

Random walks in a sparse random environment

by

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ABSTRACT

We introduce random walks in a sparse random environment on the integer lattice \mathbb{Z} and investigate such fundamental asymptotic property of this model as recurrence-transience criteria, the existence of the asymptotic speed and a phase transition for its value, limit theorems in both transient and recurrent regimes. The new model combines features of several existing models of random motion in random media and admits a transparent physics interpretation. More specifically, the random walk in a sparse random environment can be characterized as a perturbation of the simple random walk by a random potential which is induced by “rare impurities” randomly distributed over the integer lattice. The “impurities” in the media are rigorously defined as a marked point process on \mathbb{Z} . The most interesting seems to be the critical (recurrent) case, where Sinai’s scaling $(\log n)^2$ for the location of the random walk after n steps is generalized to basically $(\log n)^\alpha$, with $\alpha > 0$ being a parameter determined by the distribution of the distance between two successive impurities of the media.

CHAPTER 1. INTRODUCTION

The main focus of this work is on random walks in a sparse random environment (RWSRE). In the reminder of this chapter, we will recall the general framework of RWRE (Section 1.1) and give a precise description of RWSRE (Section 1.2). The chapter also contains a brief survey of basic asymptotic results for classical RWRE (Section 1.3) and a short overview of the thesis (Section 1.4).

1.1 Random walks in a random environment in dimension one

Let $\Omega = [0, 1]^{\mathbb{Z}}$ and let \mathcal{F} be the Borel σ -algebra of subsets of Ω . A *random environment* is a random element (sequence) $\omega = (\omega_i)_{i \in \mathbb{Z}}$ in (Ω, \mathcal{F}) . Throughout this work we will denote by P the probability measure associated with ω in (Ω, \mathcal{F}) . We will use the notation E_P for the corresponding expectation operator.

A random walk on \mathbb{Z} in a random environment $\omega = (\omega_n)_{n \in \mathbb{Z}} \in \Omega$ is a Markov chain $X = (X_n)_{n \geq 0}$ on \mathbb{Z} governed by the following transition law:

$$P_\omega(X_{n+1} = j | X_n = i) = \begin{cases} \omega_i & \text{if } j = i + 1, \\ 1 - \omega_i & \text{if } j = i - 1, \\ 0 & \text{otherwise.} \end{cases} \quad (1.1)$$

Thus X is an element of $\mathbb{Z}^{\mathbb{Z}_+}$, where \mathbb{Z}_+ stands for the set of non-negative integers. We will often refer to $\mathbb{Z}^{\mathbb{Z}_+}$ as the *space of trajectories* of the random walk. The law of the random walk in a fixed environment is usually referred to as a *quenched law* of the random walk. We will use the notation P_ω^x for the quenched probability law when $X_0 = x$ and E_ω^x for the corresponding expectation operator acting in the space of trajectories.

The so called *annealed* or *averaged* distributions \mathbb{P}^x of the random walk are obtained by av-

eraging the quenched distributions over all possible environments, that is $\mathbb{P}^x(\cdot) = \int P_\omega^x(\cdot) P(d\omega)$. More precisely, let \mathcal{G} be the cylinder σ -algebra of the space of trajectories $\mathbb{Z}^{\mathbb{Z}_+}$. Note that for each $G \in \mathcal{G}$, $P_\omega^x(G) : \Omega \rightarrow [0, 1]$ is a \mathcal{F} -measurable function of ω . The joint probability distribution \mathbb{P}^x of the random walk and the environment on the product space $(\Omega \times \mathbb{Z}^{\mathbb{Z}_+}, \mathcal{F} \otimes \mathcal{G})$ is defined by $\mathbb{P}^x(F \times G) = \int_F P_\omega^x(G) P(d\omega)$, $F \in \mathcal{F}$, $G \in \mathcal{G}$. The projection of \mathbb{P}^x on the space of trajectories $\mathbb{Z}^{\mathbb{N}}$ is the annealed law of the random walk. The expectation under the law \mathbb{P}^x is denoted by \mathbb{E}^x . We will usually assume that $X_0 = 0$ and often omit the upper index x in the notations for the underlying probability laws and expectation operators when $x = 0$. For instance, we will write P_ω for P_ω^0 and \mathbb{E} for \mathbb{E}^0 .

Random walks in random environment (RWRE) have been shown to exhibit an interesting and surprising behavior. In a seminal paper, Solomon [56] identified a recurrent/transient criterion for one-dimensional RWRE and obtained an explicit formula for the limiting velocity $V_P = \lim_{n \rightarrow \infty} \frac{X_n}{n}$. From these results it is easy to construct explicit examples of RWRE that are transient to $+\infty$ even though $\mathbb{E}(X_1) < 0$ or that are transient with asymptotically zero velocity ($V_P = 0$). Limit laws for one-dimensional RWRE were obtained by Kesten, Kozlov and Spitzer [33] in the transient regime and by Sinai [53] in the recurrent case. In both cases, the asymptotic behavior of the RWRE turns out to be considerably different from that of the corresponding regular random walk. Although many refinements of the above results have been obtained, the one-dimensional RWRE is still an active area of research with several interesting questions remaining open. RWRE in higher dimensions is a more challenging model and is understood considerably less than its one-dimensional counterpart.

In the classical RWRE model, ω is a stationary and ergodic sequence under P [66]. Let

$$\rho_n = \frac{1 - \omega_n}{\omega_n}, \quad n \in \mathbb{Z}. \quad (1.2)$$

It turns out (see [33, 53, 59, 66] and, for instance, [3, 6, 12] and [16, 17, 45] for some recent advances in, respectively, recurrent and transient cases) that asymptotic results for one-dimensional RWRE can usually be stated in terms of certain averages of functions of ρ_0 and explained by means of typical “landscape features” (such as *traps* and *valleys*, cf. [59, 66]) of the *random potential* $(V_n)_{n \in \mathbb{Z}}$, which is associated with the random environment as follows:

$V_0 = 0$ and

$$V_n = \begin{cases} \sum_{k=1}^n \log \rho_k & \text{if } n > 0, \\ -\sum_{k=1}^{|n|} \log \rho_{-k} & \text{if } n < 0. \end{cases} \quad (1.3)$$

The role played by the process $V_n = \sum_{i=0}^n \log \rho_{-i}$ in the theory of one-dimensional RWRE stems from the explicit form of harmonic functions (cf. [66, Section 2.1]; see also Section 3.2 in this thesis), which allows one to relate hitting times of the RWRE to those associated with the random walk V_n . This general paradigm can serve to provide a heuristic explanations to most of the results about random walks in random environment, including those discussed in this thesis.

1.2 Sparse environment

The main topic of this thesis is random walks in a sparse (in general, non-stationary) random environment. In dimension one, the model is defined as follows. Let $(d_k)_{k \in \mathbb{Z}}$ be a sequence of i.i.d. strictly positive integer-valued random variables. For $n \in \mathbb{Z}$ let

$$a_n = \begin{cases} \sum_{k=1}^n d_k & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\sum_{k=n+1}^0 d_k & \text{if } n < 0. \end{cases}$$

The random variables a_n serve as locations of random impurities in the, otherwise homogeneous, random media. Let P_d denote the probability law of d_0 . Further, let P_e be any Borel probability measure on the interval $[0, 1]$. A *sparse random environment* on the integer lattice \mathbb{Z} , associated with the pair (P_d, P_e) is defined as follows. Let $(\lambda_n)_{n \in \mathbb{Z}}$ be an i.i.d. sequence of random variables, each one distributed according to P_e . Let $(\omega_n)_{n \in \mathbb{Z}}$ be a sequence of reals taking values in the interval $[0, 1]$ such that

$$\omega_n = \begin{cases} \lambda_k & \text{if } n = a_k \text{ for some } k \in \mathbb{Z}, \\ 1/2 & \text{otherwise.} \end{cases}$$

Notice that the sparse random environment ω is in general not stationary. We refer to a random walk in the random environment ω as a *random walk in a sparse random environment* and abbreviate it as RWSRE.

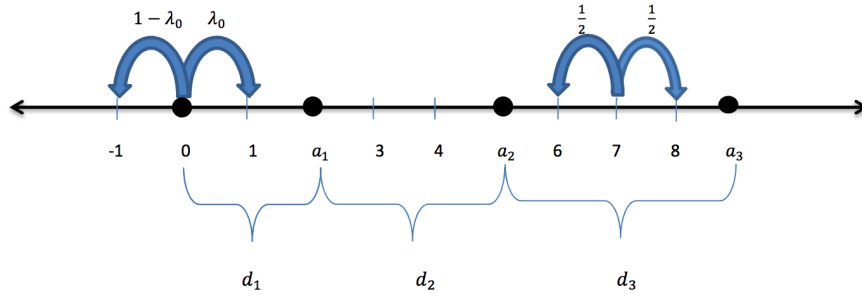


Figure 1.1 Sparse random environment

The RWSRE defined above is a novel model introduced in this dissertation. The model allows for extra flexibility in modeling random motion in random media and also exhibits a richer type of asymptotic behavior than its classical counterpart. The point of view on RWRE as random walk in the random potential (1.3) places RWSRE into the family of processes describing random motion in a disordered media, where the random media is modeled by an infinite “cloud” of soft obstacles. A typical model of this class is Brownian motion among Poisoanin cloud of obstacles intensively discussed in [60], see [20] for a related RWRE model.

In general, RWRE and random walks “around obstacles” or “with traps” are very challenging models to investigate in higher dimensions. In this thesis we focus on RWSRE in dimension one and obtain recurrence and transience criteria, compute the asymptotic speed of the random walk, and prove limit theorems for the location of the walker in both transient and recurrent regimes. These results will be also reported in a paper in preparation [50].

In a continuous setting, a close relative of our model is the multi-skewed Brownian motion introduced in [47]. A direct discrete-time analogue of the multi-skewed Brownian motion is a *multi-skewed random walk*, which can be introduced as a quenched variant of our model with λ_n being a certain deterministic sequence of constants. Appealing to a physical motivation of the model, the marked sites (those in the set $\mathcal{A} := \{a_n : n \in \mathbb{Z}\}$) are called in [47] interfaces while the long stretches of the “regular” sites divided by the interfaces are referred to as layers.

We note that certain random environments consisting of alternating stretches of sites of two different kinds and inducing a sub-linear growth rates on underlying random processes defined in such environments, have been considered in [46, 57] and, in a different context, for instance,

in [8]. The overlap between either results or proof methods in this work and those in [46] and [57] is minuscule.

Somewhat related to our work is the study of [40, 41], where it is in particular shown that an $X_n \sim (\log n)^\alpha$ asymptotic behavior of the random walk can occur in a perturbation of an i.i.d. recurrent environment $(\omega_n)_{n \in \mathbb{Z}}$ in the form $\omega_n^{\text{new}} = \omega_n + f_n$, where $E_P(\log \frac{1-\omega_0}{\omega_0}) = 0$ and f_n converges to zero in probability as $|n| \rightarrow \infty$.

1.3 Basic asymptotic results for the classical RWRE

In this section we recall some basic asymptotic results for random walks in a stationary and ergodic environment. Criteria for transience and recurrence of the one-dimensional RWRE were provided by Solomon [56] in the case where ω_n is an i.i.d. sequence and extended to stationary and ergodic environments by Alili [2] (see also [42]). Let

$$R(\omega) = 1 + \sum_{n=1}^{\infty} \rho_0 \rho_{-1} \cdots \rho_{-n+1} = 1 + \sum_{n=1}^{\infty} \exp \left\{ \sum_{i=0}^{n-1} \log \rho_{-i} \right\}, \quad (1.4)$$

Let $T_0 = 0$ and for $n \in \mathbb{N}$,

$$T_n = \inf\{k \geq 0 : X_k = n\} \quad \text{and} \quad \tau_n = T_n - T_{n-1}. \quad (1.5)$$

The walk X_n is a.s. transient if $E_P(\log \rho_0) \neq 0$ and is a.s. recurrent if $E_P(\log \rho_0) = 0$. If $E_P(\log \rho_0) < 0$ then $\mathbb{P}(\lim_{n \rightarrow \infty} X_n = +\infty) = 1$, T_n are a.s. finite, $\{\tau_n\}_{n \geq 1}$ is a stationary and ergodic sequence (but not i.i.d. since the walk returns to sites already visited with a positive probability), and

$$v_P := \lim_{n \rightarrow +\infty} \frac{X_n}{n} = \lim_{n \rightarrow +\infty} \frac{n}{T_n} = \frac{1}{2E_P(R) - 1}, \quad \mathbb{P} - \text{a. s.} \quad (1.6)$$

Thus, the transient walk X_n has a deterministic speed $v_P = \lim_{n \rightarrow \infty} X_n/n$ which may be zero. In the case of i.i.d. environments, Solomon's result, $v_P = 0$ if $E(\rho_0) \geq 1$ and $v_P = \frac{1 - E_P(\rho_0)}{1 + E_P(\rho_0)}$ otherwise, is recovered from (1.6). It follows from (1.6), by Jensen's inequality, that the random walk X_n is always slower than the usual random walk with deterministic probability p of moving to the right given by $\log \left(\frac{1-p}{p} \right) = E_P(\log \rho_0)$, cf. [46, 62]. This phenomenon happens because, in contrast to a homogeneous medium, random environments may create atypical segments (traps) which retain the random walk for abnormally long periods of time.

For the classical RWRE model, asymptotic limit theorems are known for both transient and recurrent regimes. These theorems measure the amplitude of the fluctuations in the deviation of X_n from its asymptotic expected value $v_P \cdot n$.

In the transient regime, Solomon's law of large numbers for random walks in i.i.d. environment was complemented by limit laws in the work of Kesten, M. Kozlov, and Spitzer [33]. The limit laws for the RWRE X_n are deduced in [33] from stable limit laws for the hitting times T_n , and the index κ of the stable distribution is determined by the condition

$$E_P(\rho_0^\kappa) = 1. \quad (1.7)$$

In particular, if $E_P(\rho^2) < \infty$ then the central limit theorem holds with the standard normalization \sqrt{n} . Mayer-Wolf, Roitershtein, and Zeitouni in [39] extended the limit laws of [33] to environments that are stochastic functionals of either Markov processes or so called *chains of infinite order*. For related quenched results in the transient regime see recent articles [16, 17, 45] and references therein.

