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COMPLEMENTARY VARIATIONAL FORMULATION OF MAXWELL'S
EQUATIONS IN POWER SERIES FORM

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Complementary variational formulation of Maxwell's

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by

Masao Shimoji

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TABLE OF CONTENTS

1. INTRODUCTION

1.1. Preamble

In many practical engineering electromagnetic problems, exact
analytical solutions do not exist. Fortunately, the capabilities of
modern computers make it possible for engineers to seek alternatives.
Among the most preferred of these alternatives are the methods of
Rayleigh-Ritz, finite difference, finite elements, and the method of
moments. These techniques as mathematical disciplines are discussed
in various textbooks such as [28,31,35,53].

The method of moments, Rayleigh-Ritz, and finite element techniques
are based on the stationary property of a variational integral [28,35,53].
(These techniques reduce the problem of finding a minimizing (or maxi-
mizing) function for the variational integral to a set of simultaneous
linear algebraic equations.) It is true that finite element techniques
can be applied directly to the method of weighted residuals [31] rather
than a stationary varitional functional. However, a quick review of
IEEE Transactions on Microwave Theory and Techniques over the past
three decades seems to indicate a dominant role of variational functionals
over the concept of weighted residuals as a fundamental tool on which
the finite element and the Rayleigh-Ritz methods are based. Furthermore,
this observation seems to be supported by the frequent appearance in
recent years of publications in the same journal dealing with the
variational formulation of Maxwell's equations.

The above discussion points to the fact that the variational principle plays a crucial role in the numerical analysis of electromagnetic problems. These facts together with some curiosity motivated the author to pursue the basic aspects of variational principle with application to electromagnetics in mind. Indeed, the main content of this thesis will be devoted to the topic of formulating Maxwell's equations as two functionals, called complementary variational integrals.

The complementary variational principles, as they are usually referred, are general methods of formulating a given boundary value problem as two variational integrals. The theory is based on some abstract concepts in linear vector space. As such, it was felt necessary to devote a good portion of the thesis to clarifying the fundamental concepts and theorems. In the following, a brief explanation of each chapter is attempted.

The two sections immediately following the present section discuss the application of variational methods in electromagnetics. Section 1.2 reviews some literature beginning with 1969. It was felt that Wexler's article [52] in that year marked the end of scalar variational formulation. Shortly after, a more powerful vector variational formulation started to get attention. The presently popular finite element technique is capable of reducing the latter formulation into a discrete algebraic problem.

Section 1.3 gives brief overview of complementary variational principles as applied to electromagnetics. The last section defines the problem pursued in this thesis.

The discussion of variational principles begins with Chapter 2. Although not essential for our purpose, some basic aspects of the classical theory are illustrated in this chapter. It is hoped that familiarity with conventional variational theory will shed some light on the complementary variational principles discussed in the succeeding chapters.

Most of Chapter 3 is devoted to the preliminary basic concepts necessary for the development of complementary variational theory. Operators and scalar products are discussed as part of the structure of Hilbert space. Complementary extremum principles in their most general form are presented as Theorem 3.5.1 at the conclusion of the chapter.

Chapter 4 covers the general topics of formulating a given boundary value problem as two complementary variational integrals. Section 4.7 points out some important aspects of the theory that could be overlooked by the reader.

In Chapter 5 difficulties are pointed out when one tries to apply complementary extremum principles directly to Maxwell's equations. It is clear from discussions in this chapter that there is a need for modification if the theory is going to be useful in electromagnetics.

Chapter 6 is devoted to the power series approach to electromagnetism. The importance of this approach to the engineering electromagnetics is stressed.

Finally, in Chapter 7 it is shown that the complementary extremum principles can be applied to the kth-order field laws in the power series. The following chapter illustrates the theory through a simple example of parallel plate capacitor analysis.

1.2. Conventional Variational Principles in Electromagnetics

The word "conventional" or "classical" as opposed to "complementary" will be used throughout the thesis. It refers to the variational theory that yields only the one-sided bound to the stationary value of the functional. The complementary variational theory is capable of yielding upper and lower bounds. With this point clarified, we are ready to begin the discourse into the main content of this section.

As pointed out by A. D. Berk, the often quoted paper [19] seems to be the first vector variational technique in engineering electromagnetics. He formulates variational expressions in terms of vector fields $\bar{E}$ and $\bar{H}$. His formulation enabled engineers to apply variational techniques to inhomogeneous as well as anisotropic regions [29]. However, as evident from the following paragraph, Berk's point of view did not become popular until 1971 when W. J. English [23] published a paper on vector variational formulation of inhomogeneously loaded waveguide structures.

In 1969, Wexler [52] discussed popular numerical techniques in engineering electromagnetics. According to the author, the standard procedure was to formulate the given boundary value problem as a scalar

variational functional. (The term "scalar" is used here because the variational functional is minimized with respect to a scalar function rather than a vector field. A functional is a function of a function assigning unique numerical value to each given function. The discussion of the general topic of reformulating boundary value problems as variational functionals can be found in various textbooks such as [13,32,35].) A trial function is then inserted into the functional. By following the standard Ritz procedure, the functional is minimized with respect to coefficient parameters in the trial function. This reduces the boundary value problem to a set of linear algebraic equations, the solutions of which determine the coefficient parameters in the trial function. The trial function, with its coefficient parameters determined, constitute the approximate solution to the original boundary value problem. Examples of applications of this technique in electromagnetics abound in literature [44,45,48].

Unfortunately, as pointed out by English and Young [24], Wexler [52], and Konrad [30], the scalar variational formulation has a serious limitation. It is useful only when the geometries are such that fields can be derived from a single scalar potential. This prompted Wexler [52] to emphasize the need for a vector variational formulation in which a functional should be minimized with respect to electric and/or magnetic fields. The first successful application of a numerical tecnhnique using all six components of E and H fields to a vector variational formulation of Maxwell's equations appeared in English's paper [23] in 1971. In his paper, the author formulated a cylindrical waveguide

problem as a vector variational integral in terms of vectors of $\overline{E}$ and $\overline{H}$ fields. He then uses six trial functions, one for each component of $\overline{E}$ and $\overline{H}$, and determines the coefficients in the trial functions by minimizing the functional by Ritz procedure. His results on field distributions, propagation constants, and cutoff frequencies agree very well with the exact values.

The above six-component vector variational formulation was succeeded by three-component vector variational formulations [6,21,24,30]. The latter formulation is desirable mainly because of its reduced matrix size as compared to the six-component formulation [24]. In his 1976 paper [30], Konrad points out the advantage of his three-component formulation. Unlike previous three-component formulations, Konrad's vector variational integral can be applied to anisotropic media and does not require the trial fields to satisfy boundary conditions.

There are other investigators whose main interests seem to lie in Hamilton's principle. In classical dynamics, Hamilton's principle is well established and can be stated as follows [34]:

Hamilton's Principle: Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval, the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.

Of course, this theory cannot give results different than those obtained

using Newton's law. Hamilton's principle is just another statement

of Newtonian dynamics which happens to be more advantageous than Newton's

formulation in certain problems.

Hamilton's principle changes the Newton's law to a variational

problem of finding coordinate functions that minimizes the functional

$$\int_{t_1}^{t_2} (T-U)dt \qquad\qquad (1.2.1)$$

where T,U are the kinetic and potential energies respectively. Although

not very well-established, such a principle is valid in electromag-

netics [21]. The papers [20], [21], [36],[37] discuss Hamilton's

principle in electromagnetics in various contexts. They derive vector

variational integrals from Hamilton's principle and show their advantages

and usefulness in engineering electromagnetics.

1.3. Complementary Variational Principles in Electromagnetics

In 1964, Rall [42] published a paper in which he formulates a

simple boundary value problem as two variational integrals. These two

integrals (or functionals) are commonly referred to as the complementary

variational formulation of a given boundary value problem. Since Rall's

first paper, the theory has been extended to cover many boundary value

problems in mathematical physics [16,18,41,43].

In electromagnetics, the first application of the theory appeared in 1969 [17]. The authors formulate static Maxwell's equations as complementary variational integrals. They also suggested how the integrals can be used to estimate capacitance of a given structure. Since this first publication, there have been numerous papers [3-5,7-9,11] demonstrating the usefulness of this relatively new variational theory in dealing with some limited classes of electromagnetic problems.

In recent years, N. Anderson and A. M. Arthurs have published three papers [2,6,10] in succession. Their point of view is closely related to Hamilton's principle. They regard two curl equations of electromagnetic fields as canonical equations in Hamilton's formulation. By working backwards, they derive a variational integral in terms of electric and magnetic fields. Furthermore, they derive two functionals, in terms of $\bar{E}$ or $\bar{H}$ alone, from the original functional. They call their theory complementary variational principles because the two functionals are derived from the original one in a complementary fashion. However, it must be stressed that their complementary integrals do not give complementary bounds as implied in [6]. In other words, complementary "stationary" principles are valid but the complementary "extremum" principles fail.

1.4. Statement of Problem

In the previous section, we cited the first paper in which authors formulated the basic Maxwell's equations of static fields as two comple-

mentary variational integrals. Unlike their recent papers mentioned
above, the authors prove that the two integrals approach the stationary
value from opposite directions. In other words, the complementary
"extremum" principle is valid in their static formulation.

It is true that the complementary variational formulation of static
Maxwell's equations, mentioned above, can be useful in certain time-
harmonic problems such as [3]. However, it must be mentioned that
in order for the theory to apply, additional terms in the time-harmonic
equation arising from the time variation must be neglected. Therefore,
the two complementary variational integrals are good only to the static
approximation. But, as it often happens in engineering electromagnetic
problems, such seemingly crude approximations can be very useful.
We are now ready to state the problem studied in this thesis.

Problem Definition: To investigate the usefulness of the
complementary extremum principles in time-varying electromagnetic
field problems. Specifically, we seek a mathematical device or
a technique through which Maxwell's equations can be formulated
as two complementary variational integrals. These integrals must
account for time variation and approach the stationary value from
both above and below.

Chapter 7 discusses one way by which sinusoidally-varying fields can
be formulated as two variational integrals that satisfy the requirements
in Problem Definition.

2. CLASSICAL VARIATIONAL PRINCIPLES

2.1. Introduction

In this chapter, some basic concepts of variational theory are illustrated through discussion of the simplest problem. We assume the "admissible functions" to be at least twice differentiable. Following the usual procedure, we substitute the trial function into the integrand. This reduces the variational problem into maximum-minimum problem of ordinary functions. After applying the stationary condition for ordinary functions, that is, the first derivative must vanish, the necessary condition of Euler and Lagrange is derived.

The basic concepts introduced here carry directly into the complementary variational theory. In complementary variational theory, a given boundary value problem is formulated as two different variational integrals. However, one needs to follow the conventional analysis techniques shown in this chapter in order to arrive at the stationary equations of these two integrals.

2.2. The Fundamental Problem

Traditionally, the calculus of variation begins with discussion of the simplest type of problem. It deals with the problem of finding a function or functions that extremize (maximize or minimize) the integral of the form

$$I(\Phi(x)) = \int_{x_0}^{x_1} L(x, \Phi(x), \Phi'(x)) dx \qquad (2.2.1)$$

The integral I is a function of a function $\Phi(x)$. Two end points of $\Phi(x)$ are assumed to be fixed: $\Phi(x_0) = \alpha$ and $\Phi(x_1) = \beta$. The symbol $\Phi'(x)$ stands for the derivative of $\Phi(x)$, and the function $L(x, \Phi(x), \Phi'(x))$ of the three variables is assumed to possess continuous derivatives up to some order, n, required by the theory. If the integral of Eq. 2.2.1 has a maximum value for some given function $\Phi(x)$, it can be changed to a minimum problem by considering the negative of the integral [13]. Therefore, it is sufficient to develop the theory for a minimizing problem only [13].

The theory depends largely on the type of functions, called "admissible functions," that are allowed to compete for minimization [40]. The most restricted class of functions for this fundamental problem requires the function to be at least twice continuously differentiable. On the other hand, a function can be picked from a much larger class where the only requirement is that the function be piecewise continous. The theory developed in Section 2.4 is based on the restriction that the admissible functions be at least twice continuously differentiable.

2.3. Types of Minima of Integrals

Let us denote the space of twice continuously differentiable functions of single variable by the symbol Ω. Before defining minima of integrals, it is necessary to introduce the concepts of "distance" and "neighborhood" in space Ω. The following discussion closely parallels that of Leitman [32].

Consider two functions, $\Phi(x):[x_0,x_1] \to R^1$ and $\widetilde{\Phi}(x):[x_0,x_1] \to R^1$, taking a closed set $[x_0,x_1]$ into a real line R^1, which are members of Ω. The distance of order zero between $\Phi(x)$ and $\widetilde{\Phi}(x)$ is

$$d_0[\Phi(x),\widetilde{\Phi}(x)] \overset{\Delta}{=} \text{L.U.B.} |\Phi(x)-\widetilde{\Phi}(x)| \qquad (2.3.1)$$
$$x \in [x_0,x_1]$$

where L.U.B. stands for the least upper bound. Two vertical bars are used to denote "absolute value of" and the symbol $\in$ signifies that x is a member of the closed set $[x_0,x_1]$. The distance of order one between $\Phi(x)$ and $\widetilde{\Phi}(x)$ is

$$d_1[\Phi(x),\widetilde{\Phi}(x)] \overset{\Delta}{=} \text{L.U.B.} |\Phi'(x)-\widetilde{\Phi}'(x)| \qquad (2.3.2)$$

Thus, the functions $\Phi(x)$ and $\widetilde{\Phi}(x)$ are "near" each other in the sense of zero order distance if their values are close to each other at every $x \in [x_0,x_1]$. They are "near" each other in the sense of first order distance if, in addition, their slopes are close to each other at every $x \in [x_0,x_1]$.

Now, we are ready to define neighborhoods. Given two functions $\Phi(x),\widetilde{\Phi}(x) \in \Omega$ and a positive real number $\delta \in R^1$, a δ-neighborhood of order zero of $\Phi(x)$ is

$$N_0[\delta,\phi(x)] \triangleq \{\widetilde{\phi}(x):[x_0,x_1]\to R^1 | d_0[\phi(x),\widetilde{\phi}(x)]<\delta\} \tag{2.3.3}$$

while a δ-neighborhood of order one of $\phi(x)$ is

$$N_1[\delta,\phi(x)] \triangleq \{\widetilde{\phi}(x):[x_0,x_1]\to R^1 | d_1[\phi(x),\widetilde{\phi}(x)]<\delta\} \tag{2.3.4}$$

In both Eqs. 2.3.3 and 2.3.4, the brackets {} read "a set of all" and the vertical bar denotes "such that."

We are now able to define minima of integrals in terms of neighborhood of a function. The arbitrary function $\phi(x)\in\Omega$ furnishes a strong local minimum of the integral of Eq. 2.2.1 if and only if there exists $\delta>0$, such that

$$I(\phi(x)) \leq I(\Phi(x)) \quad \forall\Phi(X)\in\Omega\cap N_0[\delta,\phi(x)] \tag{2.3.5}$$

The symbol $\forall$ denotes "for all," while $\cap$ signifies the intersection of two sets. In a similar fashion, the arbitrary function $\phi(x)\in\Omega$ furnishes a weak local minimum of the integral (Eq. 2.2.1) if and only if there exists $\delta>0$ such that

$$I(\phi(x))\leq I(\Phi(x)) \quad \forall\Phi(x)\in\Omega\cap N_1[\delta,\phi(x)] \tag{2.3.6}$$

The difference between Eqs. 2.3.5 and 2.3.6 lies in the nature of the neighborhood of $\phi(x)$. In addition to the two types of local minima introduced above, we define global minimum in the following way: the function $\phi(x)$ furnishes the global minimum of the integral (Eq. 2.2.1) if and only if

$$I(\phi(x))\leq I(\Phi(x)) \quad \forall\Phi(x)\in\Omega \tag{2.3.7}$$

When the integral of Eq. 2.2.1 is maximum instead of minimum, we can speak of maxima defined in an analogous fashion with inequalities reversed. However, as pointed out earlier, it suffices to talk about

minimum only, since a function $\phi(x)$ that minimizes $I(\Phi(x))$ of Eq. 2.2.1 also maximizes $I(\Phi(x))$.

In the definitions of global, strong local, and weak local minima, the function $\phi(x)$ is compared to members of successively smaller sets of functions. Thus, we conclude that a global minimum → a strong local minimum → a weak local minimum [32]. In other words, a weak local minimum is necessary for a strong local minimum. In turn, a condition that is necessary for a strong local minimum is necessary for a global minimum.

2.4. Euler-Lagrange Theory

Consider a function $\eta(x) \in \Omega$ with $\eta(x_0)=0$ and $\eta(x_1)=0$. It is then possible to represent an arbitrary function $\Phi(x) \in \Omega$ in the form

$$\Phi(x) = \phi(x) + \epsilon\eta(x) \tag{2.4.1}$$

where $\phi(x)$ is some fixed function, ϵ is a constant, and $\eta(x)$ is adjusted to satisfy Eq. 2.4.1 [51]. Geometrically, it represents a curve as drawn in Fig. 2.4.1, where $\Phi(x)$ assumes α and β at x_0 and x_1, respectively.

We can consider Eq. 2.4.1 to be a function of a parameter ϵ for a given x. This leads to

$$\lim_{\epsilon \to 0} \Phi(x,\epsilon) = \lim_{\epsilon \to 0}[\phi(x)+\epsilon\eta(x)] = \phi(x) \tag{2.4.2}$$

$$\lim_{\epsilon \to 0} \partial\Phi(x\epsilon)/\partial x = \lim_{\epsilon \to 0}[\phi'(x)+\epsilon\eta'(x)] = \phi'(x) \tag{2.4.3}$$
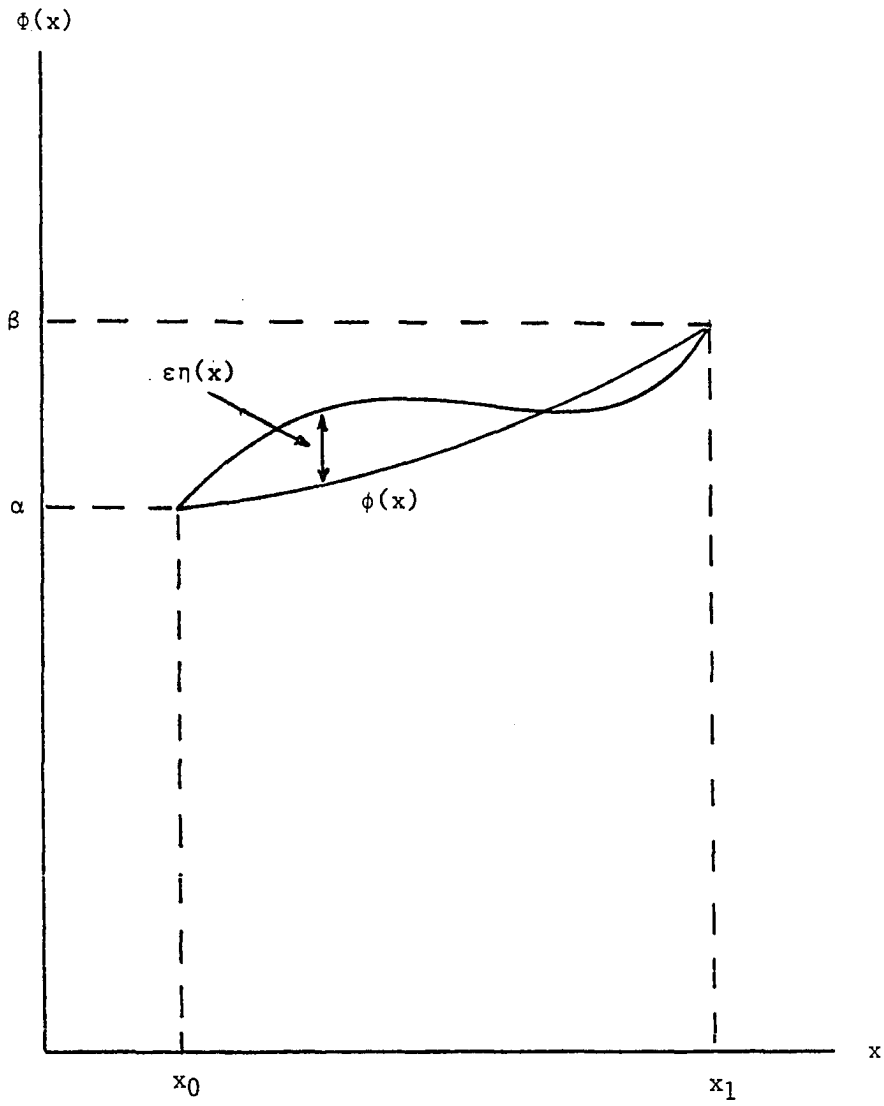
Figure 2.4.1.   An arbitrary function $\Phi(x)$

showing that function $\Phi(x)$ belongs to the δ-neighborhood of order 1

for $\phi(x)$ for sufficiently small ϵ [40].

Now, we can proceed with the following. The integral (Eq. 2.2.1),

which is a function of a function $\Phi(x)$, can be changed to a function

$I(\epsilon)$ of a single variable ϵ. This is done by substituting the particular

representation of $\Phi(x)$ (Eq. 2.4.1) into the integrand and performing

the integration. Also, if the integral yields the minimum when

$\Phi(x) = \phi(x0$, then $I(\epsilon)$ assumes minimum for $\epsilon=0$. Therefore, unlike

maximum-minimum problems in ordinary functions, the stationary point,

$\epsilon=0$, is known in advance [51].

Mathematically, the above reasoning translates essentially into

the following equations

$$I(\epsilon) = I(\phi(x)+\epsilon\eta(x)) = \int_{x_0}^{x_1} L(x,\phi(x)+\epsilon\eta(x),\phi'(x)+\epsilon\eta'(x))dx \qquad (2.4.4)$$

$$\frac{dI(\epsilon)}{d\epsilon} = \int_{x_0}^{x_1} [\,\eta(x)\frac{\partial}{\partial\phi(x)} L(x,\phi(x)+\epsilon\eta(x),\phi'(x)+\epsilon\eta'(x)) +$$

$$\eta'(x) \frac{\partial}{\partial\phi'(x)} L(x,\phi(x)+\epsilon\eta(x),\phi'(x)+\epsilon\eta'(x))]dx \qquad (2.4.5)$$

Now, suppose $I(\Phi(x))$ yields minimum at $\phi(x)$. It then follows that

$$\frac{dI(\epsilon)}{d\epsilon}\bigg|_{\epsilon=0} = \int_{x_0}^{x_1} \eta \frac{\partial L}{\partial\phi} + \dot{\eta} \frac{\partial L}{\partial\phi'} \, dx = 0 \qquad (2.4.6)$$

where the new arguments in the integrand are suppressed. Note that

functions $\partial L/\partial \phi$ and $\partial L/\partial \phi'$ are now evaluated at x, $\phi(x)$, and $\phi'(x)$.

