MINIMIZATION OF SIMILITUDE PARAMETERS IN PHYSICAL PHENOMENA

by

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I. INTRODUCTION

A. Statement of Problem

One of the chief benefits of dimensional analysis is the reduction of the number of independent variables (pi terms) that must be investigated. However, every student of dimensional analysis has an uneasy feeling that the benefits it yields are far out of proportion to the small effort it demands. Yet, once interested, every student of dimensional analysis usually wonders if additional techniques are possible so that conventional dimensional analysis can be made to yield even more information. Many instances are known, for example, in which the number of pi terms yielded by dimensional analysis is still larger than the number experimentally verified. These superfluous pi terms have been the cause of much controversy and much needless laboratory work. Dimensional analysis needs a technique that will minimize (or detect) the occurrence of such pi terms.

The aims of this thesis are two-fold. They are, (a) to establish the reason for existence of superfluous pi terms, and (b) to describe a technique which will prevent their formation. Basically the technique consists of increasing (maximizing) the number of dimensions that are used in a problem. Before utilizing such a technique two important questions must be answered. Can the <u>number</u> of dimensions ever be a variable? If yes, under what circumstances?

B. Review of Literature

Two important questions were raised at the end of the preceeding section. Publications of Fourier (10)[1], Bridgman (3), Buckingham (4,5), Brand (2), Birkhoff (1), Drobot (6), and Murphy (19) shall be reviewed for possible answers. The publications of several other authors offer little additional information concerning these questions (7, 9, 12, 14, 15, 24, 25).

Fourier initiated the concepts of dimensional formulas and dimensional homogeneity. It seems proper, therefore, to start with a review of his views. He made the following remark in discussing the nature of the quantities he used in his book on heat (10).

> "In order to measure these quantities and express them numerically, they must be compared with different kinds of units, five in number, namely the unit of length, the unit of time, that of temperature,...."

A few lines later he wrote

> "every undetermined magnitude or constant has one <u>dimension</u> proper to itself, and ... the terms of one and the same equation could not be compared, if they had not the same <u>exponent</u> <u>of</u> <u>dimension</u>. ...Numbers such as S which represent surfaces or solids are of two dimensions in the first case, and of three dimensions in the second. Angles, sines, and other trigonometrical functions, logarithms or exponents of powers, are, according to the principles of analysis, <u>absolute</u> numbers which do not change with the unit <u>of length</u>; their dimension must therefore be taken equal to 0, ..."

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[1]Numbers in parentheses refer to similarly numbered references in the literature cited.

Fourier clearly had two distinct ideas in mind for the words 'unit' and 'dimension'. A unit, in the terminology of Fourier, seems to be a particular magnitude of a unique abstract concept. Examples are 1 hour, 1 minute, and 1 second, which are different magnitude units of the abstract concept 'time'. The collection of five abstract concepts mentioned by Fourier is (apparently) his definition of <u>all</u> measurable qualities possessed by specimens of the real world. Fourier apparently believed that these five 'measurable qualities' form fundamental universal 'dimensions' which are independent of man's measuring systems. A measurable quality is a property of a 'real world' specimen such that two specimens of the real world can be ordered in this quality through the subterfuge of comparing a pair of logically assigned, real positive numbers. Thus Fourier used the word 'unit' to discuss dual concepts; (a) abstract concepts that many people today call 'dimensions', (b) specific measures (units) of these 'dimensions'.

The word 'dimension' in Fourier's work can be interpreted to refer to the numerical exponent that accompanies a unit. Thus Fourier says that the <u>dimension</u> of surface in the unit of length is two, and the dimension of sine (trigonometry) is zero. Dimension, when used in this fashion, implies no information about the identity or number of abstract concepts which are employed as measurable qualities.

Bridgman accepts Fourier's definition of the word 'dimension'. He says (3, p. 23)

> "the exponents of the powers are a matter of vital importance. The exponent of the power of any particular primary quantity is by definition the <u>dimension</u> of the secondary quantity in that particular primary quantity."

Unlike Fourier, Bridgman rarely uses the word 'unit' in his discussion of 'dimension'. In its place he uses the terms 'primary quantity' and 'secondary quantity'. He wrote (3, p. 18)

> "the arguments fall into two groups, depending on the way in which the numbers are obtained physically. The first group of quantities we call primary quantities. These are the quantities which, according to the particular set of rules of operation by which we assign numbers characteristic of the phenomenon, are regarded as fundamental and of an irreducible simplicity. Thus in the ordinary systems of mechanics, the fundamental quantities are taken as mass, length, and time."

A few lines later Bridgman says

> "in general, it is characteristic of primary quantities that there are certain rules of procedure by which it is possible to measure any primary quantity directly in terms of <u>units</u> of its own kind."

Bridgman distinguishes between quantities of a secondary nature and those of a primary nature as follows (3, p. 19).

> "Quantities of the second kind are measured by making measurements of certain quantities of the first kind associated with the quantity under consideration, and then combining the measurements of the associated primary quantities according to certain rules which give a number that is <u>defined</u> as the measure of the secondary quantity in question."

This statement gives the impression that primary and secondary quantities are permanently separated by <u>measurement technique</u>, but on the next page Bridgman makes the following statement (3, p. 20).

> "It is particularly to be noticed that the line of separation between primary and secondary quantities is not a hard and fast one imposed by natural conditions, but is to a large extent arbitrary, and depends on the particular set of rules of operation which we find convenient to adopt in defining our system of measurement."

The primary quantities of Bridgman stand in place of the units of Fourier. Bridgman makes this point very clear (3, p. 23).

> "It is to be noticed that the dimensions of any primary quantity are by a simple extension of the definition above merely the dimensional symbol of the corresponding primary quantity itself."

Bridgman believes that the choice of a 'correct' set of primary quantities is determined by the variables which appear in each problem. This idea represents a significant philosophical shift from Fourier's view that there are exactly five units with which all measurable quantities must be compared. Bridgman's approach is that the number of primary quantities is determined by whatever variables are assumed to be pertinent, and by the number of dimensional constants that must be used to correlate experimental measures of these variables. He says (3, p. 54)

> "we may define our fundamental units with this relation in view, thus obtaining a system of units in which the dimensional constant has disappeared but in which the number of units which may be regarded as primary has been restricted in such a way that all units belonging to the system automatically bear the experimental relation to each other. <u>The system of units so obtained is of value</u>

> <u>only in treating that group of phenomena to which the law in question applies</u>...(author's underlining).... These considerations as to the possible systems of units answer the question previously raised ... as to the number and kinds of units which we shall take as primary. The answer depends entirely upon the particular problem, and will involve the physical relations which are necessary to a complete expression of the motion of the parts."

Bridgman justifies this procedure by its usefulness.

Buckingham's views on 'dimension' (4,5) are hard to extract from his papers. In the article of 1914 he wrote (5, p. 347)

> "Let k be the number of arbitrary fundamental units <u>needed</u> as a basis for the absolute system... by which the Q's are measured. Then principle... there is always, among the n units [Q], at least one set of k which may be used as fundamental units.."

In a subsequent article he said (4, p. 290)

> "To measure n kinds of quantities we require n units, but these need not all be adopted arbitrarily for they can in general be derived from, i.e., described or defined in terms of, some smaller number of fundamental units. ...In mechanics all the necessary units can be derived from only three, such as force, length, time, or work, speed, density."

These remarks are not clarified by Buckingham in his proof of the Pi Theorem. In the proof he mentions the possibility of using a group of variables of the problem as a set of fundamental units, and states that it is always possible to describe the other variables in these units. However, in his examples he uses measures of length, mass, and time as fundamental units whether or not a specific mass or a specific time is a variable in the example. Consideration of all evidence from these two papers leads to the conclusion that Buckingham's view of 'dimensions' lies somewhere between those of Fourier and Bridgman.

Both Garrett Birkhoff (1, p. 89) and Louis Brand (2, p. 36) use the word 'dimension' in the fashion of Fourier and Bridgman. That is, they define the dimension of a variable in a fundamental unit to be the exponent that appears with this units conversion factor when changing the magnitude of the fundamental unit. They both say it is correct to speak of the 'dimensions' of a variable only with respect to a given set of fundamental units.

