Bootstrapping the sample quantile based on weakly dependent observations

by

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For the Major Program

To my parents and my teachers

With gratitude

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CHAPTER 1 GENERAL INTRODUCTION

1.1 Background

Let $\{X_i\}_{i\in\mathbb{Z}}$ be a sequence of random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$ with common distribution function F, where $\mathbb{Z} \equiv \{0, \pm1, \pm2, \dots\}$ denotes the set of all integers. Let F^{-1} be the corresponding quantile function, defined by

$$F^{-1}(t) = \inf\{u : F(u) \geq t\}, \quad 0 < t < 1. \tag{1.1}$$

For a sample $X_1, \cdots, X_n, n \geq 1$, let F_n denote the empirical distribution function, putting mass $1/n$ on each X_i, i.e.,

$$F_n(x) = n^{-1}\sum_{i=1}^{n} I(X_i \leq x), \quad x \in \mathbb{R},$$

where $I(\cdot)$ denotes the indicator function, with $I(S) = 0$ or 1 according as the statement S is false or true. Then, F_n^{-1} is the sample quantile function. If $X_1, \cdots, X_n$ are independently and identically distributed (i.i.d.) random variables and F is differentiable at the population quantile $F^{-1}(p)$ with a positive derivative $f(F^{-1}(p))$ for some $p \in (0,1)$, then the p-th sample quantile is asymptotically normal. More precisely, the centered and scaled sample quantile,

$$Z_n \equiv \sqrt{n}(F_n^{-1}(p) - F^{-1}(p)), \tag{1.2}$$

converges in distribution to $N(0, \sqrt{p(1-p)}/f^2(F^{-1}(p)))$ as the sample size $n \to \infty$.

Like population means, population quantiles are also very important parameters. For statistical inference about the population parameter $F^{-1}(p)$ (e.g., for setting a confidence interval for $F^{-1}(p)$ using the limiting normal distribution), estimation of the asymptotic variance $\sqrt{p(1-p)}/f^2(F^{-1}(p))$ is an important problem. The classical Jackknife method is known to be ineffective in this problem, as the resulting Jackknife variance estimator is inconsistent (cf. Efron (1982)). As an alternative, bootstrap approximations for the distribution and the asymptotic variance of the sample quantiles have been extensively studied in the literature in the i.i.d. situation. Bickel and Freedman (1981) and Singh (1981) proved that, in the i.i.d. set up, under some regularity conditions on F, the bootstrap approximation to the distribution of the statistic Z_n is strongly consistent. Furthermore, in a significant work, Ghosh, Parr, Singh and Babu (1984) showed that under some mild moment and smoothness conditions, the bootstrap variance estimator of the normalized sample quantile was strongly consistent. As a result, in the i.i.d. set up, the bootstrap method is superior to the classical Jackknife method for estimation of asymptotic variances of sample quantiles.

In the dependent case, under suitable mixing conditions on the process $\{X_i\}_{i\in\mathbb{Z}}$ and under mild regularity conditions on the one-dimensional marginal distribution function F, the centered and scaled p-th sample quantile, $Z_n \equiv \sqrt{n}(F_n^{-1}(p) - F^{-1}(p))$, is also asymptotically normal with mean zero and variance given by

$$\tau_\infty^2 \equiv \Big[\sum_{i=-\infty}^{\infty} Cov\Big(I(X_1 \leq F^{-1}(p)), I(X_{i+1} \leq F^{-1}(p))\Big)\Big] / f^2(F^{-1}(p)) \tag{1.3}$$

(see, for example, Sen (1972) or Theorem 2.1 below). Thus, under dependence, the asymptotic variance of the p-th sample quantile not only involves the density of the random variable X_1 at the population quantile $F^{-1}(p)$, but at the same time, an infinite series of lag-covariances of the transformed sequence $\{I(X_i \leq F^{-1}(p))\}_{i\in\mathbb{Z}}$. We shall show that, in spite of the more complicated form of the limit distribution of Z_n, the simple blocking mechanism of the moving block bootstrap method (cf. Section 1.2 below) captures both the effect of the dependence structure of the process $\{X_i\}_{i\in\mathbb{Z}}$ (given by the infinite series in the numerator of τ_∞^2) and the effect of the nonlinear nature of the sample quantile (quantified by the *density* function of individual X_i's) on the limit distribution of Z_n.

Although properties of the bootstrap method for sample quantiles in the independent case is well studied in the literature, no work seems to be available on properties of bootstrap approximations for sample quantiles when the observations are dependent. The main objective of this dissertation is to investigate asymptotic properties of bootstrap methods for estimating the sampling distributions and the asymptotic variances of the sample quantiles under weak dependence.

1.2 Literature Review

1.2.1 Moving Block Bootstrap Method

It is well-known that, in the i.i.d. set up, compared to the classical normal approximation method, Efron's (1979) bootstrap resampling procedure provides more accurate approximations to the distributions of many regular statistics, e.g., smooth functions of sample means (cf. Singh (1981), Babu (1986)). However, it fails to provide valid approximations in the situations when the observations are dependent. In his Remark 2.1, Singh (1981) pointed out that, even in the simple m-dependence case, Efron's (1979) bootstrap approximation to the distribution of the normalized sample mean is invalid. Block bootstrap methods for dependent data have been put forward by several authors. Carlstein (1986) initiated the idea of nonoverlapping block bootstrap (NBB). Politis and Romano (1992) proposed the circular block bootstrap (CBB) rule. Künsch (1989) and Liu and Singh (1992) independently proposed a substantially important resampling procedure, called the moving block bootstrap (MBB). Recently, Paparoditis and Politis (2001) proposed a resampling method called the tapered block bootstrap. See Lahiri (2003) for a detailed account of bootstrap methods in the dependence case. For definiteness and conciseness, in this dissertation, we shall exclusively concentrate on the moving block bootstrap method of Künsch (1989) and Liu and Singh (1992). Similar conclusions can be proved for certain other variants of the block bootstrap method, as pointed out in Remark 3.3 of Chapter 3 below.

We now briefly describe the MBB method, which is frequently used for estimating the unknown distributions of statistics based on weakly dependent data. Let $X_1, \cdots, X_n$ denote the observations from the stationary process $\{X_i\}_{i \in \mathbb{Z}}$. For ℓ, a positive integer between 1 and n, we define the overlapping blocks of size ℓ as

$$B_i = (X_i, \cdots, X_{i+\ell-1}), i = 1, \cdots, n - \ell + 1.$$

Let $B_1^*, \cdots, B_b^*$ be a random sample of blocks from $\{B_1, \cdots, B_N\}$, where $N = n - \ell + 1$, $b = \lfloor n/\ell \rfloor$, i.e., $B_1^*, \cdots, B_b^*$ are independently and identically distributed as Uniform$\{B_1, \cdots, B_N\}$. Here and in the following, for any real number x, we denote by $\lfloor x \rfloor$ the largest integer not exceeding x, and by $\lceil x \rceil$ the smallest integer not less than x. The observations in the resampled block B_i^* are denoted by $X_{(i-1)\ell+1}^*, \cdots, X_{i\ell}^*$, $1 \le i \le b$. Then, $X_1^*, \cdots, X_\ell^*, \cdots, X_{n_1}^*$ is the MBB sample, where $n_1 = b\ell$. Let

$$T_n \equiv t_n(X_1, \ldots, X_n; \theta) \tag{1.4}$$

be a random variable of interest that is a function of the random variables $\{X_1, \ldots, X_n\}$ and of some unknown (possibly vector valued) population parameter θ. Then, the MBB version of T_n is defined as

$$T_n^* = t_{n_1}(X_1^*, \ldots, X_{n_1}^*; \hat{\theta}_n), \tag{1.5}$$

where $\hat{\theta}_n$ is a suitable estimator of θ based on $\{X_1, \ldots, X_n\}$. The MBB estimator of the distribution of T_n is given by the conditional distribution of T_n^*, given $\mathcal{X}_n \equiv \{X_1, \cdots, X_n\}$.

