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Steady-state well-flow theory for a horizontal confined aquifer with
arbitrary conditions on the outer boundary

by

Rienk Rindert van der Ploeg

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

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Approved:

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In Charge of Major Work

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INTRODUCTION

In this thesis we shall determine flow nets and flux for saturated steady-state water flow to one or more wells in a horizontal, confined aquifer of finite extent and of uniform thickness and permeability. The shape of the aquifer can be arbitrary. The distribution of the hydraulic head along the outer boundary of the aquifer (positive boundary) must be known, and also the location of impervious parts on the outer boundary, through which water cannot enter the aquifer (negative boundary). As boundary condition at the well(s) we take the height of the standing water in the well above a reference level, or we take the well discharge as a boundary condition to be satisfied. We are trying to solve two-dimensional mixed boundary value problems of potential flow. We will work with polar coordinates and we are looking for solutions to Laplace's equation in polar coordinates.

We will use a modified Gram-Schmidt method to solve the various boundary value problems. Numerical calculations have always been carried out by means of a digital computer, (IBM 360/365). The computer was also used for drawing the flow nets, but these computer drawings were traced by a draftsman. A solution to a particular problem is given in the form of a function which gives the hydraulic head above an arbitrary reference level at any point in the flow region. After the theory has been worked out for hypothetical aquifers of selected geometry, the newly developed theory will be applied to the Ames aquifer. The Ames aquifer is a confined horizontal aquifer of rather uniform thickness and permeability. As outer boundary, the channel beds of two surface streams (the Skunk River

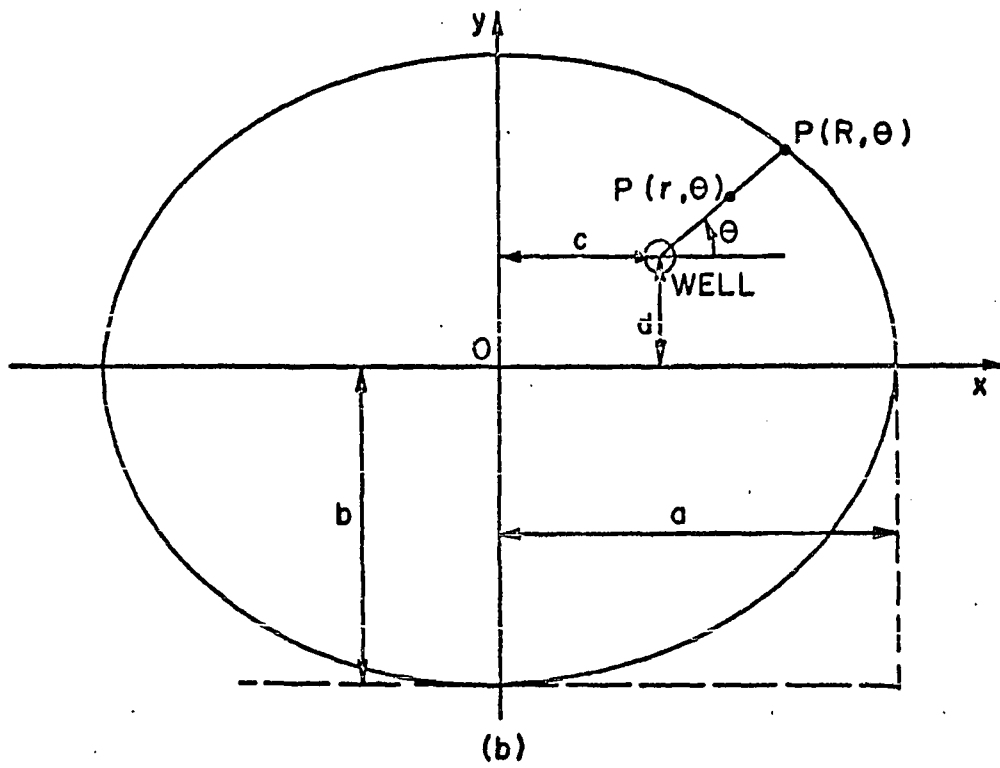
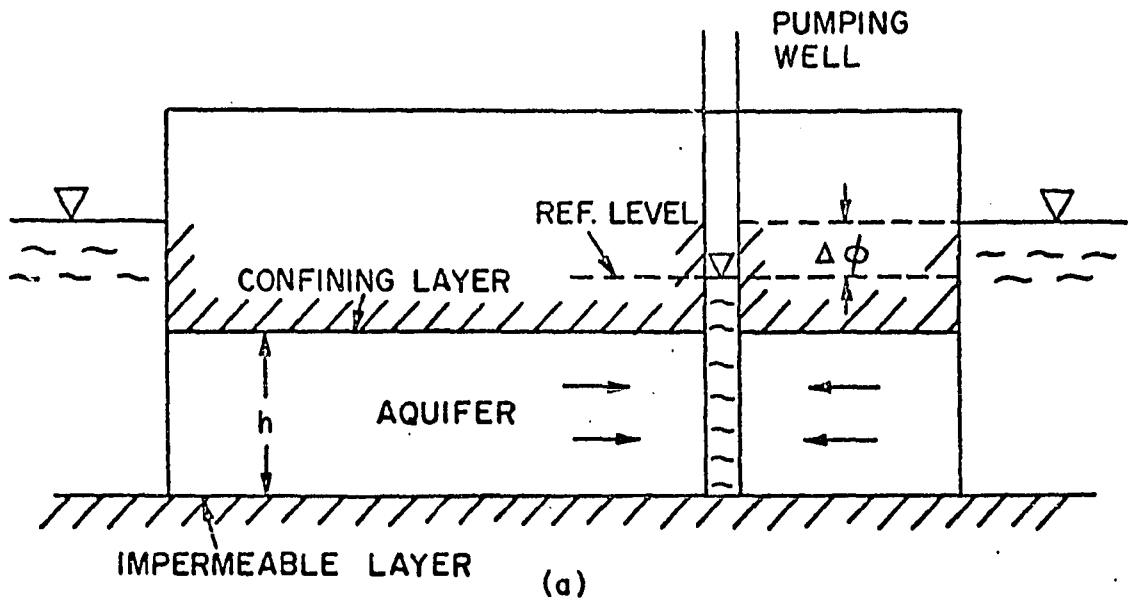
and Squaw Creek) can be considered. For the Ames aquifer we calculate the hydraulic head throughout the aquifer and calculate the streamlines. Also we calculate the well discharge for a particular drawdown and location of well(s).

Notation (see Figure 1)

A_{Nm} ,	constants, each having two subscripts ($N = 0, 1, 2, \dots$; $m = 0, 1, 2, \dots$) ;
D_m ,	constant for Bessel's inequality, dimensionless;
K ,	hydraulic conductivity of the aquifer, L/T ;
N ,	integer $0, 1, \dots$, indicating order of approximation;
O ,	origin of x, y Cartesian coordinates;
$P(r, \theta)$,	point in polar coordinates r, θ with origin at center of the well, L , dimensionless;
Q ,	well discharge, L^3/T ;
R ,	distance from origin of polar coordinate system (center of well) to aquifer boundary as in Figure 1b, L ;
a ,	major axis of the ellipse, L ;
b ,	minor axis of the ellipse, L ;
c, d ,	values of the x and y coordinates of the well center referred to the ellipse center, L, L (Figure 1b);
h ,	aquifer thickness, L ;
m ,	integer $0, 1, 2, \dots$;
n ,	integer $0, 1, 2, \dots$, $n \leq m$;
q ,	well discharge per unit aquifer thickness, L^3/T ;
r ,	polar radial coordinate, L (Figure 1b);

- r_w , radius of well, L;
 x, y , rectangular coordinates measured from the center 0 of the ellipse and coinciding with the major and minor axes, respectively, L, L (Figure 1b);
 ϕ , hydraulic head (potential) referred to the level of water standing in the well, L; in equation 6a ϕ is a dimensionless integration variable;
 $\Delta\phi$, head difference between outer boundary of ellipse and well, L (Figure 1a);
 $\frac{\phi}{\Delta\phi}$, dimensionless hydraulic head;
 Σ , summation sign, operator;
 Φ , velocity potential, equal to $K\phi$, L^2/T ;
 θ , polar angular coordinate, dimensionless, radians, or degrees;
 π , approximately 3.1416;
 ψ , stream function, L^2/T .

Figure 1. Geometrical representation of an elliptically shaped aquifer, with arbitrary well location. (a) cross-section. (b) plan view.



LITERATURE REVIEW

We will illustrate saturated water flow to a well in a confined horizontal aquifer of uniform thickness by use of Figure 2. Figure 2a shows a plan view, and Figure 2b shows a cross-sectional view of the aquifer. The thickness of the aquifer is h , the hydraulic conductivity is K (hence the transmissibility, $T = Kh$), and the discharge of the pumped well is Q . In many books on groundwater hydrology, one can find a discussion of the problem of well flow in a circular, horizontal, confined aquifer as shown in Figure 2 (Harr, 1962, p. 255 or Todd, 1967, p. 82). It is commonly assumed that the aquifer is homogeneous and isotropic. The sought-for expressions to describe the flow to the well are the velocity potential Φ (or the hydraulic head, ϕ , $\Phi = K\phi$), and the stream function ψ . With these quantities known, the well discharge, Q , can be calculated for a predescribed amount of drawdown in the well. The sought-for expressions of the velocity potential Φ and the stream function ψ can be found by solving Laplace's equation, subject to the boundary conditions as found at the well, and at the outer boundary of the aquifer. Laplace's equation for the velocity potential Φ in cylindrical coordinates can be given as:

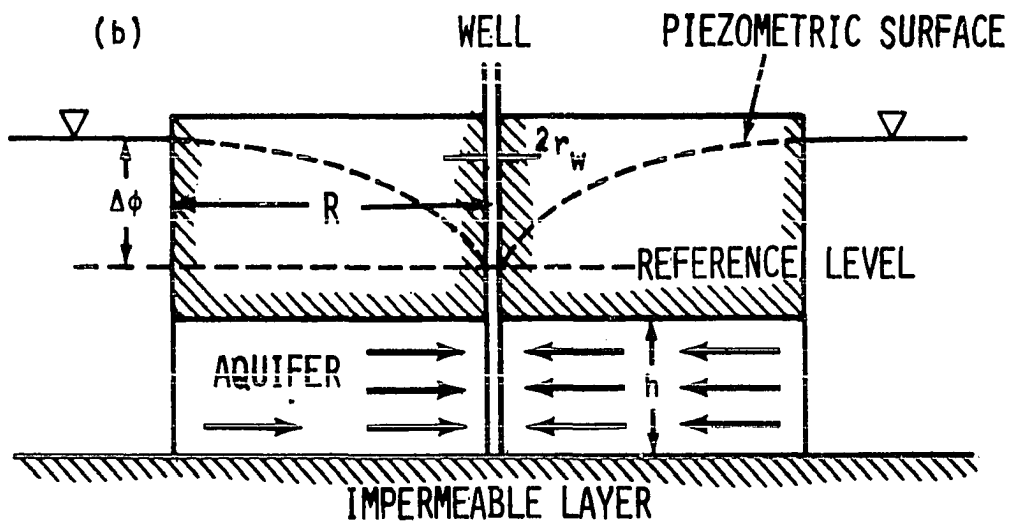
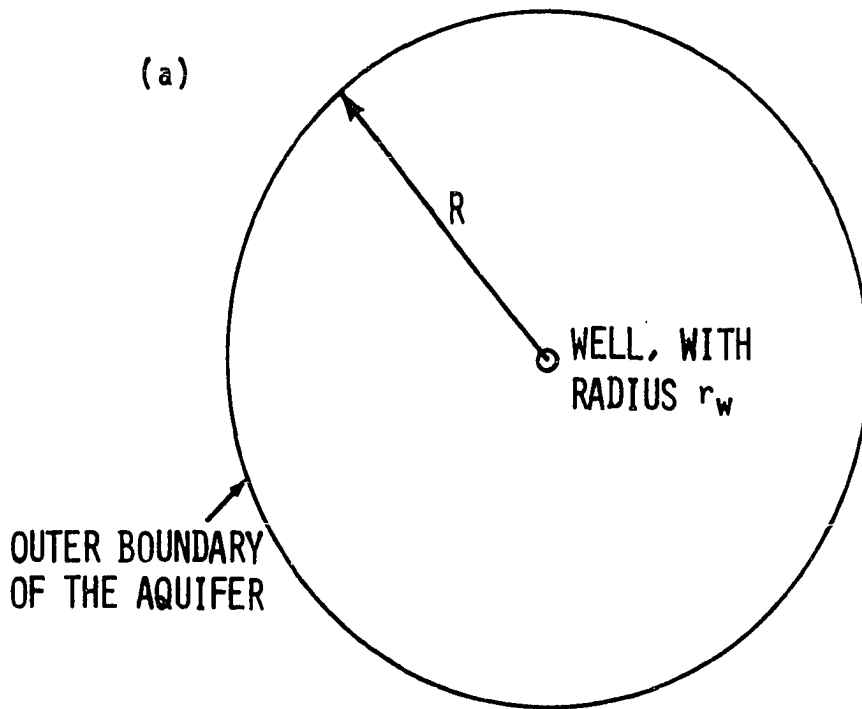
$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} = 0 \quad (1)$$

For radial flow in the horizontal plane there is no variation of Φ with the angle θ and the above equation reduces to:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) = 0,$$

which has as general solution:

Figure 2. Geometrical representation of a circular, horizontal confined aquifer, with a well in the center of the circle. (a) in plan view. (b) in cross-sectional view.



$$\Phi = C_1 \ln r + C_2$$

If r_w is the radius of the well one finds, by use of the boundary conditions, the following logarithmic expression for Φ (Harr, 1962, p. 255):

$$\Phi = \frac{Q}{2\pi Kh} \ln \frac{r}{r_w}, \quad (2)$$

For the streamfunction one then finds:

$$\psi = \frac{Q}{2\pi Kh} \theta, \quad (3)$$

and, for the well discharge Q , one finds, if $\Delta\phi$ is the uniform hydraulic head difference across the circular aquifer of radius R , the expression:

$$Q = \frac{-2\pi Kh \Delta\phi}{\ln(R/r_w)} \quad (\text{where the minus sign is usually omitted}) \quad (4)$$

In this thesis we will try to solve for these quantities Φ (or ϕ), ψ , and Q , for more complicated geometries than the one of Figure 2. Before we do so we will discuss some work that already exists in the groundwater literature.

For a circular horizontal confined aquifer with a fully penetrating well, but with the well not located in the center of the aquifer, solutions in the groundwater literature may be found in Muskat (1946, p. 172) or in Harr (1962, pp. 253-255). For the well discharge in this case one finds:

$$Q = \frac{2Kh \Delta\phi \pi}{\ln(a^2 - c^2) - \ln(ar_w)} \quad (5)$$

Similar solutions exist for equivalent problems in electrostatics, for example in Smythe (1939, p. 76, equation 5) and in Bewley (1948, p. 47, equation 85).

Polubarinova-Kochina (1962, pp. 366-368) was able to calculate the discharge from a well located in the center of an elliptical confined

aquifer by using conformal mapping. She provides an equation that yields the ratio Q_E/Q_C of the discharge of a well in an elliptical aquifer and the discharge of a well in a circular aquifer. Her equation, see Van der Ploeg, et al. (1971), is:

$$\frac{Q_E}{Q_C} = \frac{\ln(R/r_w)}{\ln(\pi C_P / 2r_w K_E k^{1/2})} \quad (6)$$

in which Q_E is the discharge from a well located in the center of an elliptical confined aquifer; Q_C is the discharge from a well located in the center of a circular confined aquifer; R is the radius of the circular aquifer; C_P is the ellipse parameter, $C_P^2 = a^2 - b^2$, where a and b refer to the semiaxes of the ellipse and subscript P is used to prevent confusion with c of our Figure 1; and k is the modulus of an elliptical function determined by:

$$K_E' / K_E = \frac{2}{\pi} \ln \frac{a+b}{a-b}$$

in which a and b again refer to the semiaxes of the ellipse and K_E and K_E' are complete elliptical integrals given by:

$$K_E = \int_0^{\pi/2} \frac{d\phi}{(1 - k^2 \sin^2 \phi)^{1/2}} \quad (6a)$$

$$K_E' = \int_0^{\pi/2} \frac{d\phi}{(1 - k'^2 \sin^2 \phi)^{1/2}}$$

where $k'^2 = 1 - k^2$ and ϕ is an integration variable (not hydraulic head). Van der Ploeg (1970) and Van der Ploeg et al. (1971) extended the solution of Polubarinova-Kochina to horizontal confined elliptical aquifers, where the well can be located at any place in the aquifer. They used a modified Gram-Schmidt method and found for the discharge of a well at an arbitrary location on an elliptical aquifer the following expression:

$$Q = \frac{2K A_{NO} h \Delta\phi \pi}{\ln (a/r_w)} \quad (7)$$

in which the coefficient A_{NO} was a constant depending on the location of the well, radius of the well, and shape of the aquifer. The formula that was used by van der Ploeg to derive equation 7 is given by:

$$Q = Kh \Delta\phi \int_0^{2\pi} \left(\frac{\partial\phi}{\partial r} \right) r d\theta, \text{ for } r = r_w \quad (7a)$$

Van der Ploeg (1970) and Van der Ploeg et al. (1971) checked numerically their expression for the well discharge against values of Todd, Harr, and Polubarinova-Kochina, that we discussed earlier. For corresponding aquifer shape, well radius, and well location, Van der Ploeg (1970) or Van der Ploeg et al. (1971) found an exact agreement of their values with those of Todd, Harr, and Polubarinova-Kochina. Van der Ploeg (1970) or Van der Ploeg et al. (1971) provide tables of dimensionless well discharge for various combinations of aquifer shape, well size, and well location. One such table is reproduced here as Table 1.

Table 1. Dimensionless well discharge $Q/Kh\Delta\phi$ for a confined elliptical aquifer with the well located on the major axis of the ellipse

r_w/a	c/a	$b/a = 1.0$	$b/a = 0.5$	$b/a = 0.2$	$b/a = 0.1$
1/100	1/2	1.455	1.607	2.039	2.585
	3/4	1.663	1.790	2.249	2.999
	7/8	1.985	2.096	2.581	3.33
1/400	1/2	1.102	1.186	1.406	1.648
	3/4	1.216	1.283	1.503	1.768
	7/8	1.381	1.433	1.638	1.93
1/4000	1/2	0.785	0.827	0.928	1.028
	3/4	0.841	0.873	0.969	1.073
	7/8	0.917	0.940	1.026	1.132

Table 1. (Continued)

r_w/a	c/a	$b/a = 1.0$	$b/a = 0.5$	$b/a = 0.2$	$b/a = 0.1$
1/40,000	1/2	0.609	0.634	0.693	0.747
	3/4	0.643	0.661	0.715	0.771
	7/8	0.687	0.699	0.744	0.801

Van der Ploeg (1970) or Van der Ploeg et al. (1971) provide also a number of flow nets for saturated water flow to a well in a confined aquifer.

So far we have discussed only horizontal confined aquifers of finite extent and with a constant head distribution along the outer boundary of the aquifer. Also, so far we have only discussed aquifers that are pumped by only one well, and that have an outer boundary which is pervious to water flow over its whole length, that is, there are no impermeable parts on the outer boundary. We will now review some literature on other aquifers.

For a horizontal confined aquifer of infinite extent, pumped by a completely penetrating well, solutions for well discharge and hydraulic head distribution are usually obtained with the assumption of a finite radius of influence, outside of which the action of the pumped well is not felt. The problem then reduces to the one illustrated by our Figure 2. Such a solution is found in Muskat (1946, p. 152).

Muskat (1946, p. 163) also discusses horizontal confined aquifers, pumped by one well, where there is an arbitrary head distribution over the outer boundary. Again he assumes a finite radius of influence.

For a well, located in a horizontal, confined infinite aquifer, with a uniform sloping piezometric head distribution, a solution can be obtained as in Rouse (1949, p. 344), where the total potential is obtained by superposition of a linear and a radial component. Rouse (1949) considers uniform flow in the negative x direction and calls the component of velocity potential due to the initially sloping piezometric surface ϕ_0 . This component ϕ_0 is equal to:

$$\phi_0 = v_0 x,$$

where $v_0 = -K \partial \phi / \partial x$. Then the total potential ϕ is given by:

$$\phi = \phi_0 + \phi_w = v_0 x + \frac{Q}{2\pi} \ln \frac{r}{r_w} \quad (8)$$

Rouse (1949) shows how to obtain ϕ graphically from ϕ_0 and ϕ_w by connecting with a smooth curve all intersections of ϕ_0 and ϕ_w of the same potential. De Wiest (1965, p. 254) shows the same for the combined stream function. Both Rouse (1949) and De Wiest (1965) make use of the fact that Laplace's equation is homogeneous, and that the sum of different solutions to Laplace's equation also satisfies Laplace's equation. Muskat (1946, p. 509) uses the same principle when he considers more than one pumped well in a horizontal, confined aquifer of infinite extent. He adds potential components, of the form:

$$\phi_0 = \frac{Q}{2\pi Kh} \ln \frac{r}{r_w} \quad (9)$$

due to individual wells together to obtain a composite potential. Again he assumes a radius of influence outside of which the action of the pumped wells is not felt. He takes an "average" head distribution over this radius of influence, in the case the head distribution is not constant.

In this thesis we will consider a finite horizontal confined aquifer, pumped by one or more wells. The wells may have different radii, each well may have a different discharge, and the actual head distribution over the outer boundary will be taken into account. Also parts of the outer boundary may be impervious (negative boundary). We will develop analytical solutions and we will use a modified Gram-Schmidt method, as first used by Powers et al. (1967), and further developed by Kirkham and Powers (1972). For the reader's benefit we will review the modified Gram-Schmidt method of Kirkham and Powers (1972) and Powers et al. (1967). Their modification consists, for a large part, in changing the development (for almost any function $f(x)$):

$$f(x) = \sum_{m=0}^N B_m \lambda_m(x)$$

to the development

$$f(x) = \sum_{m=0}^N A_{Nm} u_m(x), \alpha < x < \beta$$

where the $\lambda_m(x)$ are the conventional Gram-Schmidt orthonormal polynomials made up of linear combination of a new auxiliary set of functions $u_m(x)$. Powers et al. (1967) found recursion formulas for the A_{Nm} in terms of $f(x)$, α and β , and N and earlier A_{Nm} , where for $N (= 0, 1, 2, \dots) \longrightarrow \infty$, $f(x)$ becomes exact. Powers et al. (1967) relate the A_{Nm} to the B_m . All formulas are suitable for rapid digital computation. In a theoretical background section we shall give the development of Powers et al. (1967) and supply some steps that they omitted. Tables of the A_{Nm} that will not be given here are in Kirkham and Powers (1972). The modified Gram-Schmidt method of Powers et al. (1967) has been used by Boast (1969), who wrote a

computer program for the modified Gram-Schmidt process, Boast (1970), Boast and Kirkham (1971), Van der Ploeg (1970), and Selim and Kirkham (1972a and 1972b). The procedure for using the modified Gram-Schmidt method for mixed boundary value problems has been given by Kirkham (1972).

THEORETICAL BACKGROUND

A Boundary Value Problem

With the use of a particular boundary value problem, solved in Van der Ploeg (1970), or in Van der Ploeg et al. (1971) we will show how a modified Gram-Schmidt method can be used in our work. Van der Ploeg (1970) or Van der Ploeg et al. (1971) provided a solution for the problem of well flow in an elliptically shaped, horizontal confined aquifer. The most general case, as far as location of the well goes, is illustrated by Figure 2. Here the well is neither located in the center of the ellipse, nor on an axis. The major and minor axis of the ellipse are denoted by a and b , and choosing a Cartesian coordinate system as shown, we can give the coordinates of the center of the well as (c, d) . Rather than using the center of the ellipse as origin of a coordinate system, Van der Ploeg et al. (1971) chose the center of the well as origin of a polar coordinate system. A point P inside the ellipse is then denoted by $P(r, \theta)$, whereas a point P on the boundary of the ellipse is denoted by $P(R, \theta)$. Rather than to work with an arbitrary location of the well, we will illustrate the Gram-Schmidt method with a particular location of the well. We choose the well location to be on the major axis of the ellipse, half way between the center of the ellipse and the outer boundary. If we look at Figure 1 this means that $c = 1/2a$ and $d = 0$. One should realize that the well is completely penetrating the aquifer, and that the aquifer is water-saturated. The problem is to find an expression for the hydraulic head ϕ that besides

satisfying Laplace's equation, also satisfies the boundary conditions. Since there is symmetry we need only to consider the upper half of the ellipse. If we take the reference level from which we measure the hydraulic head ϕ to be the water level in the well, if we take a unit hydraulic gradient across the aquifer, and if we denote the well radius by r_w , we can state the boundary conditions as follows:

Boundary condition 1

$$\phi = 0 \quad \text{for } r = r_w \quad 0 \leq \theta \leq \pi$$

Boundary condition 2

$$\phi/\Delta\phi = 1 \quad \text{for } r = R \quad 0 \leq \theta \leq \pi \quad (10)$$

Boundary condition 3

$$\partial\phi/\partial\theta = 0 \quad \text{for } \theta = 0 \quad r_w < r < (a - c)$$

$$\partial\phi/\partial\theta = 0 \quad \text{for } \theta = \pi \quad r_w < r < (a + c)$$

To find the coordinates of a point $P(R, \theta)$ on the outer boundary of the elliptical aquifer we recall the equation of an ellipse, with respect to the center of the well:

$$[(R \cos \theta + c)^2/a^2] + [(R \sin \theta)^2/b^2] = 1, \quad (11)$$

which is a quadratic equation in R . When solved for R the positive value of R should be taken. As mentioned before, Laplace's equation in polar coordinates is given by:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0$$

By the method of separation of variables a number of solutions to this equation can be obtained. Byerly (1959) gives eight particular solutions to Laplace's equation in polar coordinates. The first and the third of the eight solutions may be written as $\phi = r^m \cos m \theta$ and $\phi = r^{-m} \cos m \theta$,

where m is an arbitrary constant. Van der Ploeg et al. (1971) combined and modified these solutions and they give as a suitable expression for the potential function ϕ for the geometry of Figure 4, and with $m = 0, 1, 2 \dots N$, $N = 0, 1, 2 \dots \infty$, the expression (see Appendix):

$$\frac{\phi}{\Delta\phi} = A_{N0} \frac{\ln (r/r_w)}{\ln (a/r_w)} + \sum_{m=1}^{N \rightarrow \infty} A_{Nm} \left\{ \frac{(r/a)^m - [r_w^2/(ar)]^m}{1 - (r_w^2/a^2)^m} \right\} \cos m\theta \quad (12)$$

In the above equation the A_{Nm} are arbitrary constants. By inspection of equation 12, one can verify that boundary conditions 1 and 3 are satisfied regardless of the A_{Nm} , and so is Laplace's equation. By using a Gram-Schmidt method as modified by Powers et al. (1967), or Kirkham and Powers (1972), one sees the coefficients A_{Nm} can be determined to satisfy the remaining boundary conditions.

The above equation is of the form:

$$\frac{\phi}{\Delta\phi} = \sum_{m=0}^N A_{Nm} u_m(r, \theta),$$

where $u_m(r, \theta)$ is defined by:

$$u_m(r, \theta) = \left\{ \frac{(r/a)^m - [r_w^2/(ar)]^m}{1 - (r_w^2/a^2)^m} \right\} \cos m\theta$$

If we want to satisfy boundary condition 2, for which $r = R$, with R given by equation 11 we can substitute $r = R$ in the expression for $u_m(r, \theta)$ and we can define $u_m(\theta)$ as:

$$u_0 = \frac{\ln (R/r_w)}{\ln (a/r_w)}, \text{ and}$$

$$u_m = \left\{ \frac{(R/a)^m - [r_w^2/(aR)]^m}{1 - (r_w^2/a^2)^m} \right\} \quad (13)$$

Thus, on the outer boundary of the aquifer, we can write:

$$(\phi/\Delta\phi)_{r=R} = \sum_{m=0}^{N \rightarrow \infty} A_{Nm} u_m(\theta), \quad 0 \leq \theta \leq \pi$$

Or, to satisfy boundary condition 2, where we wish to have $\phi_{r=R} = 1$, we may write:

$$1 = \sum_{m=0}^{N \rightarrow \infty} A_{Nm} u_m(\theta) \quad (14)$$

To satisfy boundary condition 2, with respect to equation 14 we can now use the modified Gram-Schmidt method (see e.g., Kirkham and Powers (1972, Chap. 4)).

The Modified Gram-Schmidt Method

Before getting into the modified Gram-Schmidt method we shall write down some preliminary equations and derive the conventional Gram-Schmidt polynomials.

Equation 14 is a special case of the equation:

$$f(x) = \sum_{m=0}^{N \rightarrow \infty} A_{Nm} u_m(x) \quad (15)$$

$$\alpha < x < \beta \quad m = 0, 1, \dots, N \quad \text{and } N \rightarrow \infty$$

and is in terms of the polar coordinate θ , rather than in terms of the rectangular coordinate x . We see that in equation 14, the $f(\theta)$, corresponding to $f(x)$ of equation 15, is equal to 1. The coefficient A_{Nm} of equation 15 will be determined later on, using the Gram-Schmidt method as modified by Powers et al. (1967). In our work the $f(x)$ and the $u_m(x)$ are always known. The $f(x)$ represents the hydraulic head distribution on the outer boundary of the aquifer, and the $u_m(x)$ are expressions, as in equation

13, modified combinations of particular solutions to Laplace's equation in polar coordinates.