Birkhoff offers a short discussion on his philosophy of fundamental units. Birkhoff's main point is that there does not exist a set of 'truly' fundamental units that will allow all laws of nature to be expressed as unit-free equations, i.e., complete equations of Bridgman. He discusses several equations which he believes (incorrectly) to be inhomogeneous, and then concludes (1, p. 98)

> "We are thus impelled irresistibly to the conclusion that there are no known 'fundamental units' with respect to which all known physical laws are unit-free. Indeed, the decision to call certain units 'fundamental' (or primary) and others 'derived' (or secondary) is one of convention and not of physical necessity."

Brand places certain restrictions on the units used to form a set of fundamental units (2, p. 41). (See chapter 1, part D.) In addition he exhibits a method for changing sets of fundamental units. Both sets, however, must have the same number of fundamental units. Brand also displays the classic heat transfer paradox of Rayleigh and Riabouchinsky. He first works the problem using four fundamental units, then subsequently using three fundamental units. The method with three fundamental units yields an additional independent variable. Brand concludes (2, p. 44) that the Pi Theorem gives one result under one physical theory, and a different result under another physical theory, and "only experiment can decide which is correct".

The Polish mathematician, S. Drobot (6), takes a decidedly different approach to the concepts of 'dimension' and 'fundamental units' than does Brand, Birkhoff, or Bridgman. Drobot constructs a 'linear space of multiplicative form' such that the elements of this space closely resemble the 'units' of Fourier. That is, each element, a 'dimensional quantity', represents a unique abstract concept which is distinct from its measure. Drobot says (6, p. 85)

> "it is intuitive to consider the dimensional quantities as elements of a space, different from ordinary numbers."

A few pages later he comments (6, p. 93)

> "The space... can be divided into disjoint classes such that all elements of the same class have the same dimension. Thus, it is natural... to identify the dimension of a given quantity with the class to which this quantity belongs."

This implies, among other things, that when a quantity is raised to a power it belongs to a different 'dimensional quantity' by reason of its being a different abstract concept, and not because of the value of its exponent. Drobot does not give the exponent of the units a special name or significance.

Drobot defines a 'system of units' as a set of n dimensionally independent elements of the dimensional space, where n is the maximum number of dimensionally independent elements contained in the space. The definition of a dimensionally independent set of elements A_1, A_2, .. A_n is a set such that

$$A_1^{a_1} A_2^{a_2} \,..\, A_n^{a_n} = \alpha \quad (\text{ a number}) \tag{1}$$

is true if, and only if

$$a_1 = a_2 = \ldots . = a_n = 0, \text{ and } \alpha = 1 \tag{2}$$

The space Drobot constructs contains more than one system of units, but each system can be obtained from the other by a linear transformation. The method for changing systems of units is precisely the one used by Brand, Bridgman, etc.

Murphy (19, p. 191) displays two dimensional analyses of a heat transfer problem which are closely related to the questions being investigated. One analysis assumes the five dimensions mass, length, time, temperature, and heat are applicable. The second analysis assumes only the four dimensions mass, length, time, and temperature are applicable (because heat can be expressed in units of mass, length, and time). Murphy's analysis of the additional pi term shows it to be a ratio of kinetic energy to thermal energy. Murphy concludes (19, p. 193)

> "That is, if transformation of thermal energy to mechanical (potential or kinetic) is not involved in the

> phenomenon, the dimension H may be considered to be independent to M, L, and T, but if conversion of energy is involved, then H, M, L, and T are not independent and only three of the four may be considered as independent dimensions."

C. Conclusions of Literature Review

Clearly the nomenclature of dimensional analysis is not uniform. This is not too surprising since the concept of 'dimension' differs widely between authors. Drobot was correct, it seems, in his appraisal (6, p. 85)

> "Some authors identify dimensional quantities with ordinary numbers, real or complex, and, as a matter of fact, they do not introduce the notions of dimension, or of dimensional quantity into the Dimensional Analysis, although they formulate theorems on these very notions."

The literature reveals at least three distinct concepts concerning the nature of 'dimensions'. One concept is characterized by Bridgman's 'primary quantities', or the 'fundamental units' of Buckingham, Birkhoff, and Brand. These are unit magnitudes (measures) of applicable 'dimensions'. These unit magnitudes are usually constructed for a specific system of physical measurement. Examples of these quantities are 1 foot, 1 acre, and 1 hour. These magnitudes can be transformed by arbitrary numerical scale changes, i.e., the 'change ratio' of Bridgman (3, p. 29). 'Fundamental units' of the Pi Theorem are required to possess the quality of arbitrary numerical scale changes. An important restriction placed on a set of fundamental units as used in the Pi Theorem of Drobot is that the set be composed of dimensionally independent units.

Another concept is that of Bridgman's 'secondary quantities', or the 'derived quantities' of Birkhoff. The measure formula of these quantities can be obtained only by multiplicative combinations of fundamental units. Examples of these quantities are 1 foot/second, 1 BTU/slug-^{o}F, and 1 rad/meter^2. The magnitudes of the 'change ratios' of these quantities are not independent, but are determined by the magnitudes of the 'change ratios' of the fundamental units. The Pi Theorem's concern with 'secondary quantities' is only as specific variables in a problem.

A third concept is that of Fourier's 'units', or Drobot's 'dimensional quantities'. These are abstract concepts whose distinguishing characteristic is the capability of being measured by logically assigned, real positive numbers. Examples of these concepts are length, mass, time, charge, and area. No sample of a 'pure' dimensional quantity exists in the real world because all such quantities are abstract. These concepts are not the subject of the Pi Theorem (except as expressed by Drobot).

The word 'variable' is often used with different meanings, but a variable is best described as a measure of a specific 'dimension'. Usually the measure changes magnitude during the phenomenon. The measure may or may not be expressed in units of its own dimension. The units to be used are specified by the measure formula of the variable. Since a variable is a measure it cannot be a fundamental unit of a problem.

The literature reveals that the definitions of several key words are not standardized. The following definitions cannot agree with all the usages expressed by past authorities, but they do coincide with current usage.

1. DIMENSION. This is a unique abstract concept whose characteristic quality is measurability in units of its own kind. Examples are mass, time, area, and temperature. This usage makes the word unrelated to a numerical exponent.

2. UNIT. This is a specific measure (magnitude) of a dimension. Examples are 1 slug, 1 hour, 1 acre, and $1^{o}F$.

3. FUNDAMENTAL UNITS. This is a set of dimensionally independent units selected for a given set of variables in a specific problem. Thus the set of fundamental units is in a 1-1 correspondence with the dimensions used in the problem.

4. VARIABLE. This is a magnitude-unit combination which is the measure of a specific example of a dimension. The measure can expressed two ways: (a) directly in units of the dimension, or (b) in a combination of units which are homogeneous to the dimension.

5. DEGREE. This is the value of the exponent that accompanies a fundamental unit of one dimension when the fundamental unit is used in the measure of a different dimension. (A great many of the original papers in dimensional analysis assign this definition to the word 'dimension'. But

in current literature the word 'dimension' has clearly come to mean an abstract concept.)

The definitions given will be used throughout the remainder of this thesis.

D. The Pi Theorem

Buckingham's statement of the Pi Theorem (5, p. 350) is essentially as follows: If an equation in n kinds of physical quantities is dimensionally homogeneous with respect to m fundamental units, it can be reduced to a relation between n - m independent dimensionless products. Bridgman showed (3, p. 43) that this useful rule is not entirely correct. In certain special cases there will be more independent dimensionless products (pi terms) than this rule allows. Although Bridgman did not specifically give a complete formal statement of the improved Pi Theorem, his remarks leave no doubt that he knew of its existence and what it should be.

Brand's paper (2) on dimensional analysis contains a form of the Pi Theorem which is much longer, but more accurate than Buckingham's statement. Brand's paper makes use of two terms which need to be carefully defined; dimensional matrix, and isobaric. (Brand's terminology 'dimensional matrix' will be replaced with the phrase 'degree matrix' to be consistent with the definitions previously adopted.) The definitions of these terms will be developed before considering the Pi Theorem.

Consider a problem in which the physical quantities X_i have positive measure x_i in a system of m fundamental units $U_1, U_2, \ldots, U_m$. Let the magnitude of any U_j of these units be changed by a positive ratio t_j to a new unit U_j', so that the positive measures x_i also change to new values x_i'. Then if under any unit magnitude change of the type

$$U_j' = U_j/t_j \qquad (t_j > 0) \tag{3}$$

the relation

$$x_i' = [t_1^{a_{i1}} t_2^{a_{i2}} \ldots t_m^{a_{im}}]x_i \tag{4}$$

is always obtained, then we say as a definition that the degrees of X_i are $(a_{i1}, a_{i2}, \ldots, a_{im})$ in the units U_1, U_2, ..., U_m. When all $a_{ij} = 0$, X_i is said to be without dimension. The degree of n quantities X_i in the m units U_j may be displayed in a rectangular n x m array.

