SPECTRALLY ARBITRARY PATTERNS: REDUCIBILITY AND THE $2n$ CONJECTURE FOR $n = 5$

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Abstract. A sign pattern Z (a matrix whose entries are elements of $\{+,-,0\}$) is spectrally arbitrary if for any self-conjugate spectrum there is a real matrix with sign pattern Z having the given spectrum. Spectrally arbitrary sign patterns were introduced in [5], where it was (incorrectly) stated that if a sign pattern Z is reducible and each of its irreducible components is a spectrally arbitrary sign pattern, then Z is a spectrally arbitrary sign pattern, and it was conjectured that the converse is true as well; we present counterexamples to both of these statements. In [2] it was conjectured that any $n \times n$ spectrally arbitrary sign pattern must have at least $2n$ nonzero entries; we establish that this conjecture is true for 5×5 sign patterns. We also establish analogous results for nonzero patterns.

Key words. Sign pattern, nonzero pattern, spectrally arbitrary sign pattern, reducible sign pattern, irreducible sign pattern, potentially nilpotent

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1. Introduction.

A *sign pattern* $Z = [z_{ij}]$ is a square matrix whose entries z_{ij} are elements of $\{+,-,0\}$. Given a real matrix $A = [a_{ij}]$, let $\mathcal{Z}(A) = [z_{ij}]$ be the sign pattern where $z_{ij} = sgn(a_{ij})$. The *qualitative class* of Z is $\mathcal{Q}(Z) = \{A : \mathcal{Z}(A) = Z\}$. The study of sign patterns arose more than fifty years ago in economics. Brualdi and Shader [1] provide a thorough mathematical treatment of sign patterns through 1995. For a current survey with an extensive bibliography, see Hall and Li [6]. A *nonzero pattern* $Z = [z_{ij}]$ is a square matrix whose entries z_{ij} are elements of $\{*, 0\}$. A nonzero pattern with k nonzero entries describes the 2^k sign patterns obtained by replacing each $*$ by $+$ or $-$; the *qualitative class* of a nonzero pattern Z is $\mathcal{Q}(Z) = \{A : a_{ij} \neq 0 \Leftrightarrow z_{ij} = *\}$. We will use the term *pattern* to mean either a sign pattern or a nonzero pattern, and *order* n pattern to mean an $n \times n$ pattern.

An order n pattern Z is a *spectrally arbitrary pattern* (*SAP*) if given any monic polynomial $q(x)$ of degree n with real coefficients, there exists a real matrix $A \in \mathcal{Q}(Z)$ such that the characteristic polynomial $p_A(x)$ of A is equal to $q(x)$ (note that necessarily $n \geq 2$). Equivalently, Z is spectrally arbitrary if given any self-conjugate multi-set σ of n complex numbers, there exists a real matrix $A \in \mathcal{Q}(Z)$ such that σ is the spectrum of A.

An order n pattern Z is *potentially nilpotent* (or *allows nilpotence*) if there exists a real matrix $A \in \mathcal{Q}(Z)$ such that A is nilpotent, i.e., $A^n = 0$. A spectrally arbitrary sign pattern is potentially nilpotent, but not conversely.

A pattern Z of order $n \geq 2$ is *reducible* provided for some integer r with $1 \leq r \leq n-1$, there exists an $r \times (n-r)$ zero submatrix that does not meet the main diagonal of Z, that is, there is a permutation matrix P such that

$$PZP^T = \begin{bmatrix} X & Y \\ O_{r,n-r} & W \end{bmatrix}.$$

Z is *irreducible* provided that Z is not reducible. A *Frobenius normal form* of Z is a block upper triangular matrix with irreducible diagonal blocks that is permutationally similar to Z; the diagonal blocks are called the *irreducible components* of Z. Analogous definitions are given for real matrices. If a reducible matrix A

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has irreducible components $A_1, \ldots, A_h$, then $p_A(x) = \Pi_{i=1}^h p_{A_i}(x) = p_{A_1 \oplus \cdots \oplus A_h}(x)$. Thus a reducible pattern is spectrally arbitrary if and only if the direct sum of its irreducible components is spectrally arbitrary.

Spectrally arbitrary sign patterns were introduced in [5], where it was stated that if a sign pattern Z is reducible and each of its irreducible components is a spectrally arbitrary sign pattern, then Z is a spectrally arbitrary sign pattern, and it was conjectured that the converse is true as well. In Section 2 we exhibit counterexamples to both of these statements.

There has been considerable interest recently in spectrally arbitrary sign patterns. Much of the work has focused on minimal spectrally arbitrary sign patterns (see, e.g., [2]). In [2] it was established that any irreducible order n spectrally arbitrary sign pattern must have at least $2n-1$ nonzero entries and conjectured that any spectrally arbitrary sign pattern must have at least $2n$ nonzero entries. (This is known as the $2n$-*conjecture*.) In [2], and also [3], the order 3 spectrally arbitrary sign patterns were classified and demonstrated to have at least six nonzero entries. In [4] it is shown that every spectrally arbitrary order 4 nonzero pattern must have at least eight nonzero entries. Thus the $2n$-conjecture is established for sign patterns of order at most 4; we establish the $2n$-conjecture for nonzero patterns of order 5, and hence for sign patterns of order 5.

For an $n \times n$ matrix A, the sum of the $k \times k$ principal minors is denoted $S_k(A)$. Note that $p_A(x) = x^n - S_1(A)x^{n-1} + \cdots + (-1)^n S_n(A)$. For a given k, a sign pattern Z is S_k-*sign-arbitrary* if there exist matrices A_+, A_0, and $A_- \in \mathcal{Q}(Z)$ such that $S_k(A_+) > 0, S_k(A_0) = 0$, and $S_k(A_-) < 0$. For an order n pattern Z to be spectrally arbitrary, it is necessary (but not sufficient [3]) that Z be S_k-sign-arbitrary for all $k = 1, \ldots, n$. For a given k, a pattern Z is S_k-*znz-arbitrary* if there exist matrices $A_*, A_0 \in \mathcal{Q}(Z)$ such that $S_k(A_*) \neq 0$, and $S_k(A_0) = 0$. Any S_k-sign-arbitrary pattern is necessarily S_k-znz-arbitrary. If Z is S_1-znz-arbitrary or S_n-znz-arbitrary, then we say Z has *znz-arbitrary trace* or *znz-arbitrary determinant*, respectively.