With the $f(x)$ and $u_m(x)$ of equation 15 known, the A_{Nm} can be determined, by the modification of Powers et al. (1967) by means of an auxiliary equation

$$f(x) = \sum_{m=0}^{N \rightarrow \infty} B_m \lambda_m(x) \quad (16)$$

$$m = 0, 1, \dots \quad (N \rightarrow \infty) \quad \alpha < x < \beta$$

where the B_m are constants that we, following Powers et al. (1967), shall find and where each λ_m , which we shall also find, is an orthonormal polynomial developed from linear combinations of the known $u_m(x)$ of equation 15. A polynomial λ_m is orthonormal, if the λ_m satisfy, for the interval $\alpha < x < \beta$, the expression

$$\int_{\alpha}^{\beta} \lambda_m(x) \lambda_n(x) dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases} \quad (17)$$

$$m, n, = 0, 1, \dots$$

When polynomials $\lambda_m(x)$ are orthonormal, they can be used to develop other functions $f(x)$ into series of the form [see Powers et al. (1967)]

$$f(x) = \sum_{m=0}^{N \rightarrow \infty} B_m \lambda_m \quad (18)$$

$$m = 0, 1, \dots \quad (N \rightarrow \infty) \quad \alpha < x < \beta$$

where the B_m have the values

$$B_m = \int_{\alpha}^{\beta} f(x) \lambda_m dx \quad (19)$$

$$\alpha \leq x \leq \beta \quad m = 0, 1, \dots \quad N \rightarrow \infty$$

To find the orthonormal polynomials λ_m we may use the Gram-Schmidt orthonormalization method. Before we show how this is done we need to explain some shorthand notations that we are going to use in the procedure. Let us consider two functions of x , say $f(x)$ and $g(x)$. Then the inner product of f and g is defined as a definite integral of the product of $f(x)$ and $g(x)$ with respect to x , where x lies between α and β . We denote this inner product by (fg) or by (f, g) , hence we have

$$(fg) = (f, g) = \int_{\alpha}^{\beta} f(x) g(x) dx \quad (20)$$

Let us consider a function of x , say $\gamma_m(x)$ that we shall relate to the $\lambda_m(x)$, and that we are going to use later on in the Gram-Schmidt orthonormalization process. The function $\gamma_m(x)$ is to be an orthogonal function, which means that if we use the inner product notation of equation 20, we can write

$$\begin{aligned} (\gamma_m \gamma_n) &= 0 \text{ for } m \neq n \\ &\neq 0, 1 \text{ for } m = n \\ m, n &= 0, 1, \dots \end{aligned} \quad (21)$$

Equation 17 which expresses the orthonormality of the $\lambda_m(x)$ can be written as

$$\begin{aligned} (\lambda_m \lambda_n) &= 0 \text{ for } m \neq n \\ &= 1 \text{ for } m = n \\ m, n &= 0, 1, \dots \end{aligned} \quad (22)$$

An orthogonal function $\gamma_m(x)$ can be made orthonormal by dividing each term of the orthogonal polynomial by its normalizing factor $D_m^{1/2}$, in which $D_m^{1/2}$ is defined as

$$D_m^{1/2} = (\gamma_m \gamma_m)^{1/2} \quad (23)$$

We now relate $\gamma_m(x)$ to $\lambda_m(x)$ by the expression

$$\lambda_m = \gamma_m / D_m^{1/2} \quad (24)$$

We now look back at equations 15 and 16. We want to develop the set of $u_m(x)$ into an orthonormal set of $\lambda_m(x)$. According to Courant and Hilbert (1924, p. 34) (see, Powers et al. (1967)) we can obtain the orthogonal set $\gamma_m(x)$ from the $u_m(x)$ by the following recursion formulas.

$$\gamma_0 = u_0, \quad \gamma_m = u_m - \sum_{n=0}^{m-1} (u_m \gamma_n) \gamma_n D_n^{-1} \quad (25)$$

$$m = 1, 2, \dots$$

$$n = 0, 1, \dots$$

By use of equation 24 we then can get the orthonormal set λ_m from equation 25 as

$$\lambda_m = \frac{u_m - \sum_{n=0}^{m-1} (u_m \lambda_n) \lambda_n}{D_m^{1/2}} \quad \begin{matrix} m = 0, 1, 2, \dots \\ n = 0, 1, \dots \end{matrix} \quad (26)$$

Equations 25 and 26 were obtained by Courant-Hilbert (1924, p. 34) and by Kirkham and Powers (1972, Chap. 4) by what is now commonly called the Gram-Schmidt method. We shall derive equations 25 and 26 for completeness here later.

As equations 25 and 26 stand they are not very useful to get the A_{Nm} of equation 15. We see this by first expanding equation 25 as follows:

From equation 25 we write down

$$\gamma_0 = u_0 \quad (27)$$

$$\gamma_1 = u_1 - \frac{u_1 \gamma_0}{D_0} \gamma_0 \quad (28)$$

$$\gamma_2 = u_2 - \frac{(u_2 \gamma_1)}{D_1} \gamma_1 - \frac{(u_2 \gamma_0)}{D_0} \gamma_0 \quad (29)$$

$$\gamma_3 = u_3 - \frac{(u_3 \gamma_2)}{D_2} \gamma_2 - \frac{(u_3 \gamma_1)}{D_1} \gamma_1 - \frac{(u_3 \gamma_0)}{D_0} \gamma_0 \quad (30)$$

We now wish to express the γ_m in terms of the u_m only, rather than having the γ_m expressed in terms of u_m and γ_{m-1} , γ_{m-2} , etc., as is done in equations 28, 29, and 30. We first repeat equation 27,

$$\gamma_0 = u_0 \quad (31)$$

We next substitute equation 27 in equation 28; then substitute equation 27 and the equation that results from 28 in equation 29, and so on. We obtain

$$\gamma_1 = u_1 - \frac{(u_1 \gamma_0)}{D_0} u_0 \quad (32)$$

Similarly we obtain

$$\gamma_2 = u_2 - \frac{(u_2 \gamma_1)}{D_1} [u_1 - \frac{(u_1 \gamma_0)}{D_0} u_0] - \frac{(u_2 \gamma_0)}{D_0} u_0,$$

which we can rewrite as

$$\gamma_2 = u_2 - \frac{(u_2 \gamma_1)}{D_1} u_1 + \frac{(u_2 \gamma_1)}{D_1} \frac{(u_1 \gamma_0)}{D_0} u_0 - \frac{(u_2 \gamma_0)}{D_0} u_0$$

which in turn we can write as

$$\gamma_2 = u_2 - \frac{(u_2 \gamma_1)}{D_1} u_1 + [\frac{(u_2 \gamma_1)}{D_1} \frac{(u_1 \gamma_0)}{D_0} - \frac{(u_2 \gamma_0)}{D_0}] u_0 \quad (33)$$

We see that in equation 33 we have expressed γ_2 as a linear combination

of u_2 , u_1 , and u_0 , with the coefficients of u_2 , u_1 , and u_0 becoming increasingly complex.

We will now find γ_3 as a linear combination of u_0 , u_1 , u_2 , and u_3 . In equation 30 we substitute equation 33 for γ_2 , we substitute equation 32 for γ_1 , and we substitute equation 31 for γ_0 . Then equation 30 becomes

$$\begin{aligned} \gamma_3 = u_3 - \frac{(u_3\gamma_2)}{D_2} \left\{ u_2 - \frac{(u_2\gamma_1)}{D_1} u_1 + \left[\frac{(u_2\gamma_1)}{D_1} \frac{(u_1\gamma_0)}{D_0} - \frac{(u_2\gamma_0)}{D_0} \right] u_0 \right\} \\ - \frac{(u_3\gamma_1)}{D_1} \left[u_1 - \frac{(u_1\gamma_0)}{D_0} u_0 \right] - \frac{(u_3\gamma_0)}{D_0} u_0, \end{aligned}$$

which can be rearranged to

$$\begin{aligned} \gamma_3 = u_3 - \frac{(u_3\gamma_2)}{D_2} u_2 + \frac{(u_3\gamma_2)}{D_2} \frac{(u_2\gamma_1)}{D_1} u_1 - \frac{(u_3\gamma_1)}{D_1} u_1 \\ - \frac{(u_3\gamma_2)}{D_2} \frac{(u_2\gamma_1)}{D_1} \frac{(u_1\gamma_0)}{D_0} u_0 + \frac{(u_3\gamma_2)}{D_2} \frac{(u_2\gamma_0)}{D_0} u_0 \\ + \frac{(u_3\gamma_1)}{D_1} \frac{(u_1\gamma_0)}{D_0} u_0 - \frac{(u_3\gamma_0)}{D_0} u_0 \end{aligned}$$

which upon rearranging can be written as

$$\begin{aligned} \gamma_3 = u_3 - \left[\frac{(u_3\gamma_2)}{D_2} \right] u_2 + \left[\frac{(u_3\gamma_2)}{D_2} \frac{(u_2\gamma_1)}{D_1} - \frac{(u_3\gamma_1)}{D_1} \right] u_1 \\ + \left[- \frac{(u_3\gamma_2)}{D_2} \frac{(u_2\gamma_1)}{D_1} \frac{(u_1\gamma_0)}{D_0} + \frac{(u_3\gamma_2)}{D_2} \frac{(u_2\gamma_0)}{D_0} \right. \\ \left. + \frac{(u_3\gamma_1)}{D_1} \frac{(u_1\gamma_0)}{D_0} - \frac{(u_3\gamma_0)}{D_0} \right] u_0 \end{aligned} \quad (34)$$

Again, as in equation 33 for γ_2 , as in equation 32 for γ_1 , and as in equation 31 for γ_0 , we have succeeded to express the γ_3 as a linear combination of u_3 , u_2 , u_1 , and u_0 . The fact that the γ_m can be expressed

as linear combinations of the u_m enabled Powers et al. (1967) to obtain more useful forms of the γ_m . For the sake of discussion we write equations 31, 32, 33, and 34, so that they can easily be compared, as

$$\gamma_0 = u_0 \quad (35)$$

$$\gamma_1 = u_1 - \left[\frac{(u_1 \gamma_0)}{D_0} \right] u_0 \quad (36)$$

$$\gamma_2 = u_2 - \left[\frac{(u_2 \gamma_1)}{D_1} \right] u_1 - \left[\frac{(u_2 \gamma_0)}{D_0} - \frac{(u_2 \gamma_1)}{D_1} \frac{(u_1 \gamma_0)}{D_0} \right] u_0 \quad (37)$$

$$\begin{aligned} \gamma_3 = u_3 - \left[\frac{(u_3 \gamma_2)}{D_2} \right] u_2 - \left[\frac{(u_3 \gamma_1)}{D_1} - \frac{(u_3 \gamma_2)}{D_2} \frac{(u_2 \gamma_1)}{D_1} \right] u_1 \\ - \left[\frac{(u_3 \gamma_0)}{D_0} - \frac{(u_3 \gamma_1)}{D_1} \frac{(u_1 \gamma_0)}{D_0} - \frac{(u_3 \gamma_2)}{D_2} \frac{(u_2 \gamma_0)}{D_0} \right. \\ \left. + \frac{(u_3 \gamma_2)}{D_2} \frac{(u_2 \gamma_1)}{D_1} \frac{(u_1 \gamma_0)}{D_0} \right] u_0 \end{aligned} \quad (38)$$

We stop for a while to look back to equations 35-38. From the way we developed γ_0 , γ_1 , γ_2 , and γ_3 they are orthogonal. We do not want to work with the set γ_m , but we want to work with the orthonormal set λ_m which can be obtained from equations 35-38, by dividing through by the corresponding normalizing factor. If, for example, we want to get λ_3 from γ_3 we have to divide γ_3 by $D_3^{1/2}$, where, by use of equation 23, $D_3^{1/2}$ is given by

$$D_3^{1/2} = (\gamma_3 \gamma_3)^{1/2} \quad (39)$$

By looking at equation 38 for γ_3 one realizes that $D_3^{1/2}$ as given by equation 39 is not easily obtained, and hence λ_3 can only be obtained after tedious and careful work. One should realize also that λ_0 , λ_1 , λ_2 , and λ_3 are only the first few terms of a set of λ_m ; for example, to obtain λ_4 , it

will take much more effort than λ_3 . Further, in view of equation 18, we not only need to determine the λ_m , but we also need the B_m , which we can get from equation 19. To continue with equations like 35-38 seems impractical, unless simple recursion formulas can be obtained from which the γ_m and then the λ_m and B_m and A_{Nm} can be obtained.

By careful inspection of equations like 27-30, and like equations 35-38, Powers et al. (1967) or Kirkham and Powers (1972) were able to derive recursion formulas, so that the Gram-Schmidt process could be made practical. We will follow Powers et al. (1967) in their derivations of various recursion formulas, but we will give some more details.

Recursion Formulas for the γ_m and λ_m

We will now derive the recursion formulas, equations 25 and 26, for the orthogonal set γ_m and the orthonormal set λ_m . We recall the notations of equations 20, 21, and 22,

$$(f, g) = (fg) = \int_{\alpha}^{\beta} fg \, dx, \quad \alpha < x < \beta, \quad (20)$$

and

$$(\gamma_m, \gamma_n) = 0, \text{ for } m \neq n; \neq 0, \text{ for } m = n, m = 0, 1, \dots, \text{(orthogonal)} \quad (21)$$

$$(\lambda_m, \lambda_n) = 0, \text{ for } m \neq n; = 1, \text{ for } m = n, m = 0, 1, \dots, \text{(orthonormal)} \quad (22)$$

The Conventional Gram-Schmidt Method

We follow the Gram-Schmidt process and assume that each of our orthonormal functions $\lambda_0, \lambda_1, \dots$, can be made up of an orthogonal function γ_m (which we seek) and a normalization factor D_m such that we may express λ_m as

$$\lambda_m = \gamma_m / D_m^{1/2} \quad (40)$$

To get the normalization factor D_m in equation 40, we multiply equation 40 through by $\lambda_m dx$, integrate each side over the range of x , and then use the inner product notation of equation 20 to find

$$(\lambda_m \lambda_m) = (\lambda_m \gamma_m) / D_m^{1/2} \quad (41)$$

We now substitute λ_m as given by equation 40 into the right side of equation 41 and find

$$(\lambda_m \lambda_m) = (\gamma_m \gamma_m) / D_m$$

showing by equation 22 that we must have D_m as

$$D_m = (\gamma_m \gamma_m) \quad (42)$$

To get orthogonal forms of $\gamma_0, \gamma_1, \dots$, we consider $m = 0, 1, \dots$ in equation 40. $m = 0$ gives

$$\lambda_0 = \gamma_0 / D_0^{1/2} \quad (43)$$

where we are free to choose a starting value of γ_0 . We take γ_0 as

$$\gamma_0 = u_0 \quad (44)$$

so that equation 43 may be written as

$$\lambda_0 = u_0 / D_0^{1/2} \quad (45)$$

We now consider $m = 1$ in equation 40 and write

$$\lambda_1 = \gamma_1 / D_1^{1/2} \quad (46)$$

We express γ_1 by (in the following a_0, b_0, b_1, \dots are constants) the linear relation

$$\gamma_1 = u_1 + a\lambda_0 \quad (47)$$

Now, since λ_1 and λ_0 must be orthogonal, we have

$$(\lambda_1 \lambda_0) = 0$$

which, in view of the right of equation 46, is equivalent to

$$(\lambda_1 \gamma_0) = 0 \quad (48)$$

So we multiply both sides of equation 47 by $\lambda_0 dx$, integrate from $x = \alpha$ to β , and then set the result equal to zero. Using the notation of equation 20 we find that equation 47 may be written as

$$(\gamma_1 \lambda_0) = (u_1 \lambda_0) + a_0 (\lambda_0 \lambda_0) \quad (49)$$

which after use of equations 22 and 48 gives

$$a_0 = - (u_1 \lambda_0) \quad (50)$$

so that equations 47 and 50 give

$$\gamma_1 = u_1 - (u_1 \lambda_0) \lambda_0 \quad (51)$$

We are now ready to get λ_2 . We define a function γ_2 by

$$\gamma_2 = u_2 + b_0 \lambda_0 + b_1 \lambda_1 \quad (52)$$

For λ_2 to be orthogonal to λ_0 and λ_1 we must, by equation 40 have γ_2 orthogonal to λ_0 and λ_1 . That is, we must have from equations 52 and 21 the two equations

$$(\gamma_2 \lambda_0) = (u_2 \lambda_0) + b_0 (\lambda_0 \lambda_0) + b_1 (\lambda_1 \lambda_0) = 0 \quad (53)$$

$$(\gamma_2 \lambda_1) = (u_2 \lambda_1) + b_0 (\lambda_0 \lambda_1) + b_1 (\lambda_1 \lambda_1) = 0 \quad (54)$$

In each of equations 53 and 54 we now use equations 21 and 22 and find

$$b_0 = -(u_2 \gamma_0), \quad b_1 = -(u_2 \gamma_1) \quad (55), (56)$$

so that equations 52, 55 and 56 yield

$$\gamma_2 = u_2 - (u_2 \lambda_0) \lambda_0 - (u_2 \lambda_1) \lambda_1 \quad (57)$$

Continuing as in equation 47 and 52 we write γ_3 as

$$\gamma_3 = u_3 + c_0 \lambda_0 + c_1 \lambda_1 + c_2 \lambda_2 \quad (58)$$

to find, as we found in equation 49, and in equations 53 and 54, the orthogonality relations

$$(\gamma_3 \lambda_0) = (\lambda_0 u_3) + c_0 + 0 + 0 = 0 \quad (59)$$

$$(\gamma_3 \lambda_1) = (\lambda_1 u_3) + 0 + c_1 + 0 = 0 \quad (60)$$

$$(\gamma_3 \lambda_2) = (\lambda_2 u_3) + 0 + 0 + c_2 = 0 \quad (61)$$

From equation 59, 60 and 61 we find

$$c_0 = (u_3 \lambda_0), \quad c_1 = (u_3 \lambda_1), \quad c_2 = (u_3 \lambda_2) \quad (62), (63), (64)$$

We put equations 62, 63, and 64 in equation 58 and find

$$\gamma_3 = u_3 - (u_3 \lambda_0) \lambda_0 - (u_3 \lambda_1) \lambda_1 - (u_3 \lambda_2) \lambda_2 \quad (65)$$

We collect equations 44, 51, 57 and 65 to write

$$\gamma_0 = u_0 \quad (66)$$

$$\gamma_1 = u_1 - (u_1 \lambda_0) \lambda_0 \quad (67)$$

$$\gamma_2 = u_2 - (u_2 \lambda_0) \lambda_0 - (u_2 \lambda_1) \lambda_1 \quad (68)$$

$$\gamma_3 = u_3 - (u_3 \lambda_0) \lambda_0 - (u_3 \lambda_1) \lambda_1 - (u_3 \lambda_2) \lambda_2 \quad (69)$$

and equations 66 - 69 yield in view of equation 40 the expressions

$$\lambda_0 = \gamma_0^{D_0^{-1/2}}$$

$$\lambda_1 = \gamma_1^{D_1^{-1/2}}$$

$$\lambda_2 = \gamma_2^{D_2^{-1/2}}$$

$$\lambda_3 = \gamma_3^{D_3^{-1/2}}$$

...

and see that these formulas may be generalized to the two equations

$$\gamma_0 = u_0, \quad m = 0 \quad (70)$$

$$\gamma_m = u_m - \sum_{n=0}^{m-1} (u_m \lambda_n) \lambda_n, \quad m = 1, 2, \dots \quad (71)$$

$$n = 0, 1, \dots$$

With the γ_m now known by equations 70 and 71, we may quickly write down, in view of equations 40, 70 and 71, our sought λ_m (where we rewrite equation 45 first) as

$$\lambda_0 = u_0 / D_0^{1/2}, \quad m = 0 \quad (72)$$

$$\lambda_m = \frac{u_m - \sum_{n=0}^{m-1} (u_m \lambda_n) \lambda_n}{D_m^{1/2}}, \quad m = 1, 2, \dots \quad (73)$$

$$n = 0, 1, \dots$$

which is equation 26.

Equation 73 is a recursion formula for the λ_m (because the right side of equation 73 contains earlier supposedly known λ_m , namely, λ_{m-1} , λ_{m-2} , etc.). Equation 73 may be compared with a formula given in Courant-Hilbert [1924, bottom p. 34]. To derive the formula of Courant-Hilbert we may take their constants c_2 and c_3^* , etc., each equal to unity and take their ϕ' equal to the numerator of the right side of their formula. To correct a misprint one should change, in the Courant-Hilbert formula, v_{n+1} to ϕ'_{n+1} .

The numerator in the right side of equation 73 is γ_m of equation 71 and may in view of equations 71 and 40 be written as

$$\gamma_m = u_m - \sum_{n=0}^{m-1} (u_m \gamma_n) \gamma_n D_n^{-1}, \quad m = 1, 2, \dots \quad (74)$$

$$n = 0, 1, \dots$$

which is equation 25.

If we introduce an abbreviation c_{mn} defined by

$$c_{mn} = (u_m \gamma_n) D_n^{-1}, \quad m = 1, 2, \dots, \quad n = 0, 1, \dots, m-1, \quad (75)$$

then equation 74 becomes

$$\gamma_m = u_m - \sum_{n=0}^{m-1} c_{mn} \gamma_n, \quad m = 1, 2, \dots, \quad n = 0, 1, \dots, m-1 \quad (76)$$

and use of equation 76 in equation 40 gives us the λ_m as

$$\lambda_m = \frac{u_m - \sum_{n=0}^{m-1} c_{mn} \gamma_n}{D_m^{1/2}}, \quad m = 1, 2, \dots, \quad n = 0, 1, \dots, m-1 \quad (77)$$

which is an alternate form of equation 73.

Normalization Factor D_m

The D_m of equations 72-77, defined so far by equation 42, are not in useful form for numerical work. We may get useful forms as follows:

For $m = 0$, equations 42 and 44 give

$$D_0 = (u_0 u_0) \quad (78)$$

For $m = 1$, we get, from equations 42 and 47, and the comma parentheses notation of the inner product notation of equation 20, the expression

$$D_1 = (u_1 + a_0 \lambda_0, u_1 + a_0 \lambda_0)$$

which, when the integrand on the right is expanded, may be written as

$$D_1 = (u_1 u_1) + 2a_0 (u_1 \lambda_0) + a_0^2 (\lambda_0 \lambda_0) \quad (79)$$

which, in view of $(\lambda_0 \lambda_0) = 1$ of equation 22 and $a_0 = - (u_1 \lambda_0)$ of equation 50, may be written as

$$D_1 = (u_1 u_1) - 2(u_1 \lambda_0)^2 + (u_1 \lambda_0)^2$$

which, after combination of the last two terms in the right, becomes finally

$$D_1 = (u_1 u_1) - (u_1 \lambda_0)^2 \quad (80)$$

As we found D_1 of equation 80, we similarly find, but now with use of equations 52, 21, 22, 55 and 56 in that order, the result

$$D_2 = (u_2 u_2) - (u_2 \lambda_0)^2 - (u_2 \lambda_1)^2 \quad (81)$$

and for D_3 , the result

$$D_3 = (u_3 u_3) - (u_3 \lambda_0)^2 - (u_3 \lambda_1)^2 - (u_3 \lambda_2)^2 \quad (82)$$

Continuing (or using directly γ_m of equation 25 for one of the γ_m in $D_m = (\gamma_m \gamma_m)$ of equation 23, and then using equations 21 and 22), one finds

$$D_m = (u_m u_m) - \sum_{n=0}^{m-1} (u_m \lambda_n)^2 \quad \begin{matrix} m = 1, 2, \dots \\ n = 0, 1, \dots \end{matrix}$$

which in view of equation 40 may be written as

$$D_m = (u_m u_m) - \sum_{n=0}^{m-1} (u_m \gamma_n)^2 D_n^{-1} \quad \begin{matrix} m = 1, 2, \dots \\ n = 0, 1, \dots \end{matrix}$$

or since from equation 75 we have

$$c_{mn}^2 D_n = (u_m \gamma_n)^2 D_n^{-1},$$

we may write

$$D_0 = u_{00}$$

$$D_1 = u_{11} - c_{10}^2 D_0$$

$$D_2 = u_{22} - c_{20}^2 D_0 - c_{21}^2 D_1$$

$$D_3 = u_{33} - c_{30}^2 D_0 - c_{31}^2 D_1 - c_{32}^2 D_2$$

$$D_4 = u_{44} - c_{40}^2 D_0 - c_{41}^2 D_1 - c_{42}^2 D_2 - c_{43}^2 D_3,$$

or, in general

$$D_m = (u_m u_m) - \sum_{n=0}^{m-1} c_{mn}^2 D_n, \quad m = 1, 2, \dots \quad (83)$$

$$n = 0, 1, \dots, m-1$$

Equation 83 subject to our being able to evaluate the c_{mn} is a recursion formula, useful in numerical work, for the D_m . To get the c_{mn} in equation 83 we need to introduce some auxiliary constants J_{mn} .

Introduction of Constants J_{mn}

We have chosen the γ_m (and by equation 40 therefore, the λ_m) to be linear functions of u_0, u_1, \dots , as is seen by putting equation 66 in equation 67, putting that result and equation 66 in equation 68, and so on. We now utilize the linear form of the γ_m to introduce valuable constants J_{mn} into our analysis by writing equations 66, 67, 68, 69, ..., as

$$\gamma_0 = u_0 \quad (84)$$

$$\gamma_1 = u_1 - J_{10} u_0 \quad (85)$$

$$\gamma_2 = u_2 - J_{20} u_0 - J_{21} u_1 \quad (86)$$

$$\gamma_3 = u_3 - J_{30} u_0 - J_{31} u_1 - J_{32} u_2 \quad (87)$$

$$\gamma_4 = u_4 - J_{40} u_0 - J_{41} u_1 - J_{42} u_2 - J_{43} u_3 \quad (88)$$

Equations 84-88, ..., are a set of equations that may be said to define the J_{mn} .

From equations 66 and 76 we may write

$$\gamma_0 = u_0 \quad (89)$$

$$\gamma_1 = u_1 - c_{10} \gamma_0 \quad (90)$$

$$\gamma_2 = u_2 - c_{20} \gamma_0 - c_{21} \gamma_1 \quad (91)$$

$$\gamma_3 = u_3 - c_{30}\gamma_0 - c_{31}\gamma_1 - c_{32}\gamma_2 \quad (92)$$

$$\gamma_4 = u_4 - c_{40}\gamma_0 - c_{41}\gamma_1 - c_{42}\gamma_2 - c_{43}\gamma_3 \quad (93)$$

. . .

So that if we place the right sides of equations 84-88, ..., in the right sides of equations 89-93, ..., and, then, in the right side of each resulting equation, collect the coefficients of u_0, u_1, \dots , we may write the then resulting equations as

$$\gamma_0 = u_0 \quad (94)$$

$$\gamma_1 = u_1 - c_{10}u_0 \quad (95)$$

$$\gamma_2 = u_2 - (c_{20} - c_{21}J_{10})u_0 - c_{21}u_1 \quad (96)$$

$$\gamma_3 = u_3 - \dots, \quad \gamma_4 = u_4 - \dots, \quad (97), (98)$$

and so on, where the right sides of equations 97 and 98 are not completed to save space.

We now equate the coefficient of u_0 in equation 25 to the coefficient of u_0 in equation 95 and find

$$J_{10} = c_{10} \quad (99)$$

We next equate the coefficient of u_0 and u_1 in equation 86 to the coefficients of u_0 and u_1 in equation 96 and find

$$J_{20} = c_{20} - c_{21}J_{10} \quad (100a)$$

$$J_{21} = c_{21} \quad (100b)$$

Continuing, we find from equations 87 and 97 expanded, the results

$$J_{30} = c_{30} - c_{31}J_{10} - c_{32}J_{20} \quad (101a)$$

$$J_{31} = c_{31} - c_{32}J_{21} \quad (101b)$$

$$J_{32} = c_{32} \quad (101c)$$

and from equations 88 and 98 expanded, the results

$$J_{40} = c_{40} - c_{41}J_{10} - c_{42}J_{20} - c_{43}J_{30} \quad (102a)$$

$$J_{41} = c_{41} - c_{42}J_{21} - c_{43}J_{31} \quad (102b)$$

$$J_{42} = c_{42} - c_{43}J_{32} \quad (102c)$$

$$J_{43} = c_{43} \quad (102d)$$

We continue

$$J_{50} = c_{50} - c_{51}J_{10} - c_{52}J_{20} - c_{53}J_{30} - c_{54}J_{40} \quad (103a)$$

$$J_{51} = c_{51} - c_{52}J_{21} - c_{53}J_{31} - c_{54}J_{41} \quad (103b)$$

$$J_{52} = c_{52} - c_{53}J_{32} - c_{54}J_{42} \quad (103c)$$

$$J_{53} = c_{53} - c_{54}J_{43} \quad (103d)$$

$$J_{54} = c_{54} \quad (103e)$$

With all the J_{mn} in view, we see that three recursion formulas are needed. The first, for J_{20}, J_{30}, \dots , is

$$J_{m+1,0} = c_{m+1,0} - \sum_{r=1}^m c_{m+1,r} J_{r,0} \quad (104)$$

$$(m = 1, 2, \dots; r = 1, 2, \dots, m)$$

The second, for $J_{31}, J_{41}, J_{42}; J_{51}, J_{52}, J_{53}; J_{61}, J_{62}, J_{63}, J_{64};$ etc., is

$$J_{m+1,n+1} = c_{m+1,n+1} - \sum_{r=n+2}^m c_{m+1,r} J_{r,n+1} \quad (105)$$

In equation 105 the commas in the subscripts are not inner product notations.

And the third "recursion" formula, for $J_{21}, J_{32}, J_{43}, J_{54},$ etc., may be written as

$$J_{mn} = c_{mn}, \quad m = 1, 2, \dots; n = m-1 \quad (106)$$

where the term "recursion" is used because we shall determine a value c_{mn} before we determine the value J_{mn} . In our equations 99-106, ..., for the J_{mn} (see also equations 83, 76 and 77 we still need to get the c_{mn} which we now obtain as follows.

Formulas for the c_{mn}

Equation 75 is

$$c_{mn} = (u_m \gamma_n) D_n^{-1}, \quad m = 1, 2, \dots, n = 0, 1, \dots, m-1 \quad (107)$$

where we now have D_n as in equation 83 but we need to know the integral $(u_m \gamma_n)$.

We define b_{mn} by

$$b_{mn} = (u_m \gamma_n), \quad m = 1, 2, \dots, n = 0, 1, \dots, m-1 \quad (108)$$

From equations 107 and 108 we have

$$c_{mn} = b_{mn} / D_n \quad m = 1, 2, \dots; n = 0, 1, \dots, m-1 \quad (109)$$

So we can immediately write down formulas for the c_{mn} if we can find formulas for the b_{mn} .

For reference, we write down the b_{mn} of equation 108 as

$$b_{10} = (u_1 \gamma_0) \quad (110)$$

$$b_{20} = (u_2 \gamma_0) \quad (111)$$

$$b_{21} = (u_2 \gamma_1) \quad (112)$$

$$b_{30} = (u_3 \gamma_0) \quad (113)$$

$$b_{31} = (u_3 \gamma_1) \quad (114)$$

$$b_{32} = (u_3 \gamma_2) \quad (115)$$

We put equation 84 in equation 110 and find

$$b_{10} = (u_1 u_0) \quad (116)$$

We put equation 84 in equation 111 and find

$$b_{20} = (u_2 u_0) \quad (117)$$

We put equation 85 in equation 112 and find, using the comma notation of equation 20, the inner product result

$$b_{21} = (u_2, u_1 - J_{10} u_0)$$

which when expanded gives

$$b_{21} = (u_2 u_1) - J_{10} (u_0 u_2) \quad (118)$$

Continuing this process with equations 113, 114, 115, ..., and assembling results, we find for $m = 1$, the equation

$$b_{10} = (u_1 u_0) \quad (119)$$

and for $m = 2$, the equations

$$b_{20} = (u_2 u_0) \quad (120)$$

$$b_{21} = (u_2 u_1) - J_{10} (u_0 u_2) \quad (121)$$

and for $m = 3$, the equations

$$b_{30} = (u_3 u_0) \quad (122)$$

$$b_{31} = (u_3 u_1) - J_{10} (u_3 u_0) \quad (123)$$

$$b_{32} = (u_3 u_2) - J_{20} (u_3 u_0) - J_{21} (u_3 u_1) \quad (124)$$

...

