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ORDER RELATIONS AND BOOLEAN METHODS IN GENERAL SET THEORY

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I. INTRODUCTION

In this dissertation Boolean algebraic as well as transfinite arithmetical techniques are used in the study of order relations such as cofinality order in sets of ordinal numbers, lattice-theoretical order in Boolean rings and partial well orders in arbitrary sets. Moreover, these order relations are used in the study of questions pertaining to inaccessible and measurable cardinal numbers.

The general nature of this research is along the lines of current investigations concerning reduced products and ultrapowers. These structures have proved to be effective in the application of set-theoretical methods to various disciplines of mathematics. In particular, they have been successfully used in the application of algebraic methods to mathematical logic, i.e., in questions on the borderline of algebra and mathematical logic.

In Section II completeness properties of ultrafilters and prime ideals of Boolean rings are considered. These are used in determining the cardinality of the set of prime ideals of certain Boolean rings, i.e., the cardinality of the Stone space of these Boolean rings.

Isomorphisms of partially ordered sets are studied in Section III with particular emphasis on partially well ordered sets and on the cofinality order in sets of ordinal numbers. Based on the concept of dimension of partially ordered sets it is shown that every set of ordinal numbers partially ordered by cofinality has dimension two or less. It is also shown that a partially well ordered set is isomorphic to a set of ordinal numbers partially ordered by cofinality if and only if its dimension is not greater than two.

Normal functions are defined in Section IV and used to show that the existence of a regular cardinal number in the range of every normal function implies the axiom schema of strong infinity which asserts the existence of a strongly inaccessible cardinal number in the range of every normal function.

In Section V the methods of the previous sections are employed in the study of ultrapowers of partially ordered sets. Necessary and sufficient conditions are given to ensure that an ultrapower inherit certain properties of the initial partially ordered set. Finally, the method of ultrapowers is used in connection with the cardinality of the set of inaccessible cardinal numbers which precede a measurable cardinal number.

II. COMPLETENESS OF ULTRAFILTERS OF BOOLEAN RINGS

A. Terminology and Basic Properties

A Boolean ring is a commutative ring in which every element is idempotent. A Boolean algebra is a Boolean ring with unit. A filter F and an ideal I of a Boolean ring B [19, pp. 11-13] are subsets of B such that

(1) $xy \in F$ if and only if $x \in F$ and $y \in F$

and

(2) $x+y+xy \in I$ if and only if $x \in I$ and $y \in I$.

An ultrafilter is a nonvoid proper filter which is maximal and a prime ideal is a nonvoid proper ideal P such that for every element x and y of B if $xy \in P$ then $x \in P$ or $y \in P$. It follows that P is a prime ideal of B if and only if $B-P$ is an ultrafilter. A nonzero element a of B is an atom of B if and only if $ab = 0$ or $ab = a$ for every element b of B.

A Boolean ring is partially ordered [9, p. 21] by the relation $\leq$ where $a \leq b$ if and only if $ab = a$. Every reference to order in a Boolean ring will be to this partial order. The definitions of an $\bar{\bar{m}}$-complete (complete) Boolean algebra, an $\bar{\bar{m}}$-complete (complete) ideal of an $\bar{\bar{m}}$-complete (complete) Boolean algebra and an $\bar{\bar{m}}$-complete (complete) filter of an $\bar{\bar{m}}$-complete (complete) Boolean algebra are those of Sikorski [19, pp. 65, 74].

A subset S of a Boolean ring B is called orthogonal if and only if $ab = 0$ for every two distinct elements a and b of S. A Boolean ring B is $\bar{\bar{m}}$-orthogonally-complete if and only if $\sup S$ exists in B for every orthogonal subset S of B with $\bar{\bar{S}} \leq \bar{\bar{m}}$, and is orthogonally-complete if and only if it is $\bar{\bar{m}}$-orthogonally-complete for every cardinal $\bar{\bar{m}}$. An ideal is defined to be $\bar{\bar{m}}$-orthogonally-complete (orthogonally-complete) in an analogous manner.

If S and T are subsets of a Boolean ring B such that $\sup S$ and $\sup T$ exist in B then, clearly, $\sup\{\sup S, \sup T\}$ exists in B. Further, if u is an upper bound of $S \cup T$ then $\sup\{\sup S, \sup T\} \leq u$. On the other hand, if $x \in S \cup T$ then $x \leq \sup S$ or $x \leq \sup T$ so that $x \leq \sup\{\sup S, \sup T\}$. Consequently, we have

$$\sup\{\sup S, \sup T\} = \sup(S \cup T). \tag{3}$$

Let S be a subset of a subset A of a Boolean ring B. If $\sup S$ exists then we say that A preserves the supremum $\sup S$ if and only if $\sup S \in A$. Moreover, we say that A is <u>suprema-preserving</u> if and only if A preserves the supremum (in B) of everyone of its subsets. Furthermore, a prime ideal P of B is called <u>suprema-preserving</u> if and only if P is a suprema-preserving subset of B.

B. Preservation of Suprema and $\bar{\bar{m}}$-completeness

Included here are two lemmas concerning prime ideals and ultrafilters of Boolean rings. Lemma 1 will be used in the discussion of the cardinality of the set of prime ideals of Boolean rings in Section II.C. Lemma 2 will be used in Section IV in the discussion of the order modulo a filter. However, they are included at this time because they are general properties of Boolean rings.

Lemma 1. <u>If</u> P <u>is a prime ideal of a Boolean ring</u> B <u>then</u> P <u>is suprema-preserving if and only if there exists an atom</u> a <u>of</u> B <u>with</u> $a \notin P$.

Proof. Let a be an atom of B with $a \notin P$. Clearly, $P = \{b : b \in B \text{ and } ab = 0\}$. Let U be any subset of P and v an upper bound of U with $v \notin P$. Since $au = 0$ for every $u \in U$ we have $(v+a)u = u$ or $v+a \geq u$ for every $u \in U$. However, $v(v+a) = v+a$ so that $v+a < v$. Thus, if sup U exists then $\sup U \in P$. Hence, P is suprema-preserving.

Conversely, assume that $a \in P$ for every atom a of B. Let $v \in B-P$ and $U = \{vp : p \in P\}$. Clearly, v is an upper bound of U. We will show that $v = \sup U$ and, therefore, that P is not suprema-preserving. To do this we show that $wv = v$ for every upper bound w of U. Let w be any upper bound of U and let $q = v+wv$. If $p \in P$ then $qp = 0$ since $vp \in U$ and $wu = u$ for every $u \in U$. But if $t \notin P$ then $q+qt \in P$ so that $q(q+qt) = 0$. Hence, $q+qt = 0$ or $qt = q$. Thus, $qb = 0$ or $qb = q$ for every element b of B and, therefore, either q is an atom of B or $q = 0$. In either case, it follows that $q \in P$ so that $qq = 0$ or $q = 0$. Thus, $wv = v$ or $v = \sup U$. Hence, P is not suprema-preserving.

Corollary 1. _If_ P _is a prime ideal of a Boolean ring_ B _then for every element_ v _of_ $B-P$ _there exists a subset_ U _of_ P _with_ $v = \sup U$ _if and only if_ P _contains every atom of_ B.

Proof. If every atom of B is an element of P then for every element v of $B-P$ we have shown that $v = \sup U$ when $U = \{vp : p \in P\}$. On the other hand, if $B-P$ contains an atom of B then P is suprema-preserving. Thus, if $v \in B-P$ and $U \subset P$ then $v \neq \sup U$.

We now consider the concept of $\bar{\bar{m}}$-completeness and present five equivalent statements concerning ultrafilters and prime ideals of an $\bar{\bar{m}}$-complete Boolean algebra.

Lemma 2. <u>For every ultrafilter</u> U <u>in an $\bar{\bar{m}}$-complete Boolean algebra</u> B <u>the following statements are pairwise equivalent</u>.

(A). U <u>is $\bar{\bar{m}}$-complete</u>.

(B). $B-U$ <u>is $\bar{\bar{m}}$-complete</u>.

(C). $B-U$ <u>is $\bar{\bar{m}}$-orthogonally-complete</u>.

(D). <u>For every orthogonal subset</u> D <u>of</u> B <u>if</u> $\bar{\bar{D}} \leq \bar{\bar{m}}$ <u>and</u> $\sup D = 1$ <u>then</u> $D \cap U \neq \emptyset$.

(E). <u>For every subset</u> H <u>of</u> B <u>if</u> $\bar{\bar{H}} \leq \bar{\bar{m}}$ <u>and</u> $\sup H = 1$ <u>then</u> $H \cap U \neq \emptyset$.

Proof. For every subset A of B if $\inf A$ exists, then $\sup_{a \in A} 1+a = 1 + \inf A$. Hence, (A) and (B) are equivalent. Further, it is clear that (B) implies (C) and that (C) implies (D). We shall prove that (D) implies (E) and that (E) implies (B).

Assume (D) and let H be any subset of B with $\bar{\bar{H}} \leq \bar{\bar{m}}$ and $\sup H = 1$. Well order H, i.e., $H = \{h_i : i < u\}$ for some ordinal u. For every $i < u$ let

$$d_i = h_i + h_i(\sup_{j<i} d_j) \tag{4}$$

Then $D = \{d_i : i < u\}$ is an orthogonal subset of B with $\overline{\overline{D}} \leq \overline{\overline{m}}$. From Statement 4 for every $i < u$ we have

$$(5) \qquad d_i h_i = d_i .$$

But also from Statement 4 it follows that if $i < u$ then $h_i d_i = h_i + h_i(\sup_{j<i} d_j)$ or $h_i(d_i + \sup_{j<i} d_j) = h_i$. By Lemma 8 of [1], since D is orthogonal, this reduces to the equation $h_i(\sup_{j<i} d_j) = h_i$. Thus, $h_i(\sup D) = h_i$ for every $i < u$ or

$$(6) \qquad (\sup H)(\sup D) = \sup H.$$

Since $\sup H = 1$ Statement 6 reduces to $\sup D = 1$. By hypothesis, $d_i \in U$ for some $i < u$. Statement 1 on page 3 and Statement 5 imply that $h_i \in U$ or that $H \cap U \neq \emptyset$. Thus, (D) implies (E).

Finally, assume that (E) is valid. If $\overline{\overline{m}}$ is finite then (B) is valid since $B-U$ is an ideal. If $\overline{\overline{m}}$ is infinite, let A be any subset of $B-U$ with $\overline{\overline{A}} \leq \overline{\overline{m}}$ and let $H = A \cup \{1+\sup A\}$. From Statement 3 on page 4 we have that $\sup H = 1$. Thus, H is a subset of B with $\overline{\overline{H}} \leq \overline{\overline{m}}$ and $\sup H = 1$. By hypothesis $H \cap U \neq \emptyset$. Since

$A \cap U = \emptyset$ we have $1+\sup A \in U$ or $\sup A \in B-U$. Hence, $B-U$ is $\bar{\bar{m}}$-complete and (E) implies (B).

C. Cardinality of the Set of Prime Ideals of Boolean Rings

Throughout this discussion the following notations will be used. For every Boolean ring B,

$A(B) = \{a : a$ is an atom of $B\}$

$M(B) = \{P : P$ is a prime ideal of $B\}$.

The following theorem uses the result of Lemma 1 on page 5 to show that certain Boolean rings have uncountably many prime ideals.

Theorem 1. *If* B *is an infinite Boolean ring such that* $A(B) = \emptyset$ *then* $M(B)$ *is uncountable*.

Proof. Assume that $M(B)$ is countable. For every $P \in M(B)$ it follows from Lemma 1 on page 5 that there exist $v_P \in B-P$ and $V_P \subset P$ such that $v_P = \sup V_P$. Since $M(B)$ is countable, Statement (iv) of [17, p. 197] implies the existence of a prime ideal Q of B such that for every $P \in M(B)$ the ideal Q preserves the supremum $v_P = \sup V_P$. However, $Q \in M(B)$ so that $V_Q \subset Q$. Thus, $v_Q \in Q$ and $v_Q \in B-Q$ which is impossible. Therefore, $M(B)$ is uncountable.

Corollary 2. *If* B *is a nonatomic Boolean ring then* $M(B)$ *is uncountable*.

Proof. Since B is nonatomic there exists a nonzero element $c \in B$ such that $ca = 0$ for every atom $a \in B$. If $C = \{cb : b \in B\}$ then C is an infinite atomless Boolean ring. Thus, from Theorem 1 we have $\overline{\overline{M(C)}} > \aleph_0$. For every $Q \in M(C)$ let $P(Q) = \{b : b \in B$ and $bc \in Q\}$. But $Q \in M(C)$ implies $P(Q) \in M(B)$ and $Q \neq Q'$ implies $P(Q) \neq P(Q')$. Hence, $\overline{\overline{M(B)}} \geq \overline{\overline{M(C)}}$ or $M(B)$ is uncountable.

If C is a subring of a Boolean ring B and $b \in B$ then let

$$(7) \qquad C+b = \{ab+c : a,c \in C\} \cup \{b+ab+c : a,c \in C\}.$$

Clearly, $C+b$ is the subring of B generated by $C \cup \{b\}$ and $C+b = C$ if and only if $b \in C$. If C is a finite subring of B and $b \notin C$ then $\overline{\overline{C+b}} = \overline{\overline{C}}\cdot 2$ so that if C is finite then $\overline{\overline{M(C+b)}} = 1+\overline{\overline{M(C)}}$ for every $b \in B-C$.

Theorem 2. *If* C *is a subring of a Boolean ring* B *then* $\overline{\overline{M(C+b)}} = 1+\overline{\overline{M(C)}}$ *for every* $b \in B-C$.

Proof. In view of the statement above we need only consider the case when C is infinite. But then $M(C)$ is infinite so that $1+\overline{\overline{M(C)}} = \overline{\overline{M(C)}}$. Let Q be any prime ideal of $C+b$. If $C \subset Q$ then $ab+c \in Q$ for every element a and c of C. Hence,

(8) <u>if</u> $C \subset Q$ <u>then</u> $Q = \{ab+c : a,c \in C\}$.

If $C \not\subset Q$ then $C \cap Q \in M(C)$. But if $P \in M(C)$ and $C \cap Q = P$ and $b \in Q$ then $ab \in Q$ for every $a \in C$. Thus, for every element a and c of C it follows that $ab+c \in Q$ if and only if $c \in P$ and $b+ab+c \in Q$ if and only if $c \in P$. Hence,

(9) <u>if</u> $P \in M(C)$ <u>and</u> $C \cap Q = P$ <u>and</u> $b \in P$ <u>then</u>

$$Q = \{ab+c : a \in C \text{ and } c \in P\} \cup$$
$$\cup \{b+ab+c : a \in C \text{ and } c \in P\}.$$

But if $C \cap Q = P$ and $b \notin Q$ then for every $a \in C$ we have $ab \in Q$ if and only if $a \in P$. Thus, if $a \in C$ and $c \in C$ then $ab+c \in Q$ if and only if $a+c \in P$ and $b+ab+c \in Q$ if and only if $a+c \notin P$. Consequently,

(10) <u>if</u> $P \in M(C)$ <u>and</u> $C \cap Q = P$ <u>and</u> $b \notin P$ <u>then</u>

$$Q = \{ab+c : a,c \in C \text{ and } a+c \in P\} \cup$$
$$\cup \{b+ab+c : a,c \in C \text{ and } a+c \notin P\}.$$

From Statements 7, 8, 9 and 10 it follows that $\overline{\overline{M(C+b)}} \leq 1+\overline{\overline{M(C)}}+\overline{\overline{M(C)}}$ for every $b \in B-C$. Hence, $\overline{\overline{M(C+b)}} \leq \overline{\overline{M(C)}}$. On the other hand, if $P \in M(C)$ then $P' = \{ab+c : a,c \in P\}$ is an ideal of $C+b$. If $c \in C-P$ then $c \notin P'$ so there exists a prime ideal Q of $C+b$ such that $P' \subset Q$ and $c \notin Q$. Since $P \subset Q$ and $C \not\subset Q$ it follows that $C \cap Q = P$. Hence, $\overline{\overline{M(C)}} \leq \overline{\overline{M(C+b)}}$. Consequently, if C is infinite then $\overline{\overline{M(C+b)}} = \overline{\overline{M(C)}}$. Thus, for every subring C of B and every element b of $B-C$ we have $\overline{\overline{M(C+b)}} = 1+\overline{\overline{M(C)}}$.