Digraphs and especially permutation digraphs are useful in analyzing whether a sign pattern is S_k-znz-arbitrary. A *digraph* is a directed graph; a digraph allows loops (1-*cycles*) but does not allow multiple edges. A directed edge is called an *arc* and denoted as an ordered pair, (v, w) or (v, v). If $v \neq w$, a digraph is permitted to have both of the arcs (v, w) and (w, v), and this pair of arcs is a 2-*cycle*, denoted by (vw) or (wv). More generally, the k-*cycle* or *cycle* $(v_1 v_2 \ldots v_k)$ is the sequence of arcs $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k), (v_k, v_1)$ with $v_1, v_2, \ldots, v_{k-1}, v_k$ distinct. The digraph of an order n pattern Z, denoted $\Gamma(Z) = (V, E)$, is the digraph having $V = \{1, \ldots, n\}$ and $E = \{(i, j) : z_{ij} \neq 0\}$. The digraph of a matrix is defined analogously. A digraph is *strongly connected* if for each vertex v and every other vertex $w \neq v$, there is a (correctly oriented) path from v to w. A pattern or matrix is irreducible if and only if its digraph is strongly connected.

Let D be a digraph. To *reverse* arc (v, w) means to replace it by arc (w, v). The digraph obtained from D by reversing all the arcs of D will be denoted by D^T. Note that for a pattern Z, $\Gamma(Z)^T = \Gamma(Z^T)$. Nonzero patterns Z_1 and Z_2 are permutationally similar if and only if their digraphs $\Gamma(Z_1)$ and $\Gamma(Z_2)$ are isomorphic. Nonzero patterns Z_1 and Z_2 are *equivalent* if Z_1 is permutationally similar to Z_2 or Z_2^T. Nonzero patterns are customarily classified up to equivalence; this is the same as classifying digraphs up to isomorphism and arc reversal, so we say two digraphs D_1 and D_2 are *equivalent* if D_1 is isomorphic to D_2 or D_2^T. When an unlabeled digraph diagram is used, the digraph is being described up to isomorphism.

Let D be a digraph of order n. A digraph P is an *order k permutation digraph of D* (for $1 \leq k \leq n$) if P has k vertices, every arc of P is an arc of D, and the set of arcs of P is a union of one or more disjoint cycles. For an order k permutation digraph P, $\pi(P)$ denotes the permutation (of a subset of $\{1, \ldots, n\}$ of cardinality k) consisting of the cycles in P. Let $\mathrm{perm}_k(D)$ denote the set of all permutations $\pi(P)$ such that P is an order k permutation digraph of D. If $A = [a_{ij}]$ is an $n \times n$ matrix, then

$$S_k(A) = \sum_{\pi \in \mathrm{perm}_k(\Gamma(A))} \mathrm{sgn}(\pi) a_{i_1 \pi(i_1)} \ldots a_{i_k \pi(i_k)},$$

where the sum over the empty set is zero. It follows that an S_k-znz-arbitrary pattern must have at least 2 permutation digraphs of order k, and thus that a spectrally arbitrary pattern must have at least 2 permutation digraphs of order k for all $k = 1, \ldots, n$.

A sign pattern Z can also be associated with a simple (undirected) graph by first constructing the digraph $\Gamma(Z)$ of the pattern, removing loops, and replacing an arc or 2-cycle by a single edge; this graph is denoted by $\mathcal{G}(Z)$.

2. Reducibility and spectrally arbitrary patterns. First we describe when a direct sum of spectrally arbitrary sign patterns is a spectrally arbitrary sign pattern and give an example to show that the direct sum of two spectrally arbitrary sign patterns is not necessarily spectrally arbitrary.

PROPOSITION 2.1. *The direct sum of sign patterns of which at least two are of odd order is not an SAP. Furthermore if the direct sum of spectrally arbitrary sign patterns has at most one odd order summand, then the direct sum is an SAP.*

Proof. Let $Z = Z_1 \oplus \cdots \oplus Z_n$. Let $A \in \mathcal{Q}(Z)$; then $A = A_1 \oplus \cdots \oplus A_n$, where $A_i \in \mathcal{Q}(Z_i), i = 1, \ldots, n$. If the direct sum Z has at least two odd order summands Z_i, then $A \in \mathcal{Q}(Z)$ must have at least two real eigenvalues and hence Z is not spectrally arbitrary.

It remains to show that if Z is a direct sum of SAPs and has at most one odd order summand, then Z is an SAP. Let the order of Z be m. Observe that any monic real polynomial $p(x)$ of degree m may be factored over the reals into a product of monic irreducible quadratic and linear factors. We denote irreducible quadratic factors by f_j and linear factors by g_j. Then $p(x) = f_1 f_2 \cdots f_k g_1 g_2 \ldots g_l$, where $2k + l = m$. Let Z_i have order m_i, so that $m_1 + \cdots + m_n = m$. Then assign to each summand Z_i of even order a product of elements from a subset of $\{f_1, f_2, \ldots, f_k, g_1, g_2, \ldots, g_l\}$ with degree m_i. If Z has an odd order summand (and thus the order of Z is odd), then assign to it the product of all remaining factors. Each Z_i is an SAP, so there is some $A_i \in \mathcal{Q}(Z_i)$ such that $p_{A_i}(x)$ realizes the polynomial assigned to Z_i. By construction, $p_{A_1 \oplus \cdots \oplus A_n}(x) = p(x)$. □

For example, $T_3 = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & + \end{bmatrix}$ is an SAP [5], but $T_3 \oplus T_3$ is not an SAP. For instance, $(1+x^2)^3$ cannot be realized as the characteristic polynomial of any matrix in $\mathcal{Q}(T_3 \oplus T_3)$.

