reduces the computational complexity of the matrix multiplication.

Notice that the Fast Fourier Transform matrix is an example of Diţă's construction, with $B = \mathcal{F}_{N-1}$, $E_1 = D$, and

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

which is also the two-by-two Discrete Fourier Transform Matrix $\mathcal{F}_2$.

2.2 Hutchinson's fractal measures

In 1981, John E. Hutchinson defined the general framework for the type of fractal we discuss here, that is, the compact invariant set of a set of contraction maps on $\mathbb{R}^n$.

Definition 2.2.1. *(18) For any metric space X with distance function d, $F : X \to X$ is a* contraction *if*

$$\sup_{x \neq y} \frac{d(F(x), F(y))}{d(x,y)} < 1$$

Definition 2.2.2. *(18) The compact set $K \subset \mathbb{R}^n$ is* invariant *if there exists a finite set $\mathcal{S} = \{S_1, \ldots, S_N\}$ of contraction maps on $\mathbb{R}^n$ such that $K = \bigcup_{i=1}^N S_i K$.*

In the context of this thesis, we will discuss contraction maps of the form $S(x) = Ax + b$, with A an $n \times n$ matrix with $||A|| < 1$ and b a vector in $\mathbb{R}^n$. Hutchinson's results apply to any finite set of contraction maps.

Theorem 2.2.3. *(18) If $\mathcal{S} = \{S_1, \ldots, S_N\}$ is a finite set of contraction maps on a complete metric space X, then there exists a unique, closed, bounded set K such that $K = \bigcup_{i=1}^N S_i K$. Also, K is compact and the closure of the set of fixed points of finite compositions $S_{i_1} \circ \cdots \circ S_{i_p}$ of members of $\mathcal{S}$. (18)*

In addition to a unique closed bounded set, the set $\mathcal{S}$ determines a set of unique Borel regular probability measures:

Theorem 2.2.4. *(18) Let $\mathcal{S}$ as above and $\rho_1, \ldots, \rho_N \in (0,1)$ with $\sum_{i=1}^N \rho_i = 1$. Then there exists a unique Borel regular measure μ with $\mu(X) = 1$ such that for any measurable set A, $\mu(A) = \sum_{i=1}^N \rho_i \mu(S_i^{-1}(A))$.*

Consequently, for any continuous function f on X,

$$\int f d\mu = \sum_{i=1}^N \left(\rho_i \int f(S_i(x)) \; d\mu(x) \right). \tag{2.4}$$

In particular, this applies when $\rho_1 = \cdots = \rho_N = 1/N$, which is the usual fractal measure considered in this context.

2.2.1 In which μ_4 is spectral but μ_3 is not

Later on, Jorgensen and Pedersen in (19) looked at the case where the contraction maps are of the form

$$S_j x = R^{-1}x + b_j,\ x \in \mathbb{R}^N, \tag{2.5}$$

where R is a real matrix with eigenvalues ξ_i all satisfying $|\xi_i| > 1$.

In particular, they discuss the case of μ_4, which is the unique probability measure of compact support on $\mathbb{R}$ such that:

$$\int f d\mu_4 = \frac{1}{2}\left(\int f\left(\frac{x}{4}\right)\ d\mu_4 + \int f\left(\frac{x}{4}+\frac{1}{2}\right) d\mu_4\right).$$

In this case the compact support of μ_4 is the Cantor-4 set, the Cantor set obtained by dividing $I = [0,1]$ into four equal subintervals and retaining only the first and third; then repeating the process with each remaining interval.

Similarly, there is also μ_3, the unique probability measure of compact support on $\mathbb{R}$ such that:

$$\int f d\mu_3 = \frac{1}{2}\left(\int f\left(\frac{x}{3}\right)\ d\mu_3 + \int f\left(\frac{x}{3}+\frac{2}{3}\right) d\mu_3\right),$$

which is supported on the standard middle thirds Cantor set.

Each of these fractal measures has a Fourier transform:

$$\widehat{\mu}(t) = \int e^{i2\pi t\cdot x} d\mu(x) \tag{2.6}$$

for any $t \in \mathbb{R}^d$.

Because of Equation 2.4, we can write:

$$\widehat{\mu}(t) = \chi_B(t)\widehat{\mu}(R^{*-1}t) \tag{2.7}$$

where

$$\chi_B(t) = \frac{1}{N}\sum_{b\in B} e^{i2\pi b\cdot t}. \tag{2.8}$$

By applying Equation 2.7 repeatedly, Jorgensen and Pedersen obtained:

$$\widehat{\mu_4}(t) = \prod_{n=0}^{\infty} \frac{1}{2}\left(1 + e^{i\frac{\pi t}{4^n}}\right) = e^{i\pi\frac{2t}{3}} \prod_{n=0}^{\infty} \cos\left(\frac{\pi t}{2\cdot 4^n}\right), \tag{2.9}$$

and

$$\widehat{\mu}_3(t) = \prod_{n=0}^{\infty} \frac{1}{2}\left(1 + e^{i\frac{4\pi t}{3^n}}\right) = e^{i\pi t} \prod_{n=0}^{\infty} \cos\left(\frac{2\pi t}{3^n}\right). \tag{2.10}$$

Jorgensen and Pedersen would show that μ_4 has a Fourier basis, that is, there is at least one set of frequencies $P \subset \mathbb{R}$ for which $\{e^{2\pi i x\cdot\lambda} : \lambda \in P\}$ is an orthonormal basis for $L^2(\mu_4)$.

The space $L^2(\mu_3)$, however, can have no such Fourier basis, since there can be no more than two orthogonal exponential functions in $L^2(\mu_3)$.

Theorem 2.2.5. *(19) Any set of μ_3 orthogonal exponentials contains at most two elements.*

Proof. From Equation 2.10, the set of $t \in \mathbb{R}$ with $\widehat{\mu}(t) = 0$ is:

$$Z(\mu_3) = \left\{\frac{3^n}{4}(1 + 2\mathbb{Z}) : n = 1, 2, 3, \ldots\right\} \tag{2.11}$$

For three exponentials, $e_{\gamma_1}, e_{\gamma_2}, e_{\gamma_3}$ to be mutually orthogonal in $L^2(\mu)$, we must have $\widehat{\mu_3}(\gamma_i - \gamma_j) = 0$ for every pair $i, j = 1, 2, 3$, $i \neq j$. That is, $\gamma_i - \gamma_j \in Z(\mu_3)$.

Let $\lambda_1 = \gamma_1 - \gamma_2$, $\lambda_2 = \gamma_2 - \gamma_3$, $\lambda_0 = \gamma_1 - \gamma_3$, and let $z_j \in \mathbb{Z}$ be such that:

$$\lambda_j = \frac{3^{n_j}}{4}(1 + 2z_j).$$

Since $\lambda_1 + \lambda_2 = \lambda_0$, we have that:

$$3^{n_1}(1 + 2z_1) + 3^{n_2}(1 + 2z_2) = 3^{n_0}(1 + 2z_0).$$

However, the left-hand side of the equation must be even, while the right-hand side must be odd, so we have a contradiction. $\square$

2.2.2 Hadamard duality

Definition 2.2.6. *(7)*

A set of vectors $B = \{b_0, \ldots, b_{n-1}\} \subset \mathbb{R}^d$ and a set of vectors $L = \{l_0, \ldots, l_{n-1}\} \subset \mathbb{R}^d$ are called a Hadamard pair if the matrix:

$$H := \left(e^{2\pi i b_j \cdot l_k}\right)_{0 \leq j,k \leq n-1} \tag{2.12}$$

is Hadamard, that is, $\frac{1}{\sqrt{n}}H$ is unitary.

Notation 1. *Let* $e_\lambda(x) := e^{2\pi i\lambda\cdot x}$

Definition 2.2.7. *(7) Fix some* $d \in \mathbb{N}$*.* (B, L, R) *is a system in Hadamard duality if:*

- B *and* L *are subsets of* $\mathbb{R}^d$ *such that* $\#B = \#L =: N$*,*
- R *is a fixed* $d \times d$ *invertible matrix over* $\mathbb{R}$ *with all eigenvalues* λ *satisfying* $|\lambda| > 1$*,*
- *the sets* $(R^{-1}B, L)$ *form a Hadamard pair.*

Then let:

- $\tau_b(x) = R^{-1}(b + x)$*,* $x \in \mathbb{R}^d$*;*
- X_B *be the unique compact subset of* $\mathbb{R}^d$ *(see Definition 2.2.2) such that*

$$X_B = \bigcup_{b\in B} \tau_b(X_B);$$

- μ_B *the measure guaranteed in Theorem 2.2.4, with* $\rho_1 = \cdots = \rho_N = 1/N$*.*
- $\tau_l(x) := (R^T)^{-1}(l + x)$*,* $x \in \mathbb{R}^d$*.*
- X_L *be the unique compact subset of* $\mathbb{R}^d$ *such that* $X_L = \bigcup_{l\in L} \tau_l(X_L)$*.*

In this context, Jorgensen and Pedersen proved an important theorem:

Theorem 2.2.8. *(19) For* B, L, R *as in Definition 2.2.7, with* $RB \subset \mathbb{Z}^d$*,* $0 \in B$*, and* (B, L, R) *in Hadamard duality, let*

$$h_X(t) := \sum_{\lambda\in\mathcal{E}} \left|\hat{\mu_B}(t - \lambda)\right|^2, \quad t \in \mathbb{R}^d, \quad \lambda \in \mathcal{E} = \{l_0 + R^* l_1 + \cdots : l_i \in L, \text{ finite sums}\} \tag{2.13}$$

Then $\{e_\lambda : \lambda \in \mathcal{E}\}$ *is an orthonormal basis for* $L^2(\mu_B)$ *if and only if* $h_X \equiv 1$ *on* R^d*.*

Later, Dutkay and Jorgensen extended this in the following construction.

Definition 2.2.9. *(7) For* B, N *as in Definition 2.2.7, let:*

$$W_B(x) = \frac{1}{N^2}\left|\sum_{b\in B} e^{2\pi i b\cdot x}\right|^2$$

Lemma 2.2.10. *For B, L, N, τ_l as in Definition 2.2.7,*

$$\sum_{l \in L} W_B(\tau_l x) = 1, \quad x \in \mathbb{R}^d$$

Proof.

$$\begin{aligned}
\sum_{l \in L} W_B(\tau_l x) &= \sum_{l \in L} \frac{1}{N^2} \left| \sum_{b \in B} e^{2\pi i b \cdot \tau_l x} \right|^2 && (2.14) \\
&= \frac{1}{N^2} \sum_{l \in L} \left(\sum_{b \in B} e^{2\pi i b \cdot (R^T)^{-1}(l+x)} \right) \left(\sum_{b' \in B} e^{-2\pi i b' \cdot (R^T)^{-1}(l+x)} \right) && (2.15) \\
&= \frac{1}{N^2} \sum_{l \in L} \sum_{b \in B} \sum_{b' \in B} e^{2\pi i b \cdot (R^T)^{-1}(l+x)} e^{-2\pi i b' \cdot (R^T)^{-1}(l+x)} && (2.16) \\
&= \frac{1}{N^2} \sum_{l \in L} \sum_{b \in B} \sum_{b' \in B} e^{2\pi i R^{-1} b \cdot (l+x)} e^{-2\pi i R^{-1} b' \cdot (l+x)} && (2.17) \\
&= \frac{1}{N^2} \sum_{l \in L} \sum_{b \in B} \sum_{b' \in B} e^{2\pi i R^{-1} b \cdot l} e^{2\pi i R^{-1} b \cdot x} e^{-2\pi i R^{-1} b' \cdot l} e^{-2\pi i R^{-1} b' \cdot x} && (2.18) \\
&= \frac{1}{N^2} \sum_{b \in B} \sum_{b' \in B} \sum_{l \in L} e^{2\pi i R^{-1} b \cdot l} e^{2\pi i R^{-1} b \cdot x} e^{-2\pi i R^{-1} b' \cdot l} e^{-2\pi i R^{-1} b' \cdot x} && (2.19) \\
&= \frac{1}{N^2} \sum_{b \in B} \sum_{b' \in B} e^{2\pi i R^{-1} b \cdot x} e^{-2\pi i R^{-1} b' \cdot x} \sum_{l \in L} e^{2\pi i R^{-1} b \cdot l} e^{-2\pi i R^{-1} b' \cdot l} && (2.20) \\
& && (2.21)
\end{aligned}$$

The inner sum is the row sum of HH^*, for H as in Equation 2.12, therefore it is equal to N for $b = b'$ and 0 otherwise. So:

$$\begin{aligned}
\sum_{l \in L} W_B(\tau_l x) &= \frac{1}{N^2} \sum_{b \in B} e^{2\pi i R^{-1} b \cdot x} e^{-2\pi i R^{-1} b \cdot x} N && (2.22) \\
&= \frac{1}{N^2} \sum_{b \in B} N = \frac{1}{N^2} N^2 = 1 && (2.23)
\end{aligned}$$

□

Definition 2.2.11. *(7) A point $x \in X_L$ is a cycle if for some $l_1, \dots, l_k \in L$, not necessarily distinct or in any particular order, $\tau_{l_1} \circ \tau_{l_2} \circ \cdots \circ \tau_{l_k}(x) = x$.*

If in addition $W_B(x) = 1$, x is called a W_B-cycle.

Definition 2.2.12 (Transversality of the zeros (7))**.** *A function W on X satisfies the transversality of the zeros condition if:*

(a) *If* $x \in X$ *is not a cycle, then there exists* $k_x \geq 0$ *such that, for* $k \geq k_x$, $\{\tau_{l_1} \circ \tau_{l_2} \circ \cdots \circ \tau_{l_k}(x) : l_1, \dots l_n = k \in L\}$ *does not contain any zeros of* W*;*

(b) *If* $\{x_0, x_1, \dots, x_p\}$ *are on a cycle with* $x_1 = \tau_l(x_0)$ *for some* $l \in L$*, then for every* $y = \tau_{l'}(x_0)$, $y \neq x_1$ *is either not on a cycle or* $W(y) = 0$.

Theorem 2.2.13. *(7)*

Suppose that:

- $\{R^{-1}B, L\}$ *form a Hadamard pair,* $\#B = \#L =: N$,
- $R^n b \cdot l \in \mathbb{Z}$, *for* $b \in B$, $l \in L$, $n \geq 0$,
- $0 \in B$, $0 \in L$.
- W_B *satisfies the transversality of the zeros condition.*

Let $\Lambda \subset \mathbb{R}^d$ *be the smallest set that contains* $-C$ *for every* W_B*-cycle* C*, and such that* $R^T \Lambda + L \subset \Lambda$. *Then*

$$\{e^{2\pi i \lambda \cdot x} | \lambda \in \Lambda\}$$

is an orthonormal basis for $L^2(\mu_B)$.

CHAPTER 3. A FAST FOURIER TRANSFORM ON $\mathcal{S}_N$

3.1 A Fast Fourier Transform on $\mathcal{S}_N$

We consider an iterated function system generated by contractions $\{\psi_0, \psi_1, \ldots, \psi_{K-1}\}$ on $\mathbb{R}^d$ of the following form:

$$\psi_j(x) = A(x + \vec{b}_j)$$

where A is a $d \times d$ invertible matrix with $\|A\| < 1$. We require A^{-1} to have integer entries, the vectors $\vec{b}_j \in \mathbb{Z}^d$, and without loss of generality we suppose $\vec{b}_0 = \vec{0}$. We then choose a second iterated function system generated by $\{\rho_0, \rho_1, \ldots, \rho_{K-1}\}$ of the form

$$\rho_j(x) = Bx + \vec{c}_j$$

where $B = (A^T)^{-1}$, with $\vec{c}_j \in \mathbb{Z}^d$, and $\vec{c}_0 = \vec{0}$. We require the matrix

$$M_1 = (e^{-2\pi i \vec{c}_j \cdot A\vec{b}_k})_{j,k} \tag{3.1}$$

be invertible (or Hadamard). Note that depending on A and $\{\vec{b}_0, \vec{b}_1, \ldots, \vec{b}_{K-1}\}$, there may not be any choice $\{\vec{c}_0, \vec{c}_1, \ldots, \vec{c}_{K-1}\}$ so that M_1 is invertible. However, for many IFS's there is a choice:

Proposition 3.1.1. *If the set $\{\vec{b}_0, \vec{b}_1, \ldots, \vec{b}_{K-1}\}$ is such that for every pair ($j \neq k$), $A\vec{b}_j - A\vec{b}_k \notin \mathbb{Z}^d$, then there exists $\{\vec{c}_0, \vec{c}_1, \ldots, \vec{c}_{K-1}\}$ such that the matrix M_1 is invertible.*

Proof. The mappings $\phi_1 : \vec{x} \mapsto e^{2\pi i \vec{x} \cdot A\vec{b}_j}$ and $\phi_2 : \vec{x} \mapsto e^{2\pi i \vec{x} \cdot A\vec{b}_k}$ are characters on $G = \mathbb{Z}^d / B\mathbb{Z}^d$. Since $A\vec{b}_j - A\vec{b}_k \notin \mathbb{Z}^d$, the characters are distinct. Thus, by Schur orthogonality, $\sum_{x \in G} \phi_1(x)\overline{\phi_2(x)} = 0$. Therefore, the matrix $M = (e^{-2\pi i \vec{x}_k \cdot A\vec{b}_j})_{j,k}$, where $\{\vec{x}_k\}$ is any enumeration of G, has orthogonal columns. Thus, there is a choice of a square submatrix of M which is invertible. $\square$

Even under the hypotheses of Proposition 3.1.1 there is not always a choice of $\vec{c}$'s so that M_1 is Hadamard; this is the case for the middle-third Cantor set, which is the attractor set for the IFS generated by $\psi_0(x) = \frac{x}{3}$, $\psi_1(x) = \frac{x+2}{3}$ (and is a reflection of the fact that μ_3 is not spectral).

Notation 2. *We define our notation for compositions of the IFS's using two distinct orderings. Let $N \in \mathbb{N}$. For $j \in \{0, 1, \ldots, K^N - 1\}$, write $j = j_0 + j_1 K + \cdots + j_{N-1} K^{N-1}$ with $j_0, \ldots, j_{N-1} \in \{0, 1, \ldots K-1\}$. We define*

$$\Psi_{j,N} := \psi_{j_0} \circ \psi_{j_1} \circ \cdots \circ \psi_{j_{N-1}}$$

$$\mathcal{R}_{j,N} := \rho_{j_0} \circ \rho_{j_1} \circ \cdots \circ \rho_{j_{N-1}}.$$

These give rise to enumerations of $\mathcal{S}_N$ and $\mathcal{T}_N$ as follows:

$$\mathcal{S}_N = \{\Psi_{j,N}(0) : j = 0, 1, \ldots K^N - 1\}$$

$$\mathcal{T}_N = \{\mathcal{R}_{j,N}(0) : j = 0, 1, \ldots K^N - 1\}.$$

We call these the "obverse" orderings of $\mathcal{S}_N$ and $\mathcal{T}_N$.

Likewise, we define

$$\widetilde{\Psi}_{j,N} := \psi_{j_{N-1}} \circ \psi_{j_{N-2}} \circ \cdots \circ \psi_{j_0}$$

$$\widetilde{\mathcal{R}}_{j,N} := \rho_{j_{N-1}} \circ \rho_{j_{N-2}} \circ \cdots \circ \rho_{j_0}$$

which also enumerate $\mathcal{S}_N$ and $\mathcal{T}_N$. We call these the "reverse" orderings.

Remark Note that for $N = 1$, $\Psi_{j,1} = \widetilde{\Psi}_{j,1}$ and $\mathcal{R}_{j,1} = \widetilde{\mathcal{R}}_{j,1}$.

We define the matrices M_N and $\widetilde{M}_N$ as follows:

$$[M_N]_{jk} = e^{-2\pi i \mathcal{R}_{j,N}(0) \cdot \Psi_{k,N}(0)}$$

and

$$[\widetilde{M}_N]_{jk} = e^{-2\pi i \widetilde{\mathcal{R}}_{j,N}(0) \cdot \widetilde{\Psi}_{k,N}(0)}.$$

Both of these are the matrix representations of the exponential functions with frequencies given by $\mathcal{T}_N$ on the data points given by $\mathcal{S}_N$. The matrix M_N corresponds to the obverse ordering on both $\mathcal{T}_N$ and $\mathcal{S}_N$, whereas the matrix $\widetilde{M}_N$ corresponds to the reverse ordering on both. Since these matrices arise from different orderings of the same sets, there exist permutation matrices P and Q such that

$$Q\widetilde{M}_N P = M_N. \tag{3.2}$$

Indeed, define for $j \in \{0, \ldots, K^N - 1\}$ a conjugate as follows: if $j = j_0 + j_1 K + \cdots + j_{N-1}K^{N-1}$, let $\tilde{j} = j_{N-1} + j_{N-2}K + \cdots + j_0 K^{N-1}$. Note then that $\tilde{\tilde{j}} = j$, and

$$\widetilde{\Psi}_{k,N} = \Psi_{\tilde{k},N} \qquad \widetilde{\mathcal{R}}_{k,N} = \mathcal{R}_{\tilde{k},N}. \tag{3.3}$$

Now, define a $K^N \times K^N$ permutation matrix P by $[P]_{mn} = 1$ if $n = \tilde{m}$, and 0 otherwise.

Lemma 3.1.2. *For P defined above,*

$$P\widetilde{M}_N P = M_N.$$

Proof. We calculate

$$\begin{aligned}
[P\widetilde{M}_N P]_{mn} &= \sum_k [P]_{mk} \sum_\ell [\widetilde{M}_N]_{k\ell}[P]_{\ell n} \\
&= [P]_{m\tilde{m}}[\widetilde{M}_N]_{\tilde{m}\tilde{n}}[P]_{\tilde{n}n} \\
&= e^{-2\pi i \widetilde{\mathcal{R}}_{\tilde{m},N}(0)\cdot\widetilde{\Psi}_{\tilde{n},N}(0)} \\
&= e^{-2\pi i \mathcal{R}_{m,N}(0)\cdot\Psi_{n,N}(0)} = [M_N]_{mn}
\end{aligned}$$

by virtue of Equation (3.3). □

Proposition 3.1.3. *For scale $N = 1$,*

$$M_1 = \widetilde{M}_1 = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \exp(2\pi i \vec{c}_1 \cdot A\vec{b}_1) & \cdots & \exp(2\pi i \vec{c}_1 \cdot A\vec{b}_{K-1}) \\ \vdots & \vdots & \vdots\vdots & \vdots \\ 1 & \exp(2\pi i \vec{c}_{K-1} \cdot A\vec{b}_1) & \cdots & \exp(2\pi i \vec{c}_{K-1} \cdot A\vec{b}_{K-1}) \end{pmatrix}.$$

Proof. The proof follows from Remark 3.1. □

Lemma 3.1.4. *For $N \in \mathbb{N}$, $0 \le j < K^N$, and $\vec{x}, \vec{y} \in \mathbb{R}^d$,*

1. $\Psi_{j,N}\left(\vec{x}+\vec{y}\right)=\Psi_{j,N}(\vec{x})+A^{N}\vec{y}$

2. $\widetilde{\Psi}_{j,N}\left(\vec{x}+\vec{y}\right)=\widetilde{\Psi}_{j,N}(\vec{x})+A^{N}\vec{y}$

3. $\mathcal{R}_{j,N}\left(\vec{x}+\vec{y}\right)=R_{j,N}(\vec{x})+B^{N}\vec{y}$

4. $\widetilde{\mathcal{R}}_{j,N}\left(\vec{x}+\vec{y}\right)=\widetilde{R}_{j,N}(\vec{x})+B^{N}\vec{y}$.

Proof. 1. We prove by induction on N. Base case, $n=1$, we have $j=0,1,\dots K-1$ and $\Psi_{n,j}=\widetilde{\Psi}_{n,j}=\psi_j$.

$j=0$:

$\psi_0\left(\vec{x}+\vec{y}\right)=A\left(\vec{x}+\vec{y}\right)=A(\vec{x})+A(\vec{y})=\psi_0(\vec{x})+A\vec{y}$

$j=1,\dots K-1$:

$\psi_j\left(\vec{x}+\vec{y}\right)=A\left(\vec{x}+\vec{y}+b_j\right)=A(\vec{x})+A(\vec{y})+Ab_j=\psi_0(\vec{x})+A\vec{y}$

Assume the equality in Item 1. holds for $N-1$. For $j=j_0+j_1K+\cdots+j_{N-1}K^{N-1}$, let $\ell=j-j_{N-1}K^{N-1}$. We have:

$$\begin{aligned}
\Psi_{j,N}\left(\vec{x}+\vec{y}\right)&=\Psi_{\ell,N-1}(\psi_{j_{N-1}}(\vec{x}+\vec{y}))\\
&=\Psi_{\ell,N-1}(\psi_{j_{N-1}}(\vec{x})+A\vec{y})\\
&=\Psi_{\ell,N-1}(\psi_{j_{N-1}}(\vec{x}))+A^{N-1}A\vec{y}\\
&=\Psi_{j,N}(\vec{x})+A^{N}\vec{y}
\end{aligned}$$

2. Proof is the same as for 1.

3. Base case: For $N=1$, $R_{j,1}=\rho_j$. $\rho_0(\vec{x}+\vec{y})=B\vec{x}+B\vec{y}$, and for $j=1,\dots,K-1$, $\rho_j(\vec{x}+\vec{y})=B(\vec{x}+\vec{y})+\vec{c}_j=\rho_j(\vec{x})+B\vec{6}$.

   Let $R_{j',N-1}=\rho_{j_1}\circ\cdots\circ\rho_{j_{N-1}}$ be the composition of $N-1$ ρ_k's with the property that $\rho_{j_0}\circ R_{j',N-1}=R_{j,N}$.

Assume true for $n < N$. Then:

$$
\begin{aligned}
R_{j,N}\left(\vec{x}+\vec{y}\right) &= \rho_{j_0} \circ \rho_{j_1} \circ \cdots \circ \rho_{j_{N-1}}\left(\vec{x}+\vec{y}\right) \\
&= \rho_{j_0}\left(R_{j',N-1}\left(\vec{x}+\vec{y}\right)\right) \\
&= \rho_{j_0}\left(R_{j',N-1}(\vec{x}) + B^{N-1}\vec{y}\right) \\
&= \rho_{j_0}(R_{j',N-1}\vec{x}) + B(B^{N-1}\vec{y}) \\
&= R_{j,N}\vec{x} + B^N\vec{y}
\end{aligned}
$$

4. Proof is the same as for 3.

$\square$

Lemma 3.1.5. *For $N \in \mathbb{N}$ and $0 \leq j < K^N$,*

1. $\Psi_{j,N}(0) = A^N\vec{z}$ *for some* $\vec{z} \in \mathbb{Z}^d$*,*
2. $\widetilde{\Psi}_{j,N}(0) = A^N\vec{z}$ *for some* $\vec{z} \in \mathbb{Z}^d$*,*
3. $\mathcal{R}_{j,N}(0) \in \mathbb{Z}^d$*,*
4. $\widetilde{\mathcal{R}}_{j,N}(0) \in \mathbb{Z}^d$*.*

Proof. 1. We prove by induction on N. Base case, $N = 1$, we have $j = 0, 1, \ldots, K-1$ and $\Psi_{j,N} = \widetilde{\Psi}_{j,N} = \psi_j$.

For $j = 0$, $\psi_0(\vec{0}) = A\vec{0} = \vec{0}$, and for $j = 1, \ldots, K-1$, $\psi_j(\vec{0}) = A(\vec{0}+\vec{b}_j) = A\vec{b}_j$, so the lemma is satisfied for $\vec{z} = \vec{b}_j$, which is in $\mathbb{Z}^d$ by definition.

Assume the equality in Item 1. holds for $N-1$. For $j = j_0 + j_1K + \cdots + j_{N-1}K^{N-1}$, let $q_j = j - j_{N-1}K^{N-1}$. We have:

$$
\begin{aligned}
\Psi_{j,N}(0) &= \psi_{j_{N-1}}\left(\Psi_{q_j,N-1}(0)\right) \\
&= A\left(A^{N-1}\vec{z} + \vec{b}_j\right) \\
&= A^N(\vec{z} + A^{-(N-1)}\vec{b}_j)
\end{aligned}
$$

Since A^{-1} is an integer matrix, so is $A^{-(N-1)}$ and thus $\vec{z}+A^{-(N-1)}\vec{b}_j \in \mathbb{Z}^d$. Item 2. is analogous. For Item 3., note first that $\rho_j(\mathbb{Z}^d) \subset \mathbb{Z}^d$, so by induction, $\rho_{j_0} \circ \cdots \circ \rho_{j_{N-1}}(0) \in \mathbb{Z}^d$. Likewise for Item 4.

□

Lemma 3.1.6. *Assume $N \geq 2$, let ℓ be an integer between 0 and $K-1$, and suppose $l \cdot K^{N-1} \leq j < (l+1)K^{N-1}$. Then,*

1. $\Psi_{j,N}(0) = \Psi_{j-l\cdot K^{N-1},N-1}(0) + A^N\vec{b}_l$,
2. $\widetilde{\Psi}_{j,N}(0) = A\widetilde{\Psi}_{j-l\cdot K^{N-1},N-1}(0) + A\vec{b}_l$,
3. $\mathcal{R}_{j,N}(0) = \mathcal{R}_{j-l\cdot K^{N-1},N-1}(0) + B^{N-1}\vec{c}_l$,
4. $\widetilde{\mathcal{R}}_{j,N}(0) = B\widetilde{\mathcal{R}}_{j-l\cdot K^{N-1},N-1}(0) + \vec{c}_l$.

Proof. For $l \cdot K^{N-1} \leq j < (l+1)K^{N-1}$, $j_{N-1} = l$, so we have:

$$\begin{aligned}\Psi_{j,N}(0) &= \psi_{j_0} \circ \psi_{j_1} \circ \cdots \circ \psi_{j_{N-2}} \circ \psi_l(0)\\ &= \psi_{j_0} \circ \psi_{j_1} \circ \cdots \circ \psi_{j_{N-2}} \left(A(0+\vec{b}_l)\right)\\ &= \Psi_{j-l\cdot K^{N-1},N-1}(0+A\vec{b}_l).\end{aligned}$$

Applying Lemma 3.1.4 Item i) to $\Psi_{j-l\cdot K^{N-1},N-1}$:

$$\Psi_{j-l\cdot K^{N-1},N-1}(0+A\vec{b}_l) = \Psi_{j-l\cdot K^{N-1},N-1}(0) + A^{N-1}A\vec{b}_l.$$

The proof of Item iii) is similar to Item i) with one crucial distinction, so we include the proof here. We have:

$$\begin{aligned}\mathcal{R}_{j,N}(0) &= \rho_{j_0} \circ \rho_{j_1} \circ \cdots \circ \rho_{j_{N-2}} \circ \rho_l(0)\\ &= \rho_{j_0} \circ \rho_{j_1} \circ \cdots \circ \rho_{j_{N-2}} (B0+\vec{c}_l)\\ &= \mathcal{R}_{j-l\cdot K^{N-1},N-1}(0+\vec{c}_l).\end{aligned}$$

Applying Lemma 3.1.4 Item iii) to $R_{j-l\cdot K^{N-1},N-1}$:

$$\mathcal{R}_{j-l\cdot K^{N-1},N-1}(0+\vec{c}_l) = \mathcal{R}_{j-l\cdot K^{N-1},N-1}(0) + B^{N-1}\vec{c}_l.$$

For Item ii), we have

$$\begin{aligned}\widetilde{\Psi}_{j,N}(0) &= \psi_\ell(\widetilde{\Psi}_{j-\ell\cdot K^{N-1},N-1}(0)) \\ &= A\widetilde{\Psi}_{j-\ell\cdot K^{N-1},N-1}(0) + A\vec{b}_\ell.\end{aligned}$$

The proof of Item iv) is analogous. □

Note that in Item i), the extra term involves A^N, whereas in Item iii) the extra term involves B^{N-1}. We are now in a position to prove our main theorem.

Theorem 3.1.7. *The matrix M_N representing the exponentials with frequencies given by $\mathcal{T}_N$ on the fractal approximation $\mathcal{S}_N$, when both are endowed with the obverse ordering, has the form*

$$M_N = \begin{pmatrix} m_{00}M_{N-1} & m_{01}D_{N,1}M_{N-1} & \dots & m_{0(K-1)}D_{N,K-1}M_{N-1} \\ m_{10}M_{N-1} & m_{11}D_{N,1}M_{N-1} & \dots & m_{1(K-1)}D_{N,K-1}M_{N-1} \\ \vdots & \vdots & \vdots\vdots & \vdots \\ m_{(K-1)0}M_{N-1} & m_{(K-1)1}D_{N,1}M_{N-1} & \dots & m_{(K-1)(K-1)}D_{N,K-1}M_{N-1} \end{pmatrix}. \tag{3.4}$$

Here, $D_{N,m}$ are diagonal matrices with $[D_{N,m}]_{pp} = e^{-2\pi i \mathcal{R}_{p,N-1}(0)\cdot A^N\vec{b}_m}$, and $m_{jk} = [M_1]_{jk}$.

Proof. Let us first subdivide M_N into blocks $B_{\ell m}$ of size $K^{N-1}\times K^{N-1}$, so that

$$M_N = \begin{pmatrix} B_{00} & \dots & B_{0(K-1)} \\ \vdots & \ddots & \vdots \\ B_{(K-1)0} & \dots & B_{(K-1)(K-1)} \end{pmatrix}.$$

Fix $0 \le j,k < K^N$ and suppose $\ell K^{N-1} \le j < (\ell+1)K^{N-1}$ and $mK^{N-1} \le k < (m+1)K^{N-1}$ with $0 \le \ell, m < K$. Let $q_j = j - \ell K^{N-1}$ and $q_k = k - mK^{N-1}$. Observe that

$$[M_N]_{jk} = [B_{\ell m}]_{q_j q_k}. \tag{3.5}$$

Using Lemma 3.1.6 Items ii) and iv), we calculate

$$\mathcal{R}_{j,N}(0)\cdot\Psi_{k,N}(0) = \left(\mathcal{R}_{q_j,N-1}(0) + B^{N-1}\vec{c}_\ell\right)\cdot\left(\Psi_{q_k,N-1}(0) + A^N\vec{b}_m\right).$$

By Lemma 3.1.5 Item i), for some $z\in\mathbb{Z}^d$,

$$B^{N-1}\vec{c}_\ell\cdot\Psi_{q_k,N-1}(0) = B^{N-1}\vec{c}_\ell\cdot A^{N-1}z = \vec{c}_\ell\cdot z \in \mathbb{Z}.$$

Note that

$$B^{N-1}\vec{c}_\ell \cdot A^N\vec{b}_m = \vec{c}_\ell \cdot A\vec{b}_m.$$

Therefore, combining the above, we obtain

$$\begin{aligned}[M_N]_{jk} &= e^{-2\pi i \mathcal{R}_{j,N}(0)\cdot\Psi_{k,N}(0)} \\ &= e^{-2\pi i \mathcal{R}_{q_j,N-1}(0)\cdot\Psi_{q_k,N-1}(0)} e^{-2\pi i \mathcal{R}_{q_j,N-1}(0)\cdot A^N\vec{b}_m} e^{-2\pi i \vec{c}_\ell\cdot A\vec{b}_m} \\ &= [M_{N-1}]_{q_j q_k} e^{-2\pi i \mathcal{R}_{q_j,N-1}(0)\cdot A^N\vec{b}_m}[M_1]_{\ell m}. \end{aligned} \tag{3.6}$$

Letting j vary between ℓK^{N-1} and $(\ell+1)K^{N-1}$ and k vary between mK^{N-1} and $(m+1)K^{N-1}$ corresponds to q_j and q_k varying between 0 and K^{N-1}. Therefore, we obtain from Equations (3.5) and (3.6) the matrix equation

$$B_{\ell m} = [M_1]_{\ell m} D_{N,m} M_{N-1}$$

where $[D_{N,m}]_{pp} = e^{-2\pi i \mathcal{R}_{p,N-1}(0)\cdot A^N\vec{b}_m}$ as claimed. □

Corollary 3.1.8. *The matrix M_N is invertible. If M_1 is Hadamard, then M_N is also Hadamard.*

Proof. If M_1 is invertible, then by induction, M_N is invertible via Proposition 2.1.1. If M_1 is Hadamard, then again by induction, M_N is Hadamard by Diţă's construction. □

Theorem 3.1.9. *The matrix $\widetilde{M}_N$ representing the exponentials with frequencies given by $\mathcal{T}_N$ on the fractal approximation $\mathcal{S}_N$, when both are endowed with the reverse ordering, has the form*

$$\widetilde{M}_N = \begin{pmatrix} m_{00}\widetilde{M}_{N-1} & m_{01}\widetilde{M}_{N-1} & \cdots & m_{0(K-1)}\widetilde{M}_{N-1} \\ m_{10}\widetilde{M}_{N-1}\widetilde{D}_{N,1} & m_{11}\widetilde{M}_{N-1}\widetilde{D}_{N,1} & \cdots & m_{1(K-1)}\widetilde{M}_{N-1}\widetilde{D}_{N,1} \\ \vdots & \vdots & \vdots\vdots & \vdots \\ m_{(K-1)0}\widetilde{M}_{N-1}\widetilde{D}_{N,K-1} & m_{(K-1)1}\widetilde{M}_{N-1}\widetilde{D}_{N,K-1} & \cdots & m_{(K-1)(K-1)}\widetilde{M}_{N-1}\widetilde{D}_{N,K-1} \end{pmatrix}. \tag{3.7}$$

Here, $\widetilde{D}_{N,q}$ is a diagonal matrix with $[\widetilde{D}_{N,\ell}]_{pp} = e^{-2\pi i c_\ell \cdot A(\widetilde{\Psi}_{p,N-1}(0))}$, and $m_{jk} = [M_1]_{jk}$.