For an example, suppose that $T_n = \sqrt{n}(\bar{X}_n - \theta)$, with $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$ and $\theta = EX_1$. Then, the above description of the MBB method suggests that the MBB version of T_n be given by $T_n^* \equiv \sqrt{n_1}(\bar{X}_n^* - \hat{\theta}_n)$, where we write $\bar{X}_n^* = n_1^{-1}\sum_{i=1}^{n_1} X_i^*$ for the bootstrap sample mean and where $\hat{\theta}_n = E_*(\bar{X}_n^*)$. Throughout this dissertation, we use P_*, E_*, and Var_* to denote, respectively, the conditional probability, the conditional expectation, and the conditional variance, given $\mathcal{X}_n$. An alternative definition of the MBB version of T_n of (1.4) is given by resampling $\lceil n/\ell \rceil$ blocks from $\{B_1, \cdots, B_N\}$, and using the first n out of the $\lceil n/\ell \rceil \cdot \ell$-many resampled values. However, the difference between the two versions is asymptotically negligible. To simplify the proofs of the main results, here we shall use the version given by (1.5) based on b *complete* resampled blocks.

Next we define the MBB version of the p-th sample quantile and of its centered and scaled version Z_n, for a given $p \in (0,1)$. Let F_n^* denote the MBB empirical distribution function, i.e., $F_n^*(x) = n_1^{-1}\sum_{i=1}^{n_1} I(X_i^* \le x)$, $x \in \mathbb{R}$. Then, the MBB version of the sample quantile $\hat{\xi}_n = F_n^{-1}(p)$ is defined as $\xi_n^* \equiv F_n^{*-1}(p)$. Similarly, the MBB version of the centered and scaled sample quantile $Z_n = \sqrt{n}(\hat{\xi}_n - \xi_p)$ is given by

$$Z_n^* \equiv \sqrt{n_1}(\xi_n^* - \tilde{\xi}_n) , \tag{1.6}$$

where $\xi_p = F^{-1}(p)$, $\tilde{\xi}_n = \tilde{F}_n^{-1}(p)$, and $\tilde{F}_n(\cdot) = E_* F_n^*(\cdot)$. Note that in the definition of the MBB version of Z_n, we center ξ_n^* by $\tilde{\xi}_n$. As in the case of the sample mean (cf. Lahiri (1992)), this appears to be the analogous centering constant for the bootstrap sample quantile. Because F_n^* is a valid distribution function for each set of resampled $\{X_1^*, \ldots, X_{n_1}^*\}$, the function $\tilde{F}_n(x) \equiv E_* F_n^*(x), x \in \mathbb{R}$, is also a valid distribution function. Hence, $\tilde{\xi}_n$ is well-defined. Let

$$G_n(x) = P(Z_n \leq x), \quad x \in \mathbb{R}, \tag{1.7}$$

denote the distribution function of Z_n. Then, the MBB estimator of $G_n(x)$ is given by the conditional distribution of Z_n^*, i.e., by

$$\hat{G}_n(x) = P_*(Z_n^* \leq x), \quad x \in \mathbb{R}. \tag{1.8}$$

Furthermore, the MBB estimator of the asymptotic variance τ_∞^2 of Z_n (cf. (1.3)) is given by the conditional variance of Z_n^*, i.e., by

$$\hat{\tau}_n^2 = Var_*(Z_n^*). \tag{1.9}$$

In the independent case, properties of the bootstrap approximation for the sample quantile have been studied by Efron (1979, 1982), Bickel and Freedman (1981), Singh (1981), Ghosh et al. (1984), Babu (1986), Hall and Sheather (1988), Hall and Martin (1988), Hall, Diciccio and Romano (1989), and Falk and Janas (1992), among others. For weakly dependent processes, properties of various block-bootstrap methods (for *smooth* functions of the data) have been studied by Lahiri (1992, 1996a, 1996b, 1999), Bühlmann (1994), Naik-Nimbalkar and Rajarshi (1994), Hall, Horowitz and Jing (1995), Götze and Künsch (1996), among others. See Lahiri (2003) for a detailed account of results on bootstrap methods in the dependent case. In this dissertation, we investigate the properties of the MBB method in the case of *nonsmooth* functionals, the sample quantiles.

1.2.2 Stationary Processes and Mixing Conditions

We now introduce some measures of dependence between the observed random variables. Suppose that random variables $\{X_i\}_{i\in\mathbb{Z}}$ are defined on the same probability space $(\Omega, \mathcal{F}, P)$. The sequence $\{X_i\}_{i\in\mathbb{Z}}$ is called *(strictly) stationary* if, for any positive integer k, any $t_1, \cdots t_k \in \mathbb{Z}$, and any $h \in \mathbb{Z}$, the distribution of $(X_{t_1+h}, \cdots, X_{t_k+h})$ is the same as the distribution of $(X_{t_1}, \cdots, X_{t_k})$. Let $\mathcal{F}_m^n = \sigma\langle X_i : m \le i \le n, i \in \mathbb{Z}\rangle$, $-\infty \le m \le n \le \infty$, i.e., $\mathcal{F}_m^n$ is a σ-algebra generated by the random variables $X_m, \cdots, X_n$. For $n \ge 1$, we define

$$\alpha(n) = \sup_{m\in\mathbb{Z}} \sup_{A\in\mathcal{F}_{-\infty}^m, B\in\mathcal{F}_{m+n}^\infty} |P(A\cap B) - P(A)P(B)|$$

and

$$\phi(n) = \sup_{m\in\mathbb{Z}} \sup_{A\in\mathcal{F}_{-\infty}^m, B\in\mathcal{F}_{m+n}^\infty} \frac{|P(A\cap B) - P(A)P(B)|}{P(A)}.$$

The sequence $\{X_i\}_{i\in\mathbb{Z}}$ is called *uniformly mixing* or *ϕ-mixing* if $\phi(n) \to 0$ as $n \to \infty$, and it is called *strongly mixing* or *α-mixing* if $\alpha(n) \to 0$ as $n \to \infty$. In this dissertation, we shall focus on the situation when the observations are from a *α-mixing* process with polynomially or exponentially decaying coefficients. It is easily seen from the definitions that ϕ-mixing implies α-mixing.

As a convention, we assume throughout this dissertation that, unless otherwise specified, limits are taken as $n \to \infty$.

1.3 Dissertation Organization

This dissertation is organized as follows. In Chapter 2, we investigate consistency properties of MBB approximations to the distribution of the scaled and centered sample quantile of weakly dependent data. Strong consistency of the MBB estimators of the asymptotic variance of the sample quantile is established in Chapter 3. In Chapter 4, a Berry-Esseen Theorem for sample quantile under weak dependence is discussed. Chapter 5 gives a refinement of the result in Chapter 2 by examining the rate of convergence of the MBB approximation. Finally, conclusions are addressed in Chapter 6.

CHAPTER 2 MBB DISTRIBUTION APPROXIMATION

2.1 Introduction

In this chapter, we investigate consistency properties of block bootstrap approximations for sample quantiles of weakly dependent data. Under mild weak dependence conditions and mild smoothness conditions on the one-dimensional marginal distribution function, we show that the moving block bootstrap (MBB) method provides a valid approximation to the distribution of the normalized sample quantile in the almost sure sense. More specifically, we show that the MBB approximation to the distribution of the centered and scaled sample quantile is strongly consistent under a mild polynomial strong mixing rate. For the proof, we develop some exponential inequalities for block bootstrap moments and some almost sure bounds on the oscillations of the empirical distribution function of strongly mixing random variables, which may be of some independent interest.

Sample quantiles have been extensively studied in the literature. In the i.i.d. set up, Bahadur (1966) introduced an elegant representation for the sample quantiles in terms of the empirical distribution function, which is usually referred to in the literature as *Bahadur representation* for sample quantiles. The Bahadur representation allows one to express asymptotically a sample quantile as a sample mean of certain (bounded) random variables, from which many important properties of the sample quantile, e.g., the central limit theorem, the law of iterated logarithm, may be easily proved. Under dependence, Sen (1972) gave the Bahadur representation for sample quantiles for the sequence of ϕ-mixing random variables. Babu and Singh (1978) established a Bahadur representation for sample quantiles under the assumption of α-mixing with exponentially decaying coefficients. A Bahadur representation result given by Yoshihara (1995) requires that the random variables are from a bounded, α-mixing sequence with a polynomially decaying rate.