	U_1	U_2	U_m
X_1	a_{11}	a_{12}....	a_{1m}
X_2	a_{21}	a_{22}....	a_{2m}
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
X_n	a_{n1}	a_{n2}....	a_{nm}

An array such as this is called a degree matrix. The degree matrix is said to be of rank r if it contains at least one non-zero determinant of order r, while all determinants of higher order which the matrix may contain are zero.

Consider next a single physical quantity X whose measure is x in a system of units U_1, U_2, ..., U_m. Let x be related to the measures x_i of other quantities by

$$x = f(x_1, x_2, \ldots, x_n) \; . \tag{5}$$

Assume Equation 5 holds for all changes of units of the type described by Equation 3. Therefore an equation such as

$$x' = f(x_1', x_2', \ldots, x_n') \tag{6}$$

will always be obtained. If X has degrees $(a_1, a_2, \ldots, a_m)$ the measures x and x' are related through Equation 4 by

$$x' = t_1^{a_1} t_2^{a_2} \ldots t_m^{a_m} \; x = f(t_1^{a_{11}} t_2^{a_{12}} \ldots t_m^{a_{1m}} x_1, t_1^{a_{21}} t_2^{a_{22}} \ldots t_m^{a_{22}} x_2, \ldots) \tag{7}$$

Therefore under the given unit transformations Equation 5 satisfies an identity in t_1, t_2, ..., t_m; namely:

$$t_1^{a_1} t_2^{a_2} \ldots t_m^{a_m} f(x_1, x_2, \ldots, x_n) = f(t_1^{a_{11}} t_2^{a_{12}} \ldots t_m^{a_{1m}} x_1, \; t_1^{a_{21}} t_2^{a_{22}} \ldots t_m^{a_{2m}} x_2, \ldots) \tag{8}$$

All such functions are termed isobaric (dimensionally homogeneous) of degrees $(a_1, a_2, \ldots, a_m)$ in units $U_1, U_2, \ldots, U_m$.

Equation 8 shows that any function that is isobaric to the m units U_1, U_2, ..., U_m is also isobaric with respect to any subset of these units, say $U_1, U_2, \ldots, U_k$ $(k < m)$. This fact is made obvious by setting $t_{k+1} = t_{k+2} = \ldots = t_m = 1$ and observing that Equation 8 is still an identity in $t_1, t_2, \ldots, t_k$. Thus there are more functions isobaric to units U_1, U_2, ..., U_k than there are functions isobaric to units U_1, U_2, ..., U_m for $m > k$. In other words, the larger the number of units, the smaller the number of corresponding isobaric functions. This simple fact lies at the root of several apparent paradoxes in dimensional analysis. Conversely a function that is known to be isobaric with respect to k units might also be isobaric with respect to m units for $m > k$. That is, there is nothing in the definition of isobaric functions which limits the number of units to be considered.

Brand's statement of the Pi Theorem (2, p. 38) is as follows:

"Let the function f in an equation with n arguments such as

$$f(x_1, x_2, \ldots., x_n) = 0 \tag{9}$$

be isobaric with respect to m fundamental units U_1, U_2, .., U_m. Then if the n x m dimensional matrix of x_1, x_2, .., x_n is of rank $r = n - k$, the given equation is equivalent to

$$f(1,1,.., 1, \pi_1, \pi_2, \ldots, \pi_k) = 0 \tag{10}$$

in which the first r arguments are 1, and the π's are $n - r$ independent and dimensionless products formed from x_1, x_2,.., x_n."

The complete proof of the theorem is not pertinent to this thesis but consideration of a special case will indicate the general method Brand used in his proof.

A function $f(x_1, x_2, \ldots, x_n)$ is defined as homogeneous of degree d (13, p. 35) if

$$f(tx_1, tx_2, \ldots, tx_n) = (t^d)\cdot f(x_1, x_2, \ldots, x_n) \quad t > 0. \tag{11}$$

In the terminology of Brand it is said that f is a function of the measures x_i of physical quantities X_i, and that f is isobaric and of dimension d in a single fundamental unit U_1. The degree matrix of this function will be a single column consisting of n entries of d (rank = 1). Equation 11 is an identity in t, therefore it is true when $t = 1/x_1$. When this substitution is made in Equation 11 the results are

$$f(x_1/x_1,\ x_2/x_1, \ldots, x_n/x_1) = (1/x_1)^d \cdot f(x_1,\ x_2, \ldots, x_n). \qquad (12)$$

By hypothesis, i.e., Equation 9, the function is of the type $f(x_1, x_2, \ldots, x_n) = 0$. Therefore it follows that

$$f(1,\ x_2/x_1,\ x_3/x_1,\ \ldots,\ x_n/x_1) = 0. \qquad (13)$$

Thus an equation such as Equation 9, whose arguments have a degree matrix with rank = 1, is equivalent to the equation

$$f(1,\ \pi_1,\ \pi_2,\ \ldots,\ \pi_{n-1}) = 0 \qquad (14)$$

in the n - 1 dimensionless products $\pi_1 = x_2/x_1$, $\pi_2 = x_3/x_1$, ..., $\pi_{n-1} = x_n/x_1$. The Pi Theorem is thus demonstrated for this special case, and in the words of Brand (2, p. 38),

> "The proof in the general case consists of a natural extension of this reasoning."

Brand's statement of the Pi Theorem restricts U_1, U_2, ..., U_m to be a set of fundamental units. The restriction for the units U_1, U_2, ..., U_m to form a set of fundamental units in a given field of application is that every variable X in that field must be expressed <u>uniquely</u> in the form

$$X = (x)\, U_1^{a_1} U_2^{a_2} \ldots U_m^{a_m} . \qquad x > 0. \tag{15}$$

Brand points out that this statement means that X is dimensionless if and only if

$$a_1 = a_2 = a_3 = \ldots = a_m = 0 . \tag{16}$$

This last statement can be used as an equivalent definition of a set of fundamental units.

E. Number of Units

It is not quite correct to assume that the Pi Theorem deals with variables and dimensions. Brand's statement of the Pi Theorem relates the terms 'arguments' (variables) and 'fundamental units' to the number of independent variables. Thus there is a flaw in the question at the beginning of this chapter. The question should read "Can the number of fundamental units ever be varied?" The answer, as determined in the review of literature and by the Pi Theorem, is yes.

The Pi Theorem, stripped of all 'real world' connotations, is merely the answer to the following mathematical question. Given an isobaric relation between n quantities (variables), each quantity being a combination of m independent items (fundamental units); how many independent terms (pi terms) can be formed from the n quantities such that every term shall have a degree of 0 on all independent items? Hence the theorem only predicts the mathematical consequences of collecting an arbitrary number of variables and fundamental units into an isobaric function.

The heat transfer example from Murphy clearly shows that the number of units can be varied, and that the pi terms thus produced can reflect restrictions in the phenomenon. Many authorities have said that the initial step in dimensional analysis is selection of the 'correct' variables, ignoring any mention of the units. Murphy's example illustrates that a correct set of units does not necessarily follow from a correct set of variables. If such were the case there would be an 'absolute measure formula' for every dimension (variable). Practically every modern writer supports Bridgman's view (3, p. 24, p. 54) that there are no 'true' units with which to measure a dimension. Langhaar quotes Max Planck (14, p. 10) as saying

> "To inquire into the 'real' dimension of a quantity has no more meaning than to inquire into the 'real' name of an object..."

It is the engineers responsibility to <u>select</u> both the variables <u>and</u> the fundamental units for each problem. The implications (and circumstances) of varying the number of fundamental units will be investigated in the next chapter.