For every ordinal u let $X(u)$ denote the set of all functions from ω_u into 2 and let

$$Y(u) = \{x : x \in X(u) \text{ and there exists an ordinal } v < \omega_u \text{ such that for every ordinal } w \text{ if } v \leq w < \omega_u \text{ then } x(w) = 0\}.$$

For every element x and x' of $X(u)$ define $x \leq x'$ if and only if $x = x'$ or there exists an ordinal $v < \omega_u$ such that $x(v) < x'(v)$ while $x(w) = x'(w)$ for every ordinal $w < v$. Clearly, $(X(u), \leq)$ is simply ordered. Further, if $x < x'$ let v be the first ordinal such that $x(v) < x'(v)$. If

$$y(t) = \begin{cases} x(t), & \text{for} \quad t < v \\ 1, & \text{for} \quad t = v \\ 0, & \text{for} \quad t > v \end{cases}$$

then $y \varepsilon Y(u)$ and $x < y \leq x'$. Thus, for every element x and x' of $X(u)$

(11) <u>if</u> $x < x'$ <u>then there exists an element</u> y <u>of</u> $Y(u)$ <u>such that</u> $x < y \leq x'$.

Moreover, let $x < x'$ and let v be the first ordinal such that $x(v) < x'(v)$ and assume that $x(w) = 0$ for some ordinal $w > v$. If

$$y(t) = \begin{cases} x(t), & \text{for} \quad t < w \\ 1, & \text{for} \quad t = w \\ 0, & \text{for} \quad t > w \end{cases}$$

then $y \varepsilon Y(u)$ and $x < y < x'$. Clearly, for every element y and y' of $Y(u)$

(12) <u>if</u> $y < y'$ <u>then there exists an element</u> y'' <u>of</u> $Y(u)$ <u>such that</u> $y < y'' < y'$.

We note finally that $\overline{\overline{X(u)}} = 2^{\aleph_u}$ and

$$(13) \qquad \aleph_u \leq \overline{\overline{Y(u)}} = \sum_{v<\omega_u} 2^{\overline{\overline{v}}} = \sup_{v<\omega_u} 2^{\overline{\overline{v}}} \leq 2^{\aleph_u}.$$

Lemma 3. For every ordinal u there exists an atomless Boolean ring $B(u)$ such that $\overline{\overline{B(u)}} = \overline{\overline{Y(u)}}$ and $\overline{\overline{M(B(u))}} \geq 2^{\aleph_u}$.

Proof. For every element x and x' of $X(u)$ let $[x,x') = \{z : z \in X(u)$ and $x \leq z < x'\}$. Let $A(u) = \{[y,y') : y,y' \in Y(u)\}$ and let $B(u)$ be the Boolean ring of subsets of $X(u)$ generated by $A(u)$ under the operations of set-theoretical intersection and symmetric difference. Clearly, $B(u)$ is the set of all finite unions of elements of $A(u)$ so that $\overline{\overline{B(u)}} = \overline{\overline{A(u)}} = \overline{\overline{Y(u)}}$. From Statement 12 and the definition of $B(u)$ it follows that $B(u)$ is atomless.

For every element x of $X(u)$ let $P(x) = \{b : b \in B(u)$ and $x \notin b\}$. Clearly, $P(x)$ is a prime ideal of $B(u)$ since $b \in P(x)$ and $c \in P(x)$ imply $b \cup c \in P(x)$ and for every element b and c of $B(u)$ we have $b \cap c \in P(x)$ if and only if $b \in P(x)$ or $c \in P(x)$. On the other hand, if $x \neq x'$ we may assume that $x < x'$.

Let x'' be the element of $X(u)$ such that $x''(v) = 1$ for every $v < \omega_u$. From Statement 11 it follows that there exist elements y and y' of $Y(u)$ such that $x < y \leq x' < y' \leq x''$. Clearly, $[y,y') \in P(x)$ but $[y,y') \notin P(x')$. Thus, $x \neq x'$ implies $P(x) \neq P(x')$. Consequently, $\overline{\overline{M(B(u))}} \geq \overline{\overline{X(u)}}$ or $\overline{\overline{M(B(u))}} \geq 2^{\aleph_u}$. Thus, the lemma is proved.

Let us observe that Statement 13 implies

(14) $$\overline{\overline{Y(u)}} = \aleph_u \ \underline{\text{if and only if}} \ v < \omega_u \ \underline{\text{implies}} \ 2^{\overline{\overline{v}}} \leq \aleph_u.$$

A cardinal $\overline{\overline{c}}$ is called <u>e-inaccessible</u> [2, p. 101] if and only if for every cardinal $\overline{\overline{m}}$ and $\overline{\overline{n}}$

$$\overline{\overline{m}} < \overline{\overline{c}} \ \underline{\text{and}} \ \overline{\overline{n}} < \overline{\overline{c}} \ \underline{\text{imply}} \ \overline{\overline{m}}^{\overline{\overline{n}}} < \overline{\overline{c}}.$$

From Theorem 2 of [2, p. 101] it follows that every infinite cardinal $\overline{\overline{c}}$ is e-inaccessible if and only if for every cardinal $\overline{\overline{n}}$

(15) $$\overline{\overline{n}} < c \ \underline{\text{implies}} \ 2^{\overline{\overline{n}}} < c.$$

Corollary 3. <u>For every ordinal</u> u <u>if</u> $\aleph_u$ <u>is e-inaccessible then there exists an atomless Boolean ring</u> $B(u)$ <u>such that</u> $\overline{\overline{B(u)}} = \aleph_u$ <u>and</u> $\overline{\overline{M(B(u))}} = 2^{\aleph_u}$.

Proof. If $v < \aleph_u$ then $\overline{\overline{v}} < \aleph_u$ so that $2^{\overline{\overline{v}}} < \aleph_u$ by Statement 15. Thus, Statement 14 implies that $\overline{\overline{Y(u)}} = \aleph_u$. By Lemma 3 there exists an atomless Boolean ring $B(u)$ such that $\overline{\overline{B(u)}} = \aleph_u$ and $\overline{\overline{M(B(u))}} \geq 2^{\aleph_u}$. Since $\overline{\overline{M(B)}} \leq 2^{\overline{\overline{B}}}$ for every Boolean ring B, the desired result follows.

In particular, since $\aleph_o$ is e-inaccessible, we have

Corollary 4. <u>There exists a denumerable atomless Boolean ring</u> B <u>such that</u> $\overline{\overline{M(B)}} = 2^{\aleph_o}$.

If the Generalized Continuum Hypothesis is assumed then for every ordinal u and v if $v < \omega_u$ then $2^{\overline{\overline{v}}} \leq \aleph_u$. Hence, Statement 14 implies that $\overline{\overline{Y(u)}} = \aleph_u$ for every ordinal u. Thus, we have

Corollary 5. <u>The Generalized Continuum Hypothesis implies that for every ordinal</u> u <u>there exists an atomless Boolean ring</u> $B(u)$ <u>such that</u> $\overline{\overline{B(u)}} = \aleph_u$ <u>and</u> $\overline{\overline{M(B(u))}} = 2^{\aleph_u}$.

We now use the above results to investigate the set $M(B)$ when B is an atomic and complete Boolean ring.

Theorem 3. <u>If</u> B <u>is an atomic and complete Boolean ring such that</u> $\overline{\overline{A(B)}} = \overline{\overline{Y(u)}}$ <u>then</u> $\overline{\overline{M(B)}} \geq 2^{\aleph_u}$.

Proof. Let $B(u)$ be the Boolean ring defined in the proof of Lemma 3 on page 14 and let $C(u) = \{b \cap Y(u) : b \in B(u)\}$. Clearly, $C(u)$ is a Boolean ring and the map $h(b) = b \cap Y(u)$ is an isomorphism of $B(u)$ onto $C(u)$. On the other hand, B is isomorphic to the Boolean ring of all subsets of $Y(u)$. Consequently, there is a subring C of B such that C is isomorphic to $C(u)$, and therefore, to $B(u)$. If $P \in M(C)$ then $0 \in P$ and $C-P$ is a multiplicative system of B. Hence, there exists $P' \in M(B)$ with $P' \cap C = P$. Thus, $\overline{\overline{M(C)}} \geq 2^{\aleph_u}$ implies $\overline{\overline{M(B)}} \geq 2^{\aleph_u}$.

Corollary 6. <u>For every e-inaccessible cardinal</u> $\aleph_u$ <u>and every atomic and complete Boolean ring</u> B <u>if</u> $\overline{\overline{A(B)}} = \aleph_u$ <u>then</u> $\overline{\overline{M(B)}} \geq \overline{\overline{B}}$.

Proof. This corollary follows from the proofs of Corollary 3 and Theorem 3 together with the observation that $\overline{\overline{B}} = 2^{\aleph_u}$.

In particular, we have

Corollary 7. <u>If</u> B <u>is an atomic and complete Boolean ring and</u> $A(B)$ <u>is denumerable then</u> $\overline{\overline{M(B)}} \geq \overline{\overline{B}}$.

From Corollary 5 and Theorem 3 we have

Corollary 8. *The Generalized Continuum Hypothesis implies that* $\overline{\overline{M(B)}} \geq \overline{\overline{B}}$ *for every infinite atomic and complete Boolean ring* B.

Finally, from Corollary 7 we conclude

Corollary 9. *If* B *is an infinite atomic and complete Boolean ring then* $\overline{\overline{M(B)}} \geq 2^{\aleph_0}$.

Proof. Let A be any denumerable subset of A(B). The subring C of B generated by sup A is an atomic and complete Boolean ring with denumerably many atoms. Thus, $\overline{\overline{M(C)}} \geq C$ by Corollary 7. Since $\overline{\overline{M(B)}} \geq \overline{\overline{M(C)}}$ and $\overline{\overline{C}} = 2^{\aleph_0}$ we have $\overline{\overline{M(B)}} \geq 2^{\aleph_0}$.

III. ISOMORPHISMS OF PARTIALLY ORDERED SETS

A. Isomorphisms of Partially Well Ordered Sets

In [4] it is shown that for every finite partially ordered set P and Q if f is a function from P onto Q and g is a function from Q onto P such that f and g are one-to-one and order-preserving then P and Q are isomorphic. This result is extended below to a larger class of partially ordered sets.

If $(P,\geq)$ is a partially ordered set and X is a subset of P then X is <u>totally unordered</u> if and only if for every element x and y of X both $x \not\geq y$ and $y \not\geq x$.

A partially ordered set $(P,\geq)$ is <u>partially well ordered</u> [21] (<u>fairly well ordered</u> [15], <u>well-partially-ordered</u> [12]) if and only if every strictly decreasing sequence in $(P,\geq)$ and every totally unordered subset of P is finite.

Motivated by the "canonical" decomposition of partially well ordered sets [21, p. 179], if $(P,\geq)$ is a partially ordered set with no infinite strictly decreasing sequence, then for every ordinal u we define

$$(16) \qquad P_u = \{x : x \text{ is minimal in } P - \cup\{P_v : v < u\}\}$$

where if $(Y,\geq)$ is a partially ordered set then y is <u>minimal</u> in Y if and only if $x \not< y$ for every element x of Y. Clearly, $P = \bigcup \{P_u : u < w\}$ for some ordinal w so that if $v \geq w$ then $P_v = \emptyset$. If $P_v = \emptyset$ for some ordinal v then $\{P_u : u < v\}$ is called a <u>minimal decomposition</u> of P.

For every ordinal u and v

(17) <u>if</u> $u < v$ <u>and</u> $x \in P_v$ <u>then there exists</u> y <u>such that</u> $y \in P_u$ <u>and</u> $y < x$.

Further, for every ordinal u the set P_u is totally unordered. Hence, if P is partially well ordered, then P_u is finite for every ordinal u.

Theorem 4. <u>Let</u> $(P,\geq)$ <u>and</u> $(Q,\geq')$ <u>be partially well ordered sets with minimal decompositions</u> $\{P_u : u < v\}$ <u>and</u> $\{Q_u : u < v\}$. <u>Let</u> f <u>be a function from</u> P <u>onto</u> Q <u>and</u> g <u>a function from</u> Q <u>onto</u> P <u>such that</u> f <u>and</u> g <u>are one-to-one and order-preserving</u>. <u>Then</u> P <u>and</u> Q <u>are isomorphic</u>.

Proof. If $x < y$ then $f(x) <' f(y)$. Thus, from Statement 17, if $f(y) \in Q_0$ then $y \in P_0$. Hence, $Q_0 \subset f[P_0]$. But similarly $P_0 \subset g[Q_0]$. Since P_0 and Q_0 are finite it follows that $f[P_0] = Q_0$ and $g[Q_0] = P_0$.

Let u be a nonzero ordinal and assume that $f[P_w] = Q_w$ and $g[Q_w] = P_w$ for every ordinal $w < u$. If $f(y) \varepsilon Q_u$ and $x < y$ then $f(x) <' f(y)$ so that $f(x) \varepsilon Q_w$ for some $w < u$. Thus, $x \varepsilon P_w$ and it follows from Statement 16 that $y \varepsilon P_u$. Hence, $Q_u \subset f[P_u]$. Similarly, $P_u \subset g[Q_u]$. Since P_u and Q_u are finite we have that $f[P_u] = Q_u$ and $g[Q_u] = P_u$. Thus, $f[P_u] = Q_u$ and $g[Q_u] = P_u$ for every ordinal u.

Let $h = g \circ f$. Then h is a one-to-one order-preserving function from P onto P with $h[P_u] = P_u$ for every ordinal u. Let $h(x) \leq h(y)$. Then for some ordinal u and w we have $h(x) \varepsilon P_u$ and $h(y) \varepsilon P_w$. Since $P_u \cup P_w$ is finite it follows from [4] that $x \leq y$.

Now if $f(x) \leq' f(y)$ then $g(f(x)) \leq g(f(y))$ or $h(x) \leq h(y)$ so that $x \leq y$. Thus, f is an isomorphism from P onto Q so that P and Q are isomorphic.

From the following example it is clear that the requirement in Theorem 4 that f and g be "onto" functions cannot be relaxed to allow even one of them to be "into".