For those situations where a Bahadur representation for sample quantiles exists, consistency properties of bootstrap approximations to the distributions of the sample quantiles will also follow from consistency results on bootstrapping the empirical process. We refer to Arcones and Giné (1992), Bühlmann (1994), Naik-Nimbalkar and Rajarshi (1994), and Radulović (1998) for some details of bootstrapped empirical processes. This chapter is focused on investigating the properties of the sample quantiles under a more general weak dependence assumption, say, α-mixing with polynomially decaying coefficients. It appears that no Bahadur representation result is available under this dependence structure . Here, we shall use the central limit theorem for triangular arrays under weak dependence (cf. Lemma 2.1) to obtain a central limit theorem for the sample quantile under fairly weak conditions. A machinery for the empirical distribution function and sample quantile is built to prove the consistency of MBB approximations to the distributions of the sample quantiles.

The rest of this chapter is organized as follows. In Section 2.2, we investigate the asymptotic normality of the centered and scaled sample quantiles based on weakly dependent observations. The consistency property of the MBB approximation to the distribution of the normalized sample quantile function is discussed in Section 2.3. A small simulation study is presented in Section 2.4.

2.2 Asymptotic Normality of Sample Quantiles Under Weak Dependence

We first introduce some basic notation. Let $C, C(\cdot)$ denote generic constants in $(0, \infty)$ that depend on their arguments (if any) but not on the variables n and x. For real numbers x and y, write $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Let $\mathbb{Z}$, $\mathbb{N}$ denote the set of all integers and the set of all positive integers, respectively. For a random variable X and a real number q, we define

$$\|X\|_q \equiv \begin{cases} (E|X|^q)^{\frac{1}{q}} & \text{if} \quad q \in [1, \infty) \\ \text{ess.sup.}(X) & \text{if} \quad q = \infty. \end{cases}$$

For random variables X and Y, we write $X =^d Y$ if X and Y have the same probability distribution. Recall that unless otherwise indicated, limits are taken by letting n tend to infinity.

We now present a result that will be used to establish the asymptotic normality of sample quantiles based on strongly mixing observations. It is an extension of Theorem 1.7 of Bosq (1998), from sequences of random variables to triangular arrays.

Lemma 2.1 *Suppose that $\{X_{n,j} : j \in \mathbb{Z}\}_{n\geq 1}$ is a double array of zero-mean real-valued (row-wise) strictly stationary strong-mixing process with (row-wise) strong-mixing coefficients $\alpha_n(\cdot)$, $n \geq 1$, such that for some $\gamma > 2$ and $C > 0$, $E|X_{n,1}|^\gamma \leq C$ and $\alpha_n(j) \leq Cj^{-\beta}$, $\quad j \geq 1, \quad n \geq 1$, where $\beta > \frac{\gamma}{\gamma-2}$. If*

$$\sigma^2 = \lim_{n\to\infty} \sum_{j=-(n-1)}^{n-1} Cov(X_{n,1}, X_{n,1+j}) > 0\,,$$

then we have

$$\frac{\sum_{j=1}^n X_{n,j}}{\sigma\sqrt{n}} \to^d N(0,1)\,.$$

Proof: Note that by Davydov's inequality (cf. Corollary 1.1 of Bosq (1998)), we have

$$\begin{aligned}
\sigma^2 &= \lim_{n\to\infty} \sum_{j=-(n-1)}^{n-1} E(X_{n,1}, X_{n,1+j}) \\
&\leq \lim_{n\to\infty} \sum_{j=-(n-1)}^{n-1} 2 \cdot \frac{\gamma}{\gamma-2} \cdot (2\alpha_n(j))^{\frac{\gamma-2}{\gamma}} ||X_{n,1}||_\gamma ||X_{n,1+j}||_\gamma \\
&\leq \lim_{n\to\infty} \sum_{j=-(n-1)}^{n-1} 2 \cdot \frac{\gamma}{\gamma-2} \cdot (2Cj^{-\beta})^{\frac{\gamma-2}{\gamma}} \cdot C^{2/\gamma} \\
&< \infty\,,
\end{aligned}$$

since $\beta > \gamma/(\gamma-2)$. Thus, σ^2 is well-defined here.

For any two sequences $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$, we write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$. Let

$$r \sim \log n, \quad p \sim n/\log n - n^{1/4}, \quad q \sim n^{1/4}\,. \tag{2.1}$$

We then construct blocks of variables as follows

$$V_{n,j} = \sum_{i=(j-1)(p+q)+1}^{jp+(j-1)q} X_{n,i}\,, \quad V'_{n,j} = \sum_{i=jp+(j-1)q+1}^{j(p+q)} X_{n,i}\,, \quad \text{for} \quad j = 1, \cdots, r\,,$$

and

$$R_n = (X_{n,r(p+q)+1} + \cdots + X_{n,n}) I(r(p+q) < n)\,.$$

By Theorem 3 of Bradley (1983), there exists a triangular array of independent random variables, $W_{n,1}, \cdots, W_{n,r}$ such that $W_{n,j} =^d V_{n,j}$, $j = 1, \cdots, r$. Here we use $X =^d Y$ to indicate that the random variables X and Y have the same distribution. And for all $j \in \{1, \cdots, r\}$, for every $\epsilon > 0$

$$P(|W_{n,j} - V_{n,j}| > \epsilon) \leq 18 \left(\frac{\|V_{n,j}\|_\gamma}{\epsilon\sigma\sqrt{n}/r} \right)^{\frac{\gamma}{2\gamma+1}} (\alpha_n(q))^{\frac{2\gamma}{2\gamma+1}}\,. \tag{2.2}$$

We define

$$\begin{aligned} \Delta_{1n} &= \frac{V_{n,1} + \cdots + V_{n,r}}{\sigma\sqrt{n}} - \frac{W_{n,1} + \cdots + W_{n,r}}{\sigma\sqrt{n}} \\ \Delta_{2n} &= \frac{W_{n,1} + \cdots + W_{n,r}}{\sigma\sqrt{n}} \\ \Delta_{3n} &= \frac{V'_{n,1} + \cdots + V'_{n,r}}{\sigma\sqrt{n}} \end{aligned} \tag{2.3}$$

By equation (2.1) and Bonferroni's inequality and Minkowski's inequality, we have

$$\begin{aligned} P(|\Delta_{1n}| > \epsilon) &\leq r \cdot 18 \left(\frac{\|V_{n,j}\|_\gamma}{\epsilon\sigma\sqrt{n}/r} \right)^{\frac{\gamma}{2\gamma+1}} (\alpha_n(q))^{\frac{2\gamma}{2\gamma+1}} \\ &\leq r \cdot 18 \left(\frac{p\|X_{n,1}\|_\gamma}{\epsilon\sigma\sqrt{n}/r} \right)^{\frac{\gamma}{2\gamma+1}} (\alpha_n(q))^{\frac{2\gamma}{2\gamma+1}} \\ &\leq r \cdot 18 \cdot C^{\frac{1}{2\gamma+1}} \epsilon^{-\frac{\gamma}{2\gamma+1}} n^{\frac{\gamma}{2(2\gamma+1)}} \cdot C^{\frac{2\gamma}{2\gamma+1}} q^{-\beta \cdot \frac{2\gamma}{2\gamma+1}} \\ &= o(1) \end{aligned} \tag{2.4}$$

We now use some moment inequality (cf. Yokoyama (1980)) and Liapounov's central limit theorem (cf. Chapter 27 of Billingsley (1995)) to prove the asymptotic normality of Δ_{2n}. It can be shown that if γ' is close to 2 and $2 < \gamma' < \gamma$,

$$E|W_{n,j}|^{\gamma'} \leq C' p^{\gamma'/2}\,, j = 1, \cdots, r\,,$$

where C' is a positive constant. For details, we refer to Yokoyama (1980).