Integrating by parts the second term in Eq. 2.4.6 we obtain

$$
I'(\varepsilon) = \eta \left. \frac{\partial L}{\partial \phi'} \right|_{x_0}^{x_1} + \int_{x_0}^{x_1} \eta \left[\frac{\partial L}{\partial \phi} - \frac{d}{dx} \left(\frac{\partial L}{\partial \phi'} \right) \right] dx
$$

$$
= \int_{x_0}^{x_1} \eta \left[\frac{\partial L}{\partial \phi} - \frac{d}{dx} \left(\frac{\partial L}{\partial \phi'} \right) \right] dx = 0 \qquad (2.4.7)
$$

because $\eta(x \cdot) = \eta(x_1) = 0$. Now we invoke the so called "fundamental

theorem of calculus of variations" [25,51] to conclude:

$$
\frac{\partial L}{\partial \phi} - \frac{d}{dx} \left(\frac{\partial L}{\partial \phi'} \right) = 0 \qquad (2.4.8)
$$

This is the necessary condition normally referred to as the Euler-Lagrange

equation. It yields a second order differential equation that can be

solved for $\phi(x)$ under the boundary conditions $\phi(x_0) = \alpha$ and $\phi(x_1) = \beta$.

It is worth pointing out that Eq. 2.4.8 under given boundary

conditions may possess multiple solutions [32]. The solution or

solutions are usually referred to as "extremal." When the integral

$I(\Phi(x))$ is shown to have minimum at $\phi(x)$, the only conclusion we can

draw from 2.4.8 is that the minimizing function must be one of the

extremals.

3. AN EXTENSION OF CLASSICAL VARIATIONAL PRINCIPLES

3.1. Introduction

There are various ways the previous Euler-Lagrange theory can be extended [14,40,51]. In this chapter, we limit our attention to one type of extension [14].

The classical variational theory approaches the stationary value from one side--either above or below. However, there are certain types of variational integrals for which a complementary integral exists. The word "complementary" is used here naturally because the integral approaches the stationary value from the opposite direction. In the following discussions, we will be concerned with this relatively new extension. The first unified theory appeared in 1964 and is called the complementary extremum principles. But before we can present the general result, Theorem 2.5.1, some relevant basic concepts must be clarified.

Most of the concepts and symbols in this chapter closely follow the book by A. M. Arthurs, <u>Complementary Variational Principles</u> [14], where he gives complete treatment to linear as well as nonlinear problems. Only some relevant topics suitable for our purpose are presented in the following sections. While doing so, efforts are made to include more details than seen in A. M. Arthurs' book.

3.2. A Class of Operators in a Vector Space of Functions

Let $\Omega(\phi)$ and $\Omega(u)$ denote two vector spaces of functions ϕ and u, respectively. We assume both spaces are complete, linear and real vector spaces. Furthermore, let each space possess a scalar product denoted by $[,]$ for $\Omega(\phi)$ and $(,)$ for $\Omega(u)$ with the following properties.

a) $[\phi, \alpha\phi_1 + \beta\phi_2] = \alpha[\phi, \phi_1] + \beta[\phi, \phi_2]$, where α and β

   are arbitrary real constants

b) $[\phi_1, \phi_2] = [\phi_2, \phi_1]$

c) $[\phi, \phi] \geq 0$ with $[\phi, \phi] = 0$ if and only if $\phi = 0$

The same properties hold for a scalar product $(,)$ in $\Omega(u)$ space. In mathematical formalism, the vector spaces under consideration are called Hilbert spaces. We designate the two Hilbert spaces by $H(\phi) = \{\Omega(\phi), [,]\}$ and $H(u) = \{\Omega(u), (,)\}$.

An operator T is a transformation from one Hilbert space to another or to itself. An operator T is linear when it satisfies

$$T(\alpha\phi + \beta\psi) = \alpha T\phi + \beta T\psi \qquad (3.2.1)$$

One important property of a linear operator that we need in development of a complementary variational theory is that it possesses a conjugate operator. Given a linear operator $T:H(\phi) \rightarrow H(u)$, there is a second operator $T^*:H(u) \rightarrow H(\phi)$ such that

$$(u, T\phi) = [T^*u, \phi] + S(u, \phi) \qquad (3.2.2)$$

for all ϕ, u in the domain of T, T^*. The adjoint of T is the operator T^* which takes an arbitrary function in $\Omega(u)$ into a different space $\Omega(\phi)$. The last term $S(u, \phi)$ is called a conjunct of u and ϕ [27].

The complementary variational principle is based on a certain class of operators. A. M. Arthurs' [14] shows five different kinds of operators belonging to that class. For the sake of illustration, three of them are discussed below. First operators are based on integration by parts.

$$\int_a^b u(x)\frac{d}{dx}\phi(x)dx = \int_a^b (-\frac{d}{dx}u(x))\phi(x)dx + [u(x)\phi(x)]_a^b \qquad (3.2.3)$$

If we define two scalar products as

$$(u,v) = \int_a^b u(x)v(x)dx \qquad (3.2.4)$$

$$[\phi,\psi] = \int_a^b \phi(x)\psi(x)dx \qquad (3.2.5)$$

Eq. 3.2.3 becomes an example of Eq. 3.2.2. Comparing the two equations, we can identify

$$T = \frac{d}{dx} \text{ and } T* = - \frac{d}{dx} \qquad (3.2.6)$$

and $S(u,\phi)$ as the boundary term in Eq. 3.2.3. Note the symbolic use of the equality sign in Eq. 3.2.6.

Second operators come from the equation

$$\overline{\nabla}\cdot(\overline{u}\phi) = \overline{u}\cdot\overline{\nabla}\phi + (\overline{\nabla}\cdot\overline{u})\phi \qquad (3.2.7)$$

Rearranging Eq. 3.2.7 and integrating both sides of equality, we obtain

$$\int_V \overline{u}\cdot\overline{\nabla}\phi dV = \int_V (-\overline{\nabla}\cdot\overline{u})\phi dV + \int_{\partial V} \overline{u}\cdot\overline{n}\phi dB \qquad (3.2.8)$$

in which integrations are performed over volume V and its boundary ∂V. By defining two scalar products as

$$(u,v) = \int \overline{u}\cdot\overline{v}dV \qquad (3.2.9)$$

$$[\phi,\psi] = \int \phi\psi dV \qquad (3.2.10)$$

and comparing Eq. 3.2.8 with Eq.3.2.2, it follows that

$$T = \text{grad} \text{ and } T* = -\text{div} \qquad (3.2.11)$$

with the boundary term in Eq. 3.2.8 corresponding to $S(u,\phi)$.

The last example of operators T, T* follows from

$$\overline{\nabla}\cdot\overline{u}x\overline{\phi} = \overline{\phi}\cdot\overline{\nabla}x\overline{u} - \overline{u}\cdot\overline{\nabla}\phi \qquad (3.2.12)$$

which yields

$$\int\overline{u}\cdot\overline{\nabla}x\overline{\phi}dV = \int_V \overline{\nabla}x\overline{u}\cdot\overline{\phi}dV + \int_{\partial V}\overline{u}\cdot\overline{n}x\overline{\phi}dB \qquad (3.2.13)$$

The vector $\overline{n}$ in the last term of Eq. 3.2.13 is an outward normal vector
to the boundary ∂V. In this example, we take as scalar products

$$(u,v) = \int_V \overline{u}\cdot\overline{v}dV \qquad (3.2.14)$$

$$[\phi,\psi] = \int_V \overline{\phi}\cdot\overline{\psi}dV \qquad (3.2.15)$$

Again, by comparison of Eq. 3.2.13 and Eq. 3.2.2, we identify

$$T = curl \text{ and } T* = curl \qquad (3.2.16)$$

Like the two preceding examples, the boundary term in Eq. 3.2.13 becomes
$S(u,\phi)$ in Eq. 3.2.2.

In the first example, $\Omega(u)$ and $\Omega(\phi)$ are both vector space of functions
of one variable. In the third example, they are vector space of vector
fields $\overline{u}$ and $\overline{\phi}$. The second example is slightly different in that one is
a space of vector field while the other is a space of scaler functions
of three variables.

All three examples have a common property in their boundary terms
$S(u,\phi)$. In view of later developments, it is convenient to express
this common property by the equation

$$(u,T\phi) = [T*u,\phi] + (u,\sigma\phi)_{\partial}V \qquad (3.2.17)$$

where σ is a linear operator $\sigma:H(\phi)\to H(u)$ on the boundary with its
conjugate $\sigma*:H(u)\to H(\phi)$ satisfying the equation

$$(u,\sigma\phi)_{\partial}V = [\sigma*u,\phi]_{\partial}V \qquad (3.2.18)$$

By properly defining σ and $\sigma*$, all three examples can be written as Eq. 3.2.17. For the first example, we define

$$(u,v)\partial V = u(a)v(a) + u(b)v(b)$$

$$[\phi,\psi]\partial V = \phi(a)\psi(a) + \phi(b)\psi(b) \qquad (3.2.19)$$

$$\sigma = \overline{n}\cdot\overline{i}, \quad \sigma* = \overline{n}\cdot\overline{I}$$

where $\overline{i}$ is the unit vector in the positive x direction and $\overline{n}=\overline{i}$ at x=b, $-\overline{i}$ at x=a. Similar definitions for the second example are

$$(u,v)\partial V = \int_{\partial V}\overline{u}\cdot\overline{v}dB$$

$$[\phi,\psi]\partial V = \int_{\partial V}\phi\psi dB \qquad (3.2.20)$$

$$\sigma\phi = \overline{n}\phi, \quad \sigma*u = \overline{n}\cdot\overline{u}$$

and, for the third example,

$$(u,v)\partial V = \int_{\partial V}\overline{u}\cdot\overline{v}dB$$

$$[\phi,\psi]\partial V = \int_{\partial V}\overline{\phi}\cdot\overline{\psi}dB \qquad (3.2.21)$$

$$\sigma\phi = \overline{n}x\overline{\phi}, \quad \sigma*u = -\overline{n}x\overline{u}$$

where in both cases $\overline{n}$ is the unit vector normal to the surface ∂V.

3.3. Derivatives of Functionals

Various isolated instances of complementary variational principles can be unified under one cohesive theory by exploiting some simple ideas of functionals [14]. The most general complementary variational theory is given in Section 3.5, but we need some preliminary results and definitions.

Let $H(\phi)=\{\Omega(\phi),[,]\}$ be a Hilbert space of functions ϕ. A functional $E(\phi)$ is a function of a function ϕ written as

$$E(\phi):D(E)\subset H(\phi)\rightarrow R \qquad\qquad (3.3.1)$$

The domain $D(E)$ of $E(\phi)$ is contained in the space $H(\phi)$. For each arbitrary

function $\phi\varepsilon D(E)$, the functional $E(\phi)$ assigns one real number. If the

functional can be written in the form

$$E(\phi+\varepsilon\xi) = E(\phi) + [\varepsilon\xi,E'(\phi)] + \tfrac{1}{2}[\varepsilon\xi,E''(\phi)\varepsilon\xi] + o_3 \qquad (3.3.2)$$

where $o_3/\varepsilon^2\rightarrow 0$ as $\varepsilon\rightarrow 0$, then we say the functional is twice differentiable.

The second and third terms are scalar products that contain the derivative

$E'(\phi)$ and the second derivative $E''(\phi)$, respectively.

A similar definition is useful for a functional of two functions ϕ

and u. We take two scalar product spaces $H(u)=\{\Omega(u),(,)\}$ and $H(\phi)=$

$\{\Omega(\phi),[,]\}$ and consider the functional

$$K(u,\phi):D(K)\subset H(u)xH(\phi)\rightarrow R \qquad\qquad (3.3.3)$$

The derivatives are defined by the equation

$$K(u+\varepsilon v,\phi+\varepsilon\xi) = K(u,\phi) + (\varepsilon v,K_u) + [\varepsilon\xi,K_\phi] + \tfrac{1}{2}(\varepsilon v,K_{uu}\varepsilon v)$$

$$+ \tfrac{1}{2}(\varepsilon v,K_{u\phi}\varepsilon\xi) + \tfrac{1}{2}[K_{\phi u}\varepsilon v,\varepsilon\xi] + \tfrac{1}{2}[\varepsilon\xi,K_{\phi\phi}\varepsilon\xi] + o_3$$

$$(3.3.4)$$

K_u and K_ϕ are partial derivatives with respect to u and ϕ in that order.

The rest of the terms contain second partials. Of course, it is assumed

that a given functional can be written in the form as Eq. 3.3.4 and

the third order term becomes $o_3/\varepsilon^2\rightarrow 0$ as $\varepsilon\rightarrow 0$.

In Eqs. 3.3.2 and 3.3.4, the terms in which ε appears only once are

called first variations, while the second order terms in ε are called

second variations. Using these and the above ideas, the stationary

property of functional can be stated as follows.

Theorem <u>3.3.1</u>. A functional E(Φ) is stationary at $\Phi=\phi$ if and only if E'(ϕ)=0.

Theorem <u>3.3.2</u>. A functional K(U,Φ) is stationary at U=u and $\Phi=\phi$ if and only if K_u=0 and K_ϕ=0.

An intuitive argument for the above theorems can be found in Reference 14. Going back to Chapter 2, we see that the Euler-Lagrange equation is a necessary consequence of Theorem 3.3.1. By expressing the integrand in Eq. 2.2.1 in Taylor series expansion and comparing the results with Eq. 3.3.2, the Euler-Lagrange equation can be seen to be the derivative of the functional I($\Phi(x)$).

3.4. Convexity of Functionals

We need to clarify one more concept before discussing complementary variational theory. First, we define a convex set. We say that a set C in a linear space Ω is convex if, given ϕ and ψ in C, all elements of the form $\lambda\phi+(1-\lambda)\psi$ with $0<\lambda<1$ are in C. Next, we define convexity.

Definition <u>3.4.1</u>. A functional F(ϕ):$\phi \in C \rightarrow R$ defined on a convex subset C of $\Omega(\phi)$ is said to be convex if

$$F(\lambda\phi+(1-\lambda)\psi) \leq \lambda F(\phi) + (1-\lambda)F(\psi) \qquad (3.4.1)$$

for all ϕ,ψ in C and all λ such that $0<\lambda<1$. If strict inequality holds in Eq. 3.4.1 for $\phi\neq\psi$, F(ϕ) is said to be strictly convex. Also, F(ϕ) is (strictly) concave if -F(ϕ) is (strictly) convex. Figure 3.4.1 illustrates intuitive ideas of convexity.
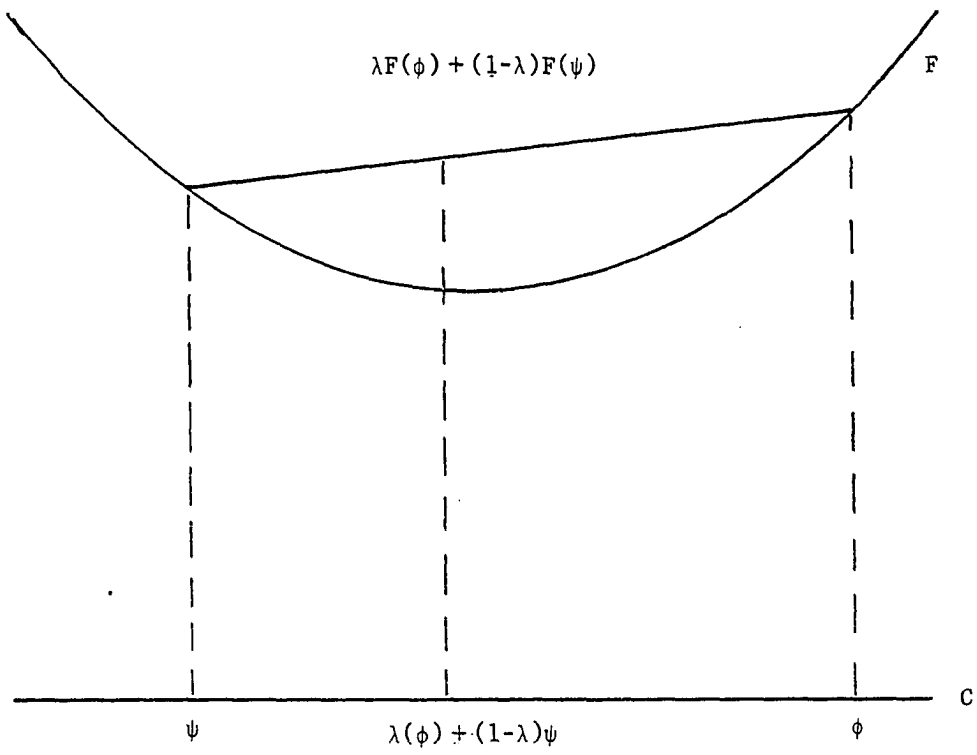
Figure 3.4.1.   Convex functional F

When the functional is differentiable, there is an equivalent statement that is more convenient for our purpose.

Lemma 3.4.1. If a functional $F(\phi)$ is differentiable in C, then $F(\phi)$ is convex in C if and only if

$$F(\phi_1)-F(\phi_2)-[\phi_1-\phi_2,F'(\phi_2)] \geq 0 \qquad (3.4.2)$$

for all ϕ_1 and ϕ_2 in C.

The proof is omitted, but a general outline is sketched in Reference 14.

For a differentiable functional, the Lemma 3.4.1 also implies

$$F(\phi_2)-F(\phi_1)-[\phi_2-\phi_1,F'(\phi_1)] \geq 0 \qquad (3.4.3)$$

By adding Eq. 3.4.2 to Eq. 3.4.3, we obtain

$$[\phi_1-\phi_2,F'(\phi_1) - F'(\phi_2)] \geq 0 \qquad (3.4.4)$$

If the derivative $F'(\phi)$ is also differentiable, Eq. 3.4.4 can be written as

$$[\phi_1 -\phi_2, F''(\widetilde{\phi})(\phi_1-\phi_2)] \geq 0 \qquad (3.4.5)$$

which implies

$$F''(\phi) \geq 0 \qquad (3.4.6)$$

where $\widetilde{\phi}=\phi_1+\eta(\phi_1-\phi_2)$, $0<\eta<1$. By reversing the process, it is not difficult to show that Eq. 3.4.6 also implies Eq. 3.4.2 or, equivalently, Eq. 3.4.3. Therefore, we obtain:

Lemma 3.4.2. A twice differentiable functional in C is convex if and only if $F''(\phi) \geq 0$.

A similar definition and lemmas for a functional of two functions are also useful. We list them below.

Definition 3.4.2. A functional $F(u,\phi):\Omega(u)x\Omega(\phi) \to R$ defined on a

convex subset B of $\Omega(u)$ is convex in u if

$$F(\lambda u_1+(1-\lambda)u_2,\phi) \leq \lambda F(u_1,\phi) + (1-\lambda)F(u_2,\phi) \qquad (3.4.7)$$

for all u_1, u_2 in B, ϕ in $\Omega(\phi)$, and all λ such that $0<\lambda<1$.

Lemma 3.4.3. If $F(u,\phi):\Omega(u)x\Omega(\phi) \to R$ is differentiable with respect

to u, then $F(u,\phi)$ is convex with respect to u in B if and only if

$$F(u_1,\phi) - F(u_2,\phi) - (u_1-u_2,Fu(u_2,\phi)) \geq 0 \qquad (3.4.8)$$

Lemma 3.4.4. A twice differentiable functional is convex with

respect to u in B if and only if $F_{uu}(u,\phi) \geq 0$.

The proof of the last lemma follows the similar argument for

Lemma 3.4.2. The same definition and lemma apply to the second variable

ϕ. We list them below for later reference.

Definition 3.4.3. A functional $F(u,\phi):\Omega(u)x\Omega(\phi) \to R$ defined on a

convex subset C of $\Omega(\phi)$ is convex in ϕ if

$$F(u,\lambda\phi_1+(1-\lambda)\phi_2) \leq \lambda F(u,\phi_1) + (1-\lambda)F(u,\phi_2) \qquad (3.4.9)$$

for all ϕ_1, ϕ_2 in C, u in $\Omega(u)$, and all λ such that $0<\lambda<1$.

Lemma 3.4.5. If $F(u,\phi):\Omega(u)x\Omega(\phi) \to R$ is differentiable with respect

to ϕ, then $F(u,\phi)$ is convex with respect to ϕ in C if and only if

$$F(u,\phi_1) - F(u,\phi_2) - [\phi_1-\phi_2,F\phi(u,\phi_2)] \geq 0 \qquad (3.4.10)$$

Lemma 3.4.6. A twice differentiable functional is convex with

respect to ϕ in C if and only if $F_{\phi\phi}(u,\phi) \geq 0$.

If strict inequality holds in Eqs. 3.4.7 or 3.4.9 for $u_1 \neq u_2$ or

$\phi_1 \neq \phi_2$ respectively, the functional $F(u,\phi)$ is called strictly convex.

Also, $F(u,\phi)$ is (strictly) concave in u or ϕ if $-F(u,\phi)$ is (strictly)

convex.

The final lemma to be presented in this section concerns a special type of functional called convex-concave saddle functional. A functional is convex-concave saddle functional if it is convex in u and concave in ϕ. If the functional is concave in u and convex in ϕ instead, we shall call it concave-convex saddle functional.

Suppose we are considering convex-concave saddle functional. It is then not difficult to see that we can write Eqs. 3.4.8 and 3.4.10 in slightly different forms:

$$F(u_i,\phi_j) - F(u_j,\phi_j) - (u_i-u_j,F_u(u_j,u_j)) \geq 0 \qquad (3.4.11)$$

and

$$-[F(u_i,\phi_j) - F(u_i,\phi_i) - [\phi_j-\phi_i,F_\phi(u_i,\phi_i)] \geq 0 \qquad (3.4.12)$$

respectively. By adding two together, we obtain the following lemma.

Lemma 3.4.7. If $F(u,\phi):\Omega(u)\times\Omega(\phi)\to R$ is differentiable, then $F(u,\phi)$

is a convex-concave saddle functional on BxC if and only if

$$F(u_i\phi_i) - F(u_j,\phi_j) - (u_i-u_j,F_u(u_j,\phi_j)) - [\phi_i-\phi_j,F_\phi(u_i,\phi_i)] \geq 0$$
$$(3.4.13)$$

Notice that the inequality in Eq. 3.4.13 will reverse if we are considering concave-convex instead of convex-concave saddle functional. This last lemma is probably the most important result for our purpose as it is shown in the next section.

3.5. Complementary Variational Principles

Finally, we are in a position to present the theorem that is the culmination of careful and systematic discussion of previous sections.

This theorem would be the most general form of complementary extremum principles we will be connected with. Because of its generality, the usefulness and implications of the theorem are not immediately obvious: we must wait until the next chapter to appreciate them. For now we will just present the theorem for the sake of completeness.