Proof. The proof proceeds similarly to the proof Theorem 3.1.7. Let us first subdivide $\widetilde{M}_N$ into $K^{N-1} \times K^{N-1}$ blocks $\widetilde{B}_{\ell m}$, so that

$$\widetilde{M}_N = \begin{pmatrix} \widetilde{B}_{00} & \dots & \widetilde{B}_{0(K-1)} \\ \vdots & \ddots & \vdots \\ \widetilde{B}_{(K-1)0} & \dots & \widetilde{B}_{(K-1)(K-1)} \end{pmatrix}.$$

Fix $0 \leq j, k < K^N$ and suppose $\ell K^{N-1} \leq j < (\ell+1)K^{N-1}$ and $mK^{N-1} \leq k < (m+1)K^{N-1}$ with $0 \leq \ell, m < K$. Let $q_j = j - \ell K^{N-1}$ and $q_k = k - mK^{N-1}$. Observe that

$$[\widetilde{M}_N]_{jk} = [\widetilde{B}_{\ell m}]_{q_j q_k}. \tag{3.8}$$

We calculate using Lemma 3.1.6 items ii) and iv):

$$\begin{aligned} \widetilde{\mathcal{R}}_{j,N}(0) \cdot \widetilde{\Psi}_{k,N}(0) &= (B\widetilde{\mathcal{R}}_{q_j,N-1}(0) + \vec{c}_\ell) \cdot (A\widetilde{\Psi}_{q_k,N-1}(0) + A\vec{b}_m) \\ &= \widetilde{\mathcal{R}}_{q_j,N-1}(0) \cdot \widetilde{\Psi}_{q_k,N-1}(0) + \vec{c}_\ell \cdot A\widetilde{\Psi}_{q_k,N-1}(0) \\ &\qquad + \widetilde{\mathcal{R}}_{q_j,N-1}(0) \cdot \vec{b}_m + \vec{c}_\ell \cdot A\vec{b}_m. \end{aligned}$$

By Lemma 3.1.5 Item iv), $\widetilde{\mathcal{R}}_{q_j,N-1}(0) \cdot \vec{b}_m \in \mathbb{Z}$. Thus,

$$[\widetilde{B}_{\ell m}]_{q_j q_k} = [M_{N-1}]_{q_j q_k} e^{-2\pi i \vec{c}_\ell \cdot A\widetilde{\Psi}_{q_k,N-1}(0)} [M_1]_{\ell m}$$

and as in the proof of Theorem 3.1.7, we have

$$\widetilde{B}_{\ell m} = [M_1]_{\ell m} \widetilde{M}_{N-1} \widetilde{D}_{N,\ell}.$$

□

3.1.1 Computational Complexity of Theorems 3.1.7 and 3.1.9

As a consequence of Proposition 2.1.2, the matrix M_N can be multiplied by a vector of dimension K^N in at most $K\mathcal{P}_{N-1} + 3K^{N+1} - 2K^N$ operations, where $\mathcal{P}_{N-1}$ is the number of operations required by the matrix multiplication for M_{N-1}. Since M_{N-1} has the same block form as M_N, $\mathcal{P}_{N-1}$ can be determined by $\mathcal{P}_{N-2}$, etc. The proof of the following proposition is a standard induction argument, which we omit. Note that this says that the computational complexity for M_N is comparable to that for the FFT (recognizing the difference in the number of generators for the respective IFS's).

Proposition 3.1.10. *The number of operations to calculate the matrix multiplication $M_N\vec{v}$ is $\mathcal{P}_N = K^{N-1}\mathcal{P}_1 + 3(N-1)K^{N+1} - 2(N-1)K^N$. Consequently, $\mathcal{P}_N = O(N \cdot K^N)$.*

The significance of Theorem 3.1.9 concerns the inverse of M_N. If P is the permutation matrix as in Lemma 3.1.2, then $M_N^{-1} = P\widetilde{M}_N^{-1}P$. By Proposition 2.1.1, $\widetilde{M}_N^{-1}$ has the form of Diţă's construction, and so the computational complexity of $\widetilde{M}_N^{-1}$ is the same as M_N. Thus, modulo multiplication by the permutation matrices P, the computational complexity of multiplication by M_N^{-1} is the same as that for M_N.

3.1.2 The Diagonal Matrices

The matrices M_N and $\widetilde{M}_N$ have the form of Diţă's construction as shown in Theorems 3.1.7 and 3.1.9. The block form of Diţă's construction involves diagonal matrices, which in Equations (3.4) and (3.7) are determined by the IFS's used to generate the matrices M_N and $\widetilde{M}_N$. As such, the diagonal matrices satisfy certain recurrence relations.

Theorem 3.1.11. *The diagonal matrices which appear in the block form of M_N (Equation (3.4)) satisfy the recurrence relation $D_{N,m} = D_{N-1,m} \otimes E_{N,m}$, where $E_{N,m}$ is the $K \times K$ diagonal matrix with $[E_{N,m}]_{uu} = e^{-2\pi i c_u \cdot A^N \vec{b}_m}$. That is:*

$$[D_{N,m}]_{pp} = [D_{N-1,m}]_{\widehat{p}\widehat{p}}\, e^{-2\pi i (c_{p_0} \cdot A^N \vec{b}_m)}$$

where $\widehat{p} = (p - p_0)/K$.

Likewise, the diagonal matrices which appear in the block form of $\widetilde{M}_N$ (Equation (3.7)) satisfy the recurrence relation $\widetilde{D}_{N,\ell} = \widetilde{D}_{N-1,\ell} \otimes \widetilde{E}_{N,\ell}$, where $\widetilde{E}_{N,\ell}$ is the $K \times K$ diagonal matrix with $[\widetilde{E}_{N,\ell}]_{uu} = e^{-2\pi i \vec{c}_\ell \cdot A^N \vec{b}_u}$. That is:

$$[\widetilde{D}_{N,\ell}]_{pp} = [D_{N-1,\ell}]_{\widehat{p}\widehat{p}}\, e^{-2\pi i \vec{c}_\ell \cdot A^N \vec{b}_{p_0}}.$$

Proof. As demonstrated in Theorem 3.1.7, for $p = 0, 1, \ldots, K^{N-1}$, $[D_{N,m}]_{pp} = e^{-2\pi i \mathcal{R}_{p,N-1}(0) \cdot A^N \vec{b}_m}$. Note that $p_{N-1} = 0$, and $\rho_0(0) = 0$. We want to cancel one power of A in $A^N\vec{b}_m$, so we factor out a B from $\mathcal{R}_{p,N-1}(0)$:

$$\mathcal{R}_{p,N-1}(0) = \rho_{p_0} \circ \rho_{p_1} \circ \cdots \circ \rho_{p_{N-2}}(0) = B\left(\rho_{p_1} \circ \cdots \circ \rho_{p_{N-2}}(0)\right) + \vec{c}_{p_0}.$$

Since $\widehat{p} = p_1 + p_2 K + \cdots + p_{N-2}K^{N-3}$, $\mathcal{R}_{p,N-1}(0) = BR_{\widehat{p},N-2}(0) + \vec{c}_{p_0}$. Thus,

$$\begin{aligned}
[D_{N,m}]_{pp} &= e^{-2\pi i \mathcal{R}_{p,N-1}(0)\cdot A^N \vec{b}_m} \\
&= e^{-2\pi i (B\mathcal{R}_{\widehat{p},N-2}(0)\cdot A(A^{N-1}\vec{b}_m))} e^{-2\pi i (\vec{c}_{p_0}\cdot A^N \vec{b}_m)} \\
&= e^{-2\pi i (\mathcal{R}_{\widehat{p},N-2}(0)\cdot (A^{N-1}\vec{b}_m))} e^{-2\pi i (\vec{c}_{p_0}\cdot A^N \vec{b}_m)} \\
&= [D_{N-1,m}]_{\widehat{p}\widehat{p}}\; e^{-2\pi i (\vec{c}_{p_0}\cdot A^N \vec{b}_m)}.
\end{aligned}$$

Similarly, as demonstrated in Theorem 3.1.9, $[\widetilde{D}_{N,\ell}]_{pp} = e^{-2\pi i \vec{c}_\ell \cdot A(\widetilde{\Psi}_{p,N-1}(0))}$. We write:

$$\begin{aligned}
\widetilde{\Psi}_{p,N-1}(0) &= \psi_{p_{N-2}} \circ \psi_{p_{N-3}} \circ \cdots \circ \psi_{p_1} \circ \psi_{p_0}(0) \\
&= \psi_{p_{N-2}} \circ \psi_{p_{N-3}} \circ \cdots \circ \psi_{p_1}(0 + A\vec{b}_{p_0}) \\
&= \widetilde{\Psi}_{\widehat{p},N-2}(0 + A\vec{b}_{p_0}) \\
&= \widetilde{\Psi}_{\widehat{p},N-2}(0) + A^{N-1}\vec{b}_{p_0}.
\end{aligned}$$

where in the last equality we use Lemma 3.1.4 item ii). Therefore:

$$\begin{aligned}
[\widetilde{D}_{N,\ell}]_{pp} &= e^{-2\pi i c_\ell \cdot A(\widetilde{\Psi}_{p,N-1}(0))} \\
&= e^{-2\pi i \vec{c}_\ell \cdot A\left(\widetilde{\Psi}_{\widehat{p},N-2}(0) + A^{N-1}\vec{b}_{p_0}\right)} \\
&= e^{-2\pi i \vec{c}_\ell \cdot A\left(\widetilde{\Psi}_{\widehat{p},N-2}(0) + A^{N}\vec{b}_{p_0}\right)} \\
&= e^{-2\pi i \vec{c}_\ell \cdot A(\widetilde{\Psi}_{\widehat{p},N-2}(0))} e^{-2\pi i \vec{c}_\ell \cdot A^N \vec{b}_{p_0}} \\
&= [\widetilde{D}_{N-1,\ell}]_{\widehat{p}\widehat{p}}\; e^{-2\pi i \vec{c}_\ell \cdot A^N \vec{b}_{p_0}}.
\end{aligned}$$

$\square$

CHAPTER 4. FOURIER BASES ON THE SKEWED SIERPINSKI GASKET

4.1 The Invariant Set $\mathcal{S}$

Now, instead of looking at fractal approximations as in Chapter 3, we construct an orthonormal basis of exponentials on $L^2(\nu_3)$, the unique Borel regular measure supported on the complete closed bounded set $\mathcal{S}$ guaranteed by Hutchinson in (18).

Definition 4.1.1. *Let* $\mathcal{S} = \bigcup_{j=0}^{2} \psi_j(\mathcal{S})$, *with:*

$$\psi_0(x,y) = \frac{1}{3}(x,y) \tag{4.1}$$

$$\psi_1(x,y) = \frac{1}{3}(x+2,y) \tag{4.2}$$

$$\psi_2(x,y) = \frac{1}{3}(x,y+2) \tag{4.3}$$

By (18), there exists a unique Borel regular measure, which we call ν_3, *supported on* $\mathcal{S}$ *and with the property that, for any continuous function* f *on* $\mathcal{S}$,

$$\int f d\,\nu_3 = \sum_{j=0}^{2} \frac{1}{3} \int f \circ \psi_j d\,\nu_3 \tag{4.4}$$

First, we examine the set $\mathcal{S}$.

Proposition 4.1.2. *Let* $\mathcal{A} = \{(x,y) \in [0,1] \times [0,1] : (x,y) = \sum_{j=1}^{\infty}(x_j,y_j)3^{-j}, (x_j,y_j) \in \{(0,0),(0,2),(2,0)\}\}$

Then $\mathcal{A}$ *is the closed invariant set for (4.1), (4.2), (4.3), that is,* $\mathcal{A} = \mathcal{S}$.

$\mathcal{S} = \{(x,y) \in [0,1] \times [0,1] : (x,y) = \sum_{j=1}^{\infty}(x_j,y_j)3^{-j}, (x_j,y_j) \in \{(0,0),(0,2),(2,0)\}\}$.

Proof. First, we show that $\mathcal{A} = \psi_0(\mathcal{A}) \cup \psi_1(\mathcal{A}) \cup \psi_2(\mathcal{A})$.

Claim 4.1.3. *For all* $(x,y) \in \mathcal{A}$, $\psi_0(x,y) \in \mathcal{A}$.

Proof. We know that $\psi_0(\sum_{j=1}^{\infty}(x_j, y_j)3^{-j}) = \sum_{j=1}^{\infty}(x_j, y_j)3^{-j-1}$.

Let $\psi_0(x, y) := (\tilde{x}, \tilde{y}) = \sum_{j=1}^{\infty}(x_j, y_j)3^{-j}$. Show that $(\tilde{x}, \tilde{y}) \in \mathcal{A}$.

Check: for $j = 1$, $\tilde{x}_1 = \tilde{y}_1 = 0$. For $j \geq 2$, $\tilde{x}_j = \tilde{x}_{j+1}$ and $\tilde{y}_j = \tilde{y}_{j+1}$, so $(\tilde{x}_j, \tilde{y}_j) \in \{(0, 0), (0, 2), (2, 0)\}$.

Therefore, $(\tilde{x}, \tilde{y}) \in \mathcal{A}$. □

Claim 4.1.4. *For all* $(x, y) \in \mathcal{A}$, $\psi_1(x, y) \in \mathcal{A}$.

Proof. We know that $\psi_1(\sum_{j=1}^{\infty}(x_j, y_j)3^{-j}) = (2 \cdot 3^{-1}, 0 \cdot 3^{-1}) + \sum_{j=1}^{\infty}(x_j, y_j)3^{-j-1}$.

Let $\psi_1(x, y) := (\tilde{x}, \tilde{y}) = \sum_{j=1}^{\infty}(x_j, y_j)3^{-j}$. Show that $(\tilde{x}, \tilde{y}) \in \mathcal{A}$.

Check: for $j = 1$, $\tilde{x}_1 = 2$, $\tilde{y}_1 = 0$; for $j \geq 2$, $\tilde{x}_j = \tilde{x}_{j+1}$ and $\tilde{y}_j = \tilde{y}_{j+1}$, so $(\tilde{x}_j, \tilde{y}_j) \in \{(0, 0), (0, 2), (2, 0)\}$.

Therefore, $(\tilde{x}, \tilde{y}) \in \mathcal{A}$. □

Claim 4.1.5. *For all* $(x, y) \in \mathcal{A}$, $\psi_2(x, y) \in \mathcal{A}$.

Proof. We know that $\psi_2(\sum_{j=1}^{\infty} x_j, y_j)3^{-j} = (0 \cdot 3^{-1}, 2 \cdot 3^{-1}) + \sum_{j=0}^{\infty}(x_j, y_j)3^{-j-1}$.

Check: for $j = 1$, $\tilde{x}_1 = 0$, $\tilde{y}_1 = 2$; for $j \geq 2$, $\tilde{x}_j = \tilde{x}_{j+1}$ and $\tilde{y}_j = \tilde{y}_{j+1}$, so $(\tilde{x}_j, \tilde{y}_j) \in \{(0, 0), (0, 2), (2, 0)\}$.

Therefore, $(\tilde{x}, \tilde{y}) \in \mathcal{A}$. □

Now show that $\mathcal{A} \subseteq \psi_0(\mathcal{A}) \cup \psi_1(\mathcal{A}) \cup \psi_2(\mathcal{A})$.

Again, let $(x, y) = \sum_{j=1}^{\infty}(x_j, y_j)3^{-j}$. Now let $(\hat{x}, \hat{y}) = (\sum_{j=2}^{\infty} x_j 3^{-j+1}, \sum_{j=2}^{\infty} y_j 3^{-j+1})$. Notice that $\hat{x}_j = x_{j+1}$, $\hat{y}_j = y_{j+1}$, so $(\hat{x}, \hat{y}) \in \mathcal{A}$.

We will show that, for all $(x, y) \in \mathcal{A}$, $\psi_k(\hat{x}, \hat{y}) = (x, y)$ for some $k \in \{0, 1, 2\}$.

There are three cases: $(x_1, y_1) = (0, 0)$, $(x_1, y_1) = (2, 0)$, and $(x_1, y_1) = (0, 2)$.

Claim 4.1.6. *When* $(x_1, y_1) = (0, 0)$, $\psi_0(\hat{x}, \hat{y}) = (x, y)$.

Proof. $\psi_0(\sum_{j=2}^{\infty}(x_j, y_j)3^{-j+1}) = \sum_{j=2}^{\infty}(x_j, y_j)3^{-j} = (x, y)$, since $x_1 = y_1 = 0$. □

Claim 4.1.7. *When* $(x_1, y_1) = (2, 0)$, $\psi_1(\hat{x}, \hat{y}) = (x, y)$.

Proof. $\psi_1(\sum_{j=2}^{\infty}(x_j, y_j)3^{-j+1}) = (2 \cdot 3^{-1}, 0 \cdot 3^{-1}) + \sum_{j=2}^{\infty}(x_j, y_j)3^{-j} = (x, y)$, since $x_1 = 2$ and $y_1 = 0$. □

Claim 4.1.8. *When* $(x_1, y_1) = (0, 2)$, $\psi_2(\hat{x}, \hat{y}) = (x, y)$.

Proof. $\psi_2(\sum_{j=2}^{\infty}(x_j, y_j)3^{-j+1}) = (0, 2 \cdot 3^{-1}) + (\sum_{j=2}^{\infty}(x_j, y_j)3^{-j} = (x, y)$, since $x_1 = 0$ and $y_1 = 2$. □

Now, show that $\mathcal{A}$ is compact. It is obvious that $\mathcal{A}$ is bounded in $\mathbb{R}^2$, we need only show that it is closed.

Let $\{(x_k, y_k)\}_{k=1}^{\infty} \in \mathcal{A}$ be a Cauchy sequence. Notice that by construction, $x_k = \sum_{j=1}^{\infty} x_{k_j} 3^{-j}$, $x_j \in \{0, 2\}$ is contained in the Cantor-3 set C_3, a well-know closed set. The same is true for y_k. Therefore, $x_k \to x$, with $x \in C_3$, and $y_k \to y$, with $y \in C_3$.

Therefore we have that $(x_k, y_k) \to (x, y)$, with $(x, y) \in C_3 \times C_3$. We need to show that $(x, y) \in \mathcal{A}$, that is, that for $(x, y) = \sum_{j=1}^{\infty}(x_j, y_j)3^{-j}$, $(x_j, y_j) \neq (2, 2)$ for any j.

Suppose that $(x_j, y_j) = (2, 2)$ for some $j \geq 1$. By assumption, $(x_{k_j}, y_{k_j}) \neq (2, 2)$ for any j.

Consider:

$$
\begin{aligned}
||(x, y) - (x_k, y_k)||^2 &= (x - x_k)^2 + (y - y_k)^2 \\
&= \left(\sum_{j=0}^{\infty}(x_j - x_{k_j})3^{-j}\right)^2 + \left(\sum_{j=0}^{\infty}(y_j - y_{k_j})3^{-j}\right)^2
\end{aligned}
$$

If $(x_j, y_j) = (2, 2)$, then for every k, we have $(x_j - x_{k_j}) = 2$ or $(j_j - y_{k_j}) = 2$, or both. So for every k:

$$
||(x, y) - (x_k, y_k)||^2 \geq (2)3^{-j}
$$

Therefore, (x_k, y_k) cannot converge to (x, y) and we have a contradiction. So $(x, y) \in \mathcal{A}$ and $\mathcal{A}$ is closed, thus compact.

Therefore, since $\mathcal{A}$ is compact and $\mathcal{A} = \psi_0(\mathcal{A}) \cup \psi_1(\mathcal{A}) \cup \psi_2(\mathcal{A})$, we must have that $\mathcal{A} = \mathcal{S}$. □

Another characterization of $\mathcal{S}$:

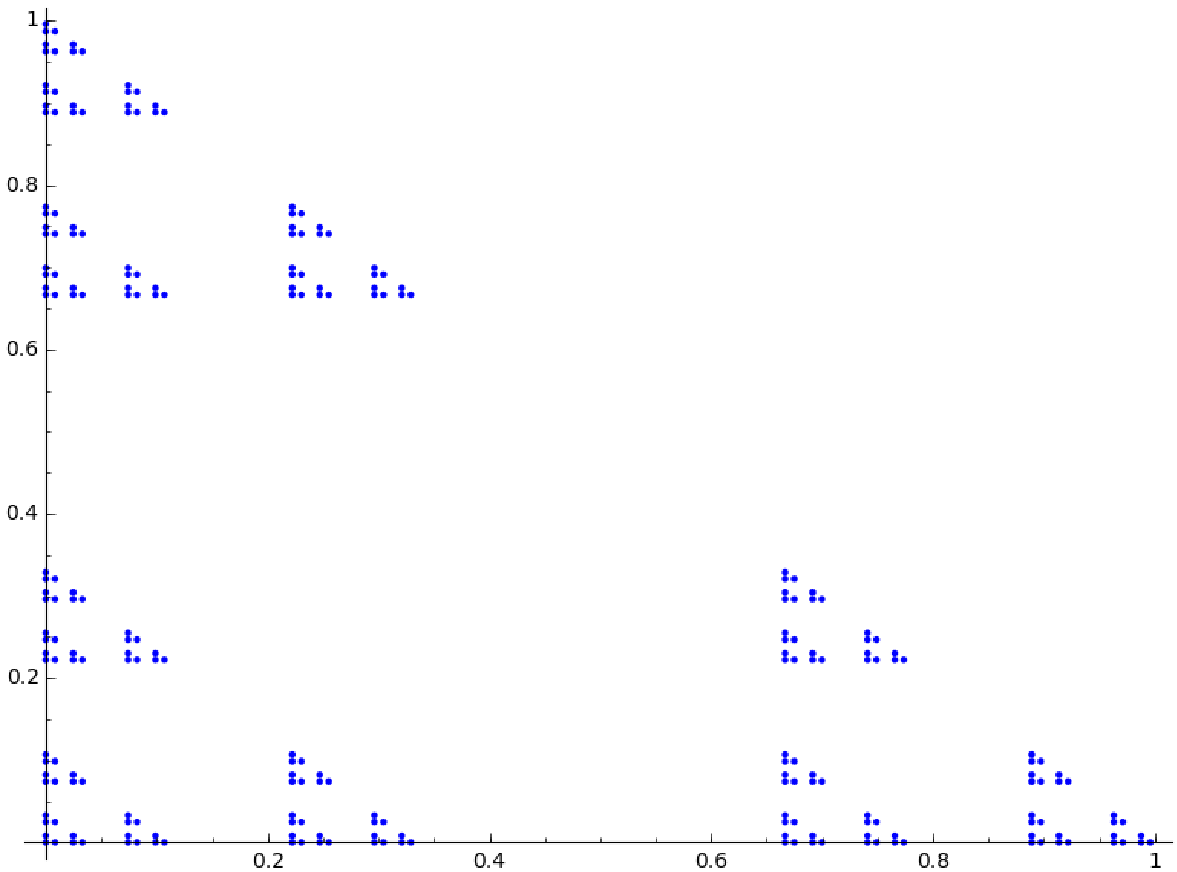

Figure 4.1 A fifth iterative approximation of $\mathcal{S}$ starting at zero.

Proposition 4.1.9. *$\mathcal{S} = \{(x,y) \in \mathbb{R}^2 : (x,y) \in C_3 \times C_3$ and $x+y \in C_3\}$ where C_3 is the standard middle-thirds Cantor-3 set mentioned earlier, the compact attractor set of $\tau_0(x) = x/3$, $\tau_1(x) = x/3 + 2/3$.*

Proof. Let $A = \{(x,y) \in \mathbb{R}^2 | x \in C_3,\ y \in C_3,\ \text{and } x+y \in C_3\}$. Again, it is sufficient to show that $A = \psi_0(A) \cup \psi_1(A) \cup \psi_2(A)$.

First, we show that $\psi_0(A) \cup \psi_1(A) \cup \psi_2(A) \subseteq A$.

Claim 4.1.10. *$\psi_j(x,y) \in C_3 \times C_3$ for $j = 0,1,2$ and $(x,y) \in \mathcal{A}$.*

Proof. Let $(x,y) \in A$

$\psi_0(x,y) = (x/3, y/3) = (\tau_0(x), \tau_0(y))$.

$\psi_1(x,y) = (x/3 + 2/3, y/3) = (\tau_1(x), \tau_0(y))$.

$\psi_2(x,y) = (x/3, y/3 + 2/3) = (\tau_0(x), \tau_1(y))$.

Since $x \in C_3$ and $y \in C_3$, $\psi_j(x,y) \in C_3 \times C_3$ for $j = 0,1,2$. □

Claim 4.1.11. *Let $\psi_j(x,y) = (\tilde{x}, \tilde{y})$, for $j = 0,1,2$, $(x,y) \in A$. We will show that $\tilde{x} + \tilde{y} \in C_3$.*

Proof. $\psi_0(x,y) = (x/3, y/3)$, $x/3 + y/3 = 1/3(x+y) = \tau_0(x+y)/3 \in C_3$.

$\psi_1(x,y) = (x/3 + 2/3, y/3)$, $x/3 + 2/3 + y/3 = (x+y)/3 + 2/3 = \tau_1(x+y) \in C_3$.

$\psi_2(x,y) = (x/3, y/3 + 2/3)$, $x/3 + y/3 + 2/3 = (x+y)/3 + 2/3\tau_1(x+y) \in C_3$. □

These two claims give us $\psi_0(A) \cup \psi_1(A) \cup \psi_2(A) \subseteq A$. Now, we show $A \subseteq \psi_0(A) \cup \psi_1(A) \cup \psi_2(A)$:

Let $(x,y) \in A$. There are three cases:

1. $x \leq 1/3$ and $y \leq 1/3$.

2. $x \geq 2/3$ and $y \leq 1/3$.

3. $x \leq 1/3$ and $y \geq 2/3$.

Since $x, y \in C_3$, neither x nor y can be in $(1/3, 2/3)$. Since $x + y \in C_3$, x and y can't both be greater than $2/3$. So the three cases given cover all of $\mathcal{A}$.

Claim 4.1.12 (Case 1)**.** *For $(x,y) \in \mathcal{A}$, $x \leq 1/3$, $y \leq 1/3$, and $(x+y) \leq 1/3$, $(x,y) \in \psi_0(\mathcal{A})$.*

Proof. Let $(\tilde{x}, \tilde{y}) = (3x, 3y)$. Since $\psi_0(3x, 3y) = (x,y)$, we will get $(x,y) \in \psi_0(A)$ if $(3x, 3y) \in A$.

For $(3x, 3y) \in A$ we need to show three things:

- $3x \in C_3$: Since $x \in C_3$, $x = \tau_0(\tilde{x})$ or $x = \tau_1(\tilde{x})$ for $\tilde{x} \in C_3$. Since $\tilde{x} \in C_3 \Rightarrow \tilde{x} \geq 0$, $\tau_1(\tilde{x}) = \tilde{x}/3 + 2/3 \geq 2/3$; since $x \leq 1/3 < 2/3$, we cannot have $\tau_1(\tilde{x}) = x$. Therefore, we must have $\tau_0(\tilde{x}) = x$; thus $\tilde{x} = 3x \in C_3$.

- $3y \in C_3$: Since $y \in C_3$, $y < 1/3$, $3y < 1$. And, $\tau_0(\tilde{y}) = \frac{1}{3}(3y) = y$. Thus, $\tilde{y} = 3y \in C_3$.

- $3x + 3y \in C_3$: We need to show that $x + y \leq 1/3$. Suppose not. Since $x \leq 1/3$, and $y \leq 1/3$, $x + y \leq 2/3$. Since $x + y \in C_3$, this means that $x + y \leq 1/3$. Therefore, $3x + 3y < 1$. Then $\tau_0(3x + 3y) = x + y \in C_3$ by assumption. Therefore, $3x + 3y \in C_3$.

□

Claim 4.1.13 (Case 2)**.** *For $(x, y) \in \mathcal{A}$, $x \geq 2/3$ and $y \leq 1/3$, $(x, y) \in \psi_1(\mathcal{A})$.*

Proof. Let $(\tilde{x}, \tilde{y}) = (3x - 2, 3y)$. Since $\psi_1(3x - 2, 3y) = \frac{1}{3}(3x - 2 + 2, 3y) = (x, y)$, if $(\tilde{x}, \tilde{y}) \in A$, $(x, y) \in \psi_1(A)$.

For $(3x - 2, 3y) \in A$ we need to show three things:

- $3x - 2 \in C_3$: Since $x \in C_3$, $x = \tau_0(\tilde{x})$ or $x = \tau_1(\tilde{x})$ for $\tilde{x} \in C_3$. Since $x \geq 2/3$ and $\tilde{x} \leq 1$, we must have that $x = \tau_1(\tilde{x})$; thus $\tilde{x} = 3x - 2 \in C_3$.

- $3y \in C_3$: Since $y \in C_3$, $y < 1/3$, $3y < 1$. And, $\tau_0(\tilde{y}) = \frac{1}{3}(3y) = y$. Thus, $\tilde{y} = 3y \in C_3$.

- $3x - 2 + 3y \in C_3$: $\tau_1(3x - 2 + 3y) = \frac{1}{3}(3x - 2 + 3y) - 2/3 = x + y \in C_3$ by assumption, so $\tilde{x} + \tilde{y} \in C_3$.

□

Claim 4.1.14 (Case 3)**.** *For $(x, y) \in \mathcal{A}$, $x \leq 1/3$ and $y \geq 2/3$, $(x, y) \in \psi_2(\mathcal{A})$.*

Proof. Take $(\tilde{x}, \tilde{y}) = (3x, 3y - 2)$. Since $\psi_2(3x, 3y - 2) = \frac{1}{3}(3x, 3y - 2 + 2) = (x, y)$, if $(\tilde{x}, \tilde{y}) \in A$, $(x, y) \in \psi_2(A)$. Show that $(3x, 3y - 2) \in A$:

For $(3x, 3y - 2) \in A$ we need to show three things:

- $3x \in C_3$: Since $x \in C_3$, $x = \tau_0(\tilde{x})$ or $x = \tau_1(\tilde{x})$ for $\tilde{x} \in C_3$. Since $\tilde{x} \in C_3 \Rightarrow \tilde{x} \geq 0$, $\tau_1(\tilde{x}) = \tilde{x}/3 + 2/3 \geq 2/3$; since $x \leq 1/3 < 2/3$, we cannot have $\tau_1(\tilde{x}) = x$. Therefore, we must have $\tau_0(\tilde{x}) = x$; thus $\tilde{x} = 3x \in C_3$.

- $3y - 2 \in C_3$: Since $y \in C_3$, $y = \tau_0(\tilde{y})$ or $y = \tau_1(\tilde{y})$ for $\tilde{y} \in C_3$. Since $y \geq 2/3$ and $\tilde{y} \leq 1$, we must have that $y = \tau_1(\tilde{y}) = \frac{1}{3}(3y - 2) + 2/3 = y$; thus $\tilde{y} = 3y - 2 \in C_3$.

- $3x + 3y - 2 \in C_3$: Since $x \geq 2/3$ and $y \leq 1/3$, $0 < 3x + 3y - 2 < 1$, and $\tau_1(3x - 2 + 3y) = \frac{1}{3}(3x - 2 + 3y) - 2/3 = x + y \in C_3$ by assumption, so $\tilde{x} + \tilde{y} \in C_3$.

□

Now, we show that $\mathcal{A}$ is compact. $\mathcal{A}$ is obviously bounded, so we need only show it is closed.

Let $\{(x_j, y_j)\}_{k=1}^{\infty} \in \mathcal{A}$ be a Cauchy sequence. Since $x_k \in C_3$, $y_k \in C_3$, $x_k \to x \in C_3$ and $y_k \to y \in C_3$, we have that $\{x_k, y_k\} \to (x, y) \in C_3 \to C_3$. Moreover, by continuity, $x_k + y_k \to x + y$, and since $x_k + y_k \in C_3$ for all k, $x + y \in C_3$ and $(x, y) \in \mathcal{A}$.

Therefore, $\mathcal{A}$ is compact, and $\mathcal{A} = \mathcal{S}$. $\square$

4.2 Our orthonormal basis

This section will use Theorem 2.2.13 to show that $\{e_{t,t/2} \mid t \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\nu_3)$.

Following Dutkay and Jorgensen (9), we begin with a dual iterated function system of the form $R(x, y) + L$, for $R = 3I_3$ and $L = \{(0, 0), (2, 1), (4, 2)\}$.

Check first that (R, B, L) are in Hadamard duality (Definition 2.2.7), that is, that the matrix $M_1 = \left(e^{2\pi i l \cdot R^{-1} b}\right)_{l \in L, b \in B}$ is Hadamard:

Claim 4.2.1. *$M_1^* M_1 = 3I_3$.*

Proof. We calculate:

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i(1/3)} & e^{2\pi i(2/3)} \\ 1 & e^{2\pi i(2/3)} & e^{2\pi i(4/3)} \end{pmatrix} \tag{4.5}$$

Proof is by basic matrix multiplication. $\square$

Recall the Definition 2.2.7. For the fractal $\mathcal{S}$, $B = \{(0, 0), (2, 0), (0, 2)\}$, therefore:
$W_B(x, y) = \frac{1}{9}\left|1 + e^{4\pi i x} + e^{4\pi i y}\right|^2 = \frac{1}{9}\left(3 + 2\cos(4\pi x) + 2\cos(4\pi y) + 2\cos(4\pi(x - y))\right)$

.

$W_B(x, y) = 1$ if and only if $e^{4\pi i x} = e^{4\pi i y} = 1$, that is, when $(x, y) \in \mathbb{Z}/2 \times \mathbb{Z}/2$.

$W_B(x, y) = 0$ if and only if $1 + e^{4\pi i x} + e^{4\pi i y} = 0$; that is, when $e^{4\pi i x} + e^{4\pi i y} = -1$; or when $(x, y) \in (1/3 + \mathbb{Z}/2, 1/6 + \mathbb{Z}/2) \cup (1/6 + \mathbb{Z}/2, 1/3 + \mathbb{Z}/2)$.

Proposition 4.2.2. *$X = X_L = \{(2t, t) : t \in [0, 1]\}$, that is, the straight line between $(0, 0)$, and $(2, 1)$.*

Proof. Let:

$$\ell_0(x,y) = (x/3, y/3) = \tau_{(0,0)}(x,y) \tag{4.6}$$

$$\ell_1(x,y) = (x/3 + 2/3, y/3 + 1/3) = \tau_{(2,1)}(x,y) \tag{4.7}$$

$$\ell_2(x,y) = (x/3 + 4/3, y/3 + 2/3) = \tau_{(1,2)}(x,y) \tag{4.8}$$

so that X_L is invariant under $\{\ell_0, \ell_1, \ell_2\}$.

It is clear that $\mathcal{T} = \{(2t,t) : t \in [0,1]\}$ is compact, so we need only show that $\mathcal{T} = \ell_0(\mathcal{T}) \cup \ell_1(\mathcal{T}) \cup \ell_2(\mathcal{T})$.

Let $(x,y) = (2t,t)$ for some $t \in [0,1]$. Then:

$\ell_0(2t,t) = (2t/3, t/3) = (2s,s)$ with $s = t/3 \in [0,1/3] \subset [0,1]$.

$\ell_1(2t,t) = (2t/3 + 2/3, t/3 + 1/3) = (2s,s)$ with $s = t/3 + 1/3$; since $t \in [0,1]$, $t/3 \in [0,1/3$ so $t/3 \in [1/3, 2/3] \subset [0,1]$.

$\ell_2(2t,t) = (2t/3 + 4/3, t/3 + 2/3) = (2s,s)$ with $s = t/3 + 2/3$; since $t \in [0,1]$, $t/3 \in [0,1/3$ so $t/3 \in [2/3, 1] \subset [0,1]$.

Now for $(x,y) = (2t,t)$ find some $(\tilde{x},\tilde{y}) = (2s,s)$, $j \in \{0,1,2\}$ with $\ell_j(\tilde{x},\tilde{y}) = (x,y)$. Case 1: $y = t \in [0,1/3)$. Take $s = 3t$; then $s \in [0,1]$ and $\ell_0(2s,s) = \ell_0(6t,3t) = (2t,t) = (x,y)$.

Case 2: $y = t \in [1/3,2/3)$, so that $3t \in [1,2)$. Take $s = 3t - 1$; then $s \in [0,1]$ and $\ell_1(2s,s) = \ell_1(6t-2, 3t-1) = (2t,t) = (x,y)$.

Case 3: $y = t \in [2/3,1]$, so that $3t \in [2,3]$. Take $s = 3t - 2$; then $s \in [0,1]$ and $\ell_2(2s,s) = \ell_1(6t-4, 3t-2) = (2t,t) = (x,y)$.

□

To find W_B-cycles on X_L, first we find which of the points in the lattice $\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ are also in X_L.

By construction, each $y \in [0,1]$ has at most one $x \in [0,2]$ with $(x,y) \in X_L$; so we really only need to check three possibilities:

$y = 0 \Rightarrow x = 2(0) = 0$: $(0,0) \in X_L$.

$y = 1/2 \Rightarrow x = 1(1/2) = 1$: $(1,1/2) \in X_L$.

$y = 1 \Rightarrow x = 2(1) = 2$: $(1,1/2) \in X_L$.

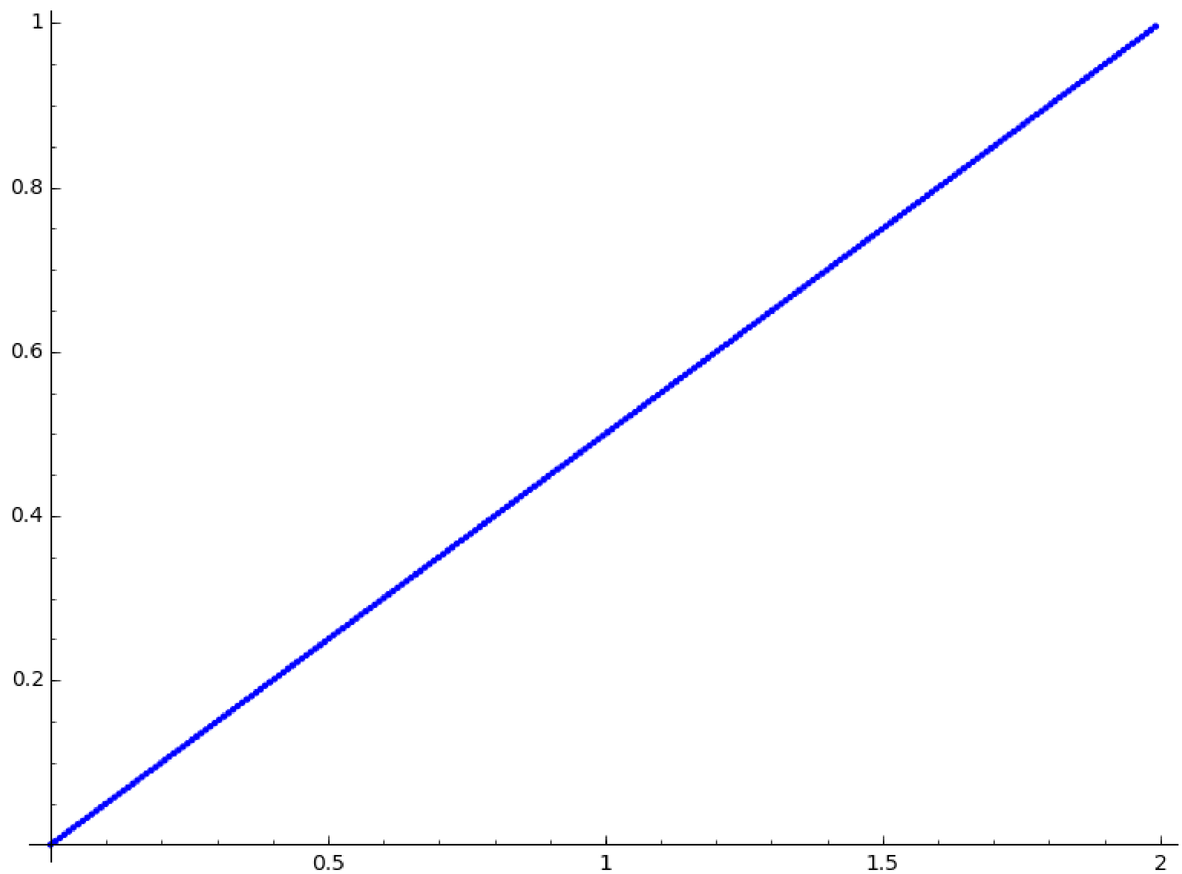

Figure 4.2 Plot of the fifth iteration of $1/3(x, y) + L$.

So the only three points in $\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \cap X_L$ are $(0,0)$, $(1,1/2)$, and $(2,1)$.