To get recursion formulas for b_{mn} we write out a few more b_{mn} (on scratch paper) up to b_{65} . From the latter equations we single out the

formula for b_{54} , b_{64} and b_{65} (where in b_{54} and b_{64} m increases by 1 with n constant and in b_{64} and b_{65} m stays constant and n increases by 1) which are

$$b_{54} = (u_5 u_4) - J_{40}(u_5 u_0) - J_{41}(u_5 u_1) - J_{42}(u_5 u_2) - J_{43}(u_5 u_3) \quad (125)$$

$$b_{64} = (u_6 u_4) - J_{40}(u_6 u_0) - J_{41}(u_6 u_1) - J_{42}(u_6 u_2) - J_{43}(u_6 u_3) \quad (126)$$

$$b_{65} = (u_6 u_5) - J_{50}(u_6 u_0) - J_{51}(u_6 u_1) - J_{52}(u_6 u_2) - J_{53}(u_6 u_3) - J_{54}(u_6 u_4) \quad (127)$$

From equations 125 and 126 we deduce

$$b_{m+1,n} = (u_{m+1} u_n) - \sum_{r=0}^{n-1} J_{nr}(u_{m+1} u_r) \quad (128)$$

$$(m = 1, 2, \dots; n = 1, 2, \dots, (m-1), n \neq 0)$$

In equation 128 we put $n = n+1$ and find, in agreement with equation 127, the formula

$$b_{m+1,n+1} = (u_{m+1} u_{n+1}) - \sum_{r=0}^n J_{n+1,r}(u_{m+1} u_r) \quad (129)$$

$$m = 1, 2, \dots; n = 0, 1, \dots; n < m,$$

To check equation 129 we put $m = 2$ and $n = 1$ in equation 129 and find

$$b_{32} = (u_3 u_2) - \sum_{r=0}^1 J_{2r}(u_3 u_r)$$

that is, we find

$$b_{32} = (u_3 u_2) - J_{20}(u_3 u_0) - J_{21}(u_3 u_1)$$

which, as it should be, is equation 124.

By inspection of equations 109 and 129 we now write down

$$c_{m+1, n+1} = \frac{(u_{m+1} u_{n+1}) - \sum_{r=0}^n J_{n+1, r} (u_{m+1} u_r)}{D_{n+1}} \quad (130)$$

$$m = 1, 2, \dots; n = 0, 1, \dots; n < m,$$

With the use of equation 97, or with use of equations 109 and 119, we will write down a few c_{mn} . In the following we shall often write u_{mn} instead of $(u_m u_n)$. It is to be understood that they mean the same. we thus write

$$u_{mn} = (u_m u_n) = \int_{\alpha}^{\beta} u_m u_n dx \quad (131)$$

$$c_{10} = u_{10} D_0^{-1}$$

$$c_{20} = u_{20} D_0^{-1}$$

$$c_{21} = [u_{21} - J_{10} u_{20}] D_1^{-1}$$

$$c_{30} = u_{30} D_0^{-1}$$

$$c_{31} = [u_{31} - J_{10} u_{30}] D_1^{-1}$$

$$c_{32} = [u_{32} - J_{20} u_{30} - J_{21} u_{31}] D_2^{-1}$$

$$c_{40} = u_{40} D_0^{-1}$$

$$c_{41} = [u_{41} - J_{10} u_{40}] D_1^{-1}$$

$$c_{42} = [u_{42} - J_{20} u_{40} - J_{21} u_{41}] D_2^{-1}$$

$$c_{43} = [u_{43} - J_{30} u_{40} - J_{31} u_{41} - J_{32} u_{42}] D_3^{-1}$$

$$c_{50} = u_{50} D_0^{-1}$$

$$c_{51} = [u_{51} - J_{10} u_{50}] D_1^{-1}$$

$$c_{52} = [u_{52} - J_{20} u_{50} - J_{21} u_{51}] D_2^{-1}$$

$$c_{53} = [u_{53} - J_{30} u_{50} - J_{31} u_{51} - J_{32} u_{52}] D_3^{-1}$$

$$c_{54} = [u_{54} - J_{40} u_{50} - J_{41} u_{51} - J_{42} u_{52} - J_{43} u_{53}] D_4^{-1}$$

etc.

Equation 130 is our sought recursion formula for the c_{mn} , and we shall see in the next paragraph that by properly tabulating J_{mn} , D_m , etc., of equation 130, we can compute the c_{mn} and hence γ_m, \dots , needed for the λ_m (needed in turn in equation 19 to get the B_m). The values of these c_{mn} , etc., will all depend on the initial choice of $u_m(x)$ and on the integrals $(u_m u_n)$ derived from the $u_m(x)$.

Table for Generating Orthonormal Functions λ_m , etc.

To generate and numerically compute the γ_m , D_m , c_{mn} , J_{mn} , and λ_m , the definite sequence of equations that we present in Table 1, which stems from Powers et al. (1967), are useful. The value of the constant γ_0 in the table is first obtained from equation 1 in the table; then D_0 , from equation 2; and λ_0 , the first orthonormal polynomial, from equation 3; and so on. Each quantity in the table, can, in turn, be computed from earlier given quantities, all stemming from $\gamma_0 = u_0$ of equation 1, and, from integrals $u_{mn} = (u_m u_n)$ of equation II in the table.

Needed u_{mn} can all be computed at the outset (and stored in the

Table 1. Sequence of formulas for computing orthonormal functions $\lambda_m(x)$ of equations 16, 17 and 18 of text from another set of functions $u_m(x)$, by use of auxiliary quantities formed from $u_m(x)$.

Definitions:

- I. $u_m(x)$ = a complete (see text) set of functions of x ; $\alpha < x < \beta$.
 $m = 0, 1, \dots, N$; $N \rightarrow \infty$; α and β , arbitrary constants.
- II. $u_{mn} = (u_m, u_n) = \int_{\alpha}^{\beta} u_m u_n dx$, $m, n = 0, 1, \dots, N$; $N \rightarrow \infty$

Sequential formulas:

1. $\gamma_0 = u_0$
 2. $D_0 = u_{00}$
 3. $\lambda_0 = \gamma_0 D_0^{-1/2}$
 4. $c_{10} = u_{10} D_0^{-1}$
 5. $J_{10} = c_{10}$

6. $\gamma_1 = u_1 - c_{10} \gamma_0$
 7. $D_1 = u_{11} - c_{10}^2 D_0$
 8. $\lambda_1 = \gamma_1 D_1^{-1/2}$
 8a. $c_{20} = u_{20} D_0^{-1}$
 8b. $c_{21} = [u_{21} - J_{10} u_{20}] D_1^{-1}$
 9a. $J_{20} = c_{20} - c_{21} J_{10}$

- 9b. $J_{21} = c_{21}$
 10. $\gamma_2 = u_2 - c_{20} \gamma_0 - c_{21} \gamma_1$
 11. $D_2 = u_{22} - c_{20}^2 D_0 - c_{21}^2 D_1$
 12. $\lambda_2 = \gamma_2 D_2^{-1/2}$
 13a. $c_{30} = u_{30} D_0^{-1}$
 13b. $c_{31} = [u_{31} - J_{10} u_{30}] D_1^{-1}$
 13c. $c_{32} = [u_{32} - J_{20} u_{30}] - J_{21} u_{31} D_2^{-1}$
 14a. $J_{30} = c_{30} - c_{31} J_{10} - c_{32} J_{20}$
 14b. $J_{31} = c_{31} - c_{32} J_{21}$
 14c. $J_{32} = c_{32}$
 15. $\gamma_3 = u_3 - c_{30} \gamma_0 - c_{31} \gamma_1 - c_{32} \gamma_2$
 16. $D_3 = u_{33} - c_{30}^2 D_0 - c_{31}^2 D_1 - c_{32}^2 D_2$
 17. $\lambda_3 = \gamma_3 D_3^{-1/2}$
 etc. *

* For additional γ_m see equations 76, 89-93, ... in the text; D_m , 78, 83; λ_m , 40; c_{mn} , 125; and J_{mn} , 99-106.

computer), or they can be computed, as need for them arises in the table. The symbol u_{mn} is defined by equation II in the table and is not to be confused with u_n . The equations from which we wrote down the tabulated equations 1, 2, 3, ..., are γ_m , 76, 89-93; D_m , 78, 83; λ_m , 40; c_{mn} , 130; and J_{mn} , 99-106.

Determination of the B_m of Equation 19; Introduction of Constants

$$E_m, G_m \text{ and } w_m$$

To get the B_m of equation 19, we shall consider equations 18 and 19 together, even though it appears that only equation 19 is needed.

We first consider equation 18 and write it as (putting a subscript N on f to indicate the number of terms in the right side) the expression

$$f_N(x) = B_0 \lambda_0 + B_1 \lambda_1 + B_2 \lambda_2 + \dots + B_N \lambda_N. \quad N \rightarrow \infty \quad (132)$$

In view of equation 40, we may write equation 132 as

$$f_N(x) = B_0 \gamma_0 D_0^{-1/2} + B_1 \gamma_1 D_1^{-1/2} + B_2 \gamma_2 D_2^{-1/2} + \dots + B_N \gamma_N D_N^{-1/2}, \quad N \rightarrow \infty \quad (133)$$

or if we define dimensionless constants E_m by

$$E_m = B_m D_m^{-1/2} \quad (134)$$

$$\text{or } B_m = E_m D_m^{1/2}$$

we may write equation 133 as

$$f_N(x) = E_0 \gamma_0 + E_1 \gamma_1 + E_2 \gamma_2 + \dots + E_N \gamma_N, \quad N \rightarrow \infty, \quad (135)$$

which in view of equations 84-88, ..., becomes

$$f_N(x) = E_0 u_0 + E_1 (u_1 - J_{10} u_0) + E_2 (u_2 - J_{20} u_0 - J_{21} u_1) + \dots, \quad N \rightarrow \infty \quad (136)$$

to which we shall return.

We now consider equation 19. We look at equation 40 and write equation 19 as

$$B_m = \int_{\alpha}^{\beta} f(x) \gamma_m D_m^{-1/2} dx, \quad m = 0, 1, \dots \quad (137)$$

or since from equation 134 we have

$$B_m = E_m D_m^{1/2}, \quad m = 0, 1, \dots \quad (138)$$

We may write equation 137 as

$$E_m D_m^{1/2} = \int_{\alpha}^{\beta} f(x) \gamma_m D_m^{-1/2} dx \quad (139)$$

from which we immediately find

$$E_m D_m = \int_{\alpha}^{\beta} f(x) \gamma_m dx \quad (140)$$

or if we define dimensionless constants G_m by

$$G_m = E_m D_m, \quad m = 0, 1, \dots, \quad (141)$$

or

$$\frac{F}{-m} = G_m D_m^{-1}$$

we may write equation 140 as

$$G_m = \int_{\alpha}^{\beta} f(x) \gamma_m dx, \quad m = 0, 1, \dots \quad (142)$$

We shall now develop a recursion formula for the G_m .

We put $m = 0$ in equation 142 and in the result use γ_0 of equation 89 to find

$$G_0 = \int_{\alpha}^{\beta} f(x) u_0 dx \quad (143)$$

We put $m = 1$ in equation 142 and in the result use γ_1 of equation 90 to find

$$G_1 = \int_{\alpha}^{\beta} f(x) [u_1 - c_{10}\gamma_0] dx \quad (144)$$

In equation 144 we have used brackets to avoid confusion with the Courant-Hilbert modified parentheses notation of equation 20.

We expand equation 144 as

$$G_1 = \int_{\alpha}^{\beta} f(x) u_1 dx - \int_{\alpha}^{\beta} f(x) c_{10} \gamma_0 dx \quad (145)$$

which, in view of equation 142 with $m = 0$, may be written as

$$G_1 = \int_{\alpha}^{\beta} f(x) u_1 dx - c_{10} G_0 \quad (146)$$

We put $m = 2$ in equation 142 and use equation 91 in the result to find

$$G_2 = \int_{\alpha}^{\beta} f(x) [u_2 - c_{20}\gamma_0 - c_{21}\gamma_1] dx \quad (147)$$

which after expansion and use of equation 142, first with $m = 0$ and then with $m = 1$, gives

$$G_2 = \int_{\alpha}^{\beta} f(x) u_2 dx - c_{20} G_0 - c_{21} G_1 \quad (148)$$

As we found G_0 and G_1 and G_2 , we now find G_3 as

$$G_3 = \int_{\alpha}^{\beta} f(x) u_3 dx - c_{30} G_0 - c_{31} G_1 - c_{32} G_2 \quad (149)$$

and so on for G_4, G_5, \dots .

We define constants w_m by

$$w_m = \int_{\alpha}^{\beta} f(x) u_m dx, \quad m = 0, 1, \dots \quad (150)$$

and use equation 150 in equations 143, 146, 148, 149, etc. to find

$$G_0 = w_0 \quad (151)$$

$$G_1 = w_1 - c_{10}G_0 \quad (152)$$

$$G_2 = w_2 - c_{20}G_0 - c_{21}G_1 \quad (153)$$

$$G_3 = w_3 - c_{30}G_0 - c_{31}G_1 - c_{32}G_2 \quad (154)$$

$$G_4 = w_4 - c_{40}G_0 - c_{41}G_1 - c_{42}G_2 - c_{43}G_3 \quad (155)$$

$$G_5 = w_5 - c_{50}G_0 - c_{51}G_1 - c_{52}G_2 - c_{53}G_3 - c_{54}G_4 \quad (156)$$

Equations 151-156 may be, by inspection, compressed to two formulas

$$G_0 = w_0, \quad m = 0 \quad (157)$$

$$G_m = w_m - \sum_{n=0}^{m-1} c_{mn}G_n, \quad m = 1, 2, \dots \quad (158)$$

$$n = 0, 1, \dots, m - 1$$

Equation 158 is a recursion formula for G_m (since in the right we have G_{m-1}, G_{m-2}, \dots). From equation 158 we can now, in view of equation 141, find a recursion formula for the E_m ($\approx G_m/D_m$) as

$$E_m = \frac{w_m - \sum_{n=0}^{m-1} c_{mn}G_n}{D_m} \quad m = 1, 2, \dots \quad (159)$$

$$n = 0, 1, \dots, m - 1$$

or
$$E_m = G_m D_m^{-1}$$

and from equations 141 and 147, find E_0 as

$$E_0 = w_0/D_0 \quad (160)$$

where by equation 150 w_0 is

$$w_0 = \int_{\alpha}^{\beta} f(x) u_0 dx \quad (161)$$

and u_0 is the first of our almost arbitrary set $u_0(x)$, $u_1(x)$, $u_2(x)$,

Recursion Formulas for the B_m

We may now get the B_m . We put the right side of equation 159 for E_m in the right of equation 138 and simplify the result to find, for $m \neq 0$, the equation

$$B_m = [w_m - \sum_{n=0}^{m-1} c_{mn} G_n] D_m^{-1/2}, \quad m = 1, 2, \dots \quad (162)$$

$$n = 0, 1, \dots$$

And for B_0 , we put $m = 0$ in equation 138; and put in the result the right side of equation 160; we simplify that result and find

$$B_0 = w_0 D_0^{-1/2} \quad (163)$$

We can get a recursion formula for B_m by expressing G_n of equation 162 in terms of B_n .

We put E_m as given by equation 134 into the right of equation 141 and find

$$G_m = G_m D_m^{-1/2} D_m^{1/2} = B_m D_m^{1/2} \quad (164)$$

We put $B_n D_n^{1/2}$ for G_n in equation 162 to find

$$B_m = (w_m - \sum_{n=0}^{m-1} c_{mn} D_n^{1/2} B_n) D_m^{-1/2} \quad (165)$$

Equation 165 is the recursion formula for the B_m when we have $m = 1, 2, \dots$.

For $m = 0$ we use equation 163.

Formulas for the A_{Nm}

Equation 15 may be expanded for N successive approximations $f_N(x)$ of $f(x)$ as

$$f_0(x) = A_{00}u_0, \quad (166)$$

$$f_1(x) = A_{10}u_0 + A_{11}u_1, \quad (167)$$

$$f_2(x) = A_{20}u_0 + A_{21}u_1 + A_{22}u_2, \quad (168)$$

$$f_3(x) = A_{30}u_0 + A_{31}u_1 + A_{32}u_2 + A_{33}u_3, \quad (169)$$

. . .

And similarly equation 136 may be expanded for N successively better approximations $f_N(x)$ of $f(x)$ as

$$f_0(x) = E_0u_0$$

$$f_1(x) = E_0u_0 + E_1[u_1 - J_{10}u_0]$$

$$f_2(x) = E_0u_0 + E_1[u_1 - J_{10}u_0] + E_2[u_2 - J_{20}u_0 - J_{21}u_1]$$

$$f_3(x) = E_0u_0 + E_1[u_1 - J_{10}u_0] + E_2[u_2 - J_{20}u_0 - J_{21}u_1]$$

$$+ E_3[u_3 - J_{30}u_0 - J_{31}u_1 - J_{32}u_2]$$

. . .

which, when coefficients of u_0, u_1, \dots , are factored out, may be written as

$$f_0(x) = E_0u_0 \quad (170)$$

$$f_1(x) = (E_0 - J_{10})u_0 + E_1u_1 \quad (171)$$

$$f_2(x) = (E_0 - E_1J_{10} - E_2J_{20})u_0 + (E_1 - E_2J_{21})u_1 + E_2u_2 \quad (172)$$

$$f_3(x) = (E_0 - E_1J_{10} - E_2J_{20} - E_3J_{30})u_0 + (E_1 - E_2J_{21} - E_3J_{31})u_1 \\ + (E_2 - E_3J_{32})u_2 + E_3u_3 \quad (173)$$

. . .

To get the A_{Nm} we now equate coefficients of equations 166 and 170; equations 167 and 171; equations 168 and 172; etc., and find

$$N = 0: A_{00} = E_0 \quad (174)$$

$$N = 1: A_{10} = E_0 - E_1 J_{10} \quad (175)$$

$$A_{11} = E_1 \quad (176)$$

$$N = 2: A_{20} = E_0 - E_1 J_{10} - E_2 J_{20} \quad (177)$$

$$A_{21} = E_1 - E_2 J_{21} \quad (178)$$

$$A_{22} = E_2 \quad (179)$$

$$N = 3: A_{30} = E_0 - E_1 J_{10} - E_2 J_{20} - E_3 J_{30} \quad (180)$$

$$A_{31} = E_1 - E_2 J_{21} - E_3 J_{31} \quad (181)$$

$$A_{32} = E_2 - E_3 J_{32} \quad (182)$$

$$A_{33} = E_3 \quad (183)$$

and so on.

By inspection of equations 175-183 we may write

$$A_{Nm} = E_m - \sum_{p=m+1}^N E_p J_{pm}, \quad m = 0, 1, \dots, N, \quad (184)$$

(N + 1 equations)

We now look back at equations 177 and 175. We see that the first two terms on the right hand side of equation 177 form the complete right hand side of equation 175. We thus can write equation 177

$$N = 2: A_{20} = A_{10} - E_2 J_{20}$$

Similarly we can write equation 178, if we substitute equation 176 into equation 178,

$$N = 2: A_{21} = A_{11} - E_2 J_{21}$$

If we carry out similar substitutions we find the following

$$N = 0: A_{00} = E_0$$

$$N = 1: A_{10} = A_{00} - E_1 J_{10}$$

$$A_{11} = E_1$$

$$N = 2: A_{20} = A_{10} - E_2 J_{20}$$

$$A_{21} = A_{11} - E_2 J_{21}$$

$$A_{22} = E_2$$

$$N = 3: A_{30} = A_{20} - E_3 J_{30}$$

$$A_{31} = A_{21} - E_3 J_{31}$$

$$A_{32} = A_{22} - E_3 J_{32}$$

$$A_{33} = E_3$$

$$N = 4: A_{40} = A_{30} - E_4 J_{40}$$

$$A_{41} = A_{31} - E_4 J_{41}$$

$$A_{42} = A_{32} - E_4 J_{42}$$

$$A_{43} = A_{33} - E_4 J_{43}$$

$$A_{44} = E_4$$

$$N = 5: A_{50} = A_{40} - E_5 J_{50}$$

$$A_{51} = A_{41} - E_5 J_{51}$$

$$A_{52} = A_{42} - E_5 J_{52}$$

$$A_{53} = A_{43} - E_5 J_{53}$$

$$A_{54} = A_{44} - E_5 J_{54}$$

$$A_{55} = E_5$$

In general we can write

$$A_{Nm} = A_{N-1,m} - E_N J_{Nm}, \quad \begin{matrix} m = 0, 1, 2, \dots, N \\ m \neq N \end{matrix} \quad (185)$$

and $A_{Nm} = E_m, \quad m = N \text{ (i.e., } A_{NN} = E_N)$

Tables for the A_{Nm} and the B_m

We now prepare a new table, Table 2, which also stems from Powers et al. (1967), to compute B_m , A_{Nm} , and auxiliary quantities for getting B_m and A_{Nm} .

Table 2. Sequence of formulas for computing a set of orthonormal functions $\lambda_m(x)$, $m = 0, 1, \dots$, $\alpha < x < \beta$, from almost any other set of functions $u_m(x)$ and for computing constants, B_0, B_1, \dots , to develop an almost arbitrary function $f(x)$ into the series $f(x) = B_0\lambda_0 + B_1\lambda_1 + \dots$ and for computing constants A_{00}, A_{10}, \dots , to develop the function $f(x)$ into successive approximations $f_N(x)$ given by the formulas $f_0(x) = A_{00}u_0$, $f_1(x) = A_{10}u_0 + A_{11}u_1$, $f_2(x) = A_{20}u_0 + A_{21}u_1 + A_{22}u_2$; etc. -- all by use of auxiliary quantities given in the body of the table and all stemming from the u_m ; tables gives A_{Nm} , B_n , and λ_n of equations 15-19 of text.

Definitions:

- I. $u_m(x)$ = a complete (see Kirkham and Powers, 1972, appen. 2, eq. k) set of function of x ; $\alpha < x < \beta$. $m = 0, 1, \dots, N$; α and β , arbitrary.
- II. $u_{mn} = (u_m u_n) = \int_{\alpha}^{\beta} u_m u_n dx$, $m, n = 0, 1, \dots, N$; $N \rightarrow \infty$.
- III. $f(x)$ = function of x , $\alpha < x < \beta$.
- IV. $w_m = \int_{\alpha}^{\beta} f(x) u_m(x) dx$; $m = 0, 1, \dots, N$; $N \rightarrow \infty$.

Table 2. (Continued)

Formulas: *	13. $\{\lambda_1 = \gamma_1 D_1^{-1/2}\}$
1. $\{\gamma_0 = u_0\}$	14a. $c_{20} = u_{20} D_0^{-1}$
2. $D_0 = u_{00}$	14b. $c_{21} = (u_{21} - J_{10} u_{20}) D_1^{-1}$
3. $\{\lambda_0 = \gamma_0 D_0^{-1/2}\}$	15a. $J_{20} = c_{20} - c_{21} J_{10}$
4. $c_{10} = u_{10} D_0^{-1}$	15b. $J_{21} = c_{21}$
5. $J_{10} = c_{10}$	16. $G_1 = w_1 - c_{10} G_0$
6. $G_0 = w_0$	17. $E_1 = G_1 D_1^{-1}$
7. $E_0 = G_0 D_0^{-1}$	18. $\{D_1^{1/2} = (\text{rhs of 12})^{1/2}\}$
8. $\{D_0^{1/2} = (\text{rhs of 2})^{1/2}\}$	19. $\{B_1 = E_1 D_1^{1/2}\}$
9. $\{B_0 = E_0 D_0^{1/2}\}$	20a. $A_{10} = E_0 - E_1 J_{10}$
10. $A_{00} = E_0$	20b. $A_{11} = E_1$
11. $\{\gamma_1 = u_1 - c_{10} \gamma_0\}$	etc. **
12. $D_1 = u_{11} - c_{10}^2 D_0$	

* Formulas in braces are not needed to get the A_{Nm} .

** For additional γ_m see equations 71 and 89-93 in the text; for addition D_m , 83; λ_m , 40; c_{mn} , 109, 119-124 or 130; J_{mn} , 99-106; G_m , 151-158; E_m , 141; B_m , 138; and A_{Nm} , 174-185.

We now go back to our boundary value problem of Figure 1 considered with the well in the major axes, and use Table 2 to complete the solution.

Return to the Boundary Value Problem

We rewrite the expression for the potential function ϕ , given by equation 12:

$$\phi = A_{NO} \frac{\ln(r/r_w)}{\ln(a/r_w)} + \sum_{m=1}^N A_{Nm} \left(\frac{(r/a)^m - [r_w^2/(ar)]^m}{1 - (r_w^2/a^2)^m} \right) \quad (12)$$

In going from equation 12 to equation 14 we showed that on the outer boundary of the aquifer equation 12 reduced to

$$1 = \sum_{m=0}^N A_{Nm} u_m(\theta) \quad (14)$$

It was noted that equation 14 was a special case of equation 15:

$$f(x) = \sum_{m=0}^N A_{Nm} u_m(x) \quad (15)$$

In equation 14 we have $u_m(\theta)$ instead of $u_m(x)$ and

$$w_m = \int_{\alpha}^{\beta} f(x) u_m(x) dx, \quad m = 0, 1, 2, \dots, N$$

$N \rightarrow \infty$

of equation 150 or of equation IV of the definitions in Table 2.

$$w_m = \int_0^{\pi} (1) u_m(\theta) d\theta$$

Van der Ploeg (1970) or Van der Ploeg et al. (1971) performed the integrations required to get the u_{mn} and the w_m numerically, by use of a digital computer.

If we look at equation 14 we see that to satisfy the remaining boundary condition exactly, the superscript N on the summation sign has to go to infinity. To keep computer costs low one can decide how large N should be by making checks on the boundary $\phi = 1$, each time a set of A_{Nm} coefficients has been determined. With the center of the well

being the origin of the polar coordinate system, Table 3 shows how with an increase in N the function $\phi = 1$ is approximated better and better along the outer boundary and we have $f(\theta) = 1$, instead of $f(x)$. By discussing the Gram-Schmidt method as modified by Kirkham and Powers (1972) we thus have shown how to determine the coefficients A_{Nm} using their (and our) Table 2. If we look at Table 2 we have to realize that there the $u_m(x)$ are our $u_m(\theta)$ where our $u_m(\theta)$ are given by

$$u_0 = \frac{\ln(R/r_w)}{\ln(a/r_w)}, \text{ and}$$

$$u_m = \left\{ \frac{(R/a)^m - [r_w^2/(aR)]^m}{1 - (r_w^2/a^2)^m} \right\} \cos m\theta$$

$$\text{The } u_{mn} [(u_m u_n)] = \int_{\alpha}^{\beta} u_m u_n dx, \quad m, n=0, 1, \dots, N,$$

$N \rightarrow \infty$

becomes, in our case of Figure 2 considered with the well on the major axes, the expression

$$u_{mn} = \int_0^{\pi} u_m(\theta) u_n(\theta) d\theta \quad m, n=0, 1, \dots, N,$$

$N \rightarrow \infty$

In Table 2, the $f(x)$ (= function of x , $\alpha < x < \beta$) becomes our

$$f(\theta) = 1, \quad 0 < \theta < \pi,$$

Table 3 shows another check in addition to the boundary check, that ϕ be equal to 1, nearly, for large N at all boundary points. It is known, (see Kirkham and Powers, 1972, Appendix 2), that the following inequality holds when $f(x)$, $\alpha < x < \beta$, has been properly developed:

$$\sum_{m=0}^N A_{mm}^2 D_m \leq \int_{\alpha}^{\beta} [f(x)]^2 dx, \quad (186)$$

Table 3. Approximations of $\phi = 1$, along outer boundary of ellipse of Figure 4 at $\pi/8$ intervals.

Bottom line shows values of Bessel's inequality.

θ	N	0	1	2	3	4	5	6	7	8	9
0		0.986830	1.05350	1.02789	1.00371	0.99611	0.99704	0.99927	1.00025	1.00022	1.00017
$\pi/8$		0.95662	1.03341	1.01468	1.00297	1.00159	1.00146	1.00044	0.99973	0.99976	0.99981
$\pi/4$		0.93829	0.99838	0.99184	0.99778	1.00259	1.00187	1.00051	1.00032	1.00030	1.00024
$3\pi/8$		0.93130	0.97373	0.97803	0.99357	0.99863	0.99810	0.99962	0.99980	0.99982	0.99987
$\pi/2$		0.94202	0.96570	0.97909	0.99620	0.99763	0.99810	0.99962	0.99980	0.99982	0.99987
$5\pi/8$		0.97400	0.97446	0.99530	1.00488	1.00071	1.00138	1.00111	1.00032	1.00034	1.00025
$3\pi/4$		1.03164	0.99672	1.02004	1.00815	1.00158	1.00111	0.99919	0.99970	0.99967	0.99977
$7\pi/8$		1.11420	1.01591	1.01901	0.98939	0.99877	0.99884	1.00139	1.00009	1.00008	0.99997
π		1.16908	1.01730	0.97003	1.01247	0.99927	0.99765	1.00150	0.99976	0.99970	0.99996
$\sum_{m=0}^N A_{mm}^2 D_m$		3.12460	3.13943	3.14054	3.14149	3.14158	3.14158	3.14159	3.14159	3.14159	3.14159
$\pi = 3.14159$											

This expression is called Bessel's inequality. In our well problem we have $f(\theta)$, instead of $f(x)$, where we have $f(\theta) = 1$ on the outer boundary of the ellipse. Equation 186 for our Figure 4 becomes

$$\sum_{m=0}^N A_{mm}^2 D_m \leq \int_0^\pi (1)^2 d\theta = \pi \quad (187)$$

The summation of $A_{mm}^2 D_m$ terms is given at the bottom of Table 3. After $N = 6$ this summation (3.14159) does not change in the fifth decimal place. By equation 187 the value that should be approached is $\pi = 3.14159$. We conclude that we can be reasonably sure that we have obtained a representation for $f(\theta)$ for each choice of R for r in the series given by equation 12.

Table 4 shows how the series of A_{Nm} coefficients looks when N goes from 0 to 9. It is with these A_{Nm} coefficients that Table 3 was calculated.

We have discussed the Gram-Schmidt method in detail and we have shown with an example how it applies to an elliptical aquifer. We will now discuss the work that was done as an extension of the work done by Van der Ploeg (1970) as reported in Van der Ploeg et al. (1971). We may note that by studying groundwater flow patterns, as in Van der Ploeg et al. (1971) one may draw certain conclusions regarding groundwater pollution. In studying groundwater pollution, one needs to know the flow pattern of the groundwater that carries the pollutants. If one knows the flow pattern in a pumped aquifer, one might be able to predict the effects of a polluting source, somewhere present in the flow region, on the quality of the pumped water. The following eight widely accepted categories of water pollutants have been listed by the United States

Table 4. A_{Nm} coefficients for $N = 0, 1, 2, \dots, 9$ for a flow configuration as in Figure 4, where $a = 1$,
 $b = 0.5$, $c = 0.5$, and $r_w = 0.0025$.