Example 1. Let $P = \omega$ and $Q = P \cup \{a\}$. Let $\geq$ be the usual order on ω and let $\geq'$ be $\geq \cup \{(a,0),(a,a)\}$. Then $(P,\geq)$ and $(Q,\geq')$ are partially well ordered sets. Let f be the insertion map of P into Q and define g from Q into P by $g(0) = 0$ and $g(a) = 1$ and if $p \neq 0$ and

$p \neq a$ then $g(p) = p+1$. Clearly f and g are one-to-one and order-preserving. However, P and Q are not isomorphic.

The next example, suggested by J. Bainbridge (Ames, Iowa, Department of Mathematics, Iowa State University, 1969, personal communication) demonstrates that in Theorem 4 the requirement that P and Q have no infinite totally unordered subset cannot be dropped.

Example 2. Let
$R = \{(m,0) : m \in \omega\} \cup \{(2k,n) : 2 < k < \omega$ and $n < k\}$ and let $S = R \cup \{(4,1)\}$. Define $(m,n) \geq (s,t)$ if and only if $m = s$ and $n \geq t$. Then $(R,\geq)$ and $(S,\geq)$ are partially ordered sets with no infinite strictly decreasing sequence. Let f be the function from R onto S such that

$$f((1,0)) = (4,1)$$
$$f((2k,m)) = (2k,m) \quad \text{for} \quad k < \omega$$
$$f((2k+3,0)) = (2k+1,0) \quad \text{for} \quad k < \omega$$

and let g be the function from S onto R such that

$g((k,0)) = (k,0)$ for $k < 4$

$g((2k,m)) = (2k+2,m+1)$ for $1 < k < \omega$

$g((4k+1,0)) = (2k+2,0)$ for $0 < k < \omega$.

$g((4k+3,0)) = (2k+3,0)$ for $0 < k < \omega$.

Clearly, f and g are one-to-one and order-preserving functions. However, R and S are not isomorphic because R has no maximal chain of two elements.

Moreover, we observe that in Theorem 4 on page 20 the requirement that P and Q must have no infinite strictly decreasing sequence cannot be dropped.

Example 3. Let Z^+ be the set of positive integers and Z^- the set of negative integers. Let $X = (Z^+ \times 3) \cup (Z^- \times 1)$ and let $Y = X \cup \{(0,0),(0,1)\}$. Define $(z,n) \geq (w,m)$ if and only if $z = w$ and $n = m$ or $z > w$. Then $(X,\geq)$ and $(Y,\geq)$ are partially ordered sets with no infinite totally unordered subset. Let f be the function from X onto Y such that $f((1,2)) = (-1,0)$ and $f((z,n)) = (z-1,n)$ otherwise. Let g be the function from Y onto X such that $g((0,1)) = (-1,0)$, if $z > 0$ then $g((z,n)) = (z,n)$ and if $z \leq 0$ then $g((z,0)) = (z-2,0)$. Clearly, f and g are one-to-one and order-preserving but X and Y are not isomorphic.

B. Cofinality of Ordinal Numbers

We consider now a special partial order on any set of ordinals. If W is a set of ordinals then $\geq$ will denote the usual order on W, where, as expected, for every element u and v of W we write $v \geq u$ if and only if $u \in v$ or $u = v$. The notations $\leq$, $>$, $\not>$, etc. will have their obvious meanings in terms of the order $\geq$.

Motivated by [11, p. 232], [13, pp. 235-236] and [3], we make the following definition:

Definition 1. <u>For every ordinal</u> u <u>and</u> v, <u>the ordinal</u> u <u>is called cofinal with</u> v (<u>denoted by</u> $u \,\mathfrak{g}\, v$) <u>if and only if there exists a function</u> f <u>from</u> v <u>into</u> u <u>such that</u>

(18) <u>for every ordinal</u> x <u>and</u> y <u>if</u> $x < y < v$ <u>then</u> $f(x) < f(y)$

<u>and</u>

(19) <u>for every ordinal</u> y, <u>if</u> $y < u$ <u>then there exists an ordinal</u> x <u>with</u> $x < v$ <u>and</u> $f(x) \geq y$.

A function f from v into u which satisfies the conditions of Statements 18 and 19 is called a cofinality map from v into u. Thus, $u \mathrel{\vartheta} v$ if and only if there exists a cofinality map from v into u.

Lemma 4. <u>For every set</u> W <u>of ordinals,</u> $(W, \mathrel{\vartheta})$ <u>is a partially ordered set</u>.

Proof. For every ordinal u, the identity map f on u satisfies Statements 18 and 19 of Definition 1. Thus, $\mathrel{\vartheta}$ is reflexive. Antisymmetry of $\mathrel{\vartheta}$ follows from the anti-symmetry of $\geq$ since $u \mathrel{\vartheta} v$ implies $u \geq v$. Finally, if $u \mathrel{\vartheta} v$ with cofinality map f and $v \mathrel{\vartheta} w$ with cofinality map g, then $g \circ f$ is clearly a cofinality map from w into u, so $u \mathrel{\vartheta} w$. Hence, $(W, \mathrel{\vartheta})$ is a partially ordered set.

We next prove two lemmas which will allow us to find a necessary and sufficient condition for any two ordinals to be comparable with respect to cofinality.

Lemma 5. <u>For every ordinal</u> w, b <u>and</u> r, <u>if</u> $w \mathrel{\vartheta} b$ <u>and</u> $\omega^r \mathrel{\vartheta} b$ <u>and</u> $\omega^r \geq w$, <u>then</u> $\omega^r \mathrel{\vartheta} w$.

Proof. Let f be a cofinality map from b into w and g a cofinality map from b into ω^r. For every ordinal x, if $x < w$ let $y(x)$ be the first ordinal (with respect to $\geq$) such that $f(y(x)) \geq x$. Let $h(x) = g(y(x)) + x$. We claim that h is a cofinality map from w into ω^r.

First, if $x < w$ then $x < \omega^r$ and $y(x) < b$. Thus, $g(y(x)) < \omega^r$ so that $g(y(x)) + x < \omega^r$ or $h(x) \varepsilon \omega^r$. Hence, h is a map from w into ω^r. Further, if $u < v < w$, then $y(u) \leq y(v)$ so that $g(y(u)) \leq g(y(v))$. Consequently, $h(u) < h(v)$ and, therefore, h satisfies Statement 18. Finally, if $u < \omega^r$ let $b(u)$ be the first ordinal such that $g(b(u)) \geq u$. Clearly $y(f(b(u))) = b(u)$ so that $h(f(b(u))) = g(b(u)) + f(b(u)) \geq v + f(b(u)) \geq v$. Hence, h satisfies Statement 19 on page 24. Thus, $\omega^r \, 9 \, w$, as desired.

Lemma 6. *For every ordinal* t, w *and* r, *if* $t \, 9 \, \omega^r$ *and* $t \, 9 \, w$ *and* $\omega^r \geq w$, *then* $\omega^r \, 9 \, w$.

Proof. Let f and g, respectively, be cofinality maps from w and ω^r into t. For every ordinal $u < w$ let $y(u)$ be the first ordinal such that $g(y(u)) \geq f(u)$ and let $h(u) = y(u) + u$.

If $u < w$ then $u < \omega^r$ and $y(u) < \omega^r$ so that $h(u) < \omega^r$. Thus, h is a map from w into ω^r. If $u < v < w$, then $y(u) \leq y(v)$ so that $h(u) \leq h(v)$ and h satisfies Statement 18 on page 24. If $u < \omega^r$, let $z(u)$ be the first ordinal such that $f(z(u)) \geq g(u)$. Then $g(y(z(u))) \geq f(z(u)) \geq g(u)$ so that $y(z(u)) \geq u$. Therefore, $h(z(u)) = y(z(u)) + z(u) \geq u$. Consequently, h satisfies Statement 19 on page 24 and $\omega^r \, 9 \, w$, as desired.

We recall, [18, p. 323], that every nonzero ordinal u has a unique normal expansion in the form

$$u = \omega^{e_1}k_1 + \ . \ . \ . + \omega^{e_n}k_n$$

where $e_1 > e_2 > \ . \ . \ . > e_n \geq 0$ and $0 < k_i < \omega$ for $1 \leq i \leq n < \omega$. Hence, every ordinal u can be written in the form $u = w + \omega^v$ where $\omega^v k$ is the last term of the normal expansion of u.

Clearly,

(20) <u>if</u> $u \mathrel{9} v$ <u>then</u> $u = 0$ <u>if and only if</u> $v = 0$.

The following lemma assumes that u and v are nonzero.

Lemma 7. <u>For every ordinal</u> u <u>and</u> v, <u>if</u> $u = a + \omega^b$ <u>and</u> $v = c + \omega^d$, <u>then</u> $u \mathrel{9} v$ <u>if and only if</u> $u \geq v$ <u>and</u> $\omega^b \mathrel{9} \omega^d$.

Proof. Clearly, if $u \mathrel{9} v$ then $u \geq v$. Let f be a cofinality map from v into u. Let w be the first ordinal such that $f(c+w) \geq a$ and let $e = c + w$. Since $e < v$ we have $w < \omega^d$. Thus, $w + \omega^d = \omega^d$ or $e + \omega^d = c + \omega^d = v$. For every $x < \omega^d$ let $h(x)$ be the ordinal for which $f(e+x) = a + h(x)$. Clearly, if $x < \omega^d$

then $f(e+x) < a + \omega^b$ or $a + h(x) < a + \omega^b$ so that $h(x) < \omega^b$. Furthermore, $x < y < \omega^d$ implies that $h(x) < h(y)$. Finally, if $z < \omega^b$ then there exists $x < \omega^d$ with $f(e+x) \geq a + z$ or $a + h(x) \geq a + z$. But then $h(x) \geq z$. Hence, h is a cofinality map from ω^d into ω^b so that $\omega^b \mathrel{\varrho} \omega^d$.

Conversely, if $u = a + \omega^b \geq c + \omega^d = v$ and $\omega^b \mathrel{\varrho} \omega^d$, let g be a cofinality map from ω^d into ω^b. Since $c < c + \omega^d \leq a + \omega^b$ it follows that $c + \omega^b \leq a + \omega^b$. For every $x < c$ let $f(x) = x$ and for every $x < \omega^d$ let $f(c+x) = \max\{a,c\} + g(x)$. Clearly, f is a cofinality map from v into u. Hence, the lemma is proved.

As usual, for every ordinal u, the least ordinal c such that $u \mathrel{\varrho} c$ is called the <u>cofinality index</u> of u and is denoted by $\mathrm{cf}(u)$. Moreover, for every ordinal u and v

(21) $\qquad u \mathrel{\varrho} v$ <u>implies</u> $\mathrm{cf}(u) = \mathrm{cf}(v)$

Remark 1. In the following, order related terms which are not prefixed will refer to the usual order among ordinals while the same terms with the prefix C- will refer to the cofinality partial order among ordinals.

The following theorem relates several basic properties of the cofinality partial order among ordinals.

Theorem 5. *For every set* S *of ordinals, the following statements are pairwise equivalent.*

(A). *There exists an ordinal* c *such that* $cf(s) = c$ *for every* $s \in S$.

(B). S *is C-bounded below.*

(C). S *is C-bounded above.*

(D). *Every two-element subset of* S *is C-bounded above.*

(E). *Every two-element subset of* S *is C-bounded below.*

Proof. It is clear that (A) implies (B) and that (C) implies (D). We will show that (B) implies (C), (D) implies (E) and (E) implies (A).

If $\overline{\overline{S}} \leq 1$ then each statement is true. Thus, we assume that $\overline{\overline{S}} > 1$. In particular, there exists $s \in S$ with $s \neq 0$.

Assume (B) and let b be any C-lower bound of S. Since $s \neq 0$ for some $s \in S$ it follows that $b \neq 0$. But if b is a nonlimit ordinal then s is nonlimit for every $s \in S$ so that $\cup S + 1$ is a C-upper bound of S. If b is a limit ordinal and $t = \omega^{\cup S+b}$ then $t \geq s$ for every $s \in S$ and the function f defined on b by $f(x) = \omega^{\cup S+x}$ is a cofinality map from b into t. Thus, for every

$s \in S$ Lemma 6 on page 26 implies that $t \wp s$. Hence, (B) implies (C).

Next, let r and s be elements of S and let t be a C-upper bound of $\{r,s\}$. From Statement 21 it follows that $cf(t) = cf(r) = cf(s)$ so that $cf(t)$ is a C-lower bound of $\{r,s\}$. Hence, (D) implies (E).

Finally, let r be a fixed element of S and let $c = cf(r)$. For every element s of S let s' be a C-lower bound of $\{r,s\}$. From Statement 21 we have $c = cf(r) = cf(s) = cf(s')$ for every $s \in S$. Hence, $cf(s) = c$ for every $s \in S$ and (E) implies (A). Thus, the theorem is proved.

The following examples show that for a set of ordinals which is C-bounded there need not exist an ordinal which is C-supremum or C-infimum of the set.

Example 4. The set $R = \{\omega^2 + \omega k : k < u\}$ where $1 < u \leq \omega$, has a C-supremum but no C-infimum. Clearly, if $w = a + \omega^b$ and $w \wp r$ for every $r \in R$ then $w \wp \omega^2$. Also, $w \geq \omega^2 + \omega k$ for every $k < u$. Hence, $w \geq \omega^2 2$. But then $w \wp \omega^2 2$ so that $\omega^2 2 = \text{C-sup } S$. On the other hand, ωk is a C-lower bound of S for $0 < k < \omega$. However, ω^2 is not a C-lower bound of S so S has no C-infimum.

Example 5. Let $S = \{\omega k : 0 < k < \omega\}$. Then $\omega = \text{C-inf } S$ while in view of Lemma 7 on page 27 the ordinals ω^2 and $\omega^2 + \omega$ are minimal C-upper bounds of S. Hence, S has a C-infimum but no C-supremum.

Example 6. The set $T = \{\omega^2\} \cup \{\omega^2 k + \omega k : 0 < k < \omega\}$ has neither a C-infimum nor a C-supremum. Clearly, s is a C-lower bound of T if and only if s is an element of the set S of Example 5. On the other hand, ω^3 and $\omega^3 + \omega^2$ are minimal C-upper bounds of S.

However, as the following theorem shows, for every finite set of ordinals which is C-bounded there exists an ordinal which is the C-supremum of the set.

Theorem 6. For every finite set S of ordinals which is C-bounded there exists a C-supremum of S.

Proof. Let $S = \{s_0, s_1, \ldots, s_{n-1}\}$ be a finite C-bounded set of ordinals where, without loss of generality, $s_i = a_i + \omega^{b_i}$ for every $i < n$. Let $a = \max\{a_0, \ldots, a_{n-1}\}$, and $b = \max\{b_0, \ldots, b_{n-1}\}$. We claim that $t = a + \omega^b$ is the C-supremum of S. Clearly, for every $s \in S$ we have $t \geq s$. From Theorem 5 on page 29 there exists an ordinal c with $cf(s) = c$ for every $s \in S$. But $b = b_i$ for some $i < n$ and $t \geq s_i$ so that $t \mathbin{\mathcal{9}} s_i$ by Lemma 7 on page 27. Thus, Statement 21 on page 28 implies that $cf(t) = c$. Also, from Lemma 7 it follows that $u + \omega^v \mathrel{9} \omega^v$ for every ordinal u and v so that $c = cf(\omega^b) = cf(\omega^{b_i})$ for every $i < n$ by Statement 21. Thus, from Lemma 5 on page 25 we have $\omega^b \mathrel{9} \omega^{b_i}$ for every $i < n$. Consequently, Lemma 7 on page 27 implies that t is a C-upper bound of S. Now let $u = w + \omega^v$ be any C-upper bound of S. Then $\omega^v \mathrel{9} \omega^b$ and $a_i < s_i \leq u$ for every $i < n$ so that $a < u$ and $\omega^b \leq \omega^v$. Thus, $a + \omega^b \leq a + \omega^v \leq w + \omega^v$ or $t \leq u$. Consequently, $u \mathrel{9} t$ by Lemma 7 and we have proved that $t = \text{C-sup } S$.