Theorem 3.5.1. Let $I(u,\phi)$ be a differentiable functional. Also, let Ω_1 and Ω_2 be the sets of functions $\Omega_1 = \{(u,\phi):Iu=0\}$ and $\Omega_2 = \{(u,\phi):I_\phi=0\}$. Then, if $I(u,\phi)$ is a concave-convex saddle functional, the complementary extremum principles

$$I(u,\phi) \leq I(u_1,\phi_1) \qquad (3.5.1)$$

and

$$I(u_2,\phi_2) \leq I(u,\phi) \qquad (3.5.2)$$

hold where (u_1,ϕ_1) and (u_2,ϕ_2) belong to Ω_1 and Ω_2, respectively, and (u,ϕ) is the intersection of Ω_1 and Ω_2. In other words, (u,ϕ) is a critical point of I. If I is a convex-concave saddle functional, instead, the inequalities in Eqs. 3.5.1 and 3.5.2 are reversed.

In the inequalities 3.5.1 and 3.5.2, subscripts 1 and 2 are added to signify that the functions belong to Ω_1 and Ω_2, respectively. The equalities hold only when ϕ_1 and u_1 are solutions of the stationary equations $I_u=0$ and $I_\phi=0$. In Fig. 3.5.1, an attempt is made to represent the theorem geometrically. It is intended only to be a visual aid in grasping essential concepts of the theorem.

Proof. Suppose $I_{(u,\phi)}$ is a concave-convex saddle functional. By Lemma 2.4.7, we then have

$$I(u_i,\phi_i) - I(u_j,\phi_j) - (u_i-u_j,I_u(u_j,\phi_j)) - [\phi_i-\phi_j,I_\phi(u_i,\phi_i)] \geq 0 \qquad (3.5.3)$$
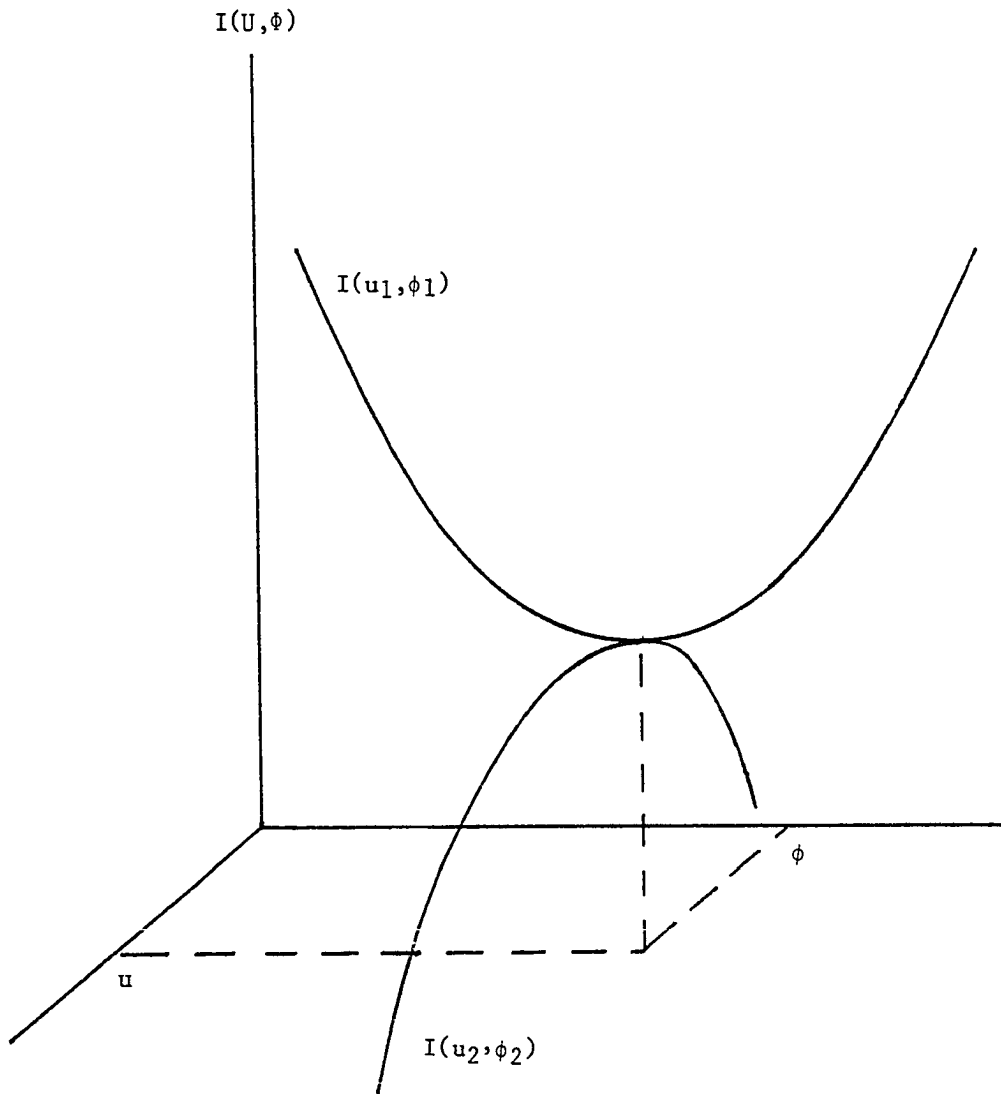
Figure 3.5.1. Concave-convex saddle functional $I(U,\Phi)$

By assumption, $I_u(u_1,\phi_1) = 0$ and $I_\phi(u_2,\phi_2) = 0$. Also, stationarity requires $I_u(u,\phi) = 0$, $I(u,\phi) = 0$. If we now let $(u_i,\phi_i) = (u,\phi)$ and $(u_j,\phi_j)=(u_1,\phi_1)$, the inequality (3.5.3) becomes

$$I(u,\phi)-I(u_1,\phi_1) \leq 0 \qquad\qquad (3.5.4)$$

which proves Eq. 3.5.1. Next, let $(u_i,\phi_i) = (u_2,\phi_2)$ and $(u_j,\phi_j) = (u,\phi)$. The result is

$$I(u_2,\phi_2) - I(u,\phi) \leq 0 \qquad\qquad (3.5.5)$$

proving (3.5.2).

4. REFORMULATION OF A CLASS OF LINEAR BOUNDARY-VALUE PROBLEMS

4.1. Introduction

In a variational problem, we are given an integral to be minimized or maximized. By changing the integral into a function of a parameter ε, we reduced the problem to a maximum-minimum problem of a function of single variable ε. This procedure yielded the necessary condition of Euler and Lagrange.

Often times, however, the problem is posed to us as differential equations of some kind with given boundary conditions. If one wants to recast the problem as variational integrals, he or she must solve the inverse problem. In the inverse problem, one begins with a differential equation and tries to find an integral whose stationary equations correspond to the problem at hand. Such a topic is discussed by various authors [13,14,35].

The main advantage of the variational formulation lies in the fact that it is well-suited in obtaining an approximate solution to the original boundary value problem. The Ritz method, for instance, yields an approximate solution that converges--at least in theory--to the exact solution [35]. There is another attractive aspect of this approach to the boundary value problem. In many problems of physical science or engineering, the stationary value itself is often an important physical quantity of great interest [25,35]. Therefore, it is important to point out that this quantity can be estimated quite accurately even

though the trial function does not even resemble the exact solution [51].
In this chapter, we will be concerned with only a certain class of
problems that can be reformulated as complementary variational integrals.
The symbols and concepts presented in this chapter closely follow those
presented by Arthurs [14].

4.2. The Inverse Problem

Consider a class of boundary value problems

$$T^*T\phi + Q\phi = f \text{ in } V \qquad (4.2.1)$$

$$\sigma\phi = \sigma\phi_B \text{ on } \partial V_1 \qquad (4.2.2)$$

$$\sigma^*T\phi + \beta\phi = \sigma^*u_B \text{ on } \partial V_2 \qquad (4.2.3)$$

Our goal is to find an integral whose stationary equations correspond
to Eq. 4.2.1 to Eq. 4.2.3. The boundary ∂V of region V consists of
two parts: ∂V_1 and ∂V_2. The operators T, T^*, σ, and σ^* are assumed
to belong to the special class discussed in the previous chapter.
Symbols Q, f, and β are given functions in V and ϕ_B, u_B are given
functions on the boundaries ∂V_1 and ∂V_2, respectively. This type of
problem occurs quite often in nature [14].

Before discussing the variational formulation of Eqs. 4.2.1 to
4.2.3, it is worth pointing out some background. The basic idea under-
lying the complementary variational principles originates from the
Hamiltonian principle in classical Newtonian dynamics [13]. In Hamilton's
principle, Newton's equations of motion become a variational integral
with two variables. The stationary equations, called canonical equations,

become the governing law for the motion of a particle. The canonical

equations are a set of coupled equations equivalent to the classical

Newtonian description of the motion. This splitting of the original

equation into a coupled set of canonical equations is the very idea

on which the complementary variational theory is based.

4.3. The Canonical Equations and Its Action Integral

The first step in our search for the desired integral--sometimes

called the action integral--is to split Eq. 4.2.1 into coupled equations.

The proper splitting is

$$T\phi = u \qquad \text{in } V \tag{4.3.1}$$

$$T^*u = f - Q\phi \tag{4.3.2}$$

$$\sigma\phi = \sigma\phi_B \quad \text{on } \partial V_1 \tag{4.3.3}$$

$$\sigma^*u + \beta\phi = \sigma^*u_B \text{ on } \partial V_2 \tag{4.3.4}$$

The new variable u is introduced here and the boundary conditions

are added for later reference.

We now turn our attention to the action integral. It must yield

Eq. 4.3.1 to Eq. 4.3.4 at its stationary point. There are systematic

general methods by which such an integral could be constructed [39].

Discourse to such a procedure, however, is outside our main purpose

and we must be content with the final result. First, we present the

action integral, then show that its stationary equations indeed reduce

to Eqs. 4.3.1 to 4.3.4.

Consider the following functional

$$I(u,\phi) = (u,T\phi)-W(u,\phi) - (u,\sigma(\phi-\phi_B))_{\partial V_1} + \tfrac{1}{2}[\phi,\beta\phi]_{\partial V_2} -$$

$$[\sigma^*u_B,\phi]_{\partial V_2} \qquad (4.3.5)$$

which, through the use of adjoint operators, can also be written as

$$I(u,\phi) = [T^*u,\phi] - W(u,\phi) + (u,\sigma\phi_B)_{\partial V_1} + [\sigma^*(u-u_B),\phi]_{\partial V_2} +$$

$$\tfrac{1}{2}[\phi,\beta\phi]_{\partial V_2} \qquad (4.3.6)$$

The second term $W(u,\phi)$ represents some arbitrary functional of variables

u and ϕ. Remember that the operators and scalar products belong to

the special class discussed in Chapter 1. As a result when written

out explicitly, Eqs. 4.3.5 and 4.3.6 become integral expressions.

By following the usual technique of replacing u and ϕ with $u+\epsilon\eta$ and $\phi+\epsilon\xi$, we

calculate the stationary equations. Each term of Eq. 4.3.5, for instance,

becomes

$$(u+\epsilon\eta,T\phi+\epsilon T\xi) = (u,T\phi) + (u,T\epsilon\xi) + (\epsilon\eta,T\phi) + (\epsilon\eta,T\epsilon\xi) \quad (4.3.7)$$

$$W(u+\epsilon\eta,\phi+\epsilon\xi) = W(u,\phi) + (\epsilon\eta,Wu(u,\phi)) + [\epsilon\xi, W_\phi(u,\phi)] + o_2 \quad (4.3.8)$$

$$(u+\epsilon\eta,\sigma(\phi-\phi_B)+\sigma\epsilon\xi)_{\partial V_1} = (u,\sigma(\phi-\phi_B))_{\partial V1} + (u,\sigma\epsilon\xi)_{\partial V_1} +$$

$$(\epsilon\eta,\sigma(\phi-\phi_B))_{\partial V_1} + (\epsilon\eta,\sigma\epsilon\xi)_{\partial V_1} \quad (4.3.9)$$

$$\tfrac{1}{2}[\phi+\epsilon\xi,\beta\phi+\epsilon\beta\xi]_{\partial V_2} = \tfrac{1}{2}[\phi,\beta\phi]_{\partial V_2} + [\epsilon\xi,\beta\phi]_{\partial V_2} + \tfrac{1}{2}[\epsilon\xi,\beta\epsilon\xi]_{\partial V_2}$$

$$(4.3.10)$$

$$[\sigma^*u_B,\phi+\epsilon\xi]_{\partial V_2} = [\sigma^*u_B,\phi]_{\partial V_2} + [\sigma^*u_B,\epsilon\xi]_{\partial V_2} \qquad (4.3.11)$$

Using these expressions the expansion of Eq. 4.3.5 about u,ϕ becomes

$$I(u+\epsilon\eta,\phi+\epsilon\xi) = I(u,\phi) + (\epsilon\eta,(T\phi-W_u)_V - \sigma(\phi-\phi_B)_{\partial V_1}) +$$

$$[\epsilon\xi,(T^*u-W_\phi)_V + (\sigma^*u+\beta\phi-\sigma^*u_B)_{\partial V_2}] + (\epsilon\eta,T\epsilon\xi) -$$

$$(\epsilon\eta,\sigma\epsilon\xi)_{\partial V_1} + \tfrac{1}{2}[\epsilon\xi,\beta\epsilon\xi] + o_2 + o_3 \qquad (4.3.12)$$

Referring back to Eq. 3.3.4, we can identify the derivatives as

$$I_u = (T\phi - Wu)_V - [\sigma(\phi - \phi_B)]_{\partial V_1} \tag{4.3.13}$$

$$I_\phi = (T^*u - W\phi)_V + [\sigma^*u + \beta\phi - \sigma^*u_B]_{\partial V_2} \tag{4.3.14}$$

Now, we invoke Theorem 3.3.2. According to this theorem, the functional of Eq. 4.3.5 is stationary at u,ϕ such that $I_u=0$ and $I_\phi=0$. Therefore, we obtain

$$T\phi = Wu$$

$$\text{in } V \tag{4.3.15}$$

$$T^*u = W\phi \tag{4.3.16}$$

$$\sigma\phi = \sigma\phi_B \qquad \text{on } \partial V_1 \tag{4.3.17}$$

$$\sigma^*u + \beta\phi = \sigma^*u_B \qquad \text{on } \partial V_2 \tag{4.3.18}$$

from the derivative expressions in Eqs. 4.3.13 and 4.3.14. Comparing Eqs. 4.3.1-4.3.4 with Eqs. 4.3.15-4.3.18, it is immediately clear that if we choose

$$Wu = u \tag{4.3.19}$$

$$W\phi = f - Q\phi \tag{4.3.20}$$

the stationary point of Eq. 4.3.5 becomes the original boundary value problem as desired. Simple calculations show that the functional $W(u,\phi)$ with above derivatives has the form

$$W(u,\phi) = \tfrac{1}{2}(u,u) - \tfrac{1}{2}[\phi,Q\phi] + [f,\phi] \tag{4.3.21}$$

Finally, we obtain the desired action integrals

$$I(u,\phi) = (u,T\phi) - \tfrac{1}{2}(u,u) + \tfrac{1}{2}[\phi,Q\phi] - [f,\phi] - (u,\sigma(\phi-\phi_B))_{\partial V_1} + \tfrac{1}{2}[\phi,\beta\phi]_{\partial V_2} - [\sigma^*u_B,\phi]_{\partial V_2} \tag{4.3.22}$$

$$I(u,\phi) = [T^*u,\phi] - \tfrac{1}{2}(u,u) + \tfrac{1}{2}[\phi,Q\phi] - [f,\phi] - (u,\sigma\phi_B)_{\partial V_1} + [\sigma^*(u-u_B),\phi]_{\partial V_2} + \tfrac{1}{2}[\phi,\beta\phi]_{\partial V_2} \tag{4.3.23}$$

where the last equation is the Eq. 4.3.6 with $W(u,\phi)$ written out explicitly.

4.4. Stationary Property of the Action Integral

In the previous section, we were able to start with a given boundary
value problem of a certain class and find its action integral. However,
nothing was mentioned about the nature of the stationary point. As
with ordinary function, a stationary point is either maximum, minimum
or saddle point. In the following, we show that added restriction
on the original problem of Eqs. 4.2.1-4.2.3 leads to complementary
extremum principles.

Referring back to Theorem 3.5.1, it is seen that either convex-
concave or concave-convex saddle functional yields complementary extremum
principles. Keeping this in mind, let us examine the action integral
of Eq. 4.3.22. Remember the expressions of Eqs. 4.3.22 and 4.3.23
are equivalent integrals in a different form. Therefore, any result
that holds for one is automatically also valid for the other. Two
lemmas (3.4.3 and 3.4.5) tell us that we look at expressions

$$I(u_1,\phi) - I(u_2,\phi) - (u_1-u_2, I_u(u_2,\phi)) \tag{4.4.1}$$

and

$$I(u,\phi_1) - I(u,\phi_2) - [\phi_1-\phi_2, I_\phi(u,\phi_2)] \tag{4.4.2}$$

where the derivatives are given by Eqs. 4.3.13 and 4.3.14 via Eqs. 4.3.19
and 4.3.20. After some manipulation and rearranging, Eqs. 4.4.1 and
4.4.2 become

$$I(u_1,\phi) - I(u_2,\phi) - (u_1-u_2, I_u(u_2,\phi)) = -\tfrac{1}{2}(u_1-u_2, u_1-u_2) \tag{4.4.3}$$

and

$$I(u,\phi_1) - I(u,\phi_2) - [\phi_1-\phi_2,Q(\phi_1-\phi_2)] = \tfrac{1}{2}[\phi_1-\phi_2,Q(\phi_1-\phi_2)] +$$

$$\tfrac{1}{2}[\phi_1-\phi_2,\beta(\phi_1-\phi_2)]_{\partial V_2} \qquad (4.4.4)$$

Therefore, the functional of Eq. 4.3.22 is always concave in u. If

we choose Q,β as

$$Q \geq 0 \text{ and } \beta \geq 0 \qquad (4.4.5)$$

the functional becomes convex in ϕ. The condition of Eq. 4.4.5 is,

therefore, the sufficient condition for the functional of Eq. 4.3.22

to be concave-convex saddle functional. We give this fact an elevated

status and state it as the following theorem.

Theorem 4.4.1. If $Q>0$ and $\beta>0$, then the action functionals

of Eqs. 4.3.22 and 4.3.23 are concave-convex saddle functionals.

4.5. Complementary Variational Integrals

Assuming that the condition of Eq. 4.4.5 is satisfied, Theorem 3.5.1

guarantees existence of two integrals--$I(u_1,\phi_1)$ and $I(u_2,\phi_2)$--which

approach the stationary value in a complementary fashion. Let us see

what these functionals look like.

Again referring to Theorem 3.5.1, we see that we need two sets of

pairs of functions, $\Omega_1 = \{(u,\phi):I_u=0\}$ and $\Omega_2 = \{(u,\phi):I_\phi=0\}$. In the

complementary variational theory, these sets are trivially generated in

the following way. In order to satisfy $I_u = 0$, we refer to Eq. 4.3.13.

Since the first and the second terms are defined in a region V and

its boundary ∂V, respectively, both terms must vanish independently.

This is accomplished first by picking ϕ arbitrarily from the set of

functions that assume the value ϕ_B on ∂V_1. This forces the boundary

term to vanish. Next, solve the first term for u in such a way that

it vanishes. The pair (u,ϕ) generated in this fashion forms a member

in the set Ω_1. In a similar manner, the set Ω_2 is generated from

Eq.4.3.14. Here, we need an assumption that the first and the second

terms can be solved for ϕ. If this is indeed the case, then $I_\phi = 0$ can

be satisfied by picking u arbitrarily, without any restriction in V

as well as on the boundary, and solving Eq. 4.3.14 for ϕ. The pair

formed in this way constitute a member in Ω_2.

With sets Ω_1 and Ω_2 constructed, we are now able to form two

complementary functionals. The functional $I(u_1,\phi_1)$ is obtained from

Eq. 4.3.22 by substituting the function u from Ω_1. The calculation

is straightforward and the result is

$$J(\phi_1) = \tfrac{1}{2}(T\phi_1,T\phi_1) + \tfrac{1}{2}[\phi_1,Q\phi_1] - [f,\phi_1] + \tfrac{1}{2}[\phi_1,\beta\phi_1]_{\partial V_2}$$
$$- [\sigma * u_B,\phi_1]_{\partial V_2} \qquad (4.5.1)$$

The subscript one is added to signify that the function ϕ is a member

of the set Ω_1. This integral gives the upper bound to the stationary

value. Substitution of ϕ from Ω_2 into Eq. 4.3.23 gives the other functional

$I(u_2,\phi_2)$. Again, after some simple calculations, one obtains the desired

integral

$$G(u_2) = -\tfrac{1}{2}(u_2,u_2) - \tfrac{1}{2}[Q^{-1}(f-T*u),f-T*u] + (u_2,\sigma\phi_B)_{\partial V_1}$$
$$- \tfrac{1}{2}[\beta^{-1}\sigma*(u_2-u_B),\sigma*(u_2-u_B)]_{\partial V_2} \qquad (4.5.2)$$

The subscript two signifies that u_2 is a member of Ω_2. In both functionals

(Eqs. 4.5.1 and 4.5.2), the designations have been changed to $J(\phi_1)$

and $G(u_2)$, respectively. This is to emphasize the fact that they are

now functions of only one variable. If we now substitute arbitrary

functions ϕ_1, u_2 to the corresponding functionals, Theorem 3.5.1

guarantees that the stationary value of the functional is always trapped

between the two functional values $J(\phi_1)$ and $G(u_2)$. The functionals

of Eqs. 4.5.1 and 4.5.2 are the desired reformulation of the boundary

value problem of Eqs. 4.2.1 through 4.2.3.