All of these are cycles, in fact, they are fixed points:

$$\ell_0(0,0) = (0,0) \tag{4.9}$$

$$\ell_1(1/2,1) = \frac{1}{3}(1/2,1) + (1/3,2/3) = (1/2,1) \tag{4.10}$$

$$\ell_2(1,1/2) = \frac{1}{3}(1,1/2) + (2/3,1/3) = (1,1/2). \tag{4.11}$$

Therefore, the W_B-cycles are exactly $(0,0)$, $(1,1/2)$ and $(1/2,1)$.

Proposition 4.2.3. *Let:*

$$\rho_0(x,y) = 3(x,y) \tag{4.12}$$

$$\rho_1(x,y) = 3(x,y) + (2,1) \tag{4.13}$$

$$\rho_2(x,y) = 3(x,y) + (4,2) \tag{4.14}$$

so that $\{\rho_j(x,y)\}_{j=0,1,2} = R(x,y) + B$.

Let $l_1(x,y) = \{\rho_0(x,y), \rho_1(x,y), \rho_2(x,y)\}$, *then for* $n \geq 2$,

$l_n(x,y) = \{\rho_0(s,t)|(s,t) \in l_{n-1}(x,y)\} \cup \{\rho_1(s,t)|(s,t) \in l_{n-1}(x,y)\} \cup \{\rho_2(s,t)|(s,t) \in l_{n-1}(x,y)\}$. *Then let* $\mathfrak{L}(x,y) = \bigcup_{n\in\mathbb{N}} l_n(x,y)$.

We will show that: $\mathfrak{L}(0,0) \cup \mathfrak{L}(-1/2,-1) \cup \mathfrak{L}(-1,-1/2) = \{(t,t/2)|t \in \mathbb{Z}\}$.

The proof is in the form of three Claims.

Claim 4.2.4. $\mathfrak{L}(0,0) = \{(2t,t)|t \in \mathbb{Z}, t \geq 0\}$.

Proof. ($\subseteq$): Let $L' = \{(2t,t)|t \in \mathbb{Z}, t \geq 0\}$. We have: $l_1(0,0) = \{(0,0),(2,1),(4,2)\} \subset L'$.

By construction, we need now only show that if $(x,y) \in L'$, each of $\rho_0(x,y), \rho_1(x,y), \rho_2(x,y)$ is in L':

- $\rho_0(2t,t) = 3(2t,t) = (6t,3t)$; with $t \geq 0$, $3t \geq 0$ and $\rho_0(2t,t) = (2(3t),3t) \in L'$.

- $\rho_1(2t,t) = 3(2t,t) + (2,1) = (6t+2,3t+1)$; with $t \geq 0$, $3t+1 \geq 0$ and $\rho_1(2t,t) = (2(3t+1),3t+1) \in L'$.

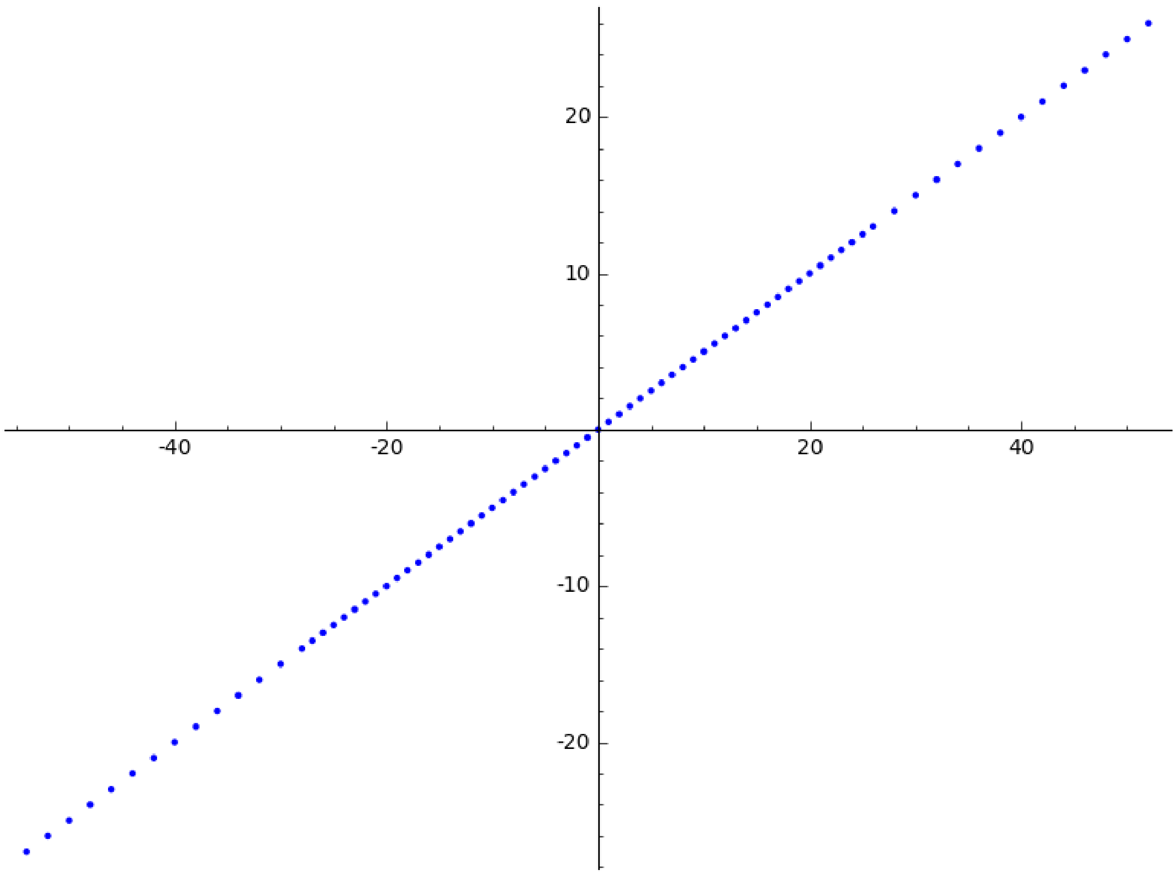

Figure 4.3 Plot of $l_3(0,0) \cup l_3(-1,-1/2) \cup l_3(-2,-1)$.

- $\rho_1(2t,t) = 3(2t,t)+(4,2) = (6t+4, 3t+2)$; with $t \geq 0$, $3t+2 \geq 0$ and $\rho_2(2t,t) = (2(3t+2), 3t+2) \in L'$.

($\supseteq$):

To show that each $(2t,t)$ with $t \geq 0$ is in $\mathfrak{L}(0,0)$, we write $t = \sum_{j=0}^{N} t_j 3^j$, $t_j \in \{0,1,2\}$, then induct on N:

Base case, $N = 0$, $t = 0, 1, 2$:

- $t = 0$: $\rho_0(0,0) = (0,0)$;
- $t = 1$: $\rho_1(0,0) = (2,1)$;
- $t = 2$: $\rho_2(0,0) = (4,2)$.

Induction step: Suppose $(2t,t) \in \mathfrak{L}(0,0)$ for all $t = \sum_{j=0}^{n} t_j 3^j$, $t_j \in \{0,1,2\}$, $n < N$; let $t = \sum_{j=0}^{N} t_j 3^j$.

Case 1: $t_0 = 0$. Then $t/3 = \sum_{j=0}^{N-1} t_{j+1} 3^j \in \mathbb{Z}$, $t/3 \geq 0$; so $(2t/3, t/3) \in \mathfrak{L}(0,0)$ by inductive assumption; then $(2t,t) = \rho_0(2t/3, t/3) \in \mathfrak{L}(0,0)$.

Case 2: $t_0 = 1$. Then $(t-1)/3 = \sum_{j=0}^{N-1} t_{j+1} 3^j \in \mathbb{Z}$, $(t-1)/3 \geq 0$, so $(2(t-1)/3, (t-1)/3) \in \mathfrak{L}(0,0)$ by inductive assumption; therefore $(2t,t) = \rho_1((2(t-1)/3, (t-1)/3) \in \mathfrak{L}(0,0)$.

Case 3: $t_0 = 2$. Then $(t-2)/3 = \sum_{j=0}^{N-1} t_{j+1} 3^j \in \mathbb{Z}$, $(t-2)/3 \geq 0$, so $(2(t-2)/3, (t-2)/3) \in \mathfrak{L}(0,0)$ by inductive assumption; therefore $(2t,t) = \rho_1((2(t-2)/3, (t-2)/3) \in \mathfrak{L}(0,0)$.

□

Claim 4.2.5. $\mathfrak{L}(-2,-1) = \{(2t,t) | t \in \mathbb{Z}, t \leq -1\}$.

Proof. ($\subseteq$):

Let $L' = \{(2t,t) | t \in \mathbb{Z}, t \leq -1\}$. We have: $l_1(-2,-1) = \{(-6,-3), (-4,-2), (-2,1)\} \subset L'$.

By construction, we need now only show that if $(x,y) \in L'$, each of $\rho_0(x,y), \rho_1(x,y), \rho_2(x,y)$ is in L':

- $\rho_0(2t,t) = 3(2t,t) = (6t,3t)$; with $t \leq -1$, $3t \leq -1$ and $\rho_0(2t,t) = (2(3t),3t) \in L'$.

- $\rho_1(2t,t) = 3(2t,t) + (2,1) = (6t+2, 3t+1)$; with $t \leq -1$, $3t+1 \leq -1$ and $\rho_1(2t,t) = (2(3t+1), 3t+1) \in L'$.

- $\rho_1(2t,t) = 3(2t,t) + (4,2) = (6t+4, 3t+2)$; with $t \leq -1$, $3t+2 \leq -1$ and $\rho_2(2t,t) = (2(3t+2), 3t+2) \in L'$.

($\supseteq$):

To show that each $(2t,t)$ with $t \leq -1$ is in $\mathfrak{L}(-2,-1)$, we write $t = -\sum_{j=0}^{N} t_j 3^j$, $t_j \in \{0,1,2\}$, then induct on N.

Base case, $N = 0$, $t = -1, -2$ (note that $t = 0$ is not in the set):

- $t = -1$: $\rho_2(-2,-1) = (-2,-1)$;

- $t = -2$: $\rho_1(-2,-1) = (-4,-2)$.

Induction step: Suppose $(2t,t) \in \mathfrak{L}(-2,-1)$ for all $t = -\sum_{j=0}^{n} t_j 3^j$, $t_j \in \{0,1,2\}$, $n < N$, $t \leq -1$; let $t = -\sum_{j=0}^{N} t_j 3^j$. Further, suppose that $t \leq -3$, since we already know $(-2,-1)$ and $(-4,-2) \in \mathfrak{L}(-2,-1)$.

Case 1: $t_0 = 0$. Then $t/3 = -\sum_{j=0}^{N-1} t_{j+1} 3^j$, and since $t \leq -3$, $t/3 \leq -1$. So we have that $(2t/3, t/3) \in \mathfrak{L}(-2,-1)$ by inductive assumption, therefore $(2t,t) = \rho_0(2t/3, t/3) \in \mathfrak{L}(-2,-1)$.

Case 2: $t_0 = 1$. Then $(t-1)/3 = \sum_{j=0}^{N-1} t_{j+1} 3^j$, and since $t \leq -3$, $(t-1)/3 \leq -1$. So we have that $(2(t-1)/3, (t-1)/3) \in \mathfrak{L}(-2,-1)$ by inductive assumption, therefore $(2t,t) = \rho_1(2t/3, t/3) \in \mathfrak{L}(-2,-1)$.

Case 3: $t_0 = 2$. Then $(t-2)/3 = \sum_{j=0}^{N-1} t_{j+1} 3^j$, and since $t \leq -3$, $(t-2)/3 \leq -1$. So we have that $(2(t-2)/3, (t-2)/3) \in \mathfrak{L}(-2,-1)$ by inductive assumption, therefore $(2t,t) = \rho_2(2t/3, t/3) \in \mathfrak{L}(-2,-1)$.

$\square$

Claim 4.2.6. $\mathfrak{L}(-1, -1/2) = \{(2t, t) | t \in \mathbb{Z}/2, t \notin \mathbb{Z}\}$.

Proof. ($\subseteq$):

Let $L' = \{(2t, t) | t \in \mathbb{Z}/2, t \notin \mathbb{Z}\}$. We have: $l_1(-2, -1) = \{(-3, -3/2), (-1, -1/2), (1, 1/2)\} \subset L'$.

By construction, we need now only show that if $(x, y) \in L'$, each of $\rho_0(x, y), \rho_1(x, y), \rho_2(x, y)$ is in L':

- $\rho_0(2t, t) = 3(2t, t) = (6t, 3t)$; with $t \in \mathbb{Z}/2, t \notin \mathbb{Z}$, $3t$ will also be in $\mathbb{Z}2 \setminus \mathbb{Z}$, so $\rho_0(2t, t) \in L'$.
- $\rho_1(2t, t) = 3(2t, t) + (2, 1) = (6t + 2, 3t + 1)$; with $t \in \mathbb{Z}/2, t \notin \mathbb{Z}$, $3t + 1$ will also be in $\mathbb{Z}2 \setminus \mathbb{Z}$, so $\rho_1(2t, t) \in L'$.
- $\rho_1(2t, t) = 3(2t, t) + (4, 2) = (6t + 4, 3t + 2)$; with $t \in \mathbb{Z}/2, t \notin \mathbb{Z}$, $3t + 2$ will also be in $\mathbb{Z}2 \setminus \mathbb{Z}$, so $\rho_2(2t, t) \in L'$.

($\supseteq$):

One way of expressing the set $\{t \in \mathbb{Z}/2, t \notin \mathbb{Z}\}$ is as the set $\{t \mid t + 1/2 \in \mathbb{Z}\}$. For our induction argument, we write the integer $t + 1/2$ in balanced ternary: $t + 1/2 = \sum_{j=0}^{N} t_j 3^j$, $t_j \in \{0, 1, -1\}$. With this expression, we can induct on N and conveniently cover all of our set L'.

Base case, $N = 0$, $t = -1/2, 1/2, -3/2$:

- $t = -1/2$: $\rho_1(-1/2, -1) = (-1/2, 1)$;
- $t = 1/2$: $\rho_2(-1/2, -1) = (1, 1/2)$;
- $t = -3/2$: $\rho_0(-1/2, -1) = (-3, -3/2)$.

Induction step: Suppose $(2t, t) \in \mathfrak{L}(-1, -1/2)$ for all $t + 1/2 = \sum_{j=0}^{n} t_j 3^j$, $t_j \in \{0, 1, -1\}$, $n < N$; let $t + 1/2 = \sum_{j=0}^{N} t_j 3^j$.

Case 1: $t_0 = -1$. Then $t + 1/2 = \sum_{j=0}^{N} t_j 3^j = -1 + \sum_{j=1}^{N} t_j 3^j$, or $t + 3/2 = \sum_{j=1}^{N} t_j 3^j$.

Therefore, $t/3 + 1/2 = \sum_{j=0}^{N-1} t_{j+1} 3^j$, and by our induction assumption, $(2t/3, t/3) \in \mathfrak{L}(-1, -1/2)$.

Thus $(2t, t) = \rho_0(2t/3, t/3) \in \mathfrak{L}(-1, -1/2)$.

Case 2: $t_0 = 0$. Then $t + 1/2 = \sum_{j=0}^{N} t_j 3^j = \sum_{j=1}^{N} t_j 3^j$, or $(t-1) + 3/2 = \sum_{j=1}^{N} t_j 3^j$.

Therefore, $(t-1)/3 + 1/2 = \sum_{j=0}^{N-1} t_{j+1} 3^j$, and by our induction assumption, $(2(t-1)/3, (t-1)/3) \in \mathfrak{L}(-1,-1/2)$.

Thus $(2t, t) = \rho_1(2(t-1)/3, (t-1)/3) \in \mathfrak{L}(-1,-1/2)$.

Case 3: $t_0 = 1$. Then $t + 1/2 = \sum_{j=0}^{N} t_j 3^j = 1 + \sum_{j=1}^{N} t_j 3^j$, or $(t-2) + 3/2 = \sum_{j=1}^{N} t_j 3^j$.

Therefore, $(t-2)/3 + 1/2 = \sum_{j=0}^{N-1} t_{j+1} 3^j$, and by our induction assumption, $(2(t-2)/3, (t-2)/3) \in \mathfrak{L}(-1,-1/2)$.

Thus $(2t, t) = \rho_2(2(t-2)/3, (t-2)/3) \in \mathfrak{L}(-1,-1/2)$.

□

Proof of Proposition 4.2.3. Putting together Claims 4.2.4, 4.2.5, 4.2.6, we get that $\mathfrak{L}(0,0) \cup \mathfrak{L}(-1/2,-1) \cup \mathfrak{L}(-1,-1/2) = \{(t, t/2) | t \in \mathbb{Z}\}$. □

Claim 4.2.7. *Let $\Lambda \subset \mathbb{R}^2$ be the smallest set that contains $-C$ for all W_B-cycles C, and such that $S\Lambda + L \subset \Lambda$. Then $\Lambda = \{(t, t/2) | t \in \mathbb{Z}\}$.*

Proof. Recall that $S = R^T = 3I$, and that $L = \{(0,0), (2,1), (4,2)\}$.

Let $L_0 = \{(t, t/2) | t \in \mathbb{Z}\}$.

By Claims 4.2.4, 4.2.5, 4.2.6, we know that $\mathfrak{L}(0,0) \cup \mathfrak{L}(-2,-1) \cup \mathfrak{L}(-1,-1/2) = L_0$.

For Λ to contain $-C$ for all W_B cycles C, and also contain $S\Lambda + L$ it must contain at least these three sets, so we have $L_0 \subseteq \Lambda$.

For $L_0 = \Lambda$, we need only show that it itself has the necessary properties; we know it contains $-C$ for all W_B cycles C, so we need only show $SL_0 + L \subseteq L_0$, that is, $\{\rho_0(t, t/2), \rho_1(t, t/2), \rho_2(t, t/2)\} \subseteq L_0$ for all $(t, t/2) \in L_0$.

Let $(t, t/2)$ with $t \in \mathbb{Z}$.

$\rho_0(t, t/2) = (3t, 3t/2)$; if $t \in \mathbb{Z}$, $3t \in \mathbb{Z}$, so $\rho_0(t, t/2) \in L_0$.

$\rho_1(t, t/2) = (3t+2, 3t/2+1) = (3t+2, (3t+2)/2)$; if $t \in \mathbb{Z}$, $3t+2 \in \mathbb{Z}$, so $\rho_1(t, t/2) \in L_0$.

$\rho_1(t, t/2) = (3t+4, 3t/2+3) = (3t+4, (3t+4)/2)$; if $t \in \mathbb{Z}$, $3t+4 \in \mathbb{Z}$, so $\rho_3(t, t/2) \in L_0$.

□

Therefore, by Theorem 2.2.13, if the transversality of the zeros condition is satisfied,

$$\{e^{2\pi i(t,t/2)\cdot(x,y)} \mid t \in \mathbb{Z}\} = \{e^{2\pi itx}e^{\pi ity} \mid t \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\nu_3)$.

4.2.1 The Transversality of the Zeros Condition

Recall the transversality of the zeros condition from Definition 2.2.12. The function W_X on X_L satisfies the *transversality of the zeros* condition if:

(a) If $x \in X_L$ is not a cycle, then there exists $k_x \geq 0$ such that, for $k \geq k_x$, $\{\tau_{l_1} \circ \tau_{l_2} \circ \cdots \circ \tau_{l_k} x : l_1, \ldots l_n \in L\}$ does not contain any zeros of W;

(b) If $\{x_0, x_1, \ldots, x_p\}$ are on a cycle with $x_1 = \tau_l(x_0)$ for some $l \in L$, then for every $y = \tau_{l'}(x_0)$, $y \neq x_1$ is either not on a cycle or $W(y) = 0$.

For the current case, $B = \{(0,0), (2,0), (0,2)\}$, so

$$W_B(x,y) = \frac{1}{9}\left|1 + e^{4\pi ix} + e^{4\pi iy}\right|^2 \tag{4.15}$$

$$= \frac{1}{9}\left(3 + 2\cos(4\pi x) + 2\cos(4\pi y) + 2\cos(4\pi(x-y))\right) \tag{4.16}$$

Therefore, $W_B(x,y) = 1$ if and only if $e^{4\pi ix} = e^{4\pi iy} = 1$, that is, when $(x,y) \in \mathbb{Z}/2 \times \mathbb{Z}/2$. Furthermore, $W_B(x,y) = 0$ if and only if $1 + e^{4\pi ix} + e^{4\pi iy} = 0$; that is, when $e^{4\pi ix} + e^{4\pi iy} = -1$; or when $(x,y) \in (1/3 + \mathbb{Z}/2, 1/6 + \mathbb{Z}/2) \cup (1/6 + \mathbb{Z}/2, 1/3 + \mathbb{Z}/2)$.

We show our set X_L satisfies Condition (b) first.

Recall from Proposition 4.2.2 that $X = X_L = \{(2t,t) : t \in [0,1]\}$.

Lemma 4.2.8. *$(2t,t) \in X_L$ is on a cycle if and only if $t = k/(3^n - 1)$ for some integer k, $n \in \mathbb{N}$.*

Proof. By definition, $(2t,t)$ is on a cycle if and only if $(2t,t) = r^n(2t,t)$ for some $n \in N$, where r is the common right inverse of ℓ_1, ℓ_2, ℓ_3:

$r(2t,t) = (2s,s)$ with $s = 3t \mod 1$.

However, there is overlap where $t = 1/3$ and $t = 2/3$; $(2/3, 1/3) = \ell_0(2,1) = \ell_1(0,0)$; $(4/3, 2/3) = \ell_2(0,0) = \ell_1(2,1)$, so r is not a well-defined function at those points: $(2/3, 1/3)$ and $(2/3, 4/3)$:

"r"$(2/3, 1/3) =$ "r"$(4/3, 2/3) = \{(0,0), (2,1)\}$.

Claim 4.2.9. *Neither (0,0) nor (2,1) is on a cycle with* $(4/3, 2/3)$*; similarly neither (0,0) nor (2,1) is on a cycle with* $(2/3, 1/3)$.

Proof. Compute: $r(0,0) = (0,0)$, and $r(2,1) = (2,1)$, so neither will go back to $(4/3, 1/3)$ or to $(2/3, 1/3)$. $\square$

Notice that both $(2/3, 1/3)$ and $(4/3, 2/3)$ have a factor of 3 in their denominators; thus they will be identified as non-cycles in the calculations below. Meanwhile, $0 = 0/(3^n - 1)$ and $1 = (3^n - 1)/(3^n - 1)$ so $(0,0)$ and $(2,1)$ fit the criteria for being cycles.

Aside from those two points, $(2t, t) = r^n(2t, t)$ if and only if $t = 3^n t \mod 1$, or $t - 3^n t = 1 \mod 1$, or $t(1 - 3^n) = k \in \mathbb{Z}$: in other words, $t = k/(1 - 3^n)$, for some $k \in \mathbb{Z}$. $\square$

Lemma 4.2.10. *If* $t = \dfrac{q}{3(3^n - 1)}$ *for some integer* q*,* $3 \nmid q$*, and* $n \in \mathbb{N}$ *then* $t \neq \dfrac{k}{3^l - 1}$ *for any integer* k*,* $l \in \mathbb{N}$.

Proof.
$$\frac{q}{3(3^n - 1)} = \frac{k}{3^l - 1} \iff q(3^l - 1) = 3k(3^n - 1)$$
Since 3 divides the right side, it must divide the left; however, we assumed $3 \nmid q$ and certainly $3 \nmid 3^l - 1$; and 3 is prime. $\square$

Proof of Condition (b). Let $x_0 = (2t, t)$ be on a cycle of length n; then $t = k/(3^n - 1)$ for some integer k. It is sufficient to show that exactly one of $\ell_0(x_0), \ell_1(x_0), \ell_2(x_0)$ is on a cycle.

There are three cases for the integer k: $k = 3m$ for some integer m, $k = 3m + 1$ for some integer m, or $k = 3m + 2$ for some integer m.

Case 1:

If $k = 3m$,

$$\begin{aligned}\ell_0(x_0) =& \ell_0\left(\frac{2(3m)}{3^n-1}, \frac{3m}{3^n-1}\right) = \left(\frac{2(3m)}{3(3^n-1)}, \frac{3m}{3(3^n-1)}\right)\\ =& \left(\frac{2(m)}{3^n-1}, \frac{m}{3^n-1}\right)\end{aligned}$$

which is on an n-cycle.

However,

$$\begin{aligned}\ell_1(x_0) &= \ell_1\left(\frac{2(3m)}{3^n-1}, \frac{3m}{3^n-1}\right) = \left(\frac{2(3m)+2}{3(3^n-1)}, \frac{3m+1}{3(3^n-1)}\right)\\ &= \left(\frac{2(3m+1)}{3(3^n-1)}, \frac{3m+1}{3(3^n-1)}\right)\end{aligned}$$

which cannot be on any cycle;

and:

$$\begin{aligned}\ell_2(x_0) &= \ell_2\left(\frac{2(3m)}{3^n-1}, \frac{3m}{3^n-1}\right) = \left(\frac{2(3m)+4}{3(3^n-1)}, \frac{3m+2}{3(3^n-1)}\right)\\ &= \left(\frac{2(3m+2)}{3(3^n-1)}, \frac{3m+2}{3(3^n-1)}\right)\end{aligned}$$

which cannot be on any cycle.

Case 2:

If $k = 3m + 1$,

$$\begin{aligned}\ell_0(x_0) =& \ell_0\left(\frac{2(3m+1)}{3^n-1}, \frac{3m+1}{3^n-1}\right) = \left(\frac{2(3m+2)}{3(3^n-1)}, \frac{3m+1}{3(3^n-1)}\right)\\ =& \left(\frac{2(3m+1)}{3(3^n-1)}, \frac{3m+1}{3(3^n-1)}\right)\end{aligned}$$

which cannot be on any cycle;

and:

$$\begin{aligned}\ell_1(x_0) &= \ell_1\left(\frac{2(3m+1)}{3^n-1}, \frac{3m+1}{3^n-1}\right) = \left(\frac{2(3m+1)+2}{3(3^n-1)}, \frac{3m+1+1}{3(3^n-1)}\right)\\ &= \left(\frac{2(3m+2)}{3(3^n-1)}, \frac{3m+2}{3(3^n-1)}\right)\end{aligned}$$

which cannot be on any cycle.

However:

$$\begin{aligned}
\ell_2(x_0) = \ell_2\left(\frac{2(3m+1)}{3^n-1}, \frac{3m+1}{3^n-1}\right) &= \left(\frac{2(3m+1)+4}{3(3^n-1)}, \frac{3m+1+2}{3(3^n-1)}\right) \\
&= \left(\frac{3(2m+2)}{3(3^n-1)}, \frac{3m+3}{3(3^n-1)}\right) \\
&= \left(\frac{2(m+1)}{3^n-1}, \frac{m+1}{3^n-1}\right)
\end{aligned}$$

which is on an n-cycle.

Case 3:

If $k = 3m+2$,

$$\begin{aligned}
\ell_0(x_0) =& \ell_0\left(\frac{2(3m+2)}{3^n-1}, \frac{3m+2}{3^n-1}\right) = \left(\frac{2(3m+2)}{3(3^n-1)}, \frac{3m+2}{3(3^n-1)}\right) \\
=& \left(\frac{2(3m+2)}{3(3^n-1)}, \frac{3m+2}{3(3^n-1)}\right)
\end{aligned}$$

which cannot be on any cycle.

However:

$$\begin{aligned}
\ell_1(x_0) = \ell_1\left(\frac{2(3m+2)}{3^n-1}, \frac{3m+2}{3^n-1}\right) &= \left(\frac{2(3m+2)+2}{3(3^n-1)}, \frac{3m+2+1}{3(3^n-1)}\right) \\
&= \left(\frac{3(2m+2)}{3(3^n-1)}, \frac{3m+3}{3(3^n-1)}\right) \\
&= \left(\frac{2(m+1)}{3^n-1}, \frac{m+1}{3^n-1}\right)
\end{aligned}$$

which is on an n-cycle.

But:

$$\begin{aligned}
\ell_2(x_0) = \ell_2\left(\frac{2(3m+2)}{3^n-1}, \frac{3m+1}{3^n-1}\right) &= \left(\frac{2(3m+2)+4}{3(3^n-1)}, \frac{3m+2+2}{3(3^n-1)}\right) \\
&= \left(\frac{2(3m+4)}{3(3^n-1)}, \frac{3m+4}{3(3^n-1)}\right) \\
&= \left(\frac{2(3(m+1)+1}{3(3^n-1)}, \frac{3(m+1)+1}{3(3^n-1}\right)
\end{aligned}$$

which cannot be on any cycle.

This concluses the proof of Condition (b). $\square$

Proof of Condition (a). We notice that there are only finitely many (x, y) that are both zeros of W_B and contained in X_L.

In particular: $\{(1/3, 1/6), (4/3, 2/3), (10/6, 5/6)\}$.

We know from Lemma 4.2.8 that none of these points are on cycles; we show also that there is no n with $r^n(1/3, 1/6) = (4/3, 2/3)$, $r^n(1/3, 1/6) = (10/6, 5/6)$, $r^n(4/3, 2/3) = (1/3, 1/6)$, $r^n(4/3, 2/3) = (10/6, 5/6)$, $r^n(10/6, 5/6) = (1/3, 1/6)$ or $r^n(10/6, 5/6) = (4/3, 2/3)$.

Now suppose for $(x, y) \in X$, (x, y) not in a cycle, $r^{-n}(x, y)$ contains $r^n(1/3, 1/6)$ for some n, that is, $(1/3, 1/6) = \ell_{j_n} \ldots \ell_{j_1}(x, y)$ with $j_1, \ldots, j_n \in \{0, 1, 2\}$. Claim: there is no $m > n$ with $(x, y) \in r^{-m}(x, y)$, that is, we cannot have $(1/3, 1/6) = \ell_{k_m} \ldots \ell_{k_n} \ldots \ell_{k_1}(x, y)$.

Suppose we do. Then:

$\ell_{j_n} \ldots \ell_{j_1}(x, y) = \ell_{k_m} \ldots \ell_{k_n} \ldots \ell_{k_1}(x, y)$.

Apply r^m to both sides:

$r^m \ell_{j_n} \ldots \ell_{j_1}(x, y) = r^m \ell_{k_m} \ldots \ell_{k_n} \ldots \ell_{k_1}(x, y)$.

And applying the left inverses: $r^{m-n}(x, y) = (x, y)$. This contradicts the assumption that (x, y) is not on a cycle.

Therefore, there is at most one $n \in \mathbb{N}$ with $(1/3, 1/6) \in r^{-n}(x, y)$. Similarly, we can show that there is at most one $n' \in \mathbb{N}$ with $(2/3, 4/3) \in r^{-n'}(x, y)$, and at most one $n'' \in \mathbb{N}$ with $(10/6, 5/6) \in r^{-n''}$. Therefore, we can take $n_{(x,y)} = \max\{n, n', n''\}$ and Condition (a) holds. $\square$

Therefore, the transversality of the zeros condition is satisfied, and

$$\{e^{2\pi i (t, t/2) \cdot (x,y)} \mid t \in \mathbb{Z}\} = \{e^{2\pi i t x} e^{\pi i t y} \mid t \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\nu_3)$.

4.3 A different proof

Now we construct a direct proof that $\mathcal{E} = \{e_{t,t/2} \mid t \in \mathbb{Z}\}$ is an orthonormal basis for $L_2(\nu_3)$ using Theorem 2.2.8. The argument in this section follows the outlines of a theorem by Dutkay, Picioroaga, and Song in (10):

Theorem 4.3.1. *Let $\mathcal{H}$ be a Hilbert space and $(S_i)_{i=0}^{N-1}$ be a representation of the Cuntz algebra $\mathcal{O}_n$. Let $\mathcal{E}$ be an orthonormal set in $\mathcal{H}$ and $f : X \to \mathcal{H}$ a norm continuous function on a topological space X with the following properties:*

(i) $\mathcal{E} = \bigcup_{i=0}^{N-1} S_i\mathcal{E}$

(ii) $\overline{span}\{f(t) : t \in X\} = \mathcal{H}$ *and* $||f(t)|| = 1$, *for all* $t \in X$.

(iii) There exists functions $\mathfrak{m}_i : X \to \mathbf{C}$, $g_i : X \to X$, $i = 0, \ldots, N-1$ *such that:*

$$S_i^* f(t) = \mathfrak{m}_i(t) f(g_i(t)), \quad t \in X \tag{4.17}$$

(iv) There exists $c_0 \in X$ *such that* $f(c_0) \in \overline{span}(\mathcal{E})$.

(v) The only continuous functions $h : X \to \mathbf{C}$ *with* $h \geq 0$, $h(c) = 1$ *for all* $c \in \{x \in X : f(x) \in \overline{span}(\mathcal{H})\}$ *and:*

$$h(t) = \sum_{i=0}^{N-1} |\mathfrak{m}_i(t)|^2 h(g_i(t)), \quad t \in X \tag{4.18}$$

are the constant functions.

Then $\mathcal{E}$ is an orthonormal basis for $\mathcal{H}$.

In this context, $\mathcal{H} = L^2(\nu_3)$, $\mathcal{X} = \mathbb{R}^2$, and $f(s,t) = e_{s,t}(x,y) = e^{2\pi i(s,t)\cdot(x,y)}$.

We will first construct a representation of $\mathcal{O}_3$ with $\mathcal{E} = \bigcup_{i=0}^{2} S_i\mathcal{E}$ to show that $\mathcal{E}$ is an orthonormal set. Then, we will construct $\mathfrak{m}_i$ and g_i and show that $h_X(s,t) := \sum_{\lambda\in\mathcal{E}} |\widehat{\nu_3}((s,t)-\lambda)|^2$ from Theorem 2.2.8 satisfies the conditions in (v). Then, through a continuity argument slightly simpler than in (v), we we show that $h_X(s,t) \equiv 1$ for all $t \in \mathbb{R}^2$, and thus that $\mathcal{E}$ is an orthonormal basis for $L_2(\nu_3)$ by Theorem 2.2.8.

4.3.1 The Cuntz algebra representation

Definition 4.3.2. *Let:* $m_0(x,y) = \frac{1}{\sqrt{3}}$

$m_1(x,y) = \frac{1}{\sqrt{3}} e^{2\pi i(x,y)\cdot(2,1)}$

$m_2(x,y) = \frac{1}{\sqrt{3}} e^{2\pi i(x,y)\cdot(4,2)}$

Lemma 4.3.3. *The matrix* $\mathcal{M}(x,y) = (m_j(\psi_k(x,y)))_{0\le j,k\le 2}$ *is unitary.*

Proof.

$$\mathcal{M}(x,y) := \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}}e^{2\pi i\left(\frac{1}{3}(x,y)\right)\cdot(2,1)} & \frac{1}{\sqrt{3}}e^{2\pi i\left(\frac{1}{3}(x,y)+(2/3,0)\right)\cdot(2,1)} & \frac{1}{\sqrt{3}}e^{2\pi i\left(\frac{1}{3}(x,y)+(0,2/3)\right)\cdot(2,1)} \\ \frac{1}{\sqrt{3}}e^{2\pi i\left(\frac{1}{3}(x,y)\right)\cdot(4,2)} & \frac{1}{\sqrt{3}}e^{2\pi i\left(\frac{1}{3}(x,y)+(2/3,0)\right)\cdot(4,2)} & \frac{1}{\sqrt{3}}e^{2\pi i\left(\frac{1}{3}(x,y)+(0,2/3)\right)\cdot(4,2)} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i\left(\frac{1}{3}(x,y)\right)\cdot(2,1)} & 0 \\ 0 & 0 & e^{2\pi i\left(\frac{1}{3}(x,y)\right)\cdot(4,2)} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}e^{2\pi i(2/3,0)\cdot(2,1)} & \frac{1}{\sqrt{3}}e^{2\pi i(0,2/3)\cdot(2,1)} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}e^{2\pi i(2/3,0)(4,2)} & \frac{1}{\sqrt{3}}e^{2\pi i(0,2/3)(4,2)} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i\left(\frac{2}{3}x+\frac{1}{3}y\right)} & 0 \\ 0 & 0 & e^{2\pi i\left(\frac{4}{3}x+\frac{2}{3}y\right)} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}e^{2\pi i(1/3)} & \frac{1}{\sqrt{3}}e^{2\pi i(2/3)} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}e^{2\pi i(2/3)} & \frac{1}{\sqrt{3}}e^{2\pi i(1/3)} \end{pmatrix}$$

Call the diagonal matrix D_1; the other matrix is $\frac{1}{\sqrt{3}}M_1$, where M_1 is same as in 4.5.

Notice D_1 is unitary for every x, y:

$$D_1D_1^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i\left(\frac{2}{3}x+\frac{1}{3}y\right)} & 0 \\ 0 & 0 & e^{2\pi i\left(\frac{4}{3}x+\frac{2}{3}y\right)} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2\pi i\left(\frac{2}{3}x+\frac{1}{3}y\right)} & 0 \\ 0 & 0 & e^{-2\pi i\left(\frac{4}{3}x+\frac{2}{3}y\right)} \end{pmatrix} = I_3$$

Meanwhile M_1 is also unitary, so $\mathcal{M}(x,y)$ is unitary for all (x,y).

□

Definition 4.3.4. *Let* $R(x,y) = 3(x,y) \mod 1$ *so that* $R \cdot \psi_j = I,\ j = 0,1,2$.

Then we can define:

$$S_j : L^2(\nu_3) \to L^2(\nu_3) \tag{4.19}$$

$$[S_jf](x,y) = \sqrt{3}m_j(x,y)f(R(x,y)). \tag{4.20}$$

Specifically:

$S_0(x,y) = f(3x, 3y \mod 1)$

$S_1(x,y) = e^{2\pi i(2x+y)}f(3(x,y) \mod 1)$

$S_2(x,y) = e^{2\pi i(4x+2y)}f(3(x,y) \mod 1)$

Claim 4.3.5. *For $j = 0, 1, 2$, S_j is an isometry in $L^2(\nu_3)$.*

Proof.

$$||S_j f||^2 = \int_{\mathcal{S}} |\sqrt{3} m_j(x,y) f(R(x,y))|^2 d(\nu_3) \tag{4.21}$$

$$= \frac{1}{3} \sum_{k=0}^{2} \int_{\mathcal{S}} 3 |m_j(\psi_k(x,y)) f(R(\psi_k(x,y)))|^2 d(\nu_3) \tag{4.22}$$

$$= \int_{\mathcal{S}} \left(\sum_{k=0}^{2} |m_j(\psi_k(x,y)|^2 \right) |f(x,y)| d(\nu_3) \tag{4.23}$$

$$\tag{4.24}$$

The $\sum_{k=0}^{2} |m_j(\psi_k(x,y))|^2$ is the square of the Euclidean norm of the jth row of the matrix $\mathcal{M}$; therefore, it is equal to 1.