C. Cofinality and Partially Well Ordered Sets

Below we show that every C-bounded set of ordinals is C-partially well ordered and then characterize the partially ordered sets which are isomorphic to sets of ordinals partially ordered by cofinality.

Theorem 7. <u>Every C-bounded set</u> W <u>of ordinals is a C-partially well ordered set</u>.

Proof. If $u \mathrel{9} v$ then $u \geq v$. Consequently, there is no infinite strictly C-decreasing sequence in W. Let $T = \{t_i : i < u\}$ be any C-totally unordered subset of W where $i < j < u$ implies $t_i < t_j$ and for every $i < u$ let $t_i = c_i + \omega^{e_i}$. From Theorem 5 on page 29 there exists an ordinal b such that $cf(t_i) = b$ for every $i < u$. Lemma 7 on page 27 implies that $cf(\omega^{e_i}) = b$ for every $i < u$. Thus, if $i < j < u$ and $e_i \leq e_j$ then $\omega^{e_j} \succ \omega^{e_i}$ by Lemma 5 on page 25. But then $t_j > t_i$ implies $t_j \succ t_i$ by Lemma 7, which contradicts the assumption that T is C-totally unordered. Hence, $e_j < e_i$ whenever $i < j < u$. Thus, $\{e_i : i < u\}$ is a strictly decreasing sequence of ordinals. Consequently, u if finite so that T is finite. Hence, $(W, \succ)$ is partially well ordered.

Remark 2. We observe that Examples 1 and 2 on pages 21 and 22 could have been given in terms of sets of ordinals partially ordered by cofinality. Let $(P,\geq)$ and $(Q,\geq)$ be the partially ordered sets of Example 1 and let $(R,\geq)$ and $(S,\geq)$ be the partially ordered sets of Example 2. If $P' = \{\omega^k + \omega : 0 < k < \omega\}$ and $Q' = P' \cup \{\omega^2\}$ then $(P', \succ)$ is isomorphic to $(P,\geq)$ and $(Q', \succ)$ is isomorphic to $(Q,\geq)$. Further, if $R' = \{\omega_m(n+1) : (m,n) \in R\}$ and $S' = \{\omega_m(n+1) : (m,n) \in S\}$ then $(R', \succ)$ is isomorphic to $(R,\geq)$ and $(S', \succ)$ is isomorphic to $(S,\geq)$.

According to Theorem 7 on page 32 every C-bounded set of ordinals is a C-partially well ordered set. In what follows we consider the natural question of representation of a partially well ordered set as a C-partially well ordered set of ordinals.

We consider first the case of partially well ordered sets which are cardinal products [6, p. 7] of two well ordered sets. That is, if X and Y are well ordered sets then we consider the cartesian product $\bar{X} \times Y$ where $(x,y) \geq (x',y')$ if and only if $x \geq x'$ and $y \geq y'$. Without loss of generality we limit ourselves to the cardinal product of ordinals.

Theorem 8. <u>For every ordinal</u> u_o <u>and</u> u_1 <u>the cardinal product</u> $(u_o \times u_1, \geq)$ <u>is isomorphic to a set of ordinals partially ordered by cofinality</u>.

Proof. Clearly, if $u_o = 0$ or $u_1 = 0$ then $u_o \times u_1 = \phi$ and $(\phi, \geq)$ is similar to $(\phi, \mathfrak{g})$. Otherwise, consider the map f from $u_o \times u_1$ where

$$f((v_o,v_1)) = \omega^{u_o+v_1} + \omega^{v_o+1}. \tag{22}$$

Clearly, if $v_1 = 0$ and $v_o + 1 = u_o$ then $f((v_o,v_1)) = \omega^{u_o} + \omega^{u_o}$. Otherwise, $f((v_o,v_1))$ is given by its normal expansion. Since $g(n) = \omega^{w}n$ is a cofinality map from ω into ω^{w+1} it follows that $cf(\omega^{w+1}) = \omega$ for every ordinal w. Hence, if $w \leq v$ then $\omega^{v+1} \mathfrak{g}\, \omega^{w+1}$ by Lemma 5 on page 25. Consequently, from Lemma 7 on page 27 we have

(23) $\quad f((v_o,v_1)) \succ f((w_o,w_1))$ <u>if and only if</u>

$v_o \geq w_o$ <u>and</u> $f((v_o,v_1)) \geq f((w_o,w_1))$.

Clearly, f is a one-to-one function and from Statements 22 and 23 it follows that if $(v_o,v_1) \geq (w_o,w_1)$ then $f((v_o,v_1)) \succ f((w_o,w_1))$. Assume that $w_1 > v_1$. Then $\omega^{u_o+w_1} > \omega^{u_o+v_1}$ so that $\omega^{u_o+w_1} \geq \omega^{u_o+v_1+1} > \omega^{u_o+v_1} + \omega^{v_o+1}$ or $f((w_o,w_1)) > f((v_o,v_1))$. However, if $f((v_o,v_1)) \succ f((w_o,w_1))$ then $f((v_o,v_1)) \geq f((w_o,w_1))$ so that $w_1 \not> v_1$. In view of Statement 23 we have that $f((v_o,v_1)) \succ f((w_o,w_1))$ implies $(v_o,v_1) \geq (w_o,w_1)$. Hence, if $W = \{f((v_o,v_1)) : (v_o,v_1) \in u_o \times u_1\}$ then f is an isomorphism from $(u_o \times u_1, \geq)$ onto $(W, \succ)$.

Remark 3. We note that if Statement 22 is replaced by

$$(24) \qquad f_k((v_o,v_1)) = \omega_k^{u_o+v_1} + \omega_k^{v_o+1}$$

for every ordinal k and if $W_k = \{f_k((v_o,v_1)) : (v_o,v_1) \in u_o \times u_1\}$ then for every ordinal k it follows that f_k is an isomorphism from $(u_o \times u_1, \geq)$ onto $(W_k, \succ)$.

The following theorem shows that Theorem 8 cannot be extended to allow cardinal products of more than two ordinals.

Theorem 9. <u>If</u> W <u>is a C-partially well ordered set of ordinals then</u> $(W, \geqslant)$ <u>is isomorphic to a subset of the cardinal product of two ordinals</u>.

Proof. If W is a C-partially well ordered set of ordinals then clearly W is the disjoint union of finitely many C-bounded sets. Let $\{W_o, \ldots, W_n\}$ be the partition of W into maximal C-bounded sets and let $u = \cup W + 1$.

Let f denote the function from W into $\omega^{u+1} \times u\omega$ such that if $w \in W_k$ then

$$f(w) = \begin{cases} (\omega^u k+w, u(n-k)+\omega^b), & \text{if } w = a + \omega^b \\ (\omega^u k, u(n-k)), & \text{if } w = 0. \end{cases} \tag{25}$$

Clearly, if $w \neq v$ then $f(w) \neq f(v)$ so f is one-to-one. Also, if $w \geqslant v$ then $w \in W_k$ if and only if $v \in W_k$. Thus, since $w \geqslant v$ implies $w \geq v$ we have $\omega^u k + w \geq \omega^u k + v$. But if $w = a + \omega^b$ and $v = c + \omega^d$ then $\omega^b \geqslant \omega^d$ so that $u(n-k) + \omega^b \geq u(n-k) + \omega^d$. Consequently, $w \geqslant v$ implies $f(w) \geq f(v)$.

On the other hand, if $f(w) \geq f(v)$ while $w \in W_k$ and $v \in W_j$ we will show that $w \geqslant v$.

Case 1. If $w = 0$ then $v = 0$ for otherwise $v = c + \omega^d$ for some ordinals c and d. But $\omega^u k \geq \omega^u j + v$ implies $k > j$ and $u(n-k) \geq u(n-j) + \omega^d$ implies $k < j$ which is impossible. Hence $f(w) \geq f(v)$ and $w = 0$ imply $v = 0$ so that $w \mathrel{9} v$.

Case 2. If $v = 0$ then $w = 0$ for otherwise $w = a + \omega^b$ for some ordinals a and b. Since $(\omega^u k + w,\ u(n-k) + \omega^b) \geq (\omega^u j,\ u(n-j))$ we have $\omega^u k \geq \omega^u j$ or $k \geq j$. But if $k > j$ then $n-k < n-j$ so that $u(n-k) + \omega^b < u(n-k) + u \leq u(n-j)$ which contradicts the assumption that $u(n-k) + \omega^b \geq u(n-j)$. Hence, $j = k$. From Theorem 5 on page 29 we have that $cf(w) = cf(v) = 0$ so that Statement 20 on page 27 implies that $w = v = 0$. Hence, $f(w) \geq f(v)$ and $v = 0$ imply $w \mathrel{9} v$.

Case 3. If $w = a + \omega^b$ and $v = c + \omega^d$ then $\omega^u k + w \geq \omega^u j + v$ implies $k \geq j$ while if $k > j$ then $n-k < n-j$ so that $u(n-k) + \omega^b < u(n-k) + u \leq u(n-j)$ or $u(n-k) + \omega^b < u(n-j) + \omega^d$. Hence, $f(w) \geq f(v)$ implies $k = j$. Consequently, $\omega^b \geq \omega^d$. Since $cf(w) = cf(v)$ we have $cf(\omega^b) = cf(\omega^d)$. Hence, Lemma 5 on page 25 and Lemma 7 on page 27 imply that $w \mathrel{9} v$.

Consequently, the function f defined by Statement 25 is an isomorphism from $(W, 9)$ onto the subset $\{f(w) : w \in W\}$ of the cardinal product $(\omega^{u+1} \times u\omega, \geq)$.

One can define [16, pp. 178-181] the *dimension* of a partially ordered set $(P,\geq)$ to be the smallest cardinal c such that $(P,\geq)$ is isomorphic to a subset of the cardinal product of c simply ordered sets.

Corollary 10. *A partially well ordered set* $(P,\geq)$ *is isomorphic to* $(W, 9)$ *for some set* W *of ordinals if and only if the dimension of* $(P,\geq)$ *is no greater than two.*

Proof. From Theorem 9 on page 36 it follows that every C-partially well ordered set of ordinals has dimension no greater than two. However, from Theorem 5 of [21, p. 177] it follows that if $(P,\geq)$ is partially well ordered and has dimension no greater than two then there exist ordinals u and v such that $(P,\geq)$ is isomorphic to a subset of the cardinal product $(u \times v,\geq)$. Hence, Theorem 8 on page 34 implies that $(P,\geq)$ is isomorphic to $(W, 9)$ for some set W of ordinals.

In fact, since $\omega^{v+1} \; 9 \; \omega$ for every ordinal v, we have from the proof of Theorem 8 and from Theorem 9 on pages 34 and 36 that

Corollary 11. *A partially well ordered set* $(P,\geq)$ *is isomorphic to* $(W, 9)$ *for some C-bounded set* W *of ordinals if and only if the dimension of* $(P,\geq)$ *is no greater than two.*

The above corollary does not preclude a two-dimensional partially well ordered set from being isomorphic to $(W, 9)$ for some set W of ordinals which is not C-bounded.

If W is a set of ordinals, let

$$(26) \qquad c(W) = \overline{\overline{\{cf(w) : w \in W\}}}.$$

Clearly, $c(W)$ is finite if and only if $(W, 9)$ is partially well ordered and $c(W) = 1$ if and only if W is C-bounded.

For every partially well ordered set $(P, \geq)$ we define the <u>cofinality dimension</u> of $(P, \geq)$, denoted $cd(P)$, by

$$(27) \qquad cd(P) = \sup\{n : \text{there exists a set } W \text{ of ordinals with } (W, 9) \simeq (P, \geq) \text{ and } c(W) = n\}$$

where "$\simeq$" means "is isomorphic to". Clearly, $cd(P) = 1$ provided that $(P, \geq)$ has dimension one and $cd(P) = 0$ if and only if the dimension of $(P, \geq)$ is either zero or greater than two.

We say that two elements p and q of a partially ordered set $(P, \geq)$ are <u>connected in P</u> if and only if there exists a finite sequence $\{p_o, \ldots, p_n\}$ in P

with $p_o = p$ and $p_n = q$ and either $p_i \geq p_{i+1}$ or $p_{i+1} \geq p_i$ for every $i < n$.

Let

(28) $E(P) = \{(p,q) : p$ and q are connected in $P\}$.

Clearly, $E(P)$ is an equivalence relation on P. Let $P/E(P)$ be the corresponding partition of P into equivalence classes.

Using Statements 26, 27 and 28 we prove

Theorem 10. <u>For every partially well ordered set</u> $(P,\geq)$ <u>of dimension one or two,</u> $cd(P) = \overline{\overline{P/E(P)}}$.

Proof. If $(P,\geq)$ is a partially well ordered set with dimension one or two then there exist ordinals u and v with $(P,\geq)$ similar to a nonvoid subset of the cardinal product $(u \times v,\geq)$. Thus, we consider only nonvoid subsets of the cardinal product of two ordinals.

Let u and v be ordinals, let P be a nonvoid subset of $u \times v$, and let $\geq$ denote the usual order on the cardinal product of u and v. Then $(P,\geq)$ is partially well ordered so that $\overline{\overline{P/E(P)}}$ is finite. Let W be any set of ordinals with $(P,\geq) \simeq (W, 9)$. Then $\overline{\overline{P/E(P)}} = \overline{\overline{W/E(W)}}$ and $\overline{\overline{W/E(W)}} \geq c(W)$. Thus, $\overline{\overline{P/E(P)}} \geq c(W)$ for every set W of ordinals with $(P,\geq) \simeq (W, 9)$. Hence, from Statement

27 it follows that $\overline{\overline{P/E(P)}} \geq cd(P)$.

If $\overline{\overline{P/E(P)}} = n$ let $P/E(P) = \{D_1, \ldots, D_n\}$. Define an ordinal valued function g from P by

$$g((a,b)) = \omega_i^{u+b} + \omega_i^{a+1} \quad \text{for} \quad (a,b) \in D_i.$$

If $W = \{g(p) : p \in P\}$ then $(W, \mathfrak{g}) \simeq (P, \geq)$. But by [3] we have $cf(\omega_i^{w+1}) = \omega_i$ for every ordinal w and every nonlimit ordinal i. Thus, $c(W) = \overline{\overline{P/E(P)}}$ so that $cd(P) \geq \overline{\overline{P/E(P)}}$. Hence, $cd(P) = \overline{\overline{P/E(P)}}$, as desired.

Theorem 11. For every partially well ordered set $(P, \geq)$ of dimension one or two, if $1 \leq k \leq cd(P)$ then there exists a set W of ordinals such that $(W, \mathfrak{g}) \simeq (P, \geq)$ and $c(W) = k$.