II. DIMENSIONAL RELATIONSHIPS

A. Dimensional Dependence

The Pi Theorem deals only with isobaric (dimensionally homogeneous) equations, and isobaric equations utilizing multi-unit variables are made possible only through the existence of dimensionally dependent relationships. Traditionally there have been two approaches to the concepts related to dimensionally dependent relationships. One approach is that there exists a specific set (or at least a specific number) of 'primary independent' dimensions, and that all other dimensions are necessarily described as products of these. The second approach, exemplified by Bridgman and Planck, is that there exists only a finite number of dimensions, but that each dimension is independent of all others. Thus both views tacitly assume that there exists only a finite number of 'truly independent' dimensions. The literature shows that these concepts are a source of constant controversy. All evidence to date indicates that the validity of either (or any) dimensionally dependent relationship cannot be determined by experiment, nor by inductive reasoning. Furthermore there is no evidence or reason to believe that there should exist only a finite number of independent dimensions. Therefore this thesis elects to resolve the questions related to dimensional-dependence by the following hypothesis (8, p. 379):

> Every dimension is an independent abstract concept, and simultaneously every dimension is uniquely related to other dimensional combinations through specific 'physical phenomena'.

The essence of this hypothesis is incorporated in all subsequent remarks. Thus a specific dimension is neither primary nor secondary until considered in the context of some physical phenomenon. This hypothesis eliminates the possibility of building an efficient permanent dimensional analysis basis with a fixed finite set of dimensions because every dimensional combination creates a new dimensional concept (dimension) which, in turn, may be independent in some other physical phenomenon. Two methods are available to achieve dimensional dependence. One method is through the use of 'defining equations', and the other is through the use of dimensional constants.

B. Defining Equations

Much work in physics and engineering is directed towards deriving functional measure relationships (in units) between differing quantities (dimensions). Dimensional homogeneity, required for dimensional analysis, can easily be satisfied after the measure relation is established by an equivalence definition. Thus after Joule developed a measure relation between a unit of mechanical work and a unit of thermal energy it seemed logical to define work $\doteq$ heat. It could even be argued that dimensional homogeneity demanded such a definition. Similarly after Maxwell's statistical derivation of the gas law showed a relation between measures of a gas molecules kinetic energy and the gas temperature it seemed logical to define kinetic energy $\doteq$ temperature. Many empirical laws describe measure relationships between dissimilar units. A few examples of such empirical laws are:

Measure-Unit Relation	Dimensional Formula	Resulting Pi Term
force = mass x acceleration	$F \doteq MA$	$F/MA \doteq 1$
work = constant x heat	$W \doteq H$	$W/H \doteq 1$
energy = constant x temperature	$E \doteq \theta$	$E/\theta \doteq 1$
area = length x width	$a \doteq L^2$	$a/L^2 \doteq 1$

The obvious fact that the symbols on the left side of the dimensional formula are different from those on the right side could be construed to mean that the relationships are dimensionally inhomogeneous. But these measure relationships can be made dimensionally homogeneous simply by an <u>equivalence definition</u> between the appropriate dimensional combinations on each side of the equation. The equations are thus 'coerced', by definition, into dimensional homogeneity.

An important point in this procedure is that the measure relationships justified in this fashion are valid only when the defining physical phenomenon is occurring. The same measure relationships, even though dimensionally homogeneous, yield incorrect measures when used in other circumstances. Thus the variable force generated by a rocket motor cannot be measured by the acceleration imparted to the mass of that motor if the motor is restrained from motion.

C. Dimensional Constants

Engineers and physicists sometimes discover measure relationships between variables (dimensions) that seem, to them, intrinsically different. The requirement for dimensional homogeneity is satisfied in these cases by the simple process of assigning appropriate products of units (hence dimensions) to some significant multiplicative constant which appears in the measure relationship. Thus dimensional constants can be made to serve as 'transfer functions' (23, p. 19) between dissimilar dimensions. Bridgman discusses an hypothetical investigation of the phenomenon of 'free fall' (3, p. 14) in which Galileo discovers the measure relationship

$$\text{distance} = \text{constant} \times \text{time}^2. \qquad (17)$$

Galileo thus could choose between two methods of achieving dimensionally homogeneity. He could either <u>define</u> the dimensional equivalence relation 'distance $\doteq$ time^2', or he could <u>assign</u> the dimensions 'distance $\div$ time^2', to the constant. Either technique is logically permissible, either will achieve dimensional homogeneity. Dimensional constants, like defining equations, yield valid measure relationships only when utilized in the defining phenomenon. (However, the concept embodied in the dimensional constant can be an independent dimension in a different physical phenomenon.) Conversely, use of either technique (defining equation or dimensional constant) to achieve dimensional homogeneity implies that the defining physical phenomenon is occurring. Thus if a measure of time is mathematically transformed into a measure of distance in the manner prescribed by Galileo's free fall relation the phenomenon called 'acceleration' is (mathematically) occurring. This example will be examined in more detail in chapter IV.

Unfortunately both methods of obtaining dimensional homogeneity are in current use. In fact the dimensionally dependent relationships which occur within one problem are sometimes the result of utilizing both techniques. A few dimensional relationships are always written with dimensional constants, a few are always written as defining equations, and some are written either way. Some measure relationships utilizing dimensional constants are listed here:

Measure-Unit Relation	Dimensional Relation	Dimensional Constant	Resulting Pi Term
length = constant x time^2	$L \doteq KT^2$	$K \doteq LT^{-2}$	$L/KT^2 \doteq 1$
force = constant x length	$F \doteq CL$	$C \doteq FL^{-1}$	$F/CL \doteq 1$
work = constant x heat	$W \doteq JH$	$J \doteq WH^{-1}$	$W/JH \doteq 1$
energy = constant x temperature	$E \doteq k\theta$	$k \doteq E\theta^{-1}$	$E/k\theta \doteq 1$

D. Dimensionless Terms

Dimensionless terms (pi terms) are deemed 'multi-unit' if they are formed by products of different kinds of variables. Thus, in one sense, such terms represent products of measures of different dimensions. Classification of multi-unit product-terms as 'dimensionless' necessarily implies a dimensionally dependent relationship exists between the dimensions of the variables. But dimensional-dependence can be accomplished only by two methods, either 'defining equations', or dimensional constants, and both methods are related to specific physical phenomenon. Consequently every multi-unit pi term can be considered, <u>ultimately</u>, to represent one of the following situations: (a) a ratio of measures of differing dimensions which are equivalent by definition, or (b) a ratio of variable products formed so that their units (dimensions) exactly 'nullify' the units (dimensions) assigned to some pertinent dimensional constant(s). In either case a specific dimensionally dependent relationship is imposed on the variables of that pi term, and the corresponding unique physical phenomena is (mathematically) invoked.

There is, however, a significant <u>practical</u> difference in <u>constructing</u> pi terms by these two methods. Successful utilization of the dimensional constant technique means that the engineer <u>knowingly</u> selects appropriate variables as dimensional constants (dimensional transfer functions), thus <u>purposefully</u> imposes pertinent physical phenomena into the dimensional analysis. This task is reasonably amenable to logic, and the presence of mathematical symbols (with their variety of units) is helpful. Phenomena thus represented are 'mathematically modeled' as individual pi terms.

It is much more difficult, however, to utilize efficiently (correctly) 'defined' dimensional equivalence relationships. The difficulty is compounded in problems which have dimensional constants (practically all multi-unit problems). The presence of dimensional constants makes it easy to use a 'defined' dimensional equivalence relationship in a situation where the defined relation is not applicable, i.e., where the affected dimensions are not dependent. Thus inappropriate physical interpretations are mathematically imposed on the problem, and these interpretations ultimately appear in the similitude parameters (pi terms). Simple one-for-one substitutions do not cause serious problems because the engineer mentally translates the written dimension, through the defining equation, back into the appropriate phenomenon.

When the inappropriate 'defined' dimensional equivalence relationship reduces the number of independent dimensions a corresponding decrease in the number of fundamental units occurs. Thus redundant, or superfluous dimensional relationships are mathematically imposed on the problem. These restrictions ultimately appear as redundant or superfluous similitude parameters (pi terms) which 'mathematically model' the physical implications contained in the redundantly related dimensions, as modified by other parameters of the problem.

Unfortunately the forms of the superfluous pi terms are not <u>readily</u> predictable because they are modified by the remaining dimensions in the 'redefined' dimensional constants. The mathematical and physical implications of utilizing redundant dimensional relationships within a dimensional constant is in the nature of imposing a 'phenomenon within a phenomenon'. The infinite number of physical dimensions when formed into an infinite number of dimensional constants can make it difficult to predict specific results of utilizing a given inappropriate dimensional equivalence relationship.