Although Theorems 3.5.1 and 4.4.1 together insure the complementary

nature of the functionals $J(\phi_1)$ and $G(u_2)$, it is reassuring to check

them directly. By following the conventional procedure of replacing

ϕ_1 with $\phi + \varepsilon\xi$, each term in Eq. 4.5.1 becomes

$$\tfrac{1}{2}(T\phi+T\varepsilon\xi,T\phi+T\varepsilon\zeta) = \tfrac{1}{2}(T\phi,T\phi) + (T\phi,T\varepsilon\xi) + \tfrac{1}{2}(T\varepsilon\xi,T\varepsilon\xi) \qquad (4.5.3)$$

$$\tfrac{1}{2}[\phi+\varepsilon\xi,Q\phi+Q\varepsilon\xi] = \tfrac{1}{2}[\phi,Q\phi] + [\phi,Q\varepsilon\xi] + \tfrac{1}{2}[\varepsilon\xi,Q\varepsilon\xi] \qquad (4.5.4)$$

$$[f,\phi+\varepsilon\xi] = [f,\phi] + [f,\varepsilon\xi] \qquad (4.5.5)$$

$$\tfrac{1}{2}[\phi+\varepsilon\xi,\beta\phi+\beta\varepsilon\xi]_{\partial V_2} = \tfrac{1}{2}[\phi,\beta\phi]_{\partial V_2} + [\phi,\beta\varepsilon\xi]_{\partial V_2} + \tfrac{1}{2}[\varepsilon\xi,\beta\varepsilon\xi]_{\partial V_2}$$

$$\qquad (4.5.6)$$

$$[\sigma^*u_B,\phi+\varepsilon\xi]_{\partial V_2} = [\sigma^*u_B,\phi]_{\partial V_2} + [\sigma^*u_B,\varepsilon\xi]_{\partial V_2} \qquad (4.5.7)$$

Grouping the terms with the same powers of ε, we obtain

$$J(\phi+\varepsilon\xi) - J(\phi) = [T^*T\phi+Q\phi-f,\varepsilon\xi] + (u,\sigma\varepsilon\xi)_{\partial V_1} +$$
$$[\sigma^*u+\beta\phi-\sigma^*u_B,\varepsilon\xi]_{\partial V_2} + \tfrac{1}{2}(T\varepsilon\xi,T\varepsilon\xi) +$$
$$\tfrac{1}{2}[\varepsilon\xi,Q\varepsilon\xi] + \tfrac{1}{2}[\varepsilon\xi,\beta\varepsilon\xi]_{\partial V_2} \qquad (4.5.8)$$

where $J(\phi)$ is the exact stationary value. Now, assuming the functional $J(\phi+\varepsilon\xi)$ is stationary at ϕ, we let the first variation vanish. The result is

$$T*T\phi + Q\phi = f \qquad \text{in } V \tag{4.5.9}$$

$$\sigma\phi = \sigma\phi_B \qquad \text{on } \partial V_1 \tag{4.5.10}$$

$$\sigma*T\phi + \beta\phi = \sigma*u_B \qquad \text{on } \partial V_2 \tag{4.5.11}$$

recovering the original boundary value problem as expected. The total variation of Eq. 4.5.8 is seen to be positive if $Q \geq 0$ and $\beta \geq 0$. This last result is what we wanted to confirm.

Following the same procedure, we can check the functional $G(u_2)$. Its total variation is calculated to be

$$G(u+\varepsilon\eta) - G(u) = -(u,\varepsilon\eta) + [Q^{-1}(f-T*u),T*\varepsilon\eta] + (\varepsilon\eta,\sigma\phi_B)_{\partial V_1} -$$
$$[\beta^{-1}\sigma*(u-u_B),\sigma*\varepsilon\eta]_{\partial V_2} - \tfrac{1}{2}(\varepsilon\eta,\varepsilon\eta) -$$
$$\tfrac{1}{2}[\varepsilon\xi,Q\varepsilon\xi] - \tfrac{1}{2}[\varepsilon\xi,\beta\varepsilon\xi]_{\partial V_2} \tag{4.5.12}$$

where $G(u)$ is the exact stationary value. The second variation is clearly negative, as predicted by Theorem 3.5.1

4.6. A Simpler Problem

So far, we have discussed the boundary value problem in which the boundary consisted of two parts, ∂V_1 and ∂V_2. This type of problem is generally referred to as the Dirichlet-Newmann problem. There is another type of problem called the Dirichlet problem. Unlike the Dirichlet-Newmann problem, the boundary is not divided in this type

of problem. In this section, a brief discussion is given for the Dirichlet problems defined by

$$T^*T\phi + Q\phi = f \quad \text{in } V \tag{4.6.1}$$

$$\sigma\phi = \sigma\phi_B \quad \text{on } \partial V \tag{4.6.2}$$

In order to reformulate the problem as complementary variational integrals, we follow the same steps illustrated in the previous section. There is no reason to repeat them here so only the final results are presented in the following paragraph.

All three functionals can be obtained from the corresponding functionals for the Dirichlet-Newmann problem by suppressing the scalar product terms on ∂V_2. We list them below for later reference.

$$I(u,\phi) = (u,T\phi) - \tfrac{1}{2}(u,u) + \tfrac{1}{2}[\phi,Q\phi] - [f,\phi] - (u,\sigma(\phi-\phi_B))_{\partial V} \tag{4.6.3}$$

$$= [T^*u,\phi] - \tfrac{1}{2}(u,u) + \tfrac{1}{2}[\phi,Q\phi] - [f,\phi] - (u,\sigma\phi_B)_{\partial V} \tag{4.6.4}$$

$$J(\phi_1) = \tfrac{1}{2}(T\phi_1,T\phi_1) + \tfrac{1}{2}[\phi_1,Q\phi_1] - [f,\phi_1] \tag{4.6.5}$$

$$G(u_2) = -\tfrac{1}{2}(u_2,u_2) - \tfrac{1}{2}[Q^{-1}(f-T^*u_2),f-T^*u_2] + (u_2,\sigma\phi_B)_{\partial V} \tag{4.6.6}$$

The last two equations give the desired complementary bounds.

4.7. Some General Aspects of the Theory

At this point, it is appropriate to point to some facts not mentioned in the previous sections. When constructing the set Ω_1, we needed to force the derivative I_u to vanish. In this process, the function ϕ had to come from a member of a certain restricted class of functions.

Therefore, the maximum principle $I(u,\phi) \leq J(\phi_1)$ is valid only when

the trial functions are picked from the class of functions that satisfy

the boundary condition $\phi_1 = \phi_B$ on ∂V. In the Dirichlet-Newmann problem,

this restriction is required only over the part of the boundary ∂V_1.

Theorem 3.5.1 also required construction of Ω_2 such that derivative

$I\phi = 0$. To meet this requirement, we had to assume that the first and the

second terms in Eq. 4.3.14 are solvable for ϕ. This is possible only when

$Q \neq 0$ and $\beta \neq 0$. The trial function u_2 is then completely arbitrary

when these conditions are satisfied. Unfortunately, if $Q = 0$, then in

both Dirichlet and Dirichlet-Newmann-type problems, the trial function u_2

must be picked from the class of functions such that $T^*u_2 = f$ in V. If

$\beta = 0$, in a Dirichlet-Newmann problem, u_2 must satisfy $\sigma^*u_2 = \sigma^*u_B$ on

∂V_2. In a Dirichlet problem, the boundary ∂V_2 is missing and u_2 is

always arbitrary on the boundary ∂V.

5. APPLICATION TO MAXWELL'S EQUATIONS

5.1. Introduction

In this chapter, we will briefly point out the difficulties encountered when one tries to apply the complementary variational theory to Maxwell's equations. We shall first discuss an attempt to formulate Maxwell's equations in their most general form. A discussion of sinusoidally-varying fields concludes the chapter.

5.2. Maxwell's Equations in General Form

Electromagnetic phenomena are governed by the vector equations

$$\bar{\nabla} x \bar{E} = - \frac{\partial \bar{B}}{\partial t} \tag{5.2.1}$$

$$\bar{\nabla} x \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J} \tag{5.2.2}$$

$$\bar{\nabla} \cdot \epsilon \bar{E} = q \tag{5.2.3}$$

$$\bar{\nabla} \cdot \bar{B} = 0 \tag{5.2.4}$$

where we are conforming with the traditional use of symbols $\bar{E}$, $\bar{H}$, $\bar{D}$, $\bar{B}$, and q. These symbols represent electric flux, magnetic flux, and charge density fields, respectively. Equation 5.2.3 can be derived from Eq. 5.2.2 through the continuity equation

$$\bar{\nabla} \cdot \bar{J} = - \frac{\partial q}{\partial t} \tag{5.2.5}$$

while taking the divergence of both sides of Eq. 5.2.1 yields Eq. 5.2.4. Therefore, Eqs. 5.2.1 and 5.2.2 are the only independent relationships [28].

There are two approaches in an attempt to formulate Eqs. 5.2.1
and 5.2.2 as complementary variational integrals. The first approach
tries to view the two curl equations as canonical equations of some
functional. In recent years, Anderson and Arthurs have discussed this
point of view [2,6,10]. There are several ways in which Maxwell's
two curl equations can be regarded as canonical equations. One of
these is to regard the two curl operators as T and T*. The next step
in the variational formulation is to find the functional W, whose partial
derivatives equal the right-hand side of Eqs. 5.2.1 and 5.2.2. However,
Anderson and Arthurs show that such a functional W does not exist [10].
In Reference 10, the authors introduce two variables, in addition to
$\overline{E}$ and $\overline{H}$, and rewrite Eqs. 5.2.1 and 5.2.2 as two pairs of coupled curl
equations. The authors were then able to derive several variational
functionals as functions of four variables.

In Reference 2, a different point of view is discussed. The
authors regard time-derivative operators in Eqs. 5.2.1 and 5.2.2 as
T and T*. By calculating the functional W, they were able to derive
a variational integral. Part of Reference 6 discusses the derivation
of two variational integrals from the original action integral presented
in Reference 2. However, these integrals do not give the dual extremum
principles as implied by the authors.

The second approach mentioned above starts with the reduction
of Eqs. 5.2.1 and 5.2.2 to a single wave equation. Specifically, we
try to see if the equation

$$\bar{\nabla} x \bar{\nabla} x \bar{E} + \mu\varepsilon \; \frac{\partial^2 \bar{E}}{\partial t^2} = -\mu \frac{\partial \bar{J}}{\partial t} \qquad\qquad (5.2.6)$$

belongs to the class of boundary value problem

$$T^*T\varphi + Q\,\phi = f \qquad\qquad (5.2.7)$$

discussed in the previous chapter. Equation 5.2.6 is written in terms

of the electric field $\bar{E}$, but a similar equation holds for the magnetic

field $\bar{\bar{H}}$.

The difficulty here is that the time derivative in the left-hand

side of Eq. 5.2.6 is, in general, not proportional to the function

$\bar{E}$. If it is, the proportionality constant can be considered as part

of Q in Eq. 5.2.7.

5.3. Time-Harmonic Wave Equation

Let us now specialize Eq. 5.2.6 to a sinusoidally-varying field

and write it as

$$\bar{\nabla} x \bar{\nabla} x \bar{E} - \omega^2 \mu\varepsilon \bar{E} = -\mu \; \frac{\partial \bar{J}}{\partial t} \qquad\qquad (5.3.1)$$

By identifying operators

$$T = \text{curl} \qquad\qquad (5.3.2)$$

$$T^* = \text{curl} \qquad\qquad (5.3.3)$$

we recognize immediately that Eq. 5.3.1 is a particular case of

Eq. 5.2.7. Therefore, the general results of Chapter 4 are applicable.

Equations 4.5.1 and 4.5.2 should yield two variational integrals.

Unfortunately, the quantity Q in Eq. 5.3.1 is

$$Q = -\omega^2 \mu\epsilon < 0 \qquad\qquad (5.3.4)$$

This violates the sufficient condition, Theorem 4.4.1, for the dual

extremum principles. It is also evident from Eq. 4.4.4 that the condition

$$Q \geq 0 \qquad\qquad (5.3.5)$$

is also a necessary condition for the dual extremum principles to be

valid if we assume boundary conditions are of nonmixed type. Therefore,

Eq. 5.3.4 implies that the boundary value problem, Eq. 5.3.1, cannot

be formulated as two complementary variational integrals. Equations

4.4.3 and 4.4.4 say that both integrals, $J(\phi)$ and $G(u)$, will approach

the stationary value from one side.

The above discussion indicates that the theory developed in Chapter 4

is not directly applicable to Maxwell's equations. In the following

chapters, we shall present an alternate approach. We will find that

the power series form of Maxwell's equations can be formulated as two

complementary variational integrals.

6. A POWER SERIES APPROACH TO SINUSOIDALLY-VARYING

ELECTROMAGNETIC FIELDS

6.1. Introduction

In the previous chapter, we saw that the complementary variational

theory in its original form does not apply to the basic electromagnetic

field laws. Even if we considered the special case of sinusoidally-

varying fields, there is difficulty. The very fact that the quantity

$\omega^2\mu\varepsilon$ is inherently positive makes $Q=-\omega^2\mu\varepsilon$ negative, violating the

assumption in Theorum 3.4.1. The result is that the stationary principles

still hold, but the complementary extremum principles fail.

It is clear then that in order for the complementary extremum

principles to be useful in the electromagnetic field theory, we must

either modify this mathematical theory--if it be possible--or rewrite

the basic electromagnetic field equations in a different form. In this

chapter, we will consider the latter option. It will be shown in the

next chapter that the complementary variational theory is applicable

to the equivalent but modified forms of Maxwell's equations.

The concepts and symbols discussed here closely follow that of

L. M. Magid [33]. In his book, Electromagnetic Fields, Energy, and

Waves, Magid carefully develops the concepts and touches upon many

insightful observations. He also gives thorough treatment of example

problems illustrating the usefulness of this approach to electromagnetic

fields. For our present purpose, we will focus only on the development
of the concepts.

6.2. Frequency Dependence of Single-Frequency
Sinusoidal Steady-State Fields

The three-dimensional space in which solutions of Maxwell's equations
are sought is sometimes referred to as the "region of fields." We will
retain the same useage. However, when the meaning is clear from the
context, we will simply refer to is as "region."

It is a physically observable fact that the sinusoidal electro-
magnetic field is a function of the spatial configuration and the
properties of the region of fields V, time t, position coordinates
x, y, z, and frequency ω. If we fix region V, time t, and the position
coordinates, the field becomes a function of frequency ω alone.

It is, therefore, legitimate to consider a Taylor series expansion
of each field quantity in ω about $\omega=0$ [33]. For example, it is possible
to write electric, magnetic, current density, and charge density fields
as follows:

$$\overline{E}(x,y,z,\tau,\omega) = \overline{e}(x,y,z,\tau) + \omega\overline{e}_1(x,y,z,\tau) +$$
$$\omega^2 \, \overline{e}_2(x,y,z,\tau) + \cdots \tag{6.2.1}$$

$$\overline{H}(x,y,z,\tau,\omega) = \overline{h}_0(x,y,z,\tau) + \omega\overline{h}_1(x,y,z,\tau) +$$
$$\omega^2 \, \overline{h}_2(x,y,z,\tau) + \cdots \tag{6.2.2}$$

$$\overline{J}(x,y,z,\tau,\omega) = \overline{j}_0(x,y,z,\tau) + \omega\overline{j}_1(x,y,z,\tau) +$$
$$\omega^2 \, \overline{j}_2(x,y,z,\tau) + \cdots \tag{6.2.3}$$

$$\rho(x,y,z,\tau,\omega) = \rho_0(x,y,z,\tau) + \omega\rho_1(x,y,z,\tau) +$$

$$\omega^2 \rho_2(x,y,z,\tau) + \cdots \qquad (6.2.4)$$

where the left-hand side explicitly shows independent variables for fixed region V. The variable τ stands for ωt. The variables ω and $\tau = \omega t$ can be considered independent variables because ω and t can be varied independent of each other [33]. Each coefficient of powers of ω can be evaluated by differentiating a required number of times and evaluating both sides of equality at $\omega = 0$. For example,

$$\overline{e}_0(x,y,z,\tau) = \overline{E}(x,y,z,\tau,\omega)\big|_{\omega=0} \qquad (6.2.5)$$

$$\overline{e}_k(x,y,z,\tau) = \frac{1}{K!} \left[\frac{\partial^k}{\partial\omega^k} E(x,y,z,\tau,\omega)\right]_{\omega=0} \qquad (6.2.6)$$

for the electric fields. However, these two expressions are hardly used in practice. The difficulty is that the total field $\overline{E}(x,y,z,\tau,\omega)$ is rarely known. If the exact field is known, there is no reason to resort to power series approach.

6.3. kth-order Field Equations

Let us now deduce the consequences of representing all the field quantities in Maxwell's equations as infinite power series in ω. First, rewrite Maxwell's equations (Eqs. 5.2.1-5.2.4) as

$$\overline{\nabla}x\overline{E} = -\omega \quad \partial\overline{B}/\partial\tau \qquad (6.3.1)$$

$$\overline{\nabla}x\overline{H} = \overline{J} + \omega \; \partial\overline{D}/\partial\tau \qquad (6.3.2)$$

$$\overline{\nabla}\cdot \overline{D} = \rho \qquad (6.3.3)$$

$$\overline{\nabla}\cdot \overline{B} = 0 \qquad (6.3.4)$$

in which $\tau=\omega t$ and $d\tau/dt$ is replaced by ω. Now, substitute a power series representation of the total field for each field quantity. For example, Eq. 6.3.1 will look like

$$\overline{\nabla}x\overline{e}_0 + \omega\overline{\nabla}x\overline{e}_1 + \omega^{-2}\,\overline{\nabla}x\overline{e}_2 + \cdots$$
$$= -\omega\,(\frac{\partial\overline{b}_o}{\partial\tau} + \omega\,\frac{\partial\overline{b}_1}{\partial\tau} + \omega^2\,\frac{\partial\overline{b}_2}{\partial\tau}\,\cdots) \qquad (6.3.5)$$

where term-by-term differentiation, with respect to the variables x,y,z,τ, is assumed to be valid. This equation can be rewritten as

$$(\overline{\nabla}x\overline{e}_0) + \omega(\overline{\nabla}x\overline{e}_1+\partial\overline{b}_0/\partial t) + \omega^2(\overline{\nabla}x\overline{e}_2+\partial\overline{b}_1/\partial t) + \cdots = 0 \qquad (6.3.6)$$

In Eq. 6.3.6, we notice that the set of functions $(1,\omega,\omega^2,\omega^3,\cdots)$ are linearly independent. Therefore, each coefficient must be separately zero in order for this equation to hold for all values of ω. Performing similar calculations for remaining Eqs. 6.3.2-6.3.4, we obtain the desired results.

$$\overline{\nabla}x\overline{e}_0 = 0 \qquad (6.3.7)$$

$$\overline{\nabla}x\overline{h}_0 = \overline{j}_0 \qquad (6.3.8)$$

$$\overline{\nabla}\cdot\overline{d}_0 = \rho_0 \qquad (6.3.9)$$

$$\overline{\nabla}\cdot b_0 = 0 \qquad (6.3.10)$$

$$\overline{\nabla}\cdot\overline{J}_0 = 0 \qquad (6.3.11)$$

for the zero-order fields and

$$\overline{\nabla}x\overline{e}_k = -\; \partial b_{k-1}/\partial\tau \qquad (6.3.12)$$

$$\overline{\nabla}x\overline{h}_k = \overline{j}_k + \partial\overline{d}_{k-1}/\partial\tau \qquad (6.3.13)$$

$$\overline{\nabla}\cdot\overline{d}_k = -\rho_k \qquad (6.3.14)$$

$$\overline{\nabla}\cdot\overline{b}_k = 0 \qquad (6.3.15)$$

$$\overline{\nabla}\cdot\overline{j}_k = -\; \partial\rho_{k-1}/\partial\tau \qquad (6.3.16)$$

for the kth-order fields where k≥1. The kth-order boundary conditions
are obtained from the original boundary conditions given in Chapter 5.
The calculation procedure is the same as above and straightforward.
There is no reason to repeat them here so we shall simply list the
results.

$$\bar{n}x(\bar{e}_I-\bar{e}_{II})_k = 0 \tag{6.3.17}$$

$$\bar{n}x(\bar{h}_I-\bar{h}_{II})_k = \bar{K}_k \tag{6.3.18}$$

$$\bar{n}\cdot(\bar{d}_I-\bar{d}_{II})_k = \eta_k \tag{6.3.19}$$

$$\bar{n}\cdot(\bar{b}_I-\bar{b}_{II})_k = 0 \tag{6.3.20}$$

$$\bar{n}\cdot(\bar{j}_I-\bar{j}_{II})_k + \bar{\nabla}_\Sigma \cdot \bar{k}_k = -\frac{\partial\eta_{k-1}}{\partial\tau} \tag{6.3.21}$$

These boundary conditions apply to fields of all orders including
zero-order fields with one minor correction. In Eq. 6.3.21, the right-
hand side of equality becomes zero for zero-order fields.

6.4. Significance of the kth-order Field Laws

Perhaps the most important feature is the fact that, like zero-order
fields, the kth-order field laws for k≥1 are not coupled anymore [33].
In the original Maxwell's equations $\bar{E}$ and $\bar{H}$, for example, are coupled
through their time derivative terms $\partial\bar{E}/\partial t$ and $\partial\bar{H}/\partial t$. A closer look
at kth-order field laws, however, shows that the right-hand side of
the curl equations contain the derivative terms on the k-1th-order
fields not on the kth-order fields. This is very significant. The
fact that the kth-order $\bar{E}$ and $\bar{H}$ are not coupled through their time
derivative terms means that the equations become much easier to

solve [33]. In fact, the time derivative terms in the kth-order

can be regarded as a source, and the resultant kth-order fields become

static-like fields [33].

Also some implications on the zero-order fields deserve to be

mentioned. The zero-order field equations have the same mathematical

form as that of the static field equations. However, with later appli-

cation in mind, it is important to point out some differences. The

zero-order fields are, first of all, time-varying fields. For a fixed

point in the region of fields, the zero-order fields vibrate sinusoidally

as time passes. This is in drastic contrast to the static fields that

are completely independent of the time variable t. Secondly, the zero-

order fields are part of the building blocks to the exact sinusoidally

varying fields through the infinite summations such as Eqs. 6.2.1-6.2.4.

The zero-order field acts as a source to the first-order field, which

in turn becomes a source to the second-order field [33].

Such a concept is entirely missing from the static fields. In

this sense, the static field is completely divorced from the time-varying

field, while the zero-order field is not. Therefore, we conclude

that the two fields, static and zero-order, are conceptually quite

different from each other. We are emphasizing these points here because

in Chapter 8, zero-order example problem is worked out as a specific

application of the complementary variational formulation of kth-order

field laws.

In his book Magid [33], works out three example problems: a

capacitor, an inductor, and a resistor. He shows that useful information

on impedance variations as a function of frequency ω are obtained from

the first three terms of the series expansions. The crudest approximation

to the true fields is to neglect every higher order terms except the

zero-order. Notice that even in this crudest approximation the time

variation factor ωt is accounted for. Once the zero-order fields

are obtained, which require no more effort than solving a static problem,

they can be used to obtain first-order fields, which in turn can be

used for obtaining second-order fields and this process can be continued

to all orders [33]. The next paragraph further illustrates the importance

of power series approach to engineering electromagnetic field problems.

This whole paragraph is quoted from the book by L. M. Magid [33].

"The quasi-static fields are defined as the time-
varying fields correct up to and including the first-order
contribution. The sinusoidally-varying quasi-static
$\overline{E}$ and $\overline{H}$ fields, for example, are given by

$$\overline{E} = \overline{e}_0 + \overline{e}_1 \qquad\qquad (6.4.1)$$

$$\overline{H} = \overline{h}_0 + \overline{h}_1 \qquad\qquad (6.4.2)$$

The quasi-static fields defined above are clearly
approximations to the corresponding exact field values
that would be given by the entire infinite series of
Eqs. 5.2.1-5.2.4. They consist, to be sure, of only
the first two terms of each of those series. Although
one's first reaction here might be to consider the above
quasi-static fields as relatively poor approximations
to the exact series solutions, this is not necessarily
the case in many low frequency (and some not-so-low
frequency) systems. The quasi-static fields frequently
offer considerable insight into the response of many

systems of practical importance to electrical engineers. In fact, the very foundation of circuit theory, ranging from the terminal current-voltage characteristics of the lumped circuit elements, R, L, and C, to Kirchhoff's laws, follows directly from Maxwell's equations as quasi-static approximations."