PROPOSITION 2.2. *The sign pattern*

$$M_4 = \begin{bmatrix} + & + & - & 0 \\ - & - & + & 0 \\ 0 & 0 & 0 & - \\ + & + & 0 & 0 \end{bmatrix}$$

is not an SAP (see also [4, Appendix C]). Moreover, M_4 realizes every characteristic polynomial $x^4 + b_3 x^3 + b_2 x^2 + b_1 x + b_0$ except those of the following form:

1. *$x^4 + b_3 x^3 + b_2 x^2$, where $b_3^2 - 4b_2 < 0$*
2. *$x^4 + b_3 x^3 + b_2 x^2 + b_0$, where $b_0 < 0$ and $b_3^2 - 4b_2 \le 0$*

Proof. Consider the family of matrices B of the form

$$B = \begin{bmatrix} a & b & -c & 0 \\ -d & -e & f & 0 \\ 0 & 0 & 0 & -g \\ h & k & 0 & 0 \end{bmatrix}$$

where variables $a, b, c, d, e, f, g, h, k$ can assume arbitrary positive values, so $\mathcal{Z}(B) = M_4$. Using a positive diagonal similarity, we can assume that variables b, c and g equal to 1, and hence

$$p_B(x) = x^4 + (e-a)x^3 + (d - ae)x^2 + (fk - h)x + (fh + dk - eh - afk).$$

Consider the system

$$\begin{aligned} b_0 &= fh + dk - eh - afk \\ b_1 &= fk - h \\ b_2 &= d - ae \\ b_3 &= e - a \end{aligned} \tag{2.1}$$

where a, d, e, f, h, k are unknowns. We need to determine those values of b_0, b_1, b_2 and b_3 for which this system has a solution where the unknowns are positive. Note that $e = a + b_3$, $d = b_2 + a(a + b_3)$, and $h = fk - b_1$. Substituting these into the first equation from (2.1) we get:

$$b_0 = f^2k - fb_1 + b_2k + a^2k + ab_3k - afk + ab_1 - b_3fk + b_3b_1 - afk,$$

and solving for k we obtain

$$k\left[(a-f)^2+b_3(a-f)+b_2\right]=b_0-b_1((a-f)+b_3). \tag{2.2}$$

We treat a and f as free variables and all other variables as defined by b_0, b_1, b_2, b_3, and a and f. To find the set of coefficients b_0, b_1, b_2, b_3 for which a positive solution exists, consider four cases.
Case 1. Suppose $b_1 \neq 0$. By (2.2), with values chosen so that the denominator is nonzero,

$$k=\frac{b_0-b_1((a-f)+b_3)}{(a-f)^2+b_3(a-f)+b_2} \tag{2.3}$$

Choose positive values a and f so that $b_0-b_1((a-f)+b_3)>0$ and $(a-f)^2+b_3(a-f)+b_2>0$. It is always possible to choose such a and f, since the first inequality has a solution for which either $a-f\in(-\infty, \frac{b_0-b_1b_3}{b_1})$ if $b_1>0$ or $a-f\in(\frac{b_0-b_1b_3}{b_1},\infty)$ if $b_1<0$. The second inequality is quadratic with respect to $a-f$ and has a positive leading coefficient, so it is satisfied for $|a-f|$ big enough. Fix $a-f=\delta$, satisfying the above inequalities; therefore we have defined $k>0$. Now k is fixed according to the difference between a and f. To guarantee a positive solution for the system (2.1), choose a sufficiently large so that

$$\begin{array}{lll} a>0, & & \\ a>\delta & \text{(hence } f>0), & \\ a>-b_3 & \text{(hence } e>0), & \\ a^2+b_3a+b_2>0 & \text{(hence } d>0), & \text{and} \\ a>\frac{b_1}{k}+\delta & \text{(hence } h>0). & \end{array} \tag{2.4}$$

Case 2. Let $b_1=0$, $b_0>0$. In this situation the numerator of (2.3) is positive. As in the previous case, choose $a-f=\delta$ satisfying $(a-f)^2+b_3(a-f)+b_2>0$, and find a satisfying the inequalities in (2.4).
Case 3. Let $b_1=0$, $b_0=0$. In this case equation (2.2) becomes $\left[(a-f)^2+b_3(a-f)+b_2\right]k=0$. The existence of a solution $k>0$ (in fact, the existence of a solution $k\neq 0$) is equivalent to requiring the above coefficient of k to be equal to 0. This is possible if and only if the quadratic equation $x^2+b_3x+b_2=0$ has a real root, i.e. $b_3^2-4b_2\geq 0$. In the case that this is satisfied, let δ be a real root, then fix $a-f=\delta$, choose arbitrary $k>0$, fix it, and proceed by choosing a satisfying the inequalities (2.4). If the condition $b_3^2-4b_2\geq 0$ is not satisfied, any values of a and f will force k to be equal to 0; therefore the system (2.1) does not have a positive solution, i.e the polynomial with given coefficients is not realizable by M_4.
Case 4. Let $b_1=0$, $b_0<0$. In this case the numerator in the equation for k (2.3) is negative, so we need to choose $\delta=a-f$ such that $(a-f)^2+b_3(a-f)+b_2<0$. It is possible if and only if $b_3^2-4b_2>0$. If this condition is satisfied, find δ and choose a satisfying conditions (2.4). Otherwise, $(a-f)^2+b_3(a-f)+b_2\geq 0$ for all values of a and f, and this forces k to be negative or undefined for any choice of a, f. Therefore this set of coefficients is also not realizable by M_4. $\square$

Corollary 2.3. *The polynomial $p(x)=x^4+b_3x^3+b_2x^2+b_1x+b_0$ can be realized as the characteristic polynomial of a matrix in $\mathcal{Q}(M_4)$ if $b_0>0$, or if $b_0=0$ and $p(x)$ has four real roots.*

Notice that it is the position of the nonzero entries, rather than their signs, that causes M_4 to fail to realize certain polynomials; the nonzero pattern

$$\begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \\ * & * & 0 & 0 \end{bmatrix}$$

derived from M_4 cannot realize x^4+x^2 and so is not spectrally arbitrary either.