Sinai [53] studied a recurrent regime and obtained the following limit theorem:

$$\frac{\sigma^2}{(\log n)^2} X_n \Rightarrow b_\infty$$

where \Rightarrow denotes convergence in law and b_∞ is a non-degenerate random variable. The density of the limiting distribution was characterized independently by Kesten [32] and Golosov [22]:

$$P(b_\infty \in dx) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left\{-\frac{(2k+1)^2 \pi^2}{8} |x|\right\} dx$$

The random variable b_∞ was characterized by Sinai as the limit of a sequence of the bottoms of “deepest valleys” associated with the random potential V_n . In the theory developed by Sinai, the unusual scaling $(\log n)^2$ (to be compared with \sqrt{n} for the simple recurrent random walk on \mathbb{Z}) is explained by the fact that the RWRE in Sinai's regime spends almost all time being localized in the “deep valleys” of the random potential.

1.4 Overview of the thesis

The thesis is divided into five chapters, including the first introductory one. We will next give a brief review of the content of each chapter.

The sparse environment ω_n is defined above as a simple functional of the *marked point process* $(a_n, \lambda_n)_{n \in \mathbb{Z}}$ and is non-stationary, in general. If $E_P(d_1) < \infty$, one can use classical Palm's dualities [63, 64] to define a “stationary dual” $\tilde{\omega}$ of the environment and study the RWSRE through the random walks in the environment $\tilde{\omega}$. Therefore, as far as $E_P(d_0) < \infty$ and basic zero-one laws (say, recurrence transience, and the asymptotic speed) are concerned, one should expect that the properties of RWSRE are similar to the corresponding features of the classical RWRE. However, a similar claim about similarity of the properties of the random walks in the environments λ and $\tilde{\omega}$ seems fairly less obvious in the case of the limit theorems. Indeed, the dependence structure of the environment plays crucial role in such theorems and it is clearly not preserved under the transformation $\lambda \mapsto \tilde{\omega}$. The dual environment is introduced and studied in Chapter 2.

In Chapter 3, we obtain recurrence and transience criteria and the law of large numbers for RWSRE on \mathbb{Z} . The chapter contains a review of the corresponding results for the classical RWRE and a brief discussion of proof techniques that have been exploited to obtain these results.

In Chapter 4, using a reduction to random walks in a dual stationary environment, we obtain stable limit theorems for the transient RWSRE. It turns out that under relatively mild conditions on the distribution of the sparse environment, its dual can be explicitly constructed as a semi-Markov environment. Using this observation, we deduce the limit laws for transient RWSRE from the corresponding results of [39] for semi-Markov environments.

Chapter 5 is devoted to an extension to our setting of Sinai's result for the recurrent RWRE. Sinai's remarkable scaling $(\log n)^2$ for the location of the random walk after n steps is generalized in our main theorem to basically $(\log n)^\alpha$, with $\alpha > 0$ being a parameter determined by the distribution of the distance between two successive impurities of the media. One interesting consequence of our results is that even extremely sparsely placed impurities in the media (when $E(d_1) = \infty$) have a dramatic impact on the motion, resulting in the localization of the moving particle in “deep valleys” of a suitably scaled random potential and in a subsequent slowdown of the particle which is translated into the unusual $(\log n)^\alpha$ normalizing factor in the limit theorem.

CHAPTER 2. DUAL STATIONARY ENVIRONMENTS

2.1 Introduction

The sparse environment introduced in Section 1.2 is in general a non-stationary sequence. The goal of this chapter is to introduce a *dual* stationary environment and relate the properties of the RWSRE to the corresponding properties of the RWRE in the dual environment. If $E_p(d_0) < \infty$, then the underlying probability space can be enlarged to include a random variable M such that the random shift $(a_{n+M}, \lambda_{n+M})_{n \in \mathbb{Z}}$ of the sequence $(a_n, \lambda_n)_{n \in \mathbb{Z}}$ is stationary and ergodic. Furthermore, the distribution of the sparse environment turns out to be the distribution of its stationary version conditioned on the event $a_0 = 0$.

The rest of this chapter is structured as follows. In Section 2.2 we introduce a dual environment to a general cycle-stationary sparse environment. In Section 2.3 we recall some additional properties of the sparse environment in the case when the underlying point process a_n is a renewal sequence, that is d_n is an i.i.d. sequence. Finally, in Section 2.4, exploiting the existence of the dual environment, we carry over to our setting the classical concept of the *environment viewed from the position of the particle*.

The results collected in this chapter shed light on the mathematical structure of the sparse random environment and appear fundamental for the theory of RWSRE. They are used directly in Chapter 4 in the proof of stable limit theorems for RWSRE and in Chapter 3 to obtain the formula for the asymptotic speed of the random walk.

2.2 Sparse environment as the dual stationary environment

In contrast to the usual RWRE, $\omega = (\omega_n)_{n \in \mathbb{Z}}$ is in general a non-stationary sequence in the RWSRE model. In fact, ω is *cycle-stationary* under P , namely

$$\theta^{d_n} \omega =_D \omega \quad \text{under } P, \quad n \in \mathbb{Z},$$

where $X =_D Y$ means that the distributions of the random variables X and Y coincide and the *shift* θ^k is a measurable mapping of (Ω, \mathcal{F}) into itself, which is defined for any $k \in \mathbb{Z}$ (possibly, random) by

$$(\theta^k \omega)_n = \omega_{n+k}, \quad n \in \mathbb{Z}. \quad (2.1)$$

If $E_P(d_1) < \infty$ one can define a “stationary dual” $\tilde{\omega}$ of the environment as follows [63, 64]. Without loss of generality, we can assume that the underlying probability space supports a random variable U which is independent of ω and is distributed uniformly on the interval $[0, 1]$ of reals. For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the integer part of x , that is $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$. Define now $(\tilde{a}_n, \tilde{\omega}_n)_{n \in \mathbb{Z}}$ by setting

$$\tilde{a}_n = a_n + \lfloor U d_0 \rfloor \quad \text{and} \quad \tilde{\omega}_n = \begin{cases} \lambda_k & \text{if } n = \tilde{a}_k \text{ for some } k \in \mathbb{Z}, \\ 1/2 & \text{otherwise.} \end{cases}$$

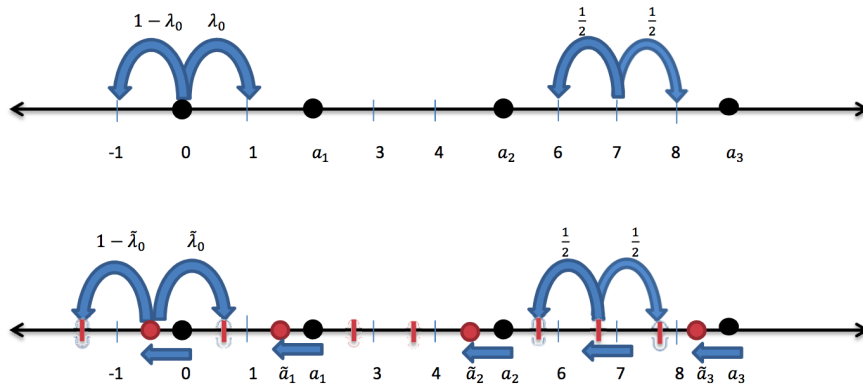


Figure 2.1 Dual stationary environment

Let $\mathcal{A} = (a_n)_{n \in \mathbb{Z}}$ be the set of marked sites of the integer lattice and let $\tilde{\mathcal{A}} = (\tilde{a}_n)_{n \in \mathbb{Z}}$ denote

its randomly shifted version introduced above. Further, let

$$e_n = \mathbf{1}_{\{n \in \mathcal{A}\}} \quad \text{and} \quad \tilde{e}_n = \mathbf{1}_{\{n \in \tilde{\mathcal{A}}\}}, \quad n \in \mathbb{Z},$$

and denote $\Upsilon := (e_n, \omega_n)_{n \in \mathbb{Z}}$, $\tilde{\Upsilon} := (\tilde{e}_n, \tilde{\omega}_n)_{n \in \mathbb{Z}}$. The following theorem is an adaptation to our setup of the classical Palm's dualities [63, Chapter 8] between the distribution of Υ under P and that of $\tilde{\Upsilon}$ under an equivalent to P measure Q .

Theorem 2.2.1 (see Theorem 2 in [64]). *Assume that $(\lambda_n, d_n)_{n \in \mathbb{N}}$ is a stationary ergodic sequence under P and $E_P(d_0) < \infty$. Define a new probability measure Q on the Borel subsets of the product set $(\{0, 1\} \times (0, 1))^{\mathbb{Z}}$ by setting*

$$\frac{dQ}{dP}(v) = \frac{d_0(v)}{E_P(d_0)}, \quad v \in (\{0, 1\} \times (0, 1))^{\mathbb{Z}}. \quad (2.2)$$

Then:

- (a) $(\tilde{e}_n, \tilde{\omega}_n)_{n \in \mathbb{Z}}$ is a stationary and ergodic sequence under Q .
- (b) $P(\Upsilon \in \cdot) = Q(\tilde{\Upsilon} \in \cdot | \tilde{e}_0 = 1)$. In particular, $P(\mathcal{A} \in \cdot) = Q(\tilde{\mathcal{A}} \in \cdot | 0 \in \tilde{\mathcal{A}})$.
- (c) $E_Q(1/d_0) = 1/E_P(d_0)$.
- (d) $Q(\tilde{a}_0 = k) = P(d_0 > k)/E_P(d_0)$.
- (e) $Q(\Upsilon \in \cdot | \tilde{a}_0 = k) = P(\Upsilon \in \cdot | d_0 > k)$, $k \geq 0$.

We remark that claims (b) and (c) above form the content of the famous Kac's recurrence lemma (see Theorem 3.3. in [15, p. 348]). The following corollary is straightforward but will be useful in Chapter 3.

Corollary 2.2.2. *Under the conditions of Theorem 2.2.1, we have:*

- (a) $E_P(d_0^2) = E_P(d_0) \cdot E_Q(d_0)$.
- (b) $E_Q(d_0) = 2E_Q(\tilde{a}_0) + 1$.

Proof. The claim in (a) is a direct consequence of the definition of Q given in (2.2). Furthermore, by part (d) of Theorem 2.2, we have:

$$\begin{aligned}
E_Q(\tilde{a}_0) &= \frac{1}{E_P(d_0)} \sum_{k=0}^{\infty} kP(d_0 > k) = \frac{1}{E_P(d_0)} \sum_{k=0}^{\infty} \sum_{i=k+1}^{\infty} kP(d_0 = i) \\
&= \frac{1}{E_P(d_0)} \sum_{i=1}^{\infty} \sum_{k=0}^{i-1} kP(d_0 = i) = \frac{1}{E_P(d_0)} \sum_{i=1}^{\infty} \frac{i(i-1)}{2} P(d_0 = i) \\
&= \frac{E_P(d_0^2) - E_P(d_0)}{2E_P(d_0)}
\end{aligned}$$

This formula along with the result in part (a) yield the claim in part (b). \square

We remark that the identity $E_Q(d_0) = 2E_Q(a_0) + 1$ can be thought as a variation of the “waiting time paradox” of the classical renewal theory [15, 64].

For a non-zero integer n let η_n be the number of marked sites within the closed interval $I_n = \text{sign}(n) \cdot [1, n]$. More precisely, let $\eta_0 = 0$ and

$$\eta_n = \chi(I_n \cap \mathcal{A}) = \begin{cases} \sum_{k=1}^n \mathbf{1}_{\{k \in \mathcal{A}\}} & \text{if } n > 0 \\ \sum_{k=1}^n \mathbf{1}_{\{-k \in \mathcal{A}\}} & \text{if } n < 0, \end{cases} \quad (2.3)$$

where $\chi(A)$ stands for the cardinality of a finite set A . We conclude this section with the following observation.

Lemma 2.2.3. *If the sequence $(d_n)_{n \in \mathbb{Z}}$ is stationary and ergodic and $E_P(d_1) < \infty$, then*

$$\lim_{|n| \rightarrow \infty} \frac{\eta_n}{|n|} = Q(0 \in \mathcal{A}) = \frac{1}{E_P(d_1)}, \quad P - \text{a. s. and } Q - \text{a. s.} \quad (2.4)$$

Proof. The identity in (2.4) under the law Q follows directly from the ergodic theorem. Since the measures Q and P are equivalent by virtue of Theorem 2.2.1, the identity holds also under the measure P . \square

2.3 Renewal theory and a Palm duality for renewal sequences

If the random variables d_n are independent, then $(a_n)_{n \in \mathbb{N}}$ is a *renewal sequence* [18, 63]. For real $t \geq 0$, let

$$N_t = \inf\{n \geq 0 : a_n > t\} = \text{the number of renewals in } [0, t]$$

and

$$A_t = t - a_{N_t-1} = \text{age at } t$$

$$B_t = a_{N_t} - t = \text{residual life at } t.$$

We have:

Theorem 2.3.1. (*Renewal theorem [18, 63]*) Let $(d_n)_{n \in \mathbb{Z}}$ be i.i.d. random variables such that $E(d_0) < \infty$. Let U be a uniform random variable on $[0, 1]$, independent of $(d_n)_{n \in \mathbb{Z}}$. Then:

(a) $\lim_{t \rightarrow \infty} N_t/t = 1/E_P(d_0)$, a.s.

(b) The random pair (A_t, B_t) converges in distribution to the pair $(U\tilde{d}_1, (1-U)\tilde{d}_1)$ as $t \rightarrow \infty$.

For sparse environments induced by a renewal sequence a_n , the dual environment can be defined equivalently in a rather explicit manner using as a functional of an auxiliary Markov chain. We will exploit this alternative construction in Chapter 4. The uniqueness of the dual environment (which implies, in particular, that the alternative construction yields the same dual) follows from the reverse “stationary to cycle-stationary” Palm duality described, for instance, in [64, Theorem 1].