6.5. Alternative Forms

Equations 6.3.12-6.3.21 can be rewritten in slightly different forms that are more convenient [33]. The time derivative terms with respect to τ in the curl equations will be replaced with terms differentiated with respect to t rather than τ. This brings the kth-order equations to a closer resemblance to the original Maxwell's equations.

For the purpose of illustration, consider Eq. 6.2.1. We define the new terms in the power series as

$$\overline{E}_k(x;y,z,\tau,\omega) \overset{\Delta}{=} \omega^k \overline{e}_k(x,y,z,\tau), \quad k \geq 0 \tag{6.5.1}$$

so the total field will look like

$$\overline{E}(x,y,z,\tau,\omega) = \overline{E}_0(x,y,z,\tau) + \overline{E}_1(x,y,z,\tau,\omega) +$$
$$\overline{E}_2(x,y,z,\tau,\omega) + \cdots \tag{6.5.2}$$

Unlike the previous kth-order fields, the new kth-order fields are dependent on frequency ω. The explicit dependence is expressed by Eq. 6.5.1.

The next step in the derivation is the multiplication of

Eq. 6.3.12 by ω^k. The result is

$$\overline{\nabla} \times \omega^k \overline{e}_k = -\omega \frac{\partial}{\partial \tau} \quad \omega^{k-1} \overline{b}_{k-1} \tag{6.5.3}$$

where in the right-hand side ω^k is split into two parts as shown.

Using the new definition, Eq. 6.5.1, it can be written

$$\overline{\nabla} \times \overline{E}_k = -\frac{\partial \overline{B}}{\partial t} \, {}_{k-1} \tag{6.5.4}$$

in which the derivative with respect to τ has been replaced by the

derivative with respect to t. This step is straightforward if we

recall that $\tau = \omega t$.

Equation 6.5.4 is the desired result. As indicated earlier,

this equation has the same form as the original Maxwell's equations,

except for the subscripts in $\overline{E}$ and $\overline{B}$. We can carry out the similar

calculations on Eqs. 6.3.13-6.3.21 with the results

$$\overline{\nabla} \times \overline{E}_0 = 0 \tag{6.5.5}$$

$$\overline{\nabla} \times \overline{H}_0 = \overline{J}_0 \tag{6.5.6}$$

$$\overline{\nabla} \cdot \overline{D}_0 = \rho_0 \tag{6.5.7}$$

$$\overline{\nabla} \cdot \overline{B}_0 = 0 \tag{6.5.8}$$

$$\overline{\nabla} \cdot \overline{J}_0 = 0 \tag{6.5.9}$$

for the zero-order fields, and

$$\overline{\nabla} \times \overline{E}_k = -\frac{\partial \overline{B}_{k-1}}{\partial t} \tag{6.5.10}$$

$$\overline{\nabla} \times \overline{H}_k = \overline{J}_k + \frac{\partial \overline{D}_{k-1}}{\partial t} \tag{6.5.11}$$

$$\overline{\nabla} \cdot \overline{D}_k = \rho_k \tag{6.5.12}$$

$$\overline{\nabla} \cdot \overline{B}_K = 0 \tag{6.5.13}$$

$$\overline{\nabla} \cdot \overline{J}_k = -\frac{\partial \rho_{k-1}}{\partial t} \tag{6.5.14}$$

for the kth-order fields where $k \geq 1$. The boundary conditions become

$$\overline{n} \times (\overline{E}_I - \overline{E}_{II})_k = 0 \qquad\qquad (6.5.15)$$

$$\overline{n} \times (\overline{H}_I - \overline{H}_{II})_k = \overline{K}_k \qquad\qquad (6.5.16)$$

$$\overline{n} \cdot (\overline{D}_I - \overline{D}_{II})_k = \eta_k \qquad\qquad (6.5.17)$$

$$\overline{n} \cdot (\overline{B}_I - \overline{B}_{II})_k = 0 \qquad\qquad (6.5.18)$$

$$\overline{n} \cdot (\overline{J}_I - \overline{J}_{II})_k + \overline{\nabla}_\Sigma \cdot \overline{k}_k = \frac{\partial \eta_{k-1}}{\partial t} \qquad\qquad (6.5.19)$$

In Eq. 6.5.19, the same comment made on Eq. 6.3.21 applies, that is,

the right-hand side of the equality becomes zero for zero-order fields.

7. COMPLEMENTARY VARIATIONAL FORMULATION OF kTH-ORDER FIELD LAWS

7.1. Introduction

We are finally in a position to discuss the application of the complementary variational principles to the electromagnetic field laws. In the following, it will be shown that the kth-order field laws discussed in the previous chapter can be formulated as two variational integrals for which dual (complementary) extremum principles hold. However, some restrictions must be placed on the region of fields before such formulations are possible.

7.2. Property of Region of Fields

As far as electromagnetic fields are concerned, a region of space--whether vacuum or filled with matter--can be characterized by three parameters [29]. These parameters are permittivity ε, permeability μ, and conductivity σ. There are certain standard terms used to describe the nature of materials that apply to many physical properties. It is common to retain this usage in describing electric and magnetic properties as well. The equivalent definitions of the following terms can be found in many textbooks such as Ref. 28.

1. If all three parameters of the material do not depend on position, the term homogeneous applies; otherwise, the material is said to be inhomogeneous.

2. If all three parameters of the medium are the same regardless of the direction of any of the field vectors, it is called isotropic. If the relations depend on field directions, the medium is anisotropic.

3. If all three parameters do not depend on the amplitude of the field, the medium is called linear; otherwise, it is nonlinear.

4. If all three parameters change with time, the medium is called time-varying; otherwise, it is time-invariant.

The solution of Maxwell's equations depend strongly on the properties of the region for which solutions are sought [33]. Normally, it is very difficult or impossible to solve field equations for the very general cases of inhomogeneous, anisotropic, nonlinear, and lossy region. For this reason, many textbooks on the subject discuss only the special cases for which exact solutions can be found. A similar difficulty seems to be true when one tries to formulate Maxwell's equations as variational integrals. Difficulties seem to multiply exponentially as one allows more properties to be general. For example, Konrad's paper in 1976 [30] appears to be the first to present a three-component variational formulation valid in a region of anisotropic media. Similarly, it was not until Chun and Chuen's work was published in 1980 [21] that the

loss in terms in variational integrals could be systematically accounted for. In their paper, they discuss the general nonself-adjoint problem and apply it to Maxwell's equations with loss.

In the following sections, it will be shown that the dual extremum formulations of kth-order field laws are possible at least in the lossless, inhomogeneous, isotropic, linear, and time-invariant region. Notice that the inhomogeneity is the only property allowed to be general while other properties are restricted. Letting other properties become general introduces difficulties unsurmountable at the present time.

However, allowing inhomogeneity of the region is very significant. There are many problems in engineering electromagnetics where all the properties of a region of fields are restricted. This is evident from the casual reading of some popular college level textbooks on the subject, such as Ref. 29. Allowing inhomogeneity will certainly enlarge the class of solvable engineering problems significantly.

7.3. Impressed Sources $\bar{J}$ and ρ

In the previous section, we indicated that the conductivity will be assumed to be zero. Under this assumption, the current density $\bar{J}$ will be considered to consist entirely of impressed source current. In other words, the only induced currents will be the electric and magnetic displacement currents, and $\bar{J}$ will be considered as a source (cause) of the field.

The variational formulation presented in the following sections depends strongly on the nature of $\bar{J}$ and ρ. To be precise, for the charge density, the zero-order term ρ_0 will be the only term allowed to take on a nonzero value. We must assume all the other terms ρ_k to be equal to zero for $k \geq 1$.

This restriction makes it possible to represent electric field $\bar{E}_k$, for $k \geq 1$ as a curl of some vector field. On the other hand, current density fields $\bar{J}_k$, for $k \geq 0$, can be allowed to be nonzero. The reason for these restrictions will become clear as one follows through the mathematical derivations in the following sections.

In order to develop some insights into the nature of sources $\bar{J}$ and ρ, let us pause for a moment and reflect on some of the implications of the power series approach. Equation 6.5.1 says that the frequency ω^k is a simple scalar multiplying factor. Frequency does not enter into $\bar{e}_k$ in any fashion. Let us consider a special class of solutions, discussed in the book by Magid [33], which are sufficient to illustrate the nature of kth-order fields. These solutions have the general form

$$\bar{E}_k(x,y,z,\tau,\omega) = \omega^k \bar{f}_k(x,y,z) \cos \omega t \qquad (7.3.1)$$

where the amplitude consists of ω^k multiplied by the spatial factor, $\bar{f}_k$. It is important to point out that, in general, the vector fields $\bar{f}_k$ will be different for different frequencies. This is evident by noting the facts that the spatial configuration of the total field changes as frequency varies and the spatial information is contained only in the term $\bar{f}_k$. Each kth-order field vibrates at single frequency ω. This situation is quite different from the usual series expansions

encountered in electrical engineering. Normally, electrical engineers talk about Fourier series in which each term in the series vibrates at integer multiple of the fundamental frequency. The equation (7.3.1) is written in terms of electric field but of course similar equations apply to other field quantities.

It is evident from Eq. 7.3.1 that the zero-order field is the only term without the frequency multiplying factor. Furthermore, the zero-order field laws (Eqs. 6.5.5-6.5.9) say that these equations are completely independent of frequency. (This statement seems to contradict the fact that zero-order fields vibrate at frequency ω. However, let us quickly remind ourselves that the theory of the power series approach is built on the rightful assumption that ω and ωt are independent.) Therefore, unlike every other term, $\overline{f}_0$ in Eq. 7.3.1 is independent of the frequency.

Now, let us go back to the sources $\overline{J}$ and ρ. As clear from the above discussion, restricting the charge density ρ to be equal to the zero-order term means that its spatial configuration as well as the amplitude multiplying factor are independent of frequency ω. Of course, they may vibrate at frequency ω. The current density, on the other hand, was allowed to have an unrestricted number of nonzero kth-order terms. This implies that the spatial distribution of the total current may be a function of frequency.

This concludes the necessary preliminary discussions and we are now ready to discuss the main topic of this chapter, the complementary variational formulation of kth-order field equations.

7.4. Formulation of Zero-order Electric Field Equations

For convenience, let us at the outset write down the zero-order electric field equations.

$$\bar{\nabla} \times \bar{E}_0 = 0 \qquad\qquad (7.4.1)$$

$$\bar{\nabla} \cdot \varepsilon\bar{E}_0 = \rho_0 \qquad\qquad (7.4.2)$$

The first equation says that $\bar{E}_0$ is curl-free. As such, it can be represented by gradient of a scalar field. So, let us write

$$\bar{E}_0 = - \bar{\nabla}\phi \qquad\qquad (7.4.3)$$

in which a minus sign is added to make the function ϕ represent a real physical quantity, voltage. This representation guarantees that Eq. 7.4.1 is identically satisfied. Now, substitute Eq. 7.4.3 into Eq. 7.4.2 to get

$$\bar{\nabla} \cdot \varepsilon\bar{\nabla}\phi = - \rho_0 \qquad\qquad (7.4.4)$$

where ε and ρ are, in general, functions of spatial coordinates. Equation 7.4.4 is equivalent to Eqs. 7.4.1 and 7.4.2 taken together, but with the restriction that $\bar{E}_0$ is represented by the special form of Eq. 7.4.3. The goal is then to find two complementary variational integrals whose stationary equations are equivalent to Eq. 7.4.4 and imposed boundary conditions not yet specified.

The first step is to invent suitable operators, T, T*, σ, σ*, as discussed in Section 3.2, satisfying the equation

$$(u,T\phi) = [T*u,\phi] + (u,\sigma\phi)_{\partial V} \qquad\qquad (7.4.5)$$

where the boundary term can also be written as:

$$(u,\sigma\phi)_{\partial V} = [\sigma*u,\phi]_{\partial V} \qquad\qquad (7.4.6)$$

To this end, consider the vector identity

$$\overline{\nabla} \cdot \varepsilon \,\overline{u}\phi = \varepsilon\overline{u} \cdot \overline{\nabla}\phi + \phi\overline{\nabla} \cdot \varepsilon\overline{u} \tag{7.4.7}$$

After integrating both sides of equality, it can be rewritten as:

$$\int (\overline{u}\cdot\overline{\nabla}\phi)\varepsilon dV = \int \frac{1}{\varepsilon}(\overline{\nabla}\cdot\varepsilon\overline{u})\phi\varepsilon dV + \int_{\partial V} (\overline{u}\cdot\overline{n}\phi)\varepsilon dB \tag{7.4.8}$$

By changing the order of $\overline{u}$ and $\overline{n}$, the last term is also equal to

$$\int_{\partial V} (\overline{u}\cdot\overline{n}\phi)\varepsilon dB = \int_{\partial V} (\overline{n}\cdot\overline{u})\phi\varepsilon dB \tag{7.4.9}$$

Comparing Eq. 7.4.8 with Eq. 7.4.5, we now identify the operators as

$$T\phi = \overline{\nabla}\phi \tag{7.4.10}$$

$$T*u = -\frac{1}{\varepsilon}\overline{\nabla}\cdot\varepsilon\overline{u} \tag{7.4.11}$$

$$\sigma\phi = \overline{n}\phi \tag{7.4.12}$$

The adjoint operator $\sigma*$ can be identified as

$$\sigma*u = \overline{n}\cdot\overline{u} \tag{7.4.13}$$

from Eqs. 7.4.6 and 7.4.9. Note that our scalar products contain a

multiplying factor ε. This is slightly different than any of the scalar

products encountered in Chapter 3. In Eqs. 7.4.11 and 7.4.13, the left-

hand sides are written in a general notation that does not distinguish a

vector function from a scalar function. This notation will be retained

throughout the remaining discussions.

Next step in the formulation is to identify the original problem

Eq. 7.4.4 as belonging to a class of problems

$$T*T\phi + Q\phi = f \tag{7.4.14}$$

Knowing what the operators T,T* for this problems look like, we see

that the left hand side of Eq. 7.4.4 must match up with the term $T*T\phi$

in Eq. 7.4.14. This can be seen by rewriting Eq. 7.4.4 as

$$-\frac{1}{\varepsilon}\overline{\nabla}\cdot\varepsilon\overline{\nabla}\phi = \frac{\rho_0}{\varepsilon} \tag{7.4.15}$$

and comparing it with Eq. 7.4.14. Now it is clear that Eq. 7.4.15

is a special case of Eq. 7.4.14 with

$$Q = 0 \qquad\qquad (7.4.16)$$

$$f = \frac{\rho_0}{\epsilon} \qquad\qquad (7.4.17)$$

Therefore, according to the theory developed in Chapter 4, the original

problem Eq. 7.4.4, or equivalently Eq. 7.4.15, can be formulated as

two complementary variational integrals.

To make the original problem complete, we must impose boundary

conditions. We will consider two such conditions.

$$\bar{n}\phi = \bar{n}\phi_B \text{ on } \partial V \qquad\qquad (7.4.18)$$

or

$$\bar{n}\phi = \bar{n}\phi_B \text{ on } \partial V_1 \qquad\qquad (7.4.19)$$

$$\bar{n} \cdot \bar{\nabla}\phi = \bar{n} \cdot \bar{u}_B \text{ on } \partial V_2 \qquad\qquad (7.4.20)$$

The first condition, Eq. 7.4.18, is the Dirichlet condition and

Eqs. 7.4.19 and 7.4.20 together are called Dirichlet-Newmann conditions.

It is felt that the above two conditions are quite general and include

many problems of interest in electrical engineering. Consequently,

the original equation (7.4.15) with one of the two boundary conditions

will completely specify our problem.

It is now easy to see that our problem belongs to the special class

of boundary value problems discussed in Chapter 4. The remaining task

is therefore is simply to apply the general results developed in that

chapter. The results we need are the Eqs. 4.6.5 and 4.66 for Dirichlet

problem and Eqs. 4.5.1 and 4.5.2 for Dirichlet-Newmann problem. For

convenience we will list the original boundary value problems together with their complementary variational equivalents.

1. Dirichlet Problem

   a) Original problem

   $$\frac{1}{\varepsilon} \overline{\nabla} \cdot \varepsilon \overline{\nabla} \phi = \frac{\rho_0}{\varepsilon} \quad \text{in } V \tag{7.4.21}$$

   $$\overline{n} \phi = \overline{n} \phi_B \quad \text{on } \partial V \tag{7.4.22}$$

   b) Complementary variational counterparts

   $$J(\phi) = \int \tfrac{1}{2} \varepsilon \overline{\nabla} \phi \cdot \overline{\nabla} \phi - \rho_0 \phi \, dV \tag{7.4.23}$$

   $$G(\overline{u}) = \int -\tfrac{1}{2} \varepsilon \, \overline{u} \cdot \overline{u} \, dV + \int_{\partial V} \overline{u} \cdot \overline{n} \phi_B \, \varepsilon dB \tag{7.4.24}$$

2. Dirichlet-Newmann Problem

   a) Original problem

   $$\frac{1}{\varepsilon} \overline{\nabla} \cdot \varepsilon \overline{\nabla} \phi = \frac{\rho_0}{\varepsilon} \quad \text{in } V \tag{7.4.25}$$

   $$\overline{n} \phi = \overline{n} \, \phi_B \quad \text{on } \partial V_1 \tag{7.4.26}$$

   $$\overline{n} \cdot \overline{\nabla} \phi = \overline{n} \cdot \overline{u}_B \quad \text{on } \partial V_2 \tag{7.4.27}$$

   b) Complementary variational counterparts

   $$J(\phi) = \int \tfrac{1}{2} \varepsilon \overline{\nabla} \phi \cdot \overline{\nabla} \phi - \rho_0 \phi \, dV - \int \overline{n} \cdot \overline{u}_B \, \phi \varepsilon dB \tag{7.4.28}$$

   $$G(\overline{u}) = \int -\tfrac{1}{2} \, \overline{u} \cdot \overline{u} \, dV + \int_{\partial V_1} \overline{u} \cdot \overline{n} \phi_B \varepsilon dB \tag{7.4.29}$$

We have therefore succeeded in transforming the boundary value problem of zero-order electric field into a variational problems.

7.5. Proof of Dual Extremum Principles

Although the theory developed in Chapter 4 guarantees complementary extremum principles, it is reassuring to prove them directly. We will

prove "minimum" principle. Let us first write down the canonical form, discussed in Chapter 4, of the original problem:

$$\overline{\nabla}\phi = \overline{u} \quad \text{in V} \tag{7.5.1}$$

$$\frac{1}{\epsilon}\overline{\nabla}\cdot\epsilon\overline{u} = \frac{\rho_0}{\epsilon} \tag{7.5.2}$$

$$\overline{n}\phi = \overline{n}\phi_B \quad \text{on } \partial V \tag{7.5.3}$$

Next, substitute $\phi + \alpha\xi$, where ϕ is assumed to be an extremal, for ϕ in Eq. 7.4.23. Each term in the integrand becomes

$$\tfrac{1}{2}\epsilon(\overline{\nabla}\epsilon+\overline{\nabla}\alpha\xi)\cdot(\overline{\nabla}\phi+\overline{\nabla}\alpha\xi) = \tfrac{1}{2}\epsilon\overline{\nabla}\phi\cdot\overline{\nabla}\phi+\epsilon\overline{\nabla}\phi\cdot\overline{\nabla}\alpha\xi+\tfrac{1}{2}\epsilon\overline{\nabla}\alpha\xi\cdot\overline{\nabla}\alpha\xi \tag{7.5.4}$$

and

$$\rho_0(\phi+\alpha\xi) = \rho_0\phi + \rho_0\alpha\xi \tag{7.5.5}$$

resulting in the equality

$$J(\phi+\alpha\xi) = \int \tfrac{1}{2}\epsilon\overline{\nabla}\phi\cdot\overline{\nabla}\phi - \rho_0 \, dV + \alpha\int\epsilon\overline{\nabla}\phi\cdot\overline{\nabla}\xi - \rho_0\xi dV +$$
$$\alpha^2\int\tfrac{1}{2}\epsilon\overline{\nabla}\xi\cdot\overline{\nabla}\xi dV \tag{7.5.6}$$

Through the use of vector identity

$$\overline{\nabla}\cdot(\xi\epsilon\overline{\nabla}\phi) = \overline{\nabla}\xi\cdot\epsilon\overline{\nabla}\phi+\xi\overline{\nabla}\cdot\epsilon\overline{\nabla}\phi \tag{7.5.7}$$

the first variation can be written as

$$\delta I = \alpha\int -\xi\overline{\nabla}\cdot\epsilon\overline{\nabla}\phi - \rho_0\xi \, dV + \alpha\int\xi\epsilon\overline{\nabla}\phi\cdot\overline{n}dB \tag{7.5.8}$$

If we pick trial functions from the class of functions satisfying the boundary condition as required by the theory, we get the relationship

$$\overline{n}\phi + \overline{n}\alpha\xi = \overline{n}\phi_B \tag{7.5.9}$$

But, the function ϕ by itself must certainly satisfy the boundary condition. Therefore, we obtain

$$\overline{n}\xi = 0 \tag{7.5.10}$$

The first variation now becomes

$$\delta I = \alpha\int -\xi(\overline{\nabla}\cdot\epsilon\overline{\nabla}\phi+\rho_0)dV \tag{7.5.11}$$

For this integral to vanish for all arbitrary ξ, the fundamental theorem of calculus of variations guarantees that the factor

$$\overline{\nabla}\cdot\epsilon\overline{\nabla}\phi + \rho_0 = 0 \qquad (7.5.12)$$

must be zero recovering the original differential equation. This proves what is sometimes referred to as the "stationary" principle. Because of the initial assumption that ϵ is an extremal Eq. 7.5.6 becomes

$$J(\phi+\alpha\xi) - J(\phi) = \alpha^2 \int \tfrac{1}{2}\epsilon\overline{\nabla}\xi\cdot\overline{\nabla}\xi dV \qquad (7.5.13)$$

where ϵ, the permittivity of region of fields, is always positive. The term $\overline{\nabla}\xi\cdot\overline{\nabla}\xi$ is square of the norm and therefore either larger than or equal to zero. This makes the total variation

$$J(\phi+\alpha\xi) - J(\phi) \geq 0 \qquad (7.5.14)$$

positive proving the minimum principle.