It was demonstrated in [5] that $T_2=\begin{bmatrix} - & + \\ - & + \end{bmatrix}$ is a spectrally arbitrary pattern.

Proposition 2.4. *There exists a spectrally arbitrary sign pattern whose direct summands are not both spectrally arbitrary. Specifically $M_4\oplus T_2$ is an SAP, while M_4 is not an SAP.*

Proof. We may write a given degree six monic polynomial in one of the following forms:

$$p(x) = \begin{cases} g_1g_2g_3g_4g_5g_6 \\ g_1g_2g_3g_4f_1 \\ g_1g_2f_1f_2 \\ f_1f_2f_3 \end{cases}$$

where each f_i is a monic irreducible quadratic factor and each g_i is a monic linear factor.

We obtain a matrix $A = A_1 \oplus A_2 \in \mathcal{Q}(M_4 \oplus T_2)$ with $p_A(x) = p(x)$ by finding a subset of the factors whose product can be realized as the characteristic polynomial of a matrix $A_1 \in \mathcal{Q}(M_4)$ and, since T_2 is an SAP, there will be a matrix $A_2 \in \mathcal{Q}(T_2)$ having the product of the remaining factor(s) as its characteristic polynomial. Note that since each f_i is assumed to be monic and irreducible, the constant of each f_i must be positive. We use Corollary 2.3 in the following cases:

Case 1. If $p(x)$ is a product of linear factors we can always choose four of the factors such that their product has a nonnegative constant term. Thus the product can be realized as the characteristic polynomial of some matrix in $\mathcal{Q}(M_4)$.

Case 2. Suppose $p(x)$ has four linear factors and one quadratic factor. If $g_i = x$ for some i, the product of the g_i can be realized by a matrix in $\mathcal{Q}(M_4)$. Otherwise, choose two g_i such that the product of their constant terms is positive. The product of these factors with f_1 can be realized as the characteristic polynomial of a matrix in $\mathcal{Q}(M_4)$.

Case 3. When $p(x) = f_1f_2g_1g_2$ or $p(x) = f_1f_2f_3$, we realize f_1f_2 as the characteristic polynomial of a matrix in $\mathcal{Q}(M_4)$. ∎

3. Reducibility and the $2n$ conjecture. In this section we develop results about reducible patterns and techniques that will be used to show, via graph classification, that any order 5 SAP must have at least ten nonzero entries, thereby establishing the $2n$ conjecture for patterns of order 5. The results in this section also lay some groundwork for any future attempt at establishing the $2n$-conjecture for order 6 patterns by graph classification.

Note that if pattern Z has znz-arbitrary trace, $\Gamma(Z)$ has at least two loops. Since any order n tree has $n-1$ edges, if the graph of a pattern is a strongly connected tree with two loops, the pattern must have $2n$ nonzero entries:

Proposition 3.1. *If an irreducible order n pattern Z has znz-arbitrary trace and $\mathcal{G}(Z)$ is a tree, then Z has at least $2n$ nonzero entries.*

Lemma 3.2. *If the pattern Z has znz-arbitrary trace and is potentially nilpotent, then $\Gamma(Z)$ must have a 2-cycle.*

Proof. Suppose the digraph of Z has no 2-cycle and $h \geq 2$ loops, at vertices $v_1, \ldots, v_h$. Let $A \in \mathcal{Q}(Z)$, and denote $a_{v_iv_i}$ by a_i. Then $S_1(A) = \sum_{i=1}^h a_i$ and $S_2(A) = \sum_{1\leq i<j\leq h}^h a_ia_j$. If $S_1(A) = 0$ then

$$S_2(A) = \frac{1}{2}\left[\left(\sum_{i=1}^h a_i\right)^2 - \sum_{i=1}^h a_i^2\right] < 0.$$

Thus Z is not potentially nilpotent. ∎

Since an order n SAP must allow the characteristic polynomial $(x-\lambda)^n$ for any real λ, any order m irreducible component Z of an SAP must allow the characteristic polynomial $(x-\lambda)^m$. By considering $(x-1)^m, (x-0)^m$, we see that Z must be S_k-znz-arbitrary for $k = 1, \ldots, m$ and so $\Gamma(Z)$ must have at least two order k permutation digraphs for $k = 1, \ldots, m$. In particular, we have the following.

Lemma 3.3. *Any order 2 irreducible component of an SAP must have four nonzero entries.*

Lemma 3.4. *Any irreducible order 3 pattern that has znz-arbitrary trace and znz-arbitrary determinant must have at least six nonzero entries. Any order 3 irreducible component of an SAP must have at least six nonzero entries.*

Proof. Let Z be an irreducible order 3 pattern that has znz-arbitrary trace and znz-arbitrary determinant. Then by Proposition 3.1, if $\mathcal{G}(Z)$ is a tree, Z must have at least six nonzero entries. If $\mathcal{G}(Z)$ is not a tree,

$\Gamma(Z)$ must contain a 3-cycle. Znz-arbitrary trace requires 2 loops, and to have less than six arcs, there must be exactly two loops and no 2-cycles. Then there is exactly one order 3 permutation digraph in $\Gamma(Z)$; znz-arbitrary determinant requires at least two order 3 permutation digraphs in $\Gamma(Z)$.

The second part follows since any order 3 irreducible component Z of an SAP must have znz-arbitrary trace and znz-arbitrary determinant. □

Note that both Lemmas 3.3 and 3.4, and Lemma 3.7 below, refer to an irreducible component of an SAP rather than to an SAP itself, and so are stronger than previous results asserting the truth of the $2n$-conjecture for $n = 2, 3, 4$, cf. [2], [3], [4].

PROPOSITION 3.5. *Any order* 5 *reducible SAP must have at least ten nonzero entries.*

Proof. A reducible order 5 SAP must have irreducible components of order 2 and order 3; if there is an order 1 irreducible component, the pattern will not be an SAP. By Lemmas 3.4 and 3.3, the entire pattern must have at least ten nonzero entries. □

LEMMA 3.6. *If Z is an order $n > 2$ irreducible component of an SAP and $\Gamma(Z)$ has exactly one* 2*-cycle and exactly two loops, then $\Gamma(Z)$ must have a* 3*-cycle. Further, unless exactly one loop is incident to the* 2*-cycle, $\Gamma(Z)$ must have at least two* 3*-cycles.*

Proof. If at least one of the loops is on a vertex of the 2-cycle, then there must be a 3-cycle to provide a second order 3 permutation digraph. If both loops are incident to the 2-cycle, then $\Gamma(Z)$ must have at least two 3-cycles.