Let Y_n be the distance from the last marked site up to n . That is,

$$Y_n = n - a_{\eta_n} = n - \sup\{k \in \mathbb{Z} : k \leq n \text{ and } k \in \mathcal{A}\}, \quad n \in \mathbb{Z}. \quad (2.5)$$

Notice that $Y_{a_k} = 0$, $k \in \mathbb{N}$, and

$$Y_{n+1} - Y_n = 1 \quad \text{if} \quad a_{\eta_n} \leq n < a_{\eta_n+1}.$$

More specifically, if $Y_n = 0$ which means that $Y_n = n - a_{\eta_n}$, then $\omega_n = \lambda_n \neq \frac{1}{2}$ is on the marked site and if $Y_n > 0$ then $\omega_n = \lambda_n = \frac{1}{2}$. Observe that if the conditions of Theorem 2.3.1 hold for the sparse environment ω , then the sequence $Y = (Y_n)_{n \in \mathbb{Z}}$ under the law P is a positive-recurrent Markov chain on \mathbb{Z}_+ whose transition kernel $H(x, y) = P(Y_{n+1} = y | Y_n = x)$ is given by

$$H(x, y) = \begin{cases} \frac{P(d_0 > x+1)}{P(d_0 > x)} & \text{if } y = x+1, \\ \frac{P(d_0 = x+1)}{P(d_0 > x)} & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases} \quad x, y \in \mathbb{Z}_+ \quad (2.6)$$

The initial state of Y under P is zero, that is $P(Y_0 = 0) = 1$. It follows then from Theorem 2.2.1 that Z under the law Q is a stationary Markov chain on \mathbb{Z}_+ whose transition kernel is $H(x, y)$ and the initial distribution is the (unique) invariant distribution of H , that is, using the notation of Theorem 2.2.1,

$$Q(Y_0 = x) = P(\lfloor Ud_0 \rfloor = x) = \frac{P(d_0 > x)}{E_P(d_0)}, \quad x \in \mathbb{Z}_+.$$

This auxiliary Markov chain will be used in Chapter 4 as a key ingredient in the proof of stable laws for a transient RWSRE. Clearly, we have

$$\text{either } (e_n)_{n \in \mathbb{Z}} =_D (\mathbf{1}_{\{Y_n=0\}})_{n \in \mathbb{Z}} \quad \text{or} \quad (\tilde{e}_n)_{n \in \mathbb{Z}} =_D (\mathbf{1}_{\{Y_n=0\}})_{n \in \mathbb{Z}}, \quad n \in \mathbb{Z}, \quad (2.7)$$

according to whether the Markov chain Y_n is considered under the initial condition $Y_0 = 0$ or assumed to be stationary.

2.4 Environment viewed from the position of the particle

. In this section we study the “environment viewed from the particle” process $(\theta^{X_n} \omega)_{n \geq 0}$ for transient RWSRE. It is not hard to see that the pair $(\theta^{X_n} \omega, X_n)$ form a Markov chain which allows to consider X_n as a functional (projection into the second coordinate) of a Markov process. Even though the state space of this Markov chain is huge, the representation is useful due to the fact the underlying Markov turns out to be stationary and ergodic in the transient regime. The concept of the environment viewed from the particle was introduced by S. Kozlov in a broader context in [36] (see also [59] and [10, 11, 61]). In Section 4 we prove the existence of the asymptotic speed $v_P := \lim_{n \rightarrow \infty} X_n$ for RWSRE associated with a stationary and ergodic environmental process $(d_n, \lambda_n)_{n \in \mathbb{Z}}$ by using a standard direct approach. In fact, using the techniques described in [66, Section 2.2] and the existence of the dual environment one can prove the following result. By analogue with (1.2), let

$$\xi_n = \frac{1 - \lambda_n}{\lambda_n}, \quad n \in \mathbb{Z}. \quad (2.8)$$

We have:

Theorem 2.4.1. *Consider a RW X_n in a stationary and ergodic sparse environment $(\lambda_n, d_n)_{n \in \mathbb{Z}}$. Assume that $E_P(\log \xi_0)$ is well-defined (possibly infinite) and $E_P(d_0) < \infty$. Then*

- (a) $v_P > 0$ if and only if there exists a stationary distribution P° equivalent to P for the Markov chain $\bar{\omega}_n = \theta^{X_n}\omega$, $n \geq 0$. If such a distribution P° exists it is unique and is given by the following formula:

$$P^\circ(B) = v_P E_Q \left[E_\omega^0 \left(\sum_{n=0}^{T_1-1} \mathbf{1}_{\{\bar{\omega}_n \in B\}} \right) \right], \quad B \in \mathcal{F}, \quad (2.9)$$

where $\bar{\omega}_n := \theta^{\xi_n}\omega$ and Q is the distribution of the dual environment.

- (b) $(\bar{\omega}_n)_{n \geq 0}$ is an ergodic process under $\mathbb{P}^\circ := P^\circ \otimes P_\omega$.

- (c) $\frac{dP^\circ}{dP} = \frac{dQ}{dP} \times \Lambda(\omega) = \frac{d_0 \cdot \Lambda(\omega)}{E_P(d_0)}$, where

$$\Lambda(\omega) := \frac{1}{\omega_0} \left[1 + \sum_{i=1}^{\infty} \prod_{j=1}^i \rho_j \right] = \frac{1}{\omega_0} \left[d_1 + \sum_{i=1}^{\infty} d_{i+1} \times \prod_{j=1}^i \xi_j \right]. \quad (2.10)$$

- (d) $v_P = 1/E_Q(\Lambda) = \frac{E_P(d_0)}{E_P(d_0\Lambda)}$.

With one exception, the proof of the claims in this theorem is verbatim along the lines of the corresponding results in [66, Section 2.1] (namely, Lemmas 2.1.18, 2.1.20, 2.1.25 and Corollary 2.1.25 there). The only exception is the proof that the existence of the environment viewed from the position of the particle actually implies $v_P > 0$. The latter can be obtained by a straightforward modification of the proofs of, for instance, [10, Theorem 3.5 (ii)] or [49, Theorem 2.3]. The proof of Theorem 2.4.1 is therefore omitted.

Remark 2.4.2. (a) Notice that $E_P(\Lambda) = E_P(R)$, where $R(\omega)$ is introduced in (1.4). The formula for the asymptotic speed v_P given by Theorem 2.4.1 is thus consistent with (1.6).

- (b) The asymptotic speed for the simple nearest-neighborhood on \mathbb{Z} with probability of jumps forward p and jumps backward $q = 1 - p$ is $(p - q) = 2p - 1$. Although v_P is not equal to $E_Q(2\omega_0 - 1)$, quite remarkably it turns out to be equivalent to $E_{P^\circ}(2\omega_0 - 1)$ (see, for instance, formula (2.1.29) in [66]).

- (c) Formula (2.9) with $B = \Omega$ shows that $v_P = 1/E_Q(E_\omega(T_1))$, and hence

$$E_Q(E_\omega(T_1)) = 1/E_Q(\Lambda) = \frac{E_P(d_0)}{E_P(d_0\Lambda)}.$$

In Chapter 3 we will give a direct derivation of this identity under some mild extra condition (see Lemma 3.4.2 and Proposition 3.3.4 below).

We conjecture (but were unable to prove it) that for i.i.d. environments, similarly to the RWRE on \mathbb{Z}^d and on strips (see [31] for $d = 1$, [61] for $d > 1$, and [49] for RWRE on a strip), the law of the environment viewed from the particle converges toward its unique stationary distribution. More precisely, we have the following:

Conjecture 2.4.3. *Assume that $(d_n, \lambda_n)_{n \in \mathbb{Z}}$ is an i.i.d. sequence such that $E_P(\log \xi_0)$ is well defined and $v_P > 0$. Then $\theta^{X_n} \omega$ converges weakly to the limiting distribution P° as n goes to infinity.*

CHAPTER 3. ASYMPTOTIC BEHAVIOR OF RWSRE

3.1 Introduction

The goal of this chapter is to obtain recurrence and transience criteria and laws of large numbers for RWSRE. Main results of this chapter are stated in Theorems 3.3.1, (3.3.2) (recurrence and transience criteria) and Theorem 3.3.3 below (formula for the asymptotic speed and the corresponding law of large numbers for T_n).

The chapter is organized as follows. In Section 3.2 we recall some related results for the regular RWRE. Our main results are stated in Section 3.3 which also contains proofs of Theorems 3.3.1 and (3.3.2). The proof of Theorem 3.3.3 is deferred to Section 3.4.

We remark that the existence of the asymptotic speed for RWSRE is evident from the results stated in Chapter 2. One then can use Theorem 2.2.1 and Corollary 2.2.2 to translate the formula for v_P given in Theorem 2.4.1 into a formula with expectations taking under the underlying probability law P . One disadvantage of this proof is that it would leave obscure the important relation between the expectations of $E_P(T_{a_1})$, $E_P(T_1)$, and the asymptotic speed. More importantly, the heavy theoretical machinery presented in Chapter 2 is specific for a motion in random media and cannot be carried out to other classes of self-interacting random walks (such as, for instance, excited random walks [35]). Furthermore, even though the formula for the speed in terms of Q is a direct consequence of the corresponding formula for the RWRE and the fact that Q and P are equivalent, its “mechanical” manipulation and translation into “under P ” terms is quite intensive computationally. We therefore prefer in this chapter, in order to prove the LLN for RWSRE, to carry over to our framework the program of Solomon and Alili and to consider the random walk directly under the non-stationary law P . The latter approach is based on a general correspondence between the path of an one-dimensional random

walk and the values of its hitting times and direct analysis of the asymptotic behavior of the latter.

3.2 Preliminaries: Asymptotic behavior of RWRE on \mathbb{Z}

In this section we recall recurrence and transience criteria and laws of large numbers (for X_n and T_n) for the regular RWRE in a stationary and ergodic environment. We also include here several auxiliary lemmas playing a key role in the Solomon's proof of the above results. Suitable extensions of these auxiliary lemmas to the RSSRE setup are cornerstones in our proofs of the main results of this chapter.

3.2.1 Recurrence and transience criteria for RWRE

Recall ρ_n from (1.2) and T_n from (1.5). Fix an environment ω such that $|\log \rho_z| < \infty$ for each $z \in \mathbb{Z}$. For any $m_-, m_+ \in \mathbb{N}$ and an integer $z \in [-m_-, m_+]$, define

$$\phi_{\omega,z}(m_-, m_+) := P_\omega^z(\{X_n\} \text{ hits } -m_- \text{ before hitting } m_+) = P_\omega^z(T_{-m_-} < T_{m_+}).$$

Due to the Markov Property, $\phi_{\omega,z}(a_-, a_+)$ as a function of z is harmonic function for the random walk. Namely, it satisfies the following equation

$$\phi_{\omega,z}(m_-, m_+) = (1 - \omega_z)\phi_{\omega,z-1}(m_-, m_+) + \omega_z\phi_{\omega,z+1}(m_-, m_+)$$

with boundary conditions $\phi_{\omega,-m_-}(m_-, m_+) = 1$ and $\phi_{\omega,m_+}(m_-, m_+) = 0$. This yields (see, for instance, [66, Section 2.1]) the following explicit formula for the hitting probabilities in a fixed environment:

$$\phi_{\omega,z}(m_-, m_+) = \frac{\sum_{i=z+1}^{m_+} \prod_{j=z+1}^{i-1} \rho_j}{\sum_{i=z+1}^{m_+} \prod_{j=z+1}^{i-1} \rho_j + \sum_{i=-m_-+1}^z \prod_{j=i}^z \rho_j^{-1}}. \quad (3.1)$$

The formula illuminates the role of the random potential $V_n = \log \prod_{j=1}^n \rho_j$ in the theory of one-dimensional RWRE.

Let

$$S(\omega) = \sum_{k=1}^{\infty} \rho_1 \rho_2 \cdots \rho_k \quad \text{and} \quad F(\omega) = \sum_{k=0}^{\infty} \rho_0^{-1} \rho_{-1}^{-1} \cdots \rho_{-k}^{-1}. \quad (3.2)$$

The following two propositions are the key to the recurrence and transience criteria for RWRE [66]. The propositions are deduced from (3.1) which relates the asymptotic behavior of the random walk to the asymptotic behavior of the exit probabilities as $m_{\pm} \rightarrow \infty$.

Proposition 3.2.1. *Let $X = (X_n)_{n \geq 0}$ be a random walk in a stationary and ergodic environment ω . Assume that $E_P(\log \rho_0)$ is well-defined, possibly infinite. Then*

- (a) $P(S(\omega) < \infty) = 1 \Rightarrow \lim_{n \rightarrow \infty} X_n = +\infty, \mathbb{P} - \text{a. s.},$
- (b) $P(F(\omega) < \infty) = 1 \Rightarrow \lim_{n \rightarrow \infty} X_n = -\infty, \mathbb{P} - \text{a. s.},$
- (c) $P(S(\omega) = F(\omega) = \infty) = 1 \Rightarrow \limsup_{n \rightarrow \infty} X_n = +\infty \text{ and } \liminf_{n \rightarrow \infty} X_n = -\infty, \mathbb{P} - \text{a. s.}$

Proposition 3.2.2. *Let $X = (X_n)_{n \geq 0}$ be a random walk in a stationary and ergodic environment ω . Assume that $E_P(\log \rho_0)$ is well-defined, possibly infinite. Then*

- (a) $P(S(\omega) < \infty) = 1 \iff E_P(\log \rho_0) < 0,$
- (b) $P(F(\omega) < \infty) = 1 \iff E_P(\log \rho_0) > 0,$
- (c) $P(S(\omega) = F(\omega) = \infty) = 1 \iff E_P(\log \rho_0) = 0.$

This leads to the following recurrence and transience criteria of [56] and [2]:

Theorem 3.2.3. *Let $X = (X_n)_{n \geq 0}$ be a random walk in a stationary and ergodic environment ω . Assume that $E_P(\log \rho_0)$ is well-defined, possibly infinite. Then*

- (a) *If $E_P(\log \rho_0) < 0$ then, $\lim_{n \rightarrow \infty} X_n = +\infty, \mathbb{P} - \text{a. s.}$*
- (b) *If $E_P(\log \rho_0) > 0$ then, $\lim_{n \rightarrow \infty} X_n = -\infty, \mathbb{P} - \text{a. s.}$*
- (c) *If $E_P(\log \rho_0) = 0$ then, $\limsup_{n \rightarrow \infty} X_n = +\infty$ and $\liminf_{n \rightarrow \infty} X_n = -\infty, \mathbb{P} - \text{a. s.}$*

3.2.2 Law of large numbers. Asymptotic speed of the random walk

In this section we review the laws of large numbers for X_n and T_n for classical RWRE. The proofs of these statements outlined below are due to [56] and [2]. Let

$$\bar{S} := \sum_{i=1}^{\infty} \frac{1}{w_{-i}} \prod_{j=0}^{i-1} \rho_{-j} + \frac{1}{w_0} = 1 + 2 \sum_{i=0}^{\infty} \prod_{j=0}^i \rho_{-j} \quad (3.3)$$

and

$$\overline{F} := \sum_{i=1}^{\infty} \frac{1}{(1-w_i)} \prod_{j=0}^{i-1} \rho_j^{-1} + \frac{1}{(1-w_0)}.$$

It is well-known (and straightforward to verify) that

$$E_P(\overline{S}) = 2E_P(R) - 1 = 2E_P(\Lambda) - 1, \quad (3.4)$$

where R and Λ are defined in (1.4) and (2.10) respectively. Similar identities hold for \overline{F} .