Proof. Assume that P is a nonvoid subset of $u \times v$ for some ordinals u and v and let $P/E(P) = \{D_1, \ldots, D_n\}$ where $n = cd(P)$. For $i = 1, \ldots, k-1$, let $E_i = D_i$ and let $E_k = \bigcup \{E_i : i = k, \ldots, n\}$. Define the ordinal valued function g_k from P by

$$g_k((a,b)) = \omega_i^{u+b} + \omega_i^{a+1} \quad \text{for} \quad (a,b) \in E_i.$$

If $W = \{g_k(p) : p \in P\}$ then from Statement 24 and Remark 3 on page 35 it follows that $(W, \mathfrak{g}) \simeq (P, \geq)$ so that $c(W) = k$, as desired.

Theorem 9 on page 36 can be extended in the following manner.

Theorem 12. *Every set W of ordinals partially ordered by cofinality has dimension no greater than two*.

Proof. Let W be a set of ordinals. If $A = \{cf(w) : w \in W\}$ then for every $a \in A$ let $W(a) = \{w : w \in W$ and $cf(w) = a\}$. Clearly, $\{W(a) : a \in A\}$ is a partition of W into C-bounded subsets. From Theorem 7 on page 32 it follows that $(W(a), \mathfrak{g})$ is a partially well ordered set for every $a \in A$. Thus, by Theorem 9 on page 36, we see that $(W(a), \mathfrak{g})$ is similar to a subset of the cardinal product of two ordinals u_a and v_a. Let $L = \cup \{u_a \times \{a\} : a \in A\}$ and $S = \cup \{v_a \times \{a\} : a \in A\}$. For every (m,a) and (k,b) in L write $(m,a) \geq (k,b)$ if and only if $a > b$ or $a = b$ and $m \geq k$. For every (s,a) and (t,b) in S write $(s,a) \geq (t,b)$ if and only if $a < b$ or $a = b$ and $s \geq t$. It then follows that $(L, \geq)$ and $(S, \geq)$ are simply ordered sets and that $(W, \mathfrak{g})$ is isomorphic to a subset of $(L \times S, \geq)$. Hence, $(W, \mathfrak{g})$ has dimension no greater than two, as desired.

The following theorem gives a necessary and sufficient condition for a partially ordered set to be isomorphic to a set of ordinals partially ordered by cofinality. By a *connected subset* of a partially ordered set P we mean a subset S of P such that every two elements of S are connected in S.

Theorem 13. *A partially ordered set* $(P,\geq)$ *is isomorphic to a set* W *of ordinals partially ordered by cofinality if and only if* $(P,\geq)$ *has dimension no greater than two and every connected subset of* P *is partially well ordered*.

Proof. First, if W is a set of ordinals then $(W, 9)$ has dimension no greater than two by Theorem 12. Further, in view of Statement 21 and Theorem 5 on pages 28 and 29, every connected subset of W is C-bounded. Thus, Theorem 7 on page 32 implies that every connected subset of W is partially well ordered. Hence, if $(P,\geq)$ is isomorphic to $(W, 9)$ for some set W of ordinals, then $(P,\geq)$ has dimension no greater than two and every connected subset of P is partially well ordered.

Conversely, assume that $(P,\geq)$ has dimension no greater than two and that every connected subset of P is partially well ordered. Let $E(P)$ be the relation defined by Statement 28 on page 40 and let $P/E(P) = \{D_i : i < n\}$ for some ordinal n. Then D_i is a connected subset of P and, hence, is partially well ordered for every $i < n$.

Further, since $D_i \subset P$ the dimension of $(D_i, \geq)$ is no greater than the dimension of $(P, \geq)$. Thus, for every $i < n$ the dimension of $(D_i, \geq)$ is no greater than two. On the other hand, however, by Corollary 11 on page 38, for every $i < n$ there is a set W_i of ordinals with $(D_i, \geq)$ isomorphic to $(W_i, \ni)$. In particular, in view of Remark 3 on page 35, for every $i < n$ we can choose W_i such that $cf(w) = \omega_i$ for every $w \in W_i$. Hence, if $W = \cup\{W_i : i < n\}$, then $(P, \geq)$ is similar to $(W, \ni)$.

IV. NORMAL FUNCTIONS AND INACCESSIBLE CARDINALS

A. Normal Functions and e-inaccessibility

An ordinal-valued function N defined for all ordinals is a <u>normal function</u> (cf. footnote 6 of [14, p. 227] if for every ordinal u and v

$$v < u \ \underline{\text{implies}} \ N(v) < N(u)$$

and for every limit ordinal u

$$N(u) = \sup_{v<u} N(v).$$

That is, N is normal if N is <u>strictly increasing</u> and <u>continuous</u>. Clearly,

(29) <u>if</u> M <u>and</u> N <u>are normal functions</u> <u>then</u> $M \circ N$ <u>is a normal function</u>.

Motivated by [14, p. 227] and [5, pp. 40-45] we introduce the notion of the v-derivative N_v of a normal function N by

(30) $$N_0 \equiv N$$

and for every ordinal u and v if $v > 0$ then $N_v(u)$ is the first ordinal such that for every ordinal w

(31) $$w < u \ \underline{\text{implies}}\ N_v(w) < N_v(u)$$

and

(32) $$w < v \ \underline{\text{implies}}\ N_w(N_v(u)) = N_v(u).$$

Theorem 14. <u>For every normal function</u> N <u>defined on the class of all ordinals and every ordinal</u> v <u>the v-derivative</u> N_v <u>is a normal function defined on the class of all ordinals</u>.

Proof. The conclusion holds for $v = 0$ by Statement 30. Assume that $v > 0$ and for every $w < v$ the w-derivative N_w is a normal function defined on the class of all ordinals. Consequently, it follows from [5, pp. 42-45] that N_w has arbitrarily large fixed points for every $w < v$. We show that N_v is defined for all ordinals, that N_v is strictly increasing and that N_v is continuous.

If $N_v(k)$ is defined for every ordinal $k < u$ then for every $w < v$ we let $a(w)$ be the smallest fixed point of N_w such that $\sup_{k<u} N_v(k) < a(w)$. Let $a(v) = \sup_{w<v} a(w)$.

Clearly, if $t < w < v$ then $a(t) \leq a(w) \leq a(v)$. Thus, for every $t < v$ we have $a(v) = \sup_{t \leq w < v} a(w)$ so that $N_t(a(v)) = N_t(\sup_{t \leq w < v} a(w))$. Since N_t is normal we have $N_t(a(v)) = \sup_{t \leq w < v} N_t(a(w))$. However, $a(w) = N_w(a(w))$ so that if $t < w$ then Statement 32 implies that $N_t(N_w(a(w))) = N_w(a(w))$. Hence, for every ordinal w if $t \leq w < v$ then $N_t(a(w)) = a(w)$. Thus, $N_t(a(v)) = a(v)$. Consequently, N_v is defined on the class of all ordinals.

Statement 31 implies that N_v is strictly increasing. If u is any limit ordinal then for every ordinal w and k if $w < v$ and $k < u$ we have $N_w(N_v(k)) = N_v(k)$. Hence, $N_w(\sup_{k<u} N_v(k)) = \sup_{k<u} N_w(N_v(k))$ since N_w is normal. Thus, $N_w(\sup_{k<u} N_v(k)) = \sup_{k<u} N_v(k)$. Therefore, $N_v(u) \leq \sup_{k<u} N_v(k)$. From this and Statement 31 on page 46 we have $N_v(u) = \sup_{k<u} N_v(k)$, i.e., N_v is continuous. Hence, N_v is a normal function defined on the class of all ordinals.

Now we define and study a particular normal function. For every ordinal u let

$$F(u) = \begin{cases} \aleph_0, & \text{if } u = 0 \\ \sup\limits_{v<u} 2^{F(v)}, & \text{if } u > 0. \end{cases} \tag{33}$$

Lemma 8. <u>The function</u> F <u>is a normal function</u>.

Proof. Clearly, F is strictly increasing. Further, $F(v+1) = 2^{F(v)}$ for every ordinal v. Hence, for every limit ordinal u we have

$$F(u) = \sup_{v<u} 2^{F(v)} = \sup_{v<u} F(v+1) = \sup_{v<u} F(v).$$

Theorem 15. <u>If</u> F <u>is the function of Statement 33 then for every cardinal</u> $\bar{\bar{c}}$ <u>there exists an ordinal</u> $v(\bar{\bar{c}})$ <u>such that</u> $\bar{\bar{c}} < F_{v(\bar{\bar{c}})}(0)$.

Proof. Clearly, $\aleph_0 < F_1(0)$ and if $\bar{\bar{m}} < F_{v(\bar{\bar{m}})}(0)$ for every cardinal $\bar{\bar{m}} < \bar{\bar{c}}$ and $v = \sup\limits_{\bar{\bar{m}}<\bar{\bar{c}}} v(\bar{\bar{m}})$ then $\bar{\bar{m}} < \bar{\bar{c}}$ implies $\bar{\bar{m}} < F_v(0)$. Hence, $\bar{\bar{c}} \leq F_v(0)$ so that $\bar{\bar{c}} < F_{v+1}(0)$.

Corollary 12. <u>For every cardinal</u> $\bar{\bar{c}}$ <u>there exists an ordinal</u> v <u>such that</u> $\bar{\bar{c}} \neq F_v(u)$ <u>for every ordinal</u> u.

Proof. This follows directly from Theorem 15 and Theorem 14.

Rewording Corollary 12 we have

Corollary 13. <u>There exists no cardinal which is in the range of every v-derivative of</u> F.

We recall the definition of an e-inaccessible cardinal given on page 15 and the definition of the function F given in Statement 33.

Theorem 16. <u>If</u> $\bar{\bar{c}} > \aleph_0$ <u>then</u> $\bar{\bar{c}}$ <u>is e-inaccessible if and only if there exists a limit ordinal</u> u <u>such that</u> $\bar{\bar{c}} = F(u)$.

Proof. Let u be any limit ordinal. If $\bar{\bar{n}} < F(u)$ then there exists an ordinal $v < u$ such that $\bar{\bar{n}} \leq F(v)$. Hence, $2^{\bar{\bar{n}}} \leq F(v+1)$ so that $2^{\bar{\bar{n}}} < F(u)$. Thus, Statement 15 on page 15 implies that F(u) is e-inaccessible.

For every ordinal v and every e-inaccessible cardinal $\bar{\bar{c}}$ if $F(v) < \bar{\bar{c}}$ then from Statement 33 on page 48 we have $F(v+1) = 2^{F(v)}$ so that $F(v+1) < \bar{\bar{c}}$. This implies that $F(v+\omega) \leq \bar{\bar{c}}$. Hence, $F(\omega)$ is the first e-inaccessible cardinal greater than $\aleph_0$. Let $\bar{\bar{c}}$ be an e-inaccessible cardinal with $\bar{\bar{c}} > F(\omega)$ and assume that for every e-inaccessible cardinal $\bar{\bar{m}}$ if $\aleph_0 < \bar{\bar{m}} < \bar{\bar{c}}$ then there exists a limit ordinal $u(\bar{\bar{m}})$ such that $\bar{\bar{m}} = F(u(\bar{\bar{m}}))$. Let $A(\bar{\bar{c}}) = \{\bar{\bar{m}} : \aleph_0 < \bar{\bar{m}} < \bar{\bar{c}}$ and $\bar{\bar{m}}$ is e-inaccessible$\}$ and let $w = \sup\{u(\bar{\bar{m}})+\omega : \bar{\bar{m}} \in A(\bar{\bar{c}})\}$. Since $A(\bar{\bar{c}}) \neq \emptyset$ it follows that w is a limit ordinal and F(w) is e-inaccessible. If $\bar{\bar{m}} \in A(\bar{\bar{c}})$ then $\bar{\bar{m}} = F(u(\bar{\bar{m}})) < \bar{\bar{c}}$ so that

$F(u(\bar{\bar{m}})+\omega) \leq \bar{\bar{c}}$. Thus, $\sup\{F(u(\bar{\bar{m}})+\omega) : \bar{\bar{m}} \in A(\bar{\bar{c}})\} \leq \bar{\bar{c}}$. But, clearly, $F(w) = \sup\{F(u(\bar{\bar{m}})+\omega) : \bar{\bar{m}} \in A(\bar{\bar{c}})\}$. Hence,

(34) $$F(w) \leq \bar{\bar{c}}.$$

On the other hand, if $F(w) < \bar{\bar{c}}$ then $F(w) \in A(\bar{\bar{c}})$ and $u(F(w)) = w$. But from the definition of w we see that $w \geq w + \omega$ which is impossible. Hence,

(35) $$F(w) \geq \bar{\bar{c}}.$$

In view of Statements 34 and 35 we have that $\bar{\bar{c}} = F(w)$ and the theorem follows by transfinite induction.

Corollary 14. <u>For every cardinal</u> $\bar{\bar{n}}$ <u>there exists an e-inaccessible cardinal</u> $\bar{\bar{c}}$ <u>such that</u> $\bar{\bar{c}} \geq \bar{\bar{n}}$.

Proof. Clearly, $F(\bar{\bar{n}}) \geq \bar{\bar{n}}$. But $F(\bar{\bar{n}})$ is e-inaccessible by Theorem 16 on page 49.

From page 27 we see that every ordinal v has a unique normal expansion. If v is a limit ordinal then each term of this expansion involves a factor of ω. Hence an ordinal v is a limit ordinal if and only if there exists an ordinal $w > 0$ such that $v = \omega w$. Thus, v is a limit ordinal if and only if $v = \omega(1+u)$ for some ordinal u.

Let L be the function which enumerates the limit ordinals. I.E., for every ordinal u,

(36) $$L(u) = \omega(1+u).$$

Clearly, it follows from Statement 36 that L is a normal function defined on the class of all ordinals.

Lemma 9. *If* L *is the function defined by Statement 36 and* u *is any ordinal then* $L(u) = u$ *if and only if* $u = \omega^{\omega}v$ *for some ordinal* $v > 0$.

Proof. If $u < \omega$ then $L(u) > u$ by Statement 36. But if $u \geq \omega$ then $1 + u = u$ so that $L(u) = \omega u$. Clearly, $\omega u = u$ if and only if $u = \omega^{\omega}v$ for some ordinal v. Hence, $L(u) = u$ if and only if $u = \omega^{\omega}v$ for some ordinal $v > 0$.

From Lemma 9 we have for every cardinal $\bar{\bar{c}} > \aleph_0$ that

(37) $$L(\bar{\bar{c}}) = \bar{\bar{c}}.$$

For every ordinal u let

(38) $E(u)$ *be the first e-inaccessible cardinal such that* $\aleph_0 < E(u)$ *and* $v < u$ *implies* $E(v) < E(u)$.

From Corollary 14 on page 50 it follows that $E(u)$ is defined for every ordinal u. In view of Statement 33 on page 48, Theorem 16 on page 49 and Statement 36 and 38 we have

$$(39) \qquad E = F \circ L.$$

Since F and L are normal functions, Statement 29 on page 45 implies

Lemma 10. <u>The function</u> E <u>defined by Statement 38 is a normal function</u>.

We now prove a theorem which shows the very strong relationship between the functions F and E.

Theorem 17. <u>If</u> F <u>is the function defined by Statement 33 on page 48 and</u> E <u>is the function defined by Statement 38 then</u> $F_v(u) = E_v(u)$ <u>for every ordinal</u> u <u>and</u> v <u>with</u> $v > 0$.