It can also be proved that (cf. proof of Theorem 2.1 below)

$$EW_{n,j}^2 = EV_{n,j}^2 = \sigma^2 p(1+o(1)) .$$

Then

$$\sum_{j=1}^{r} \frac{E|W_{n,j}|^{\gamma'}}{(Var \sum_{j=1}^{r} W_{n,j})^{\gamma'/2}} \leq O\left(r \cdot \frac{C' p^{\gamma'/2}}{(\sigma^2 rp)^{\gamma'/2}}\right) = o(1)$$

indicating that the triangular array $\{W_{n,j}\}_{j=1}^{r}$ satisfies Liapounov's condition. Thus we apply Liapounov's central limit theorem to $\{W_{n,j}\}_{j=1}^{r}$ and get

$$\Delta_{2n} = \left(\frac{rp}{n}\right)^{1/2} \cdot \frac{W_{n,1} + \cdots + W_{n,r}}{\sigma\sqrt{rp}} \to^{d} N(0,1) . \tag{2.5}$$

By (2.2)-(2.5) and Slusky's lemma, Lemma 2.1 will follow if we can show the following

$$\Delta_{3n} = o_p(1), \quad R_n = o_p(1) .$$

Note that by the same arguments above, we may show that

$$\frac{V'_{n,1} + \cdots + V'_{n,r}}{\sigma\sqrt{qr}} \to^{d} N(0,1) .$$

Hence

$$\Delta_{3n} = \frac{V'_{n,1} + \cdots + V'_{n,r}}{\sigma\sqrt{qr}} \cdot \sqrt{\frac{qr}{n}} = o_p(1) .$$

Finally, if $r(p+q) < n$,

$$\begin{aligned} P(R_n > \epsilon) &\leq \frac{E|X_{n,r(p+q)+1} + \cdots + X_{n,n}|^2}{\sigma^2 n} \\ &= \frac{\sigma^2(n - r(p+q) - 1)(1+o(1))}{\sigma^2 n} \\ &= o(1) , \end{aligned}$$

and conclude that $R_n = o_p(1)$ by Chebyschev's inequality. Thus we complete the proof of Lemma 2.1. $\square$

Now, we are ready to give conditions for the asymptotic normality of the centered and scaled p-th sample quantile

$$Z_n = \sqrt{n}(\hat{\xi}_n - \xi_p),$$

for a given $p \in (0,1)$. Recall that, $\hat{\xi}_n = F_n^{-1}(p)$ and $\xi_p = F^{-1}(p)$. The proof is based on Lemma 2.1.

Theorem 2.1 *Suppose that F is differentiable at ξ_p with a positive derivative $f(\xi_p) > 0$ and that $\alpha(n) \leq Cn^{-\beta}$ for some $C > 0$ and $\beta > 1$. Then,*

$$Z_n = \sqrt{n}(\hat{\xi}_n - \xi_p) \to^d N(0, \tau_\infty^2),$$

where τ_∞^2 is as defined in (1.3).

Proof of Theorem 2.1: For a real number x, define $Y_i(x) = I(X_i \leq x), \quad i = 1, 2, \cdots$. Then $Y_1(x), Y_2(x), \cdots$ is a strictly stationary sequence with mean $F(x)$ and auto-covariance function $R_j(x) = Cov(Y_1(x), Y_{1+j}(x)), \quad j \in \mathbb{Z}$. Further, by Billingsley's inequality (cf. Corollary 1.1 of Bosq (1998)), $|R_j(x)| \leq 4\alpha(j) \leq 4j^{-\beta}, \quad j \in \mathbb{Z}$. Thus,

$$\sigma_\infty^2(x) \equiv \sum_{j=-\infty}^{\infty} Cov(I(X_1 \leq x), I(X_{1+j} \leq x)) \equiv \sum_{j=-\infty}^{\infty} Cov(Y_1(x), Y_{1+j}(x)) \tag{2.6}$$

converges absolutely. Let $\bar{Y}_n(x) = \frac{1}{n}\sum_{i=1}^n Y_i(x)$. Then, $\bar{Y}_n(x) = F_n(x)$, where $F_n(x)$ is the empirical distribution function of the sample $X_1, \cdots, X_n$.

By the stationarity of $\{X_i\}$,

$$\begin{aligned}
Var(\sqrt{n}\bar{Y}_n(x)) &= \frac{1}{n} Var\left(\sum_{i=1}^n Y_i(x)\right) \\
&= \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n Cov(Y_i(x), Y_j(x)) \\
&= \sum_{j=-(n-1)}^{n-1} (1 - |j|/n) R_j(x) \\
&\equiv {\sigma_n}^2(x) \quad \text{(say)}.
\end{aligned}$$

Note that by Kronecker's lemma,

$$
\begin{aligned}
|\sigma_n{}^2(x) - \sigma_\infty^2(x)| &\leq \sum_{j\geq n} |R_j(x)| + \frac{1}{n} \sum_{j=-(n-1)}^{n-1} |j||R_j(x)| \\
&\leq 4\sum_{j\geq n} \alpha(j) + \frac{1}{n} \sum_{j=-(n-1)}^{n-1} |j|\alpha(j) \\
&\to 0
\end{aligned}
$$

implying that $\sigma_n{}^2(x)$ converges uniformly to $\sigma_\infty^2(x)$. Thus, we have

$$
\lim_{n\to\infty} Var(\sqrt{n}\bar{Y}_n(x)) = \lim_{n\to\infty} \sigma_n{}^2(x) = \sigma_\infty^2(x) \quad \text{uniformly in} \quad x\,. \tag{2.7}
$$

Next we show that $\sigma_\infty^2(x)$ is continuous at $x = \xi_p$. Because $\alpha(n) = O(n^{-\beta})$ for some $\beta > 1$, it follows that for any $\epsilon > 0$, there exists a positive integer N such that

$$
\sum_{|j|=N+1}^{\infty} \alpha(j) < \epsilon/12\,.
$$

Also note that for any j, $R_j(x)$ is continuous at $x = \xi_p$. Then for the same ϵ, there exists a $\delta > 0$ such that for arbitrary real number x_1 satisfying $|x_1 - \xi_p| < \delta$, we have

$$
\left|\sum_{j=-N}^{N} R_j(x_1) - \sum_{j=-N}^{N} R_j(\xi_p)\right| < \epsilon/3.
$$

Hence,

$$
\begin{aligned}
|\sigma_\infty^2(x_1) - \sigma_\infty^2(\xi_p)| &\leq \left|\sum_{j=-N}^{N} R_j(x_1) - \sum_{j=-N}^{N} R_j(\xi_p)\right| + \sum_{|j|=N+1}^{\infty} |R_j(x_1)| \\
&\quad + \sum_{|j|=N+1}^{\infty} |R_j(\xi_p)| \\
&< \epsilon/3 + 2\sum_{|j|=N+1}^{\infty} 4\alpha(j) \\
&< \epsilon\,,
\end{aligned}
$$

which shows that $\sigma_\infty^2(x)$ is continuous at $x = \xi_p$.

Next, let $x_n = \xi_p + x/\sqrt{n}$. Then, for any $x \in \mathbb{R}$, by the uniform convergence of $\sigma_n^2(x)$ to $\sigma_\infty^2(x)$ and the continuity of $\sigma_\infty^2(x)$ at $x = \xi_p$, we have

$$
\lim_{n\to\infty} Var(\sqrt{n}F_n(\xi_p + x/\sqrt{n})) = \lim_{n\to\infty} \sigma_n^2(x_n) = \sigma_\infty^2(\xi_p) \equiv \sigma_\infty^2 \;(say). \tag{2.8}
$$

We define $Y_{n,j} = I(X_j \leq x_n) - F(x_n), \quad j \in \mathbb{Z}, \quad n = 1, 2, \cdots$. Then

$$E|Y_{n,1}|^\gamma \leq 1 \quad \forall \gamma > 0, \quad n \geq 1.$$

And the variables $\{Y_{n,j} : j \in \mathbb{Z}\}$ is a strictly stationary α-mixing process with coefficients $\alpha_n(j)$ satisfying

$$\alpha_n(j) \leq 4\alpha(j) \leq 4Cj^{-\beta}, \quad j \geq 1, n \geq 1 .$$

Now applying Lemma 2.1 to the array $\{Y_{n,j} : j \in \mathbb{Z}\}_{n \geq 1}$ and using (2.8), we have

$$\frac{\sqrt{n}(F_n(x_n) - F(x_n))}{\sigma_\infty} = \frac{\sum_{j=1}^n Y_{n,j}}{\sigma_\infty \sqrt{n}} \to^d N(0,1) .$$

Also note that by the differentiability of f at ξ_p,

$$\sqrt{n}(p - F(x_n)) \to -xf(\xi_p)$$

for any $x \in \mathbb{R}$. Hence, it follows that

$$\begin{aligned} P(F_n(\xi_p + x/\sqrt{n}) \geq p) &= P\left(\frac{\sqrt{n}(F_n(x_n) - F(x_n))}{\sigma_\infty} \geq \frac{\sqrt{n}(p - F(x_n))}{\sigma_\infty}\right) \\ &\to \Phi(xf(\xi_p)/\sigma_\infty) . \end{aligned}$$

By similar arguments, for any $x \in \mathbb{R}$,

$$P(F_n(\xi_p + x/\sqrt{n}) > p) \to \Phi(xf(\xi_p)/\sigma_\infty) .$$

By the definition of $\hat{\xi}_n$,

$$P(F_n(\xi_p + x/\sqrt{n}) > p) \leq P(\sqrt{n}(\hat{\xi}_n - \xi_p)) \leq x) \leq P(F_n(\xi_p + x/\sqrt{n}) \geq p)$$

Thus, it follows that

$$\sqrt{n}(\hat{\xi}_n - \xi_p) \to^d N(0, \sigma_\infty^2/f^2(\xi_p)) .$$

This completes the proof of Theorem 2.1. □

Note that the conditions imposed on the dependence structure of the X_i's and on the marginal distribution of X_1 here are fairly non-restrictive. Asymptotic normality of Z_n for mixing random variables under stronger conditions also follows from the results of Sen (1972), Babu and Singh (1978), and Yoshihara (1995), who obtained Bahadur-representations for sample quantiles.