For the asymptotic speed of the random walk, we have:

Theorem 3.2.4. [56, 2] *Let $X = (X_n)_{n \geq 0}$ be a random walk in a stationary and ergodic environment ω . Assume that $E_P(\log \rho_0)$ is well-defined, possibly infinite. Then*

- (a) *If $E_P(\overline{S}) < \infty$, then $\lim_{n \rightarrow \infty} \frac{X_n}{n} = \frac{1}{E_P(\overline{S})}$, \mathbb{P} - a. s.*
- (b) *If $E_P(\overline{F}) < \infty$, then $\lim_{n \rightarrow \infty} \frac{X_n}{n} = -\frac{1}{E_P(\overline{F})}$, \mathbb{P} - a. s.*
- (c) *If $E_P(\overline{S}) = \infty$ and $E_P(\overline{S}) = \infty$, then $\lim_{n \rightarrow \infty} \frac{X_n}{n} = 0$, \mathbb{P} - a. s.*

Furthermore, the three cases listed above are exhausting all the possibilities.

Corollary 3.2.5. *Let the conditions of Theorem 3.2.4 hold. If, in addition, ω is an i.i.d. environment then:*

- (a) *If $E_P(\rho_0) < 1$, then $\lim_{n \rightarrow \infty} \frac{X_n}{n} = \frac{1-E_P(\rho_0)}{1+E_P(\rho_0)}$, \mathbb{P} - a. s.*
- (b) *If $E_P(\rho_0^{-1}) < 1$, then $\lim_{n \rightarrow \infty} \frac{X_n}{n} = -\frac{1-E_P(\rho_0^{-1})}{1+E_P(\rho_0^{-1})}$, \mathbb{P} - a. s.*
- (a) *If $E_P(\rho_0)^{-1} \leq 1 \leq E_P(\rho_0^{-1})$, then $\lim_{n \rightarrow \infty} \frac{X_n}{n} = 0$, \mathbb{P} - a. s.*

Notice that, since the process learns about the environment as time passes according to the Bayes' rule, $\{X_n\}$ is in general not a Markov chain under the annealed measure \mathbb{P} and the increments $\{X_n - X_{n-1}\}$ is in general not even stationary. Thus one cannot apply the ergodic theorem to X_n/n directly. The key idea of the proof of Theorem 3.2.4 given by [56] and [2] is to analyze first the hitting time T_n and then deduce the claim for the random walk from the corresponding result for T_n . More precisely, the proof is based on the following three auxiliary results (see [66] for the details):

Lemma 3.2.6. *Let the conditions of Theorem 3.2.4 hold. Then:*

$$(a) \mathbb{E}(T_1) = E_P(\bar{S})$$

$$(b) \mathbb{E}(T_{-1}) = E_P(\bar{F})$$

Lemma 3.2.7. *Let the conditions of Theorem 3.2.4 hold. Suppose that $\limsup_{n \rightarrow \infty} X_n = +\infty$, \mathbb{P} -a.s. Then the sequence $\{\tau_n\}_{n \geq 1}$ defined in (1.5) is stationary and ergodic. In particular,*

$$\lim_{n \rightarrow \infty} \frac{T_n}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \tau_i = \mathbb{E}(\tau_1) = E_P(\bar{S}), \quad \mathbb{P} - \text{a.s.}$$

Lemma 3.2.8. *Let the conditions of Theorem 3.2.4 hold and let $\alpha \in (0, +\infty]$. Then*

$$(a) \text{ If } E_P(\log \rho_0) < 0 \text{ and } \lim_{n \rightarrow \infty} \frac{T_n}{n} = \alpha, \text{ then } \lim_{n \rightarrow \infty} \frac{X_n}{n} = \frac{1}{\alpha}, \mathbb{P} - \text{a.s.}$$

$$(b) \text{ If } E_P(\log \rho_0) > 0 \text{ and } \lim_{n \rightarrow \infty} \frac{T_{-n}}{n} = \alpha, \text{ then } \lim_{n \rightarrow \infty} \frac{X_n}{n} = -\frac{1}{\alpha}, \mathbb{P} - \text{a.s.}$$

$$(c) \text{ If } E_P(\log \rho_0) = 0 \text{ and } \lim_{n \rightarrow \infty} \frac{T_n}{n} = \lim_{n \rightarrow \infty} \frac{T_{-n}}{n} = \infty, \text{ then } \lim_{n \rightarrow \infty} \frac{X_n}{n} = 0, \mathbb{P} - \text{a.s.}$$

3.3 Asymptotic behavior of RWSRE

In this section we obtain recurrence and transience criteria and laws of large numbers for RWSRE.

3.3.1 Recurrence and transience criteria

Let $\sigma_0 = 0$ and

$$\sigma_n = \inf\{k \in \mathbb{N} : k > \sigma_{n-1} \text{ and } X_k \in \mathcal{A}\}.$$

Thus σ_n are the times of successive visits of X_n to the random point set \mathcal{A} . Define a nearest-neighbor random walk $(\mathbb{X}_n)_{n \geq 0}$ on \mathbb{Z} by setting

$$\mathbb{X}_n = k \quad \text{if and only if} \quad X_{\sigma_n} = a_k. \quad (3.5)$$

Taking in account the solution of the gambler's ruin problem for the usual simple symmetric random walk, note that \mathbb{X}_n is a RWRE with quenched transition probabilities given by

$$P_\omega(\mathbb{X}_{n+1} = j | \mathbb{X}_n = i) = \begin{cases} \lambda_i \cdot \frac{1}{d_i} & \text{if } j = i + 1 \\ (1 - \lambda_i) \cdot \frac{1}{d_{i-1}} & \text{if } j = i - 1 \\ \lambda_i \cdot \frac{d_i - 1}{d_i} + (1 - \lambda_i) \cdot \frac{d_{i-1} - 1}{d_{i-1}} & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases} \quad (3.6)$$

Furthermore, clearly the following holds \mathbb{P} - a. s. :

$$\limsup_{n \rightarrow \infty} X_n = \limsup_{n \rightarrow \infty} \mathbb{X}_n \quad \text{and} \quad \liminf_{n \rightarrow \infty} X_n = \liminf_{n \rightarrow \infty} \mathbb{X}_n. \quad (3.7)$$

Thus, under very mild conditions, recurrence-transience criteria for the RWSRE X_n can be derived directly from that for the RWRE \mathbb{X}_n (see, for instance, [66, Theorem 2.1.2] for the latter). More precisely, we have:

Theorem 3.3.1. *Suppose that the following three conditions are satisfied:*

1. *The sequence of pairs $(d_n, \lambda_n)_{n \in \mathbb{Z}}$ is stationary and ergodic*
2. *$E_P(\log \xi_0)$ exists (possibly infinite)*
3. *$E_P(\log d_0) < +\infty$.*

Then:

- (a) *$E_P(\log \xi_0) < 0$ implies $\lim_{n \rightarrow \infty} X_n = +\infty$, \mathbb{P} - a. s.*
- (b) *$E_P(\log \xi_0) > 0$, implies $\lim_{n \rightarrow \infty} X_n = -\infty$, \mathbb{P} - a. s.*
- (c) *$E_P(\log \xi_0) = 0$ implies $\liminf_{n \rightarrow \infty} X_n = -\infty$ and $\limsup_{n \rightarrow \infty} X_n = +\infty$, \mathbb{P} - a. s.*

Proof. It follows from (3.6) that under the condition of the theorem,

$$E_P(\log \rho_0) = E_P(\log \xi_0).$$

The claim then follows from the equivalence (3.7) and Proposition 3.2.2¹. □

¹Notice the the quenched kernel of the RWRE \mathbb{X} allows $P_\omega(\mathbb{X}_{n+1} = i | \mathbb{X}_n = i) > 0$ in contrast to the RWRE model formally defined in Section 1.1. However, the non-zero probability of staying put is effectively equivalent to the introducing "holding times" in the model and clearly cannot change its recurrence (or transience) behavior. In particular, Proposition 3.2.2 remains true for this more general model with the random characteristics ρ_i defines as follows: $\rho_i := P_\omega(\mathbb{X}_{n+1} = i - 1 | \mathbb{X}_n = i) / P_\omega(\mathbb{X}_{n+1} = i + 1 | \mathbb{X}_n = i)$. See [66, Section 2.1] for details.

The above result implies that as long $E_P(\log d_0)$ is finite, the sparse environment ω induces the same recurrence-transience behavior as the underlying random environment λ . The following theorem shows that the opposite phenomenon (namely, the property of λ are essentially irrelevant to the basic asymptotic behavior of X_n) occurs when $E_P(\log d_0) = +\infty$.

Theorem 3.3.2. *Suppose that the following conditions hold:*

1. *The sequence of pairs $(d_n, \lambda_n)_{n \in \mathbb{Z}}$ is stationary and ergodic*
2. *The random variables d_n are i.i.d.*
3. *$E_P(|\log \xi_0|) < +\infty$ while $E_P(\log d_0) = +\infty$.*

Then, $\liminf_{n \rightarrow \infty} X_n = -\infty$ and $\limsup_{n \rightarrow \infty} X_n = +\infty$, $\mathbb{P} - \text{a.s.}$

Note that it is not assumed in the conditions of the theorem that $(d_n)_{n \in \mathbb{N}}$ is necessarily independent of $(\lambda_n)_{n \in \mathbb{Z}}$.

Proof of Theorem 3.3.2. Recall η_n from (2.3). Denote

$$S(\omega) = \sum_{k=1}^{\infty} \rho_1 \rho_2 \cdots \rho_k \quad \text{and} \quad F(\omega) = \sum_{k=0}^{\infty} \rho_0^{-1} \rho_{-1}^{-1} \cdots \rho_{-k}^{-1}. \quad (3.8)$$

To prove Theorem 3.3.2 it suffices to show (see, for instance, the proof of [66, Theorem 2.1.2]²) that the conditions of the theorem imply

$$P(S(\omega) = F(\omega) = +\infty) = 1 \quad (3.9)$$

Toward this end, note that $\eta_{a_n} = \eta_{a_n+1} = \dots = \eta_{a_{n+1}-1} = n$ for $n \geq 0$, and hence

$$S(\omega) = \sum_{k=1}^{\infty} \xi_1 \xi_2 \cdots \xi_{\eta_k} = (d_1 - 1) + \sum_{n=1}^{\infty} \xi_1 \xi_2 \cdots \xi_n \cdot d_{n+1}, \quad (3.10)$$

where, to claim the first identity, we used a standard convention that $\xi_1 \xi_2 \cdots \xi_{\eta_k} = 1$ if $\eta_k = 0$.

Similarly, $\eta_{a_n} = \eta_{a_n+1} = \dots = \eta_{a_{n+1}-1} = n + 1$ for $n < 0$, and hence

$$F(\omega) = \sum_{k=1}^{\infty} \xi_0^{-1} \xi_{-1}^{-1} \xi_{-2}^{-1} \cdots \xi_{-\eta_k}^{-1} = \sum_{n=0}^{\infty} \xi_0^{-1} \xi_{-1}^{-1} \cdots \xi_{-n}^{-1} \cdot d_{-n}. \quad (3.11)$$

²This claim is merely a statement that by virtue of (3.1), the condition in (3.9) implies that for any $k > 0$, $1 - \lim_{m \rightarrow \infty} \phi_{\omega,0}(-m, k) = \lim_{m \rightarrow \infty} \phi_{\omega,0}(-k, m) = 0$ and hence $P_{\omega}^0(\limsup X_n = -\liminf X_n = +\infty) = 1$.

Furthermore, the condition $E_P(\log d_0) = +\infty$ implies that $\sum_{n=1}^{\infty} P(\log d_0 > M \cdot n) = \infty$ for any $M > 0$. Thus, since d_n are i.i.d., it follows from the second Borel-Cantelli lemma that $P(\log d_n > M \cdot n \text{ i. o.}) = 1$ for any $M > 0$. Hence, the ergodic theorem along with the condition $E_P(|\log \xi_0|) < +\infty$ imply that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left(d_{n+1} \cdot \prod_{k=1}^n \xi_k \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sum_{k=1}^n \log \xi_k + \log d_{n+1} \right) = +\infty, \quad P - \text{a. s.},$$

which yields $P(S(\omega) = +\infty) = 1$. A similar argument shows that, under the conditions of the theorem, $P(F(\omega) = +\infty) = 1$ and hence (3.9) holds, as desired. \square

3.3.2 Transient RWSRE: asymptotic speed

We now turn to the law of the large numbers for X_n . Whenever it exists, $\lim_{n \rightarrow \infty} X_n/n$ is referred to as the *asymptotic speed* of the random walk. Let

$$\bar{S}_\lambda = 1 + 2 \sum_{i=0}^{\infty} \prod_{j=0}^i \xi_j \quad \text{and} \quad \bar{F}_\lambda = 1 + 2 \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \xi_{-j}^{-1}. \quad (3.12)$$

We have:

Theorem 3.3.3. *Let the conditions of Theorem 3.3.1 hold. Suppose in addition that $(d_n)_{n \in \mathbb{Z}}$ is independent of $(\lambda_n)_{n \in \mathbb{Z}}$ under P . Then the asymptotic speed of the RWSRE exists $\mathbb{P} - \text{a. s.}$ Moreover,*

$$\mathbb{P} \left(\lim_{n \rightarrow \infty} X_n/n = v_P \right) = \mathbb{P} \left(\lim_{n \rightarrow \infty} T_n/n = 1/v_P \right) = 1,$$

where $v_P \in (-1, 1)$ is a constant whose reciprocal v_P^{-1} is equal to

$$\begin{aligned} \frac{1}{v_P} &= \mathbf{1}_{\{\lim_{n \rightarrow \infty} X_n = +\infty\}} \left[\frac{\text{VAR}_P(d_0)}{E_P(d_0)} + E_P(\bar{S}_\lambda) \cdot E_P(d_0) \right] \\ &\quad - \mathbf{1}_{\{\lim_{n \rightarrow \infty} X_n = -\infty\}} \cdot \left[\frac{\text{VAR}_P(d_0)}{E_P(d_0)} + E_P(\bar{F}_\lambda) \cdot E_P(d_0) \right], \quad \mathbb{P} - \text{a. s.} \end{aligned} \quad (3.13)$$