N	A_{N0}	A_{N1}	A_{N2}	A_{N3}	A_{N4}	A_{N5}	A_{N6}	A_{N7}	A_{N8}	A_{N9}
0	1.09498									
1	1.12251	0.12171								
2	1.12775	0.08493	-0.04740							
3	1.13108	0.08409	-0.12336	-0.06179						
4	1.13135	0.08849	-0.13464	-0.10648	-0.02614					
5	1.13135	0.08810	-0.13538	-0.10176	-0.01662	0.00432				
6	1.13135	0.08783	-0.13881	-0.10094	0.10829	0.04133	0.01370			
7	1.13135	0.08795	-0.13969	-0.10442	0.01605	0.06718	0.03993	0.00821		
8	1.13135	0.08794	-0.13970	-0.10430	0.01626	0.06656	0.03816	0.00675	-0.00040	
9	1.13135	0.08794	-0.13975	-0.10436	0.01714	0.06800	0.03542	-0.00105	-0.00649	-0.00158

Senate Select Committee on Water Resources (U.S. Senate Select Committee, 1960):

- | | |
|-----------------------|-----------------------------|
| 1) Salts and minerals | 5) Infectious agents |
| 2) Sediment | 6) Organic chemical exotics |
| 3) Plant nutrients | 7) Radioactivity |
| 4) Organic wastes | 8) Heat |

These eight pollutants can in general be transported by groundwater (Goldberg, 1970; LeGrand, 1970), and it is therefore important to know the flow pattern in an aquifer, particularly if the aquifer is being pumped by a well. Sometimes wells are used to dispose of pollutants (Hoopes and Harleman, 1967a, 1967b). Here again it is important to know the groundwater flow pattern. In some following flow net figures we have indicated some general conclusions that can be drawn from the flow nets regarding pollution in an aquifer. These figures were used in a presentation about groundwater pollution. The remarks on the figures and in the text about pollution are not intended to detract from the use of the figures for the rational development of groundwater supplies from pumped wells in confined aquifers.

This completes our Literature Review and Theoretical Background. We now go to Results. These Results are an extension of the work of Van der Ploeg (1970) and Van der Ploeg et al. (1971).

RESULTS

Theory for Ellipse-Shaped Aquifers for Arbitrary Position of Well

We have been referring to Figure 1 with the well on the major axis. We will refer again to Figure 1, but now for the general case, that is with an arbitrary location of the well on the aquifer. In Figure 1, part a represents a vertical section of a horizontal confined aquifer of uniform thickness h being pumped steadily by a well. We let ϕ represent the hydraulic head at any point in the aquifer and let $\Delta\phi$ represent the head difference across the aquifer. In the initial analysis, we take $\Delta\phi$ to be unity as measured between any point on the external boundary of the flow region and a point on the well boundary. In Figure 1, part b represents the horizontal cross-section of the aquifer when it is of the elliptical shape. The theory for a pumped well in an elliptical confined aquifer has been given by Van der Ploeg and others (1971). This theory will be recalled here and developed further to include almost any shape of external boundary. For the figure as we recall the following symbols:

$\phi/\Delta\phi$ = dimensionless potential or head

$\Delta\phi$ = constant head difference between well and external boundary

x, y = rectangular coordinates with origin 0 at center of the ellipse

r, θ = polar coordinates with origin at center of the well

r_w = radius of well

a, b = semimajor and semiminor axes of ellipse

c, d = x and y coordinates of well center

K = hydraulic conductivity

$P(r, \theta)$ = a point in the flow medium

$P(R, \theta)$ = a point on boundary of the ellipse

h = thickness of aquifer

The dimensionless potential $\phi/\Delta\phi$ or ϕ is given by

$$\begin{aligned} \frac{\phi}{\Delta\phi} = & A_{NO} \frac{\ln(r/r_w)}{\ln(a/r_w)} \\ & + \sum_{m=2,4,\dots}^N A_{Nm} \left\{ \frac{(r/a)^{m/2} - [r_w^2/(ar)]^{m/2}}{1 - (r_w^2/a^2)^{m/2}} \right\} \cos \frac{m}{2}\theta \\ & + \sum_{m=1,3,\dots}^N A_{Nm} \left\{ \frac{(r/a)^{(m+1)/2} - [r_w^2/(ar)]^{(m+1)/2}}{1 - (r_w^2/a^2)^{(m+1)/2}} \right\} \sin \frac{m+1}{2}\theta \quad (188) \end{aligned}$$

where the last value of the value of N on a summation sign may be either even or odd and indicates the order of approximation of the solution.

For $N \rightarrow \infty$, the solution becomes exact. The constants A_{Nm} in equation 188 are obtained by the modified Gram-Schmidt method, as described in Van der Ploeg and others (1971).

The stream function ψ corresponding to ϕ is

$$\begin{aligned} \frac{\psi}{\Delta\phi} = & K \left\{ \frac{A_{NO}\theta}{\ln(a/r_w)} + \sum_{M=2,4,\dots}^N A_{Nm} \frac{(r/a)^{m/2} + [r_w^2/(ar)]^{m/2}}{1 - (r_w^2/a^2)^{m/2}} \sin \frac{m}{2}\theta \right. \\ & \left. - \sum_{m=1,3,\dots}^N A_{Nm} \frac{(r/a)^{(m+1)/2} + [r_w^2/(ar)]^{(m+1)/2}}{1 - (r_w^2/a^2)^{(m+1)/2}} \cos \frac{m+1}{2}\theta \right\} \quad (189) \end{aligned}$$

The stream function ψ is such that, if we consider two points $P(r_1, \theta_1)$ and $P(r_2, \theta_2)$ on two different streamlines, then the amount of water per unit time that flows between the two streamlines that pass through these respective

points is $\psi(r_2, \theta_2) - \psi(r_1, \theta_1)$.

When the well is on the major axis (similarly for the minor axis), the potential function, equation 188, reduces, because of symmetry, to the form

$$\frac{\phi}{\Delta\phi} = A_{NO} \frac{\ln(r/r_w)}{\ln(a/r_w)} + \sum_{m=1}^N A_{Nm} \left\{ \frac{(r/a)^m - [r_w^2/(ar)]^m}{1 - (r_w^2/a^2)^m} \right\} \cos m\theta \quad (190)$$

where $m = 1, 2, \dots$. The stream function corresponding to ϕ of equation 190 is given by

$$\frac{\psi}{\Delta\phi} = K \left\{ \frac{A_{NO}\theta}{\ln(a/r_w)} + \sum_{m=1}^N A_{Nm} \left\{ \frac{(r/a)^m + [r_w^2/(ar)]^m}{1 - (r_w^2/a^2)^m} \right\} \sin m\theta \right\} \quad (191)$$

When the well is at the center of the ellipse, there is additional symmetry, and equation 190 becomes (m is changed to $2m$) the expression

$$\frac{\phi}{\Delta\phi} = A_{NO} \frac{\ln(r/r_w)}{\ln(a/r_w)} + \sum_{m=1}^N A_{Nm} \left\{ \frac{(r/a)^{2m} - [r_w^2/(ar)]^{2m}}{1 - (r_w^2/a^2)^{2m}} \right\} \cos 2m\theta \quad (192)$$

and similarly, equation 191 becomes

$$\frac{\psi}{\Delta\phi} = K \left\{ \frac{A_{NO}\theta}{\ln(a/r_w)} + \sum_{m=1}^N A_{Nm} \left\{ \frac{(r/a)^{2m} + [r_w^2/(ar)]^{2m}}{1 - (r_w^2/a^2)^{2m}} \right\} \sin 2m\theta \right\} \quad (193)$$

For all the flow configurations corresponding to equations 188-193 there is a single equation for the well flux. If we denote this flux by Q (and the thickness of the aquifer is h and head difference $\Delta\phi$), then Q is given by

$$Q = \frac{-2\pi Kh A_{NO} \Delta\phi}{\ln(a/r_w)} \quad (194)$$

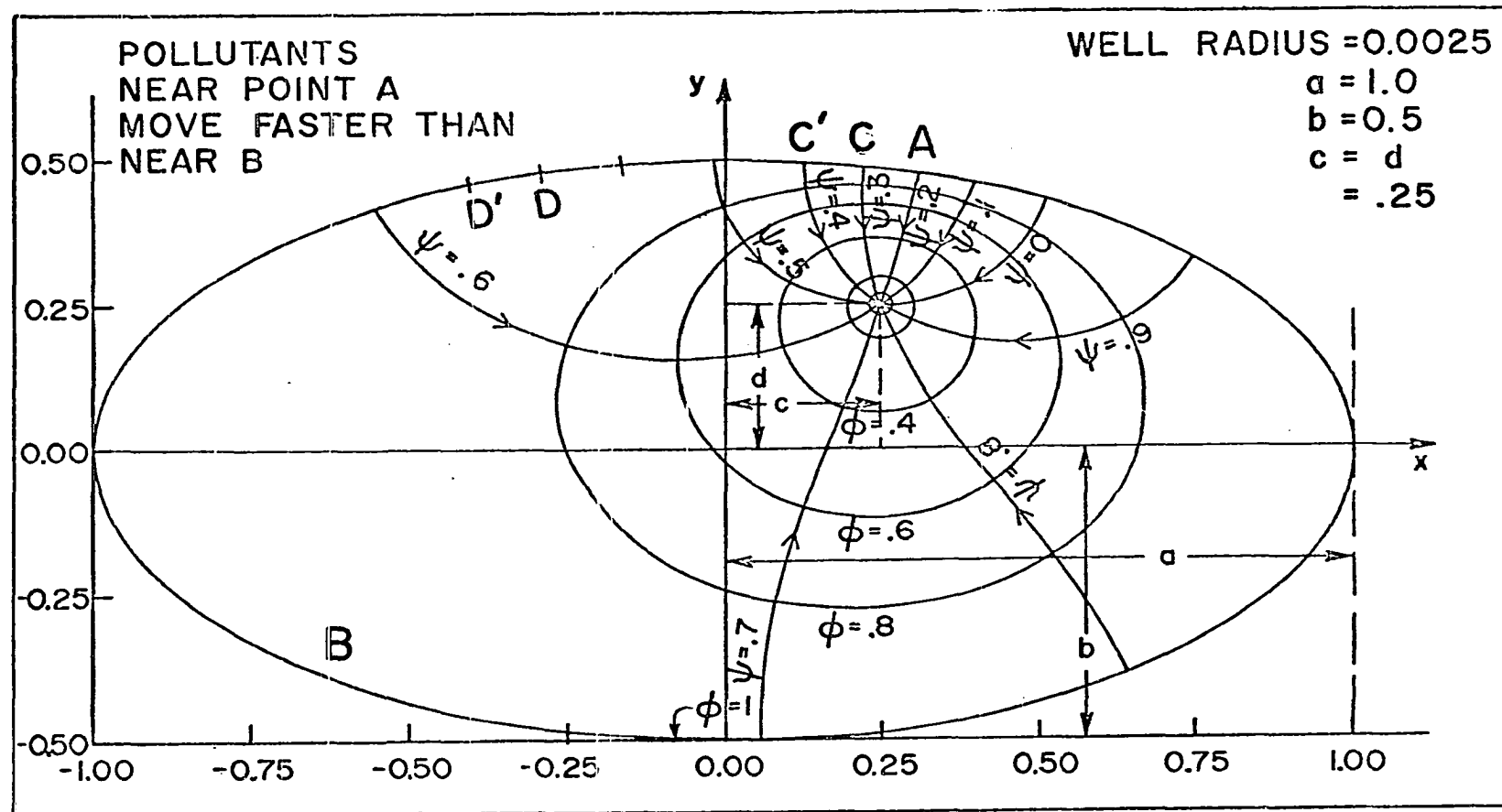
where the minus indicates that the flow is toward the well and is usually omitted. If h is unity Q is often in the literature replaced by q . In equation 194 we note that Q depends only on the zeroth coefficient A_{N0} in the modified Gram-Schmidt method; coefficients A_{N1} , A_{N2} , \dots A_{NN} are not needed.

Before going over to the development of the theory for any shape of aquifer, we shall show a few flow nets calculated from equations 188-193 for some ellipse-shaped aquifers.

Flow Nets for Ellipse-Shaped Aquifers

Figure 3 shows the flow net for a well that is neither at the center nor on an axis of the ellipse. Equal amounts of flow pass between adjacent streamlines. When the streamlines are far apart, the velocity of the water is small. Thus, pollutants near a point A will move much faster to the well than pollutants in the neighborhood of point B. Stated otherwise, a steady-state source of pollutant of a certain concentration and located along a unit length of aquifer boundary near point B would have a small effect on the water quality of the well as compared with the steady-state source of a pollutant of the same concentration located along a unit length of aquifer boundary near point A. We may be more specific. Consider in the figure the approximately equal distances CC' and DD' on the aquifer boundary. We keep in mind, as the figure shows, that equal amounts of water per day reach the well between adjacent streamlines. If the water at CC' is polluted, then the streamline pattern shows that 1/10 of the water reaching the well per day will be polluted. (We are neglecting dispersion and diffusion phenomena.) If the water at DD' is polluted,

Figure 3. Flow net for a confined elliptical aquifer, with the well not located in the center of the ellipse nor on an axis, with $a = 1$, $b = 0.5$, $c = d = 0.25$ and $r_w = 0.0025$



the streamline pattern shows that only $1/40$ of the water reaching the well per day will be polluted. Thus, the locating of the source at DD' rather than at CC' decreases the pollution concentration in the well by a factor of four.

Figure 4 shows the flow net for a well on the major axis of the ellipse. The net shows that a polluting source near B, where the streamlines are far apart, would be much less dangerous than a similar steady-state pollutant inside the aquifer near point A.

Figure 5 is a flow net for a well on the minor axis of the ellipse. Near point A, the flow is fast; near point B, the flow is slow. Pollutants near point B are less harmful than pollutants near point A.

Figure 6 is a flow net when the well is at the center of the ellipse. The velocities are high near point A as compared with point B.

Figure 7 is a flow net for a well in a circular aquifer. A circle is a special case of an ellipse when the major and minor axis are equal. A pollutant near point A would be more dangerous than an equal pollutant near point B.

Flow Theory for an Irregularly-Shaped Aquifer

To obtain the theory for an irregularly-shaped aquifer, with R known as a function of θ with a constant head on the outer boundary, we may use the same form of potential function ϕ as given by equation 188. One of the simpler forms, equations 190 and 192, cannot be used since there is no symmetry, as is evident from the example of Figure 8.

In the modified Gram-Schmidt method we have been using (Kirkham and Powers, 1972), we need two quantities. The first quantity is the potential

Figure 4. Flow net for a confined elliptical aquifer with the well located on the major axis of the ellipse, where $a = 1$, $b = 0.5$, $c = 0.5$ and $r_w = 0.02$.

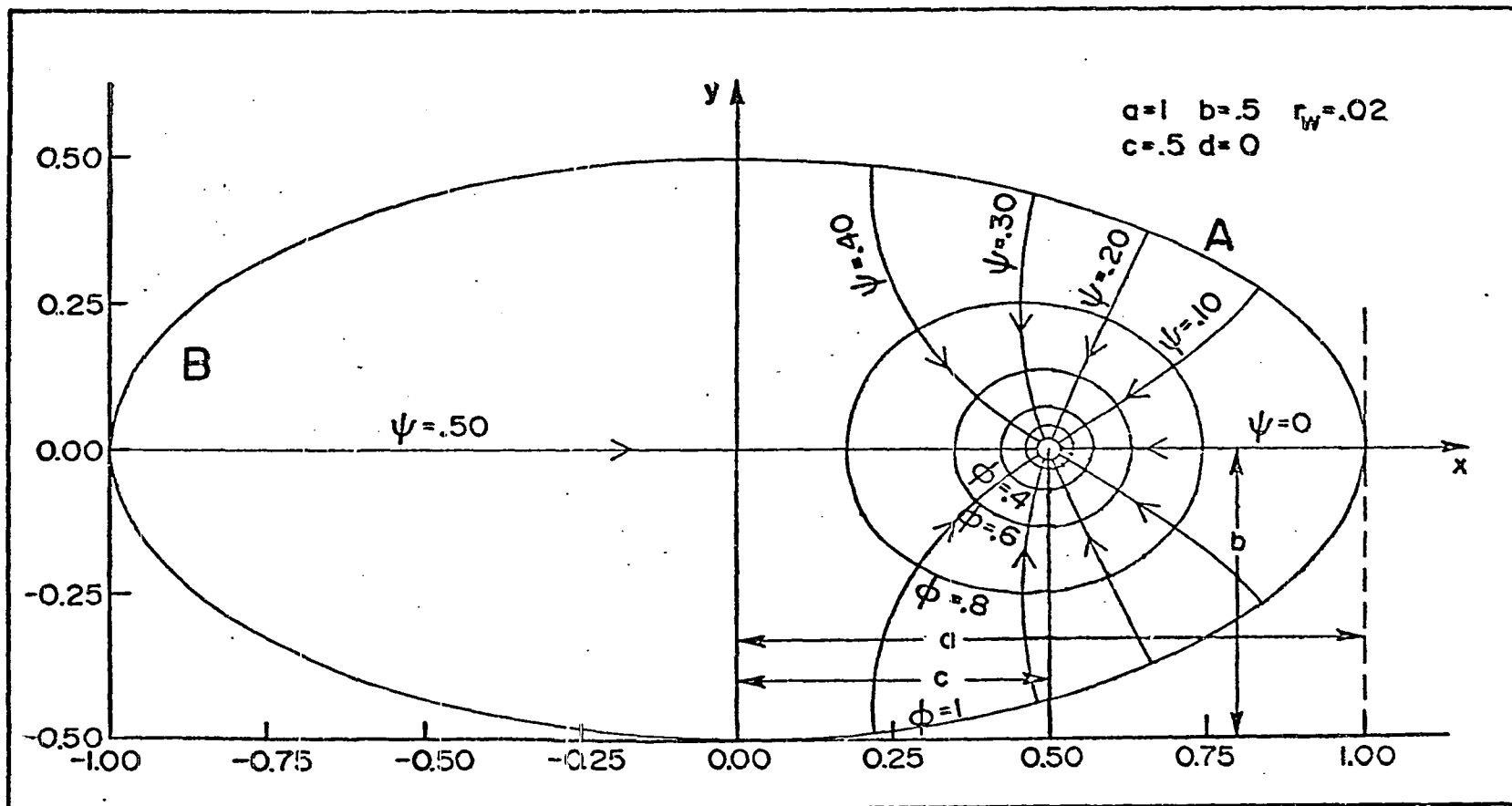


Figure 5. Flow net for a confined elliptical aquifer, with the well located on the minor axis of the ellipse, where $a = 1$, $b = 0.5$, $d = 0.25$ and $r_w = 0.0025$.

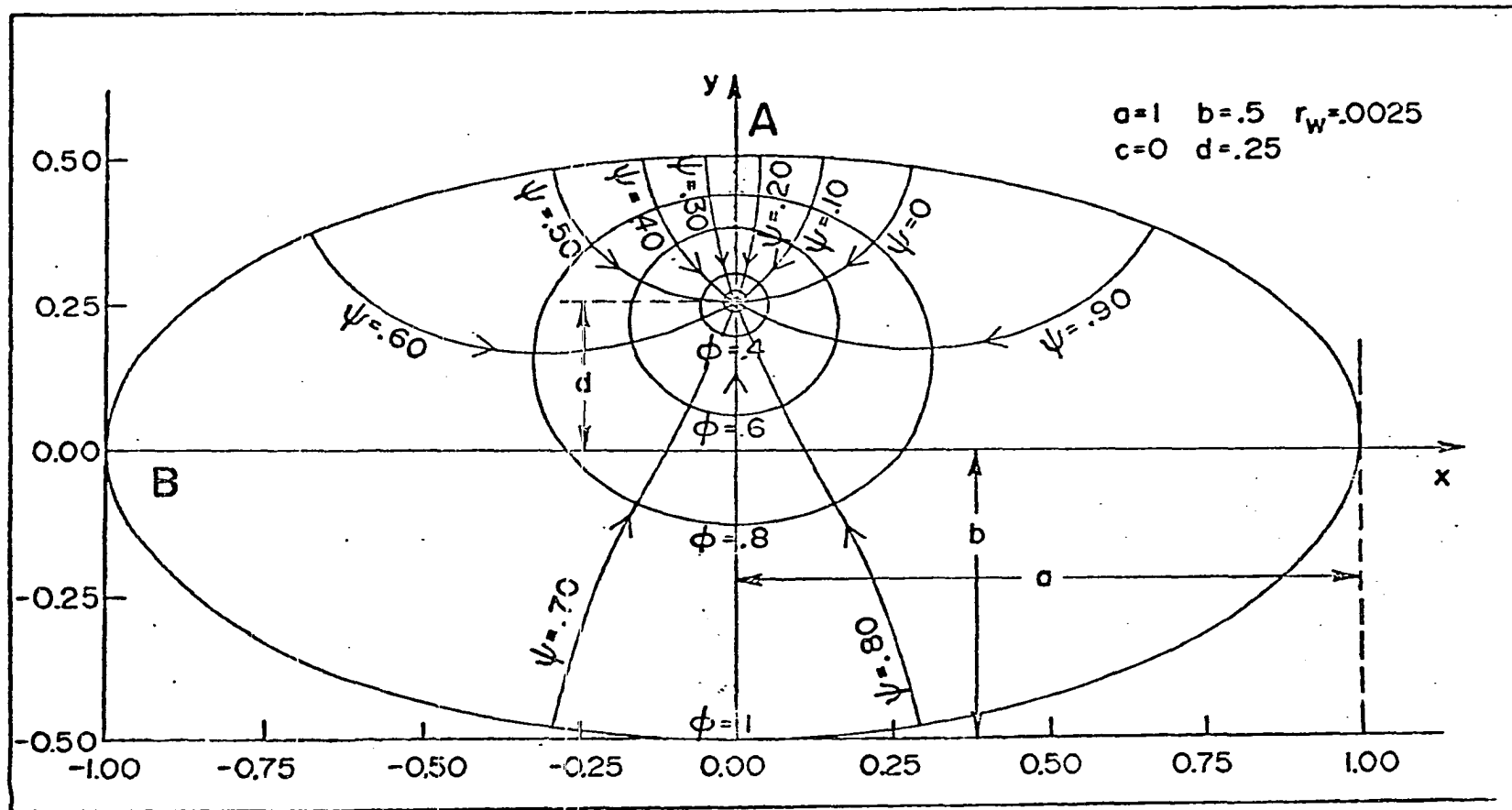


Figure 6. Flow net for a confined elliptical aquifer, with the well located at the center of the ellipse, where $a = 1$, $b = 0.5$ and $r_w = 0.02$.

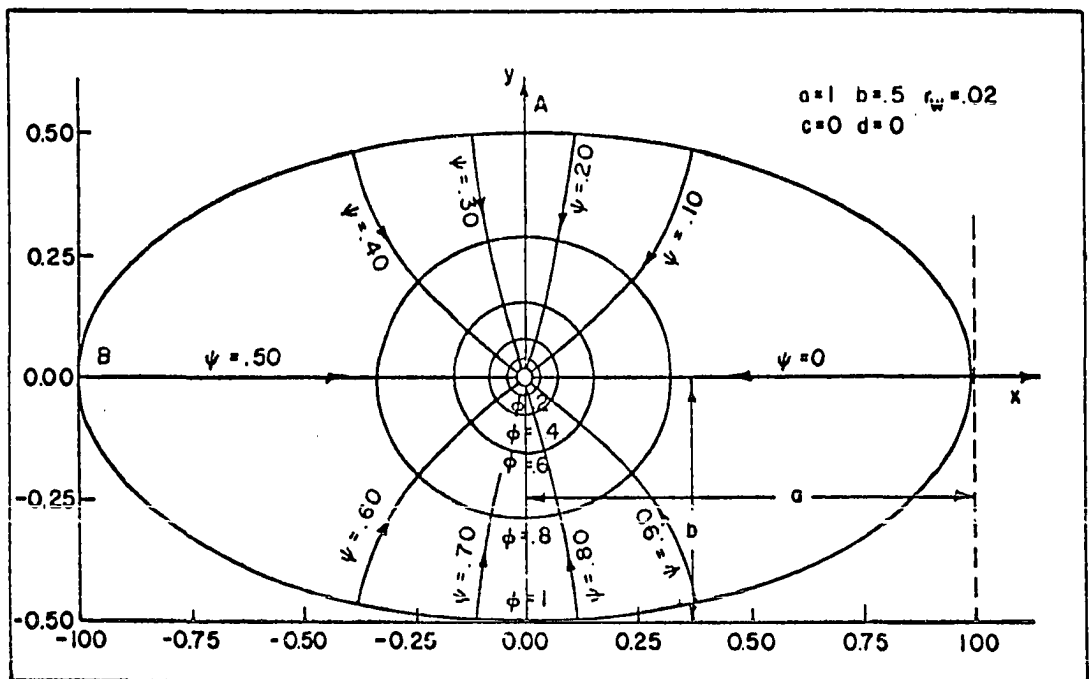


Figure 7. Flow net for a confined circular aquifer, with the well located halfway between the center of the circle and the outer boundary, with $a = 1$, $r_w = 0.02$.

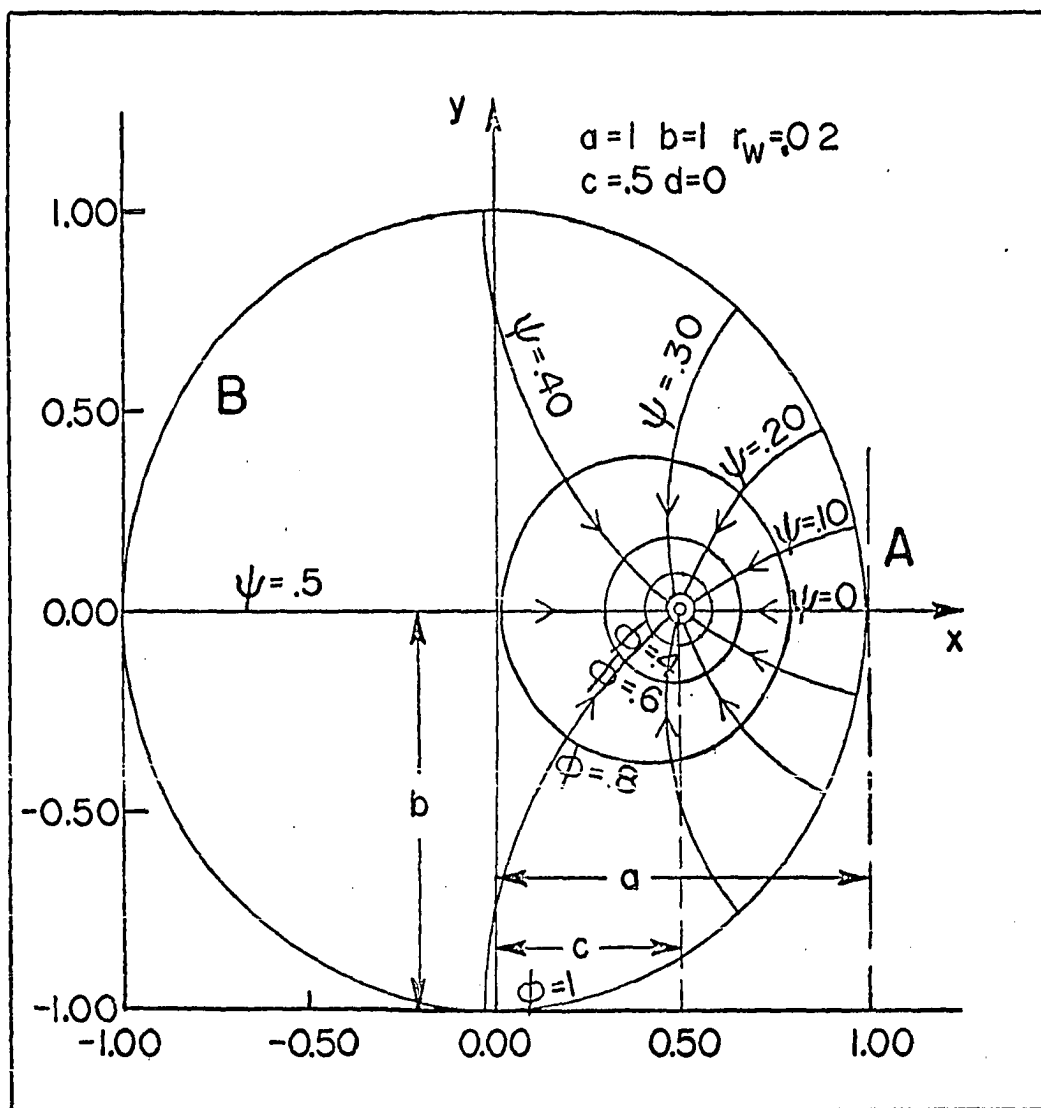
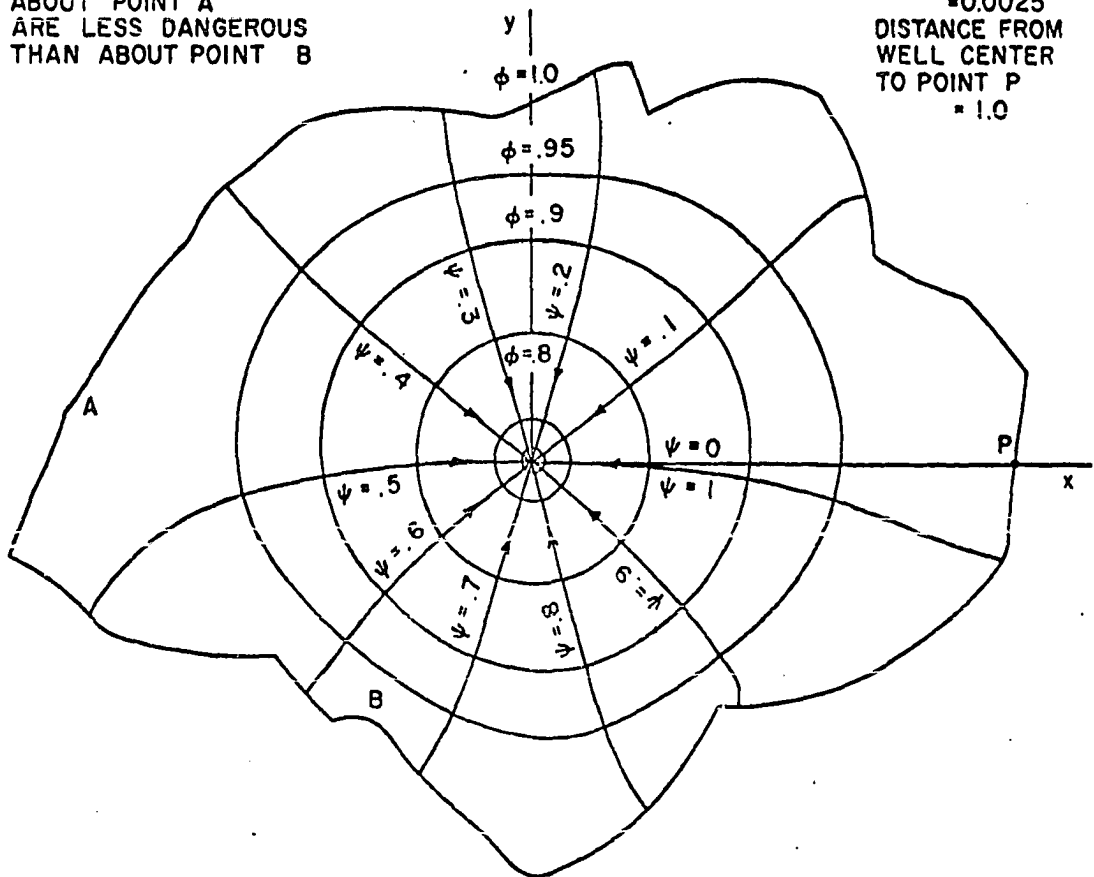


Figure 8. Flow net for an irregular shaped confined aquifer. The distance of the well center to the point P is taken as 1, the well radius $r_w = 0.0025$ times this unit distance.