Now suppose loops are at vertices r, s disjoint from the 2-cycle (ij), and let $A \in \mathcal{Q}(Z)$. Then $S_1(A) = a_{rr} + a_{ss}$, $S_2(A) = a_{rr}a_{ss} - a_{ij}a_{ji}$, and $S_3(A) = (\text{any 3-cycle products}) - S_1(A)a_{ij}a_{ji}$. If there is exactly one 3-cycle then $S_1(A) = 0$ forces $S_3(A) \neq 0$; thus Z is not potentially nilpotent.

Suppose $\Gamma(Z)$ does not have a 3-cycle. In order to realize the polynomial $(x-1)^n$ we would need $S_1(A) = n$, $S_2(A) = \binom{n}{2}$ and $S_3(A) = \binom{n}{3}$. Considering $S_1(A)$ and $S_3(A)$, we get $a_{ij}a_{ji} = -\frac{(n-1)(n-2)}{6}$; hence, using $S_2(A)$, $a_{rr}a_{ss} = \frac{1}{3}(n^2-1)$. Since $a_{rr} + a_{ss} = n$, a_{rr} and a_{ss} are roots of the function $f(x) = x^2 - nx + \frac{1}{3}(n^2-1)$, which has no real roots for $n > 2$. Therefore the polynomial $(x-1)^n$ is not realizable. □

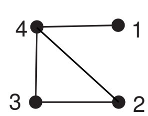
In the next section, a graph classification technique is used to establish the $2n$ conjecture for order 5 patterns; the following two results may be useful if one wishes to use the same techniques to establish the $2n$ conjecture for higher order patterns.

LEMMA 3.7. *Any order* 4 *irreducible component of an SAP must have at least eight nonzero entries.*

Proof. Let Z be an order 4 irreducible component of an SAP; Z must have znz-arbitrary trace and be potentially nilpotent. Therefore, $\Gamma(Z)$ must be strongly connected, have at least two loops, and by Lemma 3.2, have a 2-cycle.

Now suppose Z has less than eight nonzero entries. By Proposition 3.1, $\mathcal{G}(Z)$ cannot be a tree, so $\mathcal{G}(Z)$ has at least four edges. Thus, $\mathcal{G}(Z)$ has exactly 4 edges, two loops and one two cycle, since it has at most seven nonzero entries. By Lemma 3.6, $\mathcal{G}(Z)$ must have a 3-cycle. So, (up to isomorphism) the only one possible graph for $\mathcal{G}(Z)$ is the graph G_1 shown in Figure 3.1.

FIG. 3.1. *The graph G_1*



Assuming $\mathcal{G}(Z) = G_1$, the 2-cycle (14) is required in the digraph $\Gamma(Z)$ by strong connectivity, and any placement of two loops cannot create more than one permutation digraph of order 4. Thus $\mathcal{G}(Z) \neq G_1$. □

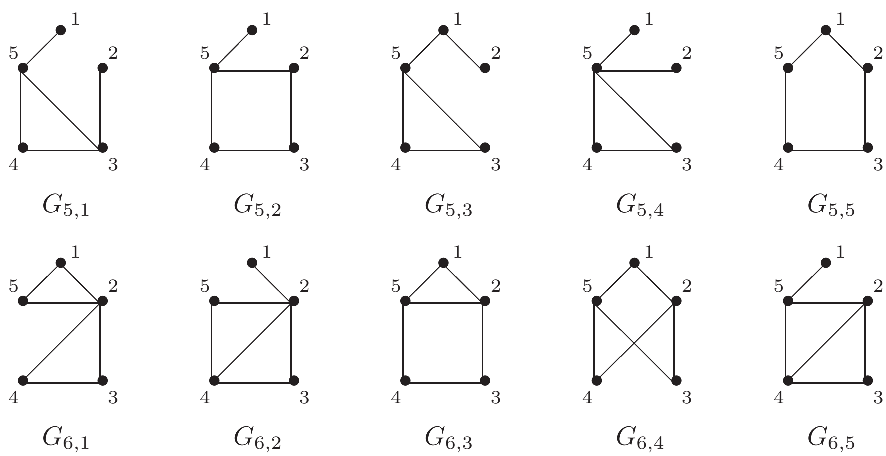
COROLLARY 3.8. *Any order* 6 *reducible SAP must have at least twelve nonzero entries.*

Proof. By Proposition 2.1, a reducible order 6 SAP must decompose into irreducible components of order 2 and 4, or three order 2 components. The result then follows from Lemmas 3.7 and 3.3. □

4. The $2n$ conjecture for order 5 patterns. In this section we show that any order 5 SAP must have at least ten nonzero entries, thereby establishing the $2n$ conjecture for patterns of order 5. In fact, we show that any order 5 irreducible component of an SAP must have at least ten nonzero entries, and as a consequence, a reducible SAP of order $n \leq 7$ or less must have at least $2n$ nonzero entries.

When looking for an order 5 SAP having less than ten nonzero entries, by Proposition 3.5 we can restrict our attention to irreducible patterns, which necessarily have strongly connected digraphs, and by Proposition 3.1 we need not consider any pattern whose graph is a tree. Any pattern described by a graph with less than five edges cannot be an SAP with less than ten nonzero entries, because in each case, the graph is either not connected or a tree. Since the digraph must have two loops and a 2-cycle, the graph associated with an order 5 pattern that has less than ten nonzero entries can have at most six edges. Figure 4.1 presents all nonisomorphic connected graphs $G_{q,r}$ of order 5 with at least five and at most six edges (see for example [7]).