The proof of this theorem is deferred to Section 3.4. Notice that if λ_i (and hence ξ_i) are i.i.d., then (3.13) is reduced to

$$\begin{aligned} \frac{1}{v_P} &= \mathbf{1}_{\{\lim_{n \rightarrow \infty} X_n = +\infty\}} \cdot \left[\frac{\text{VAR}_P(d_0)}{E_P(d_0)} + E_P(d_0) \cdot \frac{1 + E_P(\xi_0)}{1 - E_P(\xi_0)} \right] \\ &\quad - \mathbf{1}_{\{\lim_{n \rightarrow \infty} X_n = -\infty\}} \cdot \left[\frac{\text{VAR}_P(d_0)}{E_P(d_0)} + E_P(d_0) \cdot \frac{1 + E_P(\xi_0^{-1})}{1 - E_P(\xi_0^{-1})} \right], \quad \mathbb{P} - \text{a. s.} \end{aligned}$$

In order to compare (3.13) with the corresponding result for the regular RWRE, note that if $\lim_{n \rightarrow \infty} X_n = +\infty$, \mathbb{P} – a. s., then (3.13) is reduced to

$$\frac{1}{v_P} = \frac{\text{VAR}_P(d_0)}{E_P(d_0)} + E_P(d_0) \cdot E_P(\bar{S}_\lambda), \quad \mathbb{P} - \text{a. s.}$$

Recall the dual environment $\tilde{\omega}$ defined in Section 2.2. Let

$$\tilde{\rho}_n = \frac{1 - \tilde{\omega}_n}{\tilde{\omega}_n}, \quad n \in \mathbb{Z}, \quad (3.14)$$

and (compare with \bar{S} defined in (3.3))

$$\tilde{S} = \sum_{i=1}^{\infty} (1 + \tilde{\rho}_{-i}) \prod_{j=0}^{i-1} \tilde{\rho}_{-j} + 1 + \tilde{\rho}_0 = 1 + 2 \sum_{i=0}^{\infty} \prod_{j=0}^i \tilde{\rho}_{-j}, \quad (3.15)$$

and recall from Theorem 3.2.4 above that the asymptotic speed of the usual RWRE (for which $\rho_n = \tilde{\rho}_n$) equals $1/E_P(\tilde{S})$. The proof following observation is immediate from (3.11) and Corollary 2.2.2.

Proposition 3.3.4. *Let the conditions of Theorem 3.3.3 hold. Suppose in addition that*

1. $\lim_{n \rightarrow \infty} X_n = +\infty$, \mathbb{P} – a. s.
2. $E_P(d_0) < \infty$.

Then, $v_P = 1/E_Q(\tilde{S})$.

Note that a similar to Proposition 3.3.4 result for the random transient to the left (that is, when $\lim_{n \rightarrow \infty} X_n = -\infty$, \mathbb{P} – a. s.) can be obtained by replacing $\tilde{\rho}_{-k}$ with ρ_k^{-1} in the formula (3.15) for \tilde{S} .

Slowdown of the random walk in a random media comparing to simple random walk with comparable parameters is a well-known general paradigm [19, 59, 66]. The phenomenon is explained heuristically by fluctuations in the associated random potential. For example, a random walk transient to the right will pass quickly stretches of the environment “pushing” it forward but will be “trapped” for a long time in atypical stretches “pushing” it backward. The slowdown phenomenon has various manifestations, including for instance large deviations-type results. To illustrate the idea, we next state a simple corollary to Theorem 3.3.3 showing in

essence that the slowdown effect on a transient RWSRE is minimal when both d_0 and λ_0 are deterministic constants rather than random variables. Similar results for the usual RWRE can be found, for instance, in [46] and [62]. More precisely, Theorem 3.3.3 yields immediately the following version of Theorem 1.3 and Corollary 1.4 in [46]. For any constants $\mu > 0$ and $\nu \geq 0$ denote by $\mathcal{P}_{\mu,\nu}^\circ$ the set of distributions $(\lambda_n, d_n)_{n \in \mathbb{Z}}$ for which the conditions of Theorem 3.3.3 hold and, moreover,

$$E_P(d_0) = \mu \quad \text{and} \quad 1/E_P(\bar{S}) = \nu.$$

We have:

Corollary 3.3.5. $\max_{P \in \mathcal{P}_{\mu,\nu}^\circ} v_P = \nu/\mu$. Furthermore, the maximum is attained at $P \in \mathcal{P}_{\mu,\nu}^\circ$ if and only if $\text{VAR}_P(d_0) = 0$.

Combining this result with [62, Theorem 4.1], we obtain the following. For any constants $\mu > 0$ and $b < 0$ denote by $\mathcal{P}_{\mu,\nu}^*$ the set of distributions $(\lambda_n, d_n)_{n \in \mathbb{Z}}$ for which the conditions of Theorem 3.3.3 hold and, moreover,

$$E_P(d_0) = \mu \quad \text{and} \quad E_P(\log \xi_0) = b.$$

Corollary 3.3.6. *We have:*

$$\max_{P \in \mathcal{P}_{\mu,\nu}^*} v_P = \frac{1}{\mu} \cdot \frac{1 - e^b}{1 + e^b}.$$

Furthermore, the maximum is attained at $P \in \mathcal{P}_{\mu,\nu}^\circ$ if and only if $\text{VAR}_P(d_0) = \text{VAR}_P(\lambda_0) = 0$, in which case $\lambda_0 = \frac{1}{1+e^b}$, P - a.s.

3.4 Proof of Theorem 3.3.3

We begin with the following version of Lemma 3.2.7 suitable to our setup.

Lemma 3.4.1. *Let the conditions of Theorem 3.3.3 hold. Suppose that $\limsup_{n \rightarrow \infty} X_n = +\infty$, \mathbb{P} - a.s. Then the sequence $\{T_{a_n} - T_{a_{n-1}}\}_{n \geq 1}$ defined is stationary and ergodic. In particular,*

$$\lim_{n \rightarrow \infty} \frac{T_n}{n} = \mathbb{E}(T_{a_1}), \quad \mathbb{P} - \text{a.s.}$$

The proof of the lemma is a straightforward but tedious modification of the proof of Lemma 3.2.7 and therefore is omitted. In fact, a stronger than the ergodicity property, namely *strong mixing*, holds for the sequence $\{T_{a_n} - T_{a_{n-1}}\}_{n \geq 1}$. Again, the proof of the corresponding statement for RWRE (see [66, Lemma 2.1.10]) goes through almost verbatim for RWSRE. Heuristically, the mixing property of the hitting times is a very general result (see, for instance, [49]) and is due to the fact that a transient to the right random walk is unlikely to go far backward and hence with a high probability $(X_k)_{k \leq n}$ and $(X_k)_{k \geq n+m}$ do not intersect their paths (and therefore are roughly independent) for large m .

The next lemma is a counterpart of Lemma 3.2.6 in the RWSRE framework.

Lemma 3.4.2. *Assume that the conditions of Theorem 3.3.1 hold and suppose, in addition, that λ and \mathcal{A} are independent under P . Then:*

$$(a) \mathbb{E}(T_{a_1}) = \text{VAR}_P(d_1) + E_P(\bar{S}_\lambda) \cdot [E_P(d_1)]^2,$$

$$(b) \mathbb{E}(T_{a_{-1}}) = \text{VAR}_P(d_1) + E_P(\bar{F}_\lambda) \cdot [E_P(d_1)]^2.$$

Proof of Lemma 3.4.2. We will only proof the result in (a), the proof of (b) being similar. To evaluate T_{a_1} , we will use a decomposition of the paths of the random walk according to its first step:

$$\begin{aligned} T_{a_1} = 1 + \mathbf{1}_{\{X_1=1\}} [\mathbf{1}_{\{\tilde{T}_0 < \tilde{T}_{a_1}\}} (\tilde{T}_0 + T'_{a_1}) + \mathbf{1}_{\{\tilde{T}_0 > \tilde{T}_{a_1}\}} \tilde{T}_{a_1}] \\ + \mathbf{1}_{\{X_1=-1\}} [\mathbf{1}_{\{\hat{T}_0 < \hat{T}_{a_{-1}}\}} (\hat{T}_0 + T''_{a_1}) + \mathbf{1}_{\{\hat{T}_0 > \hat{T}_{a_{-1}}\}} (\hat{T}_{a_{-1}} + T'_0 + T'''_{a_1})], \end{aligned} \quad (3.16)$$

where

$$\begin{aligned} \tilde{T}_0 &= \inf\{n > T_1 : X_n = 0\}, & \tilde{T}_0 + T'_{a_1} &= \inf\{n > \tilde{T}_0 : X_n = a_1\}, \\ \tilde{T}_{a_1} &= \inf\{n > T_1 : X_n = a_1\}, & \hat{T}_0 &= \inf\{n > T_{-1} : X_n = 0\}, \\ \hat{T}_0 + T''_{a_1} &= \inf\{n > \hat{T}_0 : X_n = a_1\}, & \hat{T}_{a_{-1}} &= \inf\{n > T_{-1} : X_n = a_{-1}\}, \\ \hat{T}_{a_{-1}} + T'_0 &= \inf\{n > \hat{T}_{a_{-1}} : X_n = 0\}, & \hat{T}_{a_{-1}} + T'_0 + T'''_{a_1} &= \inf\{n > T_{a_{-1}} + T'_0 : X_n = a_1\}. \end{aligned}$$

Let \mathcal{P}_x denote the law of the (simple symmetric) nearest-neighbor random walk on \mathbb{Z} with equal probabilities of jumping to the left and to the right, starting deterministically at $x \in \mathbb{Z}$. Let \mathcal{E}_x denote the corresponding expectation operator. Taking quenched expectations $E_\omega(\cdot)$ in

both sides of (3.16) yields

$$\begin{aligned} E_\omega(T_{a_1}) &= 1 + \lambda_0[\mathcal{E}_1(T_0 \wedge T_{a_1}) + \mathcal{P}_1(T_0 < T_{a_1})E_\omega(T_{a_1})] \\ &+ (1 - \lambda_0)[\mathcal{E}_{-1}(T_0 \wedge T_{a_{-1}}) + E_\omega(T_{a_1}) + \mathcal{P}_{-1}(T_{a_{-1}} < T_0)E_{\omega, a_{-1}}(T_0)]. \end{aligned}$$

Using the solution of the gambler's ruin problem under \mathcal{P} , we obtain

$$\begin{aligned} \frac{\lambda_0}{a_1} E_\omega(T_{a_1}) &= 1 + \lambda_0(a_1 - 1) + (1 - \lambda_0)(|a_{-1}| - 1) + \frac{1 - \lambda_0}{|a_{-1}|} E_{\omega, a_{-1}}(T_0) \\ &= \lambda_0 a_1 + (1 - \lambda_0)|a_{-1}| + \frac{1 - \lambda_0}{|a_{-1}|} E_{\omega, a_{-1}}(T_0). \end{aligned}$$

Thus,

$$\frac{1}{a_1} E_\omega(T_{a_1}) = a_1 + \xi_0 |a_{-1}| + \xi_0 \cdot \frac{1}{|a_{-1}|} E_{\omega, a_{-1}}(T_0).$$

Iterating yields

$$\frac{1}{a_1} E_\omega(T_{a_1}) = a_1 + 2 \sum_{k=0}^{\infty} \xi_0 \xi_{-1} \cdots \xi_{-k} \cdot d_{-k}.$$

Taking expectation with respect to P and using a truncation argument similar to that given in the proof of [66, Lemma 2.1.12] in order to verify when $\mathbb{E}(T_{a_1}) < \infty$, we obtain

$$\mathbb{E}(T_{a_1}) = \text{VAR}_P(d_1) + [E_P(d_1)]^2 \cdot E_P(\bar{S}_\lambda),$$

as desired. This completes the proof of (a) of the lemma. Part (b) can be derived along the same lines, and hence its proof is omitted. \square

In view of Lemma 3.4.2, we are now in a position to finish the proof of Theorem 3.3.3.

Lemma 3.4.3. *Assume that the conditions of Theorem 3.3.1 hold and suppose, in addition, that $\lim_{n \rightarrow \infty} \frac{T_{a_n}}{n} = \alpha$, \mathbb{P} -a.s., for some constant $\alpha \leq \infty$. Then,*

$$\lim_{n \rightarrow \infty} \frac{T_n}{n} = \frac{\alpha}{E_P(d_1)} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{X_n}{n} = \frac{E_P(d_1)}{\alpha}, \quad \mathbb{P} - \text{a. s.}$$

Proof of Lemma 3.4.3. First, observe that (2.3) implies

$$a_{\eta_n} \leq n < a_{\eta_n+1}, \quad \mathbb{P} - \text{a. s.}$$

Thus, in view of (2.3),

$$\lim_{n \rightarrow \infty} \frac{T_n}{n} = \lim_{n \rightarrow \infty} \frac{T_{a_{\eta_n}+1}}{n} = \lim_{n \rightarrow \infty} \frac{T_{a_{\eta_n}}}{\eta_n} \cdot \frac{\eta_n}{n} = \frac{\alpha}{E_P(d_1)}, \quad \mathbb{P} - \text{a. s.}$$

Let now $\zeta(n) \in \mathbb{Z}$ be the unique nonnegative random number such that

$$T_{a_{\zeta(n)}} \leq n < T_{a_{\zeta(n)+1}}. \quad (3.17)$$

Since X_n is transient to the right, $\mathbb{P}(\lim_{n \rightarrow \infty} \zeta(n) = \infty) = 1$. Furthermore, (3.17) implies that

$$X_n < a_{\zeta(n)+1} \quad \text{and} \quad X_n \geq a_{\zeta(n)} - (n - T_{a_{\zeta(n)}}).$$

Thus,

$$\frac{a_{\zeta(n)}}{n} - \left(1 - \frac{T_{a_{\zeta(n)}}}{n}\right) \leq \frac{X_n}{n} < \frac{a_{\zeta(n)+1}}{n}.$$

But (3.17) along with the existence of $\lim_{n \rightarrow \infty} \frac{n}{T_n}$ yield

$$\lim_{n \rightarrow \infty} \frac{a_{\zeta(n)}}{n} = \lim_{n \rightarrow \infty} \frac{a_{\zeta(n)}}{T_{a_{\zeta(n)}}} = \lim_{n \rightarrow \infty} \frac{n}{T_n} = \frac{E_P(d_1)}{\alpha}, \quad \mathbb{P} - \text{a. s.}$$

Hence,

$$\frac{E_P(d_1)}{\alpha} \leq \liminf_{n \rightarrow \infty} \frac{X_n}{n} \leq \limsup_{n \rightarrow \infty} \frac{X_n}{n} \leq \frac{E_P(d_1)}{\alpha},$$

which implies the result in Lemma 3.4.3. □

CHAPTER 4. LIMIT LAWS FOR TRANSIENT RWSRE

4.1 Introduction

The aim of this chapter is to derive non-Gaussian limit laws for transient random walks in a sparse random environment. The existence of the stationary dual environment suggests that the limit theorems can be first obtained for the random walk in the dual environment and then translated into the corresponding result for the RWSRE. In what follows we adopt this approach even though it has a shortcoming of restricting the consideration to a class of i.i.d. environments for which stable laws in the dual setting (with a very specific underlying Markov chain) are known. It appears plausible that alternative methodologies (which would be considerably more technically involved), such as a direct generalization of the “branching process” approach of [33, 39] or an adaptation of the “random potential” method developed in [17] would allow to extend the results presented in this chapter to a larger class of i.i.d. environments (and perhaps also to some Markov-dependent environments).