POLLUTANTS
ABOUT POINT A
ARE LESS DANGEROUS
THAN ABOUT POINT B

WELL RADIUS
= 0.0025
DISTANCE FROM
WELL CENTER
TO POINT P
= 1.0



function $\phi(r, \theta)$ as found on the boundary. Because, on the boundary, the value of r , say R , for every value of θ must be assumed known, we may say that $\phi(r, \theta)$ goes over to a function $\phi(\theta)$, which we denote by $f(\theta)$. Our $f(\theta)$ corresponds to the $f(x)$ of Kirkham and Powers (1972). Because we have specified that head on the outer boundary of the aquifer of Figure 7 is a unit height greater than the head at the well ($\Delta\phi = 1$), we have the relation

$$f(\theta) = 1, \quad 0 < \theta < 2\pi \quad (195)$$

Equation 195 is the useful representation of the first quantity needed.

The second quantity needed in the modified Gram-Schmidt method is a set of coefficients $u_m(r, \theta)$ of the A_{Nm} of equation 188. These coefficients must, however, be evaluated on the boundary of the flow medium where r becomes R (see Figure 1b) so that the second quantity $u_m(r, \theta)$ may be expressed as $u_m(\theta)$, $m = 0, 1, 2, \dots, N$, $0 < \theta < 2\pi$, corresponding to $u_m(x)$, $\alpha < x < \beta$, of Kirkham and Powers (1972). From our definition of $u_m(\theta)$ and equation 188 we may write for the irregularly shaped aquifer problem, the expression

$$u_0(\theta) = \frac{\ln(R/r_w)}{\ln(a/r_w)}, \quad (m = 0) \quad (196)$$

from $m = 0$; and

$$u_m(\theta) = \frac{(R/a)^{m/2} - [r_w^2/(aR)]^{m/2}}{1 - (r_w^2/a^2)^{m/2}} \cos \frac{m}{2} \theta \quad (197)$$

for $m = 2, 4, \dots$; and

$$u_m(\theta) = \frac{(R/a)^{(m+1)/2} - [r_w^2/(aR)]^{(m+1)/2}}{1 - (r_w^2/a^2)^{(m+1)/2}} \sin \frac{m+1}{2} \theta \quad (198)$$

for $m = 1, 3, \dots$.

In equations 196, 197, and 198 as applied to Figure 8, R is the distance from the well center to a point on the boundary of angular coordinate value θ . The angle θ is measured counterclockwise from a line connecting the well center to the point P shown.

With the two functions, $f(\theta)$ and $u_m(\theta)$, we can now evaluate the A_{Nm} of equation 188, and these A_{Nm} can then be related to the $u_m(\theta)$ [see Kirkham and Powers, 1972, appendix 2, equation (i)] by

$$f_N(\theta) = A_{N0} u_0(\theta) + A_{N1} u_1(\theta) + \dots + A_{NN} u_N(\theta) \quad (199)$$

where $f_N(\theta)$ is the N th approximation to $f(\theta)$ of equation 195. For $N \rightarrow \infty$, the right side of equation 199 becomes equal to $f(\theta)$ of equation 188 for every value of θ .

The way one finds the A_{Nm} of equations 199 and 188 is shown in Kirkham and Powers (1972) (see also our section, Theoretical Background) and need not be gone into here except to say that two sets of constants are first to be obtained from $f(\theta)$ and $u_m(\theta)$. The first set of constants is denoted by w_m , defined by

$$w_m = \int_0^{2\pi} f(\theta) u_m(\theta) d\theta \quad (200)$$

and the second set of constants is denoted by u_{mn} , defined by

$$u_{mn} = \int_0^{2\pi} u_m(\theta) u_n(\theta) d\theta \quad (201)$$

$$n = 0, 1, 2, \dots, N, \quad n \leq m$$

In equation 200, $f(\theta)$ is replaced, in accordance with equation 195 by one, and in equations 200 and 201, the functions $u_m(\theta)$ and $u_n(\theta)$ are replaced by equations 196, 197, and 198 as they are applicable. When the aquifer

is of irregular shape, the integrations needed to get the w_m and u_{mn} are carried out by digital computation numerically. For the flow net of Figure 8, the interval $0 < \theta < 2\pi$ was divided into increments of 1° .

With the w_m and u_{mn} determined (the approximation $N = 15$ was found to be good enough for Figure 8), the A_{Nm} were determined, also by the digital computer, from the formulas given at the end of Appendix 2 of Kirkham and Powers (1972). With the A_{Nm} known, the potential function ϕ and stream function ψ of equations 188 and 189 were, in turn, computed by the digital computer for a sufficient number of values to plot the equipotentials shown in Figure 8. The cost of computing the w_m , the u_{mn} , A_{Nm} , sufficient values of $\phi(r, \theta)$ and $\psi(r, \theta)$ for the flow net, and for the plotting of the flow net itself by the automatic plotter was, for Figure 8, 16.32 dollars.

The method we have described for getting the flow net of Figure 8 should work when the outer boundary of the flow region has an arbitrary potential value $\phi(\theta) = f(\theta)$, other than the value $f(\theta) = 1$ given by equation 195. For arbitrary $f(\theta)$, one would place each known head value $f(\theta)$ in equation 195, instead of the value 1, for the numerical integration, and let the digital computer go to work.

Flow Theory for an Aquifer When Some of the Boundaries are Impervious

When parts of the aquifer boundary (in the vertical direction) are impervious, the boundary conditions become more complicated than in the problems we have considered. A so-called mixed boundary value problem is involved. Kirkham (1972) has shown how to apply the modified Gram-

Schmidt method to the solution of mixed boundary value problems. We shall follow his procedure.

When a part of the boundary is impervious, we have the zero flow relation $-K \partial\phi/\partial n = 0$ or

$$\partial\phi/\partial n = 0 \quad (202)$$

where ϕ is the hydraulic head as given by equation 188, and n is a distance measured in the normal direction to the boundary at the boundary point in question.

We may also write equation 202, in vector dot product notation, as

$$\nabla\phi \cdot n = 0 \quad (203)$$

where $\nabla\phi$ (the gradient of ϕ) is to be evaluated in the polar coordinate form from equation 188 and the unit normal must also be expressed in the polar coordinate form. In polar coordinates, we have gradient of ϕ given by

$$\nabla\phi = \frac{\partial\phi}{\partial r} r_0 + \frac{1}{r} \frac{\partial\phi}{\partial\theta} \theta_0 \quad (204)$$

where r_0 is a unit vector in the outward r direction, and θ_0 is a unit vector (a distance not an angle) in the θ (counterclockwise) direction.

Let α be the angle between the normal to a boundary point in question and the outward r direction. Then the unit vector n of equation 203 may be expressed as

$$n = 1(\cos \alpha) r_0 + 1(\sin \alpha) \theta_0 \quad (205)$$

Using equations 204 and 205 to obtain the vector dot product, we find the result

$$\nabla\phi \cdot n = \frac{\partial\phi}{\partial r} \cos \alpha + \frac{1}{r} \frac{\partial\phi}{\partial\theta} \sin \alpha \quad (206)$$

Equations 202, 203, and 206 give for our boundary condition on the impervious barrier

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial r} \cos \alpha + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \sin \alpha = 0 \quad (207)$$

The way to introduce the boundary condition, equation 207, into the modified Gram-Schmidt method of Kirkham and Powers (1972) may be illustrated by considering the circular aquifer of Figure 9 where one-fourth of the circular boundary is impervious. Here, it is convenient to take half of the impervious arc in the fourth quadrant and half in the first. Because of symmetry, we need consider only the flow in the upper half of the circle. Taking the circle to be of radius a the boundary conditions are seen to be

$$\partial \phi / \partial n = 0, \quad r = a, \quad 0 < \theta < \pi/4 \quad (208)$$

$$\phi / \Delta \phi = 1, \quad r = a, \quad \pi/4 < \theta < \pi \quad (209)$$

$$\partial \phi / \partial \theta = 0, \quad \theta = 0, \quad r_w < r < a \quad (210)$$

$$\partial \phi / \partial \theta = 0, \quad \theta = \pi, \quad r_w < r < a \quad (211)$$

$$\phi = 0, \quad r = r_w, \quad 0 < \theta < \pi \quad (212)$$

The appropriate potential is given by equation 190, where we see that boundary condition equations 210, 211, and 212 are satisfied for any values of A_{NO} and the set A_{Nm} . We shall now evaluate the A_{NO} and the A_{Nm} by using the modified Gram-Schmidt method.

Equations 207 and 208 give, because α is zero, the result

$$\frac{\partial \phi}{\partial r} = 0, \quad r = a, \quad 0 < \theta < \pi/4 \quad (213)$$

and we rewrite the boundary condition of equation 209 as

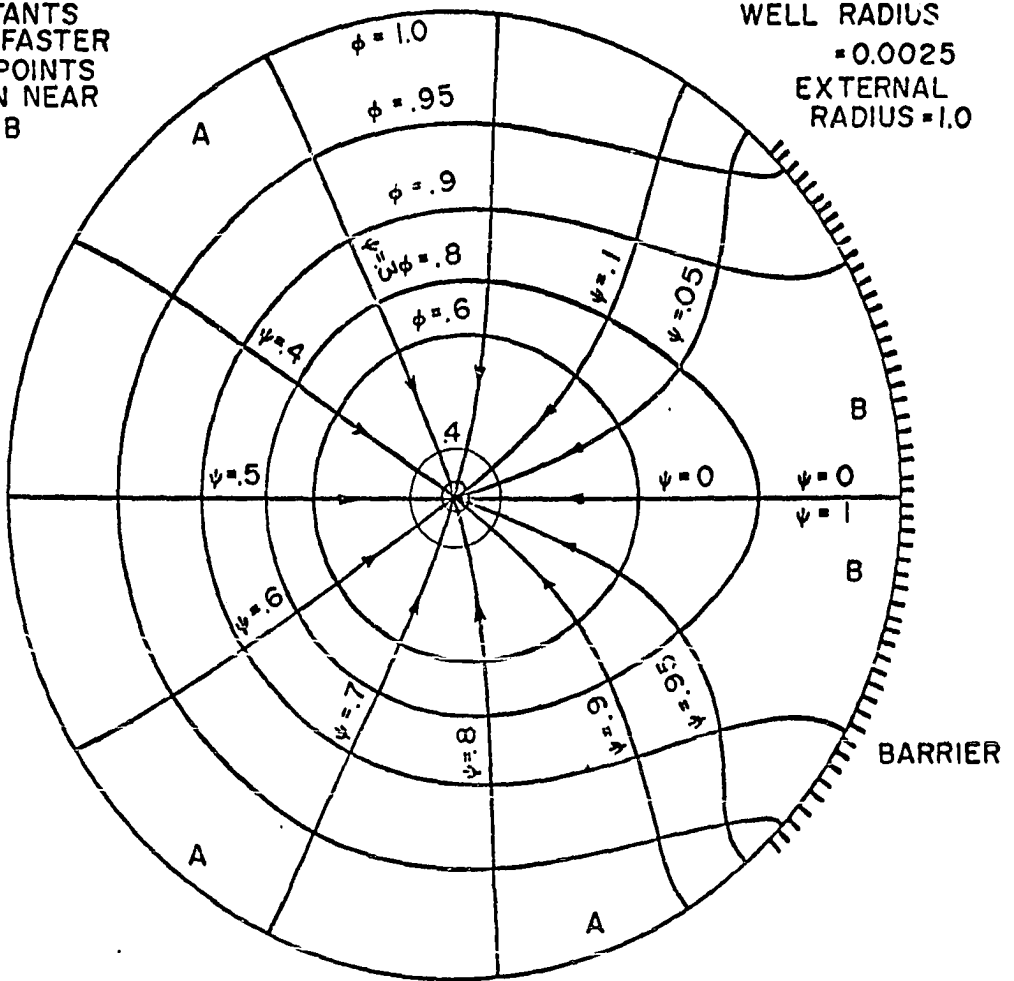
$$\phi / \Delta \phi = 1, \quad r = a, \quad \pi/4 < \theta < \pi \quad (214)$$

In potential flow theory, equations 213 and 214 represent a mixed boundary condition.

Figure 9. Flow net for a confined circular aquifer, of which one quadrant of the outer boundary is impervious, and in which $a = 1$, $r_w = 0.0025$.

POLLUTANTS
MOVE FASTER
NEAR POINTS
A THAN NEAR
POINT B

WELL RADIUS
= 0.0025
EXTERNAL
RADIUS = 1.0



Differentiating equation 190 with respect to r gives

$$\frac{\partial \phi}{\partial r} = A_{NO} \frac{1/r}{\ln(a/r_w)} + \sum_{m=1}^N A_{Nm} \frac{m(1/r) [(r/a)^m + (r_w^2/ar)^m]}{[1 - (r_w^2/a^2)]^m} \cos m\theta \quad (215)$$

Using boundary condition equation 213 in equation 215 (put $r = a$ and put the left side zero in equation 215) yields after multiplying both sides of the result by a , the expression

$$0 = A_{NO} \frac{1}{\ln(a/r_w)} + \sum_{m=1}^N A_{Nm} \frac{m[1 + (r_w^2/a^2)^m]}{1 - (r_w^2/a^2)^m} \cos m\theta, \quad 0 < \theta < \pi/4 \quad (216)$$

Using boundary condition equation 214 in equation 190 yields

$$1 = A_{NO} + \sum_{m=1}^N A_{Nm} \cos m\theta, \quad \pi/4 < \theta < \pi, \quad (217)$$

To use the modified Gram-Schmidt method, we look at the left sides of equations 216 and 217 and specify the boundary condition function $f(\theta)$ as

$$f(\theta) = \begin{cases} 0, & 0 < \theta < \pi/4 \\ 1, & \pi/4 < \theta < \pi \end{cases} \quad (218)$$

And we look at the right sides of equations 216 and 217 and find the function $u_m(\theta)$ (the coefficient of A_{Nm}) as

$$u_0(\theta) = \begin{cases} 1/\ln(a/r_w), & 0 < \theta < \pi/4 \\ 1, & \pi/4 < \theta < \pi \end{cases} \quad (219)$$

for $m = 0$; and for $m = 1 \ 2 \ \dots$ (with D as defined below)

$$u_m(\theta) = \begin{cases} \{m[1 + (r_w^2/a^2)^m]/D\} \cos m\theta, & 0 < \theta < \pi/4 \\ \cos m\theta, & \pi/4 < \theta < \pi \end{cases} \quad (220)$$

$$D = 1 - (r_w^2/a^2)^m$$

With equations 218, 219, and 200 now providing $f(\theta)$ and $u_m(\theta)$, we can now find, by equations 200 and 201, the w_m and u_{mn} and, hence, the A_{Nm} as before. With the A_{Nm} thus found (with the help of the digital computer), the digital computer then evaluates the ϕ function of equation 190 and the ψ function of equation 191 at enough points so that the plotter of the digital computer can draw the flow net that has been traced and presented already as Figure 9. For the flow net of Figure 9, N in equations 190 and 191, etc., was taken to be 100. The high value of N was found necessary because of the discontinuities in the right sides of equations 219 and 220 at $\theta = \pi/4$ and $r = a$. The cost (at $N = 100$) for all the computations and automatic plotting of Figure 9 was 23.98 dollars. This expense gives ϕ to 1% accuracy at all points on the boundary except that at the two points where the impermeable boundary meets the permeable boundary the error is 4%. If one is not interested in details at the boundary discontinuity, one may use $N = 15$ with a cost of about 2.00 dollars for all the computer work.

Figure 10 gives a flow net like that of Figure 9 except that half the circle rather than one quarter is impervious.

In Figures 9 and 10, one sees that a source of pollutant at a point A would be more dangerous than at point B.

A comment may be made for the well discharge for Figures 9 and 10 and for a circular aquifer with no portion of the boundary impervious. We let $Q(0)$, $Q(\pi/4)$, and $Q(\pi/2)$ be the flux when $r_w = 0.0025$ and $a = 1$ and for an impervious boundary of 0, $a\pi/4$, and $a\pi/2$ length. We find

$$Q(0):Q(\pi/4):Q(\pi/2) \text{ as } 1.049:1.020:0.938 \quad (221)$$

so one conclusion is: The well discharge is not as much affected by the impermeable boundaries as is the flow net.

Figure 10. Flow net for a confined circular aquifer, of which half of the outer boundary is impervious, and in which $a = 1$, $r_w = 0.0025$.

SANITARY LAND FILL
SHOULD NOT BE OVER
POINTS A; WOULD
BE SAFER OVER
POINTS B

WELL RADIUS
=0.0025
EXTERNAL
RADIUS
=1.000

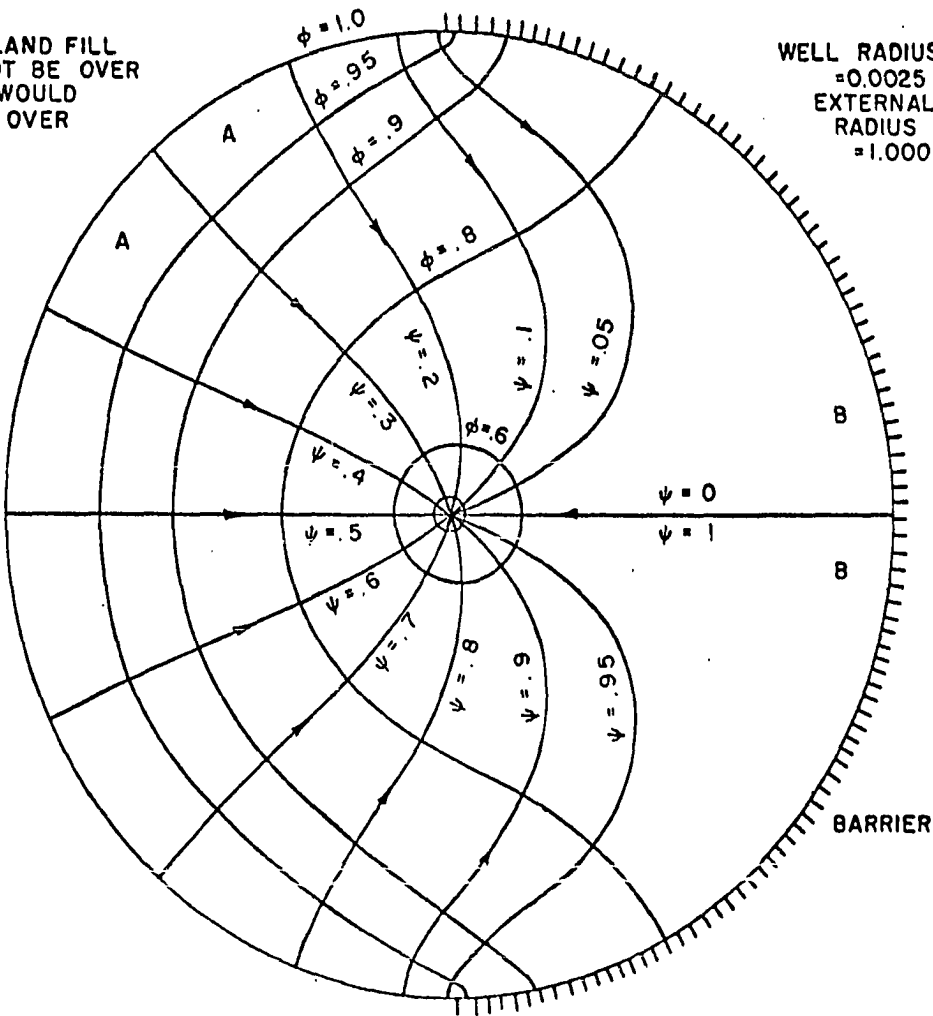


Figure 11 is like Figure 9, but now the well is not located in the center of the circular aquifer, but halfway between the circle and the outer boundary.

The aquifers that we have dealt with in Figures 3 through 11 are, except for Figure 8, more or less idealized, to illustrate the procedure.

We can further use our procedures to analyze the Ames aquifer, located at Ames, Iowa, where steady-state conditions are supplied by two rivers (Ver Steeg, 1968; Akhavi, 1970). A source of pollution in this confined aquifer is being studied.

The Ames Aquifer

The City of Ames, Iowa, depends mainly for its water supply on the Ames aquifer. The Ames aquifer has been studied quite extensively in recent years by Sendlein and his students (Ver Steeg, 1968; Akhavi, 1970). These and other studies have indicated that the Ames aquifer is approximately a horizontal, confined aquifer of rather uniform thickness and permeability, consisting of sand and gravel and about 60 ft. thick. The Ames aquifer is fed by two surface streams, the Skunk River and Squaw Creek, through the channel beds of these streams. Figure 12 (from Akhavi, 1970) shows the surface geography of the Ames aquifer area. The Ames aquifer is located between Squaw Creek and the Skunk River at about 100 ft depth. The figure also shows the elevation above sea level of the piezometric surface in the Ames aquifer when it is not pumped.

Figure 13 (Akhavi, 1970) shows a cross-section of the aquifer as prepared by Akhavi from field data.

Figure 14 is a combination of Figures 12 and 13 and shows a schematic

Figure 11. Flow net for a confined circular aquifer, of which 1 quadrant is impervious. The well is located halfway between the center of the circle and the outer boundary. As in Figure 10 $a = 1$, $r_w = 0.0025$.

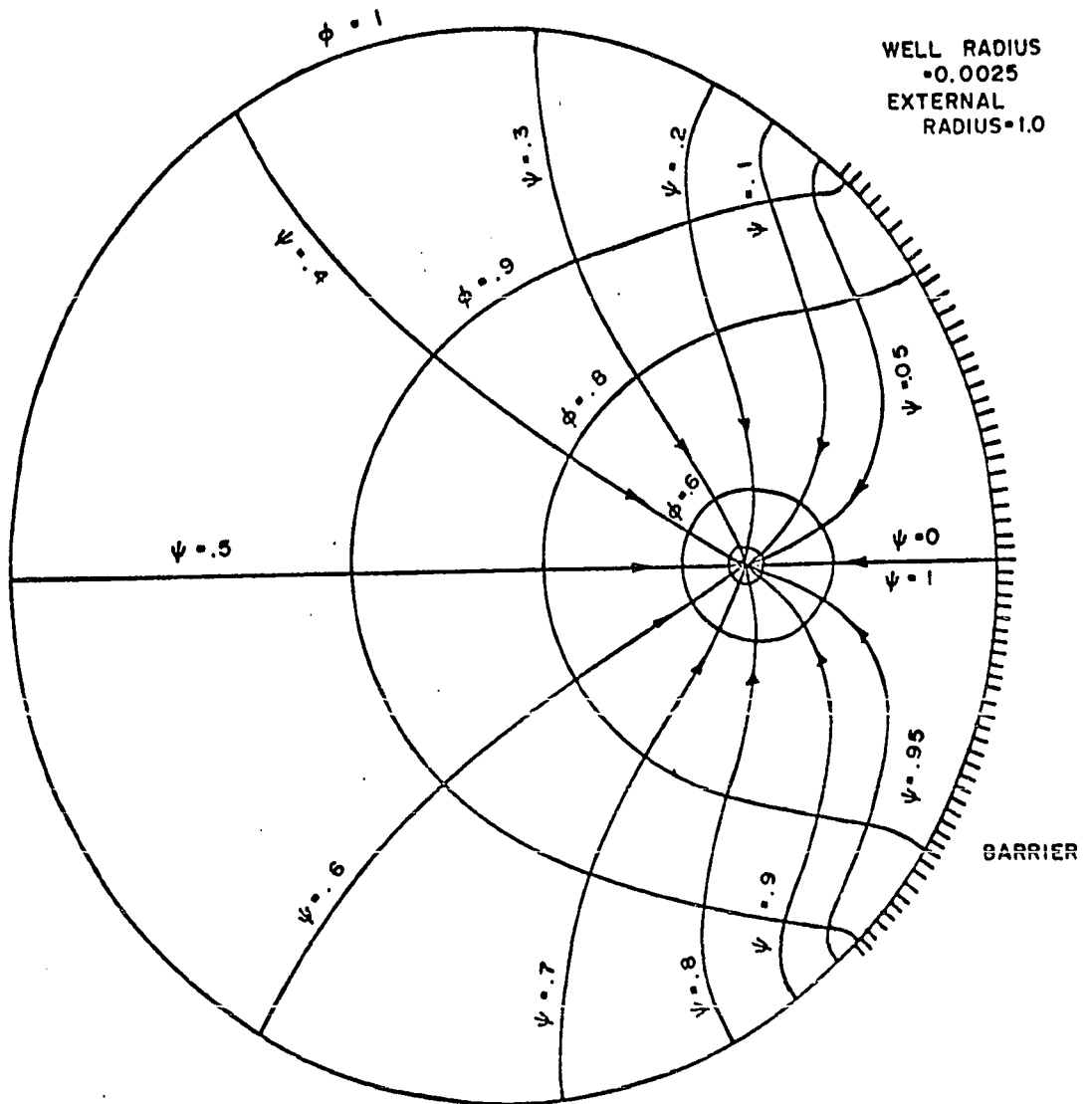
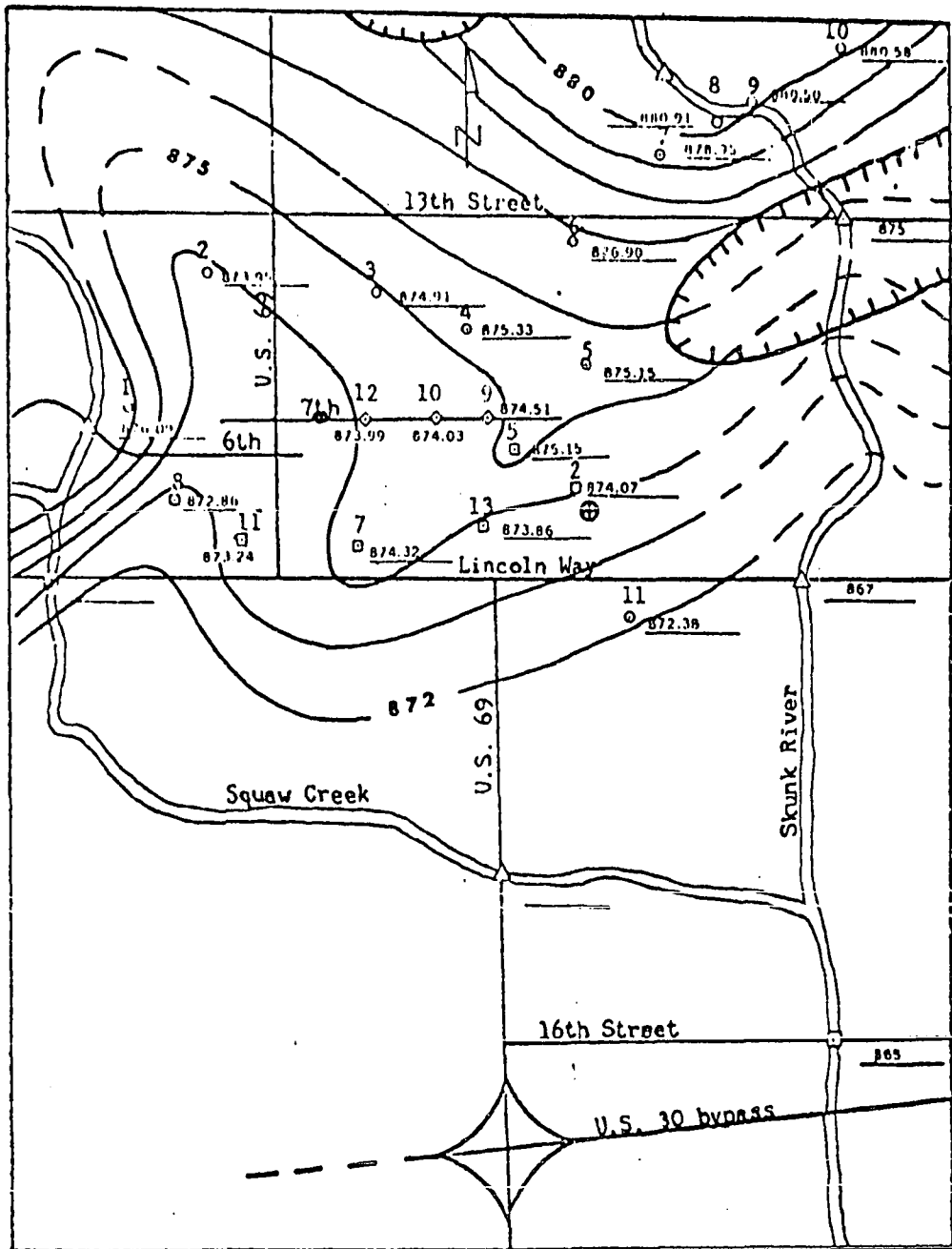


Figure 12. Location of the Ames aquifer, and piezometric surface map of the aquifer, before pumping. (Akhavi 1970).



- ⊕ Source of Phenol
- ▬ Till Boundary
- △ Staff Gage
- Observation Well
- USGS Recorder
- ◇ City Well, pumping
- ◻ City Well, stand-by

PIEZOMETRIC SURFACE MAP,
BEFORE PUMPING

Scale: 1" = 1500'
Contour Interval 1 Foot

Figure 13. A block diagram through the Ames aquifer, as prepared from field data. (Akhavi, 1970).