FIG. 4.1. *Connected Order 5 graphs with 5 or 6 edges*



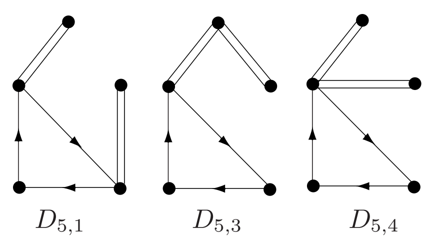
THEOREM 4.1. *Any order* 5 *irreducible component of an SAP must have at least ten nonzero entries.*

Proof. Suppose Z is an irreducible component of an SAP of order 5 with at most nine nonzero entries such that $\Gamma(Z)$ has two loops, a 2-cycle and is strongly connected. Suppose further that $\mathcal{G}(Z)$ is not a tree. In each case we derive a contradiction for any pattern described by one of the ten graphs in Figure 4.1. To derive such a contradiction, one of the following properties (which prevent Z from being an irreducible component of an SAP) is established for each possible pattern:

- $\Gamma(Z)$ does not have at least two order k permutation digraphs for some k.
- Z does not allow a nilpotent matrix.
- Z does not allow $x(1+x^2)^2$ as the characteristic polynomial of any matrix in $\mathcal{Q}(Z)$.

  (By Proposition 2.1, if the order of Z is 5 and $Z \oplus Z'$ is an SAP, then the order of Z' must be even, say $2m$. If Z does not allow $x(1+x^2)^2$, then $x(1+x^2)^{m+2}$ will not be the characteristic polynomial of any matrix in $\mathcal{Q}(Z \oplus Z')$. Hence a pattern Z that does not allow $x(1+x^2)^2$ cannot be an irreducible component of an SAP.)

FIG. 4.2. *Forced placement of non-loop arcs*



We begin by considering patterns Z such that $\mathcal{G}(Z)$ has five edges. First note that the digraphs (with the loops suppressed) for patterns $G_{5,1}$, $G_{5,3}$, $G_{5,4}$ must be as shown in Figure 4.2 in order to be strongly connected.

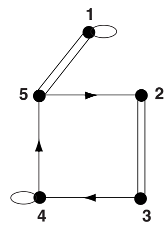
Case 1. Suppose $\mathcal{G}(Z) = G_{5,1}$ or $G_{5,4}$. Then the digraph $\Gamma(Z)$ (without loops) must be $D_{5,1}$ (respectively, $D_{5,4}$) as shown in Figure 4.2. Inserting the loops will account for nine arcs. But any placement of two loops will allow for at most one order 5 permutation digraph.

Case 2. Suppose $\mathcal{G}(Z) = G_{5,2}$. Since $\Gamma(Z)$ is strongly connected, we must have the 2-cycle (15) and the 4-cycle (2345) (or its reverse) in $\Gamma(Z)$; this accounts for six arcs. That leaves at most three additional arcs available, of which two must be loops. If $\Gamma(Z)$ has only one 2-cycle and at most three loops, then $\Gamma(Z)$ contains at most one order 5 permutation digraph. Thus $\Gamma(Z)$ must have exactly two 2-cycles and two loops. Considering the need for two order 5 permutation digraphs, this forces $\Gamma(Z)$ to be equivalent to the digraph in Figure 4.3. Given $A \in \mathcal{Q}(Z)$, $S_1(A) = a_{11} + a_{44}$ and

$$S_3(A) = -a_{11}a_{23}a_{32} - a_{44}a_{23}a_{32} - a_{44}a_{15}a_{51} = -S_1(A)a_{23}a_{32} - a_{44}a_{15}a_{51}.$$

Thus if $S_1(A) = 0$, then $S_3(A) = -a_{44}a_{15}a_{51} \neq 0$. Therefore Z does not allow a nilpotent matrix.

FIG. 4.3. *Forced digraph* $\Gamma(Z)$ *for* $G_{5,2}$



Case 3. Suppose $\mathcal{G}(Z) = G_{5,3}$. Except for the placement of the two loops, $\Gamma(Z)$ is $D_{5,3}$ in Figure 4.2. Thus the cycle structure for an order 5 permutation digraph must be either (345)(12) or (345)(1)(2) (since we cannot have more than two loops). This forces $\Gamma(Z)$ to be equivalent to the digraph in Figure 4.4. Thus if $A \in \mathcal{Q}(Z)$,

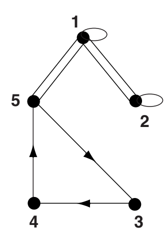
$$S_5(A) = a_{11}a_{22}a_{34}a_{45}a_{53} - a_{12}a_{21}a_{34}a_{45}a_{53} = (a_{11}a_{22} - a_{12}a_{21})a_{34}a_{45}a_{53}.$$

Since all the a_{ij} in this expression are nonzero, if $S_5(A) = 0$, then $a_{11}a_{22} - a_{12}a_{21} = 0$ and so

$$S_2(A) = a_{11}a_{22} - a_{12}a_{21} - a_{15}a_{51} = -a_{15}a_{51} \neq 0.$$

Therefore Z does not allow a nilpotent matrix.

FIG. 4.4. *Forced digraph* $\Gamma(Z)$ *for* $G_{5,3}$

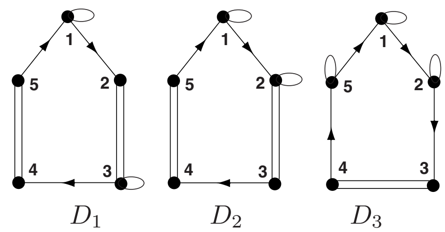


Case 4. Suppose $\mathcal{G}(Z) = G_{5,5}$. Since $\Gamma(Z)$ is strongly connected, it has a 5-cycle, but it does not have a 3-cycle or 4-cycle. To obtain a second order 5 permutation digraph, $\Gamma(Z)$ must have either two 2-cycles and two loops (with the 2-cycles and one loop disjoint) or one 2-cycle and three loops (disjoint). Thus $\Gamma(Z)$ is equivalent to one of the digraphs in Figure 4.5.