So we have: $||S_j f||^2 = ||f||^2$, thus, S_j is an isometry. □

Lemma 4.3.6. *The adjoint to S_j, S_j^*, is given by:*

$$[S_j^* f](x,y) = \frac{1}{\sqrt{3}} \sum_{k=0}^{2} \overline{m_j(\psi_k(x,y))} f(\psi_k(x,y)) \tag{4.25}$$

Proof. Calculate:

$$\langle S_j f, g \rangle = \int_{\mathcal{S}} \sqrt{3} m_j(x,y) f(R(x,y)) \overline{g(x,y)} d\nu_3 \tag{4.26}$$

$$= \frac{1}{3} \sum_{k=0}^{2} \int_{\mathcal{S}} \sqrt{3} m_j(\psi_k(x,y)) f(R(\psi_k(x,y))) \overline{g(\psi_k(x,y))} d\nu_3 \tag{4.27}$$

$$= \int_{\mathcal{S}} f(x,y) \left(\frac{\sqrt{3}}{3} \sum_{k=0}^{2} m_j(\psi_k(x,y)) \overline{g(\psi_k(x,y))} \right) d\nu_3 \tag{4.28}$$

$$= \int_{\mathcal{S}} f(x,y) \overline{\left(\frac{1}{\sqrt{3}} \sum_{k=0}^{2} \overline{m_j(\psi_k(x,y))} g(\psi_k(x,y)) \right)} d\nu_3 \tag{4.29}$$

$$= \langle f, S_j^* g \rangle \tag{4.30}$$

□

Theorem 4.3.7. *$\{S_0, S_1, S_2\}$ is a representation of the Cuntz algebra $\mathcal{O}_3$.*

Proof. Condition 1: $S_j^* S_k = \delta_{jk} I$:

$$[S_j^* S_k f](x,y) = \frac{1}{\sqrt{3}} \sum_{l=0}^{2} \overline{m_j(\psi_l(x,y))} [S_k f](\psi_l(x,y)) \tag{4.31}$$

$$= \frac{1}{\sqrt{3}} \sum_{l=0}^{2} \overline{m_j(\psi_l(x,y))} \sqrt{3} m_k(\psi_l(x,y)) f(R(\psi_l(x,y))) \tag{4.32}$$

$$= \left(\sum_{l=0}^{2} \overline{m_j(\psi_l(x,y))} m_k(\psi_l(x,y)) \right) f(x,y) \tag{4.33}$$

This sum is the scalar product of the kth row with jth row of matrix $\mathcal{M}$, which is unitary. So the sum is $\delta_{jk} I$.

Condition 2: $\sum_{k=0}^{2} S_k S_k^* f = f$, for all $f \in \mathbf{L}^2(\nu_3)$:

Let $f, g \in L^2(\nu_3)$. Calculate:

$$\begin{aligned}
\left\langle \sum_{k=0}^{2} S_k S_k^* f, g \right\rangle &= \sum_{k=0}^{2} \langle S_k^* f, S_k^* g \rangle \\
&= \sum_{k=0}^{2} \int_{\mathcal{S}} \frac{1}{\sqrt{3}} \sum_{\ell=0}^{2} \overline{m_k(\psi_\ell(x,y))} f(\psi_\ell(x,y)) \overline{\left(\frac{1}{\sqrt{3}} \sum_{n=0}^{2} \overline{m_k(\psi_n(x,y))} g(\psi_n(x,y)) \right)} d\nu_3 \\
&= \sum_{l=0}^{2} \sum_{n=0}^{2} \frac{1}{3} \int_{\mathcal{S}} \left(\sum_{k=0}^{2} \overline{m_k(\psi_\ell(x,y))} m_k(\psi_n(x,y)) \right) f(\psi_\ell(x,y)) \overline{g(\psi_n(x,y))} d\nu_3 \\
&= \sum_{l=0}^{2} \sum_{n=0}^{2} \frac{1}{3} \int_{\mathcal{S}} \delta_{ln} f(\psi_\ell(x,y)) \overline{g(\psi_n(x,y))} d\nu_3 \\
&= \sum_{n=0}^{2} \frac{1}{3} \int_{\mathcal{S}} f(\psi_n(x,y)) \overline{g(\psi_n(x,y))} d\nu_3 \\
&= \int_{\mathcal{S}} f(x,y) \overline{g(x,y)} d\nu_3 \\
&= \langle f, g \rangle
\end{aligned}$$

□

Lemma 4.3.8. *$\mathcal{E} = \bigcup_{j=0}^{2} S_j \mathcal{E}$, where the union is disjoint.*

Proof.

$$\begin{aligned}[S_0 e_{t,t/2}](x,y) &= \sqrt{3}\left(\frac{1}{\sqrt{3}} m_0(x,y)\right) e_{t,t/2}(3x,3y) \\ &= e^{2\pi i t(3x)} e^{2\pi i \frac{t}{2}(3y)} \\ &= e^{2\pi i (3t)x} e^{2\pi i \frac{3t}{2}(y)} \\ &= e_{(3t,\frac{3t}{2})}(x,y)\end{aligned}$$

Therefore, for any $t \in \mathbb{Z}$, with $t = 3s$ for some $s \in \mathbb{Z}$, $e_{(t,t/2)} \in S_0\mathcal{E}$.

$$\begin{aligned}[S_1 e_{t,t/2}](x,y) &= \sqrt{3}\left(\frac{1}{\sqrt{3}} m_1(x,y)\right) e_{t,t/2}(3x,3y) \\ &= \sqrt{3}\left(\frac{1}{\sqrt{3}} e^{2\pi i(2x+y)}\right)\left(e^{2\pi i(3t)x} e^{2\pi i \frac{3t}{2} y}\right) \\ &= e^{2\pi i[2t+2]x} e^{2\pi i\left[\frac{3t}{2}+1\right]y} \\ &= e_{\left(3t+2,\frac{3t}{2}+1\right)}(x,y)\end{aligned}$$

Therefore, for any $t \in \mathbb{Z}$, with $t = 3s+2$ for some $s \in \mathbb{Z}$, $e_{(t,t/2)} \in S_1\mathcal{E}$.

$$\begin{aligned}[S_2 e_{t,t/2}](x,y) &= \sqrt{3}\left(\frac{1}{\sqrt{3}} m_2(x,y)\right) e_{t,t/2}(3x,3y) \\ &= \sqrt{3}\left(\frac{1}{\sqrt{3}} e^{2\pi i(4x+2y)}\right)\left(e^{2\pi i(3t)x} e^{2\pi i \frac{3t}{2} y}\right) \\ &= e^{2\pi i[3t+4]x} e^{2\pi i[\frac{3t}{2}+2]y} \\ &= e_{(3t+4,\frac{3t}{2}+2)}(x,y)\end{aligned}$$

Therefore, for any $t \in \mathbb{Z}$, with $t = 3s+1 = 3(s-1)+4$ for some $s \in \mathbb{Z}$, $e_{(t,t/2)} \in S_2\mathcal{E}$.

Since each $t \in \mathbb{Z}$ is in exactly one coset mod 3, we have that $\mathcal{E} = \cup_{j=0}^{2} S_j\mathcal{E}$, where the union is disjoint.

$\square$

Theorem 4.3.9. *$S_j^* e_{(s,t)} = \mathfrak{m}_j(s,t) e_{g_j(s,t)}$, where:*

$$\begin{aligned}
g_0(s,t) &= (\frac{s}{3}, \frac{t}{3}) \\
\mathfrak{m}_0(s,t) &= \frac{1}{3}\left(1 + e^{2\pi i \frac{2s}{3}} + e^{2\pi i \frac{2t}{3}}\right) \\
g_1(s,t) &= (\frac{s-2}{3}, \frac{t-1}{3}) \\
\mathfrak{m}_1(s,t) &= \frac{1}{3}\left(1 + e^{-2\pi i \frac{4}{3}} e^{2\pi i \frac{2s}{3}} + e^{-2\pi i \frac{2}{3}} e^{2\pi i \frac{2t}{3}}\right) \\
g_2(s,t) &= (\frac{s-4}{3}, \frac{t-2}{3}) \\
\mathfrak{m}_2(s,t) &= \frac{1}{3}\left(1 + e^{-2\pi i \frac{8}{3}} e^{2\pi i \frac{2s}{3}} + e^{-2\pi i \frac{4}{3}} e^{2\pi i \frac{2t}{3}}\right)
\end{aligned}$$

Proof. Calculate:

$$\begin{aligned}
[S_0^* e_{s,t})](x,y) &= \frac{1}{\sqrt{3}} \sum_{k=0}^{2} \overline{m_0(\psi_k(x,y))} e_{s,t}(\psi_k(x,y)) \\
&= \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{3}} e_{s,t}\left(\frac{x}{3}, \frac{y}{3}\right) + \frac{1}{\sqrt{3}} e_{s,t}\left(\frac{x+2}{3}, \frac{y}{3}\right) + \frac{1}{\sqrt{3}} e_{s,t}\left(\frac{x}{3}, \frac{y+2}{3}\right)\right) \\
&= \frac{1}{3}\left(e^{2\pi i (\frac{sx}{3} + \frac{ty}{3})} + e^{2\pi i (\frac{sx+2s}{3} + \frac{ty}{3})} + e^{2\pi i (\frac{sx}{3} + \frac{ty+2t}{3})}\right) \\
&= \frac{1}{3}\left(e^{2\pi i \frac{sx}{3}} e^{2\pi i \frac{ty}{3}} + e^{2\pi i \frac{sx}{3}} e^{2\pi i \frac{2s}{3}} e^{2\pi i \frac{ty}{3}} + e^{2\pi i \frac{sx}{3}} e^{2\pi i \frac{ty}{3}} e^{2\pi i \frac{2t}{3}}\right) \\
&= \frac{1}{3} e^{2\pi i \frac{sx}{3}} e^{2\pi i \frac{ty}{3}} \left(1 + e^{2\pi i \frac{2s}{3}} + e^{2\pi i \frac{2t}{3}}\right) \\
&= \frac{1}{3} e^{2\pi i (x,y)\cdot(\frac{s}{3}, \frac{t}{3})} \left(1 + e^{2\pi i \frac{2s}{3}} + e^{2\pi i \frac{2t}{3}}\right) \\
&= \frac{1}{3} e_{\frac{s}{3}, \frac{t}{3}}(x,y) \left(1 + e^{2\pi i \frac{2s}{3}} + e^{2\pi i \frac{2t}{3}}\right)
\end{aligned}$$

so $g_0(s,t) = (\frac{s}{3}, \frac{t}{3})$ and $\mathfrak{m}_0(s,t) = \frac{1}{3}\left(1 + e^{2\pi i \frac{2s}{3}} + e^{2\pi i \frac{2t}{3}}\right)$.

$$
\begin{aligned}
[S_1^* e_{s,t})](x,y) &= \frac{1}{\sqrt{3}} \sum_{k=0}^{2} \overline{m_1(\psi_k(x,y))} e_{s,t}(\psi_k(x,y)) \\
&= \frac{1}{\sqrt{3}} \sum_{k=0}^{2} \frac{1}{\sqrt{3}} e^{-2\pi i \psi_k(x,y)\cdot(2,1)} e_{s,t}(\psi_k(x,y)) \\
&= \frac{1}{3} \left(e^{-2\pi i \left(\frac{x}{3},\frac{y}{3}\right)\cdot(2,1)} e_{s,t}\left(\frac{x}{3},\frac{y}{3}\right) + e^{-2\pi i \left(\frac{x+2}{3},\frac{y}{3}\right)\cdot(2,1)} e_{s,t}\left(\frac{x+2}{3},\frac{y}{3}\right) \right. \\
&\qquad \left. + e^{-2\pi i \left(\frac{x}{3},\frac{y+2}{3}\right)\cdot(2,1)} e_{s,t}\left(\frac{x}{3},\frac{y+2}{3}\right)\right) \\
&= \frac{1}{3} \left(e^{-2\pi i \left(\frac{2x}{3}+\frac{y}{3}\right)} e^{2\pi i (\frac{sx}{3}+\frac{ty}{3})} + e^{-2\pi i \left(\frac{2x+4}{3}+\frac{y}{3}\right)} e^{2\pi i (\frac{sx+2s}{3}+\frac{ty}{3})} + e^{-2\pi i \left(\frac{2x}{3}+\frac{y+2}{3}\right)} e^{2\pi i (\frac{sx}{3}+\frac{ty+2t}{3})}\right) \\
&= \frac{1}{3} \left(e^{-2\pi i \frac{2x}{3}} e^{-2\pi i \frac{y}{3}} e^{2\pi i \frac{sx}{3}} e^{2\pi i \frac{ty}{3}} + e^{-2\pi i \frac{2x}{3}} e^{-2\pi i \frac{4}{3}} e^{-2\pi i \frac{y}{3}} e^{2\pi i \frac{sx}{3}} e^{2\pi i \frac{2s}{3}} e^{2\pi i \frac{ty}{3}} \right. \\
&\qquad \left. + e^{-2\pi i \frac{2x}{3}} e^{-2\pi i \frac{y}{3}} e^{-2\pi i \frac{2}{3}} e^{2\pi i \frac{sx}{3}} e^{2\pi i \frac{ty}{3}} e^{2\pi i \frac{2t}{3}} \right) \\
&= \frac{1}{3} e^{-2\pi i \frac{2x}{3}} e^{-2\pi i \frac{y}{3}} e^{2\pi i \frac{sx}{3}} e^{2\pi i \frac{ty}{3}} \left(1 + e^{-2\pi i \frac{4}{3}} e^{2\pi i \frac{2s}{3}} + e^{-2\pi i \frac{2}{3}} e^{2\pi i \frac{2t}{3}}\right) \\
&= \frac{1}{3} e^{2\pi i (-\frac{2x}{3}-\frac{y}{3}+\frac{sx}{3}+\frac{ty}{3})} \left(1 + e^{-2\pi i \frac{4}{3}} e^{2\pi i \frac{2s}{3}} + e^{-2\pi i \frac{2}{3}} e^{2\pi i \frac{2t}{3}}\right) \\
&= \frac{1}{3} e^{2\pi i (x,y)\cdot(\frac{s-2}{3},\frac{t-1}{3})} \left(1 + e^{-2\pi i \frac{4}{3}} e^{2\pi i \frac{2s}{3}} + e^{-2\pi i \frac{2}{3}} e^{2\pi i \frac{2t}{3}}\right) \\
&= \frac{1}{3} e_{\frac{s-2}{3},\frac{t-1}{3}}(x,y) \left(1 + e^{-2\pi i \frac{4}{3}} e^{2\pi i \frac{2s}{3}} + e^{-2\pi i \frac{2}{3}} e^{2\pi i \frac{2t}{3}}\right)
\end{aligned}
$$

so $g_1(s,t) = (\frac{s-2}{3}, \frac{t-1}{3})$, $\mathfrak{m}_1(s,t) = \frac{1}{3}\left(1 + e^{-2\pi i \frac{4}{3}} e^{2\pi i \frac{2s}{3}} + e^{-2\pi i \frac{2}{3}} e^{2\pi i \frac{2t}{3}}\right)$.

$$
\begin{aligned}
[S_2^* e_{s,t})](x,y) &= \frac{1}{\sqrt{3}} \sum_{k=0}^{2} \overline{m_2(\psi_k(x,y))} e_{s,t}(\psi_k(x,y)) \\
&= \frac{1}{\sqrt{3}} \sum_{k=0}^{2} \frac{1}{\sqrt{3}} e^{-2\pi i \psi_k(x,y)\cdot(4,2)} e_{s,t}(\psi_k(x,y)) \\
&= \frac{1}{3}\left(e^{-2\pi i\left(\frac{x}{3},\frac{y}{3}\right)\cdot(4,2)} e_{s,t}\left(\frac{x}{3},\frac{y}{3}\right) + e^{-2\pi i\left(\frac{x+2}{3},\frac{y}{3}\right)\cdot(4,2)} e_{s,t}\left(\frac{x+2}{3},\frac{y}{3}\right)\right. \\
&\qquad \left. + e^{-2\pi i\left(\frac{x}{3},\frac{y+2}{3}\right)\cdot(4,2)} e_{s,t}\left(\frac{x}{3},\frac{y+2}{3}\right)\right) \\
&= \frac{1}{3}\left(e^{-2\pi i\left(\frac{4x}{3}+\frac{2y}{3}\right)} e^{2\pi i\left(\frac{sx}{3}+\frac{ty}{3}\right)} + e^{-2\pi i\left(\frac{4x+8}{3}+\frac{2y}{3}\right)} e^{2\pi i\left(\frac{sx+2s}{3}+\frac{ty}{3}\right)} + e^{-2\pi i\left(\frac{4x}{3}+\frac{2y+4}{3}\right)} e^{2\pi i\left(\frac{sx}{3}+\frac{ty+2t}{3}\right)}\right) \\
&= \frac{1}{3}\left(e^{-2\pi i\frac{4x}{3}} e^{-2\pi i\frac{2y}{3}} e^{2\pi i\frac{sx}{3}} e^{2\pi i\frac{ty}{3}} + e^{-2\pi i\frac{4x}{3}} e^{-2\pi i\frac{8}{3}} e^{-2\pi i\frac{2y}{3}} e^{2\pi i\frac{sx}{3}} e^{2\pi i\frac{2s}{3}} e^{2\pi i\frac{ty}{3}}\right. \\
&\qquad \left. + e^{-2\pi i\frac{4x}{3}} e^{-2\pi i\frac{2y}{3}} e^{-2\pi i\frac{4}{3}} e^{2\pi i\frac{sx}{3}} e^{2\pi i\frac{ty}{3}} e^{2\pi i\frac{2t}{3}}\right) \\
&= \frac{1}{3} e^{-2\pi i\frac{4x}{3}} e^{-2\pi i\frac{2y}{3}} e^{2\pi i\frac{sx}{3}} e^{2\pi i\frac{ty}{3}} \left(1 + e^{-2\pi i\frac{8}{3}} e^{2\pi i\frac{2s}{3}} + e^{-2\pi i\frac{4}{3}} e^{2\pi i\frac{2t}{3}}\right) \\
&= \frac{1}{3} e^{2\pi i\left(-\frac{4x}{3}-\frac{2y}{3}+\frac{sx}{3}+\frac{ty}{3}\right)} \left(1 + e^{-2\pi i\frac{8}{3}} e^{2\pi i\frac{2s}{3}} + e^{-2\pi i\frac{4}{3}} e^{2\pi i\frac{2t}{3}}\right) \\
&= \frac{1}{3} e^{2\pi i(x,y)\cdot\left(\frac{s-4}{3},\frac{t-2}{3}\right)} \left(1 + e^{-2\pi i\frac{8}{3}} e^{2\pi i\frac{2s}{3}} + e^{-2\pi i\frac{4}{3}} e^{2\pi i\frac{2t}{3}}\right) \\
&= \frac{1}{3} e_{\frac{s-4}{3},\frac{t-2}{3}}(x,y) \left(1 + e^{-2\pi i\frac{8}{3}} e^{2\pi i\frac{2s}{3}} + e^{-2\pi i\frac{4}{3}} e^{2\pi i\frac{2t}{3}}\right)
\end{aligned}
$$

so $g_2(s,t) = (\frac{s-4}{3}, \frac{t-2}{3})$, $\mathfrak{m}_2(s,t) = \frac{1}{3}\left(1 + e^{-2\pi i\frac{8}{3}} e^{2\pi i\frac{2s}{3}} + e^{-2\pi i\frac{4}{3}} e^{2\pi i\frac{2t}{3}}\right)$. □

Lemma 4.3.10. $\sum_{j=0}^{2} |\mathfrak{m}_j(s,t)| = 1.$

Proof.

$$
\begin{aligned}
|\mathfrak{m}_0(s,t)|^2 &= \left|\frac{1}{3}\left(1 + e^{2\pi i\frac{2s}{3}} + e^{2\pi i\frac{2t}{3}}\right)\right|^2 \\
&= \frac{1}{9}\left(1 + e^{2\pi i\frac{2s}{3}} + e^{2\pi i\frac{2t}{3}}\right)\left(1 + e^{-2\pi i\frac{2s}{3}} + e^{-2\pi i\frac{2t}{3}}\right) \\
&= \frac{1}{9}\left(1 + e^{-2\pi i\frac{2s}{3}} + e^{-2\pi i\frac{2t}{3}} + e^{2\pi i\frac{2s}{3}} + 1 + e^{2\pi i\left(\frac{2s}{3}-\frac{2t}{3}\right)} + e^{2\pi i\frac{2t}{3}} + e^{2\pi i\left(\frac{2t}{3}-\frac{2s}{3}\right)} + 1\right) \\
&= \frac{1}{9}\left(3 + 2\cos\left(4\pi\frac{s}{3}\right) + 2\cos\left(4\pi\frac{t}{3}\right) + 2\cos\left(\frac{4\pi}{3}(s-t)\right)\right)
\end{aligned}
$$

$$\begin{aligned}
|\mathfrak{m}_1(s,t)|^2 &= \left|\frac{1}{3}\left(1+e^{-2\pi i\frac{4}{3}}e^{2\pi i\frac{2s}{3}}+e^{-2\pi i\frac{2}{3}}e^{2\pi i\frac{2t}{3}}\right)\right|^2\\
&= \frac{1}{9}\left(1+e^{-2\pi i\frac{4}{3}}e^{2\pi i\frac{2s}{3}}+e^{-2\pi i\frac{2}{3}}e^{2\pi i\frac{2t}{3}}\right)\left(1+e^{2\pi i\frac{4}{3}}e^{-2\pi i\frac{2s}{3}}+e^{2\pi i\frac{2}{3}}e^{-2\pi i\frac{2t}{3}}\right)\\
&= \frac{1}{9}\left(1+e^{2\pi i\frac{4}{3}}e^{-2\pi i\frac{2s}{3}}+e^{2\pi i\frac{2}{3}}e^{-2\pi i\frac{2t}{3}}+e^{-2\pi i\frac{4}{3}}e^{2\pi i\frac{2s}{3}}+1+e^{2\pi i\left(-\frac{2}{3}+\frac{2s}{3}-\frac{2t}{3}\right)}\right.\\
&\qquad\left.+e^{-2\pi i\frac{2}{3}}e^{2\pi i\frac{2t}{3}}+e^{2\pi i\left(\frac{2}{3}+\frac{2t}{3}-\frac{2s}{3}\right)}+1\right)\\
&= \frac{1}{9}\left(3+2\cos\left(\frac{4\pi}{3}\left(2-s\right)\right)+2\cos\left(\frac{4\pi}{3}\left(1-t\right)\right)+2\cos\left(\frac{4\pi}{3}\left(-1+s-t\right)\right)\right)
\end{aligned}$$

$$\begin{aligned}
|\mathfrak{m}_2(s,t)|^2 &= \frac{1}{3}\left|\left(1+e^{-2\pi i\frac{8}{3}}e^{2\pi i\frac{2s}{3}}+e^{-2\pi i\frac{4}{3}}e^{2\pi i\frac{2t}{3}}\right)\right|^2\\
&= \frac{1}{9}\left(1+e^{-2\pi i\frac{8}{3}}e^{2\pi i\frac{2s}{3}}+e^{-2\pi i\frac{4}{3}}e^{2\pi i\frac{2t}{3}}\right)\left(1+e^{2\pi i\frac{8}{3}}e^{-2\pi i\frac{2s}{3}}+e^{2\pi i\frac{4}{3}}e^{-2\pi i\frac{2t}{3}}\right)\\
&= \frac{1}{9}\left(1+e^{2\pi i\frac{8}{3}}e^{-2\pi i\frac{2s}{3}}+e^{2\pi i\frac{4}{3}}e^{-2\pi i\frac{2t}{3}}+e^{-2\pi i\frac{8}{3}}e^{2\pi i\frac{2s}{3}}+1+e^{-2\pi i\frac{8}{3}}e^{2\pi i\frac{2s}{3}}e^{2\pi i\frac{4}{3}}e^{-2\pi i\frac{2t}{3}}\right.\\
&\qquad\left.+e^{-2\pi i\frac{4}{3}}e^{2\pi i\frac{2t}{3}}+e^{-2\pi i\frac{4}{3}}e^{2\pi i\frac{2t}{3}}e^{2\pi i\frac{8}{3}}e^{-2\pi i\frac{2s}{3}}+1\right)\\
&= \frac{1}{9}\left(3+2\cos\left(\frac{4\pi}{3}(4-s)\right)+2\cos\left(\frac{4\pi}{3}(2-t)\right)+2\cos\left(\frac{4\pi}{3}(-2+s-t)\right)\right)
\end{aligned}$$

Therefore:

$$\begin{aligned}
\sum_{j=0}^{2}|\mathfrak{m}_j|^2 &= \frac{1}{9}\left(3+2\cos\left(4\pi\frac{s}{3}\right)+2\cos\left(4\pi\frac{t}{3}\right)+2\cos\left(\frac{4\pi}{3}(s-t)\right)\right.\\
&\qquad+3+2\cos\left(\frac{4\pi}{3}\left(2-s\right)\right)+2\cos\left(\frac{4\pi}{3}\left(1-t\right)\right)+2\cos\left(\frac{4\pi}{3}\left(-1+s-t\right)\right)\\
&\qquad\left.+3+2\cos\left(\frac{4\pi}{3}(4-s)\right)+2\cos\left(\frac{4\pi}{3}(2-t)\right)+2\cos\left(\frac{4\pi}{3}(-2+s-t)\right)\right)\\
&= 1+\frac{2}{9}\left(\cos\left(4\pi\frac{s}{3}\right)+\cos\left(\frac{4\pi}{3}\left(2-s\right)\right)+\cos\left(\frac{4\pi}{3}(4-s)\right)\right)\\
&\qquad+\frac{2}{9}\left(\cos\left(4\pi\frac{t}{3}\right)+\cos\left(\frac{4\pi}{3}\left(1-t\right)\right)+\cos\left(\frac{4\pi}{3}(2-t)\right)\right)\\
&\qquad+\frac{2}{9}\left(\cos\left(\frac{4\pi}{3}(s-t)\right)+\cos\left(\frac{4\pi}{3}\left(-1+s-t\right)\right)+\cos\left(\frac{4\pi}{3}(-2+s-t)\right)\right)\\
&= 1
\end{aligned}$$

The last line is because:

$$\begin{aligned}\cos\left(\frac{4\pi}{3}s\right)+\cos\left(\frac{4\pi}{3}s-\frac{8\pi}{3}\right)+\cos\left(\frac{4\pi}{3}s-\frac{16\pi}{3}\right) &= \mathrm{Re}\left(e^{\frac{4\pi}{3}si}+e^{\frac{4\pi}{3}si-\frac{8\pi}{3}i}+e^{\frac{4\pi}{3}si-\frac{16\pi}{3}i}\right)\\ &= \mathrm{Re}\left(e^{\frac{4\pi}{3}si}\left(1+e^{-\frac{8\pi}{3}i}+e^{-\frac{16\pi}{3}i}\right)\right)\\ &= \mathrm{Re}\left(e^{\frac{4\pi}{3}si}\left(1+e^{-\frac{2\pi}{3}i}+e^{-\frac{4\pi}{3}i}\right)\right)\\ &= \mathrm{Re}(0)=0\end{aligned}$$

□

4.3.2 The function h_X

Recall from Theorem 2.2.8 that:

$$h_X(t) := \sum_{\lambda\in\mathcal{E}} \left|\hat{\mu_B}(t-\lambda)\right|^2, \quad t\in\mathbb{R}^d, \quad \lambda\in\mathcal{E}.$$

For $\mathcal{E}=\{e_{u,u/2}|u\in\mathbb{Z}\}$, and ν_3 as in Definition 4.1.1,

$$h_X(s,t) = \sum_{(u,u/2)\in\mathbb{Z}} |\widehat{\nu_3}(s-u,t-u/2)|^2. \tag{4.34}$$

Theorem 4.3.11. *For $\mathcal{E}=\{e_{u,u/2}|u\in\mathbb{Z}\}$ and ν_3 as in Definition 4.1.1, $h_X(s,t)=\sum_{u\in\mathbb{Z}}|\left\langle e_{s,t},e_{u,u/2}\right\rangle|^2$.*

Proof. We compute:

$$\begin{aligned}\left\langle e_{s,t},e_{u,u/2}\right\rangle &= \int e^{2\pi i(sx+ty)}e^{-2\pi i(ux+uy/2)}\ d\nu_3(x,y)\\ &= \int e^{2\pi i(sx+ty-ux-uy/2)}\ d\nu_3(x,y)\\ &= \int e^{2\pi i((s-u)x+(t-u/2))y}\ d\nu_3(x,y)\\ &= \int e^{-2\pi i((u-s)x+(u/2-t)y)}\ d\nu_3(x,y)\\ &= \widehat{\nu_3}(u-s,u/2-t)\end{aligned}$$

Therefore, $|\left\langle e_{s,t},e_{u,u/2}\right\rangle|^2 = |\widehat{\nu_3}(s-u,t-u/2)|^2$ and

$$h_X(s,t)=\sum_{\lambda\in\mathcal{E}}|\left\langle e_{s,t},e_{u,u/2}\right\rangle|^2 = \sum_{(u,u/2)\in\mathcal{Z}}|\widehat{\nu_3}(s-u,t-u/2)|^2. \tag{4.35}$$

□

Lemma 4.3.12.

$$h_X(s,t) = \sum_{j=0}^{2} |\mathfrak{m}_j(s,t)|^2 h_X(g_j(s,t)) \tag{4.36}$$

Proof. Recall Lemma 4.3.8:

$$\mathcal{E} = \{e_{u,u/2} | u \in \mathbb{Z}\} = \bigcup_{j=0}^{2} S_j(\mathcal{E})$$

where the union is disjoint.

As a consequence:

$$\begin{aligned}
h_X(s,t) &= \sum_{u\in\mathbb{Z}} |\left\langle e_{s,t}, e_{u,u/2}\right\rangle|^2 && (4.37)\\
&= \sum_{j=0}^{2}\sum_{u\in\mathbb{Z}} |\left\langle e_{s,t}, S_j e_{u,u/2}\right\rangle|^2 && (4.38)\\
&= \sum_{j=0}^{2}\sum_{u\in\mathbb{Z}} |\left\langle S_j^* e_{s,t}, e_{u,u/2}\right\rangle|^2 && (4.39)\\
&= \sum_{j=0}^{2}\sum_{u\in\mathbb{Z}} |\left\langle \mathfrak{m}_j(s,t) e_{g_j(s,t)}, e_{u,u/2}\right\rangle|^2 && (4.40)\\
&= \sum_{j=0}^{2} |\mathfrak{m}_j(s,t)|^2 \sum_{u\in\mathbb{Z}} |\left\langle e_{g_j(s,t)}, e_{u,u/2}\right\rangle|^2 && (4.41)\\
&= \sum_{j=0}^{2} |\mathfrak{m}_j(s,t)|^2 h_X(g_j(s,t)) && (4.42)
\end{aligned}$$

Line (4.40) follows from (4.39) by Theorem 4.3.9.

□

Theorem 4.3.13. $\widehat{\nu}_3(s,t) = \prod_{j=1}^{\infty} \frac{1}{3}\left(1 + e^{-2\pi i(2s/3^j)} + e^{-2\pi i(2t/3^j)}\right)$.

Proof. We use equation (2.4).

$$\begin{aligned}
\widehat{\nu_3}(s,t) &= \int e^{-2\pi i(sx+ty)}\ d\nu_3(x,y) \\
&= \frac{1}{3}\left(\int e^{-2\pi i(s(x/3)+t(y/3))}\ d\nu_3(x,y) + \int e^{-2\pi i(s(x+2)/3+ty/3)}\ \nu_3(x,y) + \int e^{-2\pi i(sx/3+t(y+2/3))}\ \nu_3(x,y)\right) \\
&= \frac{1}{3}\left(\int e^{-2\pi i((s/3)x+(t/3)y)}\ d\nu_3(x,y) + \int e^{-2\pi i((s/3)x+(t/3)y)}e^{-2\pi i 2s/3}\ \nu_3(x,y) + \right. \\
&\qquad \left. \int e^{-2\pi i((s/3)x+(t/3)y)}e^{-2\pi i 2t/3}\ \nu_3(x,y)\right) \\
&= \frac{1}{3}\left(\widehat{\nu_3}\left(\frac{s}{3},\frac{t}{3}\right) + \widehat{\nu_3}\left(\frac{s}{3},\frac{t}{3}\right)e^{-2\pi i(2s/3)} + \widehat{\nu_3}\left(\frac{s}{3},\frac{t}{3}\right)e^{-2\pi i(2t/3)}\right) \\
&= \frac{1}{3}\left(1+e^{-2\pi i(2s/3)}+e^{-2\pi i(2t/3)}\right)\widehat{\nu_3}\left(\frac{s}{3},\frac{t}{3}\right)
\end{aligned}$$

By induction:

$$\begin{aligned}
\widehat{\nu_3}(s/3^n,t/3^n) &= \int e^{-2\pi i(s/3^n x+t/3^n y)}\ d\nu_3(x,y) \\
&= \frac{1}{3}\left(\int e^{-2\pi i(s/3^n(x/3)+t/3^n(y/3))}\ d\nu_3(x,y) + \int e^{-2\pi i(s/3^n(x+2)/3+t/3^n y/3)}\ \nu_3(x,y) \right. \\
&\qquad \left. + \int e^{-2\pi i(s/3^n x/3+t/3^n(y+2/3))}\ \nu_3(x,y)\right) \\
&= \frac{1}{3}\left(\int e^{-2\pi i((s/3^{n+1})x+(t/3^{n+1})y)}\ d\nu_3(x,y) + \int e^{-2\pi i((s/3^{n+1})x+(t/3^{n+1})y)}e^{-2\pi i 2s/3^{n+1}}\ \nu_3(x,y) + \right. \\
&\qquad \left. \int e^{-2\pi i((s/3^{n+1})x+(t/3^{n+1})y)}e^{-2\pi i 2t/3^{n+1}}\ \nu_3(x,y)\right) \\
&= \frac{1}{3}\left(\widehat{\nu_3}\left(\frac{s}{3^{n+1}},\frac{t}{3^{n+1}}\right) + \widehat{\nu_3}\left(\frac{s}{3^{n+1}},\frac{t}{3^{n+1}}\right)e^{-2\pi i(2s/3^{n+1})} + \widehat{\nu_3}\left(\frac{s}{3^{n+1}},\frac{t}{3^{n+1}}\right)e^{-2\pi i(2t/3^{n+1})}\right) \\
&= \frac{1}{3}\left(1+e^{-2\pi i(2s/3^{n+1})}+e^{-2\pi i(2t/3^{n+1})}\right)\widehat{\nu_3}\left(\frac{s}{3^{n+1}},\frac{t}{3^{n+1}}\right)
\end{aligned}$$

It remains to show that the product $\prod_{j=1}^{\infty}\frac{1}{3}\left(1+e^{-2\pi i(2s/3^j)}+e^{-2\pi i(2t/3^j)}\right)$ converges for all $(s,t)\in\mathbb{R}^2$.

The product

$$\prod_{j=1}^{\infty}\frac{1}{3}\left(1+e^{-2\pi i(2s/3^j)}+e^{-2\pi i(2t/3^j)}\right)$$

converges if

$$\sum_{j=1}^{\infty}\left|\frac{1}{3}\left(1+e^{-2\pi i(2s/3^j)}+e^{-2\pi i(2t/3^j)}\right)-1\right|<\infty,$$

by Corollary 8.1.8 in (14).

We compute:

$$\begin{aligned}
\left|\frac{1}{3}(1+e^{-2\pi i(2s/3^j)}+e^{-2\pi i(2t/3^j)})-1\right| &= \left|\frac{1}{3}+\frac{1}{3}e^{-2\pi i(2s/3^j)}+\frac{1}{3}e^{-2\pi i(2t/3^j)}-1\right| \\
&= \left|\frac{e^{-2\pi i(2s/3^j)}-1}{3}+\frac{e^{-2\pi i(2t/3^j)}-1}{3}\right| \\
&\leq \frac{1}{3}\left|e^{-2\pi i(2s/3^j)}-1\right|+\frac{1}{3}\left|e^{-2\pi i(2t/3^j)}-1\right| \\
&= \frac{1}{3}\left|\cos\left(2\pi(2s/3^j)\right)-1+i\sin\left(2\pi(2s/3^j)\right)\right|+\frac{1}{3}\left|\cos\left(2\pi(2t/3^j)\right)-1+i\sin\left(2\pi(2t/3^j)\right)\right| \\
&\leq \frac{1}{3}\left|\cos\left(2\pi(2s/3^j)\right)-1\right|+\frac{1}{3}\left|\sin\left(2\pi(2s/3^j)\right)\right|+\frac{1}{3}\left|\cos\left(2\pi(2t/3^j)\right)-1\right|+\frac{1}{3}\left|\sin\left(2\pi(2t/3^j)\right)\right|
\end{aligned}$$

Now let $f(x)=\cos(4\pi(x))$, so that $\cos\left(2\pi(2s/3^j)\right)-1=f(s/3^j)-f(0/3^j)$. By the Mean Value Theorem, $|f(s/3^j)-f(0/3^j)|=|f'(x_s)(s/3^j-0/3^j)|$, for some $x_s\in(0,s/3^j)$. Since $f'(x)=-4\pi\sin(4\pi x)$, $|f'(x)|\leq 4\pi$, and therefore, $|f(s/3^j)-f(0/3^j)|\leq|4\pi(s/3^j-0/3^j)|=\frac{4\pi|s|}{3^j}$.

By the exact same argument, $\left|\cos\left(2\pi(2t/3^j)\right)-1\right|\leq\frac{4\pi|t|}{3^j}$.

We proceed similarly for the sines. Let $g(x)=\sin(4\pi(x))$, so that $\sin\left(2\pi(2s/3^j)\right)=g(s/3^j)-g(0/3^j)$. By the Mean Value Theorem, $|g(s/3^j)-g(0/3^j)|=|g'(x_s)(s/3^j-0/3^j)|$, for some $x_s\in(0,s/3^j)$. Since $g'(x)=4\pi\cos(4\pi x)$, $|g'(x)|\leq 4\pi$, and therefore, $|g(s/3^j)-g(0/3^j)|\leq|4\pi(s/3^j-0/3^j)|=\frac{4\pi|s|}{3^j}$. By the exact same argument, $\left|\sin\left(2\pi(2t/3^j)\right)-1\right|\leq\frac{4\pi|t|}{3^j}$.

Therefore:

$$\begin{aligned}
&\sum_{j=1}^{\infty}\left|\frac{1}{3}\left(1+e^{-2\pi i(2s/3^j)}+e^{-2\pi i(2t/3^j)}\right)-1\right| \\
&\quad\leq\sum_{j=1}^{\infty}\left(\frac{1}{3}\left|\cos\left(2\pi(2s/3^j)\right)-1\right|+\frac{1}{3}\left|\sin\left(2\pi(2s/3^j)\right)\right|+\frac{1}{3}\left|\cos\left(2\pi(2t/3^j)\right)-1\right|+\frac{1}{3}\left|\sin\left(2\pi(2t/3^j)\right)\right|\right) \\
&\quad\leq\sum_{j=1}^{\infty}\left(\frac{1}{3}\frac{4\pi|s|}{3^j}+\frac{1}{3}\frac{4\pi|s|}{3^j}+\frac{1}{3}\frac{4\pi|t|}{3^j}+\frac{1}{3}\frac{4\pi|t|}{3^j}\right)<\infty
\end{aligned}$$

□

Claim 4.3.14. *$h_X(s,t)\neq 0$ for $(s,t)=(-\frac{1}{2},-1)$.*

Proof. It is sufficient to show $\widehat{\nu_3}(u-(-\frac{1}{2}),u/2-(-1))\neq 0$ for at least one $u\in\mathbb{Z}$.