Proof. For every ordinal $u > 0$ both $E(u)$ and $F(u)$ are cardinals greater than $\aleph_0$. In view of Statement 37 if $E(u) = u$ or $F(u) = u$ then $L(u) = u$. Thus, $F(L(u)) = F(u)$ sc that Statement 39 implies $E(u) = F(u) = u$. The desired result then follows from Statements 30, 31 and 32 on pages 45 and 46.

B. Regularity and Strong Inaccessibility

An infinite cardinal $\bar{\bar{c}}$ is called <u>regular</u> [13, p. 308] if and only if

(40) $$\mathrm{cf}(\bar{\bar{c}}) = \bar{\bar{c}}.$$

Further, $\bar{\bar{c}}$ is called <u>strongly inaccessible</u> [13, p. 311], or <u>p-inaccessible</u> [2, p. 101] if and only if $\bar{\bar{c}}$ satisfies both Statement 15 on page 15 and

(41) $$\bar{\bar{n}} < \bar{\bar{c}} \text{ and } \bar{\bar{c}}_i < \bar{\bar{c}} \text{ for every } i < \bar{\bar{n}}$$

$$\underline{\text{imply}} \sum_{i<\bar{\bar{n}}} \bar{\bar{c}}_i < \bar{\bar{c}}.$$

From Theorem 1 of [2, p. 100] it follows that Statements 40 and 41 are equivalent. Hence, we have

Lemma 11. <u>Every infinite cardinal is strongly inaccessible if and only if it is e-inaccessible and regular</u>.

In [14, p. 227] the following axiom schema of strong infinity is introduced.

M <u>Every normal function defined for all ordinals has at least one strongly inaccessible cardinal in its range</u>.

It is shown [14, p. 228] that M is equivalent to the conjunction of the axiom

T *There exist arbitarily large strongly inaccessible cardinals*

and the axiom schema

RI *Every normal function defined for all ordinals has at least one regular cardinal in its range*.

We shall prove that RI implies T and, hence, that RI implies M. However, we will also give a direct proof of the latter.

Corresponding to Theorem 1 of [14] we have

Theorem 18. RI *is equivalent to each of the following axiom schemata*.

RI' *Every normal function defined for all ordinals has at least one fixed point which is regular*.

RI" *Every normal function defined for all ordinals has arbitrarily large fixed points which are regular*.

Proof. Clearly, RI" implies RI' and RI' implies RI. Let N be any normal function defined for all ordinals and for every ordinal v and u let $M^v(u) = N_1(v+u)$ where N_1 is the 1-derivative of N. Clearly, for every ordinal v Statements 30, 31 and 32 on pages 45 and 46 together with Theorem 14 on page 46 imply that M^v is a normal function defined for all ordinals. Thus, RI implies that for every ordinal v there exists an ordinal u such that $M^v(u)$ is regular. However, Statement 32 on page 46 and the definition of M^v imply $M^v(u) = N(M^v(u))$. Further, $M^v(u) = N_1(v+u)$ and $N_1(v+u) \geq v+u$ so that $M^v(u) \geq v$. Hence, RI implies RI" and the theorem is proved.

Theorem 19. <u>The axiom schema</u> RI <u>implies the axiom</u> T.

Proof. Let E be the function defined by Statement 38 on page 51. By Lemma 10 on page 52 and from the axiom schema RI" of Theorem 18 it follows that E has arbitrarily large fixed points which are regular. On the other hand, $E(u)$ is e-inaccessible for every ordinal u by Statement 38 on page 51. Hence, Lemma 11 on page 53 implies that E has arbitrarily large fixed points which are strongly inaccessible.

Theorem 20. <u>The axiom schema</u> RI <u>is equivalent to the axiom schema</u> M.

Proof. In view of Lemma 11 on page 53 it follows that M implies RI. On the other hand, let N be any normal function defined for all ordinals and for every ordinal u let $N'(u) = N(E(u))$ where E is the function defined by Statement 38 on page 51. From Statement 29 on page 45 it follows that N' is a normal function defined for all ordinals. By Theorem 18 on page 54 there exists a fixed point of N' which is regular. That is, $N'(u) = u = cf(u)$ for some ordinal u. But $u \leq E(u)$ and $E(u) \leq N'(u)$ so that $E(u) = u$. Thus, $N'(u) = N(u)$ or $N(u) = u$. From the definition of E it follows that $E(u)$ is e-inaccessible. Thus, Lemma 11 on page 53 implies that u is strongly inaccessible. Since $N(u) = u$ it follows that RI implies M.

We prove below that the axiom schema RI implies the usual axiom I of Infinity of ZF which states

I   <u>There exists a set</u> W <u>such that</u> $\emptyset \in W$ <u>and for every set</u> x <u>if</u> $x \in W$ <u>then</u> $x \cup \{x\} \in W$.

However, first we note that the definition of a finite ordinal can be given independently of the axiom I. Indeed, an <u>ordinal</u> u is any set such that

(42) $x \in u$ *implies* $x \subset u$ *for every set* x,

and

(43) u *is well ordered by inclusion*,

and

(44) $x \notin x$ *for every* $x \in u$.

A *finite ordinal* is a finite set which is an ordinal (i.e., u is a finite ordinal if and only if u is a finite set which satisfies Statements 42, 43 and 44.

Theorem 21. *The axiom schema* RI *implies the axiom* I.

Proof. As usual, $w+1 = w \cup \{w\}$ and $w+2 = (w+1)+1$ for every ordinal w. Also, $0 = \emptyset$. Hence, we will show that the axiom schema RI implies the existence of an ordinal v such that $0 \in v$ and $x \in v$ implies $x+1 \in v$.

For every ordinal u let

$$R(u) = \begin{cases} u+2, & \text{if } u \text{ is finite} \\ u, & \text{otherwise.} \end{cases} \qquad (45)$$

Clearly, R is a normal function defined for all ordinals so that the schema RI implies that $R(u)$ is regular for some ordinal u. We see from Statement 45 that $R(v) > 1$ for every ordinal v so that $0 \varepsilon R(u)$. Also, if $x \varepsilon R(u)$ then either $x+1 = R(u)$ or $x+1 \varepsilon R(u)$. If $x + 1 = R(u)$ then $\{(0,x)\}$ is a cofinality map from $R(u)$ into 1. Hence, $cf(R(u)) = 1$ so that $cf(R(u)) < R(u)$. Since $R(u)$ is regular we conclude that $x \varepsilon R(u)$ implies $x+1 \varepsilon R(u)$. Thus, $R(u)$ is the desired ordinal.

V. ORDER MODULO A FILTER

In this section we establish some criteria that must be satisfied by an ultrafilter U of a Boolean algebra 2^X in order that a partially ordered set to be derived from the set Y^X and the ultrafilter U reflect some of the properties of the partially ordered set Y.

A. The Almost Everywhere Order

For every partially ordered set $(Y,\leq)$ and every non-void set X let t and s be the functions from $Y^X \times Y^X$ into 2^X such that for every element f and g of Y^X and every element x of X

$$(46)\qquad t(f,g)(x) = \begin{cases} 1, & \text{if } f(x) \leq g(x) \\ 0, & \text{otherwise.} \end{cases}$$

and

$$(47)\qquad s(f,g)(x) = \begin{cases} 1, & \text{if } f(x) = g(x) \\ 0, & \text{otherwise.} \end{cases}$$

Motivated by [20, pp. 99-100], [8] and [7, p. 198] we introduce the following definitions.

Definition 2. Let $(Y,\leq)$ be a partially ordered set and let F be a nonvoid proper filter in the Boolean algebra 2^X. For every element f and g of Y^X we say that f is less than or equal to g almost everywhere with respect to the filter F, in symbols

$$f \underset{\text{Fae}}{\leq} g,$$

if and only if $t(f,g) \in F$.

Definition 3. Let $(Y,\leq)$ be a partially ordered set and let F be a nonvoid proper filter in the Boolean algebra 2^X. For every element f and g of Y^X we say that f is equal to g almost everywhere with respect to the filter F, in symbols

$$f \underset{\text{Fae}}{=} g,$$

if and only if $s(f,g) \in F$.

Remark 4. Definitions 2 and 3 could be extended to require only that $\leq$ be a relation on Y. However, if $\leq$ fails to be antisymmetric then there exist distinct elements y and y' of Y with $y \leq y'$ and $y' \leq y$. If f and g are the constant functions from X into Y with values

y and y', respectively, then $f \underset{Fae}{\leq} g$ and $g \underset{Fae}{\leq} f$ but $f \underset{Fae}{\neq} g$. However, if $\leq$ is antisymmetric then it follows that $f \underset{Fae}{=} g$ if and only if $f \underset{Fae}{\leq} g$ and $g \underset{Fae}{\leq} f$.

We now prove the following theorem.

Theorem 22. _If_ $(Y,\leq)$ _is a partially ordered set and_ F _is a nonvoid proper filter in the Boolean algebra_ 2^X, _then_ $\underset{Fae}{\leq}$ _is a reflexive and transitive relation on_ Y^X.

Proof. Since F is nonvoid, Statement 1 on page 3 implies the unit of 2^X is an element of F. Hence, $t(f,f) \in F$ for every element f of Y^X. Thus, $\underset{Fae}{\leq}$ is reflexive by Definition 2.

For every element f, g and h of Y^X, if $f \underset{Fae}{\leq} g$ and $g \underset{Fae}{\leq} h$ then $t(f,g) \in F$ and $t(g,h) \in F$ so that $(t(f,g))(t(g,h)) \in F$ by Statement 1 on page 3. For every element x of X, if $(t(f,g))(t(g,h))(x) = 1$ then $f(x) \leq g(x)$ and $g(x) \leq h(x)$. Thus, $f(x) \leq h(x)$ or $t(f,g)(x) = 1$. Hence,
$(t(f,h))((t(f,g))(t(g,h))) = (t(f,g))(t(g,h))$ so that $t(f,h) \in F$. Consequently, $f \underset{Fae}{\leq} h$ and therefore the relation $\underset{Fae}{\leq}$ is transitive.

If $(Y,\leq)$ is a partially ordered set and F is a nonvoid proper filter in the Boolean algebra 2^X for some set X then let

$$(48)\quad E(F,X,Y) = \{(f,g) : (f,g) \in Y^X \times Y^X \text{ and } f \underset{Fae}{=} g\}.$$

For every element f of Y^X let $[f] = \{g : (f,g) \in E(F,X,Y)\}$. In view of Remark 4 above and Theorem 3 of [6, p. 4], the set $E(F,X,Y)$ of Statement 48 is an equivalence relation on the set Y^X and if $[F,X,Y]$ is the set of equivalence classes of Y^X with respect to the relation $E(F,X,Y)$, i.e.,

$$[F,X,Y] = \{[f] : f \in Y^X\},$$

then $[F,X,Y]$ is partially ordered by the relation $\underset{Fae}{\leq'}$ where

$$[f] \underset{Fae}{\leq'} [g] \text{ if and only if } f \underset{Fae}{\leq} g.$$

In the sequel we will use the notation $\underset{Fae}{\leq}$ to denote both the order on Y^X and the corresponding order on $[F,X,Y]$.

We have proved the following corollary to Theorem 22 on page 61.

Corollary 15. <u>If</u> $(Y,\leq)$ <u>is a partially ordered set and</u> F <u>is a nonvoid proper filter in the Boolean algebra</u> 2^X, <u>then</u> $([F,X,Y], \underset{Fae}{\leq})$ <u>is a partially ordered set</u>.

B. The Sets $(Y,\leq)$ and $([F,X,Y], \underset{Fae}{\leq})$

For every element y of Y the symbol (y) will denote the constant function from X into Y whose value is y. The map from Y into $[F,X,Y]$ which maps y into $[(y)]$ for every element y of Y is an isomorphism of $(Y,\leq)$ into $([F,X,Y], \underset{Fae}{\leq})$. We wish to add conditions on F so that $([F,X,Y], \underset{Fae}{\leq})$ will reflect certain properties of $(Y,\leq)$.

Theorem 23. <u>If</u> $(Y,\leq)$ <u>is a partially ordered set and</u> F <u>is a nonvoid proper filter in the Boolean algebra</u> 2^X, <u>then</u> $[F,X,Y] = \{[(y)] : y \in Y\}$ <u>if and only if</u> F <u>is</u> $\overline{\overline{Y}}$<u>-complete</u>.

Proof. Clearly, every nonvoid proper filter of 2^X is $\overline{\overline{Y}}$-complete if $\overline{\overline{Y}} \leq 1$. On the other hand, if $\overline{\overline{Y}} \leq 1$ then $\overline{\overline{Y^X}} = \overline{\overline{Y}}$ so that $[F,X,Y] = \{[(y)] : y \in Y\}$. Thus, we may assume that $\overline{\overline{Y}} \geq 2$.

If $\bar{\bar{Y}} \geq 2$ and F is $\bar{\bar{Y}}$-complete then it follows that F is an ultrafilter. Let f be any function from X into Y. Let s be the function defined by Statement 47 on page 59 and let $S = \{s(f,(y)) : y \in Y\}$. Clearly, $\sup S = 1$ and $\bar{\bar{S}} \leq \bar{\bar{Y}}$ so from Lemma 2 on page 7 it follows that $S \cap F \neq \emptyset$. Thus, there exists $y \in Y$ with $s(f,(y)) \in F$. Hence, $[f] = [(y)]$ and $\lceil F,X,Y] = \{[(y)] : y \in Y\}$.

Conversely, we assume that $\bar{\bar{Y}} \geq 2$ and $[F,X,Y] = \{[(y)] : y \in Y\}$. Let D be any orthogonal subset of 2^X such that $\bar{\bar{D}} \leq \bar{\bar{Y}}$ and $\sup D = 1$. Let k be a one-to-one function from D into Y. Since D is orthogonal and $\sup D = 1$ it follows that there exists a function f from X into Y given by

$$f(x) = k(d) \ \underline{\text{if and only if}} \ d(x) = 1.$$

If s is the map defined by Statement 47 on page 59 we have that

$$s(f,(y)) = \begin{cases} d, & \text{if } y = k(d) \text{ for some } d \in D \\ 0, & \text{otherwise.} \end{cases}$$

But $[f] = [(y)]$ for some element y of Y and therefore $s(f,(y)) \in F$ for some $y \in Y$. Since $0 \notin F$ it follows that $d \in F$ for some element d of D. Hence, $D \cap F \neq \emptyset$ for every orthogonal subset D of 2^X such that $\bar{\bar{D}} \leq \bar{\bar{Y}}$ and $\sup D = 1$. In particular, if $a \in 2^X$ and $A = \{a, 1+a\}$ then $A \cap F \neq \emptyset$. Thus, F is an ultrafilter, and from Lemma 2 on page 7 it follows that F is $\bar{\bar{Y}}$-complete.

Corollary 16. <u>If</u> $(Y,\leq)$ <u>is a partially ordered set,</u> a <u>is an atom of the Boolean algebra</u> 2^X <u>and</u> U <u>is the ultrafilter in</u> 2^X <u>with</u> $a \in U$, <u>then</u> $[U,X,Y] = \{[(y)] : y \in Y\}$.