III. METHOD

Chapters I and II developed the theory that 'superfluous' or redundant similitude parameters (pi terms) are created as the result of an inconsistency in the structure and application of dimensional analysis. (Superfluous pi terms associated with excess variable selection are not the subject of this thesis.) Basically the inconsistency is that dimensional homogeneity (hence a dimensionally dependent relationship) can be, and often is, achieved by two methods. Each method is valid, but each yields correct measure relationships only in a restricted set of circumstances. These circumstances are (theoretically) always brought to mind by use of the 'dimensional constant' method. Trouble can develop, however, when <u>defined</u> dimensionally dependent relationships are used outside of their defining context. Therefore a permanent method for eliminating superfluous pi terms is to stop using 'defining equations' and require that the dimensional constant method be uniformly used. This suggestion is not practical, however, because the 'defining equation' technique is too far entrenched in the existing structure of dimensional analysis.

An alternative method is to realize the implications associated with using 'defining equations', then utilize this knowledge when selecting dimensions (and sometimes variables). Thus the method boils down to a problem-by-problem search to prevent inadvertant introduction of improper (out of context) defined dimensionally dependent relationships. In most other respects current dimensional analysis techniques can be employed (19, p. 65).

The first step of the method is to define clearly the limits of the problem of interest. This entails two major decisions. The first decision is to select the variable which is to be predicted by a measure relationship, and the second decision is to select the primary driving function within the region of interest. These two choices delineate the phenomenological boundaries of the problem and, therefore, are a great assistance in selecting pertinent variables. Proper (wise) selections are not always as easy as appears at first glance. After all, every driving function has its own driving function....ad infinitum. In defining the problem the engineer can also profitably utilize his experiences concerning the relative magnitudes of the phenomena occurring within the boundaries of the problem. Phenomena which apply but whose effect is known to be insignificant in the region of interest can be discarded.

The second step of the method is to select the primary transport or transfer mechanisms that relate the driving function to the variable to be predicted. This step always involves selection of dimensional transfer functions, i.e., dimensional constants. Methods of transmission fall into two main categories; (a) space-time relationships of the components of the system (external geometry), and (b) properties of the materials involved in the transfer mechanisms (internal geometry). These two categories often compliment one another in defining phenomena transmission and transfer mechanisms that occur within a given problem.

The final step in the method is to translate the selected pertinent physical phenomena into the symbolism required for dimensional analysis. The first part of this step is to select the remaining variables required to describe the phenomena, and the next part is to select the fundamental units (dimensions) with which to express these variables. Selection of units should <u>not</u> be made from a table, but should be made with each pertinent phenomenon in mind. This is especially true of the variables associated with the transfer mechanisms of the problem, usually properties of materials. It is important that the <u>origin</u> of the <u>definition</u> of a property of the material being utilized be analyzed in terms of the physical phenomena involved in the definition of that property. Selection can then be made at the definitional level of the 'correct' dimensions for the property of the material as applied to the specific problem. A good rule is: NEVER REPLACE A DIMENSION IN A 'DIMENSIONAL TRANSFER VARIABLE' (SECONDARY QUANTITY) WITH DIMENSIONS DEFINED AS EQUIVALENT.

The method outlined here will eliminate superfluous similitude parameters (pi terms), although admittedly it sounds too vague and too similar to existing dimensional analysis techniques to accomplish the task. A few examples will indicate the strength and scope of the hypothesis of chapter II concerning dimensionally dependent relationships, and will illustrate how the method herein described, based on the hypothesis, will eliminate superfluous pi terms.

IV. EXAMPLES

A. Dimensional Dependence-Phenomenon Relations

HYPOTHESIS: Every dimension is an independent abstract concept, and simultaneously every dimension is uniquely related to other dimensional combinations through specific 'physical phenomena'.

Webster (26) gives the following definition of the word hypothesis:

> "A tentative theory or supposition provisionally adopted to explain certain facts and to guide in the investigation of others."

A few examples are offered, in the spirit of Webster's definition, to support the hypothesis adopted in this thesis.

Space and time are universally regarded as independent dimensions, but Bridgman describes a hypothetical experiment by Galileo (3, p. 13) which leaves their dimensional independence in doubt. Bridgman assumes that Galileo wishes to establish a measure relation for bodies in free fall. In particular Galileo would like to relate the height of the free fall to the time required for the fall. His experimental equipment includes a platform of variable height, a height-measurer, a time-measurer, and a multitude of disposable missiles. His height-measurer and time-measurer are calibrated in arbitrary units. A large number of free fall tests are made in which he records the measures of height and time of fall. After a great deal of research Galileo discovers that the measure of the height of fall, called h, when divided by the square of the measure of time of fall, called τ, always yields a constant number k. Or $s/\tau^2 = k$. Galileo could define a new unit of time t, such that $(\sqrt{k}\cdot\tau) = t$. His empirical measure relationship for free fall then becomes simply $s = t^2$. A valid measure relationship is thus obtained between concepts ordinarily assumed independent, and the concepts (dimensions) have not explicitly entered the problem. The only assumption required in the problem is that the methods of measuring s and t are well defined and always repeatable. The symbols s and t can be looked upon as a 'book keeping' device which serve only to tell Galileo which measure is to be squared and which is to be predicted. The equation is dimensionally homogeneous because it contains no dimensions. Application of the Pi Theorem reveals 2 variables, 0 units, hence 2 pi terms. This is obviously the case for the equation $s = t^2$, or $\pi_1 = f(\pi_2)$.

But suppose Galileo decided that the measures which went into the free fall 'law' were somehow 'different' from those which went into his astronomical observations, and he decided to differentiate between them by giving them special designations, or 'dimensions'. The measure relationship $s = t^2$ should have its measures dimensionally identified (hence be made isobaric) to insure that both measures associated with free fall would not be inadvertantly mixed with astronomical measures. Dimensional identification and homogeneity is easily accomplished by <u>defining</u> the dimensions on each side of the measure relationship to be equivalent. Thus the measure t (called time) is defined to be dimensionally equivalent to the square root of the measure s (called height). The corresponding dimensional relationship is $L^{\frac{1}{2}} \doteq T$, or $L \doteq T^2$. Application of the Pi Theorem reveals 2 variables, 1 unit, hence 1 pi term. This is obviously the case because $s/t^2 = 1$, or π_1 = constant. Thus the dimensions of length and time have been made dimensionally dependent one on the other by <u>definition</u>. (Furthermore the <u>units</u> have been made equivalent because the calibration of the time measure was adjusted to yield unity when the height measure was unity under the phenomenon of free fall.) The dimensional system so constructed is logically coherent and yields valid measure relationships. The system will always yield correct measure relationships for objects moving under a uniform linear acceleration. However, a numerical constant would have to be introduced into the measure relationship if either the length unit, the time unit, or the original (calibration) acceleration were changed.

Thus the dimensional definition $L \doteq T^2$ plus 'normalization' of the units of L and T have simplified Galileo's free fall measure relationship formula. Future free fall measures (predictions) can be easily calculated. But he (the hypothetical Galileo) paid a dear price for this simplification. He has defined the abstract quantity 'time' to be dependent upon the abstract quantity 'length'; and furthermore, to be dependent in a particular fashion. (The fashion of dependence is not nearly so important as the dependence itself.) This hypothetical Galileo might foresee some difficulties and choose another method of obtaining dimensional homogeneity. After all, if it is wise to separate free fall measures from astronomical measures might it not also be wise to differentiate between the various elements within one measure relationship?

The empirical measure relationship of the free fall phenomenon is $s = t^2$ when expressed in the 'normalized' units of height and time. Galileo could elect to express this 'law' as $s = c \cdot t^2$, where c is understood to have (normally) the magnitude 'unity'. But introduction of the symbol c gives an additional degree of freedom in achieving dimensional homogeneity. For example, the symbol c can <u>arbitrarily</u> be assigned a ratio of dimensions (units) such that the free fall measure relationship is dimensionally homogeneous, (or $c \doteq LT^{-2}$), and still maintain the independence of the dimensions of length and time. Application of the Pi Theorem to this scheme reveals 3 variables, 2 units, hence 1 pi term. This is obviously the case because $s/ct^2 = 1$, or π_1 = constant. If this fictitious Galileo had forseen this result he could have saved himself the trouble of 'normalizing' his original unit of time. In fact a 'fringe benefit' of this method of achieving dimensional homogeneity is that changes in the length unit or the time unit can be compensated automatically by corresponding changes in the magnitude of c.