2.3 Validity of the MBB Approximation

In this section, we consider the validity of block bootstrap approximation to the distribution of the sample quantiles under dependence. The following main result asserts the consistency of the MBB approximation for the distribution function of Z_n.

Theorem 2.2 *Assume that $0 < p < 1$ and that F has a positive and continuous derivative f in a neighborhood $\mathcal{N}_p$ of ξ_p with $f(\xi_p) > 0$. Also, suppose that $\alpha(n) \leq Cn^{-\beta}$ for some $C \in (0, \infty)$ and $\beta > 9.5$, and that $\ell = O(n^{1/2-\eta})$ for some $\eta \in (5/(2+4\beta), 1/2)$. Then,*

$$\sup_{x \in \mathbb{R}} |P_*(\sqrt{n}(\xi_n^* - \tilde{\xi}_n) \leq x) - P(\sqrt{n}(\hat{\xi}_n - \xi_p) \leq x)| = o(1) \quad a.s.$$

Theorem 2.2 shows that the MBB method provides a valid approximation to the distribution of the centered and scaled sample quantile Z_n in the almost sure sense under a polynomial strong mixing rate and under a mild condition on the block length ℓ. Note that for $\beta > 9.5$, $5/(2+4\beta) < 1/8$. Hence, for any $\beta > 9.5$, the MBB approximation remains valid for block sizes ℓ that grow to infinity at the rate $O(n^{3/8})$. Furthermore, Theorem 2.2 allows the block size ℓ to grow at a rate $O(n^{1/2-\epsilon})$ for an arbitrarily small $\epsilon > 0$, provided that β is appropriately large (viz., $\beta > [5(\epsilon)^{-1} - 2]/4$). In the independent case, validity of Efron's (1979) bootstrap for the sample quantiles was established by Bickel and Freedman (1981) and Singh (1981). Theorem 2.2 is an extension of the basic consistency result to the case of weakly dependent random variables for the MBB method.

As in the case of block bootstrap methodology for smooth functions of the data, performance of the block bootstrap distribution function and variance estimators of Z_n critically depends on the block size ℓ. There has been some work on the choice of the optimal block length in approximating the distributions of statistics based on sample means (cf. Hall, Horowitz and Jing (1995)). We shall discuss the optimal block length and the optimal rate of convergence of Theorem 2.2 in Chapter 5.

The main tools used for proving the strong consistency result are (i) an 'exponential inequality' for sums of block variables (i.e., for block-bootstrap moments, cf. Lemma 2.2, Section

2.4), and (ii) a bound on oscillations of the empirical distribution function of strongly mixing random variables under a polynomial strong mixing condition (cf. Lemma 2.5, Section 2.4), which may be of some independent interest. For proving Theorem 2.2, we shall first develop some results which are presented in the form of following lemmas.

Lemma 2.2 *(An exponential inequality for sums of block variables). Let $\{\tilde{X}_i\}_{i\in\mathbb{Z}}$ be a sequence of random variables (not necessarily stationary) on a probability space $(\Omega, \mathcal{F}, P)$ and let $\{\ell\} \equiv \{\ell_n\}_{n\geq 1}$ be a sequence of positive integers satisfying $1 \leq \ell \leq n/8$ for all $n \geq 1$. For each $n \geq 1$, let $\{W_{nj} : 1 \leq j \leq N\}$ be a collection of zero mean random variables such that W_{nj} is $\sigma\langle\{\tilde{X}_i : j \leq i \leq j+\ell-1\}\rangle$-measurable for all $1 \leq j \leq N$, where $N = n - \ell + 1$. Also, let $\{d_n\}_{n\geq 1} \subset \mathbb{N}$ and $\{\epsilon_n\}_{n\geq 1} \subset (0,\infty)$ be two sequences of real numbers with $2\ell \leq d_n \leq n/4$ for all $n \in \mathbb{N}$. Then, for any $q \in [0,\infty]$, there exist positive constants $C_k, k = 1,2$, depending only on q, such that for all $n \geq 8$,*

$$\begin{aligned}
& P\Big(\Big|\sum_{j=1}^{N} W_{nj}\Big| > N\epsilon_n\Big) \\
\leq\ & C_1 \exp\Big(- \frac{C_2[n/d_n]^2\epsilon_n^2}{w_{n,2}^2 + [n/d_n]^{1/2} w_{n,\infty}\epsilon_n}\Big) \\
& + C_1 \cdot [n/d_n] \cdot \max\Big\{1, \frac{w_{n,q}}{\epsilon_n}\Big\}^{\frac{q}{2q+1}} [\alpha(d_n/2)]^{\frac{2q}{2q+1}},
\end{aligned}$$

where $w_{n,a} = \max_{1\leq j\leq N} \|W_{nj}\|_a$, $a \in [1,\infty]$. For $q = \infty$, the exponents $q/[2q+1]$ and $2q/[2q+1]$ are interpreted as $1/2$ and 1, respectively.

Proof: Let $K_n = \lceil N/d_n \rceil$ and $S_n(k) \equiv \sum_{j=1}^{N} W_{nj} I((k-1)d_n+1 \leq j \leq kd_n)$, $\quad k = 1, \ldots, K_n$. Note that for any $1 \leq k < k+r \leq K_n$, the variables $S_n(k)$ and $S_n(k+r)$ are functions of $\{\tilde{X}_j : (k-1)d_n+1 \leq j \leq kd_n+\ell-1\}$ and $\{\tilde{X}_j : (k+r-1)d_n+1 \leq j \leq (k+r)d_n+\ell-1 \wedge n\}$, respectively and are separated by $[(k+r-1)d_n - (kd_n+\ell-1)] = [(r-1)d_n - \ell+1]$-many $\tilde{X}_j$ variables. Let $\mathcal{K}_{n,1} = \{k \in \mathbb{N} : k \text{ is odd}, k \leq K_n\}$ and let $\mathcal{K}_{n,2} = \{k \in \mathbb{N} : k \text{ is even}, k \leq K_n\}$. Then, by definition,

$$\sum_{j=1}^{N} W_{nj} = \sum_{i=1}^{2} \sum_{k\in\mathcal{K}_{n,i}} S_n(k). \tag{2.9}$$

For each $i \in \{1,2\}$, by Theorem 3 of Bradley (1983), as in the proof of his Theorem 4, there exist random variables $\{S_n^0(k) : k \in \mathcal{K}_{n,i}\}$ such that

(i) $S_n^0(k) =^d S_n(k)$ for all $k \in \mathcal{K}_{n,i}$,

(ii) $\{S_n^0(k) : k \in \mathcal{K}_{n,i}\}$ are independent,

(iii) for any $\epsilon \in (0, \|S_n(k)\|_q)$,

$$P\Big(\Big|S_n^0(k) - S_n(k)\Big| > \epsilon\Big) \le 18\Big(\frac{\|S_n(k)\|_q}{\epsilon}\Big)^{\frac{q}{2q+1}} [\alpha(d_n - \ell)]^{\frac{2q}{2q+1}}, \tag{2.10}$$

for every $k \in \mathcal{K}_{n,i}$.