The rest of the chapter is organized as follows. Section 4.2 constitutes a reminder general stable limit theorems for partial sums of independent and mixing random variables. In Section 4.3 we collect background facts regarding RWRE in i.i.d. and Markov environments. Finally, the main results of this chapter are stated and proved in Section 4.4.

4.2 Stable limit theorems for partial sums of random variables

A non-degenerate random variable \mathcal{W} is said to have a *stable distribution* if it has a *domain of attraction*, i.e. if there is a sequence of i.i.d. random variables $(W_n)_{n \geq 1}$ and sequences of

positive numbers $(b_n)_{n \geq 1}$ and real numbers $(a_n)_{n \geq 1}$ such that $\lim_{n \rightarrow \infty} b_n = \infty$ and

$$S_n/b_n - a_n \Rightarrow \mathcal{W} \quad \text{where} \quad S_n = \sum_{n=1}^n W_n. \quad (4.1)$$

Such random variables W_n are said to be in the domain of attraction of the stable law. For equivalent definitions of stable laws and, in particular, for an explicit form of their characteristic functions see for instance [52, Chapter 1].

Definition 4.2.1. *Let $(W_n)_{n \geq 1}$ be a sequence of (not necessarily i.i.d.) random variables. A stable limit theorem (SLT) is said to hold for the sequence $(W_n)_{n \geq 1}$ if (4.1) holds for some numbers $b_n \nearrow \infty$, $a_n \in \mathbb{R}$, and random variable \mathcal{W} with a stable distribution.*

It is well-known (see e.g. [9, Chapter 9] or [21]) that a SLT holds for an i.i.d. sequence $(W_i)_{i \geq 1}$ if and only if either

$$\lim_{t \rightarrow \infty} \frac{P(W_1 > t)}{P(|W_1| > t)} = \theta \in [0, 1] \quad \text{and} \quad \lim_{a \rightarrow \infty} \frac{P(|W_1| > \lambda t)}{P(|W_1| > t)} = \lambda^{-\kappa} \quad \forall t > 0, \quad (4.2)$$

for some $\kappa \in (0, 2)$, which is the *index* of the limit law, or

$$\lim_{t \rightarrow \infty} \frac{t^2 P(|W_1| > t)}{E(W_1^2 \mathbf{1}_{\{W_1 \leq t\}})} = 0. \quad (4.3)$$

In the latter case the limit random variable \mathcal{W} has a non-degenerate normal law, to which we refer as a stable law of index 2. Condition (4.3) is satisfied if $E(W_1^2) < \infty$, and this leads to the usual central limit theorem for i.i.d. variables.

Non-Gaussian stable limit theorems are known for many dependent sequences (see e.g. [1, 13, 14, 24, 25, 29, 30, 34, 37, 38, 65, 51]). In crude terms these theorems state that if a stationary sequence $(W_n)_{n \geq 1}$ is “mixing enough” and its marginal distribution is such that an i.i.d. sequence with such common distribution is in the domain of attraction of a stable law, then (4.1) holds with the same a_n and b_n as in the i.i.d. case (for the proper choice of the normalizing constants see e.g. [15, p. 153] or [21, p. 175]). Typically, “mixing” conditions of general SLT’s with $\kappa \in (0, 2)$ include not only assumptions of “asymptotic independence” but also some limitations on the bivariate correlations.

4.3 Limit theorems for transient RWRE on \mathbb{Z}

Transient random walks in i.i.d. environments have a natural renewal structure which is defined (for X_n transient to the right) by the sequence of sites $z_n \in \mathbb{N}$ where the random walk never moves to the left. When the environment is an i.i.d. sequence, the pieces of the trajectory between times T_{z_n} and $T_{z_{n+1}} - 1$ are independent. In particular, the random variables $T_{z_n} - T_{z_1}$ can be represented as partial sums of an i.i.d. sequence. This renewal structure has been exploited by many authors and also can be carried over to higher dimensions but has a drawback which makes it difficult to use it for exact computations of the parameters of RWRE. Namely, T_{z_n} are not stopping times with respect to the natural filtration of the random walk. The existence of the renewal structure for X_n can nevertheless serve as a “strong supporting evidence” for the stable limit theorem for the random walk.

In [33] the limit laws for RWRE are derived from stable limit laws for the hitting times T_n , and the index κ of the stable distribution is determined by the condition

$$E_P(\rho_0^\kappa) = 1. \quad (4.4)$$

The following is a summary of the main results of [33]. For $\kappa \in (0, 2]$ and $b > 0$ we denote by $\mathcal{L}_{\kappa,b}$ the stable law of index κ with the characteristic function

$$\log \widehat{\mathcal{L}}_{\kappa,b}(t) = -b|t|^\kappa \left(1 + i \frac{t}{|t|} f_\kappa(t) \right), \quad (4.5)$$

where $f_\kappa(t) = -\tan \frac{\pi}{2}\kappa$ if $\kappa \neq 1$, $f_1(t) = 2/\pi \log t$. With a slight abuse of notation we use the same symbol for the distribution function of this law. If $\kappa < 1$, $\mathcal{L}_{\kappa,b}$ is supported on the positive reals, and if $\kappa \in (1, 2]$, it has zero mean but is not symmetric [52, Chapter 1].

Theorem 4.3.1. [33] *Let the environment $(\omega_n)_{n \in \mathbb{Z}}$ be an i.i.d. sequence such that*

- (i) *There exists a constant $\kappa > 0$ such that (4.4) holds and $E_P(\rho_0^\kappa \log^+ \rho_0) < \infty$, where we denote $\log^+(x) = \max\{\log x, 0\}$.*
- (ii) *The distribution of $\log \rho_0$ is non-lattice (its support is not contained in any proper sublattice of \mathbb{R}).*

Then, there exist positive constants $a_n, b_n, \bar{a}_n, \bar{b}_n$ (depending on the value of κ) such that $b_n \uparrow \infty, \bar{b}_n \uparrow \infty$, and:

(a) $T_n/b_n - a_n$ converges to a stable law \mathcal{L}_κ which is normal if $\kappa \geq 2$ and is of index κ if $\kappa \in (0, 2)$

(b) $X_n/\bar{b}_n - \bar{a}_n$ converges to a non-degenerate law $\bar{\mathcal{L}}_\kappa$.

Since the function $\lambda \rightarrow E_P(\rho_0^\lambda)$ is convex, the parameter κ is uniquely determined by the conditions of the theorem. By Jensen's inequality, $E_P(\log \rho_0) < 0$ and hence X_n is transient to the right. The law $\bar{\mathcal{L}}_\kappa$, defined precisely in Proposition 4.3.7, is closely related to \mathcal{L}_κ , whose characteristic function is given in (4.5). The normalizing sequences $(a_n, b_n)_{n \in \mathbb{N}}$ and $(\bar{a}_n, \bar{b}_n)_{n \in \mathbb{N}}$ are defined explicitly in Proposition 4.3.7 and Theorem 4.3.6 respectively. We remark the pairs of coefficients (a_n, b_n) can be chosen the same as in the corresponding limit theorems for partial sums of i.i.d. sequences.

The limit laws of [33] was extended in [39] to environments which are semi-Markov process. By semi-Markov we mean environments (ω_n) to which a Markov chain $(x_n)_{n \in \mathbb{Z}}$ can be associated in such a way that the bivariate process (x_n, ω_{-n}) is a Markov chain (called *Hidden Markov Model*) whose transitions depend only on the position of x_n . Similarly to [33], the limit laws for X_n are derived from stable laws of index $\kappa \in (0, 2]$ for T_n where κ is determined by (compare with (1.7)):

$$\Lambda(\kappa) = 0, \quad \text{where } \Lambda(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log E_P \left(\prod_{i=0}^{n-1} \rho_i^\beta \right). \quad (4.6)$$

We note that if x_n is an irreducible Markov chain in a finite state space, then under the assumptions of [39] κ is uniquely determined from the condition $\lambda_\kappa = 1$, where λ_β is the Perron-Frobenius eigenvalue of the matrix $H_\beta(x, y) = H(x, y)E_P(\rho_{-1}^\beta | x_0 = x, x_1 = y)$.

Definition 4.3.2. Let $(\mathcal{S}, \mathcal{T})$ be a measurable space and let $(x_n)_{n \in \mathbb{Z}}$ be a stationary Markov chain with transition kernel $H(x, \cdot)$ defined on it.

A Hidden Markov Model associated with the Markov chain $(x_n)_{n \in \mathbb{Z}}$ is a stationary Markov chain $(x_n, \zeta_n)_{n \in \mathbb{Z}}$ defined on a product space $(\mathcal{S} \times \Upsilon, \mathcal{T} \otimes \Xi)$, whose transitions depend only on the position of (x_n) . That is, for any $n \in \mathbb{Z}, x \in \mathcal{S}, A \in \mathcal{T}, B \in \Xi$,

$$P(x_n \in A, \zeta_n \in B | \sigma((x_i, \zeta_i) : i < n)) = \int_A H(x, dy) \mathbb{G}(x, y, B) \Big|_{x=x_{n-1}}, \quad (4.7)$$

where $\mathbb{G}(x, y, \cdot) = P(\zeta_1 \in \cdot | x_0 = x, x_1 = y)$ is a transition kernel on $(\mathcal{S} \times \mathcal{S} \times \Xi)$.

The second component of the hidden Markov model, i.e. the process $(\zeta_n)_{n \in \mathbb{Z}}$ on (Υ, Ξ) , is called a semi-Markov process associated with the Markov chain (x_n) .

The basic example of hidden Markov models are the random sequences $(x_n, f(x_n))_{n \in \mathbb{Z}}$, where $(x_n)_{n \in \mathbb{Z}}$ is the underlying Markov chain and $(f(x_n))_{n \in \mathbb{Z}}$ is its point-wise transformation.

We will next state the assumptions made in [39] on the underlying Markov chain (x_n) and on the semi-Markov process ζ_n associated with it.

Assumption 4.3.3. Let $(\mathcal{S}, \mathcal{T})$ be a measurable space and $(x_n)_{n \in \mathbb{Z}}$ be a Markov chain on $(\mathcal{S}, \mathcal{T})$ with transition kernel $H(x, \cdot)$ such that:

- (A1) The σ -field \mathcal{T} is countably generated, i.e. generated by a denumerable class of sets.
- (A2) The kernel $H(x, \cdot)$ is irreducible, i.e. there exists a non-zero σ -finite measure φ on $(\mathcal{S}, \mathcal{T})$ such that for all $x \in \mathcal{S}$, $\sum_{n=1}^{\infty} H^n(x, A) > 0$ whenever $\varphi(A) > 0$.
- (A3) There exist a probability measure μ on $(\mathcal{S}, \mathcal{T})$, a number $m_1 \in \mathbb{N}$, and a measurable density kernel $h(x, y) : \mathcal{S}^2 \rightarrow [0, \infty)$ such that

$$H^{m_1}(x, A) = \int_A h(x, y) \mu(dy),$$

and the family of functions $\{h(x, \cdot) : \mathcal{S} \rightarrow [0, \infty)\}_{x \in \mathcal{S}}$ is uniformly integrable with respect to the measure μ .

We turn now to the semi-Markov process ζ_n in Definition 4.3.2 which, thought of as the environment in the RWRE context could be either ω_{-n} or ρ_{-n} ; the latter will be more convenient (since one is a one-to-one transformation of the other, it doesn't really matter). The main reason why we are interested in the properties of the reversed process $(\rho_{-n})_{n \in \mathbb{Z}}$ rather than on those of $(\rho_n)_{n \in \mathbb{Z}}$ lies in the definition of an auxiliary branching process associated with the RWRE (cf. [33, 39]).

Assumption 4.3.4.

- (A4) $P(c_\rho^{-1} < \rho_0 < c_\rho) = 1$ for some $c_\rho > 1$.
- (A5) $\limsup_{n \rightarrow \infty} \frac{1}{n} \log E \left(\prod_{i=0}^{n-1} \rho_{-i}^{\beta_1} \right) \geq 0$ and $\limsup_{n \rightarrow \infty} \frac{1}{n} \log E \left(\prod_{i=0}^{n-1} \rho_{-i}^{\beta_2} \right) < 0$ for some constants $\beta_1 > 0$ and $\beta_2 > 0$.

(A6) The process $q_n = \log \rho_{-n}$ is non-arithmetic relative to (x_n) in the following sense: there do not exist a constant $\alpha > 0$ and a measurable function $\gamma : \mathcal{S} \rightarrow [0, \alpha]$ such that

$$P(q_0 \in \gamma(x_{-1}) - \gamma(x_0) + \alpha \cdot \mathbb{Z}) = 1.$$

The above assumptions define the class of environments ω consider in [39].

Definition 4.3.5. A random sequence $(\omega_n)_{n \in \mathbb{Z}}$ is said to be an admissible semi-Markov environment associated with a stationary Markov chain $x = (x_n)_{n \in \mathbb{Z}}$ if

- (i) x satisfies Assumption 4.3.3.
- (ii) $(\rho_{-n})_{n \in \mathbb{Z}}$, is a semi-Markov process associated with x satisfying Assumption 4.3.4.