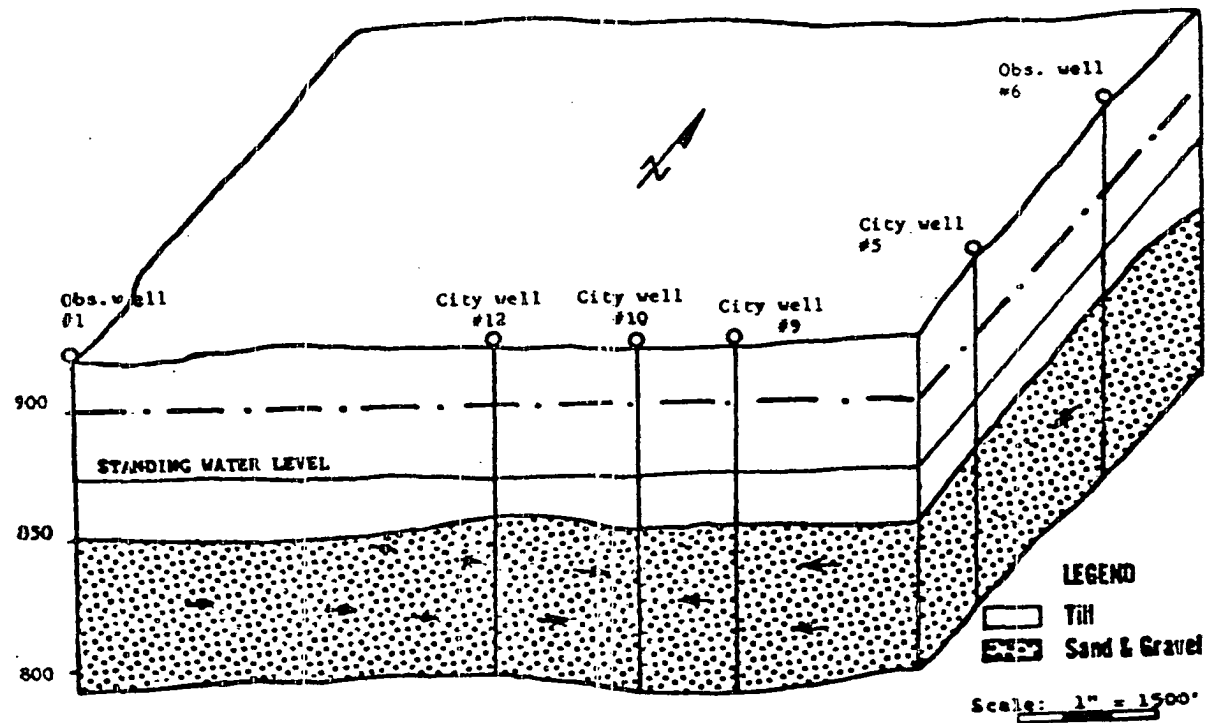
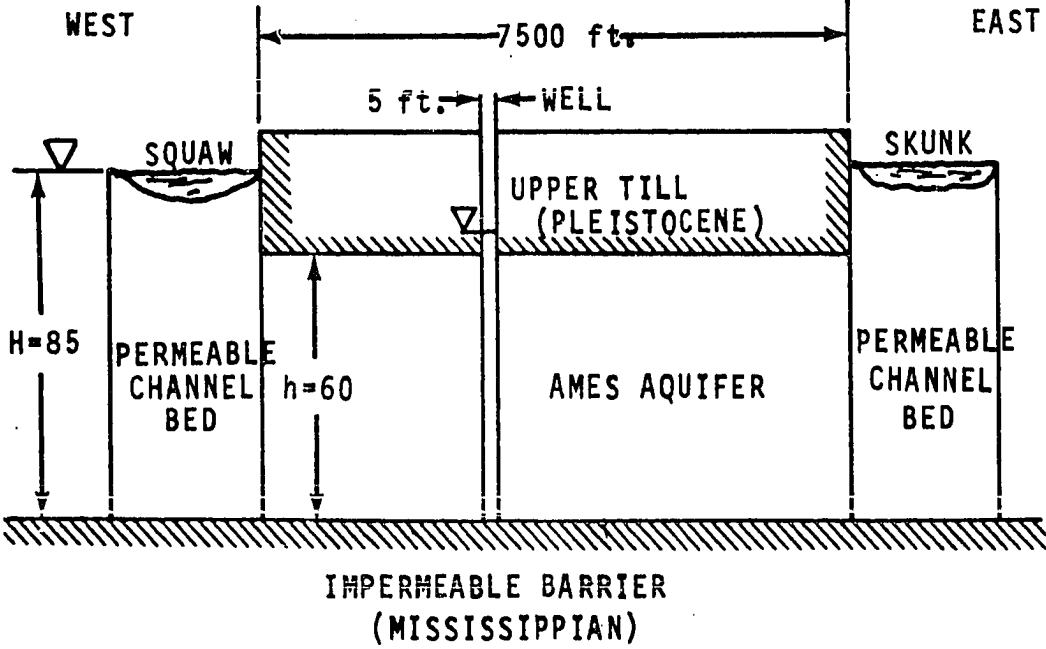


Figure 14. A schematic east-west cross-section through the Ames aquifer.

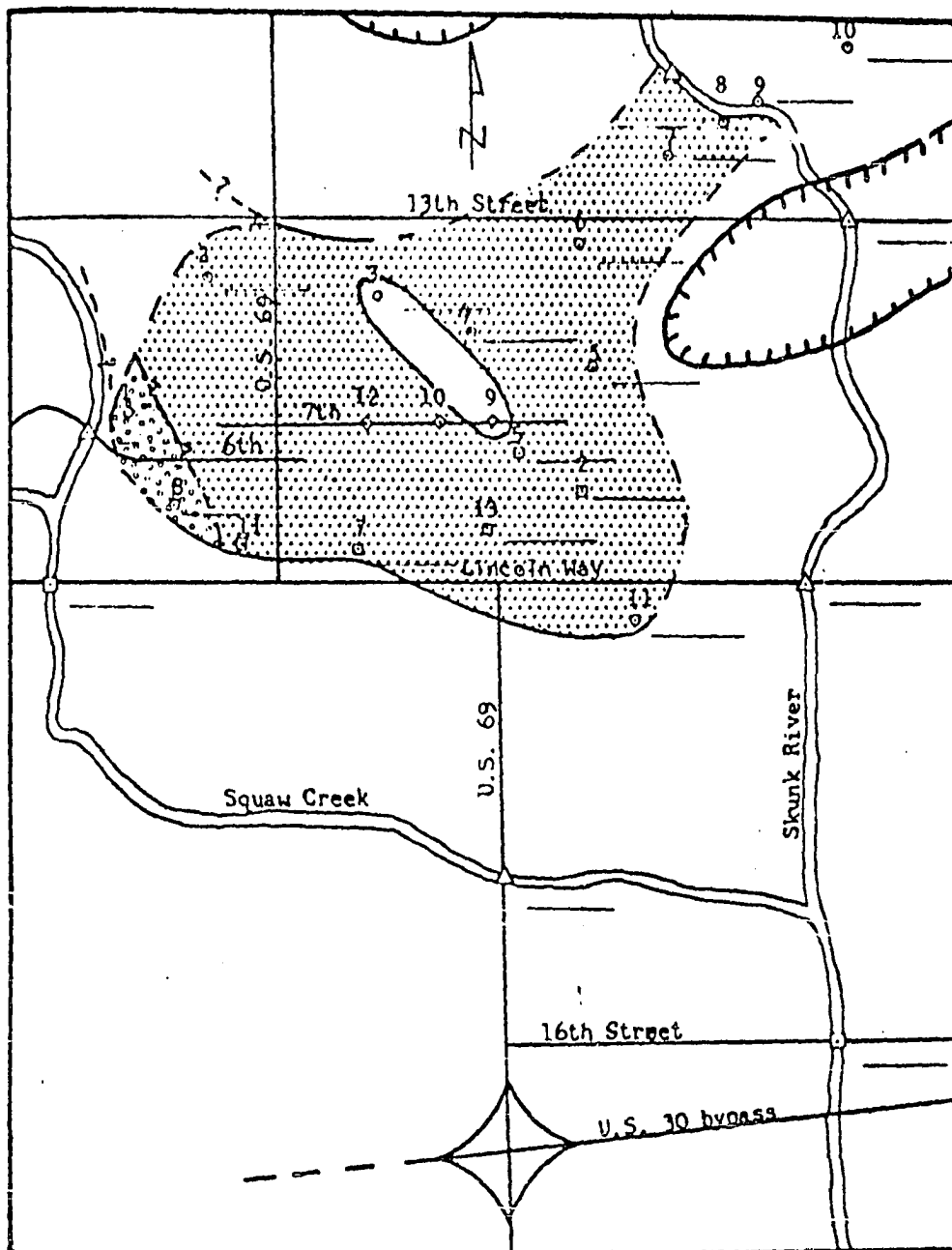


cross-section of the Ames aquifer in east-west direction. One can see that the upper confining layer is the upper till and that the lower confining layer is Mississippian bedrock. One can see also that the hydraulic head in the channel beds of either the Skunk River or Squaw Creek is well above the upper confining layer of the Ames aquifer.

Figure 15 (Akhavi, 1970), which is also prepared from field data, shows the rather uniform transmissibility of the Ames aquifer. In our theoretical analysis later on, we use a transmissibility of 200,000 gpd/ft.

Figure 16 (Akhavi, 1970) shows the elevation of the piezometric surface in the Ames aquifer after a pumping test. During this pumping test, water was pumped from the Ames city wells numbers 9, 10, and 12 at a capacity of 3000 gpm. From this pumping test, from other pumping tests (Akhavi, 1970; Backsen, 1963), and from Figure 12, it appears that the boundary of the Ames aquifer corresponds to the dashed line shown in Figure 17. At the south of the aquifer and at the east, the boundary of the aquifer is formed by the floodplains of Squaw Creek and Skunk River. An impervious till nose, shown in Figures 12 and 16 at the east, thus falls just outside the aquifer limits. A till nose of large extent on the north of the aquifer, however, is included as an impervious boundary of the aquifer. From Figures 12 and 16 and other field evidence, we conclude that the head distribution along the boundary of the Ames aquifer, whether pumped or not, is such that, at the north, the hydraulic head is 880 ft (except at the impervious boundary), and that, at the south, the hydraulic head is 870 ft. Along Squaw Creek and the Skunk River, the head drops linearly from 880 ft to 870 ft. With the limits of the aquifer and the head distribution along the limits thus established, we decided to

Figure 15. Transmissibility distribution map of the Ames aquifer (Akhavi, 1970).



--- Till

Boundary

△ Staff Gage

○ Observation Well

□ USGS Recorder

◇ City Well, pumping

■ City Well, stand-by

TRANSMISSIBILITY DISTRIBUTION MAP

AMES, IOWA

Scale: 1" = 1500'

TRANSMISSIBILITY gpd/ft.

$T < 154,000$

$(\mu - \sigma)$

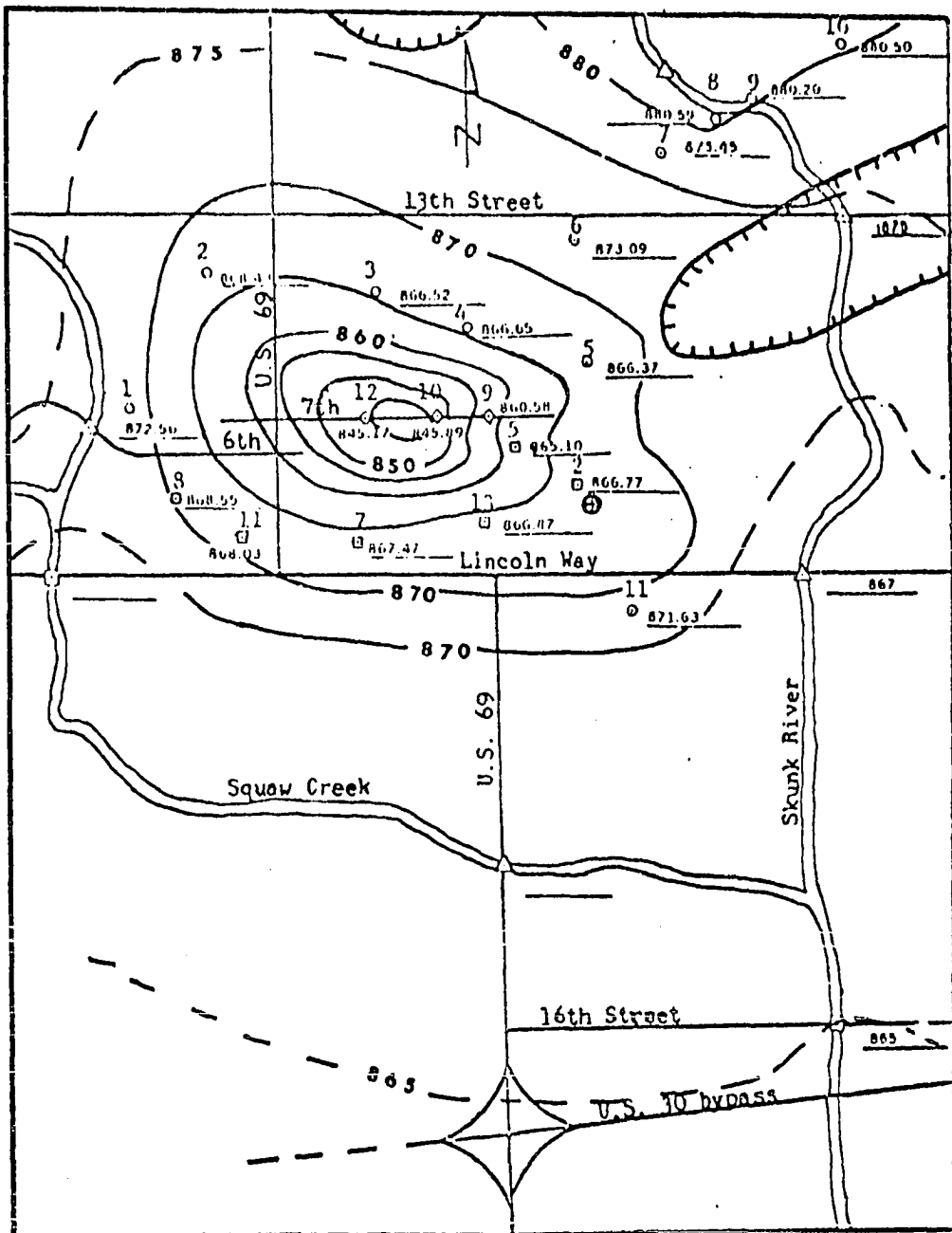
$154,000 < T < 245,000$

$(\mu - \sigma) \quad (\mu + \sigma)$

$T > 245,000$

$(\mu + \sigma)$

Figure 16. Piezometric surface map in the Ames aquifer during an extensive pumping test (Akhavi, 1970).

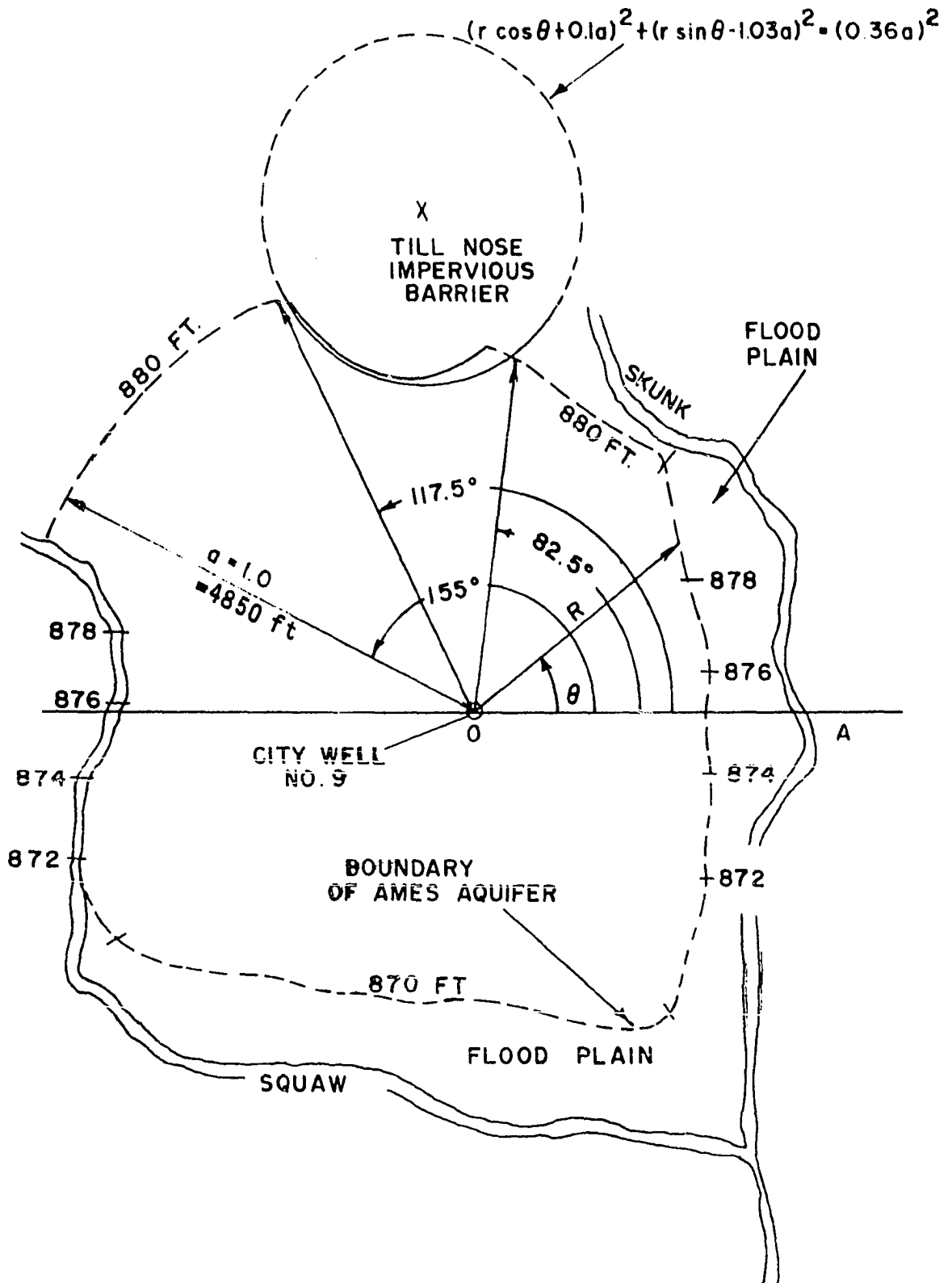


- ⊕ Source of Phenol
- ▬ Till Boundary
- △ Staff Gage
- Observation Well
- USGS Recorder
- ◇ City Well, pumping
- ◻ City Well, stand-by

PIEZOMETRIC SURFACE MAP,
DURING PUMPING

Scale: 1" = 1500'
Contour Interval 5 Feet

Figure 17. Boundaries of the Ames aquifer as determined from pumping tests, and distribution of the hydraulic head on the pervious part of the boundary.



analyze a pumping test, as described by Ver Steeg (1968, p. 67), on the Ames city well number 9. The drawdown in the pumped well at equilibrium was 9.2 ft. The pumping rate was 1280 gpm, and the well radius, including gravel pack, is 2.5 ft. The drawdown is such that the water level in the well is above the upper confining layer of the aquifer such that confined radial flow is assured

Analysis of Ames Aquifer

The analysis is much the same as used by Van der Ploeg et al. (1971) or as discussed earlier in this thesis. That is, we use a modified Gram-Schmidt method as in Powers et al. (1967), or as in Kirkham and Powers (1972).

An expression is sought that enables one to calculate the hydraulic head at every point in the flow region. From the potential function, the stream function (that is, the flow lines to the well) can be calculated.

As in Van der Ploeg (1970), or as in the previous sections of this thesis a system of polar coordinates (r, θ) is chosen with the origin in the center of the well; here city well number 9. The angle θ is measured as shown in Figure 17. The reference level for the hydraulic head is the equilibrium drawdown water level in the well. In the pumping test by Ver Steeg (1968), the drawdown in the well was 9.2 ft. Using this 9.2-ft well drawdown and the piezometric surface of Figure 12, the reference level thus is at $874.5 \text{ ft} - 9.2 \text{ ft} = 865.3 \text{ ft}$. The hydraulic head difference across the aquifer to the well, hence, varies from 14.7 ft in the north to 4.7 ft in the south. The head distribution is indicated in Figure 17. A line of longest distance from the well to the aquifer boundary is denoted

by the symbol a . This line makes an angle of 155° with the line $\theta = 0$ in Figure 17. The length a is taken for convenience as the unit length. In reality, however, $a = 4850$ ft. Since the well radius r_w which includes the gravel pack, is 2.5 ft, the ratio r_w/a , which is used in the analysis, is equal to $r_w/a = 0.00052$. The location of the impervious till nose, sticking into the aquifer from the north, can now be determined. Field data indicate that the nose is nearly circular to the south. Taking the nose as aquifer boundary between $\theta = 82.5^\circ$ and $\theta = 117.5^\circ$ (as shown in Figure 17), the aquifer boundary here can be described as a circle. The equation of this circle referred to an origin at city well number 9 can be given as

$$(r \cos \theta + 0.1a)^2 + (r \sin \theta - 1.03a)^2 - (0.36a)^2 = 0 \quad (222)$$

where r is the distance from the city well to a point on the nose and θ the angle between r and the line OA in Figure 17. (The line OA, when extended to the left, passes through city wells numbers 10 and 12). The problem now can be stated as: a hydraulic head function ϕ , has to be found, such that it satisfies Laplace's equation in polar coordinates for horizontal flow, namely

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0$$

subject to the boundary conditions (BC's)

$$\text{BC 1: } \phi = 0 \quad \text{for } r = r_w, \quad 0^\circ \leq \theta \leq 360^\circ$$

$$\text{BC 2a: } \phi = \text{a constant value, as measured or determined,}$$

ranging from 4.7 ft to 14.7 ft, and shown in

Figure 17 along the outer boundary of the

aquifer, for $0^\circ \leq \theta \leq 82.5^\circ$ and $117.5^\circ \leq \theta \leq 360^\circ$

BC 2b: $\partial\phi/\partial n = 0$ at the impervious boundary in the north of the aquifer for $82.5^\circ < \theta < 117.5^\circ$

Boundary condition 2b states that no flow occurs across the impervious boundary in a direction normal to that boundary; $\partial\phi/\partial n$ is the normal derivative.

As expression for the hydraulic head function ϕ , the same expression is used as was used for the irregularly shaped confined aquifer. The expression is

$$\begin{aligned}
 (\phi/\phi_0) = & A_{NO} \frac{\ln(r/r_w)}{\ln(a/r_w)} \\
 & + \sum_{m=1,3,\dots}^N A_{Nm} \left\{ \frac{(r/a)^{(m+1)/2} - [r_w^2/(ar)]^{(m+1)/2}}{1 - (r_w^2/a^2)^{(m+1)/2}} \right\} \sin \frac{m+1}{2} \theta \\
 & + \sum_{m=2,4,\dots}^N A_{Nm} \left\{ \frac{(r/a)^{m/2} - [r_w^2/(ar)]^{m/2}}{1 - (r_w^2/a^2)^{m/2}} \right\} \cos \frac{m}{2} \theta \quad (223)
 \end{aligned}$$

in which ϕ_0 is a unit hydraulic head, N is an integer, which goes to infinity for an exact solution, and in which the coefficients A_{Nm} have to be determined. In our analyses, we have taken $N = 15$. The way the coefficients A_{Nm} have to be determined is described in Van der Ploeg et al. (1971) or Kirkham and Powers (1972), and will be discussed here only where the method is different from the procedure as used by Van der Ploeg et al. (1971). The coefficients A_{Nm} have to be determined by use of boundary conditions 2a and 2b. This can be done by considering the hydraulic head on the outer boundary of the aquifer as a function of the angle θ and by considering the value of the normal derivative ($= 0$) on the impervious boundary also as a function of θ .

Let $f(\theta)$ be the value of the dimensionless hydraulic head as we are given it by boundary condition 2a. That is, we may write

$$f(\theta) = \text{value of } \phi/\phi_0$$

as given over the pervious part of the boundary.

If we let the subscript p stand for pervious boundary, we can write the last expression as

$$f(\theta) = \phi/\phi_{0p}, \quad (0^\circ \leq \theta \leq 82.5^\circ) \\ 117.5^\circ \leq \theta < 360^\circ) \quad (224)$$

In the same way, if we let the subscript i stand for the impervious boundary, we can write

$$f(\theta) = \left[\frac{\partial(\phi/\phi_0)}{\partial(n/a)} \right]_i \quad 82.5^\circ < \theta < 117.5^\circ \quad (225)$$

where we know physically that the left side of equation 225 is 0.

Equations 224 and 225 considered together define a step function for the range $0^\circ \leq \theta < 360^\circ$. This step function can be developed into a function of the form

$$f(\theta) = \sum_{m=0}^N A_{Nm} u_m(\theta), \quad (m = 0, 1, 2, \dots) \quad (226)$$

by the modified Gram-Schmidt method of Powers et al. (1967) as we have shown earlier. The A_{Nm} in equation 226 are the A_{Nm} of equation 223. Each $u_m(\theta)$ of equation 226 is a 3-step function. The value of each $u_m(\theta)$ for two of its steps is indicated by the θ ranges and the $f(\theta)$ value of equation 224. The third step in each $u_m(\theta)$ is for the range of θ given at the right of equation 225, and the value of each $u_m(\theta)$ for this range ($82.5^\circ < \theta < 117.5^\circ$) is (seen from equation 225) to be given by the coefficient of A_{Nm} of equation 223 after its dimensionless derivative [namely

$\partial(\phi/\phi_o/\partial(n/a))$ has been taken and evaluated on the impervious boundary.

The part of $u_m(\theta)$ for the two ranges of equation 224 is easily obtained. In equation 223 we substitute for r the measured distance R from the well to the outer aquifer boundary where R varies as θ changes. Then for $r = R$, and θ in the range, $0^\circ \leq \theta \leq 82.5^\circ$ and $117.5^\circ \leq \theta < 360^\circ$, we obtain $u_o(\theta)$ and $u_m(\theta)$ from equation 223 as

$$u_o(\theta) = \frac{\ln(R/r_w)}{\ln(a/r_w)}, \quad (227)$$

$$u_m(\theta) = \frac{(R/a)^{(m+1)/2} - [r_w^2/(aR)]^{(m+1)/2}}{1 - (r_w^2/a^2)^{(m+1)/2}} \sin \frac{m+1}{2} \theta$$

$$m = 1, 3, \dots, \quad \text{and} \quad (228)$$

$$u_m(\theta) = \frac{(R/a)^{m/2} - [r_w^2/(aR)]^{m/2}}{1 - (r_w^2/a^2)^{m/2}} \cos \frac{m}{2} \theta, \quad m = 2, 4, \dots,$$

For equations 227 and 228 values of R are determined graphically from a large drawing such as Figure 17. The values of R were drawn for each 2.5° . For the impervious portion of the boundary (discussed in the next few paragraphs), values of R were obtained similarly.

The part of $u_m(\theta)$ for the impervious part of the boundary is not as easy to obtain as equations 227 and 228. We shall now outline the steps for getting the value $u_m(\theta)$ for the range of θ where the boundary is impervious.

At the impermeable boundary, for $r = R$, and θ as $82.5^\circ < \theta < 117.5^\circ$, the function $u_m(\theta)$ is governed by the dimensionless normal derivative

$$\frac{\partial(\phi/\phi_o)}{\partial(n/a)} = 0 \quad (229)$$

which, for brevity, we may write as

$$\partial\phi/\partial n = 0 \quad (230)$$

By vector notation equation 230 can be written as

$$\frac{\partial\phi}{\partial n} = \nabla\phi \cdot \bar{n} \quad (231)$$

In other words, $\partial\phi/\partial n$ is equal to the dot product of the gradient of ϕ and the unit normal vector to the circular impervious barrier. In polar coordinates the gradient of ϕ may, from the geometry (or see Kaplan, 1959, p. 155), be written as

$$\nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r} \frac{\partial\phi}{\partial\theta} \right) \quad (232)$$

To obtain the unit normal vector to the circular element of the impervious barrier, we may denote the left side of equation 222 as $S(r, \theta)$; i.e.,

$$S(r, \theta) = (r \cos \theta + 0.1a)^2 + (r \sin \theta - 1.30a)^2 - (0.36a)^2 = 0 \quad (233)$$

The normal to this circle is obtained (Kaplan, 1959, p. 103) by taking the gradient of S . That is, n is given (or defined by) the expression

$$n = \nabla S = \left(\frac{\partial S}{\partial r}, \frac{1}{r} \frac{\partial S}{\partial\theta} \right) \quad (234)$$

To get the unit normal vector, defined as \bar{n} , from n of equation 234, we must divide the vector n by its length $|n|$. The unit normal vector thus can be written as

$$\bar{n} = \left(\frac{\frac{\partial S}{\partial r}}{|n|}, \frac{\frac{1}{r} \frac{\partial S}{\partial\theta}}{|n|} \right) \quad (235)$$

where the first term at the right is the component \bar{n} in the r direction, and the second term at the right is the component of \bar{n} in the θ direction.

We can get $|n|$ as follows. We find $\partial S/\partial r$ from equation 233 as

$$\partial S/\partial r = 2r + 0.2a \cos \theta - 2.06a \sin \theta \quad (236)$$

And we find $(1/r)\partial S/\partial\theta$ from equation 233 as

$$(1/r)\partial S/\partial\theta = -(0.2a \sin \theta + 2.06a \cos \theta) \quad (237)$$

The magnitude of the vector n of equation 234 is known (Kaplan, 1959, p.

109) to be given by

$$|n| = \{(\partial S/\partial r)^2 + [(1/r)\partial S/\partial\theta]^2\}^{1/2} \quad (238)$$

so that use of equations 236 and 237 in equation 238 gives us immediately an expression for $|n|$ that we need not write here.