Proof. Since 2^X is a complete Boolean algebra it follows that $\sup S$ exists for every subset S of 2^X. However, the prime ideal 2^X-U is suprema-preserving by Lemma 1 on page 5. Consequently, 2^X-U is complete and, from Lemma 2 on page 7, we have that U is complete. Hence, the desired result follows from Theorem 23 on page 63.

Corollary 17. <u>If</u> $(Y,\leq)$ <u>is a finite partially ordered set and</u> U <u>is an ultrafilter in the Boolean algebra</u> 2^X, <u>then</u> $[U,X,Y] = \{[(y)] : y \in Y\}$.

Proof. For every finite cardinal n and every ulta-filter U it follows that U is n-complete. Hence, $[U,X,Y] = \{[(y)] : y \in Y\}$ by Theorem 23 on page 63.

Theorem 24. <u>If</u> $(Y,\leq)$ <u>is a simply ordered set with at least two elements and</u> F <u>is a nonvoid proper filter in the Boolean algebra</u> 2^X, <u>then</u> $([F,X,Y], \underset{Fae}{\leq})$ <u>is a simply ordered set if and only if</u> F <u>is an ultrafilter</u>.

Proof. Let F be an ultrafilter in 2^X. In view of Corollary 15 on page 63 we need only show that for every element $[f]$ and $[g]$ of $[F,X,Y]$ either $[f] \underset{Fae}{\leq} [g]$ or $[g] \underset{Fae}{\leq} [f]$. If $[f] \underset{Fae}{\not\leq} [g]$ then $t(f,g) \notin F$. Hence, $(1+t(f,g)) \in F$. But if $(1+t(f,g))(x) = 1$ then $t(f,g)(x) = 0$ so that $f(x) \not\leq g(x)$. Consequently, $g(x) \leq f(x)$ or $t(g,f)(x) = 1$. Thus, $t(g,f)(1+t(f,g)) = 1+t(f,g)$ so that $t(g,f) \in F$ by Statement 1 on page 3. Hence $[g] \underset{Fae}{\leq} [f]$ and $([F,X,Y], \underset{Fae}{\leq})$ is simply ordered as desired.

Conversely, if $([F,X,Y], \underset{Fae}{\leq})$ is simply ordered then for every element f and g of Y^X either $[f] \underset{Fae}{\leq} [g]$ or $[g] \underset{Fae}{\leq} [f]$. Thus, either $t(f,g) \in F$ or $t(g,f) \in F$. Let r be any element of 2^X and let y and y' be elements of Y with $y < y'$. Define functions f and g from X into Y by

$$f(x) = \begin{cases} y, & \text{if } r(x) = 1 \\ y', & \text{if } r(x) = 0 \end{cases} \quad \underline{\text{and}} \quad g(x) = \begin{cases} y', & \text{if } r(x) = 1 \\ y, & \text{if } r(x) = 0. \end{cases}$$

Since $t(f,g) = r$ and $t(g,f) = 1+r$, it follows that $r \in F$ or $1+r \in F$. Hence, F is an ultrafilter.

Theorem 25. *If* $(Y,\leq)$ *is an infinite partially well ordered set and* U *is an ultrafilter in the Boolean algebra* 2^X, *then* $([U,X,Y], \underset{Uae}{\leq})$ *is a partially well ordered set if and only if* U *is* σ*-complete*.

Proof. Let U be a σ-complete ultrafilter in the Boolean algebra 2^X. Corollary 15 on page 63 implies that $([U,X,Y], \underset{Uae}{\leq})$ is partially ordered. Let $\{[f_n] : n \in \omega\}$ be any infinite sequence in $[U,X,Y]$ and let $A = \{t(f_m,f_n) : m \in \omega$ and $n \in \omega$ and $m < n\}$. If $m < n$ implies that $[f_m] \underset{Uae}{\not\leq} [f_n]$ then A is a countable subset of $2^X - U$ and by Lemma 2 on page 7 we have $\sup A \neq 1$. Thus, for some element x of X it follows that $t(f_m,f_n)(x) = 0$ whenever $m < n$ or $f_m(x) \not\leq f_n(x)$ whenever $m < n$. Hence, there is an infinite subsequence of $\{f_n(x) : n \in \omega\}$ which is either strictly decreasing or totally unordered. However, this is impossible since

$(Y,\leq)$ is partially well ordered. Therefore, there exist natural numbers m and n with $m < n$ and $[f_m] \underset{Uae}{\leq} [f_n]$. Consequently, $[U,X,Y]$ has no infinite strictly decreasing sequence and no infinite totally unordered subsets. Thus, $([U,X,Y], \underset{Uae}{\leq})$ is a partially well ordered set.

Conversely, if U is not σ-complete then by Lemma 2 on page 7 there exists a denumerable orthogonal subset C of 2^X-U with $\sup C = 1$. Let $C = \{c_n : n \in \omega\}$. Let $\{y_n : n \in \omega\}$ be an infinite subset of Y with $y_n < y_{n+1}$ for every $n \in \omega$. For every $x \in X$ let n be that unique natural number such that $c_n(x) = 1$ and for every natural number k, let

$$f_k(x) = \begin{cases} y_{n-k}, & \text{if } k \leq n \\ y_o, & \text{if } k > n. \end{cases}$$

Let k and m be natural numbers with $k < m$. If $x \in X$ let n be the natural number such that $c_n(x) = 1$. If $m \leq n$ then $f_m(x) = y_{n-m} < y_{n-k} = f_k(x)$. If $k < n < m$ then $f_m(x) = y_o < y_{n-k} = f_k(x)$. If $n \leq k$ then $f_m(x) = y_o = f_k(x)$. Thus, $f_k(x) \leq f_m(x)$ if and only if $n \leq k$ so that $t(f_k,f_m) = \sup\{c_n : n \leq k\}$. Since $C \subset 2^X-U$

we have $t(f_k, f_m) \in 2^X - U$ for every natural number k and m with $k < m$. Hence, if $k < m < \omega$ then $[f_k] \underset{Uae}{\not\leq} [f_m]$. Theorem 1 of [21, p. 176] implies that $([U,X,Y], \underset{Uae}{\leq})$ is not partially well ordered.

Theorem 26. <u>If</u> $(Y,\leq)$ <u>is an infinite well ordered set and</u> U <u>is an ultrafilter in the Boolean algebra</u> 2^X, <u>then</u> $([U,X,Y], \underset{Uae}{\leq})$ <u>is a well ordered set if and only if</u> U <u>is</u> <u>σ-complete</u>.

Proof. By Theorem 24 on page 66 the set $([U,X,Y], \underset{Uae}{\leq})$ is simply ordered. By Theorem 25 on page 67 it is partially well ordered if and only if U is σ-complete. However, a simply ordered set is well ordered if and only if it is partially well ordered. Hence, $([U,X,Y], \underset{Uae}{\leq})$ is well ordered if and only if U is σ-complete.

If $(Y,\leq)$ is a partially ordered set and F is a nonvoid proper filter in the Boolean algebra 2^X then $([F,X,Y], \underset{Fae}{\leq})$ is a partially ordered set by Corollary 15 on page 63. As usual, we define the relation $\underset{Fae}{<}$ on $[F,X,Y]$ by

$[f] \underset{Fae}{<} [g]$ <u>if and only if</u> $[f] \underset{Fae}{\leq} [g]$ <u>and</u> $[f] \neq [g]$.

In terms of the functions t and s given, respectively, by Statements 46 and 47 on page 59, it follows that

(49) $[f] \underset{Fae}{<} [g]$ <u>if and only if</u>

$t(f,g) \in F$ <u>and</u> $s(f,g) \notin F$.

For every element f and g of Y^X we have $(t(f,g))(s(f,g)) = s(f,g)$ so that $t(f,g) + s(f,g) = (t(f,g))(1 + s(f,g))$. Thus, if $(t(f,g) + s(f,g)) \in F$ then $t(f,g) \in F$ and $s(f,g) \notin F$ and therefore $[f] \underset{Fae}{<} [g]$.

Lemma 12. <u>If</u> $(Y,\leq)$ <u>is a partially ordered set with at least two distinct comparable elements and if</u> F <u>is a non-void proper filter in the Boolean algebra</u> 2^X, <u>then for every element</u> f <u>and</u> g <u>of</u> Y^X

$$([f] \underset{Fae}{<} [g]) \ \underline{\text{implies}} \ (t(f,g) + s(f,g)) \in F$$

<u>if and only if</u> F <u>is an ultrafilter</u>.

Proof. Assume that $([f] \underset{Fae}{<} [g])$ implies that $(t(f,g) + s(f,g)) \in F$ for every element f and g of Y^X. Let y and y' be elements of Y with $y < y'$ and let a be any element of 2^X. Let $g \in Y^X$ be given by

$$g(x) = \begin{cases} y, & \text{if } a(x) = 1 \\ y', & \text{if } a(x) = 0. \end{cases}$$

Then $t((y),g) = 1$ and $s((y),g) = a$ so that $t((y),g) + s((y),g) = 1{+}a$. If $a \notin F$ then $[(y)] \underset{Fae}{<} [g]$ by Statement 49. By hypothesis, $1{+}a \in F$. Hence, for every element a of 2^X either $a \in F$ or $1{+}a \in F$ and, consequently, F is an ultrafilter.

Conversely, if F is an ultrafilter and $[f] \underset{Fae}{<} [g]$ then $t(f,g) \in F$ and $s(f,g) \notin F$ by Statement 49. Thus, $1{+}s(f,g) \in F$ and therefore $(t(f,g))(1 + s(f,g)) \in F$ or $(t(f,g) + s(f,g)) \in F$.

In a partially ordered set $(Y,\leq)$ if y and y' are elements of Y such that $y < y'$ and for every element z of Y

$$y < z \ \underline{\text{and}} \ z \leq y' \ \underline{\text{implies}} \ z = y'$$

then y' is called an <u>immediate successor</u> of y. Clearly, an element y of the set Y may have one immediate successor, more that one immediate successor or no immediate successors.

Lemma 13. *If $(Y,\leq)$ is a partially ordered set, F is a nonvoid proper filter in the Boolean algebra 2^X, and y' is an immediate successor of y in $(Y,\leq)$, then $[(y')]$ is an immediate successor of $[(y)]$ in $([F,X,Y], \underset{Fae}{\leq})$ if and only if F is an ultrafilter.*

Proof. Let F be an ultrafilter in 2^X and let f be any function from X into Y such that $[(y)] \underset{Fae}{<} [f]$ and $[f] \underset{Fae}{\leq} [(y')]$. Clearly, $t(f,(y')) \in F$ and from Lemma 12 we have $(t((y),f) + s((y),f)) \in F$. Hence, $(t(f,(y')))(t((y),f) + s((y),f)) \in F$. Since y' is an immediate successor of y it follows that $s(f,(y')) = (t(f,(y')))(t((y),f) + s((y),f))$. Thus, $[f] = [(y')]$.

On the other hand, if F is not an ultrafilter, then there exists an element $b \in 2^X-F$ such that $1+b \in 2^X-F$. Let g be the function from X into Y given by

$$g(x) = \begin{cases} y, & \text{if } b(x) = 1 \\ y', & \text{if } b(x) = 0. \end{cases}$$

Then, $t((y),g) = 1$ and $s((y),g) = b$ while $t(g,(y')) = 1$ and $s(g,(y')) = 1+b$. Hence, $[(y)] \underset{\text{Fae}}{<} [g]$ and $[g] \underset{\text{Fae}}{<} [(y')]$. Consequently, $[(y')]$ is not an immediate successor of $[(y)]$ in $([F,X,Y], \underset{\text{Fae}}{\leq})$.

For every partially ordered set $(Y,\leq)$ an element m of Y is a <u>minimal</u> element of Y if and only if $y \leq m$ implies $y = m$ for every element y of Y. Similarly, m is a <u>minimum</u> element of Y if and only if $m \leq y$ for every element y of Y. The terms <u>maximal</u> and <u>maximum</u> have their obvious dual meanings.

Lemma 14. <u>If</u> $(Y,\leq)$ <u>is a partially ordered set and</u> F <u>is a nonvoid proper filter in the Boolean algebra</u> 2^X, <u>then for every element</u> y <u>of</u> Y <u>it follows that</u> y <u>is a minimal</u> (<u>minimum, maximal, maximum</u>) <u>element of</u> $(Y,\leq)$ <u>if and only if</u> $[(y)]$ <u>is a minimal</u> (<u>minimum, maximal, maximum</u>) <u>element of</u> $([F,X,Y], \underset{\text{Fae}}{\leq})$.

Proof. If y is minimal in $(Y,\leq)$ and $f \in Y^X$ then $t(f,(y))(x) = 1$ if and only if $f(x) \leq y$ and therefore $t(f,(y)) = s(f,(y))$. Thus, $[f] \underset{\text{Fae}}{\leq} [(y)]$ implies $[f] = [(y)]$ or $[(y)]$ is minimal in $([F,X,Y], \underset{\text{Fae}}{\leq})$. On the other hand, if y is not minimal in $(Y,\leq)$ then $z < y$ for some $z \in Y$. Hence, $[(z)] \underset{\text{Fae}}{<} [(y)]$ and $[(y)]$ is not minimal in $([F,X,Y], \underset{\text{Fae}}{\leq})$.

If y is a minimum element of $(Y,\leq)$ then $y \leq z$ for every element z of Y. Hence, $t((y),f) = 1$ for every $f \in Y^X$ so that $[(y)]$ is a minimum element of $([F,X,Y], \underset{\text{Fae}}{\leq})$. On the other hand, if $y \nleq z$ then $[(y)] \underset{\text{Fae}}{\nleq} [(z)]$. The remainder of the proof is similar.

For every element y of a partially ordered set $(Y,\leq)$ let $I(y)$ denote the <u>initial segment</u> of $(Y,\leq)$ determined by y. That is, let
$I(y) = \{z : z \in Y \text{ and } z < y\}$.

Theorem 27. <u>If</u> $(Y,\leq)$ <u>is a partially ordered set and</u> U <u>is an ultrafilter in the Boolean algebra</u> 2^X <u>then for every element</u> y <u>of</u> Y

$$I([(y)]) = \{[(z)] : z \in I(y)\}$$

if and only if u *is* $\overline{\overline{I(y)}}$*-complete*.

Proof. For every element y of Y and f of Y^X we have $t(f,(y)) + s(f,(y)) = \sup\{s(f,(z)) : z \in I(y)\}$. Thus, if $[f] \in I([(y)])$ and U is $\overline{\overline{I(y)}}$-complete it follows from Lemma 12 on page 70 and Lemma 2 on page 7 that $s(f,(z)) \in U$ or $[f] = [(z)]$ for some $z \in I(y)$. Hence, if U is $\overline{\overline{I(y)}}$-complete then $I([(y)]) = \{[(z)] : z \in I(y)\}$.

Conversely, if U is not $\overline{\overline{I(y)}}$-complete then Theorem 23 on page 63 implies that $[U,X,I(y)] \neq \{[(z)] : z \in I(y)\}$. However, if $f \in I(y)^X$ then $f \in Y^X$ and $[f] \in I([(y)])$. On the other hand, $s(f,(z)) \in 2^X-U$ for every $z \in I(y)$ so that $[f] \notin \{[(z)] : z \in I(y)\}$. Hence, $I([(y)]) = \{[(z)] : z \in I(y)\}$ if and only if U is $\overline{\overline{I(y)}}$-complete.