Next, note that by Minkowski's inequality, for any $a \in [0,\infty]$, $\|S_n(k)\|_a \le d_n w_{n,a}$ for all $k = 1, \ldots, K_n$. Hence, by (2.9), (2.10), and by Bernstein's inequality for sums of independent random variables (cf. Shorack and Wellner (1986), pp. 855), we have

$$\begin{aligned}
& P\Big(\Big|\sum_{j=1}^N W_{nj}\Big| > N\epsilon_n\Big) \\
\le\ & \sum_{i=1}^2 P\Big(\Big| \sum_{k \in \mathcal{K}_{n,i}} S_n(k)\Big| > N\epsilon_n/2\Big) \\
\le\ & \sum_{i=1}^2 P\Big(\Big| \sum_{k \in \mathcal{K}_{n,i}} S_n^0(k)\Big| > N\epsilon_n/2\Big) \\
& + \sum_{k=1}^{K_n} P\Big(\Big|S_n^0(k) - S_n(k)\Big| > \frac{d_n\epsilon_n}{4} \wedge \|S_n(k)\|_q\Big) \\
\le\ & C_1 \exp\Big(- \frac{C_2[N\epsilon_n]^2}{[d_n w_{n,2}]^2 + (d_n w_{n,\infty})(N\epsilon_n)[K_n]^{-1/2}}\Big) \\
& + C_1 K_n \Big(\max\{1, \frac{w_{n,q}}{\epsilon_n}\}\Big)^{\frac{q}{2q+1}} [\alpha(d_n - \ell)]^{\frac{2q}{2q+1}},
\end{aligned}$$

which yields Lemma 2.2 after some simple algebra. This completes the proof of Lemma 2.2. □

The next result is an extension of Bernstein's inequality, from the independent set up, to the situation where the random variables are strongly mixing. It can be proved by easily modifying the proof of Theorem 1.4 of Bosq (1998).

Lemma 2.3 *Suppose that $\{X_{n,j} : j \in \mathbb{Z}\}_{n\geq 1}$ is a double array of a zero-mean real-valued row-wise strictly stationary α-mixing random variables with mixing coefficients $\alpha_n(\cdot)$, $n \geq 1$. Also, suppose that there exists a constant $c > 0$ such that*

$$E|X_{n,1}|^k \leq c^{k-2}k!E|X_{n,1}|^2 < \infty, \quad n \geq 1, \quad k = 3, 4, \cdots.$$

Then for each $n \geq 2$, $\epsilon > 0$, each integer $q \in [1, n/2]$ and every $k \geq 3$,

$$P\left(\left|\sum_{j=1}^{n} X_{n,j}\right| > n\epsilon\right) \leq a_1 \exp\left\{-\frac{q\epsilon^2}{25m_2^2 + 5c\epsilon}\right\} + a_2(k)\alpha_n\left(\left\lfloor\frac{n}{q+1}\right\rfloor\right)^{\frac{2k}{2k+1}},$$

where

$$a_1 = 2\frac{n}{q} + 2\left(1 + \frac{\epsilon^2}{25m_2^2 + 5c\epsilon}\right), \quad \text{with} \quad m_2^2 = \max_{1\leq j\leq n} E|X_{n,j}|^2$$

$$a_2(k) = 11n\left(1 + \frac{5m_k^{\frac{k}{2k+1}}}{\epsilon}\right), \quad \text{with} \quad m_k = \max_{1\leq j\leq n} \|X_{n,j}\|_k.$$

Proof: For each integer $q \in [1, n/2]$, let $r = \lfloor\frac{n}{q+1}\rfloor$. We define the blocks as the following

$$Z_{n,j} = \sum_{i=1}^{q} X_{n,(i-1)r+j}, \quad j = 1, \cdots, r, \quad R_n = (X_{n,qr+1} + \cdots + X_{n,n})I(qr < n). \tag{2.11}$$

Then, by Bonferonni's inequality

$$P\left(\left|\sum_{j=1}^{n} X_{n,j}\right| > n\epsilon\right) \leq \sum_{j=1}^{r} P\left(|Z_{n,j}| > \frac{4n\epsilon}{5r}\right) + P\left(|R_n| > \frac{n\epsilon}{5}\right). \tag{2.12}$$

Note that for any random variable X, arbitrary $t > 0$, $a \in \mathbb{R}$, we have, by applying Markov's inequality

$$P(X \geq a) = P(e^{tX} \geq e^{ta}) \leq e^{-ta}Ee^{tX}. \tag{2.13}$$

Let $t = \epsilon/(5m_2^2 + c\epsilon)(n - qr)$, then $tc(n - qr) = c\epsilon/(5m_2^2 + c\epsilon) = \delta < 1$. We now apply inequality (2.13) to R_n with t given as above and $a = n\epsilon/5$

$$
\begin{aligned}
P\left(R_n > \frac{n\epsilon}{5}\right) &\le e^{-tn\epsilon/5} E e^{tR_n} \\
&\le e^{-tn\epsilon/5}\left(1 + \sum_{s=2}^{\infty} \frac{t^s}{s!} E|R_n|^s\right) \\
&\le e^{-tn\epsilon/5}\left(1 + \sum_{s=2}^{\infty} \frac{t^s}{s!} (n-qr)^s c^{s-2} s! m_2^2\right) \\
&= e^{-tn\epsilon/5}\left(1 + t^2 (n-qr)^2 m_2^2 \cdot \frac{1}{1-\delta}\right) \\
&\le \exp\left(-\frac{q\epsilon^2}{5(5m_2^2 + c\epsilon)}\right) \cdot \left(1 + \frac{\epsilon^2}{5(5m_2^2 + c\epsilon)}\right).
\end{aligned}
$$

Hence,

$$
P\left(|R_n| > \frac{n\epsilon}{5}\right) \le 2\left(1 + \frac{\epsilon^2}{5(5m_2^2 + c\epsilon)}\right) \exp\left(-\frac{q\epsilon^2}{5(5m_2^2 + c\epsilon)}\right) \tag{2.14}
$$

Next, let $c_0 = m_k + 2\epsilon/5$, then, by Minkowski's inequality

$$
\|X_{n,(i-1)r+1} + c_0\|_k \ge c_0 - \|X_{n,(i-1)r+1}\|_k \ge 2\epsilon/5 > 0 .
$$

Thus, by Lemma 1.2 of Bosq (1998), there exists a triangular array of independent random variables $\{Y_{n,i}\}_{i=1}^{q}$ such that $Y_{n,i} =^d X_{n,(i-1)r+1}$ and

$$
\begin{aligned}
P\left(|Y_{n,i} - X_{n,(i-1)r+1}| > \frac{2\epsilon}{5}\right) &\le 11\left(\frac{\|X_{n,(i-1)r+1} + c_0\|_k}{2\epsilon/5}\right)^{\frac{k}{2k+1}} (\alpha_n(r))^{\frac{2k}{2k+1}} \\
&\le 11\left(1 + \frac{5m_k}{\epsilon}\right)^{\frac{k}{2k+1}} \left(\alpha_n\left(\left\lfloor \frac{n}{q+1} \right\rfloor\right)\right)^{\frac{2k}{2k+1}}
\end{aligned}
$$

which together with Bonferronni's inequality and Bernstein's inequality, leads to

$$
\begin{aligned}
P\left(|Z_{n,j}| > \frac{4n\epsilon}{5r}\right) &\le P\left(|\sum_{i=1}^{q} |Y_{n,i}| > \frac{2q\epsilon}{5}\right) + \sum_{i=1}^{q} P\left(|Y_{n,i} - X_{n,(i-1)r+1}| > \frac{2\epsilon}{5}\right) \\
&\le 11q\left(1 + \frac{5m_k}{\epsilon}\right)^{\frac{k}{2k+1}} \left(\alpha_n\left(\left\lfloor \frac{n}{q+1} \right\rfloor\right)\right)^{\frac{2k}{2k+1}} \\
&\quad + 2\exp\left(-\frac{q\epsilon^2}{5(5m_2^2 + c\epsilon)}\right).
\end{aligned} \tag{2.15}
$$

Thus, Lemma 2.3 follows from (2.12), (2.14), and (2.15). $\square$

Lemma 2.4 *Assume that $\alpha(n) \le Cn^{-\beta}$ for some $C \in (0,\infty)$ and $\beta > 1$ and that $\ell = o(n^{1/2})$. Then we have the following*

(i) $\sup_{x\in\mathbb{R}} |F_n(x) - \tilde{F}_n(x)| = O(\ell/n)$, *a.s.;*

(ii) $|F_n(t) - F(t)| = O\left(n^{-\frac{1}{2}+\frac{9}{10+8\beta}} \log n\right)$ *a.s.* $t \in \mathbb{R}$*;*