Next we prove "maximum" principle. We follow similar steps as above and first write a trial function in the form $\overline{u} + \alpha\overline{v}$ where $\overline{u}$ is assumed to be an extremal. The Eq. 7.4.24 then becomes

$$G(\overline{u}+\alpha\overline{v}) = \int -\tfrac{1}{2}\overline{u}\cdot\overline{u}\epsilon dV + \int_{\partial V} \overline{u}\cdot\overline{n}\phi_B\epsilon dB - \alpha\int \overline{u}\cdot\overline{v}\epsilon dV +$$
$$\alpha \int_{\partial V} \overline{v}\cdot\overline{n}\phi_B\epsilon dB - \alpha^2 \int \tfrac{1}{2} \overline{v}\cdot\overline{v}\epsilon dV \qquad (7.5.15)$$

after some rearranging. Now suppose $\overline{u}$ satisfies

$$\overline{u} = \overline{\nabla}\phi \qquad (7.5.16)$$

where ϕ is an extremal for the functional $J(\phi)$. Substituting this particular form for $\overline{u}$, the first variation takes the form

$$\rho I = -\alpha\int\overline{\nabla}\phi\cdot \epsilon\overline{v}dV + \alpha \int_{\partial V} \overline{v}\cdot\overline{n}\phi B \ \epsilon dB$$

$$= \alpha \int \phi \overline{\nabla}\cdot\epsilon\overline{v}dV - \alpha \int_{\partial V} \epsilon\overline{v}\phi\cdot\overline{n}dB + \alpha\int_{\partial V} \overline{v}\cdot\overline{n}\phi B\epsilon dB \qquad (7.5.17)$$

The last two terms cancel because $\phi = \phi B$ at the boundary. We recall

now that the trial function must satisfy Eq. 7.5.2. As a consequence,

we obtain the relationship

$$-\overline{\nabla} \cdot \varepsilon \overline{u} - \overline{\overline{\nabla}} \cdot \varepsilon \alpha \overline{v} = \rho_0 \tag{7.5.18}$$

which gives

$$\overline{\nabla} \cdot \varepsilon \overline{v} = 0 \tag{7.5.19}$$

This last equation forces the first variation to vanish. Therefore,

the functional $G(\overline{u})$ is stationary at the exact solution of the original

problem. Furthermore, the second variation in Eq. 7.5.15 is seen to

be negative. Thus, we conclude

$$G(\overline{u+\alpha v}) - G(\overline{u}) \leq 0 \tag{7.5.20}$$

proving the maximum principle.

Before leaving the Dirichlet problem, let us compare the exact

stationary values of the two functionals. At the stationary point,

Eq. 7.4.23 becomes

$$J(\phi) = \int \tfrac{1}{2}\varepsilon \overline{\nabla}\phi \cdot \overline{\nabla}\phi - \phi \overline{\nabla} \cdot \varepsilon \overline{\nabla}\phi \, dv \tag{7.5.21}$$

because of Eqs. 7.4.21 and 7.5.1. The second term in the integral

can be replaced by its identity resulting in the expression

$$J(\phi) = \int \tfrac{1}{2}\varepsilon \overline{u} \cdot \overline{u} - \varepsilon \overline{u} \cdot \overline{u} \, dV + \int_{\partial V} \varepsilon \overline{u}\phi \cdot \overline{n} \, dB$$

$$= \int -\tfrac{1}{2} \, \varepsilon \overline{u} \cdot \overline{u} \, dV + \int_{\partial V} \overline{u} \cdot \overline{n}\phi B\varepsilon \, dB \tag{7.5.22}$$

Therefore, at the stationary point the functional $J(\phi)$ is identical

to its complementary function $G(\overline{u})$.

The proof of Eqs. 7.4.28 and 7.4.29 are quite similar to what

has been shown above. Consider Eq. 2.4.28. This equation is the same

as Eq. 7.4.23 we just proved except the additional boundary term has

been added. Its total variation becomes

$$J(\phi + \alpha\xi) - J(\phi) = \alpha\int\epsilon\overline{\nabla}\phi \cdot\overline{\nabla}\xi - \rho_0\,\xi\,dV - \alpha\int_{\partial V_2}\overline{n}\cdot\overline{u}\xi\epsilon dB +$$
$$\alpha^2\int\tfrac{1}{2}\epsilon\overline{\nabla}\xi\cdot\overline{\nabla}\xi\ dV \qquad\qquad (7.5.23)$$

The first variation can be rewritten as

$$\delta I = \alpha\int -\xi\overline{\nabla}\cdot\epsilon\overline{\nabla}\phi - \rho_0\xi dV + \alpha\int_{\partial V_1}\xi\,\epsilon\overline{\nabla}\phi\cdot\overline{n}dB + \alpha\int_{\partial V_2}\xi\epsilon\overline{\nabla}\phi\cdot\overline{n}dB -$$
$$\alpha\int_{\partial V_2}\overline{n}\cdot\overline{u}B\xi\epsilon dB$$
$$= \alpha\int -\xi\,(\overline{\nabla}\cdot\epsilon\overline{\nabla}\phi + \rho_0)dV + \alpha\int_{\partial V_2}\xi\epsilon\overline{n}\cdot(\overline{\nabla}\phi - \overline{u}B)dB \qquad (7.5.24)$$

through the use of vector identity, Eq. 7.5.7, and Eq. 7.5.10. Setting

$\delta I = 0$ results in

$$\nabla\cdot\epsilon\nabla\phi + \rho_0 = 0 \qquad \text{in } V$$

$$\overline{n}\phi = \overline{n}\phi B \qquad \text{on } \partial V_1$$

$$\overline{n}\cdot\overline{\nabla}\phi = \overline{n}\cdot\overline{u}B \qquad \text{on } \partial V_2$$

recovering the original boundary value problem. Since the second

variation in Eq. 7.5.23 is positive, we obtain

$$J(\phi + \alpha\xi) - J(\phi) \geq 0 \qquad\qquad (7.5.25)$$

proving the minimum principle.

Next, we prove the maximum principle. Consider Eq. 7.4.29. Note

that this equation is identical in form to that of Dirichlet problem.

The only difference is that in Eq. 7.4.29, the boundary term is integrated

over part of the whole boundary. The total variation looks like

$$G(\overline{u} + \alpha\overline{v}) - G(\overline{u}) = -\alpha\int \overline{u}\cdot\overline{v}\epsilon dV + \alpha\int\overline{v}\cdot\overline{n}\phi BdB - \alpha^2\int\tfrac{1}{2}\overline{v}\cdot\overline{v}\epsilon dV \qquad (7.5.26)$$

Now suppose $\overline{u} = \overline{\nabla}\phi$. Then, by the same reasons explained for the Dirichlet

case, the first variation becomes

$$\delta G = -\alpha \int \overline{\nabla} \phi \cdot \overline{v} \epsilon \, dV + \int_{\partial V_1} \overline{v} \cdot \overline{n} \phi B \epsilon \, dB$$

$$= \alpha \int \phi \overline{\nabla} \cdot \epsilon \overline{v} \, dV - \alpha \int_{\partial V_1} \phi \overline{v} \cdot \overline{n} \epsilon \, dB - \alpha \int_{\partial V_2} \phi \overline{v} \cdot \overline{n} \epsilon \, dB +$$

$$\alpha \int_{\partial V_1} \overline{v} \cdot \overline{n} \phi B \epsilon \, dB$$

$$= \int \phi \overline{\nabla} \cdot \epsilon \overline{v} \, dV - \int_{\partial V_2} \phi \overline{v} \cdot \overline{n} \epsilon \, dB \qquad (7.5.27)$$

Recall that the theory in Chapter 4 requires trial functions to satisfy not only Eq. 7.5.2, but also

$$\overline{n} \cdot (\overline{u} + \alpha \overline{v}) = \overline{n} \cdot \overline{u} B \quad \text{on} \quad \partial V_2 \qquad (7.5.28)$$

yielding the result

$$\overline{n} \cdot \overline{v} = 0 \qquad \text{on} \ \partial V_2 \qquad (7.5.29)$$

Therefore, the boundary term in Eq. 7.5.27 must vanish while the previous result, Eq. 7.5.19, forces the first term to zero. This proves that the functional $G(\overline{u})$ is stationary at the exact solution of the original boundary value problem. In Eq. 7.5.26, the second variation is seen to be negative. Therefore, the maximum principle

$$G(\overline{u} - \alpha \overline{v}) - G(\overline{u}) \leq 0 \qquad (7.5.30)$$

also holds.

As required by the general theory developed in Chapter 4, it has been demonstrated that the two functionals for Dirichlet problem assume the same value at their stationary points. We should be able to show the same results for Dirichlet-Newmann problem. By use of the vector identity, Eq. 7.5.7, it is easy to see that the functional $J(\phi)$ is equivalent to:

$$J(\phi) = \int \tfrac{1}{2}\varepsilon\overline{\nabla}\phi \cdot \overline{\nabla}\phi - \overline{\nabla}\phi \cdot \varepsilon\overline{\nabla}\phi \, dV + \int_{\partial V_1} \phi\varepsilon\overline{\nabla}\phi \cdot \overline{n}dB +$$

$$\int_{\partial V_2} \phi\varepsilon\overline{\nabla}\phi \cdot \overline{n}dB - \int_{\partial V_2} \overline{n}\cdot\overline{u}B\phi\varepsilon \, dB$$

$$= \int -\tfrac{1}{2}\varepsilon\overline{u}\cdot\overline{u}dV + \int_{\partial V_1} \overline{u}\cdot\overline{n}\phi B\varepsilon \, dB$$

$$= G(\overline{u}) \qquad\qquad\qquad (7.5.31)$$

confirming what is predicted by the general theory.

7.6. Formulation of $\overline{E}_k$, $\overline{H}_k$, and $\overline{H}_0$

The kth-order electric fields for $k \geq 1$ are divergence-free under our assumption $k = 0$ for $k \geq 1$. The kth-order magnetic fields are also divergence-free for all orders of k. Therefore, we should be able to represent them as a curl of some vector field.

Before discussing the variational formulations, let us first write down the field laws under consideration:

$$\overline{\nabla}\times\overline{E}_k = -\frac{\partial \overline{B}_{k-1}}{\partial t} \qquad\qquad (7.6.1)$$

$$k \geq 1$$

$$\overline{\nabla}\cdot\varepsilon\overline{E}_k = 0 \qquad\qquad (7.6.2)$$

for electric fields;

$$\overline{\nabla}\times\overline{H}_0 = \overline{J}_0 \qquad\qquad (7.6.3)$$

$$\overline{\nabla}\cdot\mu\overline{H}_0 = 0 \qquad\qquad (7.6.4)$$

for zero-order magnetic field; and

$$\bar{\nabla}x\bar{H}_k = \bar{J}_k + \frac{\partial \bar{D}_{k-1}}{\partial t} \qquad (7.6.5)$$

$$k \geq 0$$

$$\bar{\nabla}\cdot\mu\bar{H}_k = 0 \qquad (7.6.6)$$

for kth-order magnetic fields. These field laws are particular instances

of the equation

$$\bar{\nabla}x\bar{A} = \bar{F} \qquad (7.6.7)$$

$$\bar{\nabla}\cdot a\bar{A} = 0 \qquad (7.6.8)$$

By interpreting $\bar{A}$, a, and $\bar{F}$ differently, the above three relationships

can be recovered. Therefore, in the following we will consider the

general case of Eqs. 7.6.7 and 7.6.8.

The first step is to represent the quantity $a\bar{A}$ as a curl of a

vector field $\bar{\phi}$ by defining

$$a\bar{A} = \bar{\nabla}x\bar{\phi} \qquad (7.6.9)$$

Next, substitute this equation into Eq. 7.6.7, obtaining

$$\bar{\nabla}x \frac{1}{a} \bar{\nabla}x\bar{\phi} = \bar{F} \qquad (7.6.10)$$

Now, we need to identify Eq. 7.6.10 as belonging to the class of problems

$$T^*T\phi + Q\phi = f \qquad (7.6.11)$$

If we can show this, then the general results presented in Chapter 4

will immediately yield the desired variational formulations.

To this end, consider the vector identity

$$\bar{\nabla}\cdot\frac{1}{a}\bar{u}x\bar{\phi} = \bar{\phi}\cdot\bar{\nabla}x\frac{1}{a}\bar{u} - \frac{1}{a}\bar{u}\cdot\bar{\nabla}x\bar{\phi} \qquad (7.6.12)$$

We can rewrite this as

$$\int \bar{u}\cdot\bar{\nabla}x\bar{\phi}\frac{1}{a}dV = \int a\bar{\nabla}x\frac{1}{a}\bar{u}\cdot\bar{\phi} \frac{1}{a}dV + \int_{\partial V} \bar{u}\cdot\bar{n}x\bar{\phi} \frac{1}{a} dB$$

$$(7.6.13)$$

after integrating both sides of the equality. The boundary term in

Eq. 7.6.13 is also equal to

$$\int_{\partial V} \overline{u} \cdot \overline{nx\phi} \, \frac{1}{a} \, dB = \int_{\partial V} -\overline{nxu} \cdot \overline{\phi} \, \frac{1}{a} \, dB \qquad (7.6.14)$$

By the same reasoning given in Section 7.5, the last two relationships

enable us to define appropriate operators for the problem under con-

sideration. These operators are

$$T\phi = \overline{\nabla x \phi} \qquad (7.6.15)$$

$$\text{in } V$$

$$T^*u = a\overline{\nabla x} \, \frac{1}{a} \, \overline{u} \qquad (7.6.16)$$

$$\sigma\phi = \overline{nx\phi} \qquad (7.6.17)$$

$$\text{on } \partial V$$

$$\sigma^*u = -\overline{nxu} \qquad (7.6.18)$$

It is now clear that if we multiply both sides of the equality

by the quantity a, Eq. 7.6.10 becomes a special case of the class of

problems defined by Eq. 7.6.11. The quantities Q and f in Eq. 7.6.11

are seen to be

$$Q = 0 \qquad (7.6.19)$$

$$f = a\overline{F} \qquad (7.6.20)$$

It is interesting to note that Eq. 7.6.10 and Eq. 7.4.15 both belong

to the same subclass of Eq. 7.6.11, in that both Q=0 and f≠0 are true.

To specify the problem completely, we must impose a boundary

condition. The general theory allows Dirichlet, Newmann, and mixed

conditions. But, we will consider only the Dirichlet condition.

Therefore, our complete problem becomes

$$a\overline{\nabla x} \, \frac{1}{a} \, \overline{\nabla x \phi} = a\overline{F} \qquad \text{in } V \qquad (7.6.21)$$

$$\overline{n}x\overline{\phi} = \overline{n}x\overline{\phi}_B \qquad\qquad \text{on } \partial V \qquad\qquad (7.6.22)$$

In many electromagnetic problems, the boundary conditions are specified

in terms of the tangential components of the field. Eq. 7.6.9 then

allows us to determine $\overline{n}x\overline{\phi}_B$.

Once we have determined that our problem falls under the special

class discussed in Chapter 4, the remaining task is to invoke the general

results. Therefore, using Eqs. 4.6.5 and 4.6.6, we can immediately

write down the desired formulations

$$J(\overline{\phi}) = \int \tfrac{1}{2} \frac{1}{a} \overline{\nabla}x\overline{\phi}\cdot\overline{\nabla}x\overline{\phi} - \overline{F}\cdot\overline{\phi}dV \qquad\qquad (7.6.23)$$

and

$$G(\overline{u}) = \int -\tfrac{1}{2}\overline{u}\cdot\overline{u}\,\frac{1}{a}\,dV + \int_{\partial V}\overline{u}\cdot\overline{n}x\overline{\phi}_B\,\frac{1}{a}\,dB \qquad\qquad (7.6.24)$$

where $J(\overline{\phi})$ and $G(\overline{u})$ yield upper and lower bounds, respectively.

For convenience, let us specialize Eqs. 7.6.23 and 7.6.24 to the

original field problems and list the results for later reference.

These results are:

$$J(\overline{\phi}) = \int \tfrac{1}{2} \frac{1}{\mu} \overline{\nabla}x\overline{\phi}\cdot\overline{\nabla}x\overline{\phi} - \overline{J}_0\cdot\overline{\phi}dV \qquad\qquad (7.6.25)$$

$$G(\overline{u}) + \int -\tfrac{1}{2}\overline{u}\cdot\overline{u}\,\frac{1}{\mu}\,dV + \int_{\partial V}\overline{u}\cdot\overline{n}x\overline{\phi}_B\,\frac{1}{\mu}dB \qquad\qquad (7.6.26)$$

for H_0, and

$$J(\overline{\phi}) = \int \tfrac{1}{2} \frac{1}{\varepsilon} \overline{\nabla}x\overline{\phi}\cdot\overline{\nabla}x\overline{\phi} + \frac{\partial\overline{B}_{k-1}}{\partial t}\cdot\overline{\phi}dV \qquad\qquad (7.6.27)$$

$$G(\overline{u}) = \int -\tfrac{1}{2}\overline{u}\cdot\overline{u}\,\frac{1}{\varepsilon}\,dV + \int_{\partial V}\overline{u}\cdot\overline{n}x\overline{\phi}_B\,\frac{1}{\varepsilon}\,dB \qquad\qquad (7.6.28)$$

for $\overline{E}_k$, and

$$J(\overline{\phi}) = \int \tfrac{1}{2} \frac{1}{\mu}\overline{\nabla}x\overline{\phi}\cdot\overline{\nabla}x\overline{\phi} - (\overline{J}_k + \frac{\partial\overline{D}_{k-1}}{\partial t})\cdot\overline{\phi}dV \qquad\qquad (7.6.29)$$

$$G(\overline{u}) = \int -\tfrac{1}{2}\overline{u}\cdot\overline{u}\,\frac{1}{\mu}\,dV + \int_{\partial V}\overline{u}\cdot\overline{n}x\overline{\phi}_B\,\frac{1}{\mu}\,dB \qquad\qquad (7.6.30)$$

for $\overline{H}_k$. Equations 7.6.26 and 7.6.30 are seen to be the same, but they

are   listed with their corresponding $J(\overline{\phi})$ for completeness.

7.7. Proof of the Validity of $J(\bar{\phi})$ and $G(\bar{u})$

The general theory of Chapter 4 guarantees the validity of dual extremum principles for $J(\bar{\phi})$ and $G(\bar{u})$. In the following, we shall prove them directly. Let us first write down the canonical equations

$$\bar{\nabla}x\bar{\phi} = \bar{u} \qquad (7.7.1)$$

$$a\bar{\nabla}x \frac{1}{a} \bar{u} = a\bar{F} \qquad (7.7.2)$$

for the original problem, Eqs. 7.6.21 and 7.6.22. Consider Eq. 7.6.23. By the same technique employed in the static case, we substitute $\bar{\nabla}x\bar{\phi}+\bar{\nabla}x\alpha\bar{\xi}$ for $\bar{\phi}$. Expanding each term and collecting proper terms together, we obtain

$$J(\bar{\phi}+\alpha\bar{\xi}) = \int \tfrac{1}{2} \tfrac{1}{a}\bar{\nabla}x\bar{\phi}\cdot\bar{\nabla}x\bar{\phi}-\bar{F}\cdot\bar{\phi}dV + \alpha\int \tfrac{1}{a} \bar{\nabla}x\bar{\phi}\cdot\bar{\nabla}x\bar{\xi}-\bar{F}\cdot\bar{\xi}dV$$
$$+ \alpha^2 \int \tfrac{1}{2} \tfrac{1}{a} \bar{\nabla}x\bar{\xi}\cdot\bar{\nabla}x\bar{\xi}dV \qquad (7.7.3)$$

Through the use of a vector identity

$$\bar{\nabla}\cdot \tfrac{1}{a} \bar{\nabla}x\bar{\phi}x\bar{\xi} = \bar{\xi}\cdot\bar{\nabla} \tfrac{1}{a}\bar{\nabla}x\bar{\phi}- \tfrac{1}{a}\bar{\nabla}x\bar{\phi}\cdot\bar{\nabla}x\bar{\xi} \qquad (7.7.4)$$

the first variation takes on a different form

$$\delta I = \alpha\int \bar{\xi}\cdot(\bar{\nabla}x \tfrac{1}{a}\bar{\nabla}x\bar{\phi}-\bar{F})dV +\alpha \int_{\partial V} \tfrac{1}{a}\bar{\nabla}x\bar{\phi}\cdot\bar{n}x\bar{\xi}dB \qquad (7.7.5)$$

But, we recall that the trial functions must satisfy the boundary condition. This restriction yields the relationship

$$\bar{n}x(\bar{\phi}+\alpha\bar{\xi}) = \bar{n}x\bar{\phi}_B \qquad \text{on } \partial V \qquad (7.7.6)$$

which implies

$$\bar{n}x\bar{\xi} = 0 \qquad \text{on } \partial V \qquad (7.7.7)$$

Therefore, only the first term in Eq. 7.7.5 survives. Finally, setting the first variation equal to zero, we obtain the stationary equation

$$\bar{\nabla} \times \frac{1}{a} \bar{\nabla} \times \bar{\phi} - \bar{F} = 0 \qquad (7.7.8)$$

which is precisely what we wanted to show. Also, the second variation

in Eq. 7.7.3 is always positive, establishing the "minimum" principle

$$J(\bar{\phi} + \alpha \bar{\xi}) - J(\bar{\phi}) \geq 0 \qquad (7.7.9)$$

Next, we prove the "maximum" principle. By substituting $\bar{u} + \alpha \bar{v}$

for $\bar{u}$ in Eq. 7.6.24, we obtain

$$G(\bar{u} + \alpha \bar{v}) = \int -\tfrac{1}{2}\bar{u} \cdot \bar{u} dV + \int_{\partial V} \bar{u} \cdot \bar{n} \times \phi_B \frac{1}{a} dB$$

$$+ \alpha \int -\bar{u} \cdot \bar{v} \frac{1}{a} dV + \alpha \int_{\partial V} \bar{v} \cdot \bar{n} \times \phi_B \frac{1}{a} dB$$

$$- \alpha^2 \int \tfrac{1}{2}\bar{v} \cdot \bar{v} \frac{1}{a} dV \qquad (7.7.10)$$

Suppose the stationary function $\bar{u}$ satisfies Eq. 7.7.1. The first

variation then becomes

$$\delta G = \alpha \int -\bar{\nabla} \times \bar{\phi} \cdot \bar{v} \frac{1}{a} dV + \alpha \int_{\partial V} \bar{v} \cdot \bar{n} \times \phi_B \frac{1}{a} dB$$

$$= \alpha \int -\bar{\phi} \cdot \bar{\nabla} \times \frac{1}{a} \bar{v} dV - \alpha \int_{\partial V} \bar{v} \cdot \bar{n} \times \bar{\phi} \frac{1}{a} dB$$

$$+ \alpha \int_{\partial V} \bar{v} \cdot \bar{n} \times \bar{\phi}_B \frac{1}{a} dB \qquad (7.7.11)$$

in which the last term was obtained through the vector identity,

Eq. 7.7.4. In Eq. 7.7.11, the boundary terms cancel because

$$\bar{n} \times \bar{\phi} = \bar{n} \times \bar{\phi}_B \qquad \text{on } \partial V \qquad (7.7.12)$$

We also recall that the general theory requires the trial functions

to satisfy the second of the canonical relationships, Eq. 7.7.2. This

leads to

$$\bar{\nabla} \times \frac{1}{a} \bar{v} = 0 \qquad (7.7.13)$$

Therefore, the first term in Eq. 7.7.11 also vanishes, making the

functional $G(\bar{u})$ stationary at the solution of the original boundary

value problem, Eqs. 7.6.21 and 7.6.22. The "maximum" principle

$$G(\overline{u+\alpha v}) - G(\bar{u}) \leq 0 \qquad\qquad\qquad (7.7.14)$$

holds because the second variation in Eq. 7.7.10 is always negative.