A summary of the three techniques employed in the empirical free fall measure relationship yields the following interpretations.

<u>Case 1</u>: $s = t^2$, and 0 units. This is an equation between pure numbers. The physical setting is that a measure of length is empirically discovered to be produced by the square of the measure of an appropriate time. In addition the unit magnitudes of s and t must be correctly specified as initial conditions.

<u>Case 2</u>: $s = t^2$, and $L \doteq T^2$. This is a measure relationship between measures of dimensionally dependent physical quantities. The physical setting is that a measure of length is being produced by a measure of time hence the time measure must necessarily be squared to produce length. In addition the unit magnitudes of s and t must be correctly specified as initial conditions.

<u>Case 3</u>: $s = ct^2$, and $c \doteq LT^{-2}$. This is a measure relationship between measures of dimensionally independent physical quantities as related through the dimensional constant c. The physical setting is that a measure of length is being produced by a measure of time through the intervention of c, and the required arithmetic operations are imposed by the

'dimensions' assigned to c. The presence of c also eliminates the initial conditions on the unit magnitudes of s and t.

The various dimensional dependences and independences imposed on the measure relationships in the three examples had one thing in common: When a measure of the abstract concept 'length' was obtained by the same arithmetic operation on a measure of the abstract concept 'time' a unique physical phenomenon (linear acceleration) was always imposed on the problem. A different combination of arithmetic operations would have imposed a different, but unique, physical phenomenon. These results are seen to always hold regardless of the dimensional dependences or independences assumed in the problem. These results thus agree with the hypothesis.

The 'physical' phenomenon imposed by defined dimensional equivalence relationships are not always as obvious as that of $s = ct^2$, i.e., linear acceleration. A prominent example in point concerns the dimensions 'length' and 'area'. Practically every writer says the dimension of area is defined to be (equivalent to) length^2, or $A \doteq L^2$. But if the hypothesis of chapter II is correct the dimensions of area and length are independent except under certain unique 'physical' situations. The unique physical situation associated with the dimensionally dependent relationship $A \doteq L^2$ must be uncovered to preserve the hypothesis. A fictitious, or 'thought', experiment will again be employed as the vehicle of investigation.

Common units of area, which include the barn, the acre, and the circular mil are neither convenient nor well defined for the following investigation. Therefore an imaginary area unit, called an 'arean', shall be devised. One arean is <u>defined</u> as the area enclosed within a certain triangle scribed on a sheet of platinum. Copies of standard decimal multiples of a unit arean are available to facilitate area measurement.

Imagine a large set of <u>randomly shaped</u> ink blots drawn upon a plane. The blots are not similar in shape or size, but (for convenience only) each blot has an easily discernible maximum breadth. Imagine that the investigator finds this maximum breadth and measures the length of it. Any convenient unit of length may be utilized. Since the area is assumed to be proportional to the product of two lengths it might be argued that another independent length measure is required for each ink blot. A second convenient length to measure will be the length of the perpendicular bisector of the maximum breadth. Thus each ink blot is characterized by the measures of two lengths taken at right angles to one another. Multiply these two measures and tabulate the products for each ink blot. Each product represents, or is related in some sense to the area of its respective ink blot. Now measure the area of each ink blot directly in 'areans' by 'overlaying' or covering each blot with units (and decimals of units) of standard 'arean' area measures. Record the number (measure) of each ink blot area expressed in areans.

Each ink blot now has two measures which 'characterize' its area. One is a direct area measure in areans, and the other is a product of two characteristic lengths (measured at right angles to each other). Attempt to find a method that will convert the area measure in areans to the measure obtained by the product of the two lengths. No amount of research or correlation will yield a single functional relationship which will perform the desired conversion for <u>every ink blot</u>. Thus, in this case, the area measured in area units is <u>not</u> related to the measure given by the product of two characteristic lengths. What important 'physical' phenomenon caused this unexpected result? The answer is simple; <u>geometrically dissimilar shapes</u>. Or, to put the question in a different light, what would have been the result if the same experiment was carried out on a set of <u>geometrically similar</u> ink blots, each of a different size? The answer is obvious. The area measure expressed in areans, when divided by the area measure expressed as the product of the two lengths would yield the same constant for every blot, or area = constant x length x width. Immediately a dimensionally dependent relationship can be <u>defined</u>, or $A \doteq L^2$. Thus a 'physical phenomenon' that must exist for this dimensional equivalence to be valid is <u>geometric similarity</u>. Hence the defined dimensionally dependent relationship $A \doteq L^2$ forces cross sectional (area) geometric similarity onto a problem. 'Geometric similarity', as discussed here, does not require similarity between two physical objects, but only similarity between two geometrical (area) abstractions, such as an ellipse.

It would be redundant to make the same argument between the dimensional quantities 'length' and 'volume'. But identical arguments could be made, and identical conclusions would be drawn. Whenever the dimension 'volume' is <u>defined</u> to be equivalent to the dimension 'length^3', or $V \doteq L^3$, <u>complete geometric similarity</u> is <u>imposed</u> on the problem.

The example concerning Galileo's free fall experiment was of more academic interest than practical use because everyone treats space and time as independent dimensions. But the dimensional independence of length, area, and volume is not widely understood. (However Bridgman (3, p. 61) worked an example that touched on the independence of volume and length.) Consequently several dimensional transfer functions of phenomena which are truly functions of area, or of volume, are usually shown as dependent upon L^2 and L^3 respectively, with the necessary result that superfluous pi terms are introduced into a dimensional analysis which utilizes these dimensional constants.

Several other defining equations were mentioned in the theoretical discussion of chapter II. Thus $F \doteq MA$ <u>defines</u> force in terms of mass x acceleration apparently because a measure of force can easily be related to the product of measures of mass and acceleration <u>when the force is allowed to impart a spacial acceleration</u> (kinetic energy) <u>to the mass</u>. But the abstract concept (dimension) force is independent of its ability to impart an acceleration to a given mass. For example force can also impart energy by straining (elongating) a solid piece of material. It would be just as valid, therefore, to <u>define</u> $F \doteq L$.

The hypothesis of chapter II leads to the conclusion that the only justification for <u>defining relationships between dimensional combinations</u> is that a resulting measure relationship related to a unique physical phenomenon is made dimensionally homogeneous without the intervention of a dimensional constant, and thus the definition is good only when (actually implies) the defining phenomenon occurs. Thus to define work as dimensionally equivalent to heat implies that a measure of work is being converted to a measure of heat. To define kinetic energy as dimensionally equivalent to temperature implies that a measure of kinetic energy is being converted to a measure of temperature. To define heat as dimensionally equivalent to temperature implies that a measure of heat is being converted to a measure of temperature. Each of these operations entails a unique physical phenomenon. The better approach is never to <u>define</u> one dimensional combination as equivalent to another, but instead to utilize dimensional constants for dimensional homogeneity.

B. Applied Problems

A few applied problems will be worked to illustrate the intrusion of superfluous pi terms. Each problem will be worked using two methods: (a) the problem will first have all variables expressed in the independent dimensions applicable to the phenomenon under discussion, and (b) the problem will then be worked with specific redundantly defined dimensionally dependent relationships inserted into the dimensional formulas (transfer functions). It will be seen that the second solution produces superfluous pi terms, and that the nature of these pi terms can be <u>predicted</u> from the nature of the dependent dimensional equivalences imposed on the problem. The first example will be taken from Bridgman (3, p. 3).

Consider the time of a small oscillation of a small drop of liquid acting under its own surface tension. The drop is assumed to be free of gravitational influence, and the oscillations refer to periodic changes of shape, such as spherical to ellipsoidal and back. The time of oscillation t is assumed to depend upon the surface tension of the liquid s, the density of the liquid d, and the radius of the stationary drop r. These variables are shown with their dimensional measure formula expressed in two sets of fundamental units (dimensions). Set (A) consists of the conventional M, L, and T set. Set (B) is a 'reduced' M and L set, which assumes that the dimensional defining equation developed in Galileo's free fall experiment, $L \doteq T^2$, is applicable to this problem.