The main result of [39] is

Theorem 4.3.6. [39] Let ω be an admissible semi-Markov environment (cf. Definition 4.3.5). Then there is a unique $\kappa > 0$ such that (4.6) and the following hold for some $b > 0$ (recall the definition (1.6) of the asymptotic speed v_P):

- (i) If $\kappa \in (0, 1)$, then $\lim_{n \rightarrow \infty} \mathbb{P}(n^{-\kappa} X_n \leq \mathfrak{z}) = 1 - \mathcal{L}_{\kappa, b}(\mathfrak{z}^{-1/\kappa})$,
- (ii) If $\kappa = 1$, then $\lim_{n \rightarrow \infty} \mathbb{P}(n^{-1}(\log n)^2(X_n - \delta(n)) \leq \mathfrak{z}) = 1 - \mathcal{L}_{1, b}(-\mathfrak{z})$, for suitable $A_1 > 0$ and $\delta(n) \sim (A_1 \log n)^{-1}n$,
- (iii) If $\kappa \in (1, 2)$, then $\lim_{n \rightarrow \infty} \mathbb{P}(n^{-1/\kappa}(X_n - nv_P) \leq \mathfrak{z}) = 1 - \mathcal{L}_{\kappa, b}(-\mathfrak{z})$.
- (iv) If $\kappa = 2$, then $\lim_{n \rightarrow \infty} \mathbb{P}((n \log n)^{-1/2}(X_n - nv_P) \leq \mathfrak{z}) = \mathcal{L}_{2, b}(\mathfrak{z})$.
- (v) If $\kappa > 2$, then $\lim_{n \rightarrow \infty} \mathbb{P}(n^{-1/2}(X_n - nv_P) \leq \mathfrak{z}) = \mathcal{L}_{2, b}(\mathfrak{z})$.

The index κ of the stable distribution is determined by the condition (4.6). For the hitting times T_n , we have:

Proposition 4.3.7. [39] Let the conditions of Theorem 4.4.2 hold. Then the following hold for some $\tilde{b} > 0$:

- (i) If $\kappa \in (0, 1)$, then $\lim_{n \rightarrow \infty} \mathbb{P}(n^{-1/\kappa} T_n \leq t) = \mathcal{L}_{\kappa, \tilde{b}}(t)$,
- (ii) If $\kappa = 1$, then $\lim_{n \rightarrow \infty} \mathbb{P}(n^{-1}(T_n - nD(n)) \leq t) = \mathcal{L}_{1, \tilde{b}}(t)$, for suitable $c_0 > 0$ and $D(n) \sim c_0 \log n$,

(iii) If $\kappa \in (1, 2)$, then $\lim_{n \rightarrow \infty} \mathbb{P}(n^{-1/\kappa}(T_n - nv_P^{-1}) \leq t) = \mathcal{L}_{\kappa, \tilde{b}}(t)$.

(iv) If $\kappa = 2$, then $\lim_{n \rightarrow \infty} \mathbb{P}((n \log n)^{-1/2}(T_n - nv_P^{-1}) \leq t) = \mathcal{L}_{2, \tilde{b}}(t)$.

(v) If $\kappa > 2$, then $\lim_{n \rightarrow \infty} \mathbb{P}(n^{-1/2}(T_n - nv_P^{-1}) \leq t) = \mathcal{L}_{2, \tilde{b}}(t)$.

The proof that Theorem 4.3.6 follows from Proposition 4.3.7 is the same as in the i.i.d. case, and is based on the observation that for any positive integers η, ζ, n

$$\{T_\zeta \geq n\} \subset \{X_n \leq \zeta\} \subset \{T_{\zeta+\eta} \geq n\} \bigcup \left\{ \inf_{k \geq T_{\zeta+\eta}} X_k - (\zeta + \eta) \leq -\eta \right\}. \quad (4.8)$$

Because the random variables $\inf_{k \geq T_{\zeta+\eta}} X_k - (\zeta + \eta)$ and $\inf_{k \geq 0} X_k$ have the same annealed distribution, the probability of the last event in (4.8) can be made arbitrary small uniformly in n and ζ by fixing η large (since the RWRE X_n is transient to the right). For $\kappa = 1$, the rest of the argument is detailed in [33, pp. 167–168], where no use of the i.i.d. assumption for ω is made at that stage, and a similar argument works for all $\kappa \in (0, 2]$.

4.4 Limit theorems for a transient RWSRE

The goal of this section is to state and prove limit theorems for a transient RWSRE. We will impose here the following set of assumptions:

Assumption 4.4.1.

(B1) $(\lambda_n)_{n \in \mathbb{Z}}$ is an i.i.d. sequence

(B2) $(d_n)_{n \in \mathbb{Z}}$ is an i.i.d. sequence independent of $(\lambda_n)_{n \in \mathbb{Z}}$

(B3) $P(\epsilon < \lambda_0 < 1 - \epsilon) = 1$ for some $\epsilon \in (0, 1/2)$.

(B4) $E_P(\xi_0^\kappa) = 1$ for some $\kappa > 0$.

(B5) There exists a constant $M > 0$ such that $P(d_0 < M) = 1$.

(B6) The distribution of $\log \xi_0$ is non-arithmetic.

Notice that (B4) implies by Jensen's inequality that $E_P(\log \xi_0) \leq 0$. In view of (B6), the inequality is strict and hence the random walk is transient to the right.

The main result of this chapter is stated as follows.

Theorem 4.4.2. *Let Assumption 4.4.1. Then the conclusions of Theorem 4.3.6 and Proposition 4.3.7 hold for the RWSRE X_n and hitting times T_n .*

Proof. To establish the claim it suffices to verify (4.6) and the conditions of Theorem 4.3.6 for the Markov chain Y_n introduced in Section 2.3 and the associated Markov hidden model $(Y_{-n}, \lambda_{-n}, d_{-n})_{n \in \mathbb{Z}}$. Clearly, the reverse chain $(Y_n)_{n \in \mathbb{Z}}$ is an irreducible Markov chain in the finite state space $\{0, 1, \dots, M-1\}$. Furthermore, condition (4.6) holds because

$$\begin{aligned} E_P\left(\prod_{i=1}^n \rho_i^\kappa\right) &= E_P\left(\prod_{i=0}^{\eta_n} \xi_i^\kappa\right) = E_P\left(E_P\left(\prod_{i=0}^{\eta_n} \xi_i^\kappa \mid \eta_n\right)\right) \\ &= E_P\left((E_P(\xi_0^\kappa))^{\eta_n}\right) = E_P(1^{\eta_n}) = 1. \end{aligned}$$

To complete the proof it remains to observe that (B6) of Assumption 4.4.1 along with the fact that ξ_n are i.i.d. trivially implies (A6) of Assumption 4.3.4. \square

Remark 4.4.3. *It follows from Theorem 4.4.2 that the value of the parameter is determined exclusively by the distribution of λ_0 and is independent of the distribution of d_0 as long as the latter satisfies the conditions of the theorem. This result might appear to be surprising, especially in view of a large deviation interpretation of κ given in [66, Section 2.4] (it is not hard to see that the rate functions of the random potentials associated with the sequences ξ_n and with ρ_n are actually different!). However, it can be in fact explained using the interpretation of κ in terms of the associated branching process. Furthermore, a careful inspection of (4.6) shows that both the parameters b and \bar{b} of the limiting distributions are decreasing functions of $E_P(d_0)$ and increasing functions of $\text{VAR}(d_n)$. This is explained heuristically by the fact that b in some rigorous sense play a role of the variance for stable laws $\mathcal{L}_{\kappa, b}$ (see, for instance, the form of the characteristic function in (4.5) and compare it to the characteristic function of a normal distribution). We refer the reader to [50] for a detailed discussion.*

CHAPTER 5. A LIMIT THEOREM FOR RECURRENT RWSRE

5.1 Introduction

In this chapter we focus on a recurrent RWSRE. The goal here is to obtain a generalization of Sinai's limit theorem for recurrent RWRE. The chapter is organized as follows. Sinai's theorem for recurrent RWRE is recalled in Section 5.2. The main result of this chapter (Theorem 5.3.1) is presented in Section 5.3. A normalized random potential for the RWSRE is introduced in Section 5.4. The notion of valley is carried over to our setup in Section 5.5. Finally, the proof of the main result is completed in Section 5.6.

5.2 Limit theorem for RWRE in Sinai's regime

Sinai [53] studied a recurrent RWRE and showed that

$$\frac{\sigma^2}{(\log n)^2} X_n \Rightarrow b_\infty,$$

where b_∞ is random variable which can be described as the “location of the deepest valley” of a Brownian motion. The proof of Sinai's uses a construction suggesting that a properly scaled recurrent RWRE can be thought as a motion of a particle in a suitably normalized random potential V_n which is introduced in (1.3). The normalized potential converges to a Brownian motion, and Sinai's result shows a remarkable slowing down of the diffusive time scale. The density function of the limit distribution b_∞ was characterized independently by Kesten [32] and Golosov [23]. We have:

$$P[b_\infty \in dx] = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left\{-\frac{(2k+1)^2 \pi^2}{8} |x|\right\} dx$$

The main findings of [53] and [23, 32] are summarized in the following theorem.

Theorem 5.2.1. *Let X_n be a random walk in a stationary and ergodic environment on \mathbb{Z} such that*

- (i) $E_P(\log \rho_0) = 0$ (the walk is recurrent)
- (ii) $\sigma_P^2 := E_P(\log^2 \rho_0) \in (0, \infty)$.
- (iii) $\frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \log \rho_k \Rightarrow W(t)$, where W is a standard Brownian motion.

Then for any $\varepsilon > 0$ and $\delta \in (0, 1)$, there is an integer n_1 such that for all $n > n_1$ there exist a set of environments $C_n \subset \Omega$ and a random variable $b_n = b_n(\omega)$ such that $P(C_n) \geq 1 - \delta$ and

$$\lim_{n \rightarrow \infty} P_\omega \left(\left| \frac{X_n}{(\log n)^2} - b_n \right| > \varepsilon \right) = 0$$

uniformly in $\omega \in C_n$. Moreover, as $n \rightarrow \infty$ the probability distribution for b_n converge weakly to the limit distribution b_∞ .

Sinai's scaling factor $(\log n)^2$ can be heuristically explained as follows. Because of the diffusive scaling limit of random potential (to a Brownian motion), the deepest valley of $W_n(s) = \sum_{i=0}^{[s]} \log \rho_i$ with $0 \leq t \leq (\log n)^2$ is of order $\log n$. Notice that the location on \mathbb{Z} is translated into the time variable of the random potential. A classical result of Iglehart [28] implies that it takes approximately e^L units of time for the random walk to pass a potential barrier of height L . Thus it takes approximately $e^{\log n} = n$ units of time to escape from the deepest valley of the potential located at the distance of order $(\log n)^2$ from the origin. Furthermore, using (3.1) one can prove that the hitting time of the level n can be reduced to the time spent by the random walk to cross the deepest valley of the potential. Correspondingly, we should expect that the random walk at time n is typically located around the bottom of the deepest valley, at distance $\sim (\log n)^2$ from the origin.

5.3 Statement of the main result. Limit theorem for a recurrent RWSRE

Recall that a function $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ is said to be *regularly varying of index $\alpha \in \mathbb{R}$* if $h(t) = t^\alpha L(t)$ for some $L : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $L(\lambda t) \sim L(t)$ for all $\lambda > 0$. We will denote

the set of all regularly varying functions of index α by \mathcal{R}_α . For instance, (4.2) implies that $P(|W_1| > t) \in \mathcal{R}_{-\kappa}$. Regularly varying functions of index zero are called *slowly varying*.

The following theorem is the main result of this chapter.

Theorem 5.3.1. *Let X_n be a RWSRE on \mathbb{Z} such that*

- (i) $(\lambda_n, d_n)_{n \in \mathbb{Z}}$ *is an i.i.d. sequence*
- (ii) $E_P(\log \rho_0) = 0$ *(the walk is recurrent)*
- (iii) $\sigma_P^2 := E_P(\log^2 \rho_0) \in (0, \infty)$ *(and hence $\frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \log \rho_k \Rightarrow W(t)$)*
- (iv) $P(d_1 > t) \sim t^{-\alpha} h(t)$ *where $\alpha \in (0, 1)$ and $h(t)$ is slowly varying.*

Then there is a function $u \in \mathcal{R}_{2/\alpha}$ such that the following holds: For any $\varepsilon > 0$ and $\delta \in (0, 1)$, there is an integer n_1 such that for all $n > n_1$ there exist a set of environments $C_n \subset \Omega$ and a random variable $b_n = b_n(\omega)$ such that $P(C_n) \geq 1 - \delta$ and

$$\lim_{n \rightarrow \infty} P_\omega \left(\left| \frac{X_n}{u(\log n)} - b_n \right| > \varepsilon \right) = 0$$

uniformly in $\omega \in C_n$. Moreover, as $n \rightarrow \infty$ the probability distribution for b_n converge weakly to the limit distribution b_∞ .

- Remark 5.3.2.**
1. *Random variables b_n and b_∞ in Theorem 5.3.1 differ from their prototypes in Theorem 5.2.1. In fact these random variables are defined in a similar manner, but while the latter are related to the valley of Brownian motion, the former are defined in terms of the process V_α introduced below in Section 5.6.*
 2. *Function u in the conclusions of the theorem is explicitly defined in terms of function h Section 5.4 below.*

We note that similar scaling factors appear in two different frameworks, which have been previously discussed in [54, 55] and [57]. The model considered in [57] is somewhat similar to our one, but their arguments are “physicist proofs”. In contrast, the main result in [54] is a general statement showing in essence that a Sinai’s-type result though with a different normalization holds for a transient random walk in a i.i.d. environment as long as a suitably

defined and normalized random potential converges to a symmetric Lévy process (in [53] it converges to a standard Brownian motion). Unfortunately, we cannot apply the results of [54] directly since sparse environment is not formed by an i.i.d. sequence.

5.4 Rescaled random potential

The aim of this section is to introduce a normalized random potential suitable for our model. This goal is accomplished in (5.2) below.

Let $D(\mathbb{R}_+)$ denote the set of real-valued càdlàg functions on \mathbb{R}_+ equipped with the Skorohod J_1 -topology. We use notation \Rightarrow to denote the weak convergence in $D(\mathbb{R})$.

Definition 5.4.1. *A continuous-time real-valued process $(X_t = X(t))_{t \geq 0}$ is called a Lévy process if*

1. *Its sample paths are right-continuous and have left limits at every time point t , that is*

$$X_t \in D(\mathbb{R}),$$

2. *It has stationary, independent increments, that is:*

(a) *For all $0 = t_0 < t_1 < \dots < t_k$, the increments $X(t_i) - X(t_{i-1})$ are independent.*

(b) *For all $0 < s < t$ the random variables $X(t) - X(s)$ and $X(t - s) - X(0)$ have the same distribution.*

The default initial condition is $X_0 = 0$. A subordinator is a real-valued Lévy process with nondecreasing sample paths. A stable process is a real-valued Lévy process $(X_t)_{t \geq 0}$ with initial value $X_0 = 0$ that satisfies the self-similarity property:

$$X_t/t^{1/\alpha} \text{ has the same distribution as } X_1 \quad \forall t > 0.$$

The parameter α is called the exponent of the process.