Let $u = 1$:

$$\widehat{\nu_3}\left(1 - \left(-\frac{1}{2}\right), 1/2 - (-1)\right) = \widehat{\nu_3}\left(\frac{3}{2}, \frac{3}{2}\right)$$
$$= \prod_{j=1}^{\infty} \frac{1}{3}\left(1 + e^{-2\pi i(3/3^j)} + e^{-2\pi i(3/3^j)}\right)$$
$$= \prod_{j=1}^{\infty} \frac{1}{3}\left(1 + 2e^{-2\pi i(3/3^j)}\right)$$

This cannot have any term equal to zero because of the factor of 2. Therefore, $\widehat{\nu_3}(1 - (-\frac{1}{2}), 1/2 - (-1)) \neq 0$ and $h_X(-\frac{1}{2}, -1) \neq 0$. $\square$

Claim 4.3.15. *The only values* $(s,t) \in \mathbb{R}^2$ *with* $\mathfrak{m}_0(s,t) = 0$ *are*

$$\left(1 + \frac{3k}{2}, \frac{1}{2} + \frac{3\ell}{2}\right) \text{ or } \left(\frac{1}{2} + \frac{3k}{2}, 1 + \frac{3\ell}{2}\right)$$

for $k, \ell \in \mathbb{Z}$.

Proof. Recall $\mathfrak{m}_0(s,t) = \frac{1}{3}\left(1 + e^{2\pi i \frac{2s}{3}} + e^{2\pi i \frac{2t}{3}}\right)$.

Suppose $\mathfrak{m}_0(s,t) = 0$ for some (s,t). Then $1 + e^{2\pi i \frac{2s}{3}} + e^{2\pi i \frac{2t}{3}} = 0$.

We rewrite this equation in rectangular coordinates. Let $e^{2\pi i \frac{2s}{3}} = x + iy$, $e^{2\pi i \frac{2t}{3}} = u + iv$ for $x, y, u, v \in \mathbb{R}$. Then we have

$$x^2 + y^2 = u^2 + v^2 = 1 \tag{4.43}$$

and

$$1 + x + iy + u + iv = 0. \tag{4.44}$$

Rewrite (4.44) as the two real equations:

$$1 + x + u = 0 \tag{4.45}$$

$$y + v = 0. \tag{4.46}$$

Use (4.45) and (4.46) to rewrite x and y in terms of u and v, then substitute into (4.43) to get: $(-u-1)^2+(-v)^2=1$ or $u^2+2u+1+v^2=1$ Using (4.43) to write $v^2=1-u^2$ we get:

$$u^2+2u+1+1-u^2=1 \iff 2u+2=1 \iff u=-\frac{1}{2}. \tag{4.47}$$

Then

$$v^2=1-u^2=1-\frac{1}{4} \iff v=\pm\frac{\sqrt{3}}{2} \tag{4.48}$$

$$y=-v=\mp\frac{\sqrt{3}}{2} \tag{4.49}$$

$$x=-u-1=-(-1/2)-1=-1/2. \tag{4.50}$$

Therefore we have:

$$x+iy=-\frac{1}{2}+i\frac{\sqrt{3}}{2} \text{ and } u+iv=-\frac{1}{2}-i\frac{\sqrt{3}}{2} \tag{4.51}$$

$$x+iy=-\frac{1}{2}-i\frac{\sqrt{3}}{2} \text{ and } u+iv=-\frac{1}{2}+i\frac{\sqrt{3}}{2}. \tag{4.52}$$

Translating back into polar form, we have either:

$$e^{2\pi i\frac{2s}{3}}=e^{2\pi i/3+k2\pi i} \text{ and } e^{2\pi i\frac{2t}{3}}=e^{4\pi i/3+\ell 2\pi i}, \text{ for some } k,l\in\mathbb{Z}, \tag{4.53}$$

which gives $4\pi is/3=2\pi i/3+k2\pi i$ or $s=1/2+3/2k$ and $4\pi it/3=4\pi i/3+\ell 2\pi i$ or $t=1+3/2\ell$; or

$$e^{2\pi i\frac{2s}{3}}=e^{4\pi i/3+k2\pi i} \text{ and } e^{2\pi i\frac{2t}{3}}=e^{2\pi i/3+k2\pi i}, \tag{4.54}$$

which gives $4\pi is/3=4\pi i/3+k2\pi i$ and $4\pi it/3=2\pi i/3+\ell 2\pi i$ or: $s=1+3/2k$ and or $t=1/2+3/2\ell$. $\square$

4.3.3 Argument by rectangles

We will show that $h_X(s,t)\equiv 1$ by first proving the result on rectangles of the form: $\mathcal{R}_{a,b}=[-a,a]\times[-b,0]$. By extending our results to bigger and bigger rectangles, we show the result holds true for all of $\{(s,t)\in\mathbb{R}^2 : t\leq 0\}$.

Lemma 4.3.16. *For any $a \geq 2$, $b \geq 1$, and for all $j = 0, 1, 2$:*

$$g_j(\mathcal{R}_{a,b}) \subseteq \mathcal{R}_{a/3+4/3,b/3+2/3} \subseteq \mathcal{R}_{a,b}. \tag{4.55}$$

Proof. Case 0: $g_0(\mathcal{R}_{a,b}) = [-a/3, a/3] \times [-b/3, 0] = \mathcal{R}_{a/3,b/3}$.

Case 1: $g_1(\mathcal{R}_{a,b}) = [-a/3 - 2/3, a/3 - 2/3] \times [-b/3 - 1/3, -1/3] = \mathcal{R}_{a/3+2/3,b/3+1/3}$.

Case 2: $g_2(\mathcal{R}_{a,b}) = [-a/3 - 4/3, a/3 - 4/3] \times [-b/3 - 2/3, -2/3] = \mathcal{R}_{a/3+4/3,b/3+2/3}$.

For $a \geq 2, b \geq 1$, $\mathcal{R}_{a/3+4/3,b/3+2/3} \subseteq \mathcal{R}_{a,b}$: $a/3 + 4/3 \leq a \iff a + 4 \leq 3a \iff 4 \leq 2a \iff 2 \leq a$; $b/3 + 2/3 \leq b \iff b + 2 \leq 3b \iff 2 \leq 2b \iff 1 \leq b$. □

Lemma 4.3.17. *$h_X \equiv 1$ on $\mathcal{R} = \mathcal{R}_{9/4,3/2}$, where $\mathcal{R}_{a,b} = [-a, a] \times [-b, 0]$ for some $a \geq 2$, $b \geq 1$.*

We know that h_X is continuous, $0 \leq h_X(s,t) \leq 1$ for all $(s,t) \in \mathbb{R}^2$, $h_X(c,d) = 1$ for all (c,d) such that $e^{2\pi i(cx+dy)} \in \overline{\text{span}}\mathcal{E}$, and:

$$h_X(s,t) = \sum_{i=0}^{2} |m_i(s,t)|^2 h_X(g_i(s,t)). \tag{4.56}$$

Let $\beta = \min\{h_X(s,t) : (s,t) \in \mathcal{R}$. Then $h_1 = h_X - \beta$ also satisfies (4.56), h_1 is continuous and for some $(s_0, t_0) \in \mathcal{R}$, $h_1(s_0, t_0) = 0$, while $h_1(s,t) \geq 0$ for all $(s,t) \in \mathcal{R}$.

Claim 4.3.18. *$h_1(2b, b) \neq 0$ for $b = 0, -1/2, -1$, unless $h_X \equiv 1$ on $\mathcal{R}$.*

Proof. Since $h_X(2b, b) = 1$ for $b = 0, -1/2, -1$, $h_1(2b, b) = h_X(2b, b) - \beta = 0$ iff $\beta = 1$. But $0 \leq h_X(s,t) \leq 1$ for all $(s,t) \in \mathbb{R}^2$, and β is defined as the minimum value of h_X on $\mathcal{R}$. If the minimum value is also the maximum value, $h_X(s,t) \equiv 1$ on all of $\mathcal{R}$. □

Now take $(s_0, t_0) \in \mathcal{R}$ with $h_1(s_0, t_0) = 0$.

Then: $h_1(s_0, t_0) = \sum_{j=0}^{2} |\mathfrak{m}_j(s_0, t_0)|^2 h_1(g_j(s_0, t_0)) = 0$.

Since $|\mathfrak{m}_j(s_0, t_0)|^2 \geq 0$ and $h_1(g_j(s_0, t_0)) \geq 0$, $h_1(s_0, t_0) = 0$ if and only if $|\mathfrak{m}_j(s_0, t_0)|^2 h_1(g_j(s_0, t_0)) = 0$ for $j = 0, j = 1$, and $j = 2$.

In particular, we must have $|\mathfrak{m}_0(s_0, t_0)|^2 h_1(g_0(s_0, t_0)) = 0$.

If $|\mathfrak{m}_0(s_0, t_0)| = 0$, we have from Claim 4.3.15 that:

$(s_0, t_0) = (1 + \frac{3k}{2}, \frac{1}{2} + \frac{3\ell}{2})$ or $(s_0, t_0) = (\frac{1}{2} + \frac{3k}{2}, 1 + \frac{3\ell}{2})$, for $k, \ell \in \mathbb{Z}$.

The only possibilities that give $(s_0, t_0) \in \mathcal{R}$ are $(-2, -1)$, $(-1/2, -1)$, $(1, -1)$, $(-1, -1/2)$, $(1/2, -1/2)$, $(2, -1/2)$

From Claim 4.3.18, $h_1(-1, -1/2) \neq 0$ and $h_1(-2, -1) \neq 0$ (or we're done).

Claim 4.3.19. *If $h_1(s_0, t_0) = 0$ for any $(s_0, t_0) \in \{(-1/2, -1), (1/2, -1/2), (2, -1/2), (1, -1)\}$, then $h_1(s, t) = 0$ for some other point $(s, t) \in \mathcal{R}$ with $\mathfrak{m}_0(s, t) \neq 0$. Therefore, $h_1(s, t) = 0$ for some (s, t) with $\mathfrak{m}_0(s, t) \neq 0$.*

Proof. By cases.

$\mathfrak{m}_2(-1/2, -1) = 1 \neq 0$. Since $\mathfrak{m}_2(-1/2, -1) \neq 0$, we can say $h_1(g_2(-1/2, -1)) = 0$. $g_2(-1/2, -1) = (-3/2, -1)$; so if $h_1(-1/2, -1) = 0$, so does $h_1(-3/2, -1)$, and $\mathfrak{m}_0(-3/2, -1) \neq 0$.

$\mathfrak{m}_1(1/2, -1/2) = 1$, therefore, if $h_1(1/2, -1/2) = 0$, then $h_1(g_1(1/2, -1/2)) = 0$. $g_1(1/2, -1/2) = (-1/2, -1/2)$, and $\mathfrak{m}_0(-1/2, -1/2) \neq 0$.

$\mathfrak{m}_1(2, -1/2) = 1$, therefore, if $h_1(2, -1/2) = 0$, then $h_1(g_1(2, -1/2)) = 0$. $g_1(2, -1/2) = (-2/3, -5/6)$, and $\mathfrak{m}_0(-2/3, -5/6) \neq 0$.

$\mathfrak{m}_2(1, -1) = 1$, therefore, if $h_1(1, -1) = 0$, then $h_1(g_2(1, -1)) = 0$. $g_2(1, -1) = (-1, 1)$, and $\mathfrak{m}_0(-1, 1) \neq 0$.

□

Proof of 4.3.17. Now let $(s_0, t_0) \in \mathcal{R}$ with $h_1(s_0, t_0) = 0$, and $\mathfrak{m}_0(s, t) \neq 0$.

Consider $g_0(s_0, t_0) = \left(\frac{s_0}{3}, \frac{t_0}{3}\right)$. Since $(s_0, t_0) \in \mathcal{R}$, $g_0(s_0, t_0) \in \mathcal{R}_{3/4,1/2}$.

Let $(s_1, t_1) = g_0(s_0, t_0)$. Since $(s_1, t_1) \in \mathcal{R}_{3/4,1/2}$, then either $\mathfrak{m}_0(s_1, t_1) \neq 0$ or $(s_1, t_1) = (1/2, -1/2)$. If $h_1(1/2, -1/2) = 0$, then $h_1(-1/2, -1/2) = 0$, as above; $(-1/2, -1/2) \in R_{3/4,1/2}$ as well, so let $(s_1, t_1) = (-1/2, -1/2)$ instead. So we again have $h_1(s_1, t_1) = 0$ and $\mathfrak{m}_0(s_1, t_1) \neq 0$; therefore $h_1(g_0(s_1, t_1)) = 0$.

Let $(s_2, t_2) = g_0(s_1, t_1)$. Since $(s_1, t_1) \in \mathcal{R}_{3/4,1/2}$, $(s_2, t_2) \in \mathcal{R}_{1/4,1/6}$. Since $(s_2, t_2) \in \mathcal{R}_{1/4,1/6}$, we have to have $\mathfrak{m}_0(s_2, t_2) \neq 0$; there are no zeros of $\mathfrak{m}_0$ inside $R_{1/4,1/6}$. Therefore, $h_1(g_0(s_2, t_2)) = 0$.

Repeat the argument above to construct $(s_n, t_n) = g_0(s_{n-1}, t_{n-1})$ for all $n \in \mathbb{N}$. Beyond $n = 1$, we cannot have $\mathfrak{m}_0(s_n, t_n) = 0$, so we always have $h_1(s_n, t_n) = 0 \Rightarrow h_1(g_0(s_n, t_n)) = 0$.

$h_1(s_n,t_n)=0$ and $(s_n,t_n)\in[-2/3^n,0]\times[0,-1/3^n]$.

Therefore, $\{(s_n,t_n)\}_{n=0}^{\infty}$ with $(s_n,t_n)\in[-3^{2-n}/4,3^{2-n}/4]\times[-3^{1-n}/2,3^{1-n}/2]$ is a sequence that converges to $(0,0)$ with $h_1(s_n,t_n)=0$ for all n. But h_1 is continuous, therefore, $h_1(0,0)=0$ and, by Claim 4.3.18, $h_X(s,t)\equiv 1$ on $\mathcal{R}$. □

4.3.4 Aside: a note on the zeros of $\mathfrak{m}_0$

Claim 4.3.20. *For $k,\ell\in\mathbb{Z}$, $\mathfrak{m}_1(s,t)=1$ for $(s,t)=\left(\frac{1}{2}+\frac{3k}{2},1+\frac{3\ell}{2}\right)$ and $\mathfrak{m}_2(s,t)=1$ for $(s,t)=\left(1+\frac{3k}{2},\frac{1}{2}+\frac{3\ell}{2}\right)$.*

Proof. $\mathfrak{m}_1(s,t)=\frac{1}{3}\left(1+e^{-2\pi i(4-2s)/3}+e^{-2\pi i(2-2t)/3}\right)=1$ iff $e^{-2\pi i(4-2s)/3}=1$ and $e^{-2\pi i(2-2t)/3}=1$ iff $\frac{4-2s}{3}\in\mathbb{Z}$ and $\frac{2-2t}{3}\in\mathbb{Z}$. Let $\frac{4-2s}{3}=k\in\mathbb{Z}$ and $\frac{2-2t}{3}=\ell\in\mathbb{Z}$, then $s=2-\frac{3k}{2}$ and $t=1-\frac{3\ell}{2}$. With a slightly different choices of integers k,ℓ, $s=\frac{1}{2}+\frac{3k}{2}$ and $t=1-\frac{3\ell}{2}$, as above.

$\mathfrak{m}_2(s,t)=\frac{1}{3}\left(1+e^{-2\pi i(8-2s)/3}+e^{-2\pi i(4-2t)/3}\right)=1$ iff $e^{-2\pi i(8-2s)/3}=1$ and $e^{-2\pi i(4-2t)/3}=1$ iff $\frac{8-2s}{3}\in\mathbb{Z}$ and $\frac{4-2t}{3}\in\mathbb{Z}$. Let $\frac{8-2s}{3}=k\in\mathbb{Z}$ and $\frac{4-2t}{3}=\ell\in\mathbb{Z}$, then $s=1-\frac{3k}{2}$ and $t=\frac{1}{2}-\frac{3\ell}{2}$. With a slightly different choices of integers k,ℓ, $s=1+\frac{3k}{2}$ and $t=\frac{1}{2}+\frac{3\ell}{2}$, as above.

□

Corollary 4.3.21. *All zeros (s,t) of $\mathfrak{m}_0$ have either $\mathfrak{m}_1(s,t)=0$ and $\mathfrak{m}_2(s,t)=1$ or vice versa.*

4.3.5 Bigger rectangles

Claim 4.3.22. *If $h_X\equiv 1$ on $\mathcal{R}_{a,b}$, for $a\geq 2,b\geq 1$, then $h_X\equiv 1$ on $\mathcal{R}_{3a-4,3b-2}$.*

Proof. As noted above, for $a\geq 2,b\geq 1$, $j\in\{0,1,2\}$, $g_j(\mathcal{R}_{a,b})\subseteq\mathcal{R}_{a/3+4/3,b/3+2/3}\subseteq\mathcal{R}_{a,b}$. Inverting these relationships, we get that $g_j(R_{3a-4,3b-2})\subseteq R_{a,b}$ for $a\geq 2$, $b\geq 1$.

Let $(s,t)\in\mathcal{R}_{3a-4,3b-2}$. Then $h_X(s,t)=\sum_{j=0}^{2}|\mathfrak{m}_j(s,t)|^2h_X(g_j(s,t))$, and $g_j(s,t)\in\mathcal{R}_{a,b}$. Then by assumption, $h_X(g_j(s,t))=1$ for all j, and $h_X(s,t)=\sum_{j=0}^{2}|\mathfrak{m}_j(s,t)|^2=1$.

□

Corollary 4.3.23. *Since $h_X\equiv 1$ on $\mathcal{R}_{9/4,3/2}$, $h_X\equiv 1$ on $\{(s,t):t\leq 0\}$.*

Proof. For $a>2$ and $a>1$, $\mathcal{R}_{3a-4,3b-2}\supsetneq\mathcal{R}_{a,b}$. Moreover, $f(a)=3a-4$, $\lim_{n\to\infty}f^n(9/4)=\infty$, and for $g(b)=3b-3$, $\lim_{n\to\infty}g^n(3/2)=\infty$. Therefore, by applying Claim 4.3.22 repeatedly to $\mathcal{R}_{9/4,3/2}$, $h_X\equiv 1$ on $\mathcal{R}_{\infty,\infty}=\{(s,t):t\leq 0\}$.

□

Claim 4.3.24. $h_X(s,t) = h_X(-s,-t)$.

Proof.

$$\begin{aligned}
h_X(s,t) &= \sum_{u\in\mathbb{Z}} \left|\left\langle e_{s,t}, e_{u,u/2}\right\rangle\right|^2 \\
&= \sum_{u\in\mathbb{Z}} \left|\int e^{2\pi i(sx+ty)} e^{-2\pi i(ux+uy/2)}\, d\nu_3\right|^2 \\
&= \sum_{u\in\mathbb{Z}} \left|\int e^{2\pi i(sx+ty)-(ux+uy/2)}\, d\nu_3\right|^2 \\
&= \sum_{u\in\mathbb{Z}} \left|\int \overline{e^{2\pi i(-(sx+ty)+(ux+uy/2))}}\, d\nu_3\right|^2 \\
&= \sum_{u\in\mathbb{Z}} \left|\overline{\int e^{2\pi i(-sx-ty)} e^{-2\pi i(-ux-uy/2))}\, d\nu_3}\right|^2 \\
&= \sum_{u\in\mathbb{Z}} \left|\overline{\left\langle e_{-s,-t}, e_{-u,-u/2}\right\rangle}\right|^2 \\
&= \sum_{u\in\mathbb{Z}} \left|\left\langle e_{-s,-t}, e_{-u,-u/2}\right\rangle\right|^2 \qquad \text{let } v = -u \\
&= \sum_{v\in\mathbb{Z}} \left|\left\langle e_{-s,-t}, e_{v,v/2}\right\rangle\right|^2 \\
&= h_X(-s,-t)
\end{aligned}$$

□

Corollary 4.3.25. $h_X(s,t) = 1$ *for all* $(s,t) \in \mathbb{R}^2$.

Corollary 4.3.26. $\mathcal{E} = \{e_{t,t/2} \mid t \in \mathbb{Z}\}$ *is an orthonormal basis for* $L_2(\nu_3)$.

4.4 A Second Spectrum

We construct a second spectrum for $\mathcal{S}$ by choosing a different set of vectors L.

For the fractal $\mathcal{S} = X_B$, $B = \{(0,0), (2,0), (0,2)\}$, and additionally

$$W_B(x,y) = \frac{1}{9}\left|1 + e^{4\pi i x} + e^{4\pi i y}\right|^2 \tag{4.57}$$

$$= \frac{1}{9}\left(3 + 2\cos(4\pi x) + 2\cos(4\pi y) + 2\cos(4\pi(x-y))\right). \tag{4.58}$$

Therefore, $W_B(x,y) = 1$ if and only if $e^{4\pi ix} = e^{4\pi iy} = 1$, that is, when $(x,y) \in \mathbb{Z}/2 \times \mathbb{Z}/2$.

These computations are independent of the choice of L.

Now, we let $L = \{(0,0),(1,2),(2,1)\}$.

Note that the matrix $M_1 = \left(e^{2\pi i R^{-1} b\cdot l}\right)_{b\in B, l\in L}$ is Hadamard:

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i(2/3)} & e^{2\pi i(4/3)} \\ 1 & e^{2\pi i(2/3)} & e^{2\pi i(2/3)} \end{pmatrix}. \tag{4.59}$$

Proposition 4.4.1. *For $L = \{(0,0),(1,2),(2,1)\}$, and X_L as in Definition 2.2.7, $X_L = \{\sum_{j=1}^{\infty}(x_j,y_j)3^{-j} \mid x_j, y_j \in \{0,1,2\}$ and $x_j + y_j \equiv 0 \mod 3\ \forall j \in \mathbb{N}\}$.*

Proof. Let:

$$\begin{aligned} \lambda_0(x,y) &= \frac{1}{3}(x,y) \\ \lambda_1(x,y) &= \frac{1}{3}(x,y) + (1/3, 2/3) \\ \lambda_2(x,y) &= \frac{1}{3}(x,y) + (2/3, 1/3) \end{aligned}$$

so that $\lambda_0 = \tau_{(0,0)}$, $\lambda_1 = \tau_{(1,2)}$, and $\lambda_1 = \tau_{(2,1)}$ in the notation of Definition 2.2.7.

Let $\mathcal{A} = \{\sum_{j=1}^{\infty}(x_j,y_j)3^{-j} \mid x_j, y_j \in \{0,1,2\}$ and $x_j + y_j \equiv 0 \mod 3\ \forall j \in \mathbb{N}\}$. We need to show that that $\mathcal{A}$ is invariant under λ_0, λ_1, and λ_2, and that $\mathcal{A}$ is compact.

Show first that $\lambda_j(\mathcal{A}) \subseteq \mathcal{A}$ for all j.

Claim 4.4.2. $\lambda_0(\mathcal{A}) \subseteq \mathcal{A}$*.*

Proof. Let $\lambda_0(x,y) := (\tilde{x},\tilde{y})$. Then $\lambda_0\left(\sum_{j=1}^{\infty}(x_j,y_j)3^{-j}\right) := \sum_{j=1}^{\infty}(\tilde{x}_j,\tilde{y}_j)3^{-j} = \sum_{j=1}^{\infty}(x_j,y_j)3^{-j-1}$.

Note that for $j=1$, $\tilde{x}_1 = \tilde{y}_1 = 0$, so $\tilde{x}_1 + \tilde{y}_1 = 0 \mod 3$; and for $j \geq 2$, $\tilde{x}_j = x_{j+1}$ and $\tilde{y}_j = y_{j+1}$; therefore $\tilde{x}_j + \tilde{y}_j = 0 \mod 3$.

Therefore, $\lambda_1(x,y) \in \mathcal{A}$. ∎

Claim 4.4.3. $\lambda_1(\mathcal{A}) \subseteq \mathcal{A}$*.*

Proof. Let $\lambda_1(x,y) := (\tilde{x},\tilde{y})$. Then $\lambda_1\left(\sum_{j=1}^{\infty}(x_j,y_j)\right) := \sum_{j=1}^{\infty}(\tilde{x}_j,\tilde{y}_j)3^{-j} = (1,2)3^{-1} + \sum_{j=1}^{\infty}(x_j,y_j)3^{-j-1}$.

Note that for $j = 1$, $\tilde{x}_1 = 1$, $\tilde{y}_1 = 2$, so $\tilde{x}_1 + \tilde{y}_1 = 0 \mod 3$; and for $j \geq 2$, $\tilde{x}_j = x_{j+1}$ and $\tilde{y}_j = y_{j+1}$; therefore $\tilde{x}_j + \tilde{y}_j = 0 \mod 3$. □

Claim 4.4.4. *$\lambda_2(\mathcal{A}) \subseteq \mathcal{A}$.*

Proof. Let $\lambda_2(x, y) := (\tilde{x}, \tilde{y})$. Then $\lambda_2\left(\sum(x_j, y_j)\right) := \sum_{j=1}^{\infty}(\tilde{x}_j, \tilde{y}_j)3^{-j} = (2, 1)3^{-1} + \sum_{j=1}^{\infty}(x_j, y_j)3^{-j-1}$.

Note that for $j = 1$, $\tilde{x}_1 = 2$, $\tilde{y}_1 = 1$, so $\tilde{x}_1 + \tilde{y}_1 = 0 \mod 3$; for $j \geq 2$, $\tilde{x}_j = x_{j+1}$ and $\tilde{y}_j = y_{j+1}$, so $\tilde{x}_j + \tilde{y}_j = 0 \mod 3$. □

Now we show that $\mathcal{A} \subseteq \lambda_0(\mathcal{A}) \cup \lambda_1(\mathcal{A}) \cup \lambda_2(\mathcal{A})$.

Again, let $(x, y) = \sum_{j=1}^{\infty}(x_j, y_j)3^{-j}$.

Let $(\hat{x}, \hat{y}) = \sum_{j=2}^{\infty}(x_j, y_j)3^{-j+1}$. Notice that $\hat{x}_j = x_{-j-1}$, $\hat{y}_j = y_{-j-1}$, so $(\hat{x}, \hat{y}) \in \mathcal{A}$.

Claim 4.4.5. *$\lambda_k(\hat{x}, \hat{y}) = (x, y)$ for some $k \in \{0, 1, 2\}$.*

Proof. Case 1: $x_1 = 0$. Then, since $x_1 + y_1 = 0 \mod 3$, we must have $y_1 = 0$ as well.

We claim that $\lambda_0(\hat{x}, \hat{y}) = (x, y)$:

$\lambda_0\left(\sum_{j=2}^{\infty}(x_j, y_j)3^{-j+1}\right) = \sum_{j=2}^{\infty}(x_j, y_j)3^{-j} = (x, y)$, since $x_1 = y_1 = 0$.

Case 2: $x_1 = 1$. Then, since $x_1 + y_1 = 0 \mod 3$, we must have $y_1 = 2$.

We claim that $\lambda_1(\hat{x}, \hat{y}) = (x, y)$:

$\lambda_1\left(\sum_{j=2}^{\infty}(x_j, y_j)3^{-j+1}\right) = (1, 2)3^{-1} + \sum_{j=2}^{\infty}(x_j, y_j)3^{-j} = (x, y)$, since $x_1 = 1$ and $y_1 = 2$.

Case 3: $x_1 = 2$. Then, since $x_1 + y_1 = 0 \mod 3$, we must have $y_1 = 1$.

We claim that $\lambda_2(\hat{x}, \hat{y}) = (x, y)$:

$\lambda_2\left(\sum_{j=2}^{\infty}(x_j, y_j)3^{-j+1}\right) = (2, 1)3^{-1} + \sum_{j=2}^{\infty}(x_j, y_j)3^{-j} = (x, y)$, since $x_1 = 2$ and $y_1 = 1$.

□

By double containment, $\mathcal{A} = \bigcup_{j=0}^{2} \lambda_j(\mathcal{A})$. Therefore, $\mathcal{A} = X_L$ if $\mathcal{A}$ is compact.

Claim 4.4.6. *$\mathcal{A} = \{\sum_{j=1}^{\infty}(x_j, y_j)3^{-j} \mid x_j, y_j \in \{0, 1, 2\}, x_j + y_j = 0 \mod 3\}$ is compact.*

Proof. Clearly, $\mathcal{A}$ is bounded because $\mathcal{A} \subset [0, 1] \times [0, 1]$. We need only show that $\mathcal{A}$ is closed.

Let $\mathcal{A}_n = \{\sum_{j=1}^{\infty}(x_j, y_j)3^{-j} \mid x_j, y_j \in \{0, 1, 2\}, x_n + y_n = 0 \mod 3\}$.

Since there are only three possibilities for (x_n, y_n), $\mathcal{A}_n$ is the union of three subsets: $A_{n,0} = \{(x, y) \mid x_n = y_n = 0\}$, $A_{1,1} = \{(x, y) \mid x_n = 1, y_n = 2\}$, $A_{n,2} = \{(x, y) \mid x_n = 2, y_n = 1\}$.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = \frac{1}{3^n}(x, y)$. Since f is continuous and $f([0, 1] \times [0, 1]) = \mathcal{A}_{n,0}$, then $\mathcal{A}_{n,0}$ is closed.

Let $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (x, y) + (1/3^n, 2/3^n)$. Since g is continuous and $f(\mathcal{A}_{n,0}) = \mathcal{A}_{n,1}$, then $\mathcal{A}_{n,1}$ is closed.

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$, $h(x, y) = (x, y) + (2/3^n, 1/3^n)$. Since h is continuous and $h(\mathcal{A}_{n,0}) = \mathcal{A}_{n,2}$, then $\mathcal{A}_{n,2}$ is closed.

Then since $\mathcal{A}_n = \mathcal{A}_{n,0} \cup \mathcal{A}_{n,1} \cup \mathcal{A}_{n,2}$, $\mathcal{A}_n$ is itself closed.

Since this holds for all n, $\mathcal{A} = \bigcap_{n=1}^{\infty} A_n$ is closed. $\square$

$\square$

To find W_B-cycles on X_L, first we find which of the points in the lattice $\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ are also in X_L. We check the possible points. We note that $(x, y) \in X_L$ if and only if $(y, x) \in X_L$, so we need only check:

$(0, 0) = (0.\overline{0}_3, 0.\overline{0}_3) \in X_L$.

$(0, 1/2) = (0.\overline{0}_3, 0.\overline{1}_3) \notin X_L$, therefore also $(1/2, 0) \notin X_L$.

$(0, 1) = (0.\overline{0}_3, 0.\overline{2}_3) \notin X_L$, therefore also $(1, 0) \notin X_L$.

$(1/2, 1/2) = (0.\overline{1}_3, 0.\overline{1}_3) \notin X_L$.

$(1/2, 1) = (0.\overline{1}_3, 0.\overline{2}_3) \in X_L$, therefore also $(1, 1/2) \in X_L$.

$(1, 1) = (0.\overline{2}_3, 0.\overline{2}_3) \notin X_L$.

So the only three points in $\left(\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}\right) \cap X_L$ are $(0, 0), (1, 1/2), (1/2, 1)$.

All of these are cycles, in fact, they are fixed points:

$$\lambda_0(0, 0) = (0, 0) \tag{4.60}$$

$$\lambda_1(1/2, 1) = \frac{1}{3}(1/2, 1) + (1/3, 2/3) = (1/2, 1) \tag{4.61}$$

$$\lambda_2(1, 1/2) = \frac{1}{3}(1, 1/2) + (2/3, 1/3) = (1, 1/2). \tag{4.62}$$

Therefore, the W_B-cycles are exactly $(0, 0)$, $(1, 1/2)$ and $(1/2, 1)$.

Therefore, by Theorem 2.2.13, we construct the frequencies for an orthonormal basis by applying $\rho_j(x, y) = 3(x, y) + l_j$, $l_j \in L$, to $(0, 0)$, $(-1, -1/2)$ and $(-1/2, -1)$.

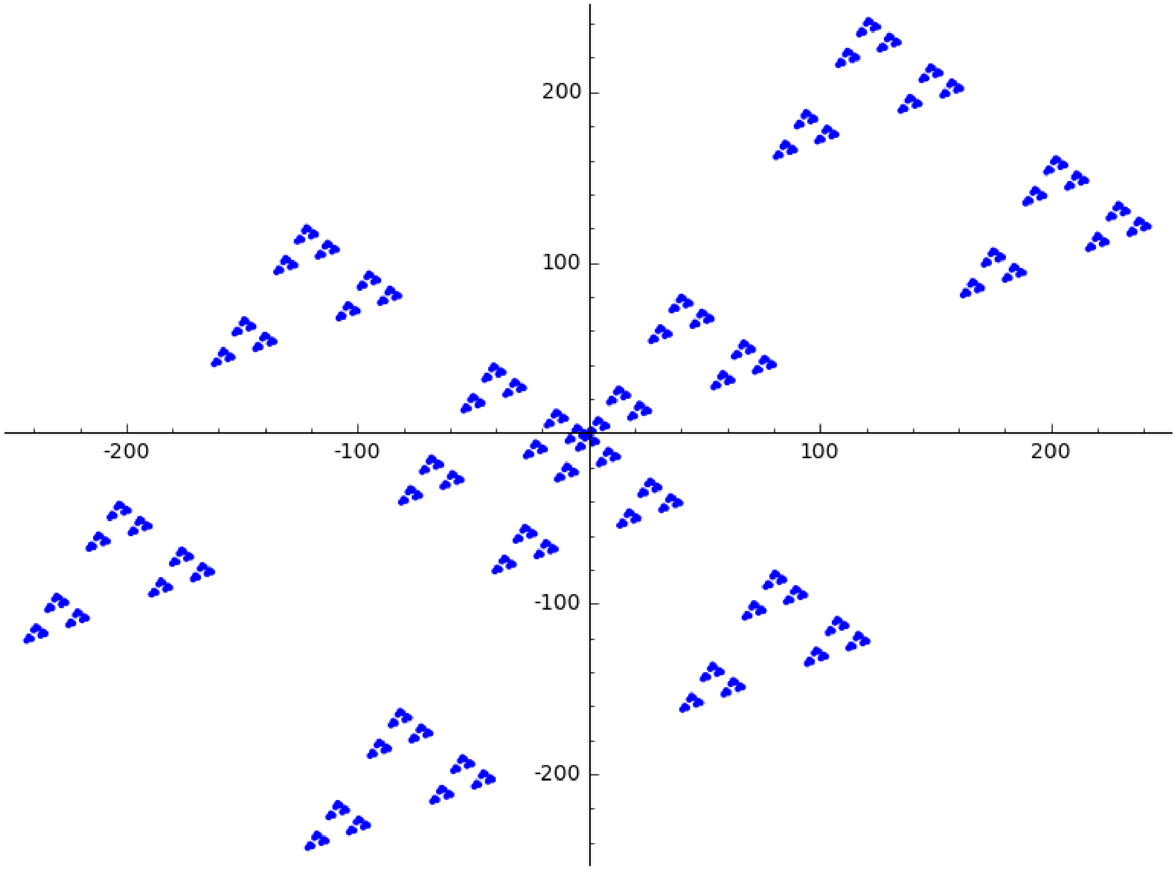

Figure 4.4 Plot of $r_5(0,0) \cup r_5(-1/2,-1) \cup r_5(-1,-1/2)$.

4.4.1 Frequencies

Let:

$$\rho_0(x,y) = 3(x,y) \tag{4.63}$$

$$\rho_1(x,y) = 3(x,y) + (1,2) \tag{4.64}$$

$$\rho_2(x,y) = 3(x,y) + (2,1). \tag{4.65}$$

Let $r_1(s,t) = \{\rho_0(s,t), \rho_1(s,t), \rho_2(s,t)\}$, then for $n \geq 2$, $r_n(s,t) = \rho_0(r_{n-1}(s,t)) \cup \rho_1(r_{n-1}(s,t)) \cup \rho 2(r_{n-1}(s,t))$.

Then let $\mathcal{R}(s,t) = \bigcup_{n \in N} r_n(s,t)$.

Theorem 4.4.7. *For $x \in \mathbb{N}$, define $d_j(x) = l_j$ where $x = \sum_{j=0}^{n} l_j 3^j$, $l_j = 0,1,2$ for all j. Then for any $(x,y) \in \mathbb{Z}^2$, $x,y > 0$, $(x,y) \in \mathcal{R}(0,0)$, if and only if $d_j(x) + d_j(y) \equiv 0 \mod 3$ for $j = 0,\ldots,n$.*

Proof. "$\Rightarrow$"

Let $(x, y) \in \mathcal{R}(0, 0)$.

Claim 4.4.8. $d_j(x) + d_j(y) \equiv 0 \mod 3$.

Proof. By induction. Base case: in $r_1(0, 0)$, $(x, y) = (1, 2)$ or $(x, y) = (2, 1)$. $d_0(1) = 1$, $d_j(1) = 0$ for $j > 0$. $d_0(2) = 2$, $d_j(2) = 0$ for $j > 0$. For $j = 0$, $1 + 2 \equiv 0 \mod 3$, and for $j > 0$, $0 + 0 \equiv 0 \mod 3$. So the base case holds.

Inductive step: Let $(x, y) \in r_n(0, 0)$. Then $(x, y) = \rho_i(x', y')$ for some $i = 0, 1, 2$ and $(x', y') \in r_{n-1}(0, 0)$.

Case 1: $(x, y) = \rho_0(x', y') = 3(x', y')$. By the inductive hypothesis, $d_j(x') + d_j(y') = 0$ for all j. Since $(x, y) = 3(x', y')$, for $j > 0$, $d_j(x) = d_{j-1}(x')$, $d_j(y) = d_{j-1}(y')$, and $d_0(x) = d_0(y) = 0$. In either case, $d_j(x) + d_j(y) \equiv 0 \mod 3$.

Case 1: $(x, y) = \rho_1(x', y') = 3(x', y') + (1, 2)$. By the inductive hypothesis, $d_j(x') + d_j(y') = 0$ for all j. Since $(x, y) = 3(x', y') + (1, 2)$, for $j > 0$, $d_j(x) = d_{j-1}(x')$, $d_j(y) = d_{j-1}(y')$, and for $j = 0$, $d_0(x) = 1$ and $d_0(y) = 2$. In either case, $d_j(x) + d_j(y) \equiv 0 \mod 3$.

Case 3: $(x, y) = \rho_1(x', y') = 3(x', y') + (2, 1)$. By the inductive hypothesis, $d_j(x') + d_j(y') = 0$ for all j. Since $(x, y) = 3(x', y') + (1, 2)$, for $j > 0$, $d_j(x) = d_{j-1}(x')$, $d_j(y) = d_{j-1}(y')$, and for $j = 0$, $d_0(x) = 2$ and $d_0(y) = 1$. In either case, $d_j(x) + d_j(y) \equiv 0 \mod 3$. $\square$

"$\Leftarrow$"

Let $(x, y) \in \mathbb{Z}^2$ and $d_j(x) + d_j(y) \equiv 0 \mod 3$ for $j = 0, \ldots, n$.