(iii) $\hat{\xi}_n = \xi_p + O\left(n^{-\frac{1}{2}+\delta}\log n\right), \quad \tilde{\xi}_n = \xi_p + O\left(n^{-\frac{1}{2}+\delta}\log n\right)$, *a.s.*
for any $\delta \in (5/(2+4\beta), 1/2)$, *provided that* $\beta > 2$*;*

(iv) Moreover, if $\beta > 9.5$,

$$F_n(\hat{\xi}_n) = p + O(n^{-1/2}(\log n)^{-2}), \quad \tilde{F}_n(\tilde{\xi}_n) = p + O(\ell/n + n^{-1/2}(\log n)^{-2}) \quad a.s.$$

Proof: For $x \in \mathbb{R}$, Let $Y_i(x)$ be defined as before, i.e., $Y_i(x) = I(X_i \le x)$. Define the block average

$$U_i^*(x) = \frac{1}{\ell}\sum_{j=1}^{\ell} I(X_{i+j}^* \le x), \quad i = 1, \cdots, b.$$

Note that $U_1^*, \cdots, U_b^*$ are conditionally i.i.d.. Hence,

$$\begin{aligned}
\tilde{F}_n(x) &= E_*\left[\frac{1}{n_1}\sum_{i=1}^{n_1} I(X_i^* \le x)\right] \\
&= E_*\left[\frac{1}{b}\sum_{i=1}^{b} U_i^*(x)\right] \\
&= E_* U_1^*(x) \\
&= E_*\left[\frac{1}{\ell}\sum_{j=1}^{\ell} I(X_j^* \le x)\right] \\
&= \frac{1}{N}\left[\sum_{i=l}^{n-\ell+1} I(X_i \le x) + \sum_{i=1}^{\ell-1}\frac{i}{\ell}(I(X_i \le x) + I(X_{n-i+1} \le x))\right] \\
&= \frac{n}{N}\bar{Y}_n(x) - \frac{1}{N\ell}\left[\sum_{i=1}^{\ell-1}(\ell - i)\left(Y_i(x) + Y_{n-i+1}(x)\right)\right]. \qquad (2.16)
\end{aligned}$$

Then for every $x \in \mathbb{R}$,

$$|F_n(x) - \tilde{F}_n(x)| = \left|\bar{Y}_n(x) - \frac{n}{N}\bar{Y}_n(x) + \frac{1}{N\ell}\sum_{i=1}^{\ell-1}(\ell - i)(Y_i(x) + Y_{n-i+1}(x))\right|$$

$$
\begin{aligned}
&\le \frac{n-N}{N}|\bar{Y}_n(x)| + \frac{1}{N\ell}\sum_{i=1}^{\ell-1}(\ell-i)\left(|Y_i(x)| + |Y_{n-i+1}(x)|\right) \\
&\le 2\frac{\ell-1}{N} \\
&= O\left(\frac{\ell}{n}\right) \quad \text{a.s.}
\end{aligned}
$$

Hence, (i) is proved.

Next we use Theorem 1.3 of Bosq (1998) to prove (ii). Let

$$
\delta = \frac{9}{10+8\beta}, \quad \epsilon = n^{-1/2+\delta}\log n, \quad q = \lfloor n^{1-2\delta}(\log n)^{-1/2}\rfloor + 1, n \ge 2\,.
$$

Then

$$
\begin{aligned}
\frac{-q\epsilon^2}{8} &\le -\frac{(\log n)^{3/2}}{8} \\
\left(1+\frac{4}{\epsilon}\right)^{1/2} &= O(n^{1/4-\delta/2}(\log n)^{-1/2}) \\
\alpha\left(\left\lfloor\frac{n}{2q}\right\rfloor\right) &= O(n^{-2\delta\beta}(\log n)^{-\beta/2})\,.
\end{aligned}
$$

Therefore, we have by Theorem 1.3 of Bosq (1998),

$$
\begin{aligned}
P(|F_n(t)-F(t)|>\epsilon) &\le 4\exp\left(-\frac{q\epsilon^2}{8}\right) + 22\left(1+\frac{4}{\epsilon}\right)^{1/2} q\alpha\left(\left\lfloor\frac{n}{2q}\right\rfloor\right) \\
&\le 4\exp\left(-\frac{(\log n)^{3/2}}{8}\right) \\
&\quad + O\left(n^{1/4-\delta/2}(\log n)^{-1/2}\cdot n^{1-2\delta}(\log n)^{-1/2}\cdot n^{-2\delta\beta}(\log n)^{-\beta/2}\right) \\
&= O(n^{-2}) + O\left(n^{5/4-5\delta/2-2\delta\beta}(\log n)^{-(1+\beta/2)}\right) \\
&= O(n^{-2}) + O(n^{-1}(\log n)^{-(1+\beta/2)})\,,
\end{aligned}
$$

implying

$$
\sum_{n=2}^{\infty} P(|F_n(t)-F(t)| > n^{-1/2+\delta}\log n) < \infty\,.
$$

Thus (ii) follows from Borel-Cantelli lemma.

Next consider (iii). Fix a $\delta \in (5/(2+4\beta), 1/2)$, and let $\gamma = -(3/2-\delta-2\delta\beta)$ and $\epsilon_1 = n^{-1/2+\delta}(\log n), n \ge 3$. Note that the lower bound on δ implies that $\gamma > 1$. We apply Lemma 2.3 to the double array

$$
X_{n,j} = I(X_j \le \xi_p + \epsilon_1) - F(\xi_p + \epsilon_1), \quad j \in \mathbb{Z}, n \ge 1,
$$

with $\epsilon = n^{-1/2+\delta}(\log n)^{1/2}$, $\quad q = \lfloor n^{1-2\delta}(\log\log n)\rfloor + 1, n \geq 3$. Then, with $p_n = F(\xi_p + \epsilon_1)$, $m_2^2 = p_n(1-p_n)$, $\quad m_k^k = p_n^k(1-p_n) + p_n(1-p_n)^k$. Further,

$$\begin{aligned} a_1 &\leq Cn^{2\delta}(\log\log n)^{-1} \\ \frac{-q\epsilon^2}{25m_2^2 + 5\epsilon} &\leq -C(\log n)(\log\log n) \\ a_2(k) &\leq Cn^{3/2-\delta}(\log n)^{-1/2}. \end{aligned}$$

Thus Lemma 2.3 leads to

$$\begin{aligned} &\sum_{n=3}^{\infty} P(|F_n(\xi_p+\epsilon_1) - F(\xi_p+\epsilon_1)| > \epsilon) \\ \leq\ &\sum_{n=3}^{\infty} \left[a_1 \exp\left\{-\frac{q\epsilon^2}{25m_2^2+5\epsilon}\right\} + a_2(k)\alpha_n\left(\left\lfloor\frac{n}{q+1}\right\rfloor\right)^{\frac{2k}{2k+1}}\right] \\ \leq\ &\sum_{n=3}^{\infty} \left[O(n^{-2}) + O\left(n^{3/2-\delta}(\log n)^{-1/2}\right)\cdot n^{-\frac{2k}{2k+1}2\delta\beta}\cdot(\log\log n)^{\frac{2k}{2k+1}\beta}\right] \\ =\ &\sum_{n=3}^{\infty} \left[O(n^{-2}) + O\left(n^{-\gamma+\frac{2}{2k+1}\delta\beta}(\log n)^{-1/2}(\log\log n)^{\frac{2k}{2k+1}\beta}\right)\right] \\ <\ &\infty, \quad \text{by taking } k \text{ sufficiently large}. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} \sum_{n=3}^{\infty} P(\hat{\xi}_n - \xi_p > \epsilon_1) &\leq \sum_{n=3}^{\infty} P(F_n(\xi_p+\epsilon_1) \leq p) \\ &= \sum_{n=3}^{\infty} P(F_n(\xi_p+\epsilon_1) - F(\xi_p+\epsilon_1) \leq p - F(\xi_p+\epsilon_1)) \\ &\leq \sum_{n=3}^{\infty} P(F_n(\xi_p+\epsilon_1) - F(\xi_p+\epsilon_1) \leq -\min_{x\in\mathcal{N}_p} f(x)\epsilon_1) \\ &\leq \sum_{n=3}^{\infty} P(|F_n(\xi_p+\epsilon_1) - F(\xi_p+\epsilon_1)| > \epsilon) \\ &< \infty. \end{aligned} \tag{2.17}$$

This proves the first part of (iii). To prove the second part, note that by (i) and the condition $\ell = o(n^{1/2})$,

$$\sum_{n=3}^{\infty} P(\tilde{\xi}_n - \xi_p > \epsilon_1) \leq \sum_{n=3}^{\infty} P(\tilde{F}_n(\xi_p+\epsilon_1) \leq p)$$

$$\begin{aligned} &\leq \sum_{n=3}^{\infty} P(F_n(\xi_p+\epsilon_1) \leq C(\ell/n)+p) \\ &\leq \sum_{n=3}^{\infty} P(|F_n(\xi_p+\epsilon_1) - F(\xi_p+\epsilon_1)| > \epsilon) \\ &< \infty . \end{aligned}$$

Thus (iii) follows from Borel-Cantelli lemma.