If $\Gamma(Z) = D_1$ or $\Gamma(Z) = D_2$ and $A \in \mathcal{Q}(Z)$ then

$$S_3(A) = -S_1(A)a_{45}a_{54} - a_{23}a_{32}a_{11}$$

FIG. 4.5. *Possible digraphs* $\Gamma(Z)$ *for* $G_{5,5}$



If $\Gamma(Z) = D_3$ and $A \in \mathcal{Q}(Z)$ then

$$S_3(A) = -S_1(A)a_{43}a_{34} + a_{11}a_{22}a_{55}.$$

In either case, if $S_1(A) = 0$ then $S_3(A) \neq 0$, so Z does not allow a nilpotent matrix.

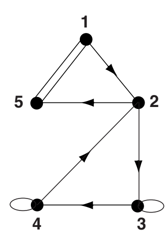
We now consider patterns Z such that $\mathcal{G}(Z)$ has six edges. If Z has less than ten nonzero entries, then Z must have exactly nine nonzero entries, since Z must have two loops and a 2-cycle (by Lemma 3.2). Thus, $\Gamma(Z)$ must have exactly two loops and one 2-cycle, and (by Lemma 3.6), at least one 3-cycle.

Case 5. Suppose $\mathcal{G}(Z) = G_{6,1}$. Notice that $\Gamma(Z)$ has no 5-cycle, and no 4-cycle, but must have two 3-cycles to be strongly connected. To have two order 5 permutation digraphs, $\Gamma(Z)$ must be equivalent to the digraph shown in Figure 4.6. Thus if $A \in \mathcal{Q}(Z)$,

$$S_4(A) = S_1(A)a_{12}a_{25}a_{51} - a_{15}a_{51}a_{33}a_{44}.$$

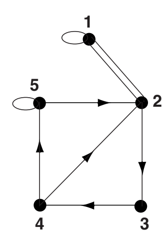
If $S_1(A) = 0$, then $S_4(A)$ is nonzero, so Z does not allow a nilpotent matrix.

FIG. 4.6. *Forced digraph* $\Gamma(Z)$ *for* $G_{6,1}$



Case 6. Suppose $\mathcal{G}(Z) = G_{6,2}$. In order for $\Gamma(Z)$ to be strongly connected, the 2-cycle must be (12). Notice that this graph has no 5-cycle. Since the 2-cycle cannot be disjoint from a 3-cycle, for permutation digraphs of order 5, we are limited to a disjoint 4-cycle and loop, or a disjoint 3-cycle and 2 loops. We cannot have two permutation digraphs consisting of a 3-cycle and two loops, as this would imply loops at vertices 1, 3, and 5, which is not possible. This means that we must have a 4-cycle. Thus $\Gamma(Z)$ is equivalent to the digraph in Figure 4.7.

FIG. 4.7. *Forced digraph* $\Gamma(Z)$ *for* $G_{6,2}$



Given $A \in \mathcal{Q}(Z)$, we have

$$S_4(A) = S_1(A)a_{23}a_{34}a_{42} - a_{23}a_{34}a_{45}a_{52}.$$

Thus, if $S_1(A) = 0$, then $S_4(A)$ is nonzero, so Z does not allow a nilpotent matrix.

Case 7. Suppose $\mathcal{G}(Z) = G_{6,3}$. Since $\Gamma(Z)$ has exactly one 2-cycle and two loops, it has at most one 3-cycle. Hence by Lemma 3.6, $\Gamma(Z)$ has exactly one 3-cycle and exactly one loop is incident to the 2-cycle. Without loss of generality, let the 3-cycle be (152).

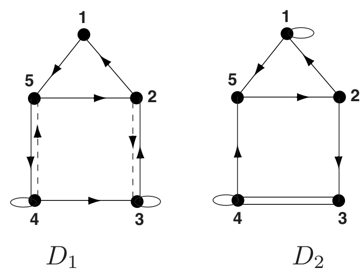
If $\Gamma(Z)$ has a 5-cycle and the 2-cycle is (25), then a 4-cycle is present and one loop must be at vertex 2 or 5. Then either $\Gamma(Z)$ has only one order 5 permutation digraph or it has only one order 4 permutation digraph.

Now suppose $\Gamma(Z)$ has a 5-cycle and the 2-cycle is not (25). Then $\Gamma(Z)$ has no 4-cycle, and (since the 2-cycle is not disjoint from both loops) the only way to obtain two order 4 permutation digraphs is to place the two loops at vertices 3 and 4. Since the 2-cycle is incident with exactly one loop, $\Gamma(Z)$ is equivalent to D_1 in Figure 4.8, where exactly one of the dashed arcs is present. For any $A \in \mathcal{Q}(Z)$,

$$S_4(A) = S_1(A)a_{15}a_{52}a_{21},$$

so $x^5 + 2x^3 + x = x(x^2 + 1)^2$ cannot be the characteristic polynomial of A.

Fig. 4.8. *Possible digraphs for $G_{6,3}$*



Now assume there is no 5-cycle. If the loops and arc $(4, 3)$ are ignored, the remaining arcs of $\Gamma(Z)$ must be as in D_2 in Figure 4.8 to make the digraph strongly connected. The only ways to obtain order 5 permutation digraphs are: disjoint 4-cycle and loop, disjoint 3-cycle and 2-cycle, disjoint 3-cycle and two loops. Since only two loops are available, it is not possible to have both (disjoint 4-cycle and loop) and (disjoint 3-cycle) two loops. Thus to obtain two order 5 permutation digraphs, the 2-cycle must be (34). By Lemma 3.6, exactly one loop must be at 3 or 4, so to obtain a second order 5 permutation digraph, the other loop must be at 1. Thus $\Gamma(Z)$ is equivalent to D_2 (since placement of a loop at 3 instead of 4 results in an equivalent digraph).