Claim 4.4.9. *$3^n \leq x, y < 3^{n+1}$ for some $n \in \mathbb{Z}$, $n \geq 0$; or $(x, y) = (0, 0)$.*

Proof. We can write $x = \sum_{j=0}^{\infty} d_j(x)3^j$ and $y = \sum_{j=0}^{\infty} d_j(y)3^j$. There must be some $n \geq 0$ for which $d_j(x) = 0$ for all $j > n$, that is, $x = \sum_{j=0}^{n} d_j(x)3^j$. By assumption, $d_j(x) + d_j(y) = 0 \mod 3$, therefore, $d_j(x) = 0$ requires that $d_j(y) = 0$. So we can write $y = \sum_{j=0}^{n} d_j(y)3^j$ as well. Therefore, if $x = 0 = \sum_{j=0}^{\infty} 03^j$, then $y = 0 = \sum_{j=0}^{\infty} 03^j$, and $(x, y) = (0, 0)$.

If $x \neq 0$, there is some $k \geq 0$ where $d_n(k) \neq 0$. Therefore, we can choose $n = \min\{n \geq 0 \mid d_j(x) = 0 \forall j > n\}$, and it then follows that $3^n \leq x, y \leq 3^{n+1}$. $\square$

Claim 4.4.10. $(0, 0) \in \mathcal{R}(0, 0)$.

Proof. $\rho_0(0,0) = 3(0,0) = (0,0)$. □

Claim 4.4.11. *Let* $(x,y) \neq (0,0)$ *and* $(x,y) = \sum_{j=0}^{n}(d_j(x), d_j(y))3^j$, *with* $d_j(x) + d_j(y) = 0 \mod 3$. *Then,*

$$(x,y) = \rho_{d_0(x)} \circ \rho_{d_1(x)} \cdots \circ \rho_{d_{n-1}(x)} \circ \rho_{d_n(x)}(0,0).$$

Proof. By induction.

Base case: for $3^0 \le x, y < 3^1$ we have that $d_j(x) = 0$ for $j \ge 1$ and $d_0(x) = 1$ or $d_0(x) = 2$. If $d_0(x) = 1$, $d_0(y) = 2$, so we want to show that $(x,y) = \rho_1(0,0) = (1,2)$; and in fact, $\rho_1(0,0) = (1,2)$. If $d_0(x) = 2$, then $d_0(y) = 1$, so we want to show that $(x,y) = \rho_2(0,0) = (2,1)$; and in fact, $\rho_2(0,0) = (2,1)$.

Inductive step: suppose $3^n \le x, y < 3^{n+1}$. We want to show that $(x,y) = \rho_0 \circ \rho_1 \cdots \circ \rho_{n-1} \circ \rho_n(0,0)$. Let $x' = \dfrac{x - d_0(x)}{3}$ and $y' = \dfrac{y - d_0(y)}{3}$. By construction, x', y' are integers with $3^{n-1} \le x', y' < 3^n$, and $d_j(x') = d_{j+1}(x)$ for $j = 0, \ldots, n-1$, similarly $d_j(y') = d_{j+1}(y)$ for $j = 0, \ldots, n-1$. Then we have $d_j(x') + d_j(y') = 0 \mod 3$ for $j = 0, \ldots, n-1$ so by the inductive hypothesis, $(x,y) = \rho_{d_0(x')} \circ \rho_{d_1(x')} \cdots \circ \rho_{d_{n-1}(x')}(0,0) = \rho_{d_1(x)} \circ \rho_{d_2(x)} \cdots \circ \rho_{d_n(x)}(0,0)$. So we need only show that $(x,y) = \rho_{d_0(x)}(x', y')$.

Case 0: $d_0(x) = d_0(y) = 0$. Then $(x', y') = (\frac{x}{3}, \frac{y}{3})$ and $\rho_0(x', y') = 3(\frac{x}{3}, \frac{y}{3}) = (x,y)$.

Case 1: $d_0(x) = 1$, $d_0(y) = 2$. Then $(x', y') = (\frac{x-1}{3}, \frac{y-2}{3})$ and $\rho_0(x', y') = 3(\frac{x-1}{3}, \frac{y-2}{3}) + (1,2) = (x,y)$.

Case 2: $d_0(x) = 2$, $d_0(y) = 1$. Then $(x', y') = (\frac{x-2}{3}, \frac{y-1}{3})$ and $\rho_0(x', y') = 3(\frac{x-1}{3}, \frac{y-2}{3}) + (2,1) = (x,y)$. □

□

Corollary 4.4.12. *For any* $x \in \mathbb{Z}$, $x \ge 0$, *there is exactly one* $y \in \mathbb{Z}$, $y \ge 0$, *with* $(x,y) \in \mathcal{R}(0,0)$.

Theorem 4.4.13. *A pair* $(x,y) \in \mathbb{Z}^2$ *is in* $\mathcal{R}(-1,-1/2)$ *if and only if* $(x,y) + 3^n(-1,-1/2) \in \mathcal{R}(0,0)$ *for some* $n \in \mathbb{N}$.

Similarly, a pair $(x,y) \in \mathcal{R}(-1/2,-1)$ *if and only if* $(x,y) + 3^n(-1/2,-1) \in \mathcal{R}(0,0)$ *for some* $n \in \mathbb{N}$.

Proof. ($\Rightarrow$)

Write $j = j_0 + 3j_1 + \cdots + 3^{N-1}j_{N-1}$ with $j_1, \ldots j_{n-1}$ all equal to 0, 1, or 2, $j_{N-1} \neq 0$. Then: $R_{j,N} = \rho_{j_0} \circ \rho_{j_1} \circ \ldots \rho_{j_{N-1}}$

Therefore, if $(x,y) \in \mathcal{R}(-1,-1/2)$, then $(x,y) = R_{j,N}(-1,-1/2)$ for some $j, n = N$.

Lemma 4.4.14. *For any $x, y \in \mathbb{R}^2$, $R_{j,N}\left(x+y\right) = R_{j,N}(x) + 3^N y$*

Proof. Prove by induction. Base case: For $N = 1$, $R_{j,1} = \rho_j$.

- $\rho_0(x+y) = 3x + 3y$.
- $\rho_1(x+y) = 3(x+y) + (1,2) = \rho_1(x) + 3y$.
- $\rho_2(x+y) = 3(x+y) + (2,1) = \rho_2(x) + 3y$.

Let $R_{j',N-1} = \rho_{j_1} \circ \cdots \circ \rho_{j_{N-1}}$ be the composition of $N-1$ ρ_k's with the property that $\rho_{j_0} \circ R_{j',N-1} = R_{j,N}$.

Assume true for $n < N$. Then:

$$
\begin{aligned}
R_{j,N}\left(x+y\right) &= \rho_{j_0} \circ \rho_{j_1} \circ \cdots \circ \rho_{j_{N-1}}\left(x+y\right) \\
&= \rho_{j_0}\left(R_{j',N-1}\left(x+y\right)\right) \\
&= \rho_{j_0}\left(R_{j',N-1}(x) + 3^{N-1}y\right) \\
&= \rho_{j_0}(R_{j',N-1}(x)) + 3(3^{N-1}y) \\
&= R_{j,N}x + 3^N y
\end{aligned}
$$

□

Therefore, $R_{j,N}((0,0) + (-1,-1/2)) = R_{j,n}(0,0) + 3^N(-1,-1/2)$. Similarly, $R_{j,n}((0,0) + (-1/2,-1)) = R_{j,N}(0,0) + 3^N(-1/2,-1)$.

($\Leftarrow$)

Suppose $(x,y) + 3^n(-1,-1/2) \in \mathcal{R}(0,0)$. Then there is some j, N with $R_{j,N}(0,0) = (x,y) + 3^N(-1,-1/2)$. Since $(0,0) = (-1,-1/2) - (-1,-1/2)$, $R_{j,N}(-1,-1/2) = R_j(0,0) - 3^N(-1,-1/2) = (x,y)$. Again, the proof works exactly the same way for $\mathcal{R}(-1/2,-1)$.

□

Theorem 4.4.15 (The Second spectrum)**.** *$\mathcal{R}(0,0)\cup\mathcal{R}(-1,-1/2)\cup\mathcal{R}(-1/2,-1)$ is a spectrum for $\mathcal{S}$.*

Proof. This follows from Theorem 2.2.13. As in Section 4.2, we have already checked that $\{R^{-1}B, L\}$ form a Hadamard pair. By construction, $0 \in B$, $0 \in L$, and it is easy to see that $R^n b \cdot l \in \mathbb{Z}$ for all $b \in B$, $l \in L$, $n \geq 0$.

By construction $\Lambda = \mathcal{R}(0,0)\cup\mathcal{R}(-1,-1/2)\cup\mathcal{R}(-1/2,-1)$ is the smallest set that contains $-C$ for all W_B-cycles C, and such that $S\Lambda + L \in \Lambda$, so it satisfies the hypotheses of Theorem 2.2.13.

By Proposition 4.4.1, $X_L = \{\sum_{j=1}^{\infty}(x_j, y_j)3^{-j} \mid x_j + y_j \equiv 0 \mod 3\ \forall j \in \mathbb{N}\}$.

Recall that $W_B(x,y) = \frac{1}{9}\left|1 + e^{4\pi i x} + e^{4\pi i y}\right|^2$, so that $W_B(x,y) = 0$ if and only if $1 + e^{4\pi i x} + e^{4\pi i y} = 0$, that is, when $(x,y) \in (1/3 + \mathbb{Z}/2, 1/6 + \mathbb{Z}/2) \cup (1/6 + \mathbb{Z}/2, 1/3 + \mathbb{Z}/2)$.

To satisfy the Transversality of the Zeros condition (Definition 2.2.12), we need to show:

(a) If $(x,y) \in X_L$ is not a cycle, then there exists $k_x \geq 0$ such that, for $k \geq k_x$, $\{\tau_{l_1}\circ\tau_{l_2}\circ\cdots\circ\tau_{l_k}x : l_1, \ldots l_n \in L\}$ does not contain any zeros of W;

(b) If $\{x_0, x_1, \ldots, x_p\}$ are on a cycle with $x_1 = \tau_l(x_0)$ for some $l \in L$, then for every $y = \tau_{l'}(x_0)$, $y \neq x_1$ is either not on a cycle or $W(y) = 0$.

Lemma 4.4.16. *$(x,y) \in X_L$ is on a cycle if and only if $x = k/(3^n - 1)$ and $y = l/(3^n - 1)$ for some integers k, l and $n \in \mathbb{N}$.*

Proof. By definition, (x,y) is on a cycle if and only if $(x,y) = r^n(x,y)$ for some $n \in N$, where $r = 3(x,y) \mod 1$ is the common right inverse of $\lambda_0, \lambda_1, \lambda_2$.

Therefore, $r^n(x,y) = 3^n(x,y) \mod 1$.

The situation is less complicated than for the previous spectrum because there are no overlap points between the ranges of $\lambda_0, \lambda_1, \lambda_2$, thus, each (x,y) has only one possible value for $r(x,y)$.

Therefore, for all $(x,y) \in X_L$, $(x,y) = r^n(x,y)$ if and only if $x = 3^n x \mod 1$ and $y = 3^n y \mod 1$, that is, $3^n x - x = 0 \mod 1$, or $x(3^n - 1) = k \in \mathbb{Z}$: in other words, $x = k/(3^n - 1)$ for some $k \in \mathbb{Z}$, and similarly, $y = l/(3^n - 1)$ for some $l \in \mathbb{Z}$. $\square$

Consider the ternary expansion of $k/(3^n - 1)$:

$$\frac{k}{3^n - 1} = \frac{k/3^n}{1 - 3^{-n}} = \sum_{j=0}^{\infty} \frac{k}{3^n}(3^{-n})^j = \sum_{j=1}^{\infty} k3^{-nj}$$

Therefore, the ternary expansion of $k/(3^n - 1)$ will repeat the ternary expansion of k in blocks of length n.

For (x, y), $x = k/(3^n - 1)$ to be on a cycle, we need $y = l/(3^n - 1)$, for some $l \in \mathbb{Z}$. This l will be $3^n - k$ mod 3. So there will be a $(k/(3^n - 1), l/(3^n - 1)) \in X_L$ for any $0 \leq k \leq 3^n - 1$.

Recall from Section 4.2.1 that $W_B(x, y) = 0$ if and only if $(x, y) \in (1/3 + \mathbb{Z}/2, 1/6 + \mathbb{Z}/2) \cup (1/6 + \mathbb{Z}/2, 1/3 + \mathbb{Z}/2)$.

Let (x, y) be on a cycle of length n. Then $(x, y) = (k/(3^n - 1), l/(3^n - 1)$ for k, l as defined above. As in Section 4.2.1, it is sufficient for (b) to show that exactly one of $\lambda_0(x, y)$, $\lambda_1(x, y)$, $\lambda_2(x, y)$ is on a cycle. We will need the result from Lemma 4.2.10. There are three cases for the integer k: $k = 3m$ for some integer m; $k = 3m + 1$ for some integer m; $k = 3m + 2$ for some integer m.

Case 1: If $k = 3m$, then $l = 3p$ for some integer p, since $l = 3^n - k$ mod 3.

$$\lambda_0(x, y) = \lambda_0\left(\frac{3m}{3^n - 1}, \frac{3p}{3^n - 1}\right) = \left(\frac{3m}{3(3^n - 1)}, \frac{3p}{3(3^n - 1)}\right) = \left(\frac{m}{3^n - 1}, \frac{p}{3^n - 1}\right)$$

which is on an n-cycle. However:

$$\begin{aligned}\lambda_1(x, y) = \lambda_1\left(\frac{3m}{3^n - 1}, \frac{3p}{3^n - 1}\right) &= \left(\frac{3m}{3(3^n - 1)} + \frac{1}{3}, \frac{3p}{3(3^n - 1)} + \frac{2}{3}\right) \\ &= \left(\frac{3m + 3^n - 1}{3(3^n - 1)}, \frac{3p + 2(3^n) - 2}{3(3^n - 1)}\right)\end{aligned}$$

Since $3m + 3^n - 1$ and $3l + 2(3^n) - 2$ are not divisible by 3, by Lemma 4.2.10, $\lambda_1(x, y)$ is not on any cycle. And:

$$\begin{aligned}\lambda_2(x, y) = \lambda_2\left(\frac{3m}{3^n - 1}, \frac{3p}{3^n - 1}\right) &= \left(\frac{3m}{3(3^n - 1)} + \frac{2}{3}, \frac{3p}{3(3^n - 1)} + \frac{1}{3}\right) \\ &= \left(\frac{m + 2(3^n) - 2}{3(3^n - 1)}, \frac{p + 3^n - 1}{3(3^n - 1)}\right)\end{aligned}$$

Since $3m + 2(3^n) - 1$ and $3p + 3^n - 1$ are not divisible by 3, by Lemma 4.2.10, $\lambda_1(x, y)$ is not on any cycle.

Case 2: If $k = 3m+1$, then $l = 3p+2$ for some integer p, since $l = 3^n - k \bmod 3$.

$$\lambda_0(x,y) = \lambda_0\left(\frac{3m+1}{3^n-1}, \frac{3p+2}{3^n-1}\right) = \left(\frac{3m+1}{3(3^n-1)}, \frac{3p+2}{3(3^n-1)}\right)$$

Since $3m+1$ and $3p+2$ are not divisible by 3, by Lemma 4.2.10, $\lambda_0(x,y)$ is not on any cycle.

$$\begin{aligned}
\lambda_1(x,y) = \lambda_1\left(\frac{3m+1}{3^n-1}, \frac{3p+2}{3^n-1}\right) &= \left(\frac{3m+1}{3(3^n-1)} + \frac{1}{3}, \frac{3p+2}{3(3^n-1)} + \frac{2}{3}\right) \\
&= \left(\frac{3m+1+3^n-1}{3(3^n-1)}, \frac{3p+2+2(3^n)-2}{3(3^n-1)}\right) \\
&= \left(\frac{3m+3^n}{3(3^n-1)}, \frac{3p+2(3^n)}{3(3^n-1)}\right) \\
&= \left(\frac{m+3^{n-1}}{(3^n-1)}, \frac{p+2(3^{n-1})}{(3^n-1)}\right)
\end{aligned}$$

Which is on a cycle.

$$\begin{aligned}
\lambda_2(x,y) = \lambda_2\left(\frac{3m+1}{3^n-1}, \frac{3p+2}{3^n-1}\right) &= \left(\frac{3m+1}{3(3^n-1)} + \frac{2}{3}, \frac{3p+2}{3(3^n-1)} + \frac{1}{3}\right) \\
&= \left(\frac{3m+1+2(3^n)-2}{3(3^n-1)}, \frac{3p+2+3^n-1}{3(3^n-1)}\right) \\
&= \left(\frac{3m+2(3^n)+1}{3(3^n-1)}, \frac{3p+2(3^n)-1}{3(3^n-1)}\right)
\end{aligned}$$

Since $3m+2(3^n)+1$ and $3p+3^n-1$ are not divisible by 3, by Lemma 4.2.10, $\lambda_2(x,y)$ is not on any cycle.

Case 3: If $k = 3m+2$, then $l = 3p+1$ for some integer p, since $l = 3^n - k \bmod 3$. Then:

$$\begin{aligned}
\lambda_0(x,y) = \lambda_0\left(\frac{3m+2}{3^n-1}, \frac{3p+1}{3^n-1}\right) &= \left(\frac{3m+2}{3(3^n-1)}, \frac{3p+1}{3(3^n-1)}\right) \\
&= \left(\frac{3m+2}{3(3^n-1)}, \frac{3p+1}{3(3^n-1)}\right)
\end{aligned}$$

Since $3m+1$ and $3p+2$ are not divisible by 3, by Lemma 4.2.10, $\lambda_0(x,y)$ is not on any cycle.

$$\begin{aligned}
\lambda_1(x,y) = \lambda_1\left(\frac{3m+2}{3^n-1}, \frac{3p+1}{3^n-1}\right) &= \left(\frac{3m+2}{3(3^n-1)} + \frac{1}{3}, \frac{3p+1}{3(3^n-1)} + \frac{2}{3}\right) \\
&= \left(\frac{3m+2+1(3^n)-1}{3(3^n-1)}, \frac{3p+1+2(3^n)-2}{3(3^n-1)}\right) \\
&= \left(\frac{3m+2(3^n)-1}{3(3^n-1)}, \frac{3p+2(3^n)+1}{3(3^n-1)}\right)
\end{aligned}$$

Proof of Condition (a). We notice that there are only finitely many (x,y) that are both zeros of W_B and contained in X_L.

Zeros of W_B contained in $[0,1]\times[0,1]$:

$(1/3,1/6),(5/6,1/6),(1/3,2/3),(5/6,2/3),(1/6,1/3),(2/3,1/3),(1/6,5/6),(2/3,5/6)$.

Zeros of W_B contained in X_L: $(1/3,1/6)$, $(1/3,2/3)$, $(5/6,2/3)$, $(1/6,1/3)$, $(2/3,1/3)$, $(2/3,5/6)$.

We know from Lemma 4.2.8 that none of these points are on cycles.

Now suppose for $(x,y)\in X$, (x,y) not in a cycle, $r^{-n}(x,y)$ contains $r^n(1/3,1/6)$ for some n, that is, $(1/3,1/6)=\ell_{j_n}\dots\ell_{j_1}(x,y)$ with $j_1,\dots,j_n\in\{0,1,2\}$. Claim: there is no $m>n$ with $(x,y)\in r^{-m}(x,y)$, that is, we cannot have $(1/3,1/6)=\ell_{k_m}\dots\ell_{k_n}\dots\ell_{k_1}(x,y)$.

Suppose we do. Then:

$\ell_{j_n}\dots\ell_{j_1}(x,y)=\ell_{k_m}\dots\ell_{k_n}\dots\ell_{k_1}(x,y)$.

Apply r^m to both sides:

$r^m\ell_{j_n}\dots\ell_{j_1}(x,y)=r^m\ell_{k_m}\dots\ell_{k_n}\dots\ell_{k_1}(x,y)$.

And applying the left inverses: $r^{m-n}(x,y)=(x,y)$. This contradicts the assumption that (x,y) is not on a cycle.

Therefore, there is at most one $n\in\mathbb{N}$ with $(1/3,1/6)\in r^{-n}(x,y)$. The proof for the other six points is identical. Therefore, we can take $n_{(x,y)}=\max\{n,n',n''\}$ and Condition (a) holds. $\square$

$\square$

4.5 A Third Spectrum

Now we choose $L=\{(0,0),(2,4),(4,2)\}$, and let:

$$F_0(x,y)=\frac{1}{3}(x,y) \tag{4.66}$$

$$F_1(x,y)=\frac{1}{3}((x,y)+(2/3,4/3)) \tag{4.67}$$

$$F_2(x,y)=\frac{1}{3}((x,y)+(4/3,2/3))\} \tag{4.68}$$

so that X_L is the invariant set of F_0,F_1,F_2.

Claim 4.5.1. *(x,y) is in the attractor set of $\{F_0,F_1,F_2\}$ if and only if $(x/2,y/2)$ is in the attractor set of $\{\lambda_0,\lambda_1,\lambda_2\}$.*

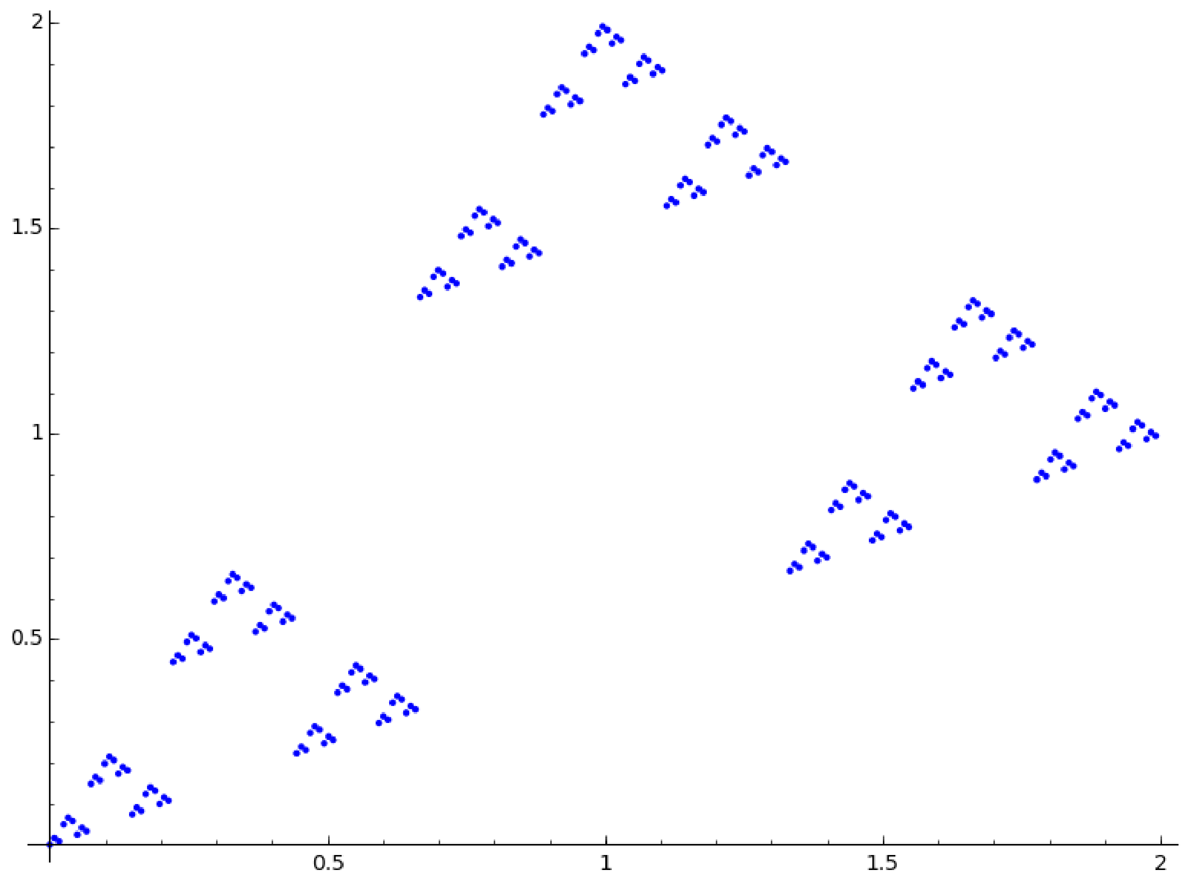

Figure 4.5 Plot of the fifth iteration of $\{F_0, F_1, F_2\}$.

Proof. We know by Proposition 4.4.1,

$\mathcal{L} = \{(x,y) = \sum_{j=1}^{\infty}(x_j, y_j)3^{-j} \in [0,1]\times[0,1] \mid x_j + y_j \equiv 0 \mod 3\ \forall j \in \mathbb{N}\}$ is the attractor set of $\{\lambda_0, \lambda_1, \lambda_2\}$.

Let $2\mathcal{L} = \{(x,y) | (x/2, y/2) \in \mathcal{L}\}$.

Let $(x,y) \in 2\mathcal{L}$.

Claim 4.5.2. $\digamma_0(x,y) \in 2\mathcal{L}$.

Proof. By definition, $\digamma_0(x,y) = (x/3, y/3)$. Then $\digamma_0(x,y)/2 = (x/6, y/6) = \lambda_0(x/2, y/2)$. □

Claim 4.5.3. $\digamma_1(x,y) \in 2\mathcal{L}$.

Proof. By definition, $\digamma_1(x,y) = (x/3 + 2/3, y/3 + 4/3)$. Then $\digamma_1(x,y)/2 = (x/6 + 1/3, y/6 + 2/3) = \lambda_1(x/2, y/2)$. □

Claim 4.5.4. $\digamma_2(x,y) \in 2\mathcal{L}$.

Proof. $\digamma_2(x,y) = (x/3 + 2/3, y/3 + 4/3)$. Then $\digamma_2(x,y)/2 = (x/6 + 2/3, y/6 + 1/3) = \lambda_2(x/2, y/2)$. □

Now we show that if $(x,y) \in 2\mathcal{L}$, there exists $(\tilde{x}, \tilde{y}) \in 2\mathcal{L}$, $j \in \{0,1,2\}$ with $\digamma_j(\tilde{x}, \tilde{y}) = (x,y)$.

Consider $(x/2, y/2)$. Since $(x/2, y/2) \in \mathcal{L}$, we have $(\hat{x}, \hat{y})$ with $\lambda_j(\hat{x}, \hat{y}) = (x/2, y/2)$ for some $j \in \{0,1,2\}$.

Case 1: $j = 0$. Then $(\hat{x}/3, \hat{y}/3) = (x/2, y/2)$, therefore, $(2\hat{x}/3, 2\hat{y}/3) = (x,y)$, and $(\tilde{x}, \tilde{y}) = (2\hat{x}, 2\hat{y})$.

Case 2: $j = 1$. Then $(\hat{x}/3+1/3, \hat{y}/3+2/3) = (x/2, y/2)$, therefore, $(2\hat{x}/3+2/3, 2\hat{y}/3+4/3) = (x,y)$, and $(\tilde{x}, \tilde{y}) = (2\hat{x}, 2\hat{y})$.

Case 3: $j = 2$. Then $(\hat{x}/3+2/3, \hat{y}/3+1/3) = (x/2, y/2)$, therefore, $(2\hat{x}/3+4/3, 2\hat{y}/3+2/3) = (x,y)$, and $(\tilde{x}, \tilde{y}) = (2\hat{x}, 2\hat{y})$.

Claim 4.5.5. $2\mathcal{L}$ *is compact.*

Proof. We showed in Proposition 4.4.1 that $\mathcal{L}$ is compact, specifically, that it was closed and bounded in $\mathbb{R}^2$. Since $\mathcal{L}$ was contained in $[0,1]\times[0,1]$, $2\mathcal{L} \subset [0,2]\times[0,2]$ and thus is bounded.

Since $\mathcal{L}$ is closed, $2\mathcal{L}$ the image of $\mathcal{L}$ under the continuous map $f(x,y) = 2(x,y)$, is also closed. Therefore, $2\mathcal{L}$ is compact. $\square$

$\square$

Now that we know what our invariant set is, we locate W_B cycles. First, we consider $2\mathcal{L} \cap (\mathbb{Z}/2 \times \mathbb{Z}/2$, that is, the set of points in $2\mathcal{L}$ where $W_B(x,y) = 1$.

We show in Section 4.4 that $\mathcal{L} \cap \mathbb{Z}/2 \times \mathbb{Z}/2 = \{(0,0), (1, 1/2), (1/2, 1)\}$. So we know automatically that $(0,0)$, $(2,1)$ and $(1,2)$ are in $2\mathcal{L}$.

However, the set $2\mathcal{L} \cap \mathbb{Z}/2 \times \mathbb{Z}/2$ corresponds to $\mathcal{L} \cap \mathbb{Z}/4 \times \mathbb{Z}/4$.

So we need additionally to check:

$(0, 1/4) = (0.\overline{0}_3, 0.\overline{02}_3) \notin \mathcal{L}$, therefore also $(1/4, 0) \notin \mathcal{L}$.

$(0, 3/4) = (0.\overline{0}_3, 0.\overline{20}_3) \notin \mathcal{L}$, therefore also $(3/4, 0) \notin \mathcal{L}$.

$(1/4, 1/4) = (0.\overline{02}_3, 0.\overline{02}_3) \notin \mathcal{L}$.

$(1/4, 1/2) = (0.\overline{02}_3, 0.\overline{1}_3) \notin \mathcal{L}$.

$(1/4, 3/4) = (0.\overline{02}_3, 0.\overline{20}_3) \notin \mathcal{L}$, therefore also $(3/4, 1/4) \notin \mathcal{L}$.

$(1/4, 1) = (0.\overline{02}_3, 0.\overline{2}_3) \notin \mathcal{L}$, therefore also $(1, 1/4) \notin \mathcal{L}$.

$(1/2, 3/4) = (0.\overline{1}_3, 0.\overline{20}_3) \notin \mathcal{L}$, therefore also $(1/2, 3/4) \notin \mathcal{L}$.

$(3/4, 3/4) = (0.\overline{20}_3, 0.\overline{20}_3) \notin \mathcal{L}$.

$(3/4, 1) = (0.\overline{20}_3, 0.\overline{2}_3) \notin \mathcal{L}$, therefore $(1, 3/4) \notin \mathcal{L}$.

Therefore, $2\mathcal{L} \cap (\mathbb{Z}/2 \times \mathbb{Z}/2) = 2(\mathcal{L} \cap \mathbb{Z}/4 \times \mathbb{Z}/4) = \{(0,0), (2,1), (1,2)\}$.

All of these turn out to be on cycles, in fact, they are fixed points:

- $F_0(0,0) = \frac{1}{3}(0,0) = (0,0)$.
- $F_2(2,1) = \frac{1}{3}((2,1) + (4,2)) = \frac{1}{3}(6,3) = (2,1)$.
- $F_1(1,2) = \frac{1}{3}((1,2) + (2,4)) = \frac{1}{3}(3,6) = (1,2)$.

Therefore, the W_B-cycles are exactly $(0,0)$, $(2,1)$, and $(1,2)$.

Therefore, by Theorem 2.2.13, we construct the frequencies for an orthonormal basis by applying $\{3(x,y), 3(x,y) + (2,4), 3(x,y) + (4,2)\}$ to $(0,0)$, $(-2,-1)$, adn $(-1,-2)$.

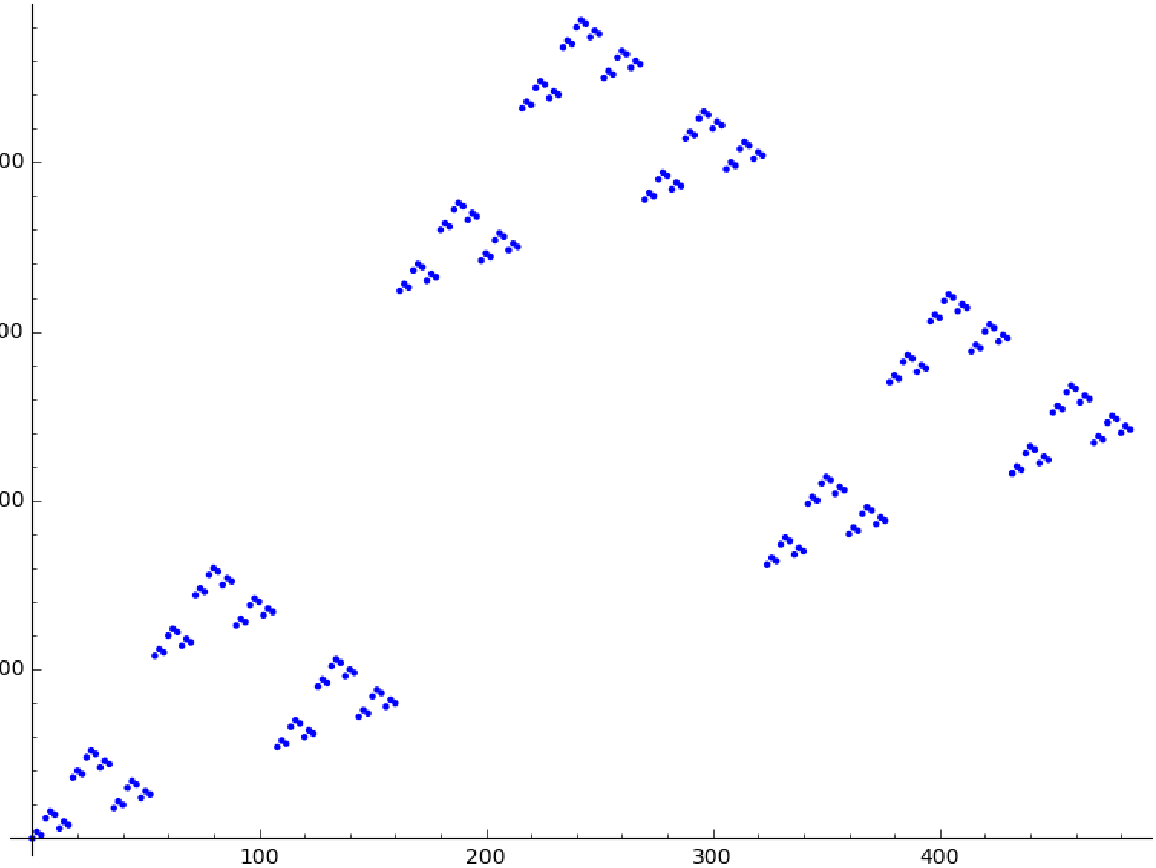

Figure 4.6 Plot of $s_5(0,0)$.

4.5.1 Frequencies

Let

$$\gamma_0(x,y) = 3(x,y) \tag{4.69}$$

$$\gamma_1(x,y) = 3(x,y) + (2,4) \tag{4.70}$$

$$\gamma_2(x,y) = 3(x,y) + (4,2) \tag{4.71}$$

.

Let $s_1(x,y) = \{\gamma_0(x,y), \gamma_1(x,y), \gamma_2(x,y)\}$, then for $n \geq 2$, let $s_n(x,y) = \{\gamma_0(s_{n-1}(x,y)), \gamma_1(s_{n-1}(x,y)), \gamma_2(s_{n-1}(x,y))\}$, with all operations applied componentwise to the pairs in the set. Then let $\mathcal{S}(x,y) = \bigcup_{n\in\mathbb{N}} s_n(x,y)$.

Following the argument in Claim 4.5.1, we argue:

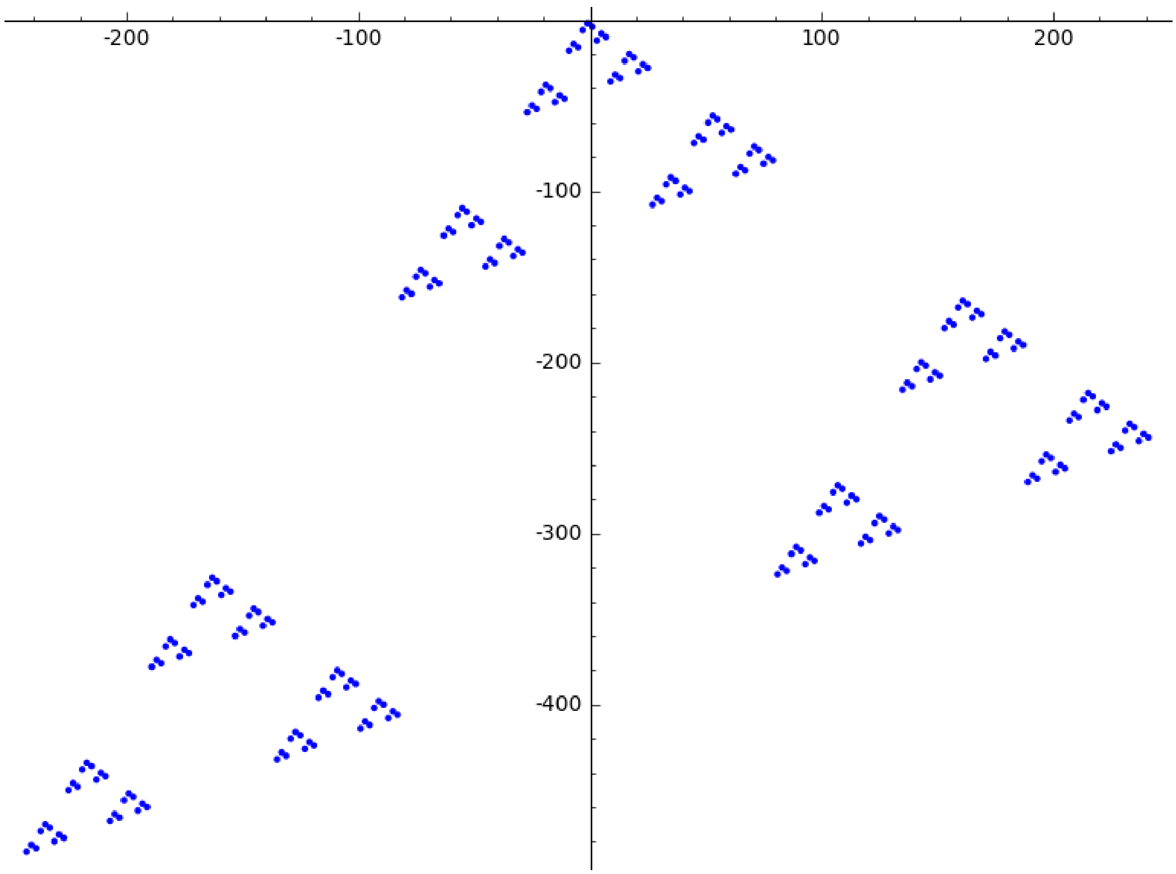

Figure 4.7 Plot of $s_5(-1,-2)$.