Corollary 18. *Let* $(Y,\leq)$ *be a partially ordered set*, $\bar{\bar{m}} = \sup_{y \in Y} \overline{\overline{I(y)}}$, *and* U *an* $\bar{\bar{m}}$*-complete ultrafilter in the Boolean algebra* 2^X. *For every* $y \in Y$ *and* $f \in Y^X$ *it follows that* $I([(y)]) = \{[(z)] : z \in I(y)\}$.

Proof. Since U is $\bar{\bar{m}}$-complete it is $\overline{\overline{I(y)}}$-complete for every element y of Y. Hence, the desired result follows from Theorem 27.

If U is an ultrafilter in the Boolean algebra 2^X then we have seen from Lemma 13 on page 72 that immediate successors in $(Y,\leq)$ are preserved in $([U,X,Y], \underset{Uae}{\leq})$ and from Lemma 14 on page 73 that minimal, maximal, minimum and maximum elements are also preserved. Theorem 27 gives a necessary and sufficient condition for an initial segment of $(Y,\leq)$ to be preserved as an initial segment of $([U,X,Y], \underset{Uae}{\leq})$. The following theorems concern the preservation of upper and lower bounds of subsets of Y.

For every subset S of a partially ordered set $(Y,\leq)$ let S^+ be the set of upper bounds of S and let S^- be the set of lower bounds of S (cf. [6, p. 58]). For every element y of Y let $y^+ = \{y\}^+$ and $y^- = \{y\}^-$. Clearly, if $(Y,\leq)$ is partially ordered and $S \subset Y$ then $S^+ = \cap \{s^+ : s \in S\}$ and $S^- = \cap \{s^- : s \in S\}$.

Theorem 28. <u>Let</u> $(Y,\leq)$ <u>be a partially ordered set and</u> U <u>an ultrafilter in the Boolean algebra</u> 2^X. <u>For every subset</u> S <u>of</u> Y <u>and every element</u> y <u>of</u> Y <u>if</u> U <u>is</u> <u>$\overline{\overline{S}}$-complete then</u>

(A). $y = \sup S$ <u>if and only if</u> $[(y)] = \sup_{s \in S} [(s)]$.

(B). y <u>is minimal in</u> S^+ <u>if and only if</u> $[(y)]$ <u>is minimal in</u> $\{[(s)] : s \in S\}^+$.

(C). $y = \inf S$ <u>if and only if</u> $[(y)] = \inf_{s \in S} [(s)]$.

<u>and</u> (D). y <u>is maximal in</u> S^- <u>if and only if</u>

$[(y)]$ <u>is maximal in</u> $\{[(s)] : s \in S\}^-$.

Proof. Clearly, we may assume that $Y \neq \phi$. If $y = \sup S$ then $y \in S^+$ so that $[(y)] \in \{[(s)] : s \in S\}^+$. Let $[f]$ be any upper bound of the set $\{[(s)] : s \in S\}$ and let $r = \inf_{s \in S} t((s),f)$. For every $s \in S$ we have $[(s)] \underset{Uae}{\leq} [f]$ or $t((s),f) \in U$. Since U is $\bar{\bar{S}}$-complete it follows that $r \in U$. But if $r(x) = 1$ then $s \leq f(x)$ for every $s \in S$ or $f(x) \in S^+$. Thus, if $r(x) = 1$ then $\sup S \leq f(x)$ or $y \leq f(x)$ and, consequently, $t((y),f)(x) = 1$. Hence, $r \in U$ implies $t((y),f) \in U$ or $[(y)] \underset{Uae}{\leq} [f]$. Thus, $[(y)] = \sup\{[(s)] : s \in S\}$. On the other hand, if $[(y)] = \sup\{[(s)] : s \in S\}$ and $z \in S^+$ then $[(y)] \underset{Uae}{\leq} [(z)]$ so that $y \leq z$ or $y = \sup S$. Thus, we have proved (A). (C) can be proved analogously,

Similarly, if $[(y)]$ is minimal in $\{[(s)] : s \in S\}^+$ and $z \in S^+$ then $[(z)] \underset{Uae}{\leq} [(y)]$ so that $z \not< y$. Thus, y is minimal in S^+. But if $[f]$ is any upper bound of the set $\{[(s)] : s \in S\}$ such that $[f] \underset{Uae}{<} [(y)]$ then let

$$r = \inf_{s \in S} ((t((s),f))(t(f,(y)) + s(f,(y)))).$$

Since U is $\bar{\bar{S}}$-complete we have $r \in U$. However $r(x) = 1$ implies that $f(x) \in S^+$ and $f(x) < y$. Hence, if $[(y)]$ is not minimal in $\{[(s)] : s \in S\}^+$ then y is not minimal in S^+. Thus, (B) has been proved and (D) follows analogously.

If $(Y, \leq)$ is well ordered then Statement (B) of Theorem 28 reduces to Statement (A) while Statements (C) and (D) are always valid since every nonvoid subset of Y has a minimum element. In the case that $(Y, \leq)$ is well ordered we present below a further condition which implies that $y = \sup S$ if and only if $[(y)] = \sup_{s \in S} [(s)]$. We recall (Theorem 8 of [3, p. 148]) that $cf(v)$ is a cardinal for every ordinal v.

Theorem 29. <u>Let</u> $(Y, \leq)$ <u>be an ordinal and let</u> U <u>be an ultrafilter in the Boolean algebra</u> 2^X. <u>For every element</u> y <u>of</u> Y <u>and every subset</u> S <u>of</u> Y <u>if</u> U <u>is</u> $\bar{\bar{m}}$<u>-complete for every</u> $\bar{\bar{m}} < cf(y)$ <u>then the following statements are equivalent</u>.

(A). $y = \sup S$ <u>if and only if</u> $[(y)] = \sup_{s \in S} [(s)]$

(B). $y \in S$ <u>or</u> U <u>is</u> $cf(y)$<u>-complete</u>.

Proof. We assume that $Y \neq \emptyset$. Clearly, if $[(y)] = \sup_{s \in S} [(s)]$ then $y = \sup S$. Further, if $y \in S$ and

$y = \sup S$ then $[(y)] \in \{[(s)] : s \in S\} \cap \{[(s)] : s \in S\}^+$ so that $[(y)] = \sup_{s \in S} [(s)]$. Hence, the condition $y \in S$ implies (A).

Next, we assume that $y = \sup S$ and $y \notin S$. Clearly, we may assume that $y > 0$ since if $y = 0$ then $S = \{[(s)] : s \in S\} = \emptyset$ and U is cf(0)-complete. But if $y > 0$ and $y \notin S$ they y and $cf(y)$ are limit ordinals. Let g be a cofinality map from $cf(y)$ into y and define the map h by $h(w)$ is the first element of S such that $g(w) \leq h(w)$ and if $v < w$ then $h(v) < h(w)$. Then h is a cofinality map from $cf(y)$ into y such that if $H = \{h(w) : w < cf(y)\}$ then $H \subset S$. Clearly, $\bar{\bar{H}} = cf(y)$ and $y = \sup H$. Thus, if U is $cf(y)$-complete then $[(y)] = \sup_{s \in H} [(s)]$ by Theorem 28 on page 76. Since $S^+ \subset H^+$ and $[(y)] \in S^+$ and $[(y)]$ is a minimum element of H^+ we have $[(y)] = \sup_{s \in S} [(s)]$. Hence, (B) implies (A).

Conversely, if U is not $cf(y)$-complete let D be an orthogonal subset of $2^X - U$ such that $\bar{\bar{D}} = cf(y)$ and $\sup D = 1$. Well order D such that $D = \{d_w : w < cf(y)\}$ and $v \neq w$ implies $d_v \neq d_w$. For every element x of X let

$$f(x) = h(w) \text{ \underline{if and only if} } d_w(x) = 1.$$

Then f is a function from X into Y with $f(x) \varepsilon S$ for every $x \varepsilon X$. Hence, $f(x) < y$ for every element x of X or $t(f,(y)) + s(f,(y)) = 1$. Hence, $[f] \underset{Uae}{<} [(y)]$. On the other hand, for every $w < cf(y)$ we have $t((h(w)),f) = 1 + \sup_{v \leq w} d_v$. But if $w < cf(y)$ then $\bar{\bar{w}} < cf(y)$ so that $\sup_{v \leq w} d_v \varepsilon 2^X - U$ for every $w < cf(y)$. Thus, $t((h(w)),f) \varepsilon U$ for every $w < cf(y)$ or $[f]$ is an upper bound of $\{[(s)] : s \varepsilon H\}$. But for every $s \varepsilon S$ there exists $s' \varepsilon H$ with $s \leq s'$ and therefore $[f]$ is an upper bound of $\{[(s)] : s \varepsilon S\}$ with $[f] \underset{Uae}{<} [(y)]$. Hence, $[(y)] \neq \sup\{[(s)] : s \varepsilon S\}$ and the theorem is proved.

C. Measurability and the Almost Everywhere Order

We recall that a cardinal $\bar{\bar{c}} > \aleph_0$ is called <u>measurable</u> if and only if there exists an ultrafilter U in the Boolean algebra $2^{\bar{\bar{c}}}$ which is σ-complete and which contains no atoms of $2^{\bar{\bar{c}}}$. Based on the results of Section V. B we prove the following theorem.

Theorem 30. *If* $\bar{\bar{c}}$ *is the first measurable cardinal and* U *is a* σ*-complete ultrafilter in the Boolean algebra* $2^{\bar{\bar{c}}}$ *which contains no atoms of* $2^{\bar{\bar{c}}}$ *then* $\{[(v)] : v < \bar{\bar{c}}\}$ *is a proper initial segment of the well ordered set* $([U,\bar{\bar{c}},\bar{\bar{c}}], \underset{Uae}{\leq})$.

Proof. Since U is σ-complete it follows from Theorem 26 on page 69 that $([U,\bar{\bar{c}},\bar{\bar{c}}], \underset{Uae}{\leq})$ is well ordered. Since the unit of $2^{\bar{\bar{c}}}$ is the supremum of the set of $\bar{\bar{c}}$ atoms we have that U is not $\bar{\bar{c}}$-complete. Hence, $[U,\bar{\bar{c}},\bar{\bar{c}}] \neq \{[(v)] : v < \bar{\bar{c}}\}$ by Theorem 23 on page 63. On the other hand, if $v < \bar{\bar{c}}$ then $I(v) = v$ so that $\overline{\overline{I(v)}} < \bar{\bar{c}}$. Consequently, Theorem 3 of [13, p. 318] implies that U is $\overline{\overline{I(v)}}$-complete. From Theorem 27 on page 74 it follows that $I([(v)]) = \{[(u)] : u \in I(v)\}$ for every $v < \bar{\bar{c}}$. Let $[f]$ be the first element of $[U,\bar{\bar{c}},\bar{\bar{c}}] - \{[(v)] : v < \bar{\bar{c}}\}$. Clearly, $I([f]) \subset \{[(v)] : v < \bar{\bar{c}}\}$. For every $v < \bar{\bar{c}}$ we have $[f] \underset{Uae}{\not\leq} [(v+1)]$ so that $[(v+1)] \underset{Uae}{\leq} [f]$ or $[(v)] \underset{Uae}{<} [f]$. Hence, $[(v)] \in I([f])$ for every $v < \bar{\bar{c}}$ or $I([f]) = \{[(v)] : v < c\}$.

The concept of measurability may be generalized in the following manner. If $\bar{\bar{c}} > \aleph_0$ and $\bar{\bar{m}}$ is any cardinal then $\bar{\bar{c}}$ is *$\bar{\bar{m}}$-measurable* if and only if there exists an $\bar{\bar{m}}$-complete ultrafilter U in the Boolean algebra $2^{\bar{\bar{c}}}$ which contains no atoms of $2^{\bar{\bar{c}}}$.

Lemma 15. <u>If</u> U <u>is an $\bar{\bar{m}}$-complete ultrafilter in the Boolean algebra</u> $2^{\bar{\bar{c}}}$ <u>and</u> $\bar{\bar{d}}$ <u>is not $\bar{\bar{m}}$-measurable then</u> U <u>is $\bar{\bar{d}}$-complete</u>.

Proof. If U contains an atom of $2^{\bar{\bar{c}}}$ then U is complete. If U contains no atoms of $2^{\bar{\bar{c}}}$ then assume that U is not $\bar{\bar{d}}$-measurable and let $D = \{d_i : i < \bar{\bar{d}}\}$ be an orthogonal subset of $2^X - U$ such that $\sup D = 1$. If $C = \{\sup A : A \subset D\}$ then C is a subalgebra of $2^{\bar{\bar{c}}}$ which is isomorphic to $2^{\bar{\bar{d}}}$. Since U is $\bar{\bar{m}}$-complete and C is complete it follows that $U \cap C$ is an $\bar{\bar{m}}$-complete ultrafilter in C. But D is the set of atoms of C so that $U \cap C$ contains no atoms of C. Consequently, there is an $\bar{\bar{m}}$-complete ultrafilter V in the Boolean algebra $2^{\bar{\bar{d}}}$. However, this contradicts the hypothesis that $\bar{\bar{d}}$ is not $\bar{\bar{m}}$-measurable. Hence, it follows that U is $\bar{\bar{d}}$-measurable.

Now we may extend Theorem 30 as follows.

Theorem 31. <u>If</u> $\bar{\bar{m}} \geq \aleph_0$ <u>and</u> $\bar{\bar{c}}$ <u>is the first $\bar{\bar{m}}$-measurable cardinal and</u> U <u>is an $\bar{\bar{m}}$-complete ultrafilter in the Boolean algebra</u> $2^{\bar{\bar{c}}}$ <u>which contains no atoms of</u> $2^{\bar{\bar{c}}}$ <u>then</u> $\{[(v)] : v < \bar{\bar{c}}\}$ <u>is a proper initial segment of the well ordered set</u> $([U, \bar{\bar{c}}, \bar{\bar{c}}], \underset{Uae}{\leq})$.

Proof. Theorem 26 on page 69 implies that $([U,\bar{\bar{c}},\bar{\bar{c}}], \underset{Uae}{\leq})$ is well ordered. Since U is not $\bar{\bar{c}}$-complete it follows from Theorem 23 on page 63 that $\{[(v)] : v < \bar{\bar{c}}\}$ is a proper subset of $[U,\bar{\bar{c}},\bar{\bar{c}}]$. For every $w < \bar{\bar{c}}$ Theorem 27 on page 74 and Lemma 15 above imply that $I([(w)]) \subset \{[(v)] : v < \bar{\bar{c}}\}$. Thus, if $[f]$ is the first element of $[U,\bar{\bar{c}},\bar{\bar{c}}] - \{[(v)] : v < \bar{\bar{c}}\}$ then $I([f]) = \{[(v)] : v < \bar{\bar{c}}\}$ so that $\{[(v)] : v < \bar{\bar{c}}\}$ is a proper initial segment of the well ordered set $([U,\bar{\bar{c}},\bar{\bar{c}}], \underset{Uae}{\leq})$.

We conclude by mentioning that based on Theorem 31 it is possible to prove (cf. [10]): if $\bar{\bar{m}} \geq \aleph_0$ and $\bar{\bar{c}}$ is the first $\bar{\bar{m}}$-measurable cardinal then there exist $\bar{\bar{c}}$ strongly inaccessible cardinals preceding $\bar{\bar{c}}$.

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