Suppose $A \in \mathcal{Q}(Z)$ is nilpotent. Then $0 = S_1(A) = a_{11} + a_{44}$, so $a_{44} = -a_{11}$. Furthermore, $0 = S_2(A) = -a_{34}a_{43} + a_{11}a_{44}$, implying that $a_{34}a_{43} = -a_{11}^2$, and $0 = S_3(A) = a_{21}a_{15}a_{52} - a_{34}a_{43}a_{11}$, implying that $a_{21}a_{15}a_{52} = -a_{11}^3$. Thus $0 = S_4(A) = -a_{23}a_{34}a_{45}a_{52} + a_{44}a_{21}a_{15}a_{52}$ implies $a_{23}a_{34}a_{45}a_{52} = a_{11}^4$, and so $S_5(A) = -a_{21}a_{15}a_{52}a_{34}a_{43} - a_{11}a_{23}a_{34}a_{45}a_{52} = -2a_{11}^5 \neq 0$, contradicting the nilpotence of A.

Case 8. $\mathcal{G}(Z) \neq G_{6,4}$ by Lemma 3.6, since $G_{6,4}$ does not have a 3-cycle.

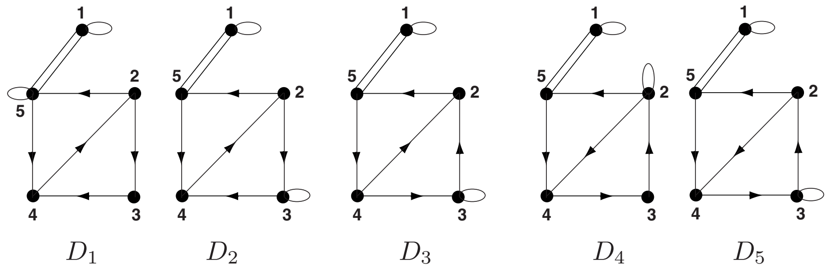
Case 9. Suppose $\mathcal{G}(Z) = G_{6,5}$. By the strong connectivity assumption, the 2-cycle (15) is forced and the remaining non-loop edges must be oriented in one of the three ways shown in Figure 4.9 (the first two and last two have the same orientation). We first show that to have two permutation digraphs of each order, it is necessary that $\Gamma(Z)$ be equivalent to one of D_1, D_2, D_3, D_4, D_5.

In D_1 and D_2, there is no 4-cycle, so the order 5 permutation digraphs must be a disjoint 3-cycle and 2-cycle or a disjoint 3-cycle and two loops. Thus the loop at 1 is forced, and the other must be disjoint from one 3-cycle. Thus the two possibilities are D_1 and D_2.

For D_3, there is one 4-cycle and one 3-cycle; the 2-cycle is not disjoint from the 3-cycle, so to obtain two order 5 permutation digraphs, loops must be placed so that one is disjoint from the 4-cycle and both are disjoint from the 3-cycle.

For D_4 and D_5, there is one 4-cycle and one 3-cycle; the 2-cycle is disjoint from the 3-cycle. So to obtain two order 5 permutation digraphs we could use a disjoint 4-cycle and loop, the disjoint 3-cycle and 2-cycle, or a disjoint 3-cycle and two loops. Since two permutation digraphs are needed, one loop must be on vertex 1. If the other loop is placed on vertex 5, there will be only one order 3 permutation digraph (the 3-cycle).

FIG. 4.9. *Possible digraphs for $G_{6,5}$*



Thus the two possibilities are D_4 and D_5 (since placement of the other loop on vertex 4 results in a digraph equivalent to D_4).

Suppose $\Gamma(Z) = D_1$ and $A \in \mathcal{Q}(Z)$. If $S_1(A) = 0$, then $S_4(A) = S_1(A)a_{23}a_{34}a_{42} + a_{11}a_{25}a_{54}a_{42} \neq 0$, so Z does not allow a nilpotent matrix.

Suppose $\Gamma(Z) = D_2$ and $A \in \mathcal{Q}(Z)$. If $S_1(A) = 0$, then $S_4(A) = S_1(A)a_{25}a_{54}a_{42} + a_{11}a_{23}a_{34}a_{42} \neq 0$, so Z does not allow a nilpotent matrix.

Suppose $\Gamma(Z) = D_3$ and $A \in \mathcal{Q}(Z)$. If $S_1(A) = 0$, then $S_4(A) = S_1(A)a_{25}a_{54}a_{42} - a_{25}a_{54}a_{43}a_{32} \neq 0$, so Z does not allow a nilpotent matrix.

Suppose $\Gamma(Z) = D_4$ (respectively, $\Gamma(Z) = D_5$) and $k = 2$ (respectively, $k = 3$). Suppose there exists $A \in \mathcal{Q}(Z)$ such that the characteristic polynomial of A is $x(x^2+1)^2 = x^5 + 2x^3 + x$. Since $0 = S_1(A)$, $a_{kk} = -a_{11}$. Then $2 = S_2(A) = -a_{15}a_{51} + a_{11}a_{kk}$ implies $a_{15}a_{51} = -a_{11}^2 - 2$. Then
$0 = S_3(A) = a_{24}a_{43}a_{32} - a_{15}a_{51}a_{kk} = a_{24}a_{43}a_{32} - a_{11}^3 - 2a_{11}$ implies $a_{24}a_{43}a_{32} = a_{11}^3 + 2a_{11}$. Then
$1 = S_4(A) = -a_{25}a_{54}a_{43}a_{32} + a_{11}a_{24}a_{43}a_{32} = -a_{25}a_{54}a_{43}a_{32} + a_{11}^4 + 2a_{11}^2$ implies $a_{25}a_{54}a_{43}a_{32} = a_{11}^4 + 2a_{11}^2 - 1$.
Finally, $S_5(A) = -a_{11}a_{25}a_{54}a_{43}a_{32} - a_{24}a_{43}a_{32}a_{15}a_{51} = 2a_{11}^3 + 5a_{11} = a_{11}(2a_{11}^2 + 5) \neq 0$. ∎

COROLLARY 4.2. *Any order 5 spectrally arbitrary sign pattern must have at least ten nonzero entries.*

COROLLARY 4.3. *Any order 7 reducible SAP must have at least fourteen nonzero entries.*

Proof. A reducible order 7 SAP must decompose into irreducible components of orders 5 and 2, 4 and 3, or 3, 2, and 2. The result then follows from Theorem 4.1 and Lemmas 3.7, 3.4, and 3.3. ∎

Acknowledgments

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