Examples of Lévy processes include Brownian motion, Cauchy process, and Poisson process.

For $n \in \mathbb{N}$, let

$$r_n = \inf \{s > 0 : P(d_1 > s) \leq 1/n\}. \tag{5.1}$$

It is not hard to see that $r_n \in \mathcal{R}_{1/\alpha}$. We have (see, for instance, [52]):

$$U_n(t) := \frac{1}{r_n} \sum_{k=1}^{\lfloor nt \rfloor} d_k \text{ converges weakly to a } G_\alpha(t).$$

where $G_\alpha(t)$ is a totally asymmetric stable process (in fact, subordinator) with exponent α and

$$E(e^{i\theta G_\alpha(1)}) = \exp\left\{-|\theta|^\alpha \left(1 - i \operatorname{sign}(\theta) \tan\left(\frac{\pi\alpha}{2}\right)\right)\right\}, \quad \theta \in \mathbb{R}.$$

The following (functional) limit theorem for the random potential is the key auxiliary lemma needed to establish the result in Theorem 5.3.1.

Lemma 5.4.2.

(a) Let the conditions of Theorem 5.3.1 hold. Then, as $n \rightarrow \infty$,

$$\frac{1}{\log n} \sum_{k=1}^{\lfloor u(\log n)t \rfloor} \log \rho_k \Rightarrow V_\alpha,$$

for some sequence $u(n) \in \mathcal{R}_{2\alpha}$.

(b) Let conditions (i)-(iii) of Theorem 5.3.1 hold. Suppose in addition that $\mu := E(d_1) < \infty$.

Then

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \log \rho_k \Rightarrow \mu \sigma_P B,$$

where B is a standard Brownian motion and $V_\alpha = \sigma_P B(G_\alpha^{-1})$.

Remark 5.4.3. We only need part (a) of the above lemma to derive Theorem 5.3.1. Part (b) of the lemma indicates (and this indeed turns out to be the case, see [50]) that Sinai's theorem is carried over to the RWSRE setting with the same scaling factor $(\log n)^2$ if $E_P(d_1) < \infty$.

Proof of Lemma 5.4.2.

(a) Let

$$U_n(t) := \frac{1}{r_n} \sum_{k=1}^{\lfloor nt \rfloor} d_k = \frac{1}{r_n} a_{\lfloor nt \rfloor} \quad \text{and} \quad R_n(t) := \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \log \xi_k.$$

It follows from the assumptions of the lemma (see, for instance, [52]) that, as $n \rightarrow \infty$,

$$U_n(t) \Rightarrow G_\alpha \quad \text{and} \quad R_n(t) \Rightarrow \sigma_P B.$$

Let $U_n^{-1} = n^{-1} \cdot \eta(\lfloor tr_n \rfloor)$ and G_α^{-1} be the inverses in $D(\mathbb{R}_+)$ of U_n and G_α , respectively. Then the convergence of U_n and R_n along with their independence of each other, imply (see, for instance, the derivation of the formula (2.29) in [26]) that in $D(\mathbb{R}_+)$,

$$\left(R_n(t), \frac{1}{n}\eta(\lfloor tr_n \rfloor)\right) \Rightarrow (\sigma_P B, G_\alpha^{-1}), \quad \text{as } n \rightarrow \infty.$$

Since the paths of the Brownian motion are continuous, it follows from a random change lemma in [5, p. 151] that $R_n(\frac{1}{n}\eta(\lfloor tr_n \rfloor)) \Rightarrow \sigma_P B(G_\alpha^{-1}(t))$ in $D(\mathbb{R}_+)$. That is,

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{\eta(\lfloor r_n t \rfloor)} \log \xi_k \Rightarrow \sigma_P B(G_\alpha^{-1}), \quad \text{as } n \rightarrow \infty,$$

and, passing to the subsequence $n_k = \log^2 k$, $k \in \mathbb{N}$, we obtain

$$\frac{1}{\log k} \sum_{i=1}^{\eta(\lfloor r_{\log^2 k} t \rfloor)} \log \xi_i = \frac{1}{\log k} \sum_{i=1}^{\lfloor r_{\log^2 k} t \rfloor} \log \rho_i \Rightarrow \sigma_P B(G_\alpha^{-1}), \quad \text{as } k \rightarrow \infty.$$

To conclude the proof of part (a), notice that $r_n \in \mathcal{R}_{1/\alpha}$ implies that $r_{\log^2 n} = u(\log n)$ for some sequence $u(n) \in \mathcal{R}_{2/\alpha}$, as desired.

(b) We now turn to the proof of part (b) of the theorem. Since $E_p(d_1) < \infty$, the renewal theorem implies

$$\frac{\eta_n}{n} \rightarrow C = \frac{1}{E_p(d_1)}.$$

Therefore (see, for instance, Theorem 14.4 in [5, p. 152]),

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \log \rho_k = \frac{1}{\sqrt{n}} \sum_{k=1}^{\eta(\lfloor nt \rfloor)} \log \xi_k = \frac{\sqrt{\eta(\lfloor nt \rfloor)}}{\sqrt{n}} \frac{1}{\sqrt{\eta(\lfloor nt \rfloor)}} \sum_{k=1}^{\eta(\lfloor nt \rfloor)} \log \xi_k \Rightarrow \mu \sigma_P B(\cdot),$$

where the convergence is the weak convergence in the Skorohod space $D(\mathbb{R}_+)$. \square

Define the normalized random potential associated with the sparse environment as follows:

$$\widehat{R}_n(t) = \text{sign}(t) \cdot \frac{1}{\log n} \sum_{k=1}^{\lfloor u(\log n)t \rfloor} \log \rho_k = \text{sign}(t) \cdot \frac{1}{\log n} \sum_{k=1}^{\eta[\lfloor u(\log n)t \rfloor]} \log \xi_k. \quad (5.2)$$

By Lemma 5.4.2, $\{\widehat{R}_n(t) : t \geq 0\}$ converges weakly in $D(\mathbb{R}_+)$ to the process V_α .

5.5 Valley of the random potential

To prove Theorem 5.3.1 we adopt Sinai's interpretation of recurrent RWRE as a particle traveling in a random potential. The goal of this section is to extend Sinai's notion of the *valley* of the random potential to the model of RWSRE. In fact, the original construction of Sinai can be carried over almost verbatim to the new setting. Recall the rescaled random potential R_n from (5.2).

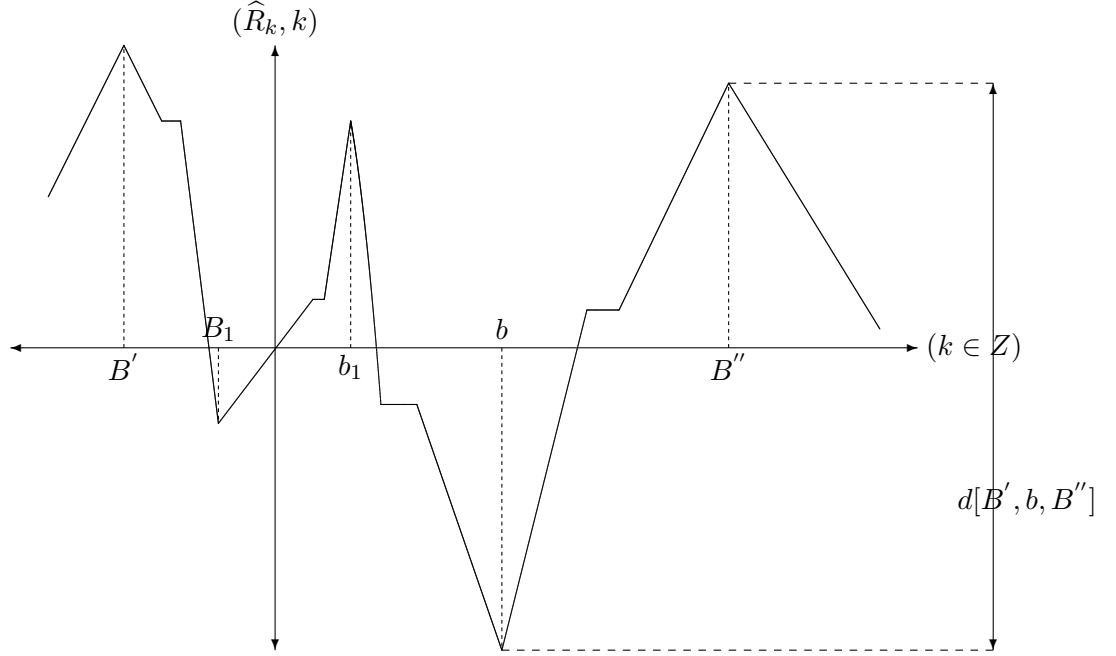


Figure 5.2. A revised valley and refinement operation

Definition 5.5.1. A triple (B', b, B'') of reals is called a valley of the path $\{\hat{R}_n(t) : t \geq 0\}$ if $B' < b < B''$ and the following three conditions hold:

$$\hat{R}_n(b) = \min_{B' \leq t \leq B''} \hat{R}_n(t), \quad \hat{R}_n(B') = \max_{B' \leq t \leq b} \hat{R}_n(t), \quad \hat{R}_n(B'') = \max_{b \leq t \leq B''} \hat{R}_n(t).$$

If b is not unique, we choose the one with the smallest absolute value.

Definition 5.5.2. The following quantity:

$$d[B', b, B''] := \min\{\hat{R}_n(B') - \hat{R}_n(b), \hat{R}_n(B'') - \hat{R}_n(b)\}.$$

is called the depth of the valley (B', b, B'') .

Following [53], we define the operation of refinement as follows:

Definition 5.5.3. *Let (B', b, B'') be a valley and let B_1 and b_1 the points and (B', B_1, b_1) and (b_1, b, B'') are again valleys such that*

$$B' < B_1 < b_1 < b \quad \text{with} \quad \hat{R}_n(b_1) - \hat{R}_n(B_1) = \max_{B' \leq x < y \leq b_1} \hat{R}_n(y) - \hat{R}_n(x).$$

We call (B', B_1, b_1) and (b_1, b, B'') a left refinement of (B', b, B'') . If B_1, b_1 is not unique, we will repeat the one such that b_1 have the smallest absolute value. In a similar way we define a right refinement.

It turns out (cf. [53, 54, 66]) that with probability one, each valley can be refined only a finite number of times. Therefore, one can apply a finite sequence of refinement to find a valley $(\hat{B}'_n, \hat{b}_n, \hat{B}''_n)$ with $0 \in [\hat{B}'_n, \hat{B}''_n]$, while $d[\hat{B}'_n, \hat{B}''_n] \geq 1$. Similarly, for any $\delta \in (0, 1)$, one can find a valley $(\hat{B}'_{n,\delta}, \hat{b}_{n,\delta}, \hat{B}''_{n,\delta})$ which is the smallest valley containing zero and such that $d[\hat{B}'_{n,\delta}, \hat{B}''_{n,\delta}] \geq 1 + \delta$.

Definition 5.5.4. *Fix any constants $\delta \in (0, 1)$, $\eta > 0$, and $J > 0$. Then $\omega \in \Omega$ is called a proper environment if it satisfies the following:*

1. $\hat{b}_n = \hat{b}_{n,\delta}$, any refinement (B', b, B'') of $(\hat{B}'_{n,\delta}, \hat{b}_{n,\delta}, \hat{B}''_{n,\delta})$ with $b \neq \hat{b}_n$ has depth $< 1 - \delta$,
2. $\min_{t \in [\hat{B}'_{n,\delta}, \hat{B}''_{n,\delta}] / [\hat{b}_n - \delta, \hat{b}_n + \delta]} (\hat{R}_n(t) - \hat{R}_n(\hat{b}_n)) > \delta^\eta$,
3. $|\hat{B}'_{n,\delta}| + |\hat{B}''_{n,\delta}| \leq J$.

Define

$$G_{n,\delta}^J := \{\omega \in \Omega : \beta(\omega) \text{ is a proper environment}\} \subset \Omega.$$

5.6 Completion of the proof of Theorem 5.3.1

We are now in a position to conclude the proof of Theorem 3.3. Once this point is reached and proper environments are introduced, one can proceed almost verbatim as in [54] in order to establish Theorem 5.3.1. In fact, the proof of the main result in [54] is an adaptation

of the original argument of Sinai [53] to a situation where the random potential \widehat{R}_n in the form given by (5.2) converges weakly to a non-degenerate symmetric stable process in $D(\mathbb{R}_+)$. Unfortunately, in order to conclude the proof of Theorem 5.3.1 one cannot refer directly to [54] since, formally, only i.i.d. environments are treated there (recall that an i.i.d. sparse environment is Markov-dependent under the stationary measure Q). This difficulty is only formal and a careful inspection of the proof of the main result of [54] shows that it works in a general setting of stationary and ergodic environments. However, checking these details is somewhat tedious and not entirely straightforward.

A shorter derivation of Theorem 5.3.1 can be obtained by adaptation of a version of Sinai's argument given in [66]. This version follows the approach of [23] and is due to Dembo, Guionnet, and Zeitouni [66, Section 2.5]. Assume (without loss of generality) that ω is such that $\hat{b}_n > 0$. Let

$$b_n = \hat{b}_n u(\log n).$$

It was shown by Sinai in [53] (see also [54]) that b_n converges in distribution to its counterpart corresponding to the limiting process V_α . We also have the following (see the proof of Theorem 2.5.3 in [66]):

Proposition 5.6.1. *Let the conditions of Theorem 5.3.1 hold. Then there exists $\eta = \eta(\alpha)$ such that*

(i)

$$\lim_{\delta \rightarrow 0} \lim_{J \rightarrow \infty} \lim_{n \rightarrow \infty} P(G_{n,\delta}^{J,\eta}) = 1.$$

(ii) For any $\delta \in (0, 1)$ and $J > 0$,

$$\sup_{\omega \in G_{n,\delta}^{J,\eta}} P_\omega \left(\left| \frac{X_n}{u(\log n)} - b_n \right| > \delta \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The proof of Theorem 5.3.1 is complete.

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