Both sets of units are summarized below:

Variable	Symbol	(A) Conventional Dimensions	(B) 'Free Fall' Dimensions
Time of oscillation	t	T	$\sqrt{L}$
Density of liquid	d	ML^{-3}	ML^{-3}
Surface tension	s	MT^{-2}	ML^{-1}
Radius of drop	r	L	L

Dimensional analysis with the conventional units, set (A), contains 4 variables, 3 fundamental units (and the rank of the degree matrix is 3), hence there is 1 pi term. By inspection $\pi_1 = st^2/dr^3$. The Pi Theorem yields the equation

$$st^2/dr^3 = \text{constant}. \qquad (18)$$

Bridgman asserts that this result is verified by experiment (3, p. 4).

Dimensional analysis with the free fall units, set (B), contains 4 variables, 2 fundamental units (and the rank of the degree matrix is 2), hence there are 2 pi terms. Various combinations of variables to form pi terms are possible, but the most instructive set is $\pi_1 = st^2/dr^3$, and $\pi_2 = r/t^2$. The Pi Theorem yields the equation

$$st^2/dr^3 = f(r/t^2). \qquad (19)$$

Of course experiment will reveal that $f(r/t^2)$ is constant, but experiment is not required. The function $f(r/t^2)$ <u>must</u> be constant in this phenomenon because the redundantly defined dimensionally dependent relationship $L \doteq T^2$ imposed $\pi_2 = r/t^2$

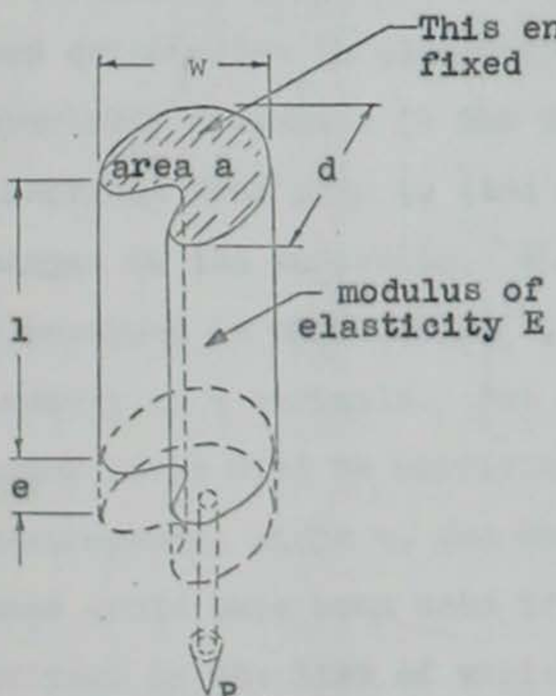
as a superfluous pi term into the dimensional analysis. The imposed dimensionally dependent relationship yields a valid measure relationship between measures of length and time only under the phenomenon of constant linear acceleration, thus the superfluous pi term (similitude parameter) requires that any pertinent measures of length must be converted to a measure of time through the relationship associated with the phenomenon of a constant linear acceleration. There is no such phenomenon in this problem.

If the free fall experiment had been conducted in a viscous medium the measure relationship s/t = constant would have been obtained. This relationship could then be used to construct a defined dimensionally dependent relationship of $L \doteq T$ between the dimensions 'length' and 'time'. If the variables of the liquid drop problem are expressed in the M and L set, with the defined dimensional equivalence $L \doteq T$ assumed to apply, an extra pi term homologous to the form s/t will appear. Dimensional analysis of this situation shows 4 variables, 2 fundamental units (and the rank of the degree matrix is 2), hence there are 2 pi terms. The Pi Theorem yields the equation

$$st^2/dr^3 = f(r/t). \qquad (20)$$

The nature of the function $f(r/t)$, in the above equation, and the explanation for its existence is completely analogous to that of the function $f(r/t^2)$ of Equation 19.

Consider next a problem dealing with loading a metal rod of unspecified cross sectional shape in tension. (The shape, assumed to be irregular, is uniform along the longitudinal axis.) The rod will be subjected to a tensile load P, and the elongation of the bar e is to be determined. First analyze the problem utilizing a set of variables and units (dimensions) which do not use redundantly defined dimensionally dependent relationships. The elongation e is assumed to be a function of the tensile load P, of the length of the bar l, of the cross sectional area a, and of the modulus of elasticity E.



The shape of the cross section can be 'characterized' by the two characteristic measurements maximum width and maximum depth, w and d respectively. A summary of the pertinent units shows:

Variable	Measure Formulas Symbol	Dimensions
elongation	e	L
tensile load	P	F
area of section	a	A
length of bar	l	L
modulus of elasticity	E	FA^{-1}

Figure 1. Tension load, bar of irregular cross section

Dimensional analysis with this set of units contains 5 variables, 3 fundamental units (and the rank of the degree matrix is 3), hence there are 2 pi terms. The pi terms are seen by inspection to be $\pi_1 = e/l$, and $\pi_2 = P/aE$. The Pi Theorem yields the equation

$$e/l = f(P/aE). \tag{21}$$

Experiment shows the relation to be simply $e/l = P/aE$.

Next analyze the problem utilizing a set of variables and units (dimensions) which employ the redundantly defined dimensionally dependent relationship $A \doteq L^2$. (This problem is used quite often in elementary engineering texts, and it is invariably presented in the following manner.) The change in dimensions from area to (the equivalent?) length^2 requires some changes in the variables. The modulus of elasticity E must now be measured in units of FL^{-2}, hence area cannot appear directly as a variable. But area is still an important factor so it will be characterized by the two 'characteristic' measurements, width w, and depth d. (Actually only one of these could have been used if the parameter 'shape' had been included in the list of variables. In this case 'shape' would be a pi term by itself.) All other variables remain unchanged. Following is a summary of the measure formulas of the variables now required:

Variable	Symbol	Dimensions
characteristic depth	d	L
characteristic width	w	L
elongation	e	L
length of the bar	l	L
tensile load	P	F
modulus of elasticity	E	FL^{-2}

Dimensional analysis with these units contains 6 variables, 2 fundamental units (and the rank of the degree matrix is 2), hence there are 4 pi terms. Four convenient pi terms can be formed by inspection. They are $\pi_1 = e/l$, $\pi_2 = d/l$, $\pi_3 = w/l$, and $\pi_4 = P/El^2$. The Pi Theorem yields the equation

$$f(1, 1, \pi_1, \pi_2, \pi_3, \pi_4) = 0 . \tag{22}$$

This equation can be rearranged and displayed in terms of the original variables. In this form it becomes

$$e/l = f_1(d/l, w/l, P/El^2) . \tag{23}$$

Much more experimental investigation is required to reduce this result to the accepted result shown following Equation 21. Two additional pi terms, d/l, and w/l (or alternately d/l, and 'shape'), have been added, and the pi term involving the ratio of P to E has been altered. These additional pi terms <u>require</u> cross sectional geometric similarity, while the correct solution does not require it. The added pi terms (similitude parameters) would be found to be superfluous by experiment, but experiment is not required. The superfluous pi terms were added because the dimensionally dependent relationship $A \doteq L^2$ imposed the 'physical phenomenon' associated with the dimensionally dependent relationship, geometric area similarity, onto the problem. Thus area similarity is imposed as the direct result of replacing a correct dimension in a dimensional transfer function of a unique physical phenomenon, the modulus of elasticity, with a dimensional combination <u>defined</u> as equivalent.

The applied problems discussed thus far have dealt with dynamic and static behavior of 'mechanical' systems. Consider next a classic problem in heat transfer, the so-called 'paradox' of Rayleigh (20, 21) and Riabouchinsky (22). Lord Rayleigh used dimensional analysis on Boussinesq's problem of steady heat transfer from a good conductor immersed in a large stream of fluid. The fluid is moving past the conductor with a fixed velocity (measured at a distance from the solid). It is assumed that the fluid is incompressible and non-viscous, and that the solid does not change shape or orientation with respect to the stream. The problem is to predict the heat transfer rate h from the solid body to the stream. Rayleigh's dimensional analysis (20) of the problem is summarized below:

Variable	Measure Formula Symbol	Dimensions
heat transfer rate	h	HT^{-1}
characteristic length of the solid	a	L
stream velocity of the fluid	v	LT^{-1}
temperature difference, solid to fluid	t	θ
specific thermal capacity of the fluid	c	$HL^{-3}\theta^{-1}$
thermal conductivity of the fluid	k	$HL^{-1}\theta^{-1}T^{-1}$

Dimensional analysis reveals 6 variables, 4 fundamental units (and the rank of the degree matrix is 4), hence there are 2 pi terms. The pi terms Rayleigh formed are $\pi_1 = h/kat$, and $\pi_2 = avc/k$. Rayleigh expressed these in the form

$$h = (kat)\cdot f(avc/k). \tag{24}$$

A short time later D. Riabouchinsky questioned Rayleigh's analysis (22).