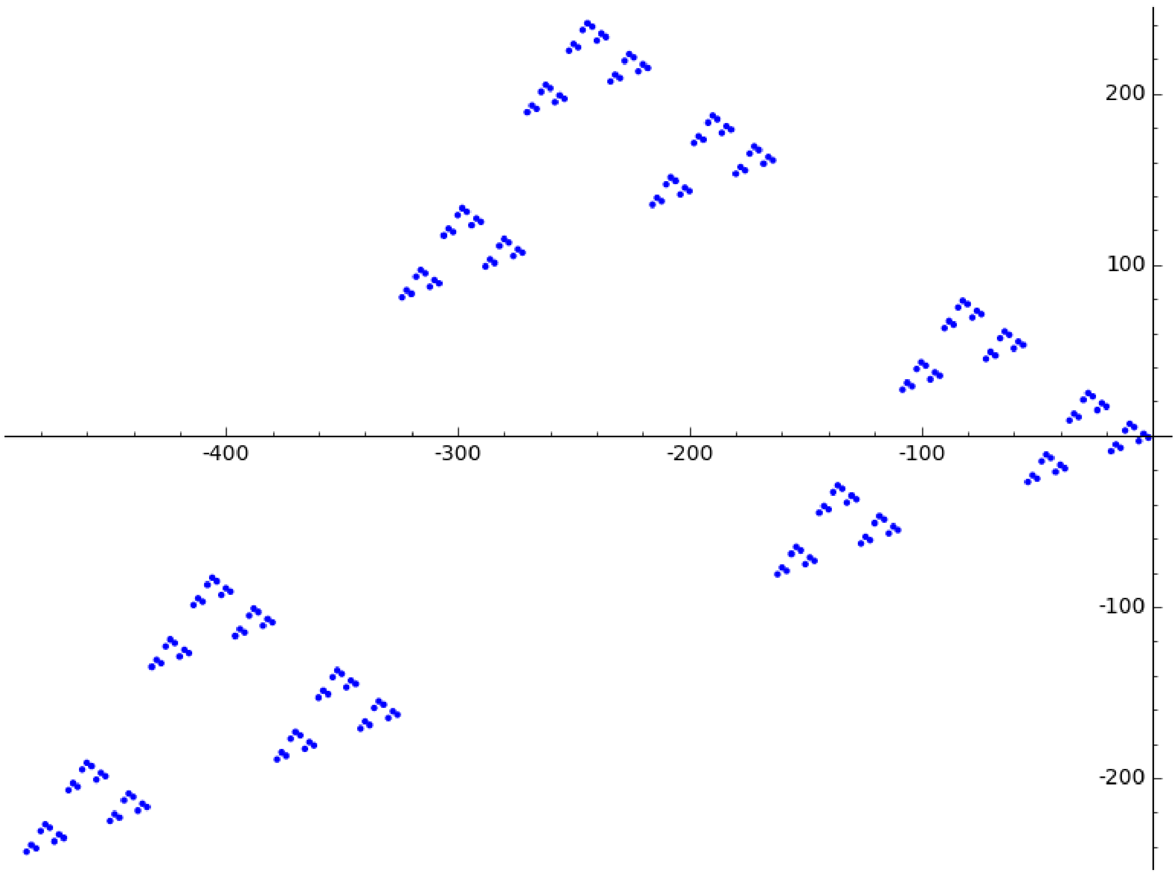

Figure 4.8 Plot of $s_5(-2,-1)$.

Claim 4.5.6. *$(x, y) \in \mathcal{S}(s, t)$ iff $(x/2, y/2) \in \mathcal{R}(s, t)$.*

Proof. We know by Proposition 4.4.7 that $(x, y) = \sum_{j=1}^{n}(x_j, y_j)3^j \in \mathcal{R}(0, 0)$ if and only if $x_j + y_j \equiv 0 \mod 3$, and $(x, y) \in \mathcal{R}(s, t)$ if and only if $(x, y) + 3^n(-1, -1/2) \in \mathcal{R}(s, t)$.

Let $2\mathcal{R}(s, t) = \{(x, y)|(x/2, y/2) \in \mathcal{R}(s, t)\}$.

Let $(x, y) \in 2\mathcal{R}(s, t)$.

Claim 4.5.7. *$\gamma_0(x, y) \in 2\mathcal{R}(s, t)$.*

Proof. By definition, $\gamma_0(x, y) = (3x, 3y)$. Then $\gamma_0(x, y)/2 = (3x/2, 3y/2) = \rho_0(x/2, y/2)$. $\square$

Claim 4.5.8. *$\gamma_1(x, y) \in 2\mathcal{R}(s, t)$.*

Proof. By definition, $\gamma_1(x, y) = (3x + 2, 3y + 4)$. Then $\gamma_1(x, y)/2 = (3x/2 + 1, 3y/2 + 2) = \rho(x/2, y/2)$. $\square$

Claim 4.5.9. *$\gamma_2(x, y) \in 2\mathcal{R}(s, t)$.*

Proof. $\gamma_2(x, y) = (3x + 2, 3y + 4)$. Then $\Gamma_2(x, y)/2 = (3x/2 + 2, 3y/2 + 1) = \rho_2(x/2, y/2)$. $\square$

Now we show that if $(x, y) \in 2\mathcal{R}(s, t)$, there exists $(\tilde{x}, \tilde{y}) \in 2\mathcal{R}(s, t)$, $j \in \{0, 1, 2\}$ with $\gamma_j(\tilde{x}, \tilde{y}) = (x, y)$.

Consider $(x/2, y/2)$. Since $(x/2, y/2) \in \mathcal{R}(s, t)$, we have $(\hat{x}, \hat{y})$ with $\rho_j(\hat{x}, \hat{y}) = (x/2, y/2)$ for some $j \in \{0, 1, 2\}$.

Case 1: $j = 0$. Then $(3\hat{x}, 3\hat{y}) = (x/2, y/2)$, therefore, $(2\hat{x}/3, 2\hat{y}/3) = (x, y)$, and $(\tilde{x}, \tilde{y}) = (2\hat{x}, 2\hat{y})$.

Case 2: $j = 1$. Then $(3\hat{x} + 1, 3\hat{y} + 2) = (x/2, y/2)$, therefore, $(6\hat{x} + 2, 6\hat{y}/3 + 4) = (x, y)$, and $(\tilde{x}, \tilde{y}) = (2\hat{x}, 2\hat{y})$.

Case 3: $j = 2$. Then $(3\hat{x} + 2, 3\hat{y} + 1) = (x/2, y/2)$, therefore, $(6\hat{x} + 4, 6\hat{y} + 2) = (x, y)$, and $(\tilde{x}, \tilde{y}) = (2\hat{x}, 2\hat{y})$.

$\square$

Theorem 4.5.10 (The third spectrum)**.** *$\mathcal{S}(0, 0) \cup \mathcal{S}(-2, -1) \cup \mathcal{S}(-1, -2)$ is a spectrum for $\mathcal{S}$.*

Proof. This follows from Theorem 2.2.13. We check first that $\{R^{-1}B, L\}$ form a Hadamard pair:

$$M_1 = \left(e^{2\pi i R^{-1}b\cdot l}\right)_{b\in B, l\in L} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i(4/3)} & e^{2\pi i(8/3)} \\ 1 & e^{2\pi i(8/3)} & e^{2\pi i(4/3)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i(1/3)} & e^{2\pi i(2/3)} \\ 1 & e^{2\pi i(2/3)} & e^{2\pi i(1/3)} \end{pmatrix},$$

which is Hadamard.

By construction, $0 \in B$, $0 \in L$, and it is easy to see that $R^n b \cdot l \in \mathbb{Z}$ for all $b \in B$, $l \in L$, $n \geq 0$.

By construction $\Lambda = \mathcal{S}(0,0) \cup \mathcal{S}(-2,-1) \cup \mathcal{S}(-1,-2)$ is the smallest set that contains $-C$ for all W_B-cycles C, and such that $S\Lambda + L \in \Lambda$, so it satisfies the hypotheses of Theorem 2.2.13 as long as the Transversality of the Zeros condition is fulfilled.

To satisfy the Transversality of the Zeros condition (Definition 2.2.12), we need to show:

(a) If $(x,y) \in X_L$ is not a cycle, then there exists $k_x \geq 0$ such that, for $k \geq k_x$, $\{\tau_{l_1} \circ \tau_{l_2} \circ \cdots \circ \tau_{l_k} x : l_1, \ldots l_n \in L\}$ does not contain any zeros of W;

(b) If $\{x_0, x_1, \ldots, x_p\}$ are on a cycle with $x_1 = \tau_l(x_0)$ for some $l \in L$, then for every $y = \tau_{l'}(x_0)$, $y \neq x_1$ is either not on a cycle or $W(y) = 0$.

In this case, $X_L = 2\mathcal{L}$, where $\mathcal{L}$ is the X_L for the second spectrum in Section 4.4.

Lemma 4.5.11. *$(x,y) \in X_L$ is on a cycle if and only if $x = k/(3^n - 1)$ and $y = l/(3^n - 1)$ for some integers $0 \leq k, l \leq 3^n$ and $n \in \mathbb{N}$.*

Since $X_L = 2\mathcal{L}$, it shares the property that there are no overlap points between the ranges of $\digamma_0, \digamma_1, \digamma_2$, hence each (x,y) has only one possible value for $r(x,y)$.

Therefore by the same computation as in Lemma 4.4.16, for all $(x,y) \in X_L$, $(x,y) = r^n(x,y)$ if and only if $x = k/(3^n - 1)$ for some $k \in \mathbb{Z}, k \leq 3^n$, and similarly $y = l/(3^n - 1)$ for some $l \in \mathbb{Z}, l \leq 3^n$.

Consider:

$$\left(\frac{k}{3^n - 1}, \frac{l}{3^n - 1}\right) \in X_L \iff \left(\frac{k}{2(3^n - 1)}, \frac{l}{2(3^n - 1)}\right) \in \mathcal{L}$$

$\square$

We leave aside the question of which such (x,y) are actually contained in X_L. It suffices for condition (b) to show that for such an (x,y), at most one of $\lambda_0(x,y)$, $\lambda_1(x,y)$, $\lambda_2(x,y)$ is on a cycle.

Proof of condition (b). Let (x,y) be on a cycle, that is, $(x,y) = (k/(3^n-1), l/(3^n-1)$ for integers $0 \leq k, l \leq 3^n$.

There are three options for k: $k = 3m$, $k = 3m+1$, and $k = 3m+2$ for some integer m.

Case 1: $k = 3m$.

$$F_0\left(\frac{3m}{3^n-1}, \frac{l}{3^n-1}\right) = \left(\frac{3m}{3(3^n-1)}, \frac{l}{3(3^n-1)}\right) = \left(\frac{m}{3^n-1}, \frac{l}{3(3^n-1)}.\right)$$

This may be on a cycle, if $l = 3p$ for some integer p.

$$F_1\left(\frac{3m}{3^n-1}, \frac{l}{3^n-1}\right) = \left(\frac{3m+2(3^n-1)}{3(3^n-1)}, \frac{l+4(3^n-1)}{3(3^n-1)}\right) = \left(\frac{3m+2(3^n)-2}{3(3^n-1)}, \frac{l+4(3^n)-4}{3(3^n-1)}\right).$$

By Lemma 4.2.10, this cannot be on a cycle.

$$F_2\left(\frac{3m}{3^n-1}, \frac{l}{3^n-1}\right) = \left(\frac{3m+4(3^n-1)}{3(3^n-1)}, \frac{l+2(3^n-1)}{3(3^n-1)}\right) = \left(\frac{3m+4(3^n)-4}{3(3^n-1)}, \frac{l+2(3^n)-2)}{3(3^n-1)}\right).$$

By Lemma 4.2.10, this cannot be on a cycle.

Case 2: $k = 3m+1$.

$$F_0\left(\frac{3m+1}{3^n-1}, \frac{l}{3^n-1}\right) = \left(\frac{3m+1}{3(3^n-1)}, \frac{l}{3(3^n-1)}\right) = \left(\frac{3m+1}{3(3^n-1)}, \frac{l}{3(3^n-1)}.\right)$$

By Lemma 4.2.10, this cannot be on a cycle.

$$F_1\left(\frac{3m+1}{3^n-1}, \frac{l}{3^n-1}\right) = \left(\frac{3m+1+2(3^n-1)}{3(3^n-1)}, \frac{l+4(3^n-1)}{3(3^n-1)}\right) = \left(\frac{3m+2(3^n)-1}{3(3^n-1)}, \frac{l+4(3^n)-4}{3(3^n-1)}\right).$$

By Lemma 4.2.10, this cannot be on a cycle.

$$F_2\left(\frac{3m+1}{3^n-1}, \frac{l}{3^n-1}\right) = \left(\frac{3m+1+4(3^n-1)}{3(3^n-1)}, \frac{l+2(3^n-1)}{3(3^n-1)}\right) = \left(\frac{3m+4(3^n)-3}{3(3^n-1)}, \frac{l+2(3^n)-2)}{3(3^n-1)}\right).$$

This may be on a cycle, if $l = 3p+2$ for some integer p.

Case 3: $k = 3m+2$.

$$F_0\left(\frac{3m+2}{3^n-1}, \frac{l}{3^n-1}\right) = \left(\frac{3m+2}{3(3^n-1)}, \frac{l}{3(3^n-1)}\right) = \left(\frac{3m+2}{3(3^n-1)}, \frac{l}{3(3^n-1)}.\right)$$

By Lemma 4.2.10, this cannot be on a cycle.

$$F_1\left(\frac{3m+2}{3^n-1},\frac{l}{3^n-1}\right)=\left(\frac{3m+2+2(3^n-1)}{3(3^n-1)},\frac{l+4(3^n-1)}{3(3^n-1)}\right)=\left(\frac{m+2(3^n)}{3^n-1},\frac{l+4(3^n)-4}{3(3^n-1)}\right).$$

This may be on a cycle, if $l=3p+1$ for some integer p.

$$F_2\left(\frac{3m+1}{3^n-1},\frac{l}{3^n-1}\right)=\left(\frac{3m+2+4(3^n-1)}{3(3^n-1)},\frac{l+2(3^n-1)}{3(3^n-1)}\right)=\left(\frac{3m+4(3^n)-2}{3(3^n-1)},\frac{l+2(3^n)-2)}{3(3^n-1)}\right).$$

By Lemma 4.2.10, this cannot be on a cycle. □

Proof of condition (a). We notice there are only finitely many (x,y) that are both zeros of W_B and contained in X_L.

Zeros of W_B contained in $[0,2]\times[0,2]$:

$(1/3,1/6),(1/6,1/3),(5/6,1/6),(1/6,5/6),(4/3,1/6),(1/6,4/6),(11/6,1/6),(1/6,11/6),$ $(1/3,2/3),(2/3,1/3),(5/6,2/3),(2/3,5/6),(4/3,2/3),(2/3,4/3),(11/6,2/3),(2/3,11/6),$ $(1/3,7/6),(7/6,1/3),(5/6,7/6),(7/6,5/6),(4/3,7/6),(7/6,4/3),(11/6,7/6),(7/6,11/6),$ $(1/3,5/3),(5/3,1/3),(5/6,5/3),(5/3,5/6),(11/6,5/3),(5/3,11/6).$

Without explicitly listing them, we observe that the number of zeros of W_B in X_L is finite, and also that none of them are cycles, by Lemma 4.2.10.

The remainder of the proof is identical to the case for Theorem 4.4.15. □

CHAPTER 5. RELATED FRACTALS AND SPECTRA

We extend our results by applying affine transformations to $\mathcal{S}$ and considering spectra for the resulting fractals.

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$, $B(x) = Ax + b$, with A a 2 by 2 invertible matrix and $b \in \mathbb{R}^2$. Let $\mathcal{S} \in \mathbb{R}^2$ a fractal as in Definition 2.2.2, with the measure ν given by Theorem 2.2.4, and let Λ be a spectrum for $L^2(\nu)$.

Let $G : \mathbb{R}^2 \to \mathbb{R}^2$, $G(x) = (A^T)^{-1}(x - b)$ so that $G^{-1}(x) = A^T x + b$.

Proposition 5.0.1. *$G(\Lambda)$ is a spectrum for $L^2(\tilde{\nu})$, the fractal measure supported on $F(\mathcal{S})$, with $\tilde{\nu}(C) = \nu(G(C))$ for $A \subset F(\mathcal{S})$, and for any continuous $f : F(\mathcal{S}) \to \mathbf{C}$, $\int f d\tilde{\nu} = \int f \circ F d\nu$.*

Proof. We use Theorem 2.2.8. We want to show that $\sum_{\lambda \in \Lambda} |\widehat{\nu}(t-\lambda)|^2 = \sum_{\lambda' \in G(\Lambda)} |\widehat{\tilde{\nu}}(t-\lambda')|^2$. First, we compute:

$$\widehat{\tilde{\nu}}(t) = \int_{F(\mathcal{S})} e^{-2\pi i x \cdot t}\ d\tilde{\nu}(x) \tag{5.1}$$

$$= \int_{\mathcal{S}} e^{-2\pi i (Ax+b)\cdot(t)} d\nu(x) \tag{5.2}$$

$$= \int_{\mathcal{S}} e^{-2\pi i (Ax\cdot t + b\cdot t)} d\nu(x) \tag{5.3}$$

$$= e^{-2\pi i (b\cdot t)} \int_{\mathcal{S}} e^{-2\pi i (Ax\cdot t)} d\nu(x) \tag{5.4}$$

$$= e^{-2\pi i (b\cdot t)} \int_{\mathcal{S}} e^{-2\pi i (x\cdot A^T t))} d\nu(x) \tag{5.5}$$

$$= e^{-2\pi i (b\cdot t)} \widehat{\nu}(A^T t) \tag{5.6}$$

Then, for $\lambda' \in G(\Lambda)$, $\widehat{\tilde{\nu}}(t-\lambda') = e^{-2\pi i (b\cdot(t-\lambda'))} \widehat{\nu}(A^T(t-\lambda')$. Since $\lambda' \in G(\Lambda)$, $G^{-1}(\lambda') =$

$A\lambda' + b \in \Lambda$. Therefore:

$$\sum_{\lambda' \in G(\Lambda)} |\widehat{\nu}(t - \lambda')|^2 = \sum_{\lambda' \in G(\Lambda)} \left| e^{-2\pi i (b \cdot (t-\lambda')} \widehat{\nu}(A^T(t - \lambda')) \right|^2 \tag{5.7}$$

$$= \sum_{\lambda' \in G(\Lambda)} \left| \widehat{\nu}(A^T t - A^T \lambda') \right|^2 \tag{5.8}$$

$$= \sum_{\lambda' \in G(\Lambda)} \left| \widehat{\nu}(A^T t + b - b - A^T \lambda') \right|^2 \tag{5.9}$$

$$= \sum_{\lambda' \in G(\Lambda)} \left| \widehat{\nu}(G^{-1}(t) - G^{-1}(\lambda')) \right|^2 \tag{5.10}$$

$$= \sum_{\lambda \in \Lambda} \left| \widehat{\nu}(G^{-1}(t) - \lambda) \right|^2 \tag{5.11}$$

$$= \sum_{\lambda \in \Lambda} |\widehat{\nu}(s - \lambda)|^2 \tag{5.12}$$

The last few lines use the fact that G is a bijection from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Therefore, since $\sum_{\lambda \in \Lambda} |\widehat{\nu}(t - \lambda)|^2 \equiv 1$ by Theorem 2.2.8, $\sum_{\lambda' \in G(\Lambda)} |\widehat{\nu}(t - \lambda')|^2 = 1$ and $G(\Lambda)$ is a spectrum for $F(\mathcal{S})$.

$\square$

Lemma 5.0.2. *If $\mathcal{S} \subset \mathbb{R}^2$ is the unique compact set invariant under the iterated function system $\{\psi_0, \psi_1, \psi_2\}$, and F is as in Proposition 5.0.1, then $F(S)$ is the unique compact set invariant under the iterated function system $\{F\psi_0 F^{-1}, F\psi_1 F^{-1}, F\psi_2 F^{-1}\}$.*

Proof. First, we show that $F(\mathcal{S}) \subseteq \bigcup_{j=0}^{2} F\psi_j F^{-1}(F(\mathcal{S}))$.

Let $x \in F(S)$. Then $x = F(y)$ for some $y \in \mathcal{S}$. Since $y \in \mathcal{S}$, $y \in \psi_j(\mathcal{S})$ for some $j \in \{0, 1, 2\}$, or $y = \psi_j(\tilde{y})$ for some $\tilde{y} \in \mathcal{S}$. Let $\tilde{x} \in F(\mathcal{S})$ with $F(\tilde{y}) = \tilde{y}$. Then $F\psi_j F^{-1}(\tilde{x}) = F\psi_j(\tilde{y}) = F(y) = x$, so $x \in F\psi_j F^{-1}(F(\mathcal{S}))$, and $F(\mathcal{S}) \subseteq \bigcup_{j=0}^{2} F\psi_j F^{-1}(F(\mathcal{S}))$.

Now, show that $\bigcup_{j=0}^{2} F\psi_j F^{-1}(F(\mathcal{S})) \subseteq F(\mathcal{S})$.

Let $x \in F\psi_j F^{-1}(F(\mathcal{S}))$ for some j, that is, $x = F\psi_j F^{-1}(\tilde{x})$ for some $\tilde{x} \in F(\mathcal{S})$. Write $\tilde{x} = F\tilde{y}$ for some $\tilde{y} \in \mathcal{S}$. Then $x = F\psi_j F^{-1}(F\tilde{y}) = F\psi_j(\tilde{y})$. Since $\tilde{y} \in \mathcal{S}$, $\psi_j(\tilde{y}) = y$ for some $y \in \mathcal{S}$. Then $x = Fy$ for some $y \in \mathcal{S}$, therefore, $x \in F(\mathcal{S})$, and $\bigcup_{j=0}^{2} F\psi_j F^{-1}(F(\mathcal{S})) \subseteq F(\mathcal{S})$.

$\square$

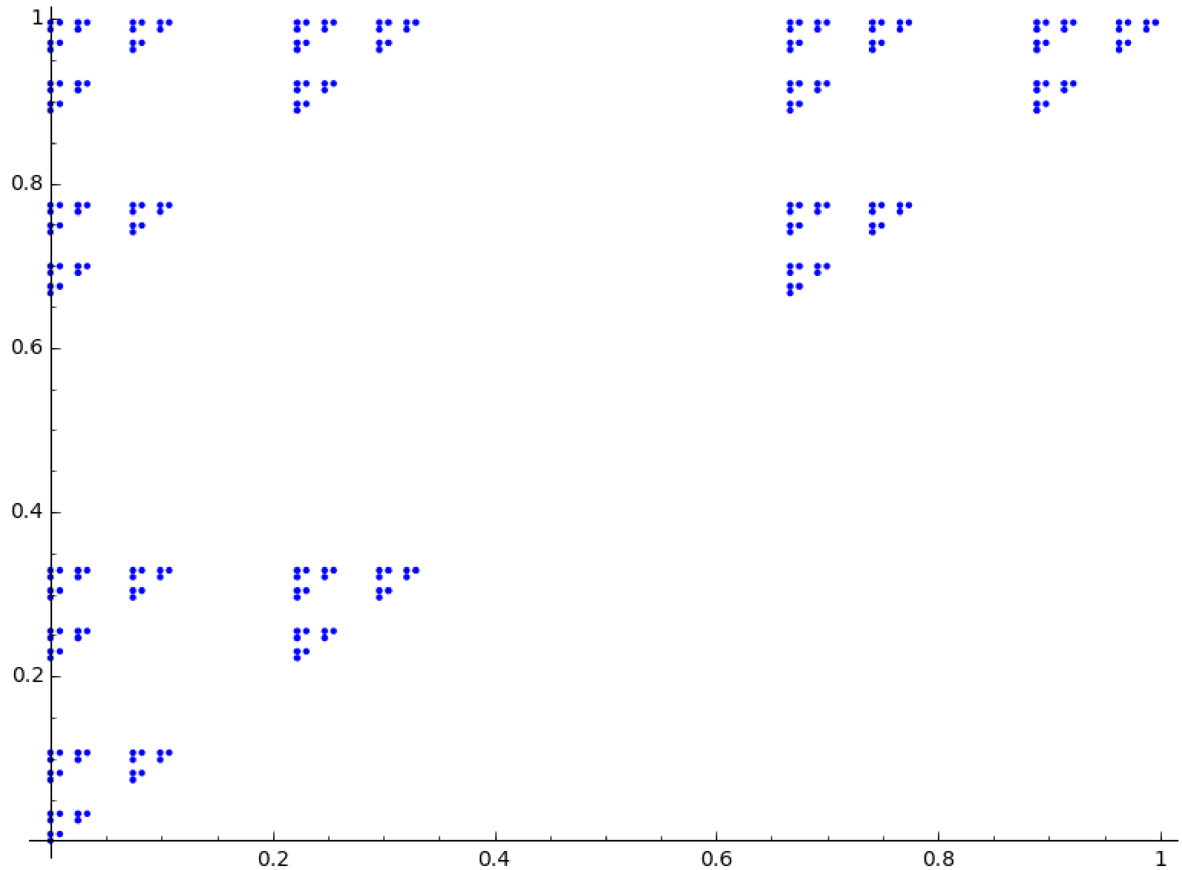

Figure 5.1 A fifth iterative approximation of $F(\mathcal{S})$ starting at zero.

5.1 The "Upper Left" fractal

Now, let:

$$F(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and $\mathcal{S}$ the fractal from Definition 4.1.1.

Geometrically, $F(x,y)$ flips $\mathbb{R}^2$ across the line $x = 1/2$.

$F(x,y)$ satisfies the conditions for Proposition 5.0.1, with

$$G(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = F(x,y).$$

By Proposition 5.0.1 and the work done on $\mathcal{S}$ in Chapter 4, we already know three spectra for $F(\mathcal{S})$, derived from the three spectra in Chapter 4.

We find another spectrum by working directly.

Claim 5.1.1. *$F(\mathcal{S})$ is the invariant set generated by the iterated function system $\{\frac{1}{3}(x,y)+B\}$, for $B=\{(0,0),(0,2),(2,2)\}$.*

Proof. Use Lemma 5.0.2. By Definition 4.1.1, $\mathcal{S}$ is invariant under the iterated function system $\{\psi_0(x,y)=\frac{1}{3}(x,y), \psi_1(x,y)=\frac{1}{3}(x+2,y), \psi_2(x,y)=\frac{1}{3}(x,y+2)\}$.

Claim 5.1.2. *$\frac{1}{3}((x,y)+(0,2)) = F\psi_0F^{-1}(x,y)$.*

Proof.

$$\begin{aligned}
F\psi_0F^{-1}(x,y) &= F\psi_0(x,-y+1)\\
&= F(\frac{1}{3}(x,-y+1))\\
&= F(\frac{x}{3},-\frac{y}{3}+\frac{1}{3}))\\
&= (\frac{x}{3},\frac{y}{3}-\frac{1}{3}+1)\\
&= (\frac{x}{3},\frac{y}{3}+\frac{2}{3})\\
&= \frac{1}{3}((x,y)+(0,2))
\end{aligned}$$

□

Claim 5.1.3. *$\frac{1}{3}((x,y)+(2,2)) = F\psi_1F^{-1}(x,y)$.*

Proof.

$$\begin{aligned}
F\psi_1F^{-1}(x,y) &= F\psi_1(x,-y+1)\\
&= F(\frac{1}{3}(x+2,-y+1))\\
&= F(\frac{x}{3}+\frac{2}{3},-\frac{y}{3}+\frac{1}{3}))\\
&= (\frac{x}{3}+\frac{2}{3},\frac{y}{3}-\frac{1}{3}+1)\\
&= (\frac{x}{3}+\frac{2}{3},\frac{y}{3}+\frac{2}{3})\\
&= \frac{1}{3}((x,y)+(2,2))
\end{aligned}$$

□

Claim 5.1.4. $\frac{1}{3}((x,y)+(0,0)) = F\psi_2 F^{-1}(x,y)$.

Proof.

$$\begin{aligned}
F\psi_2 F^{-1}(x,y) &= F\psi_2(x,-y+1) \\
&= F(\frac{1}{3}(x,-y+1+2)) \\
&= F(\frac{x}{3}, -\frac{y}{3}+\frac{3}{3})) \\
&= (\frac{x}{3}, \frac{y}{3}-\frac{3}{3}+1) \\
&= (\frac{x}{3}, \frac{y}{3}) \\
&= \frac{1}{3}((x,y)+(0,0))
\end{aligned}$$

□

Since $\mathcal{S}$ is compact and F is continuous, $\mathcal{F}(S)$ is compact, thus it is the unique invariant set generated by this iterated function system.

□

Proposition 5.1.5. *$F(\mathcal{S}) = \{(x,y) | x \in C_3, y \in C_3, y - x \in C_3\}$, where C_3 is the middle-thirds Cantor set discussed in Proposition 4.1.9.*

Proof. Let $(x,y) \in \mathcal{S}$. We know by Proposition 4.1.9 that $x \in C_3, y \in C_3$ and $x + y \in C_3$.

Let $F(x,y) = (s,t) = (x, 1-y)$. We want to show that $s \in C_3$, $t \in C_3$ and $t - s \in C_3$. We have that $s = x \in C_3$ by assumption. $t = 1 - y \in C_3$ because the Cantor set is symmetric across the point $1/2$ and $y \in C_3$. $t - s = (1-y) - x = 1 - (y+x) \in C_3$ because $x + y \in C_3$ and the Cantor set is symmetric across the point $1/2$.

□

Next, we construct a basis for $F(\mathcal{S})$ using the method of Theorem 2.2.13.

First we compute:

$$W_B(x,y) = \frac{1}{N^2}\left|\sum_{b\in B} e^{2\pi i b\cdot(x,y)}\right|^2 = \frac{1}{9}\left|1 + e^{4\pi i y} + e^{4\pi i(x+y)}\right|^2. \tag{5.13}$$

$W_B(x,y) = 1$ if and only if $e^{4\pi i x} = e^{4\pi i(x+y)} = 1$, that is, when $y \in \mathbb{Z}/2$ and $x+y \in \mathbb{Z}/2$; that is, for $(x,y) \in \mathbb{Z}/2 \times \mathbb{Z}/2$.

We choose a set L to generate our orthonormal basis. Let $L = \{(0,0),(1,1),(2,2)\}$.

We verify that:

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i(2/3)} & e^{2\pi i(4/3)} \\ 1 & e^{2\pi i(4/3)} & e^{2\pi i(8/3)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i(2/3)} & e^{2\pi i(1/3)} \\ 1 & e^{2\pi i(1/3)} & e^{2\pi i(2/3)} \end{pmatrix}$$

is Hadamard.

Proposition 5.1.6. *X_L, the invariant set of $\{\frac{1}{3}(x,y), \frac{1}{3}(x+1,y+1) + \frac{1}{3}(x+2,y+2)\}$, is $\{(t,t) : t \in [0,1]\}$.*

Proof. Let $\mathcal{A} = \{(t,t) : t \in [0,1]\}$. Clearly, $\mathcal{A}$ is compact.

Claim 5.1.7. *For $(t,t) \in A$, $\frac{1}{3}(t,t) \in \mathcal{A}$.*

Proof. $\frac{1}{3}(t,t) = (t/3, t/3)$. Let $s = t/3$. For $t \in [0,1]$, $s \in [0,1/3] \subset [0,1]$. Then $\frac{1}{3}(t,t) = (s,s) \in \mathcal{A}$. $\square$

Claim 5.1.8. *For $(t,t) \in A$, $\frac{1}{3}((t,t)+(1,1)) \in \mathcal{A}$.*

Proof. $\frac{1}{3}((t,t)+(1,1)) = (t/3+1/3, t/3+1/3)$. Let $s = t/3+1/3$. For $t \in [0,1]$, $s \in [1/3,2/3] \subset [0,1]$. Then $\frac{1}{3}((t,t)+(1,1)) = (s,s) \in \mathcal{A}$. $\square$

Claim 5.1.9. *For $(t,t) \in A$, $\frac{1}{3}((t,t)+(2,2)) \in \mathcal{A}$.*

Proof. $\frac{1}{3}((t,t)+(2,2)) = (t/3+2/3, t/3+2/3)$. Let $s = t/3+2/3$. For $t \in [0,1]$, $s \in [2/3,1] \subset [0,1]$. Then $\frac{1}{3}((t,t)+(2,2)) = (s,s) \in \mathcal{A}$. $\square$

Claim 5.1.10. *For $(t,t) \in A$, there is some $(s,s) \in A$ with $\frac{1}{3}((s,s)) = (t,t)$ or $\frac{1}{3}((s,s)+(1,1)) = (t,t)$ or $\frac{1}{3}((s,s)+(2,2)) = (t,t)$*

Proof. There are three cases: $0 \le t < 1/3$, $1/3 \le t < 2/3$, and $2/3 \le t \le 1$.

Case 1:

Let $(s, s) = (3t, 3t)$. Since $0 \leq t \leq 1/3$, we have $0 \leq s \leq 1$, so $(s, s) \in \mathcal{A}$, and $1/3(s, s) = (t, t)$.

Case 2:

Let $(s, s) = (3t - 1, 3t - 1)$. Since $1/3 \leq t < 2/3$, $0 \leq s < 1$, so $(s, s) \in \mathcal{A}$, and $1/3((s, s) + (1, 1)) = (t, t)$.

Case 3:

Let $(s, s) = (3t - 2, 3t - 2)$. Since $2/3 \leq t \leq 1$, $0 \leq s < 1$, so $(s, s) \in \mathcal{A}$, and $1/3((s, s) + (2, 2)) = (t, t)$. □

□

To find W_B-cycles, we look at the values in $\mathbb{Z}/2 \times \mathbb{Z}/2$ which are also in X_L, namely, $\{(0, 0), (1/2, 1, 2), (1, 1)\}$.

Then we check if these are on cycles. In fact, they are all fixed points:

- $\frac{1}{3}(0, 0) = (0, 0)$.
- $\frac{1}{3}((1/2, 1/2) + (1, 1)) = \frac{1}{3}(3/2, 3/2) = (1/2, 1/2)$.
- $\frac{1}{3}((1, 1) + (2, 2)) = \frac{1}{3}(3, 3) = (1, 1)$.

So our W_B-cycles are $\{(0, 0), (1/2, 1/2), (1, 1)\}$.

Therefore, to get our spectrum, we apply $\{3(x, y), 3(x, y) + (1, 1), 3(x, y) + (2, 2)\}$ to $\{(0, 0), (-1/2, -1/2), (-1, -1)\}$.

Definition 5.1.11. *Let $r_1(x, y) = \{3(x, y), 3(x, y) + (1, 1), 3(x, y) + (2, 2)\}$, then for $n \geq 2$, let $r_n(x, y) = \{3(s_{n-1}(x, y)), 3(s_{n-1}(x, y) + (1, 1)), 3(s_{n-1}(x, y)) + (2, 2)\}$, with all operations applied componentwise to the pairs in the set. Then let $\mathcal{R}(x, y) = \bigcup_{n \in \mathbb{N}} r_n(x, y)$.*

Theorem 5.1.12. $\mathcal{R}(0, 0) \cup \mathcal{R}(-1, -1)\mathcal{R}(-1/2, -1/2) = \{(t, t) : t \in \mathbb{Z}/2\}$.

Proof. We show this in three Propositions:

Proposition 5.1.13. $\mathcal{R}(0, 0) = \{(t, t) : t \in \mathbb{Z}, t \geq 0\}$.

Proof. Let $\mathcal{A} = \{(t,t) : t \in \mathbb{Z}, t \geq 0\}$.

We show first that $\mathcal{R}(0,0) \subseteq \mathcal{A}$.

First, we notice that $(0,0) \in \mathcal{A}$. It is sufficient then to show the following three claims:

Claim 5.1.14. *If* $(t,t) \in \mathcal{A}$, *then* $3(t,t) \in \mathcal{A}$.

Proof. $3(t,t) = (3t, 3t)$. If $t \in \mathbb{Z}$, $t \geq 0$, $3t \in \mathbb{Z}$, $3t \geq 0$. □

Claim 5.1.15. *If* $(t,t) \in \mathcal{A}$, *then* $3(t,t) + (1,1) \in \mathcal{A}$.

Proof. $3(t,t) + (1,1) = (3t+1, 3t+1)$. If $t \in \mathbb{Z}$, $t \geq 0$, $3t+1 \in \mathbb{Z}$, $3t+1 \geq 0$. □

Claim 5.1.16. *If* $(t,t) \in \mathcal{A}$, *then* $3(t,t) + (2,2) \in \mathcal{A}$.

Proof. $3(t,t) + (2,2) = (3t+2, 3t+2)$. If $t \in \mathbb{Z}$, $t \geq 0$, $3t+2 \in \mathbb{Z}$, $3t+2 \geq 0$. □

We then show that $\mathcal{R}(0,0) \supseteq \mathcal{A}$.

Proof by induction. Base case:

$(0,0) = 3(0,0) \in \mathcal{R}(0,0)$.

Now we $(t,t) \in \mathcal{A}$ and suppose that $(s,s) \in \mathcal{R}(0,0)$ for all $0 \leq s \leq t$. There are three cases for t: $t = 3m$ for some $m \in \mathbb{Z}$, $m \geq 0$, $t = 3m+1$, $t = 3m+2$.

Case 1: $t = 3m$. Notice that $3(m,m) = (t,t)$, and since by inductive assumption, $(m,m) \in \mathcal{R}(0,0)$, $(t,t) \in \mathcal{R}(0,0)$.

Case 2: $t = 3m+1$. Notice that $3(m,m)+(1,1) = (t,t)$, and since by inductive assumption, $(m,m) \in \mathcal{R}(0,0)$, $(t,t) \in \mathcal{R}(0,0)$.

Case 3: $t = 3m+2$. Notice that $3(m,m)+(2,2) = (t,t)$, and since by inductive assumption, $(m,m) \in \mathcal{R}(0,0)$, $(t,t) \in \mathcal{R}(0,0)$. □

Therefore, by induction, $\mathcal{R}(0,0) = \mathcal{A}$. □

Proposition 5.1.17. $\mathcal{R}(-1,-1) = \{(t,t) : t \in \mathbb{Z}, t \leq -1\}$.

Proof. Let $\mathcal{B} = \{(t,t) : t \in \mathbb{Z}, t \leq -1\}$. We show first that $\mathcal{R}(-1,-1) \subseteq B$.

We note that $\mathcal{R}(-1,-1) \in \mathcal{B}$. It is sufficient then to show the following three claims:

Claim 5.1.18. *If* $(t,t) \in \mathcal{B}$ *then* $3(t,t) \in \mathcal{B}$.

Proof. $3(t,t) = (3t, 3t)$. If $t \in \mathbb{Z}$, $t \leq -1$, $3t \in \mathbb{Z}$, $3t \leq -1$. $\square$

Claim 5.1.19. *If* $(t,t) \in \mathcal{B}$*, then* $3(t,t) + (1,1) \in \mathcal{B}$.

Proof. $3(t,t) + (1,1) = (3t+1, 3t+1)$. If $t \in \mathbb{Z}$, $t \leq -1$, $3t+1 \in \mathbb{Z}$, $3t+1 \leq -1$. $\square$

Claim 5.1.20. *If* $(t,t) \in \mathcal{B}$*, then* $3(t,t) + (2,2) \in \mathcal{B}$.

Proof. $3(t,t) + (2,2) = (3t+2, 3t+2)$. If $t \in \mathbb{Z}$, $t \leq -1$, $3t+2 \in \mathbb{Z}$, $3t+2 \leq 0$. $\square$

We then show that $\mathcal{R}(-1,-1) \supseteq \mathcal{B}$.

Proof by induction. Base case:

$(-1,-1) = 3(0,0) + (2,2) \in \mathcal{R}(-1,-1)$.

Now we let $(t,t) \in \mathcal{B}$ and suppose that $(s,s) \in \mathcal{R}(-1,-1)$ for all $-1 \geq s \geq t$. There are three cases for t: $t = 3m$ for some $m \in \mathbb{Z}$, $m \leq -1$; $t = 3m+1$; or $t = 3m+2$.

Case 1: $t = 3m$. Notice that $3(m,m) = (t,t)$, and since by inductive assumption, $(m,m) \in \mathcal{R}(-1,-1)$, $(t,t) \in \mathcal{R}(-1,-1)$.