8. SIMPLE APPLICATIONS

8.1. Introduction

In the previous chapter, we derived complementary variational integrals for Maxwell's equations in power series form. We have also proved the validity of each integral directly. These variational integrals are equivalent to the k-th order field equations in point form. As such, they must be applicable to any sinusoidally-varying electromagnetic field problems.

In this chapter, we will apply our formulations to a simple analysis of a parallel-plate capacitor. We will derive variational approximations to the D.C. capacitance of the structure and compare them with the exact value. Also, the effect of the magnetic field (produced by the changing electric field) on capacitance will be estimated by keeping the first three terms in the power series solution.

8.2. Definition of the Problem

Figure 8.2.1 shows a parallel-plate capacitor with dimensions and coordinate system as indicated. We assume that the capacitor has an air dielectric between the plates and is excited by the distributed sinusoidal source, v_s, at $z = -\ell$. Solution is sought under the non-fringing assumption. This problem is described by Magid in Ref. 33.
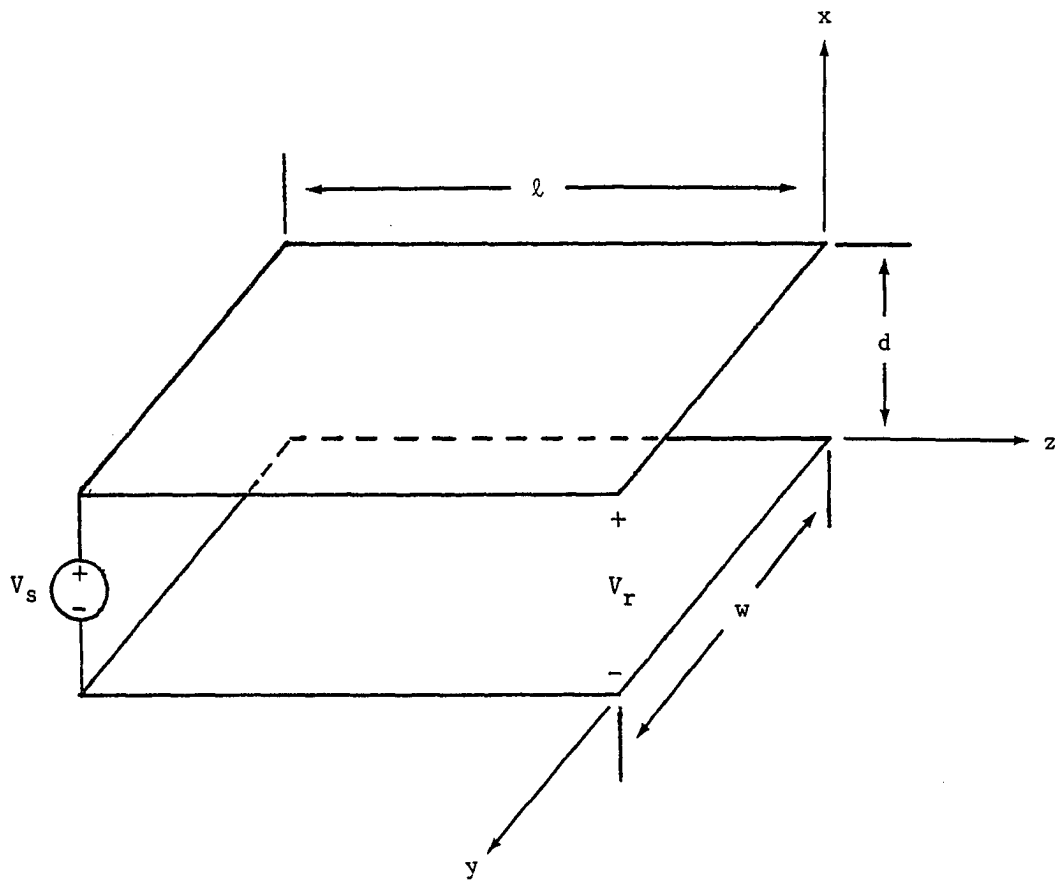
Figure 8.2.1. Parallel-plate capacitor

For the purpose of comparing different solutions at different frequencies, we must fix one reference quantity at a chosen reference point [33]. In order to make the problem precisely the same as that described by Magid, we will choose the reference point at z=0 and fix the voltage at this point

$$v_s = A_r \cos\omega t \qquad (8.2.1)$$

with the constant magnitude A. We shall refer to it as the "reference voltage." This means that at each frequency, the amplitude of the source must be adjusted until the magnitude of the voltage reads A at z=0. This reference requirement enables us to compare the relative size of each kth-order field. Our problem is to estimate D.C. and frequency-dependent capacitances of the system using complementary variational techniques, and to compare the results with exact solutions.

8.3. Exact Solutions

Before we proceed with the variational formulations, let us write down the exact solutions. Our main purpose in this chapter is to illustrate the variational methods--not the mathematical techniques leading to the exact solutions. In fact, we shall skip the entire mathematical details and simply list the final results for later reference. For our purpose, it is sufficient to know the first three terms in the power series. These fields are

$$\overline{E}_0 = -\overline{i}_x \, A/d \, \cos\omega t \qquad (8.3.1)$$

$$\overline{H}_0 = 0 \qquad (8.3.2)$$

$$\overline{E}_1 = 0 \tag{8.3.3}$$

$$\overline{H}_1 = -\overline{i}_y \; \frac{\epsilon_0 \omega A}{d} \; z \; \sin\omega t \tag{8.3.4}$$

$$\overline{E}_2 = \overline{i}_x \; \frac{\omega^2 \mu_0 \epsilon_0 A}{2d} \; z^2 \; \cos\omega t \tag{8.3.5}$$

$$H_2 = 0 \tag{8.3.6}$$

in which μ_0 and ϵ_0 are the permittivity and permeability of free space. Magid's book [33] discusses the mathematical details in obtaining these exact solutions.

Notice that the odd-term electric field is zero, while it is the two even terms that vanish for the magnetic field. This situation continues to hold true for all orders of k. Therefore, for both $\overline{E}_k$ and $\overline{H}_k$, the nonzero terms appear in an alternating fashion. This trend is typical of the power series approach [33].

8.4. D.C. Capacitance

In general, the D.C. capacitance of a two-conductor structure can be defined in terms of stored electric energy by the equation

$$c_{D.C.} = 2/A^2 \; W_E \tag{8.4.1}$$

The quantity A stands for applied D.C. voltage and w_E stands for energy stored in the electric field. For the parallel plates in our problem, this D.C. capacitance can be calculated to be

$$c_{D.C.} = \epsilon_0 \; \frac{\ell w}{d} \tag{8.4.2}$$

showing that it is independent of the applied voltage. Therefore, we should be able to apply an arbitrary D.C. voltage A without violating the equality in Eq. 8.4.1.

Let us now consider exact zero-order electric field solutions for this parallel-plate capacitor. We see from Eq. 8.3.1 that the only difference between static and zero-order fields is that the zero-order field vibrates sinusoidally at frequency ω. However, even though the zero-order field is vibrating, it is not capable of producing a magnetic field. This is evident from the zero-order electric field equations. Therefore, at each fixed time, the zero-order electric field is equivalent to the static electric field. (There is no magnetic field associated with a static field.) It is clear then that Eq. 8.4.1 should remain valid when static voltage A is replaced by the zero-order reference voltage, $A\cos\omega t$, oscillating at an arbitrary frequency. In our problem, the source and reference voltages are the same because $\overline{E}_0$ is independent of the coordinates y and z.

8.5. Zero-order Variational Formulation of D.C. Capacitance

The governing laws for the zero-order electric field are

$$\overline{\nabla} \times \overline{E}_0 = 0 \tag{8.5.1}$$

$$\overline{\nabla} \cdot \varepsilon_0 \overline{E}_0 = 0 \tag{8.5.2}$$

where the zero-order charge density is zero for our system. The problem is therefore a special case of the more general situation discussed in Section 7.4. By representing $\overline{E}_0$ as a gradient field

$$\overline{E}_0 = - \overline{\nabla}\phi \tag{8.5.3}$$

we obtain the governing equation for ϕ

$$\overline{\nabla} \cdot \varepsilon_0 \overline{\nabla}\phi = 0 \tag{8.5.4}$$

$$\bar{n}\phi = \bar{n}\phi_B \tag{8.5.5}$$

where a boundary condition has been added for convenience. The first step in complementary variational formulation of Eqs. 8.5.4 and 8.5.5 is to write them as coupled canonical equations

$$\bar{\nabla}\phi = \bar{u} \tag{8.5.6}$$

$$\text{in } V$$

$$-\frac{1}{\epsilon_0}\bar{\nabla}\cdot\epsilon_0\bar{u} = 0 \tag{8.5.7}$$

$$\bar{n}\phi = \bar{n}\phi_B \qquad \text{on } \partial V \tag{8.5.8}$$

We could have formulated the problem as the Dirichlet-Newmann-type problem. Our choice of the Dirichlet condition has no particular reason.

The general results of Section 7.4 immediately yield the desired variational formulation. These integrals are

$$J(\phi) = \int \tfrac{1}{2} \epsilon_0 \bar{\nabla}\phi\cdot\bar{\nabla}\phi \, dV \tag{8.5.9}$$

$$G(\bar{u}) = \int -\tfrac{1}{2}\bar{u}\cdot\bar{u} \, dV + \int_{\partial V} \bar{u}\cdot\bar{n}\phi_B \epsilon_0 \, dB \tag{8.5.10}$$

One immediately realizes that the integrand in Eq. 8.5.9 is the usual definition of electric energy density. We will interpret it as the "zero-order electric energy." Therefore, in view of the equality of both Eqs. 8.5.9 and 8.5.10 at their stationary point, we conclude that $J(\dot\phi)$ and $G(\bar{u})$ represent the exact energy at the stationary point. Equation 8.4.1 then says that we can write D.C. capacitance

$$C_{D.C.} = \frac{2}{A^2\cos^2\omega t} \int \tfrac{1}{2}\epsilon_0 \bar{\nabla}\phi\cdot\bar{\nabla}\phi \, dV \tag{8.5.11}$$

$$C_{D.C.} = \frac{2}{A^2\cos^2 t} \int -\tfrac{1}{2}\bar{u}\cdot\bar{u} \, dV + \int_{\partial V} \bar{u}\cdot\bar{n}\phi_B \epsilon_0 \, dB \tag{8.5.12}$$

which are the desired complementary variational formulations of the DC capacitance.

8.6. Approximate Calculation of D.C. Capacitance

The exact solutions for our problem can be determined to be

$$\phi = \left(\frac{A}{d}\right) x \cos\omega t \qquad (8.6.1)$$

$$\overline{E}_0 = - i_x \left(\frac{A}{d}\right) \cos\omega t \qquad (8.6.2)$$

under the Dirichlet boundary conditions

$$\phi_B = \begin{cases} 0 \text{ at } x=0 \\ A \cos\omega t \text{ at } x=d \\ (A/d) x \cos\omega t \text{ at } y=0,w \\ \qquad\qquad\qquad z=0,-\ell \end{cases}$$

For the purpose of illustrating the variational formulations, Eqs. 8.5.11 and 8.5.12, we shall perturb the exact solution slightly and evaluate approximate capacitances.

First, let us evaluate Eq. 8.5.11 through the use of a trial function

$$\phi + \alpha\xi = A/d \ x \cos\omega t + \alpha x(x-d)y(y-w)z(z+\ell)\cos\omega t \qquad (8.6.3)$$

This particular choice of the second term is consistent with the requirement that the trial function must satisfy the Dirichlet boundary condition. The functional $J(\phi+\alpha\xi)$ is calculated to be

$$J(\phi+\alpha\xi) = \int \frac{1}{2} \varepsilon_0 \left(\frac{A}{d}\right)^2 \cos^2\omega t \ dV$$

$$+ \alpha \int \varepsilon_0 \left(\frac{A}{d}\right) (2x-d)y(y-w)z(z+\ell)\cos^2\omega t \ dV$$

$$+ \alpha^2 \int [(2x-d)^2 y^2(y-w)^2 z^2(z+\ell)^2$$

$$+ x^2(x-d)^2(2y-w)^2z^2(z+\ell)^2$$

$$+ x^2(x-d)^2y^2(y-w)^2(2z+\ell)^2]\cos^2\omega t \; dV \qquad (8.6.4)$$

in which the first variation becomes

$$\delta J \;=\; \alpha\epsilon_0 \left(\frac{A}{d}\right) [x^2-dx]_0^d \; [\frac{y^3}{3}-\frac{wy^2}{2}]_0^w \; [\frac{z^3}{3}+\frac{\ell z^2}{2}]_{-\ell}^0 \cos^2\omega t$$

$$= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (8.6.5)$$

This is in agreement with the general theory since the first term of

our trial function is the exact solution. After performing the

integration, Eq. 8.6.4 takes the form

$$J(\phi+\alpha\xi) \;=\; \tfrac{1}{2}\epsilon_0\left(\frac{A}{d}\right)^2 \; w\ell d\cos^2\omega t$$

$$+ \tfrac{1}{2} \; \epsilon_0\alpha^2 \left(\frac{d^3w^5\ell^5+d^5w^3\ell^5+d^5w^5\ell^3}{2700}\right)w\ell d\cos^2\omega t \qquad (8.6.6)$$

which results in the approximate capacitance expression

$$C'_{D.C.} \;=\; \frac{2}{A^2\cos^2\omega t} \; J(\phi+\alpha\xi)$$

$$= \; \epsilon_0 \frac{w\ell}{d} + \epsilon_0(\alpha/A)^2\beta_1 \; w\ell d \qquad (8.6.7)$$

The quantity β_1 in Eq. 8.6.7 represents the bracketed factor in the

second term of Eq. 8.6.6.

Notice that Eq. 8.6.7 is an equation of a parabola in the variable

α. At $\alpha=0$, it assumes the exact D.C. capacitance value, the first term

of the equation, as predicted by the theory. When $\alpha\neq0$, Eq. 8.6.7 is

always larger than its exact stationary value, exhibiting the "maximum"

principle. The error term in Eq. 8.6.7 is seen to be directly pro-

portional to the square of the ratio α/A. This certainly agrees with

what one would expect: the larger the ratio α/A, the farther away the trial function is from the exact solution.

Next, we will evaluate Eq. 8.5.12. To do this, let us first calculate the functional $G(\bar{u}+\alpha\bar{v})$. We choose our trial function to be

$$\bar{u} + \alpha\bar{v} = \bar{i}_x \, A/d \, \cos\omega t + \bar{i}_x\alpha y \, \cos\omega t \qquad (8.6.8)$$

where the first term is the exact stationary function. This particular choice of trial function satisfies the required condition, Eq. 8.5.7. After some calculation, both terms in the functional become

$$\int - \tfrac{1}{2} \, \varepsilon_0 \, (\overline{u+\alpha v}) \cdot (\overline{u+v}) \, dV = \tfrac{1}{2} \, \varepsilon_0 \, (A/d)^2 \, w\ell d \, \cos^2\omega t$$

$$- \alpha\varepsilon_0 \, A(w^2/2)\ell\cos^2\omega t - \alpha^2 \, \tfrac{1}{2} \, \varepsilon_0 \, d \, (w^3/3)\ell\cos^2\omega t \qquad (8.6.9)$$

and

$$\int_{\partial V} \bar{u}\cdot\bar{n}\phi_B \, \varepsilon_0 \, dB = (A^2/d) \, \varepsilon_0 w\ell\cos^2\omega t + \alpha A \, \varepsilon_0 \, (w^2/2)\ell\cos^2\omega t$$

$$(8.6.10)$$

The only contribution to the surface integral, Eq. 8.6.10, comes from the upper plate. Notice that the terms containing α, in both equations, are negatives of each other resulting in cancellation when we add. Again, this vanishing of the first variation is what we expect because our trial function is built by slightly perturbing the exact solution. The desired functional then becomes

$$G(\overline{u}+\alpha\overline{v}) = \tfrac{1}{2}\varepsilon_0 \ (A^2/d) \ w\ell\cos^2\omega t - \alpha^2 \ \tfrac{1}{2}\varepsilon_0 d \ (w^3/3)\ell\cos^2\omega t$$

$$(8.6.11)$$

leading to the second approximate capacitance expression

$$C''_{D.C.} = \frac{2}{A^2\cos^2\omega t} \ G(\overline{u}+\alpha\overline{v})$$

$$= \varepsilon_0 \ (w\ell/d) - (\alpha/A)^2 \ \varepsilon_0 d \ (w^3\ell/3) \qquad (8.6.12)$$

This equation is the same as Eq. 8.6.7 except for the negative multiplying

factor in the second term. Therefore, the comments given for Eq. 8.6.6

are applicable except that the stationary value of Eq. 8.6.12 is larger

than that of any other approximate D.C. capacitance value $C''_{D.C.}$. In

other words, Eq. 8.6.12 exhibits "maximum" principle as expected.

Figure 8.6.1 is a plot of two approximate capacitances $C'_{D.C.}$

and $C''_{D.C.}$. The figure shows that the relationship

$$C''_{D.C.} \leq C'_{D.C.} \qquad (8.6.13)$$

is always true as predicted by the general theory. We can always take

the average of $C'_{D.C.}$ and $C''_{D.C.}$ as the best approximation.

This section illustrated how the zero-order electric field can

be used to obtain an approximate D.C. capacitance of the parallel-plate

capacitor. The important step was to justify the validity of D.C.

capacitance expression, Eq. 8.4.1, when a D.C. voltage was replaced

by the zero-order reference voltage, $A\cos\omega t$. Also, the two functionals,

$J(\phi)$ and $G(\overline{u})$, were recognized as the zero-order energy of the system.

Since the complementary variational formulation is valid in general,

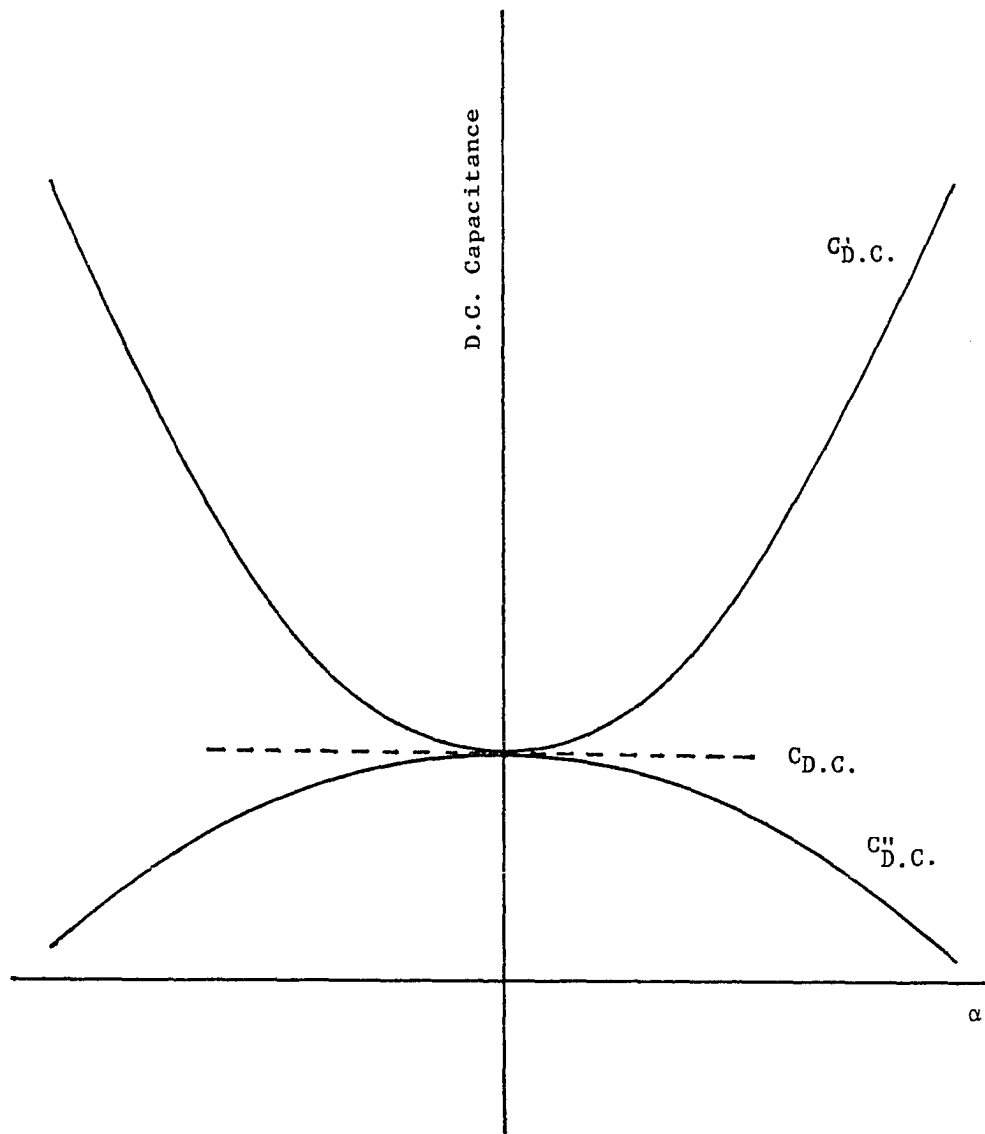the above techniques must also apply to any arbitrary capacitance

configuration.

Figure 8.6.1. Variational approximation of the D.C. capacitance

We must remember that the parallel-plate capacitor is a very special case. Normally, in a practical problem, a person will not be able to construct the trial function in the form of Eq. 8.6.3 or Eq. 8.6.8 because the exact solution is not known in advance. Usually, in practice, one builds a trial function laden with many parameters. These parameters are then adjusted until each functional assumes its minimum or maximum value.