Turning now to $\partial\phi/\partial n$ of equation 231, we write, from vector analysis, $\partial\phi/\partial n$ in view of equations 231, 232, and 234 as

$$\frac{\partial\phi}{\partial n} = \frac{\frac{\partial\phi}{\partial r} \frac{\partial S}{\partial r}}{|n|} + \frac{\frac{1}{r} \frac{\partial\phi}{\partial\theta} \frac{1}{r} \frac{\partial S}{\partial\theta}}{|n|} \quad (239)$$

where we now know $|n|$, $\partial S/\partial r$ and $(1/r)\partial S/\partial\theta$ from equations 236, 237, and 238. We need $\partial\phi/\partial r$ and $(1/r)\partial\phi/\partial\theta$. We find $\partial\phi/\partial r$ in dimensionless form from equation 223 as

$$\begin{aligned} \frac{\partial(\phi/\phi_o)}{\partial(r/a)} &= A_{NO} \frac{a}{r} \frac{1}{\ln(a/r_w)} \\ &+ \sum_{m=1,3,\dots}^N A_{Nm} \frac{m+1}{2} \frac{a}{r} \frac{\left(\frac{r}{a}\right)^{(m+1)/2} + \left(\frac{r_w}{ar}\right)^{(m+1)/2}}{1 - (r_w^2/a^2)^{(m+1)/2}} \sin \frac{m+1}{2} \theta \\ &+ \sum_{m=2,4,\dots}^N A_{Nm} \frac{m}{2} \frac{a}{r} \frac{\left(\frac{r}{a}\right)^{m/2} + \left(\frac{r_w}{ar}\right)^{m/2}}{1 - (r_w^2/a^2)^{m/2}} \cos \frac{m}{2} \theta \end{aligned} \quad (240)$$

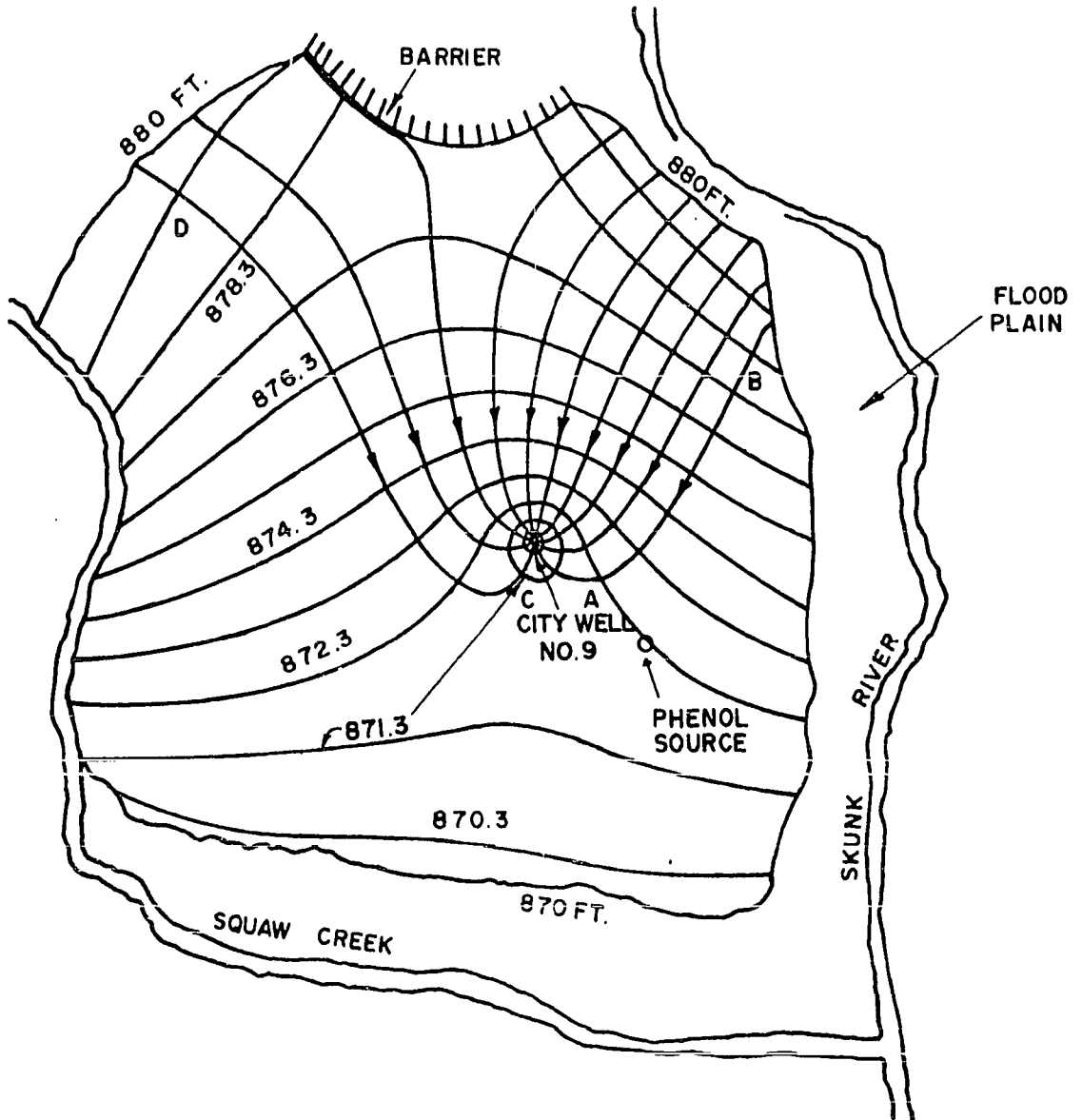
And we find the dimensionless potential form of $(1/r)\partial\phi/\partial\theta$ from equation 223 as

$$\begin{aligned}
\frac{1}{r} \frac{\partial(\phi/\phi_o)}{\partial\theta} = & \sum_{m=1, 3, \dots}^N A_{Nm} \frac{m+1}{2} \frac{1}{r} \frac{\left(\frac{r}{a}\right)^{(m+1)/2} - \left(\frac{r_w}{a}\right)^{(m+1)/2}}{1 - (r_w^2/a^2)^{(m+1)/2}} \cos \frac{m+1}{2} \theta \\
& - \sum_{m=2, 4, \dots}^N A_{Nm} \frac{m}{2} \frac{1}{r} \frac{\left(\frac{r}{a}\right)^{m/2} - \left(\frac{r_w}{a}\right)^{m/2}}{1 - (r_w^2/a^2)^{m/2}} \sin \frac{m}{2} \theta \quad (241)
\end{aligned}$$

We can now put values of $\partial S/\partial r$, $(1/r)\partial S/\partial\theta$, $\{n\}$, $\partial\phi/\partial r$, and $(1/r)\partial\phi/\partial\theta$ found from equations 236, 237, 238, 240, and 241 in equation 239 and find an expression (which we do not need to write) for $\partial\phi/\partial n$ from which by multiplication by (ϕ_o/a) we immediately find the dimensionless normal derivative $\partial(\phi/\phi_o)/\partial(n/a)$ in terms of the A_{Nm} . The coefficients of the A_{Nm} in this expression are the $u_m(\theta)$ for the impermeable boundary. We now know (in view of equations 227 and 228) the $u_m(\theta)$ for the complete range $0 \leq \theta < 360^\circ$; and the procedure (in which we must also use equations 224 and 225 to get the A_{Nm} of equation 226 is now straightforward as described in Powers et al. (1967). With the A_{Nm} now known, we can substitute these A_{Nm} in equation 223 and determine the hydraulic head throughout the aquifer. We can then also determine the stream function ψ and the well discharge Q .

Figure 18 shows the flow net for the Ames aquifer when the aquifer is pumped by city well number 9. The drawdown at the well is 9.2 ft, and the water in the well is standing at 865.3 ft (above sea level). Equipotentials are shown at 2-ft intervals. The streamlines shown are such that, between each adjacent pair, there flows 0.1 of the total well discharge. The figure shows that almost all the well discharge flows to the well from the north. This is a fortunate circumstance because a pollution source

Figure 18. Flow net for the Ames aquifer, when the aquifer is pumped by city well number 9. Phenol source is shown southeast of the well.



of phenol is present in the aquifer southeast from city well number 9. If this well were the only pumped well in the aquifer and if the drawdown would not exceed the present 9.2 ft, the quality of the well water would not be much affected by the phenol source. When city wells numbers 9, 10, and 12 (Figure 16) are pumped all together, however, such that the drawdown at city well number 9 exceeds the present 9.2 ft considerably, then the quality of the water of well number 9 might well be and is found influenced by the polluting source. When city well number 9 is pumped alone, as in Figure 18, we see that the phenol source is in a relatively stagnant area between the streamlines labeled AB and CD. The reason the area is stagnant is that 1/10 of the total flow enters the well from the area to the south of the lines AB and CD, and this is a large area compared with the area between, and north of, the lines AB and CD where 9/10 of the flow originates for the well.

If in the future, additional wells have to be drilled into the aquifer it seems desirable, both from a qualitative and quantitative standpoint, to locate such new wells north of the present ones (city wells number 9, 10, and 12).

A check on our method is readily made. When the test pumping of city well number 9 was actually performed by Ver Steeg (1968), there were 1280 gpm pumped from the well, with a drawdown of 9.2 ft. For this amount of drawdown, our theoretical analysis yields 1163 gpm, less than 10-percent difference, for engineering purposes acceptable.

Two Wells of Equal Strength in a Circular Aquifer

In practice, most aquifers are usually not pumped by only one well, but by more than one well. In what follows we will consider some horizontal, confined aquifers of finite extent that are pumped by two wells. We will indicate how the theory, that we next will derive, can be extended to a confined aquifer, pumped by more than two wells. Although we could consider aquifers that have a varying head distribution along the outer boundary we will not do so. We will only consider a constant head distribution along the outer boundary. Also we will not consider impervious parts on the outer boundary. From the well flow theory that we have discussed so far, plus what we will discuss in the next sections, it should become clear to the reader (provided we have our theory well presented) that we could easily include a varying head distribution along the outer boundary, or could include the condition that part of the boundary be impervious.

Except for the Ames aquifer, where we had a varying head distribution along the outer boundary, we have taken the dimensionless hydraulic head $\phi/\Delta\phi$ along the outer boundary of the aquifers that we have considered as 1, that is $\phi/\Delta\phi = 1$, at the outer boundary where the reference level for ϕ is the level of the water in the well and $\Delta\phi$ is the difference in head between the well and the outer boundary.

The condition $\phi/\Delta\phi = 1$ (or $\phi = \Delta\phi$) along the outer boundary, was always one of the boundary conditions that we had to satisfy. For the other boundary conditions we now pay some attention to Figures 3 and 8. Figures 3 and 8 both show a general flow pattern; the aquifer has no symmetry lines. Besides the boundary condition $\phi/\Delta\phi = 1$ at the outer boundary, we only need

to specify one other boundary condition. We have done so, by specifying ϕ at the well, and we always have chosen, $\phi = 0$, at the well. We were then able to find a solution, ϕ to Laplace's equation that satisfied both the boundary conditions $\phi/\Delta\phi = 1$ at the outer boundary, and $\phi = 0$, at the well. From the expression for ϕ we were then able to calculate the well discharge Q , for the drawdown, $\Delta\phi$ that we considered.

We could have stated the boundary conditions of Figures 3 and 8 differently. Rather than specifying the drawdown at the well, we could have specified the well discharge Q . With the solution for ϕ thus obtained, we then could have determined the amount of drawdown $\Delta\phi$ at the well, caused by the discharge Q . This drawdown $\Delta\phi$ would be with respect to the level of the constant water head operating over the outer boundary of the aquifer.

In deriving the theory for a horizontal, confined aquifer pumped by 2 wells (or more), we could have stated the boundary conditions in the same way as we did for aquifers pumped by only 1 well. That is, we could have stated the boundary conditions for a two-well system in terms of ϕ at the boundaries, that is at the outer boundary and at the wells. From the solution for ϕ we then could have obtained the well discharge for each well. Rather than doing so, we decided to state each well discharge as a boundary condition instead, and then calculate later on the drawdown at each well, due to its own discharge and due to interference of the other well(s).

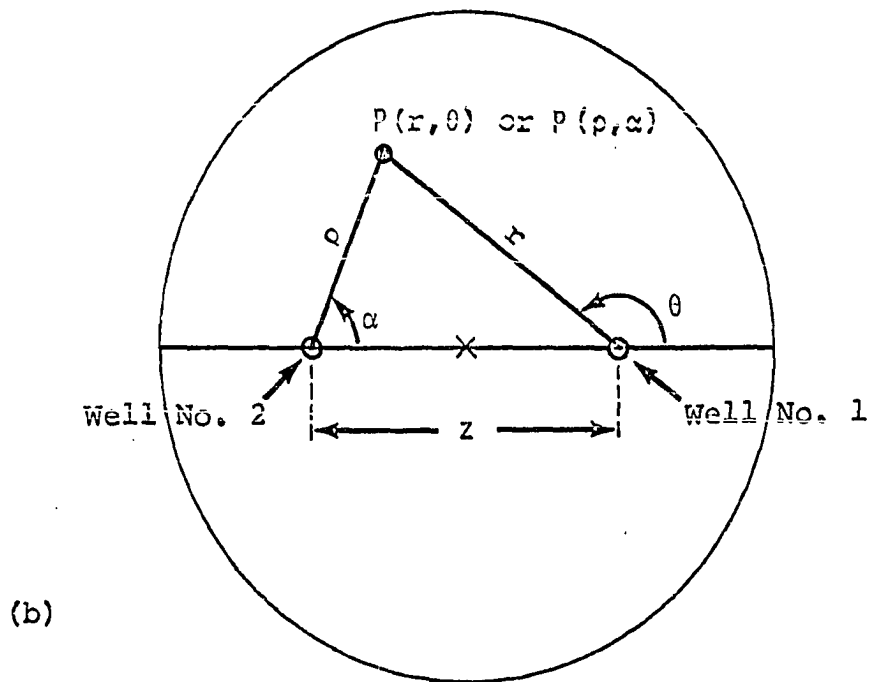
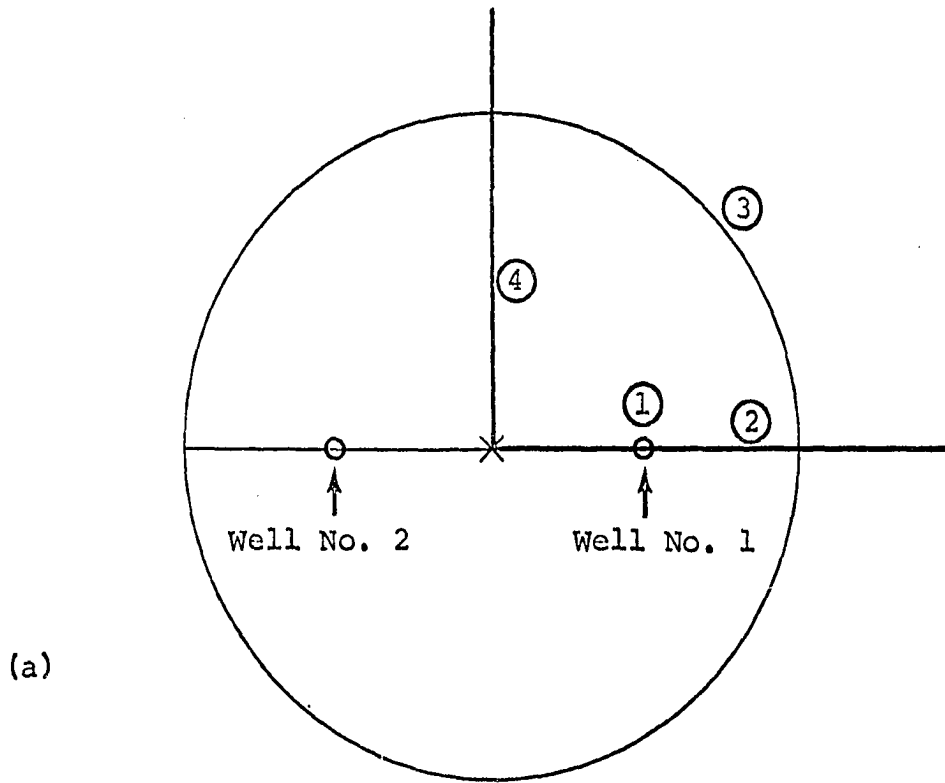
We will start our analysis of a multiple well system with a circular aquifer. We will choose two wells, initially both equal and symmetrically placed in the aquifer. We can solve the problem in two different ways.

- a) by reducing the problem to a flow system having only one pumping well, so that our previously discussed theory applies. We state the boundary conditions in terms of ϕ at the outer boundary, and ϕ at the well.
- b) by developing a new theory. We will still state the boundary condition at the outer boundary in terms of ϕ (we take, as before, $\phi/\Delta\phi = 1$), but we prescribe at the wells the flux Q , rather than the hydraulic head ϕ .

The circular aquifer of Figure 19 is chosen to start with. As usual, the radius a of the aquifer is taken as unity, that is $a = 1$. In the analysis 2 wells are taken into consideration. Both wells are located on a line through the center of the circular aquifer, and each well is located at a distance of $1/2 a$ from the center. Hence, as can be seen from Figure 19a or 19b there is symmetry with respect to the center of the circle. The distance between the 2 wells is called z , and we have here $z = a = 1$. Both wells have a radius $r_w = 0.0025$, and from each well a same amount Q is pumped. Let us denote the discharge from the well on the right side as Q_R and the discharge of the well on the left side as Q_L .

The problem illustrated by Figure 19 is solved in 2 different ways. We first look at Figure 19a. We realize, that because of symmetry, we need only to consider one quadrant of the circle. We can consider the upper right quadrant. We now have reduced the problem of having 2 wells in the flow region to a problem of only 1 well in the flow region. So far in this thesis we have dealt exclusively with flow regions having only one pumped well, and we know how to solve this problem. The boundary conditions of this problem, in which we take the static water level in the

Figure 19. Wells of equal strength in a circular aquifer. (a) Quadrant representation. (b) Upper half plane representations.



well as reference level for the hydraulic head ϕ , and in which we take a head difference $\Delta\phi$ across the aquifer, and in which further we take the center of the well (well no. 1) as origin of a polar coordinate system (r, θ) , are indicated with encircled numerals in Figure 19a. They are

B.C. 1 $\phi = 0$ at the well

B.C. 2 $\partial\phi/\partial\theta = 0$ $\theta = 0$ or $\theta = \pi$

In B.C. 2 remember the apex of the angle θ is at the center of well no. 1.

B.C. 3 $\phi/\Delta\phi = 1$ along the outer curved boundary

B.C. 4 $\partial\phi/\partial n = 0$ at left boundary

Boundary condition 4 indicates that there is no flow across the symmetry line indicated by the encircled number 4, in a direction perpendicular to that line. We will indicate now how this problem can be solved. As function for the hydraulic head ϕ , we propose the same equation (equation 12) that was used for Figure 3 which we will rewrite here:

$$\frac{\phi}{\Delta\phi} = A_{NO} \frac{\ln r/r_w}{\ln a/r_w} + \sum_{m=1}^N A_{Nm} \left\{ \frac{(r/a)^m - [r_w^2/(ar)]^m}{1 - (r_w^2/a^2)^m} \right\} \cos m\theta \quad (12)$$

As one may check, equation 12 satisfies boundary conditions 1 and 2 of Figure 19a, regardless of the values of the coefficients A_{Nm} . We have to use the modified Gram-Schmidt method again to satisfy boundary conditions 3 and 4. Boundary no. 3 is not much of a problem, since, as in equation 13, the step function part of $u_m(\theta)$ can be determined quite easily for the range of boundary no. 3. We have as in equation 13, for the θ region of boundary 3, the expressions

$$u_0 = \frac{\ln(R/r_w)}{\ln(a/r_w)}, \quad \text{and}$$

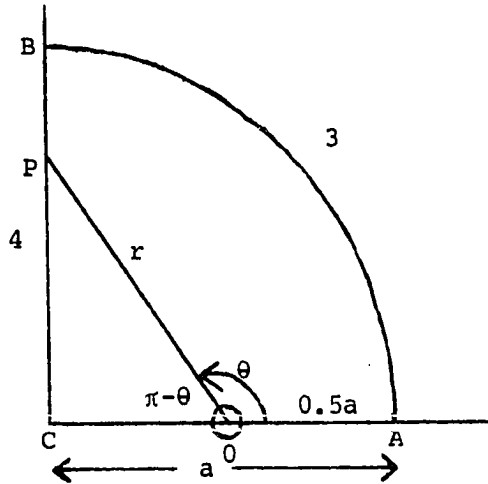
$$u_m = \left\{ \frac{(R/a)^m - [r_w^2/(aR)]^m}{1 - (r_w^2/a^2)^m} \right\} \cos m\theta \quad (13)$$

In this equation we can calculate R from the equation of the circle, with respect to the well center.

We will now look at boundary no. 4 where the step function part of $u_m(\theta)$ for that boundary is not so easily obtained. We rewrite $\partial\phi/\partial n$ as

$$\frac{\partial\phi}{\partial n} = \nabla\phi \cdot n$$

where n is the unit normal to boundary no. 4



We have to find the equation of the line segment BC , in terms of r and θ .

We note that

$$\cos(\pi - \theta) = OC/OP = OC/r \quad (242)$$

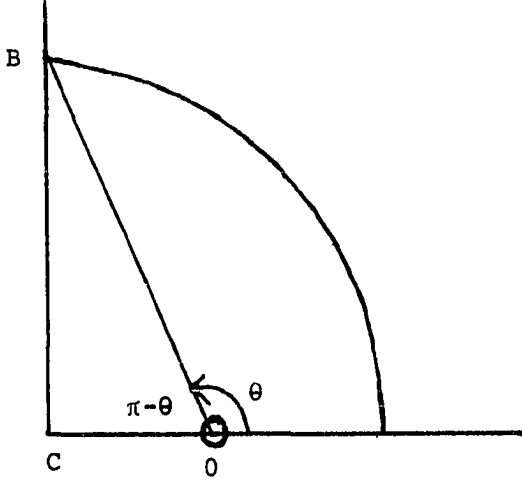
in which $OC = \text{constant} = 0.5a$

Hence, equation 242 can be written as

$$-\cos \theta = 0.5a/r,$$

$$\text{or } r = -0.5/\cos \theta \quad (243)$$

In equation 243 the angle θ has only a limited range. The lower range of θ can be found as follows:



$$OC = 0.5a$$

$$BC = a$$

$$\tan (\pi - \theta) = BC/OC = a/0.5a = 2, \text{ or}$$

$$\tan (\pi - \theta) = 2, \text{ or}$$

$$\pi - \theta = 63^{\circ}26'$$

$$\theta = \pi - 63^{\circ}26' = 116^{\circ}34' \quad (244)$$

Hence equation 243 rewritten, gives

$$r = -0.5a/\cos \theta, \quad 116^{\circ}34' \leq \theta \leq 180^{\circ} \quad (245)$$

We can rewrite equation 245 as

$$S = r + 0.5a/\cos \theta = 0 \quad (246)$$

To find the normal to S, we take

$$\nabla S = \partial S/\partial r, \quad 1/r \partial S/\partial \theta \quad (247)$$

$$\partial S/\partial r = 1 \quad (248)$$

$$\partial S/\partial \theta = \frac{(-0.5a)(-\sin \theta)}{\cos^2 \theta} = \frac{0.5a \sin \theta}{\cos^2 \theta} \quad (249)$$

From equation 249 we find $1/r \partial S/\partial \theta$ as

$$\frac{1}{r} \frac{\partial S}{\partial \theta} = \frac{0.5a \sin \theta}{r \cos^2 \theta} \quad (250)$$

Hence equation 247 can be written as:

$$\nabla S = \left\{ 1, \frac{0.5a \sin \theta}{r \cos^2 \theta} \right\} \quad (251)$$

To find the unit normal, we have to divide the two members of equation 251 by

$$\sqrt{(1)^2 + \left(\frac{0.5a \sin \theta}{r \cos^2 \theta} \right)^2} \quad (252)$$

Thus we can write the unit normal to boundary 4 (from expression 251 and 252) as,

$$\bar{n} = \left\{ \frac{1}{(252)}, \frac{\frac{0.5 a \sin \theta}{r \cos^2 \theta}}{(252)} \right\} \quad (253)$$

To find $\partial\phi/\partial n$ of B.C. 4 we now must find $\partial\phi/\partial r$ and $1/r \partial\phi/\partial\theta$, using equation 12.

$$\begin{aligned} \frac{\partial(\phi/\Delta\phi)}{\partial r} &= \frac{A_{NO}}{r \ln(a/r_w)} + \sum_{m=1}^N A_{Nm} \left\{ \frac{\frac{r^{m-1}}{a^m} + m \frac{r_w^{2m}}{a^m} \frac{1}{r^{m+1}}}{1 - \left(\frac{r_w}{a}\right)^{2m}} \right\} \cos m\theta \\ &= \frac{A_{NO}}{r \ln(a/r_w)} + \sum_{m=1}^N A_{Nm} \frac{m}{r} \left\{ \frac{\left(\frac{r}{a}\right)^m + \left(\frac{r_w}{ar}\right)^m}{1 - \left(\frac{r_w}{a}\right)^{2m}} \right\} \cos m\theta \end{aligned} \quad (254)$$

From equation 12 we now find $1/r \partial\phi/\partial\theta$ as,

$$\frac{1}{r} \frac{\partial(\phi/\Delta\phi)}{\partial \theta} = 0 - \sum_{m=1}^N A_{Nm} \frac{m}{r} \left\{ \frac{\left(\frac{r}{a}\right)^m - \left(\frac{r_w}{ar}\right)^m}{1 - \left(\frac{r_w}{a}\right)^{2m}} \right\} \sin m\theta \quad (255)$$

Thus $\partial\phi/\partial n$ of B.C. 4 is given by

$$\frac{\partial(\phi/\Delta\phi)}{\partial n} = \frac{1}{(252)} \times (254) + \frac{\frac{0.5 a \sin \theta}{r \cos^2 \theta}}{(252)} \times (255) \quad (256)$$

We will not write out equation 256 in its full length. From equation 256 we can get then the $u_m(\theta)$ over the range of boundary no. 4. We will not further discuss the details of the calculations, but we just give the well discharge of Figure 19a, for a unit hydraulic gradient across the aquifer. For the conditions of Figure 19a we find the discharge Q , using equation 7a, as

$$Q = 1.060 Kh\Delta\phi \quad (257)$$

where we remember h is the thickness of the aquifer.

Alternate Solution for Two Equal Wells

We will now try to solve the same problem in a different way, referring to Figure 19b. In the analysis, both wells will be taken into consideration. Taking both wells into consideration will prepare us for solving the problems when the two wells do not have equal discharge and equal drawdown. Instead of considering one quadrant of the flow region, we will now consider the upper half of the circle. We now state the boundary conditions. As mentioned before we rather specify the flux Q at each well than the hydraulic head ϕ . We want to have a discharge Q from each well in Figure 19b that is equal to the discharge Q of the well in Figure 19a. Thus we want: $Q_R = Q_L = 1.060 Kh\Delta\phi$. We choose the origin of coordinates of an r, θ system at the center of well no. 1 (at right) and write down the boundary conditions for Figure 19b as

$$\text{B.C. 1: } Q_R = Q_L = 1.060 Kh\Delta\phi$$

B.C. 2: $\phi/\Delta\phi = 1$, at the outer boundary

B.C. 3: $\partial\phi/\partial\theta = 0$ for $\theta = 0$ or $\theta = \pi$

We are looking again for a solution to Laplace's equation in polar coordinates, subject to the boundary conditions just specified. The hydraulic head function ϕ that we will find should be such that it gives a drawdown equal to $\Delta\phi$ at either well, because this is the drawdown we have in the solution for Figure 19a, where we found $Q = 1.060 Kh\Delta\phi$.

We propose the following expression for ϕ in which the constants F and G will be discussed in a moment and other symbols, ρ and α , are in Figure 19b:

$$\begin{aligned} \frac{\phi}{\Delta\phi} = & A_{NO} \left\{ F \frac{\ln(r/r_w)}{\ln(a/r_w)} + G \frac{\ln(\rho/r_w)}{\ln(a/r_w)} \right\} \\ & + \sum_{m=1}^N A_{Nm} \left\{ F \frac{(r/a)^m - [r_w^2/(ar)]^m}{1 - (r_w^2/a^2)^m} \cos m\theta \right. \\ & \left. + G \frac{(\rho/a)^m - [r_w^2/(a\rho)]^m}{1 - (r_w^2/a^2)^m} \cos m\alpha \right\} \end{aligned} \quad (258)$$

In equation 258 we need to consider only one well radius, since both our wells have the same radius r_w . It may be observed from Figure 19b that we can express ρ and α of equation 258 in terms of r and θ if we wish. We note the expressions

$$\rho = (z^2 + r^2 + 2 zr \cos \theta)^{1/2}, \text{ and} \quad (259)$$

$$\alpha = \sin^{-1} \left(\frac{r \sin \theta}{\rho} \right) \quad (260)$$

If we now calculate the discharge Q_R of the well on the right, we find by use of the equation

$$Q_R = Kh \int_0^{2\pi} \frac{\partial \phi}{\partial r} r d\theta, \quad r = r_w,$$

the following expression for Q_R

$$Q_R = \frac{2FK h A_{NO} \pi \Delta\phi}{\ln(a/r_w)} \quad (261)$$

For Q_L we find, by use of the equation

$$Q_L = Kh \int_0^{2\pi} \frac{\partial \phi}{\partial \rho} \rho d\alpha$$

the following expression for Q_L

$$Q_L = \frac{2 GK h A_{NO} \pi \Delta\phi}{\ln(a/r_w)} \quad (262)$$

In equations 261 and 262 we assume Q_R and Q_L are given so that if we know F and G , the drawdown $\Delta\phi$, if it is considered unknown can be determined.

As stated in the boundary conditions, we want to have that $Q_R = Q_L = 1.060 Kh\Delta\phi$. If we compare equations 261 and 262 we see that the only difference in the equations is that in equation 261 the symbol F occurs and in equation 262, the symbol G . We conclude that the unknown constants F and G must be equal to each other. We will now see how we have to determine the constants F and G , such that $Q_R = Q_L = 1.060 Kh\Delta\phi$.

We compare equations 258 and 13 and see that in equation 258 we find $u_0(\theta)$ and $u_m(\theta)$ as

$$u_0(\theta) = F \frac{\ln r/r_w}{\ln a/r_w} + G \frac{\ln \rho/r_w}{\ln a/r_w}, \quad \text{and} \quad (263)$$

$$\begin{aligned}
u_m(\theta) = F \frac{(r/a)^m - [r_w^2/(ar)]^m}{1 - (r_w^2/a^2)^m} \cos m\theta \\
+ G \frac{(\rho/a)^m - [r_w^2/(a\rho)]^m}{1 - (r_w^2/a^2)^m} \cos m\alpha,
\end{aligned} \tag{263a}$$

in which equations we can express r , ρ , and α in terms of θ . In order to use the Gram-Schmidt method and Table 2 we need to assign a certain value to F and G . As starting value we may take $F = G = 1$. When we do so we can satisfy the boundary condition $\phi/\Delta\phi = 1$ at the outer boundary all right, but for Q_R and Q_L we find

$$Q_R = Q_L = 0.522 \text{ Kh}\Delta\phi \tag{264}$$

We see that this discharge is much smaller than we wish to have it.

We want $Q_R = Q_L = 1.060 \text{ Kh}\Delta\phi$ and we find $Q_R = Q_L = 0.522 \text{ Kh}\Delta\phi$.

To get a higher value for Q_R (and Q_L) we take a higher value for F and G in equation 258. We take $F = G = 2$. We now calculate the discharge of each well again, and find

$$Q_R = Q_L = 1.044 \text{ Kh}\Delta\phi \tag{265}$$

From equations 261 and 262 it follows that Q_R (and Q_L) go up linearly with F and G . We then determine the next values for F and G that we should put into equation 258 as $F = G = 2.032$. We then determine the discharge of each well again, and indeed we find

$$Q_R = Q_L = 1.060 \text{ Kh}\Delta\phi \tag{266}$$

We then determine, by use of equation 258, the hydraulic head ϕ at the well at the right (well no. 1), and we find, for $N = 10$ in equation 258 that ϕ at the well at the right is equal to

$$\phi = 0.0009 \Delta\phi \quad (267)$$

This is very near to $\phi = 0$, as we should have found.

With the coefficients F and G determined, and also with the A_{Nm} coefficients determined, we can by use of equation 258 and the Cauchy-Riemann relations, as in Van der Ploeg et al. (1971) determine the stream-function ψ . We find for ψ the expression

$$\begin{aligned} \frac{\psi}{K\Delta\phi} = & A_{NO} \left(F \frac{\theta}{\ln(a/r_w)} + G \frac{\alpha}{\ln(a/r_w)} \right) \\ & + \sum_{m=1}^N A_{Nm} \left\{ F \frac{(r/a)^m + [r_w^2/(ar)]^m}{1 - (r_w^2/a^2)^m} \cos m\theta \right. \\ & \left. + G \frac{(\rho/a)^m + [r_w^2/(\rho a)]^m}{1 - (r_w^2/a^2)^m} \cos m\alpha \right\} \end{aligned} \quad (268)$$

With ϕ (given by equation 258) and ψ (given by equation 268) we then can prepare a flow net which we present as Figure 20. In Figure 20 the streamlines are drawn such that between each adjacent pair of streamlines there flows 0.1 of the combined discharge of both wells. Thus between each adjacent pairs of streamlines there flows 0.1 ($Q_R + Q_L$).

We thus have solved the problem depicted by Figure 19b in two different ways. Both solutions give the same result. We will now attack a problem like the one of Figure 19b, except that now we let the wells be of unequal strength.

Two Wells of Unequal Strength in a Circular Aquifer

Figure 21 shows a flow net for a circular horizontal confined aquifer pumped by two wells. The wells are located such that they are at equal distance from the center of the circle. The radius a , of the circular

Figure 20. Flow net for two equal wells in a circular aquifer.