Case 2: $t = 3m+1$. Notice that $3(m,m)+(1,1) = (t,t)$, and since by inductive assumption, $(m,m) \in \mathcal{R}(-1,-1)$, $(t,t) \in \mathcal{R}(-1,-1)$.

Case 3: $t = 3m+2$. Notice that $3(m,m)+(2,2) = (t,t)$, and since by inductive assumption, $(m,m) \in \mathcal{R}(-1,-1)$, $(t,t) \in \mathcal{R}(-1,-1)$.

Therefore, by induction, $\mathcal{R}(-1,-1) = \mathcal{B}$. $\square$

Claim 5.1.21. $\mathcal{R}(-1/2,-1/2) = \{(t,t) : t \in \mathbb{Z}/2, t \notin \mathbb{Z}\}$.

Proof. Let $\mathcal{C} = \{(t,t) : t \in \mathbb{Z}/2, t \notin \mathbb{Z}\}$. We show first that $\mathcal{R}(-1/2,-1/2) \subseteq \mathcal{C}$. We note that $(-1/2,-1/2) \in \mathcal{C}$. It is sufficient then to show the following three claims:

Claim 5.1.22. *If* $(t,t) \in \mathcal{C}$ *then* $3(t,t) \in \mathcal{C}$.

Proof. $3(t,t) = (3t,3t)$. If $t \in \mathbb{Z}/2$, $t \notin \mathbb{Z}$, $3t \in \mathbb{Z}/2$, $3t \notin \mathbb{Z}$. $\square$

Claim 5.1.23. *If* $(t,t) \in \mathcal{C}$*, then* $3(t,t) + (1,1) \in \mathcal{C}$.

Proof. $3(t,t) + (1,1) = (3t+1, 3t+1)$. If $t \in \mathbb{Z}/2$, $t \notin \mathbb{Z}$, $3t+1 \in \mathbb{Z}/2$, $3t+1 \notin \mathbb{Z}$. $\square$

Claim 5.1.24. *If* $(t,t) \in \mathcal{C}$*, then* $3(t,t)+(2,2) \in \mathcal{C}$*.*

Proof. $3(t,t)+(2,2) = (3t+2, 3t+2)$. If $t \in \mathbb{Z}/2$, $t \notin \mathbb{Z}$, then $3t+2 \in \mathbb{Z}/2$, $3t+2 \notin \mathbb{Z}$. □

We then show that $\mathcal{R}(-1/2,-1/2) \subseteq \mathcal{C}$.

One way of expressing the set $\{t \in \mathbb{Z}/2, t \notin \mathbb{Z}\}$ is as the set $\{t \mid t+1/2 \in \mathbb{Z}\}$.

For our induction argument, we write the integer $t+1/2$ in balanced ternary: $t+1/2 = \sum_{j=0}^{N} t_j 3^j$, $t_j \in \{0,1,-1\}$. With this expression, we can induct on N and conveniently cover all of our $\mathcal{C}$.

Base case, $N=0$, $t=-1/2, 1/2, -3/2$:

- $t=-1/2$: $3(-1/2,-1/2)+(1,1) = (-1/2,-1/2) \in \mathcal{R}(-1/2,-1/2)$;
- $t=1/2$: $3(-1/2,-1/2)+(2,2) = (1/2,1/2) \in \mathcal{R}(-1/2,-1/2)$;
- $t=-3/2$: $3(-1/2,-1/2) = (-3/2,-3/2) \in \mathcal{R}(-1/2,-1/2)$.

Induction step: Suppose $(t,t) \in \mathcal{R}(-1/2,-1/2)$ for all $t+1/2 = \sum_{j=0}^{n} t_j 3^j$, $t_j \in \{0,1,-1\}$, $n<N$; let $t+1/2 = \sum_{j=0}^{N} t_j 3^j$.

Case 1: $t_0 = -1$. Then $t+1/2 = \sum_{j=0}^{N} t_j 3^j = -1 + \sum_{j=1}^{N} t_j 3^j$, or $t+3/2 = \sum_{j=1}^{N} t_j 3^j$.

Therefore, $t/3+1/2 = \sum_{j=0}^{N-1} t_{j+1} 3^j$, and by our induction assumption, $(t/3, t/3) \in \mathcal{R}(-1/2,-1/2)$.

Thus $(t,t) = 3(t/3,t/3) \in \mathcal{R}(-1/2,-1/2)$.

Case 2: $t_0 = 0$. Then $t+1/2 = \sum_{j=0}^{N} t_j 3^j = \sum_{j=1}^{N} t_j 3^j$, or $(t-1)+3/2 = \sum_{j=1}^{N} t_j 3^j$.

Therefore, $(t-1)/3+1/2 = \sum_{j=0}^{N-1} t_{j+1} 3^j$, and by our induction assumption, $((t-1)/3, (t-1)/3) \in \mathcal{R}(-1/2,-1/2)$.

Thus $(t,t) = 3(2(t-1)/3, (t-1)/3) + (1,1) \in \mathcal{R}(-1/2,-1/2)$.

Case 3: $t_0 = 1$. Then $t+1/2 = \sum_{j=0}^{N} t_j 3^j = 1 + \sum_{j=1}^{N} t_j 3^j$, or $(t-2)+3/2 = \sum_{j=1}^{N} t_j 3^j$.

Therefore, $(t-2)/3+1/2 = \sum_{j=0}^{N-1} t_{j+1} 3^j$, and by our induction assumption, $((t-2)/3, (t-2)/3) \in \mathcal{R}(-1/2,-1/2)$.

Thus $(t,t) = 3(2(t-2)/3, (t-2)/3) + (2,2) \in \mathcal{R}(-1/2,-1/2)$.

□

Theorem 5.1.25. *$\{e_{t,t} : (t,t) : t \in \mathbb{Z}/2\}$ is an orthonormal basis for $L^2(\mu_B)$.*

By Proposition 5.1.6, $X_L = \{(t,t) : t \in [0,1]\}$.

To satisfy the Transversality of the Zeros condition (Definition 2.2.12), we need to show:

(a) If $(x,y) \in X_L$ is not a cycle, then there exists $k_x \geq 0$ such that, for $k \geq k_x$, $\{\tau_{l_1} \circ \tau_{l_2} \circ \cdots \circ \tau_{l_k} x : l_1, \ldots l_n \in L\}$ does not contain any zeros of W;

(b) If $\{x_0, x_1, \ldots, x_p\}$ are on a cycle with $x_1 = \tau_l(x_0)$ for some $l \in L$, then for every $y = \tau_{l'}(x_0)$, $y \neq x_1$ is either not on a cycle or $W(y) = 0$.

We note from the computations in Proposition 5.1.6 that the common inverse of $\frac{1}{3}(t,t)$, $\frac{1}{3}((t,t)+(1,1))$, and $\frac{1}{3}((t,t)+(2,2))$ is $r(x,y) = 3(x,y) \mod 1$.

Therefore we have:

Lemma 5.1.26. *A point (t,t) in X_L is on a cycle if and only if $t = \frac{1}{3^n - 1}$ for some $n \in \mathbb{N}, 0 \leq l \leq 3^n + 1$.*

Proof. It follows directly from the condition $r^n(x,y) = 3^n(x,y) \mod 1 = (x,y)$ that $t = 1/(3^n - 1)$ for some $n \in \mathbb{N}$, $l \in \mathbb{Z}$, and since $0 \leq t \leq 1$ we must have $0 \leq l \leq 3^n + 1$. See the proof of Lemma 4.2.8 for details.

As in the case of Lemma 4.2.8 there are two points in X_L that are in the image of two different generating functions:

- $(1/3, 1/3) = \frac{1}{3}(1,1) = \frac{1}{3}((0,0)+(1,1)$; therefore $''r''(1/3,1/3) = \{(0,0),(1,1)\}$;
- $(2/3, 2/3) = \frac{1}{3}((1,1)+(1,1)) = \frac{1}{3}((0,0)+(2,2)$; therefore $''r''(1/3,1/3) = \{(0,0),(1,1)\}$.

Claim 5.1.27. *Neither (0,0) nor (1,1) is on a cycle with $(1/3,1/3)$; similarly, neither (0,0) nor (1,1) is on a cycle with $(2/3,2/3)$.*

Proof. Compute: $r(0,0) = (0,0)$, and $r(1,1) = (1,1)$, so neither will go back to $(1/3,1/3)$ or to $(2/3,2/3)$. Therefore, $(1/3,1/3)$ and $(2/3,2/3)$ cannot be on cycles. Since they also cannot be written in the form $(l/(3^n - 1), 1/(3^n - 1))$ they match the general case. □

Proof of condition (b). Let (x, y) be on a cycle of length n. Then $(x, y) = (k/(3^n - 1), k/(3^n - 1))$ for $0 \leq k \leq 3^n - 1$. As in Section 4.2.1, it is sufficient to show that exactly one of $\frac{1}{3}(x, y)$, $\frac{1}{3}((x, y) + (1, 1))$, $\frac{1}{3}((x, y) + (2, 2))$ is on a cycle. We will need the result from Lemma 4.2.10. There are three cases for the integer k: $k = 3m$ for some integer m; $k = 3m + 1$ for some integer m; $k = 3m + 2$ for some integer m. Case 1: $k = 3m$.

$$\frac{1}{3}(x, y) = \frac{1}{3}\left(\frac{3m}{3^n - 1}, \frac{3m}{3^n - 1}\right) = \left(\frac{3m}{3(3^n - 1)}, \frac{3m}{3(3^n - 1)}\right) = \left(\frac{m}{3^n - 1}, \frac{m}{3^n - 1}\right)$$

which is on an n-cycle. However:

$$\begin{aligned}\frac{1}{3}\left((x, y) + (1, 1)\right) &= \frac{1}{3}\left(\frac{3m}{3^n - 1}, \frac{3p}{3^n - 1} + (1, 1)\right) = \left(\frac{3m}{3(3^n - 1)} + \frac{1}{3}, \frac{3m}{3(3^n - 1)} + \frac{1}{3}\right)\\ &= \left(\frac{3m + 3^n - 1}{3(3^n - 1)}, \frac{3m + 3^n - 1}{3(3^n - 1)}\right)\end{aligned}$$

Since $3m + 3^n - 1$ is not divisible by 3, by Lemma 4.2.10, $\frac{1}{3}\left((x, y) + (1, 1)\right)$ is not on any cycle. And:

$$\begin{aligned}\frac{1}{3}\left((x, y) + (2, 2)\right) &= \frac{1}{3}\left(\frac{3m}{3^n - 1}, \frac{3p}{3^n - 1} + (2, 2)\right) = \left(\frac{3m}{3(3^n - 1)} + \frac{2}{3}, \frac{3m}{3(3^n - 1)} + \frac{2}{3}\right)\\ &= \left(\frac{3m + 2(3^n) - 2}{3(3^n - 1)}, \frac{3m + 2(3^n) - 2}{3(3^n - 1)}\right)\end{aligned}$$

Since $3m + 2(3^n) - 2$ is not divisible by 3, by Lemma 4.2.10, $\frac{1}{3}\left((x, y) + (2, 2)\right)$ is not on any cycle.

Case 2: $k = 3m + 1$.

$$\frac{1}{3}(x, y) = \frac{1}{3}\left(\frac{3m + 1}{3^n - 1}, \frac{3m + 1}{3^n - 1}\right) = \left(\frac{3m + 1}{3(3^n - 1)}, \frac{3p + 2}{3(3^n - 1)}\right)$$

Since $3m + 1$ is not divisible by 3, by Lemma 4.2.10, $\frac{1}{3}(x, y)$ is not on any cycle.

$$\begin{aligned}\frac{1}{3}\left((x, y) + (1, 1)\right) &= \frac{1}{3}\left(\frac{3m + 1}{3^n - 1}, \frac{3m + 1}{3^n - 1} + (1, 1)\right) = \left(\frac{3m + 1}{3(3^n - 1)} + \frac{1}{3}, \frac{3m + 1}{3(3^n - 1)} + \frac{1}{3}\right)\\ &= \left(\frac{3m + 1 + 3^n - 1}{3(3^n - 1)}, \frac{3m + 1 + 3^n - 1}{3(3^n - 1)}\right)\\ &= \left(\frac{3m + 3^n}{3(3^n - 1)}, \frac{3m + 3^n}{3(3^n - 1)}\right)\\ &= \left(\frac{m + 3^{n-1}}{(3^n - 1)}, \frac{m + 3^{n-1}}{(3^n - 1)}\right),\end{aligned}$$

which is on a cycle.

$$\begin{aligned}\frac{1}{3}\left((x,y)+(2,2)\right) &= \frac{1}{3}\left(\frac{3m+1}{3^n-1},\frac{3m+1}{3^n-1}+(2,2)\right) = \left(\frac{3m+1}{3(3^n-1)}+\frac{2}{3},\frac{3m+1}{3(3^n-1)}+\frac{2}{3}\right)\\ &= \left(\frac{3m+1+2(3^n)-2}{3(3^n-1)},\frac{3m+1+2(3^n)-2}{3(3^n-1)}\right)\\ &= \left(\frac{3m+2(3^n)+1}{3(3^n-1)},\frac{3m+2(3^n)+1}{3(3^n-1)}\right)\end{aligned}$$

Since $3m+2(3^n)+1$ is not divisible by 3, by Lemma 4.2.10, $\frac{1}{3}\left((x,y)+(2,2)\right)$is not on any cycle.

Case 3: $k=3m+2$.

$$\frac{1}{3}(x,y)=\frac{1}{3}\left(\frac{3m+2}{3^n-1},\frac{3m+2}{3^n-1}\right)=\left(\frac{3m+1}{3(3^n-1)},\frac{3p+2}{3(3^n-1)}\right)$$

Since $3m+2$ is not divisible by 3, by Lemma 4.2.10, $\frac{1}{3}(x,y)$ is not on any cycle.

$$\begin{aligned}\frac{1}{3}\left((x,y)+(1,1)\right) &= \frac{1}{3}\left(\frac{3m+2}{3^n-1},\frac{3m+2}{3^n-1}+(1,1)\right) = \left(\frac{3m+2}{3(3^n-1)}+\frac{1}{3},\frac{3m+2}{3(3^n-1)}+\frac{1}{3}\right)\\ &= \left(\frac{3m+2+3^n-1}{3(3^n-1)},\frac{3m+2+3^n-1}{3(3^n-1)}\right)\\ &= \left(\frac{3m+3^n+1}{3(3^n-1)},\frac{3p+3^n+1}{3(3^n-1)}\right)\end{aligned}$$

Since $3m+3^n+1$ is not divisible by 3, by Lemma 4.2.10, $\frac{1}{3}\left((x,y)+(1,1)\right)$ is not on any cycle.

$$\begin{aligned}\frac{1}{3}\left((x,y)+(2,2)\right) &= \frac{1}{3}\left(\frac{3m+2}{3^n-1},\frac{3m+2}{3^n-1}+(2,2)\right) = \left(\frac{3m+2}{3(3^n-1)}+\frac{2}{3},\frac{3m+2}{3(3^n-1)}+\frac{2}{3}\right)\\ &= \left(\frac{3m+2+2(3^n)-2}{3(3^n-1)},\frac{3m+2+2(3^n)-2}{3(3^n-1)}\right)\\ &= \left(\frac{3m+2(3^n)}{3(3^n-1)},\frac{3m+2(3^n)}{3(3^n-1)}\right)\\ &= \left(\frac{m+2(3^{n-1})}{(3^n-1)},\frac{m+2(3^{n-1})}{(3^n-1)}\right),\end{aligned}$$

which is on a cycle. □

Proof of condition (a). Recall that $W_B(x,y)=\frac{1}{9}\left|1+e^{4\pi ix}+e^{4\pi i(x+y}\right|^2$, so that $W_B(x,y)=0$ if and only if $1+e^{4\pi ix}+e^{4\pi i(x+y)}=0$, that is, when $x\in 1/3+\mathbb{Z}/2$ and $x+y\in 1/6+\mathbb{Z}/2$, that is, $x\in 1/3+\mathbb{Z}/2$ and $y\in 1/6+\mathbb{Z}/2$; or $x\in 1/6+\mathbb{Z}/2$ and $x+y\in 1/3+\mathbb{Z}/2$, that is, $x\in 1/6+\mathbb{Z}/2$ and $y\in 1/3+\mathbb{Z}/2$.

We notice that there are no points (x,y) that are both zeros of W_B and contained in X_L.

Zeros of W_B contained in $[0,1] \times [0,1]$:

$(1/3, 1/6), (5/6, 1/6), (1/3, 2/3), (5/6, 2/3), (1/6, 1/3), (2/3, 1/3), (1/6, 5/6), (2/3, 5/6)$.

Zeros of W_B contained in $X_L = \{(t,t) : t \in [0,1]\}$: none.

Therefore condition (a) is true trivially.

$\square$

Since the transversality of the zeros condition is satisfied, we have that $\{e_{t,t} : (t,t) \in \mathbb{Z}/2\}$ is an orthonormal basis for $L^2(\mu_B)$.

$\square$

5.2 The "Lower Right" fractal

Now, let:

$$F(x,y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and $\mathcal{S}$ the fractal from Definition 4.1.1.

Geometrically, $F(x,y)$ flips $\mathbb{R}^2$ across the line $y = 1/2$.

$F(x,y)$ satisfies the conditions for Proposition 5.0.1, with

$$G(x,y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = F(x,y).$$

Proposition 5.2.1. *$F(\mathcal{S})$ is the invariant set generated by the iterated function system $\{\frac{1}{3}(x,y)+B\}$, for $B = \{(0,0),(2,0),(2,2)\}$.*

Proof. Use Lemma 5.0.2. By Definition 4.1.1, $\mathcal{S}$ is invariant under the iterated function system $\{\psi_0(x,y) = \frac{1}{3}(x,y), \psi_1(x,y) = \frac{1}{3}(x+2,y), \psi_2(x,y) = \frac{1}{3}(x,y+2)\}$.

Claim 5.2.2. $\frac{1}{3}((x,y)+(2,0)) = F\psi_0F^{-1}(x,y)$.

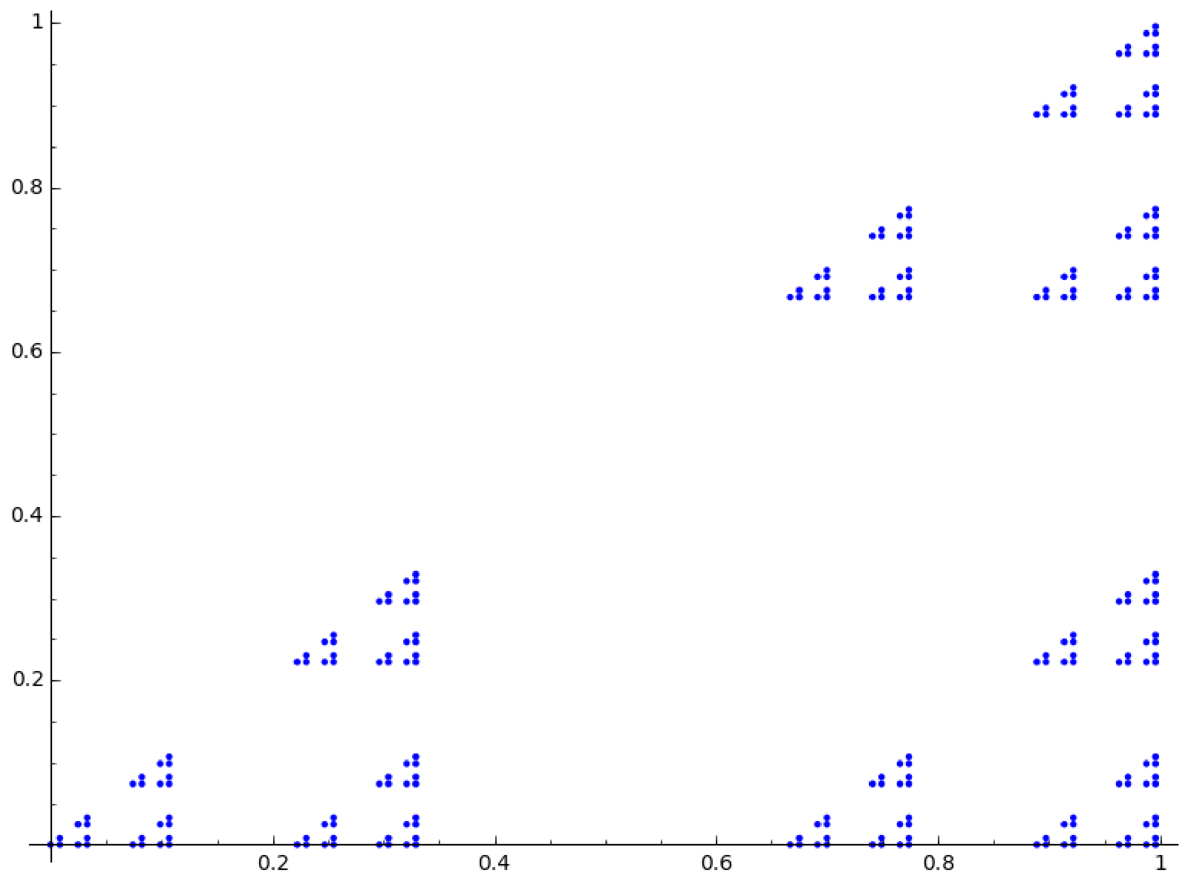

Figure 5.2 A fifth iterative approximation of $F(\mathcal{S})$ starting at zero.

Proof.

$$\begin{aligned}
F\psi_0F^{-1}(x,y) &= F\psi_0(-x+1,y)\\
&= F\left(\frac{1}{3}(-x+1,y)\right)\\
&= F\left(-\frac{x}{3}+\frac{1}{3},\frac{y}{3}\right)\\
&= \left(\frac{x}{3}-\frac{1}{3}+1,\frac{y}{3}\right)\\
&= \left(\frac{x}{3}+\frac{2}{3},\frac{y}{3}\right)\\
&= \frac{1}{3}((x,y)+(2,0))
\end{aligned}$$

□

Claim 5.2.3. $\frac{1}{3}((x,y)+(0,0)) = F\psi_1F^{-1}(x,y)$.

Proof.

$$\begin{aligned}
F\psi_1F^{-1}(x,y) &= F\psi_1(-x+1,y)\\
&= F\left(\frac{1}{3}(-x+1,y)\right)\\
&= F\left(-\frac{x}{3}+\frac{3}{3},\frac{y}{3}\right)\\
&= \left(\frac{x}{3}-1+1,\frac{y}{3}\right)\\
&= \left(\frac{x}{3},\frac{y}{3}\right)\\
&= \frac{1}{3}((x,y)+(0,0))
\end{aligned}$$

□

Claim 5.2.4. $\frac{1}{3}((x,y)+(2,2)) = F\psi_2F^{-1}(x,y)$.

Proof.

$$\begin{aligned}
F\psi_2F^{-1}(x,y) &= F\psi_2(-x+1,y)\\
&= F\left(\frac{1}{3}(-x+1,-y+2)\right)\\
&= F\left(-\frac{x}{3}+\frac{1}{3},\frac{y}{3}+\frac{2}{3}\right)\\
&= \left(\frac{x}{3}-\frac{1}{3}+1,\frac{y}{3}+\frac{2}{3}\right)\\
&= \left(\frac{x}{3}+\frac{2}{3},\frac{y}{3}+\frac{2}{3}\right)\\
&= \frac{1}{3}((x,y)+(2,2))
\end{aligned}$$

□

Since $\mathcal{S}$ is compact and F is continuous, $\mathcal{F}(S)$ is compact, thus it is the unique compact set invariant under this iterated function system.

□

Proposition 5.2.5. *$F(\mathcal{S}) = \{(x,y)|x \in C_3, y \in C_3, x-y \in C_3\}$, where C_3 is the middle-thirds Cantor set discussed in Proposition 4.1.9.*

Proof. Let $(x,y) \in \mathcal{S}$. We know by Proposition 4.1.9 that $x \in C_3, y \in C_3$ and $x+y \in C_3$.

Let $F(x,y) = (s,t) = (1-x,y)$. We want to show that $s \in C_3$, $t \in C_3$ and $s-t \in C_3$. We have that $s = x \in C_3$ by assumption. $t = 1-y \in C_3$ because the Cantor set is symmetric across the point $1/2$ and $y \in C_3$. $s-t = (1-x)-y = 1-(x+y) \in C_3$ because $x+y \in C_3$ and the Cantor set is symmetric across the point $1/2$.

□

Theorem 5.2.6. *$\{(t,t) : t \in \mathbb{Z}/2\}$ is a spectrum for $L^2(\mu_B)$.*

Proof. Let $\mathcal{T}$ be the fractal defined in Claim 5.3.1, $F : \mathbb{R}^2 \to \mathbb{R}$ with $F(x,y) = (y,x)$. Note that

$$F(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

satisfies the conditions for Proposition 5.0.1 with $G(x,y) = F(x,y)$.

Note also that $F(\mathcal{T}) = F(\{(x,y) : x \in C_3, y \in C_3, y - x \in C_3\}) = \{(x,y) : x \in C_3, y \in C_3, y - x \in C_3\} = X_B$, for $B = \{(0,0),(2,0),(2,2)\}$.

So by Proposition 5.0.1, $\Lambda = \{(t,t) : t \in \mathbb{Z}/2\}$, which we proved in Proposition 5.1.25 is a spectrum for $\mathcal{T}$. Therefore, $G(\Lambda) = \{(t,t) : t \in \mathbb{Z}/2\}$ is a spectrum for $F(\mathcal{T})$. $\square$

5.3 The "Upper Right" fractal

Now, let:

$$F(x,y) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.14}$$

and $\mathcal{S}$ the fractal from Definition 4.1.1.

Geometrically, $F(x,y)$ flips $\mathbb{R}^2$ across the line $y = 1/2 - x$.

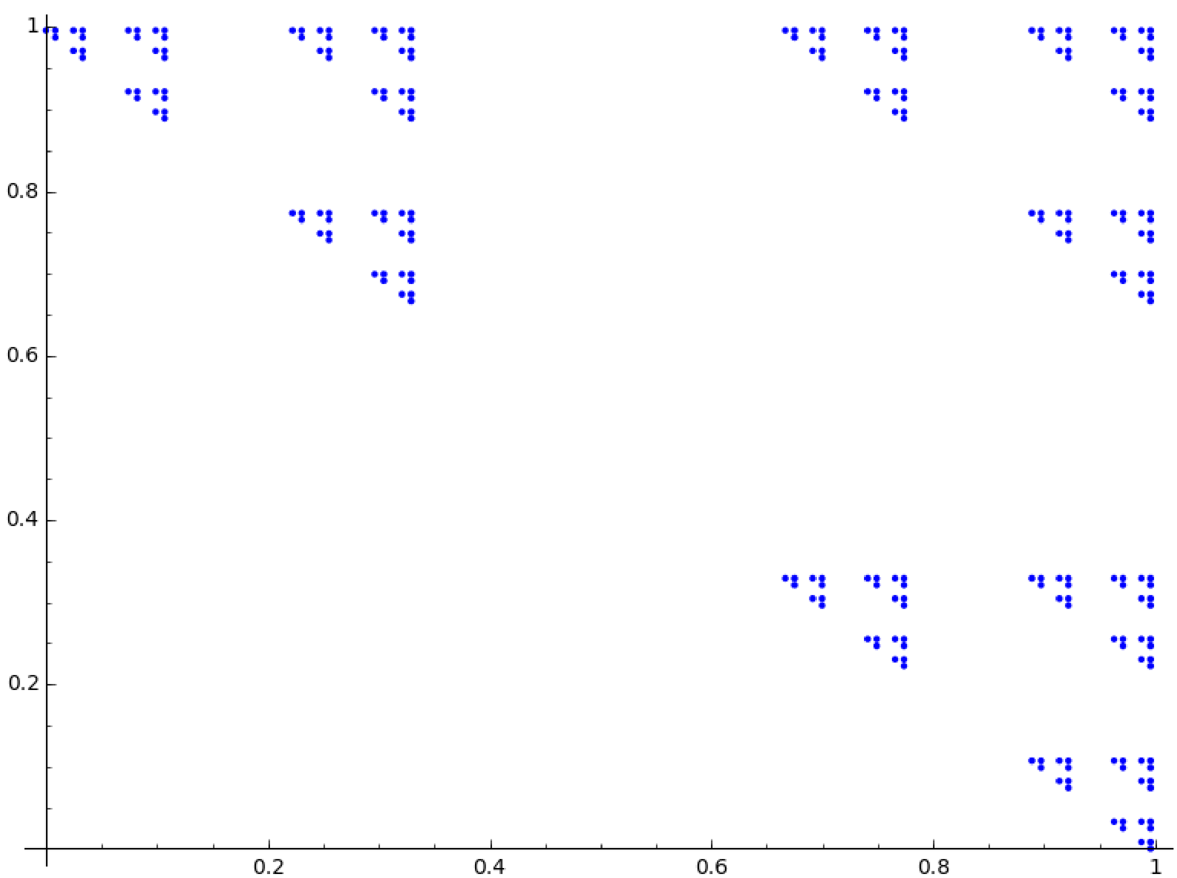

Figure 5.3 A fifth iterative approximation of $F(\mathcal{S})$ starting at zero.

$F(x, y)$ satisfies the conditions for Proposition 5.0.1, with

$$G(x,y) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = F(x,y). \tag{5.15}$$

By Proposition 5.0.1 and the work done on $\mathcal{S}$ in Chapter 4, we already know three spectra for $F(\mathcal{S})$, derived from the three spectra in Chapter 4.

We find another spectrum by working directly.

Claim 5.3.1. *$F(\mathcal{S})$ is the invariant set generated by the iterated function system $\{\frac{1}{3}(x, y)+B\}$, for $B = \{(2, 2), (0, 2), (2, 0)\}$.*

Proof. Use Lemma 5.0.2. By Definition 4.1.1, $\mathcal{S}$ is invariant under the iterated function system $\{\psi_0(x, y) = \frac{1}{3}(x, y), \psi_1(x, y) = \frac{1}{3}(x + 2, y), \psi_2(x, y) = \frac{1}{3}(x, y + 2)\}$.

Claim 5.3.2. $\frac{1}{3}((x, y) + (2, 2)) = F\psi_0 F^{-1}(x, y)$.

Proof.

$$\begin{aligned} F\psi_0F^{-1}(x,y) &= F\psi_0(-x+,-y+1) \\ &= F(\frac{1}{3}(-x+1,-y+1)) \\ &= F(-\frac{x}{3}+\frac{1}{3},-\frac{y}{3}+\frac{1}{3})) \\ &= (-\frac{x}{3}-\frac{1}{3}+1,\frac{y}{3}-\frac{1}{3}+1) \\ &= (\frac{x}{3}+\frac{2}{3},\frac{y}{3}+\frac{2}{3}) \\ &= \frac{1}{3}((x,y)+(2,2)) \end{aligned}$$

□

Claim 5.3.3. $\frac{1}{3}((x, y) + (0, 2)) = F\psi_1 F^{-1}(x, y)$.

Proof.

$$\begin{aligned}
F\psi_1 F^{-1}(x,y) &= F\psi_1(-x+1,-y+1)\\
&= F(\frac{1}{3}(-x+3,-y+1))\\
&= F(-\frac{x}{3}+1,-\frac{y}{3}+\frac{1}{3}))\\
&= (\frac{x}{3}-1+1,\frac{y}{3}-\frac{1}{3}+1)\\
&= (\frac{x}{3},\frac{y}{3}+\frac{2}{3})\\
&= \frac{1}{3}((x,y)+(0,2))
\end{aligned}$$

□

Claim 5.3.4. $\frac{1}{3}((x,y)+(2,0)) = F\psi_2 F^{-1}(x,y)$.

Proof.

$$\begin{aligned}
F\psi_2 F^{-1}(x,y) &= F\psi_2(-x+1,-y+1)\\
&= F(\frac{1}{3}(-x+1,-y+1+2))\\
&= F(-\frac{x}{3}+\frac{1}{3},-\frac{y}{3}+\frac{3}{3}))\\
&= (\frac{x}{3}-\frac{1}{3}+1,\frac{y}{3}-\frac{3}{3}+1)\\
&= (\frac{x}{3}+\frac{2}{3},\frac{y}{3})\\
&= \frac{1}{3}((x,y)+(2,0))
\end{aligned}$$

□

Since $\mathcal{S}$ is compact and F is continuous, $\mathcal{F}(S)$ is compact, thus it is the unique invariant set generated by this iterated function system.

□

Proposition 5.3.5. $F(\mathcal{S}) = \{(x,y)|x \in C_3, y \in C_3, 1-(x+y) \in C_3\}$*, where* C_3 *is the middle-thirds Cantor set discussed in Proposition 4.1.9.*

Proof. Let $(x,y) \in \mathcal{S}$. We know by Proposition 4.1.9 that $x \in C_3, y \in C_3$ and $x+y \in C_3$.

Let $F(x,y) = (s,t) = (1-x, 1-y)$. We want to show that $s \in C_3$, $t \in C_3$ and $t - s \in C_3$. We have that $s = 1 - x \in C_3$ and $t = 1 - y \in C_3$ because the Cantor set is symmetric across the point $1/2$ and $x, y \in C_3$. $1 - (t-s) = 1 - (1 - y + 1 - x) = 1 - (x+y) \in C_3$ because $x + y \in C_3$ and the Cantor set is symmetric across the point $1/2$.

$\square$

Next, we construct a basis for $F(\mathcal{S})$ using the method of Theorem 2.2.13.

First we compute:

$$W_B(x,y) = \frac{1}{N}|m_B(x,y)|^2 = \frac{1}{9}\left|e^{4\pi i(x+y)} + e^{4\pi ix} + e^{4\pi iy}\right|^2 \tag{5.16}$$

Lemma 5.3.6. *$W_B(x,y) = 1$ if and only if $(x,y) \in \mathbb{Z}/2 \times \mathbb{Z}/2$.*

Proof. $W_B(x,y) = 1$ if and only if $|e^{4\pi i(x+y)} + e^{4\pi ix} + e^{4\pi iy}|$, that is, if all three terms have the same argument, i.e.:

$2x = 2y = 2x+2y \mod 1$, which implies that $x = y = x+y \mod 1/2$. The third condition is redundant because: $x = x + y \mod 1/2 \Rightarrow 0 = y \mod 1/2$

$y = x + y \mod 1/2 \Rightarrow 0 = x \mod 1/2$

Therefore, it is sufficient to require that $(x,y) \in \mathbb{Z}/2 \times \mathbb{Z}/2$. $\square$

Next, we choose a set L to generate our orthonormal basis: let $L = \{(0,0),(2,1),(4,2)\}$

We verify that:

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ e^{2\pi i(4/3)} & e^{2\pi i(2/3)} & e^{2\pi i(6/3)} \\ e^{2\pi i(8/3)} & e^{2\pi i(4/3)} & e^{2\pi i(12/3)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ e^{2\pi i(1/3)} & e^{2\pi i(2/3)} & 1 \\ e^{2\pi i(2/3)} & e^{2\pi i(1/3)} & 1 \end{pmatrix}$$

is Hadamard. We already know from Proposition 4.2.2 that X_L, the invariant set of $\{\frac{1}{3}(x,y), \frac{1}{3}(x+2, y+1) + \frac{1}{3}(x+4, y+2)\}$ is $\{(2t,t) : t \in [0,1]\}$.

From Lemma 4.2.8, we know that $(2t,t) \in X_L$ is on a cycle if $t = \frac{1}{3^n-1}$ for some n; we also know that $W_B(2t,t) = 1$ if and only if $t \in \{0, 1/2, 1\}$. So our W_B-cycles are $\{(0,0),(1,1/2),(2,1)\}$, the same as they were in Section 4.2.

Therefore, to get our spectrum, we apply $\{3(x,y), 3(x,y)+(2,1), 3(x,y)+(4,2)$ to $\{(0,0),(-1,-1/2),(-2,-1)\}$, and get the same result as in Proposition 4.2.3.

Since the X_L has not changed, the proof of the transversality of the zeros condition in Section 4.2.1 also still holds.

Therefore:

Theorem 5.3.7. *$\{(2t, t) : t \in \mathbb{Z}/2\}$ is a spectrum for $L^2(\mu_B)$.*

We note that this is not the spectrum for $F(\mathcal{S})$ given by Proposition 5.0.1. That spectrum would be $\{G((2t, t)) : t \in \mathbb{Z}/2\}$, where $G(x, y)$ is the function from (5.15):

Corollary 5.3.8. *$\{(s, 2s - 1) : s \in \mathbb{Z}/2\}$ is a spectrum for $L^2(\mu_B)$.*

Proof. By Proposition 5.0.1, $\{G((2t, t)) : t \in \mathbb{Z}/2\}$ is a spectrum for $L^2(\mu_B)$. We calculate:

$$G((2t, t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \left(\begin{pmatrix} 2t \\ t \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 2t - 1 \\ t - 1 \end{pmatrix}$$
$$= (1 - 2t, 1 - t), t \in \mathbb{Z}/2$$

Rewritten in terms of $s = 1 - t$, this becomes: $\{(s, 2s - 1) : s \in \mathbb{Z}/2\}$ □

5.3.1 Another shared spectrum

After noticing that $\{(2t, t) : t \in \mathbb{Z}/2\}$ is a spectrum for both $L^2(\mathcal{S})$ and $L^2(F(\mathcal{S}))$, for F as in (5.14), we naturally wondered whether they might share other spectra, in particular, the spectrum $\mathcal{R}(0, 0) \cup \mathcal{R}(-1, -1/2) \cup \mathcal{R}(-1/2, -1)$ from Theorem 4.4.

Theorem 5.3.9. *$\Lambda = \mathcal{R}(0, 0) \cup \mathcal{R}(-1, -1/2) \cup \mathcal{R}(-1/2, -1)$ is a spectrum for $L^2(F(\mathcal{S}))$.*

Recall that the L which generates Λ is $\{(0, 0), (1, 2), (2, 1)\}$ and check that $(R^{-1}B, L)$ is a Hadamard pair for $\{(2, 2), (0, 2), (2, 0)\}$:

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ e^{2\pi i(6/3)} & e^{2\pi i(4/3)} & e^{2\pi i(2/3)} \\ e^{2\pi i(6/3)} & e^{2\pi i(2/3)} & e^{2\pi i(4/3)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ e^{2\pi i(1/3)} & e^{2\pi i(2/3)} & 1 \\ e^{2\pi i(2/3)} & e^{2\pi i(1/3)} & 1 \end{pmatrix}$$

We have already computed $W_B(x, y)$ in (5.16), and verified that $W_B(x, y) = 1$ if and only if $(x, y) \in \mathbb{Z}/2, \mathbb{Z}/2)$. Moreover, since the matrix $A = \frac{1}{3}I$ is the same for both $\mathcal{S}$ and $F(\mathcal{S})$, and the set L is the same, X_L is the same for both.

Therefore, the W_B-cycles will be the same as in (4.60): $\{(0,0),(1,1/2),(1/2,1)\}$; and W_B and X_L will continue to satisfy the transversality of the zeros condition in Definition 2.2.12.

Therefore, by Theorem 2.2.13, $\Lambda = \mathcal{R}(0,0) \cup \mathcal{R}(-1,-1/2) \cup \mathcal{R}(-1/2,-1)$ is a spectrum for $L^2(F(\mathcal{S}))$.

It is our hope in the future to use these shared spectra to generate a spectrum for $\mathcal{S} \cup F(S)$.

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