> "If we suppose only three of these quantities are 'really independent' we obtain a different result. For example if the temperature is defined as the mean kinetic energy of the molecules, the principle of similitude allows us only to affirm that
>
> $$h = (kat)\cdot F(v/ka^2,\ ca^3)." \tag{25}$$

Riabouchinsky apparently assumed that heat (H) is also dimensionally equivalent to energy, for if he had replaced only the unit of temperature there would still be 4 fundamental units, i.e., H, L, T, and $\underline{M}$ versus H, L, T, and $\underline{\theta}$. A quick check reveals that either of these sets of 4 fundamental units yields Rayleigh's original answer. It is further assumed that Riabouchinsky chose to omit mass from his analysis because Rayleigh specifically omitted it. Thus Riabouchinsky's argument must run essentially as follows: heat $\doteq$ energy, and temperature $\doteq$ energy, therefore heat $\doteq$ temperature. Consequently the units used by Riabouchinsky can be obtained from Rayleigh's units by <u>defining</u> the dimensionally dependent relationship $\theta \doteq H$. This is a redundant dimensionally dependent relationship and it implies that a specific physical phenomenon is occurring.

Namely that a measure of temperature can be converted directly into a measure of heat. Thus the physical situation which allows a measure of temperature to be converted directly into a measure of heat must appear as a superfluous similitude parameter (pi term) in Riabouchinsky's analysis. Riabouchinsky's (apparent) dimensional analysis of the problem is summarized below:

Measure Formula

Variable	Symbol	Dimensions
heat transfer rate	h	HT^{-1}
characteristic length of the solid	a	L
stream velocity of the fluid	v	LT^{-1}
temperature difference, solid to fluid	t	H
specific thermal capacity of the fluid	c	L^{-3}
thermal conductivity of the fluid	k	$L^{-1}T^{-1}$

Since $\theta \doteq H$ only one symbol can be retained. It is H.

Dimensional analysis reveals 6 variables, 3 fundamental units (and the rank of the degree matrix is 3), hence there are 3 pi terms. The pi terms Riabouchinsky formed are $\pi_1 = h/kat$, $\pi_2 = v/ka^2$, and $\pi_3 = ca^3$. Riabouchinsky's results can be made to appear more like Rayleigh's by replacing $\pi_2 = v/ka^2$ with a new π_4 term, where π_4 is defined by the relation $\pi_4 = \pi_2 \cdot \pi_3$. The result of Riabouchinsky's analysis then takes the form

$$h = (kat) \cdot F(avc/k,\ ca^3). \tag{26}$$

This form of Riabouchinsky's answer shows it to be identical to Rayleigh's with the exception of an additional (superfluous) pi term. The pi term could be eliminated by experiment, but experiment is not required. The pi term (similitude parameter) results from the redundant dimensional definition of $\theta \doteq H$ so it must reflect the 'physical phenomenon' required to convert a measure of θ into a measure of H. This means that a body with a fixed volumetric thermal capacity does not change volume during the heat transfer process (which is Rayleigh's problem); or, if it does change volume then its volumetric thermal capacity must change inversely with the volume. (This is seen to be precisely the similitude requirement imposed by Riabouchinsky's pi term ca^3.) There is no such material in the real world, certainly not in the world discussed by Rayleigh and Riabouchinsky. Thus Brand's conclusion concerning the 'paradox' (2, p. 44) "only experiment can decide which is correct" is not true. Application of the hypothesis of chapter II reduced the two possible answers to only one, and <u>without experiment</u>.

V. SUMMARY AND CONCLUSIONS

This thesis had two objectives. They were (a) to establish the reason for existence of superfluous pi terms (not associated with extraneous variables), and (b) to describe a technique which will prevent their formation. Review of the literature and analysis of the Pi Theorem indicated superfluous pi terms were formed because of insufficient knowledge of the role played by dimensionally dependent relationships. A lack of agreement on the terminology used in the Pi Theorem contributed to this situation. The Pi Theorem is a mathematical statement that could easily be stripped of all 'real world' meanings, but the usual interpretations placed on elements of the Pi Theorem clearly showed the theorem dealt with relationships between variables (which are here defined to be measures of specific examples of dimensions), and units (which are here defined to be specific magnitudes of a dimension). Thus, ultimately, the Pi Theorem predicts the consequences of 'measuring' a set of n dimensions by using a subset, m, of these dimensions, where $n > m$. Whenever measures of a set of dimensions are expressed in terms of a proper subset of those dimensions, and simultaneously dimensional homogeneity is enforced, then there must be some dimensionally dependent relationships expressed or implied between the dimensions of the set.

Consideration of dimensionally dependent relationships has historically been a controversial subject. A review of methods of achieving dimensional homogeneity, hence dimensionally dependent relationships, revealed that it has been accomplished in two ways: (a) the dimensional 'defining equation', and (b) the 'dimensional transfer variable'. It has not been generally understood that dimensional 'defining equations' are merely one means of achieving dimensional homogeneity. Both techniques were seen to <u>require the use of empirical measure relationships</u> which relate measures of abstract concepts (dimensions) through <u>unique</u> physical phenomena. The resulting measure relationships are then 'manipulated' so that they are dimensionally homogeneous. Thus dimensionally dependent relationships are always constructed so that they yield valid measure relationships <u>only</u> under restricted physical conditions. It is seen that all dimensionally dependent relationships can be neatly summarized in the following hypothesis:

> Every dimension is an independent abstract concept, and simultaneously every dimension is uniquely related to other dimensional combinations through specific 'physical phenomena'.

The fact that dimensionally dependent relationships can be 'constructed' in two different fashions explains why the number of fundamental units can be a variable. Differing similitude parameters are produced whenever the number or type of dimensionally dependent relationships <u>assumed</u> to be pertinent is varied. The validity of these similitude parameters (pi terms) depends directly upon the validity of the assumed dimensionally dependent relationships, (which in turn depends upon the validity of the assumed physical phenomena).

Thus the first objective of this thesis has been accomplished. Superfluous pi terms are seen to be the result of utilizing dimensionally dependent relationships out of context with their defining physical phenomenon. The second goal of this thesis, a technique to prevent the formation of superfluous pi terms, follows naturally from this knowledge. That is, superfluous pi terms are eliminated when the engineer uses only the dimensionally dependent relationships that are appropriate to the physical phenomena under investigation. This simple rule, though correct, is not easy to apply. Two guide lines that will help the engineer from inadvertently creating superfluous pi terms are (a) avoid the use of 'defining equations' whenever possible, (b) never replace a dimension in a 'dimensional transfer variable' with dimensions defined as equivalent. It is the author's opinion that the sciences would greatly benefit if the concept of dimensional-defining equations was struck from the literature, but this is impractical because the system is well entrenched. Thus the engineer perpetually faces the task of ferreting out redundantly defined dimensional relationships buried within commonly defined 'dimensional transfer variables', i.e., the 'secondary quantities' of Bridgman. An example was cited in Murphy in which it was shown that changing the number of fundamental units changed the physical phenomena that had to be simulated. Many other examples are shown in the literature (11, 16, 17, 18). Most of these are based on the concept of independent directed lengths, or so-called 'vector dimensional analysis'. The use of directed lengths is only a special case of dimensional independence, and the concept is too narrow to eliminate all superfluous pi terms.

This thesis points to two tasks that could profitably be undertaken by engineers. The first is for an engineer to re-examine the physical definitions of all commonly used 'dimensional transfer variables', usually properties of materials. The object would be to list the dimensions of these properties in terms of the phenomena implied by the definition of that property, insuring that no 'defined dimensional-equivalence' relationships accidently made their way into the list. The second task is an investigation into the 'true' phenomenological meaning of common similitude parameters (pi terms). Murphy points out (19, p. 170) that this has been accomplished for most of the common fluid flow similitude parameters. They have been reduced to ratios of pertinent forces. This thesis shows that phenomena involved in redundantly defined dimensionally dependent relationships will appear in the dimensional analysis as similitude parameters. It may be possible to study all dimensional analysis problems as ratios of certain phenomena which are necessarily contained within the 'dimensional transfer variables' selected for a given problem.

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