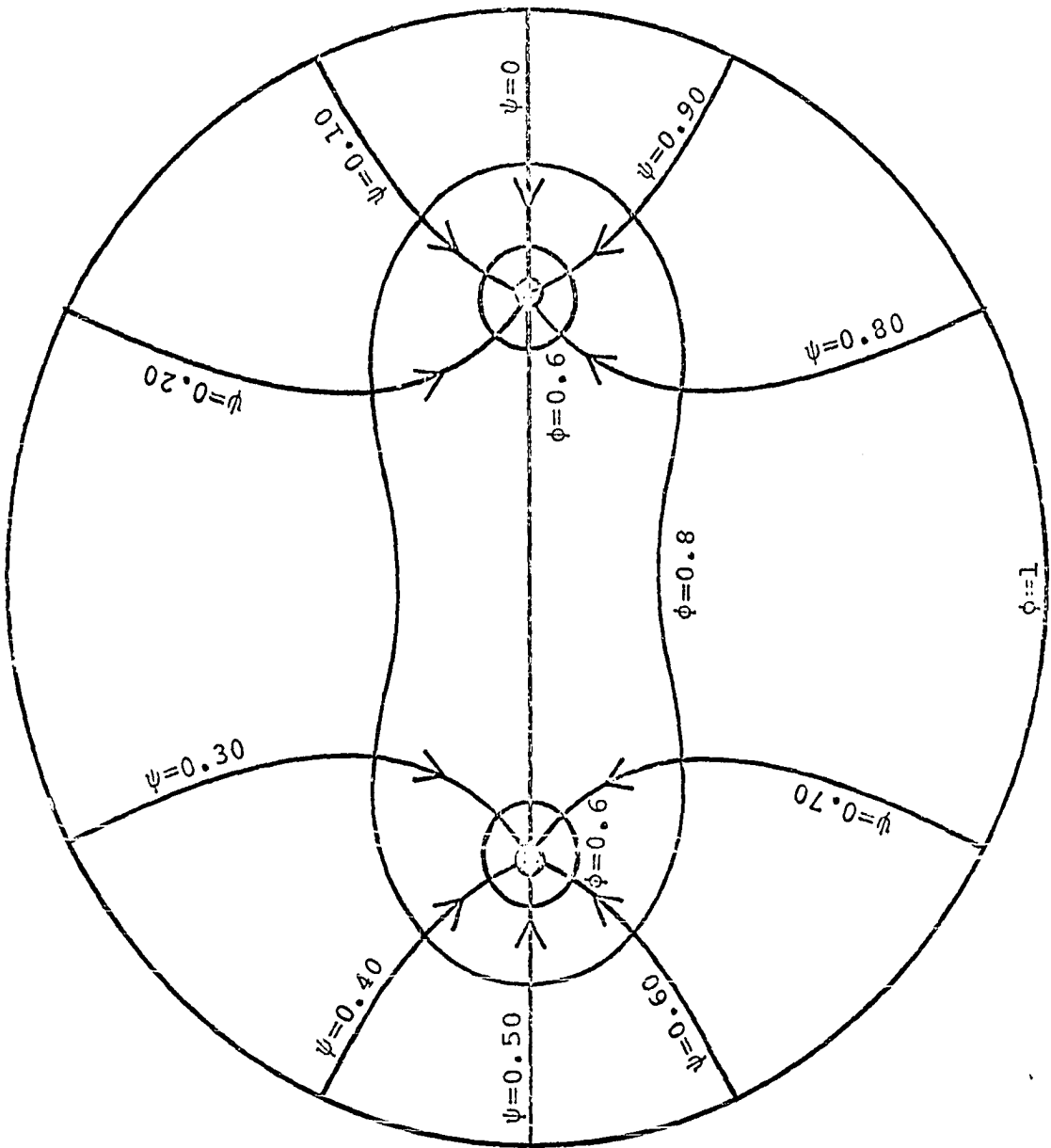
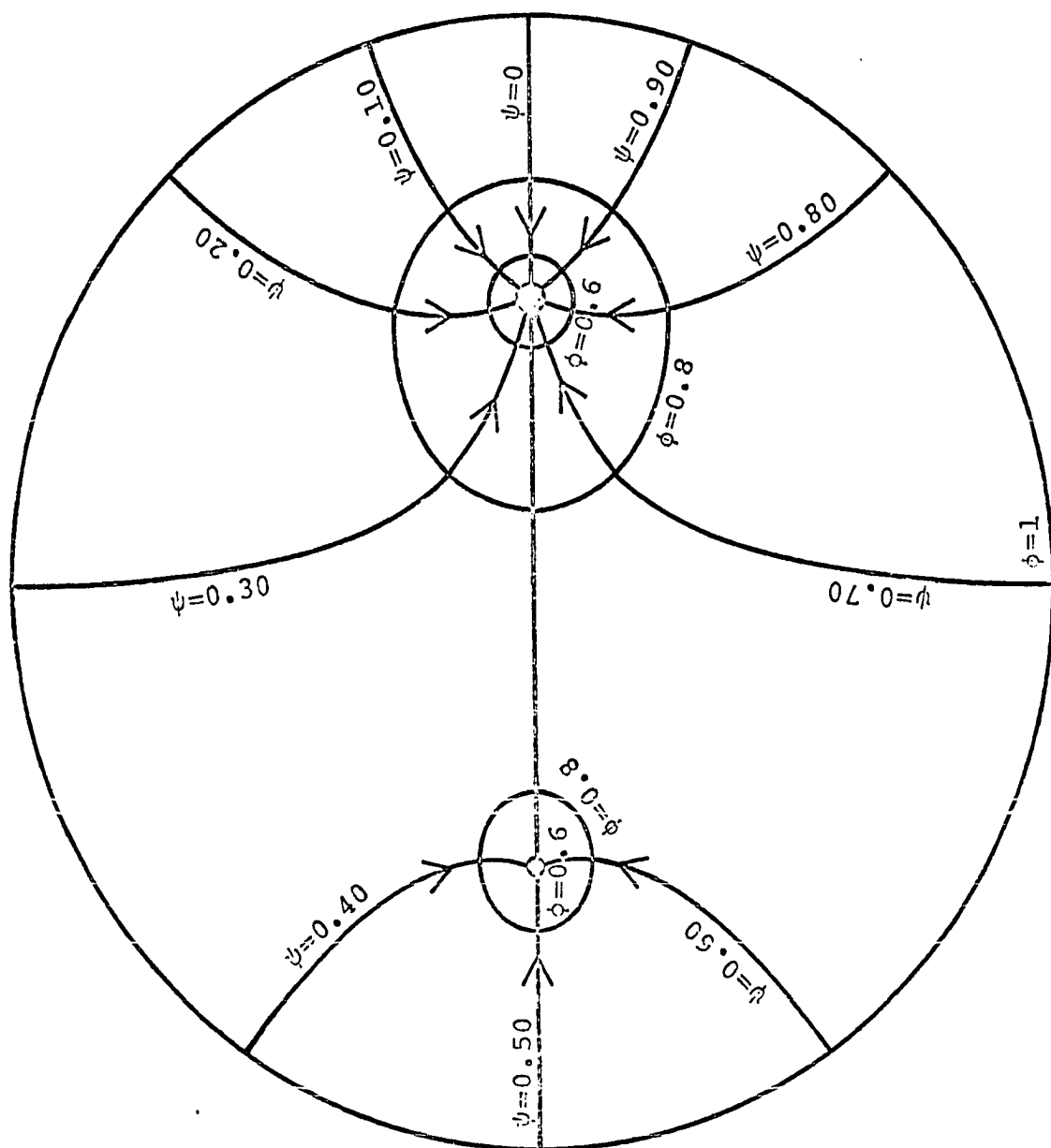


Figure 21. Flow net for two wells with the same radius but with well at the right (top) having twice the discharge of the one at the left (bottom).



aquifer is taken as unity again, hence $a = 1$. Both wells have the same radius, $r_w = 0.0025$. The discharge from the well on the right hand we take twice as large as the discharge from the well on the left hand, and additionally we take, to compare drawdowns, the discharge of the well at the right, Q_R to be the same as the discharge of the well at the right of Figure 20. We thus require the relations

$$2Q_L = Q_R = 1.060 Kh\Delta\phi$$

Again as before we take as boundary condition at the outer boundary the condition $\phi/\Delta\phi = 1$. In this expression $\Delta\phi$ now is taken to be the head difference between the outer boundary and the well with the largest drawdown.

As expression for the dimensionless hydraulic head is now selected as

$$\begin{aligned} \frac{\phi}{(\Delta\phi)_R} = & A_{NO} \left\{ F \frac{\ln(r/r_w)}{\ln(a/r_w)} + G \frac{\ln(\rho/r_w)}{\ln(a/r_w)} \right\} \\ & + \sum_{m=1}^N A_{Nm} \left\{ F \frac{(r/a)^m - r_w^2/(ar)^m}{1 - (r_w^2/a^2)^m} \cos m\theta \right. \\ & \left. + G \frac{(\rho/a)^m - [r_w^2/(a\rho)]^m}{1 - (r_w^2/a^2)^m} \cos m\alpha \right\} \end{aligned} \quad (269)$$

where F and G will not have the same values as in equation 258.

If we want to have an expression for the well discharge we can use equation 7a. We find for the well at the right

$$Q_R = \frac{2FK h A_{NO} \pi (\Delta\phi)_R}{\ln(a/r_w)} \quad (270)$$

and for the discharge of the well at the left we find

$$Q_L = \frac{2GK h A_{NO} \pi (\Delta\phi)_R}{\ln(a/r_w)} , \quad (271)$$

where, in equations 270 and 271, we have $(\Delta\phi)_R$ as the head difference between the outer boundary and the well at the right. Although Q_R in equation 270 is equal to Q of the problem of two equal wells, the drawdown $(\Delta\phi)_R$ is not equal to the drawdown $\Delta\phi$ of the two equal well problem.

To get Q_L equal to half the value of Q_R we see that we need to have

$$G = 1/2 F \quad (272)$$

After some trials we found that for $F = 1.517$ (and hence $G = 1/2 F = 0.7535$) we get a discharge Q_R for the well at the right, as

$$Q_R = 1.062 Kh\Delta\phi \quad (273)$$

We decided that this discharge value was close enough to the value of Figure 20 ($Q = 1.060 Kh\Delta\phi$) and we then calculated the flow net of Figure 21.

After we had determined the constants F and G , we calculated the hydraulic head ϕ at the right well, by applying equation 269 to a point on the well at the right where we had $\theta = \pi$ and $\alpha = 0$ and $\rho = a = 1$ approximately. We found

$$\phi \text{ (at the well at the right)} = 0.017 \Delta\phi \quad (274)$$

In equation 269 we took N equal to 10. If we had chosen N large enough we should have obtained a better approximation to $\phi = 0$, in equation 274.

Now that we have discussed Figures 20 and 21 we will discuss the most general case that we have solved, that is an aquifer of arbitrary shape, pumped by two wells of different radius, and of different discharge.

Two Wells of Unequal Strength in an Aquifer of Arbitrary Shape

Figure 22 shows a flow net for an irregularly-shaped horizontal confined aquifer. The aquifer is pumped by two wells. The radius of the well at the right, which we denote by r_{w1} is given by $r_{w1} = 0.0025$. The radius of the well at the left, r_{w2} is equal to $r_{w2} = 0.01$. The horizontal distance a from the well at the right to the aquifer boundary (at the right of the well) is taken as unity, that is $a = 1$. The distance z between the two wells is such that $z = 0.3$. The aquifer of Figure 22 is the same as the aquifer of Figure 8 except for an additional well. For the single well of Figure 8 we computed the discharge Q (a result we have not given before) as

$$Q = 1.102 \text{ Kh}\Delta\phi$$

For Figure 22 we decided that we wished for the well at the right (well no. 1) to have the same amount of discharge as the one well of Figure 8 had. That is, for Figure 22 we take

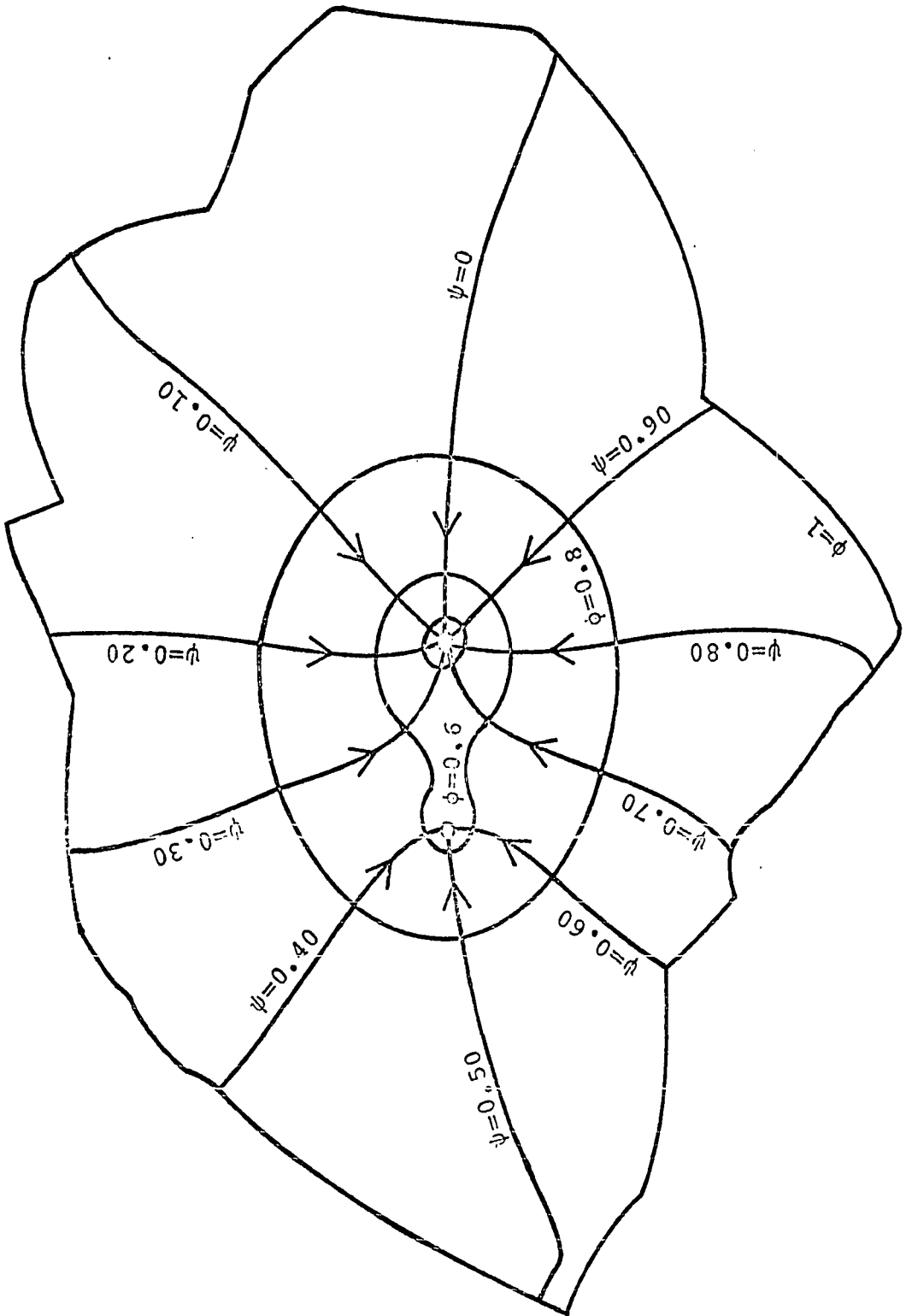
$$Q_R = 1.102 \text{ Kh}\Delta\phi \quad (274a)$$

where $\Delta\phi$ is the $\Delta\phi$ of Figure 8. For Figure 22 we also decided to take the discharge Q_L of the well at the left to be such that $Q_L = 1/2 Q_R$

We will give now the expressions for the hydraulic head function ϕ and the streamfunction ψ that we used to prepare the flow net of Figure 22. After we give these expressions we will explain how we chose the constants F and G in the equations for ϕ so as to satisfy the discharge relations $Q_L = 1/2 Q_R$ ($= Q$ of Figure 8).

For the hydraulic head ϕ for Figure 22 we propose the expression (in which $(\Delta\phi)_R$ is the unknown drawdown from the level of the outside aquifer boundary at the right well).

Figure 22. Flow net for an irregularly shaped confined horizontal aquifer pumped by two wells of unequal strength and unequal radius.



$$\begin{aligned}
\frac{\phi}{(\Delta\phi)_R} = & A_{NO} \left\{ F \frac{\ln(r/r_{w1})}{\ln(a/r_{w1})} + G \frac{\ln(\rho/r_{w2})}{\ln a/r_{w2}} \right\} \\
& + \sum_{m=2,4}^N A_{Nm} \left\{ F \frac{(r/a)^{m/2} - [r_{w1}^2/(ar)]^{m/2}}{1 - (r_{w1}^2/a^2)^{m/2}} \cos \frac{m}{2} \theta \right. \\
& + G \frac{(\rho/a)^{m/2} - [r_{w2}^2/(a\rho)]^{m/2}}{1 - (r_{w2}^2/a^2)^{m/2}} \cos \frac{m}{2} \alpha \} \\
& + \sum_{m=1,3}^N A_{Nm} \left\{ F \frac{(r/a)^{\frac{m+1}{2}} - [r_{w1}^2/(ar)]^{\frac{m+1}{2}}}{1 - (r_{w1}^2/a^2)^{\frac{m+1}{2}}} \sin \frac{m+1}{2} \theta \right. \\
& + G \frac{(\rho/a)^{\frac{m+1}{2}} - [r_{w2}^2/(a\rho)]^{\frac{m+1}{2}}}{1 - (r_{w2}^2/a^2)^{\frac{m+1}{2}}} \sin \frac{m+1}{2} \alpha \}, \tag{275}
\end{aligned}$$

The streamfunction ψ corresponding to ϕ (or $K\phi$) of equation 275 is found as

$$\begin{aligned}
\frac{\psi}{(\Delta\phi)_R} = & K A_{NO} \left(F \frac{\theta}{\ln(a/r_{w1})} + G \frac{\alpha}{\ln(a/r_{w2})} \right) \\
& + \sum_{m=2,4}^N A_{Nm} \left\{ F \frac{(r/a)^{m/2} + [r_{w1}^2/(ar)]^{m/2}}{1 - (r_{w1}^2/a^2)^{m/2}} \sin m/2 \theta \right.
\end{aligned}$$

$$\begin{aligned}
& + G \frac{(\rho/a)^{m/2} + [r_{w2}^2/(a\rho)]^{m/2}}{1 - (r_{w2}^2/a^2)^{m/2}} \sin \frac{m}{2} \alpha \\
& - \sum_{m=1,3,\dots}^N A_{Nm} \left\{ F \frac{(r/a)^{\frac{m+1}{2}} + [r_{w1}^2/(ar)]^{\frac{m+1}{2}}}{1 - (r_{w1}^2/a^2)^{\frac{m+1}{2}}} \cos \frac{m+1}{2} \theta \right. \\
& \left. + G \frac{(\rho/a)^{\frac{m+1}{2}} + [r_{w2}^2/(a\rho)]^{\frac{m+1}{2}}}{1 - (r_{w2}^2/a^2)^{\frac{m+1}{2}}} \cos \frac{m+1}{2} \alpha \right\}
\end{aligned} \tag{276}$$

If we calculate the well discharge Q_R and Q_L by use of equation 7a and equation 267, we find

$$Q_R = \frac{2FK h A_{NO} (\Delta\phi)_R \pi}{\ln(a/r_{w1})}, \tag{276a}$$

and for Q_L we find

$$Q_L = \frac{2GK h A_{NO} (\Delta\phi)_R \pi}{\ln(a/r_{w2})}$$

To get $Q_L = 1/2 Q_R$ we need to take the constant G (seen from $Q_L = 1/2 Q_R$ and the last two displayed equations) as

$$G = 1/2 \frac{\ln(a/r_{w2})}{\ln(a/r_{w1})} F \tag{277}$$

To get the A_{Nm} of equation 276 we used a trial and error method. We assumed a starting value, for F which we took as $F = 1$ and from this found a value of G from equation 277. We then put the values of $F = 1$ and the determined value of G in equation 275 and determined a set of A_{Nm} . This set included a value of A_{NO} which for our case was $A_{15,0}$ because we used $N = 15$. With this value, $A_{15,0}$ and the starting value $F = 1$ we computed from equation 276a a value of Q_R . We wanted this value of Q_R to be the

same as given by equation 274a. The value $F = 1$ did not give equation 274a, but by trying different values of F , and proceeding as above, we found the values of F and G that give Q_R as in equation 274a. The flow net that resulted is given in Figure 22.

We do not present flow nets for horizontal confined aquifers of finite extent that are pumped by more than 2 wells. For an arbitrarily shaped aquifer, as that of Figure 8 or Figure 22 it is obvious now how our method can be extended to more than two wells. For an aquifer pumped by three wells we can use as expression for the hydraulic head ϕ an equation like equation 275, but we would have to add to the F -component in (r, θ) and the G -component in (ρ, α) , a third component, say a H -component in (σ, β) , where σ, β would be the polar coordinates of a point P in the flow region, with respect to the center of the third well. Theoretically, we should be able to present a flow net for the Ames aquifer, that is pumped by the three city wells, no. 9, 10, and 12. By continuing the procedure for 2 and 3 wells, it appears that our method of solution can be extended to aquifers pumped by more than 3 wells.

SUMMARY

We have solved analytically a number of steady-state boundary value problems. We have dealt with saturated flow in confined horizontal aquifers of finite extent, pumped by one or more wells. At the outer boundary of an aquifer we assumed the hydraulic head function to be known and also the location of impervious parts on the outer boundary condition that we considered we had to find a solution to Laplace's equation and the appropriate boundary conditions. We first reviewed some flow nets for elliptically shaped aquifers, pumped by one well, and the corresponding expressions for the hydraulic head ϕ and the stream function ψ . We then showed that the same expressions for ϕ and ψ of an elliptically shaped aquifer, pumped by one well, for an arbitrary location of the well, can also be used for an irregularly shaped aquifer, pumped by one well. Up till this point we considered only aquifers that had outer boundaries that were completely pervious over the whole length, and the hydraulic head at the outer boundary was taken as a constant. We next considered circular aquifers of which part of the boundary was impervious. The extent of the impervious part on the outer boundary was varied and also the location of the well. It was found that impervious parts on the outer boundary may change the flow pattern drastically in an aquifer, but that the discharge of a well is not so much affected. For example, it is found that the well discharge of a circular aquifer, with the well in the center of the circle, and with a well radius of 0.0025 times the radius of the aquifer, yields the discharge ratios 1.000:0.972:0.894 when none of the outer boundary is impervious, one quadrant is impervious, or 2 quadrants are impervious,

respectively, when the discharge for none of the boundary being impervious is taken as unity. Then the Ames aquifer was considered. The Ames aquifer is a confined horizontal aquifer of finite extent. Part of its boundary is impervious. The hydraulic head distribution along its boundary is not a constant, but a function of space. A flow net was prepared for the Ames aquifer, and the theoretically calculated discharge (1163 gpm) agreed well with the actually pumped rate (1280 gpm). Next we considered again a circular aquifer, but now pumped by two wells. First the wells were of the same strength, then the wells were of unequal strength. Finally an arbitrarily shaped aquifer was considered, pumped by two wells of different radius and with a different discharge. At the end it was indicated how our method could be extended to horizontal confined aquifers of finite extent, such as the Ames aquifer, pumped by more than two wells, for which the outer boundary has mixed boundaries (potential given over part of the boundary, and zero flow is given over the remainder of the boundary).

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APPENDIX

Solution of Laplace's Equation in Polar Coordinates

Laplace's equation in polar coordinates is given by the expression

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0 \quad (1)$$

We can solve Laplace's equation in polar coordinates by the method of separation of variables, that is, we assume a solution to equation 1 of the form

$$\phi(r, \theta) = R(r) \theta(\theta), \quad (2)$$

in which R is a function of r only, and θ is a function of θ only.

Then equation 1 can be written as

$$R''\theta + \frac{1}{r} R'\theta + \frac{1}{r^2} R\theta'' = 0, \quad (3)$$

in which R' and R'' denote first order and second order derivatives respectively of the function R with respect to r , and θ'' denotes the second order derivative of the function θ with respect to θ .

We can divide equation 3 by $R\theta$ and find

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\theta''}{\theta} = 0, \text{ or} \quad (4)$$

$$r^2 \frac{R''}{R} + r \frac{R'}{R} + \frac{\theta''}{\theta} = 0, \text{ or} \quad (5)$$

$$r^2 \frac{R''}{R} + r \frac{R'}{R} = - \frac{\theta''}{\theta} \quad (6)$$

We see that the left hand side of equation 6 is a function of r only, and the right hand side of equation 6 is a function of θ only. This is only possible if both sides are equal to a constant, say λ . We then can write equation 6 as

$$r^2 \frac{R''}{R} + r \frac{R'}{R} = -\frac{\theta''}{\theta} = \lambda, \text{ or}$$

$$r^2 \frac{R''}{R} + r \frac{R'}{R} = \lambda \quad (7a)$$

$$\frac{\theta''}{\theta} = -\lambda \quad (7b)$$

To solve equations 7a and 7b we have to examine different values of the constant λ . We need to examine

$$a) \quad \lambda < 0$$

$$b) \quad \lambda = 0$$

$$c) \quad \lambda > 0$$

We first start with case a, $\lambda < 0$. For convenience we take λ such that we can write $-\lambda = m^2$. Equation 7a then becomes

$$r^2 \frac{R''}{R} + r \frac{R'}{R} = -m^2, \text{ or} \quad (8)$$

$$r^2 R'' + r R' + m^2 R = 0, \quad (9)$$

and equation 7b becomes

$$\frac{\theta''}{\theta} = m^2, \text{ or} \quad (10)$$

$$\theta'' - m^2 \theta = 0 \quad (11)$$

As solution to equation 11 we have

$$\theta = C_1 e^{m\theta} + C_2 e^{-m\theta}, \quad (m^2 > 0) \quad (12)$$

From physical considerations we realize (see for example Figure 4) that the function θ is periodic, with period 2π . From equation 12 we see that in order to have a periodic function θ we need to have

$$C_1 = C_2 = 0 \quad (13)$$

but this gives us as solution to equation 12

$$\theta = 0 \quad (14)$$

and this yields the trivial solution

$$\phi(r, \theta) = 0, \quad (15)$$

for equation 2. We are not interested in a trivial solution and therefore we decide that the possibility $\lambda < 0$ must be disregarded.

We next try case b, $\lambda = 0$. Equations 7a and 7b then become

$$r^2 \frac{R''}{R} + r \frac{R'}{R} = 0, \text{ and} \quad (16)$$

$$\frac{\theta''}{\theta} = 0 \quad (17)$$

We can then rewrite equation 16 as

$$r R'' + R' = 0, \text{ or} \quad (18)$$

$$\frac{d}{dr}(rR') = 0 \quad (19)$$

From equation 19 we then find, that

$$rR' = C_1 \text{ (a constant), and} \quad (20)$$

$$R = C_1 \ln r + C_2 \quad (21)$$

We now solve equation 17, which we can rewrite as

$$\theta'' = 0 \quad (22)$$

As solution we find

$$\theta = C_3 \theta + C_4 \quad (23)$$

As before we want θ to be such that it is periodic, with a period of 2π .

We see that in equation 23 we need to have

$$C_3 = 0, \quad (24)$$

so that equation 23 becomes

$$\theta = C_4 \quad (25)$$

We now combine equations 21 and 25 and find as solution for $\phi(r, \theta)$ of equation 2 the expression

$$\phi(r, \theta) = (C_1 \ln r + C_2)(C_4), \text{ or}$$

if we lump the constants together, so that we get new constants C_1 and C_2 we can write

$$\phi(r, \theta) = C_1 \ln r + C_2 \quad (26)$$

This then is one useful solution to Laplace's equation for the problems we consider in this thesis.

We next have to consider case c, $\lambda > 0$. For convenience we take $\lambda = m^2$. Equations 7a and 7b now become

$$r^2 \frac{R''}{R} + r \frac{R'}{R} = m^2, \text{ and} \quad (27)$$

$$\frac{\theta''}{\theta} = -m^2 \quad (28)$$

We first look at equation 27 which we rewrite as

$$r^2 R'' + r R' - m^2 R = 0 \quad (29)$$

As solutions to equation 29 we can have

$$R = C_3 r^m, \text{ or } R = C_4 r^{-m},$$

and we also may combine these two solutions, if we wish, to

$$R = C_3 r^m + C_4 r^{-m} \quad (30)$$

As one may verify, equation 30 satisfies equation 29.

We now look at equation 28 which can be rewritten as

$$\theta'' + m^2 \theta = 0 \quad (31)$$

which has as solution

$$\theta = C_5 \sin m\theta \text{ or } \theta = C_6 \cos m\theta, \quad (32)$$

which we can combine, as we did in equation 30, if we wish, to

$$\theta = C_5 \sin m\theta + C_6 \cos m\theta \quad (33)$$

As before, we realize that θ should be periodic with period 2π , and therefore m has to be chosen such that $m = 1, 2, 3, \dots$ (remember $\lambda = m^2 > 0$). We now combine equations 30 and 33 to find as additional solution to equation 26 the following expression for $\phi(r, \theta)$,

$$\begin{aligned} \phi(r, \theta) = R(r) \theta(\theta) &= (C_3 r^m + C_4 r^{-m}) (C_5 \sin m\theta + C_6 \cos m\theta) \\ m &= 1, 2, \dots \end{aligned} \quad (34)$$

Equation 34 yields infinite many solutions to Laplace's equation, all of which we may want to use if we have to satisfy certain boundary conditions.

Combining equations 26 and 34 yields

$$\begin{aligned} \phi &= C_1 \ln r + C_2 \\ &+ \sum_{m=1}^{N \rightarrow \infty} (C_3 r^m + C_4 r^{-m}) (C_5 \sin m\theta + C_6 \cos m\theta) \end{aligned} \quad (35)$$

Equation 35 is a general solution to Laplace's equation in polar coordinates; it can be modified to suit a particular boundary value problem. For Figure 4, in which there is symmetry and for which we state as one boundary condition

$$\phi = 0, \text{ for } r = r_w \text{ and } 0 \leq \theta < 2\pi \quad (36)$$

we can modify equation 35 to

$$\frac{\phi}{\Delta\phi} = A_{NO} \frac{\ln(r/r_w)}{\ln(a/r_w)} + \sum_{m=1}^{N \rightarrow \infty} A_{Nm} \frac{(r/a)^m - [r_w^2/(ar)]^m}{1 - (r_w^2/a^2)^m} \cos m\theta \quad (37)$$

Equation 37 is our earlier equation 12 of page 18. It turns out that we can write equation 37, and hence equation 12 as

$$\frac{\phi}{\Delta\phi} = \sum_{m=0}^{N \rightarrow \infty} A_{Nm} \frac{(r/a)^m - [r_w^2/(ar)]^m}{1 - (r_w^2/a^2)^m} \cos m\theta \quad (38)$$

One may check, that equation 38, with the use of l'Hospital's rule for $m = 0$, gives equation 37. We often use the notation of equation 38, but the reader should realize that equation 38 stems from equation 37, and that equation 37 is obtained by considering different values of the separation constant in the method of separation of variables used to solve Laplace's equation in polar coordinates.