8.7. Complementary Variational Formulation of $\overline{E}_2$

Referring back to Fig. 8.2.1, our problem in this section is to formulate the second-order electric field between two parallel plates as complementary variational integrals. The governing equations are

$$\overline{\nabla} \times \overline{E}_2 = -\frac{\partial \mu_0 \overline{H}_1}{\partial t} \tag{8.7.1}$$

$$\overline{\nabla} \cdot \epsilon_0 \overline{E}_2 = 0 \tag{8.7.2}$$

By defining

$$\epsilon_0 \overline{E}_2 = \overline{\nabla} \times \overline{\phi} \tag{8.7.3}$$

we obtain the differential equation

$$\overline{\nabla} \times \frac{1}{\epsilon_0} \overline{\nabla} \times \overline{\phi} = -\frac{\partial \mu_0 \overline{H}_1}{\partial t} \qquad \text{in } V \tag{8.7.4}$$

$$\overline{n} \times \overline{\phi} = \overline{n} \times \overline{\phi}_B \qquad \text{on } \partial V \tag{8.7.5}$$

for vector potential field $\overline{\phi}$. The boundary condition, Eq. 8.7.5, is listed for convenience. The corresponding canonical equations are

$$\overline{\nabla} \times \overline{\phi} = \overline{u} \tag{8.7.6}$$

$$\overline{\nabla} \times \overline{u} = -\epsilon_0 \frac{\partial \mu_0 \overline{H}_1}{\partial t} \tag{8.7.7}$$

$$\overline{n}x\overline{\phi} = \overline{n}x\phi_B \qquad\qquad (8.7.8)$$

leading to the complementary variational formulation

$$J(\overline{\phi}) = \int \tfrac{1}{2} \frac{1}{\varepsilon_0} \overline{\nabla}x\overline{\phi}\cdot\overline{\nabla}x\overline{\phi} + \frac{\partial\mu_0\overline{H}_1}{\partial t}\ dV \qquad\qquad (8.7.9)$$

$$G(\overline{u}) = \int -\tfrac{1}{2}\overline{u}\cdot\overline{u}\frac{1}{\varepsilon_0}dV + \int_{\partial V} \overline{u}\cdot\overline{n}x\overline{\phi}_B \frac{1}{\varepsilon 0}dB \qquad\qquad (8.7.10)$$

8.8. Evaluation of $J(\overline{\phi})$ and $G(\overline{u})$

For the purpose of illustrating the basic structure of the functionals, $J(\overline{\phi})$ and $G(\overline{u})$, let us evaluate them at the trial functions which are slightly different from the exact solutions. Our specific choice of these trial functions are

$$\overline{\phi}+\alpha\overline{\xi} = -\overline{i}_y\frac{1}{6}\beta z^3\cos\omega t + \overline{i}_y\alpha x(x-d)y(y-w)z(z+\ell)\cos\omega t \qquad (8.8.1)$$

and

$$\overline{u}+\alpha\overline{v} = \overline{i}_x\tfrac{1}{2}\beta z^2\cos\omega t + \overline{i}_y\alpha y \qquad\qquad (8.8.2)$$

where

$$\beta = \frac{\mu_0\varepsilon_0^2\omega^2 A}{d} \qquad\qquad (8.8.3)$$

The first terms in both equations, which are the exact $\overline{\phi}$ and $\overline{u}$, were determined from the relationships $\overline{u}=\varepsilon_0\overline{E}_2$ and $\overline{\nabla}x\overline{\phi}=\overline{u}$ where $\overline{E}_2$ is known. Let us evaluate $J(\overline{\phi})$. The first term becomes

$$\tfrac{1}{2}\frac{1}{\varepsilon_0}\overline{\nabla}x\overline{\phi}\cdot\overline{\nabla}x\overline{\phi} = \left(\frac{\partial\phi_y}{\partial z}\right)^2 + \left(\frac{\partial\phi_y}{\partial x}\right)^2 \qquad\qquad (8.8.4)$$

After some algebraic manipulations, each term in Eq. 8.8.4 becomes

$$\frac{1}{2\varepsilon_0}\left(\frac{\partial\phi_y}{\partial z}\right)^2 = \frac{1}{8\varepsilon_0}\beta^2 z^4 \cos^2\omega t$$

$$- \frac{1}{2\varepsilon_0}\alpha\beta x(x-d)y(y-w)(2z^3+\ell z^2)\cos\omega t$$

$$+ \frac{1}{2\varepsilon_0}\alpha^2 x^2(x-d)^2 y^2(y-w)^2(2z+\ell)^2\cos^2\omega t$$

$$= \alpha^2 f_1 + \alpha f_2 + f_3 \qquad (8.8.5)$$

and

$$\frac{1}{2\varepsilon_0}\left(\frac{\partial\phi_y}{\partial x}\right)^2 = \frac{1}{2\varepsilon_0}\alpha^2(2x-d)^2 y^2(y-2)^2 z^2(z+\ell)^2\cos^2\omega t$$
$$= \alpha^2 f_4 \qquad (8.8.6)$$

The last term of the integrand is calculated to be

$$\frac{\partial\mu_0\overline{H}_1}{\partial t}\cdot\overline{\phi} = \frac{1}{6\varepsilon_0}\beta^2 z^4\cos^2\omega t - \frac{1}{\varepsilon_0}\alpha\beta x(x-d)y(y-w)(z^3+\ell z^2)\cos^2\omega t$$

$$= \alpha f_5 + f_6 \qquad (8.8.7)$$

Now, we can write down the total variations as

$$J(\overline{\phi}+\alpha\overline{\xi}) = \int f_3+f_6 dV + \alpha\int f_2+f_5 dV + \alpha^2\cdot\int f_1+f_4 dV \qquad (8.8.8)$$

in which the integrations should be performed over the entire volume

between the two parallel plates in Fig. 8.2.1. The first variation

$$\delta J = \frac{-1}{2\varepsilon_0} \alpha\beta \left[\frac{-d^3}{6}\right] \left[\frac{-w^3}{6}\right] \left[\frac{-\ell^4}{6}\right] \cos^2\omega t$$

$$- \frac{1}{\varepsilon_0} \alpha\beta \left[\frac{-d^3}{6}\right] \left[\frac{-w^3}{6}\right] \left[\frac{\ell^4}{12}\right] \cos^2\omega t$$

$$= 0 \tag{8.8.9}$$

as predicted by the general theory. Each term in the second variation integrates to be

$$\int f_1 dV = \frac{1}{2\varepsilon_0} \left[\frac{d^5}{30}\right]\left[\frac{w^5}{30}\right]\left[\frac{\ell^5}{3}\right] \tag{8.8.10}$$

$$\int f_4 dV = \frac{1}{2\varepsilon_0} \left[\frac{d^3}{3}\right]\left[\frac{w^5}{30}\right]\left[\frac{\ell^5}{30}\right] \tag{8.8.11}$$

while the exact stationary value becomes

$$\int f_3 + f_6 dV = \frac{7}{24} \frac{1}{\varepsilon_0}\beta^2 wd \left[\frac{\ell^5}{5}\right] \cos^2\omega t \tag{8.8.12}$$

The pair of Eqs. 8.8.10 and 8.8.11 say that the second variation is a positive constant. Therefore, we can write

$$\bar{J}(\bar{\phi}+\alpha\bar{\xi}) = \frac{7}{24} \frac{1}{\varepsilon_0} \beta^2 wd\left[\frac{\ell^5}{5}\right] \cos^2_\omega t + \alpha^2\beta_2 \tag{8.8.13}$$

in which β_2 represents the sum of Eqs. 8.8.10 and 8.8.11. The result of Eq. 8.8.13 is plotted as part of Fig. 8.8.1 and is commented on at the end of this section.

Next, we calculate $G(\bar{u})$. Somewhat tedious but straightforward algebraic calculations show

$$\int -\tfrac{1}{2} \frac{1}{\varepsilon_0}(\overline{u}+\alpha\overline{v})\cdot(\overline{u}+\alpha\overline{v})dV = \frac{-1}{8}\frac{1}{\varepsilon_0}\beta^2 wd\ [\frac{\ell^5}{5}]\cos^2\omega t$$

$$-\alpha^2\ \tfrac{1}{2}\ \frac{1}{\varepsilon_0}d\ell\ [\frac{w^5}{3}] \qquad . \qquad (8.8.14)$$

and

$$\int_{\partial V}(\overline{u}+\alpha\overline{v})\cdot\overline{n}x\overline{\phi}_B\frac{1}{\varepsilon_0}dB = \frac{5}{12}\frac{1}{\varepsilon_0}\beta^2 wd[\frac{\ell^5}{5}]\cos^2\omega t \qquad (8.8.15)$$

The only contribution to the surface integral of the last equation comes from the integration over the surface at $z=-\ell$. The total variation becomes

$$G(\overline{u}+\alpha\overline{v}) = \frac{7}{24}\frac{1}{\varepsilon_0}\beta^2 wd[\frac{\ell^5}{5}]\cos^2\omega t-\alpha^2\frac{1}{2}\frac{1}{\varepsilon_0}\ d\ell\ [\frac{w^3}{3}] \qquad (8.8.16)$$

whose first term agrees with that of $J(\overline{\phi}+\alpha\overline{\xi})$. Equation 8.8.16 is also plotted in Fig. 8.8.1.

Some comments are now in order. As predicted by the general theory, Fig. 8.8.1 clearly shows the complementary nature of both functionals $J(\overline{\phi}+\alpha\overline{\xi})$ and $G(\overline{u}+\alpha\overline{v})$. At $\alpha=0$, the trial functions become exact and both $J(\overline{\phi}+\alpha\overline{\xi})$ and $G(\overline{u}+\alpha\overline{v})$ assume the same value. When $\alpha\neq0$, $J(\overline{\phi}+\alpha\overline{\xi})$ is always larger, while $G(\overline{u}+\alpha\overline{v})$ is always smaller than the exact functional value.

In most practical applications, however, the exact solutions are not known and it is highly improbable that one can choose the same
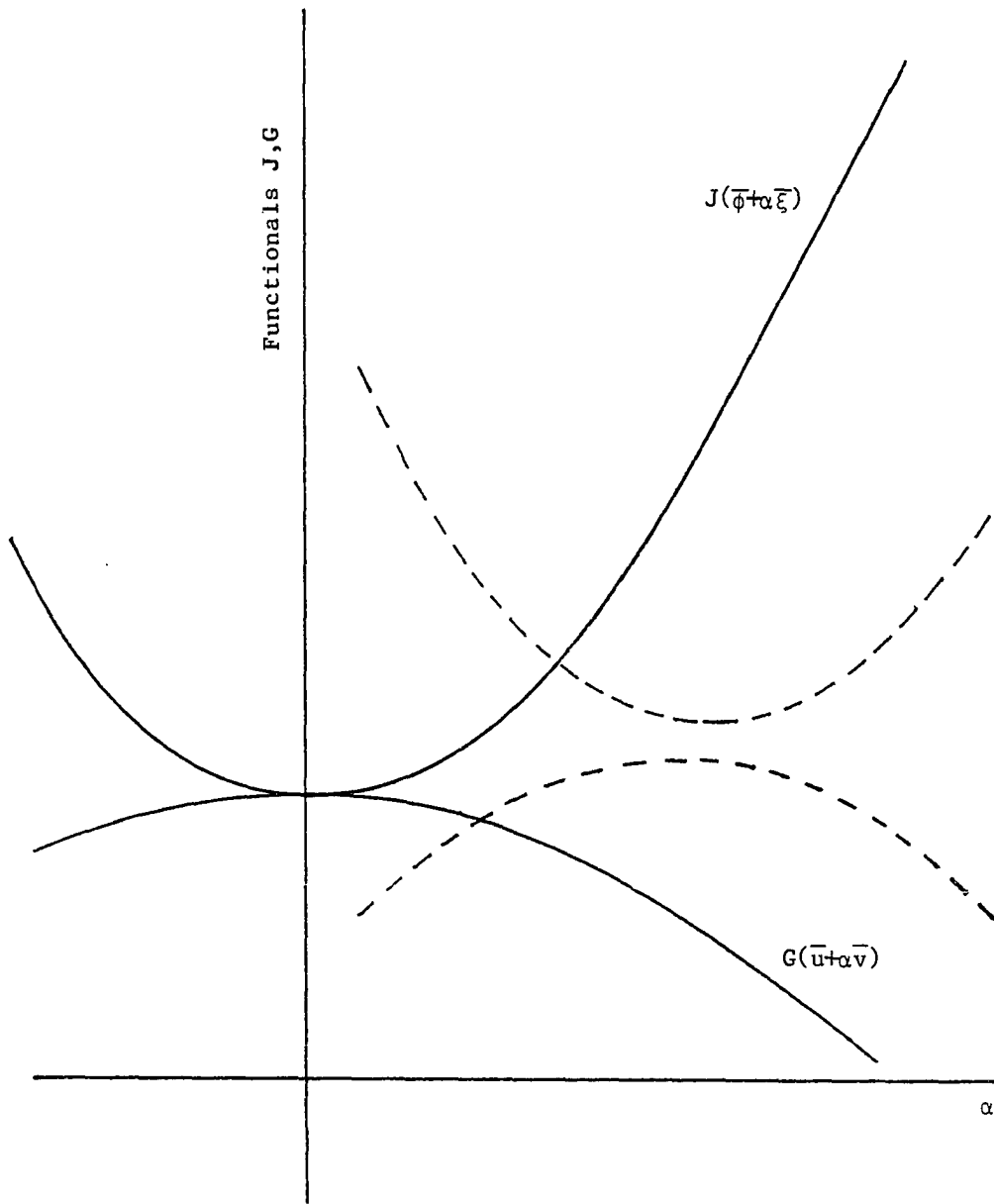
Figure 8.8.1. Complementary variational functionals J and G

form of trial functions as ours. In such situations, the two functionals never assume the same value. One such possible contour is also drawn in Fig. 8.8.1.

8.9. A.C. Capacitance Calculation

The A.C. capacitance is defined through the equation

$$i_s = C_{A.C.} \frac{dv_s}{dt} \qquad (8.9.1)$$

where i_s and v_s are the time-domain current and voltage, respectively. This capacitance is frequency-dependent because of the existence of magnetic field produced by the changing electric field.

In order to determine the nature of the frequency dependence through a power series approach, one needs to keep calculating higher order fields until the frequency variable ω enters into the amplitude of the field. Then, one should be able to calculate the frequency-dependent terminal voltage and current. Subsequent calculation of impedance should allow one to identify the equation for the capacitor.

Let us use our trial function to calculate the A.C. capacitance for the parallel-plate capacitor. The results should suggest how our variational techniques can be applied to similar problems. First, we shall calculate the second-order electric field. We are free to use either $\overline{\phi} + \alpha \overline{\xi}$ or $\overline{u} + \alpha \overline{v}$, but we will arbitrarily choose $\overline{\phi} + \alpha \overline{\xi}$. The approximate $\overline{E}_2$ becomes

$$\bar{E}_2 = \frac{1}{\varepsilon_0} \bar{\nabla} \times \bar{\phi}$$

$$= \bar{i}_x \left(\frac{1}{2\varepsilon_0} \beta z^2 \cos\omega t - \alpha x(x-d)y(y-w)(2z+\ell)\cos\omega t \right)$$

$$+ \bar{i}_z \ \alpha \frac{1}{\varepsilon_0} (2x-d)y(y-w)z(z+\ell)\cos\omega t \qquad (8.9.2)$$

The total electric field $\bar{E}$ is

$$\bar{E} = \bar{E}_0 + \bar{E}_1 + \bar{E}_2 \qquad (8.9.3)$$

where $\bar{E}_0$ and $\bar{E}_1$ are given as Eqs. 8.3.1 and 8.3.3, respectively. We need to evaluate this total electric field at $z = -\ell$ in order to calculate the source voltage. Only the x-component of $\bar{E}$ contributes to the source voltage. This x-component is

$$\bar{i}_x \cdot (\bar{E}_0 + \bar{E}_1 + \bar{E}_2)\big|_{z=-\ell} = \left[\left(-\frac{A}{d} + \frac{1}{2\varepsilon_0} \beta \ell^2 \right) \right.$$

$$\left. + \alpha \frac{1}{\varepsilon_0} x(x-d)y(y-w)\ell \right] \cos\omega t \qquad (8.9.4)$$

which gives the source voltage

$$v_s = -\int_0^d \bar{E} \cdot \bar{a}_x dx$$

$$= \left[A(1 - \tfrac{1}{2}\mu_0\varepsilon_0\omega^2\ell^2) + \alpha \frac{1}{6\varepsilon_0} d^3 y(y-w)\ell \right] \cos\omega t \qquad (8.9.5)$$

where the second term is seen to be a function of coordinate y. This is not consistent with our nonfringing assumption where $\bar{E}$ and $\bar{H}$ cannot vary with y [33]. This implies that the source voltage must also be a function of y. However, we are using a perturbed $\bar{E}$ field, and, therefore, cannot expect to obtain the correct result. Only when α goes to zero does Eq. 8.9.5 yield the correct result given by Magid [33].

Next, we need to calculate the terminal current i_s. To this end, we need the magnetic field

$$\bar{H} = \bar{H}_0 + \bar{H}_1 + \bar{H}_2$$

$$= -\bar{i}_y \frac{\varepsilon_0 \omega A}{d} z \sin\omega t. \tag{8.9.6}$$

where $\bar{H}_0$ and $\bar{H}_2$ are zero. The terminal current then becomes

$$i_s = \int_0^w -\bar{i}_x \times \bar{H} \cdot \bar{i}_z dy$$

$$= -(\frac{\varepsilon_0 w\ell}{d}) \omega A \sin\omega t \tag{8.9.7}$$

By transforming Eqs. 8.9.5 and 8.9.7 to phasors, the impedance

is calculated to be

$$\frac{v_s}{I_s} = \frac{1}{j\omega \frac{C_{D.C.}}{(1-\frac{1}{2}\mu_0\varepsilon_0\ell^2\omega^2)}} + \frac{1}{j\omega \left(\frac{C_{D.C.}}{\alpha \frac{d^3 y(y-w)\ell}{6\varepsilon_0}}\right)} \tag{8.9.8}$$

allowing us to identify the A.C. capacitance as

$$C_{A.C.} = \frac{C_{D.C.}}{\left(1- \alpha\frac{d^3 w^2\ell}{24\varepsilon_0}\right) - \frac{1}{2}\mu_0\varepsilon_0\ell^2\omega^2} \tag{8.9.9}$$

We evaluated the α-term (error term) in the denominator at $y=w/2$.

This corresponds to the largest error term.

The above calculations suggest how one might proceed in a practical

problem. When the exciting source is electric in nature, such as the

voltage source in the parallel-plate capacitor, the zero-order magnetic

field is always zero [33]. The vanishing zero-order magnetic field

results in a vanishing first-order electric field which, in turn, forces

the second-order magnetic field to be zero. This process continues,
resulting in alternating zero and nonzero kth-order terms.

Therefore, one can always choose the trial functions of the form

$$\bar{E} = \bar{e}_0 + \omega^2 \bar{e}_2 + \omega^4 \bar{e}_4 + \cdots \tag{8.9.10}$$

$$\bar{H} = \bar{h}_1 + \omega^3 \bar{h}_2 + \omega^5 \bar{h}_5 + \cdots \tag{8.9.11}$$

in which coefficients of ω^k may contain many parameters to be adjusted.
Several numerical techniques, mentioned in Chapter 1, can be employed
to optimize these parameters. After determining approximate $\bar{E}$ and
$\bar{H}$ fields, one can then use them to calculate the source voltage and
current. Since ω^k are constants, they must be somehow imbedded in
voltage and current expressions. As a result, when the impedance is
calculated, the factor ω^k must show up in the capacitance portion
of the impedance. Therefore, one is able to determine the functional
form of the frequency dependence of A.C. capacitance.

The fact that the error term does not contain the frequency factor
ω^2, in our result for the approximate A.C. capacitance, originates
from the trial function $\bar{\phi} + \alpha \bar{\xi}$. Equation 8.8.1 shows that the error
term $\alpha \bar{\xi}$ in our trial function is in the form of a zero-order field.
There is certainly nothing wrong with this choice. But, we paid a
price for it, in the sense that the error term in A.C. capacitance
expression did not contain a frequency factor. If we had chosen a
trial function of the form $\bar{\phi} + \alpha \omega^2 \bar{\xi}$, where the frequency factor has
explicitly been entered, we would have obtained a slightly different
result,

$$C_{A.C.} = \frac{C_{D.C.}}{1-\omega^2\left(\frac{1}{3}\mu_0\varepsilon_0\ell^2 + \alpha\,\frac{d^3w^2\ell}{24\varepsilon_0}\right)} \qquad (8.9.12)$$

8.10. Practical Applications

There are many problems in electrical engineering where the

configuration of fields is similar to static fields [33]. Even in

the microwave frequency range, some problems can be regarded as

almost-static [3,33]. In addition, present microwave integrated

circuit technology has extended the almost-static problems well into

the microwave frequency range. The microscopic size of some high-frequency

circuits makes even the conventional circuit theory valid at tens of

giga-Hertz and even more. The technology of miniaturizing circuit

components will certainly continue to advance, as evidenced by the

booming microelectronics area. It is therefore foreseeable that in

many microwave engineering problems, the almost-static analysis will

yield sufficient accuracy for engineering purposes.

Although the power series approach to electromagnetics is valid

in general, it is most suitable for analysis of almost-static

problems [33]. The advantages are obvious. First, the useful results

can be obtained by keeping only the first few terms in the series.

Second, because of the alternating zero and nonzero fields (the typical

feature of the power series approach), one needs to perform the actual

calculation only for the nonzero fields [33]. Vanishing kth-order

terms are known in advance. Furthermore, as pointed out in Chapter 6, the zero-order problem is a static problem. Magid also points out that each kth-order field becomes static-like and could be solved with no more effort than that needed in a static problem.

We found in Chapter 7 that each kth-order field equation can be formulated as a complementary variation problem. Therefore, we come to the conclusion that many useful problems in engineering electromagnetics can now be posed as two variational integrals yielding upper and lower bounds to the stationary value. This new variational formulation can be more advantageous in certain problems than the conventional variational formulation yielding just a one-sided bound. Existing numerical techniques, such as finite element methods, can be employed to reduce the two integrals into discrete algebraic problems. The advantage of such numerical procedures based on our new variational functionals are yet to be studied.

9. CONCLUSIONS

Maxwell's equations in power series form have been formulated as complementary variational integrals. It has been found that each kth-order static-like fields can be formulated as two variational integrals. One of these integrals yields an upper bound to the stationary value while the other integral closes in from below. The general theory guarantees that the exact stationary value is always between the two integral values.

An illustrative example of a parallel-plate capacitor is discussed. It is shown that the zero-order trial field could be used to estimate the D.C. capacitance. Recognizing that the two variational functionals are "zero-order electric eneries," the D.C. capacitance was formulated as a quantity proportional to the functional value. Also, some general procedures were suggested whereby one can use the higher-order fields to estimate the frequency dependence of the capacitance.

The applicability of the dual extremum principles appears to include many problems of interest in electrical engineering. The continuing trend of miniaturizing circuit components allows circuit theory to be applied at tens of giga-Hertz and even more because of the small dimensions of the circuit compared to the wavelength. This seems to imply that in many microwave problems, useful information can be obtained by regarding the problems as almost static. These problems can then be recast as two variational integrals yielding both upper and lower bounds. Finally, it seems possible that numerical techniques, such as finite

element methods, can be developed based on the two complementary

functionals; this could lead to significant advantages over the existing

methods of analyzing microwave problems.

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