To show (iv), first we prove that

$$\sum_{n=2}^{\infty} P(|F_n(\hat{\xi}_n) - p| \geq n^{-1/2}(\log n)^{-2}) < \infty .$$

Let $\epsilon_n = n^{-1/2}(\log n)^{-2}$, $\delta_n = n^{-3/8}\log n$, $\eta_n = n^{-5/2-1/8}(\log n)^{-3}$, $m_n = \lceil \delta_n/\eta_n \rceil$, and $I_{r,n} = [\xi_p + r\eta_n, \xi_p + (r+1)\eta_n)$, $r = -m_n, \cdots, m_n$. Then

$$\mathcal{I}_n = [\xi_p - \delta_n, \xi_p + \delta_n] \subset \left(\bigcup_{r=-m_n}^{m_n} I_{r,n} \right) \bigcap \mathcal{N}_p ,$$

if n is large enough. Note that for $j \geq 1$ and for any $n > 1$,

$$\begin{aligned} P(X_1 = X_j, X_1 \in \mathcal{I}_n) &\leq \sum_{r=-m_n}^{m_n} P(X_1 = X_j, X_1 \in I_{r,n}, X_j \in I_{r,n}) \\ &\leq \sum_{r=-m_n}^{m_n} [(P(X_1 \in I_{r,n}))^2 + 4\alpha(j-1)] \\ &\leq \sum_{r=-m_n}^{m_n} [(d_1\eta_n)^2 + 4\alpha(j-1)] , \end{aligned} \tag{2.18}$$

where $d_1 = \max_{x \in \mathcal{N}_p} f(x)$. Also, note that for $\beta > 9.5$, $\frac{5}{2+4\beta} < \frac{1}{8}$. By part (iii), with $\delta = 1/8$, we have

$$\sum_{n=2}^{\infty} P(|\hat{\xi}_n - \xi_p| > \delta_n) < \infty .$$

Hence, by (2.18) and the inequality above,

$$\begin{aligned} &\sum_{n=2}^{\infty} P(F_n(\hat{\xi}_n) \geq p + \epsilon_n) \\ \leq\ &\sum_{n=2}^{\infty} P(X_{(\lceil np \rceil)} = \cdots = X_{(\lfloor n(p+\epsilon_n) \rfloor)}) \\ \leq\ &\sum_{n=2}^{\infty} \left[P\Big(X_{(\lceil np \rceil)} \in \mathcal{I}_n, X_{(\lceil np \rceil)} = \cdots = X_{(\lfloor n(p+\epsilon_n) \rfloor)}\Big) + P(X_{(\lceil np \rceil)} \notin \mathcal{I}_n) \right] \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{n=2}^{\infty} P(\exists i,j\in\{1,\cdots,n\} \quad \text{with} \quad j-i\ge n\epsilon_n-2, \quad \text{such that} \quad X_i=X_j\in\mathcal{I}_n) \\
&\qquad\qquad +O(1) \\
&= \sum_{n=2}^{\infty}\sum_{i=1}^{n-1}\sum_{j\ge i+n\epsilon_n-2} P(X_i=X_j, X_i\in\mathcal{I}_n)+O(1) \\
&\le \sum_{n=2}^{\infty}\sum_{i=1}^{n-1}\sum_{k=\lceil n\epsilon_n\rceil-2}^{n} P(X_1=X_{1+k}, X_1\in\mathcal{I}_n)+O(1) \\
&\le \sum_{n=2}^{\infty}\sum_{j=\lceil n\epsilon_n\rceil-1}^{n} nP(X_1=X_j, X_1\in\mathcal{I}_n)+O(1) \\
&\le \sum_{n=2}^{\infty}\sum_{j=\lceil n\epsilon_n\rceil-1}^{n} n\left[\sum_{r=-m_n}^{m_n}(d_1\eta_n)^2+\sum_{r=-m_n}^{m_n}4\alpha(j-1)\right]+O(1) \\
&\le \sum_{n=2}^{\infty} n^2(2m_n+1)(d_1\eta_n)^2+\sum_{n=2}^{\infty}\sum_{j=\lceil n\epsilon_n\rceil-1}^{n} n(2m_n+1)4\alpha(j-1)+O(1) \\
&\le CA_2^2\sum_{n=2}^{\infty} n^{-1}(\log n)^{-2}+C\sum_{n=2}^{\infty} nm_n(n\epsilon_n)^{-\beta+1}+O(1) \\
&= O(1)+C\sum_{n=2}^{\infty} n(n^{-3/8}\log n)[n^{-5/2-1/8}(\log n)^{-3}]^{-1}[n^{1/2}(\log n)^{-2}]^{-\beta+1} \\
&\le O(1)+C\sum_{n=2}^{\infty} n^{13/4-(\beta-1)/2}(\log n)^{2+2\beta} \\
&< \infty,
\end{aligned}
$$

provided that $\beta > 9.5$.

Likewise, we may show that

$$\sum_{n=1}^{\infty} P(F_n(\hat{\xi}_n)\le p-\epsilon_n)<\infty\,.$$

Hence, by Bore-Cantelli lemma, we have, for $\beta > 9.5$,

$$F_n(\hat{\xi}_n)=p+O(n^{-1/2}(\log n)^{-2}) \quad \text{a.s.}$$

Using part (i) and similar arguments, we get

$$\tilde{F}_n(\tilde{\xi}_n)=p+O(\ell/n+n^{-1/2}(\log n)^{-2}) \quad \text{a.s.}$$

This completes the proof of Lemma 2.4. □

It follows from the proof above that, unlike in the i.i.d. case, the MBB empirical distribution function $F_n^*(x)$ for a given $x \in \mathbb{R}$ is no longer an unbiased estimator of the empirical distribution function based on the original sample, and the bias depends on the choice of block length ℓ. In fact, the bound on Lemma 2.4 (i) can be refined as $O(\sqrt{\ell}/n)$ (cf. Götze and Künsch (1996)).

It is known that, when the observations are independent,

$$\hat{\xi}_n = \xi_p + O(n^{-1/2}(\log\log n)^{1/2}), \quad F_n(t) = F(t) + O(n^{-1/2}(\log\log n)^{1/2}).$$

Lemma 2.4 (ii), (iii) extends these results from the i.i.d. set up to the weakly dependent case. The bounds in those results may be modified to be in the uniform sense by blocking the random variables, applying the exponential inequality within each block, and then applying Bonferroni's inequality to the union of all those blocks. However, we do not pursue such extensions. These results stated above are sufficient to establish the consistency of the MBB approximations.

Lemma 2.5 *Assume that the α-mixing coefficient of $\{X_i\}_{i\in\mathbb{Z}}$ satisfies $\alpha(n) \le Cn^{-\beta}$, $n \ge 1$, for some positive constant C and for some $\beta > 7.5$. Also, suppose that the marginal distribution function $F(x)$ is continuously differentiable with derivative $f(x)$ in an open neighborhood $\mathcal{N}_p$ of ξ_p such that*

$$0 < d_1 = \inf\{f(x) : x \in \mathcal{N}_p\} \le d_2 = \sup\{f(x) : x \in \mathcal{N}_p\} < \infty.$$

Then

$$\Delta_n \equiv \sup_{x\in\mathcal{I}_n} |F_n(x) - F(x) - F_n(\xi_p) + p| = O(n^{-1/2}(\log n)^{-2}) \quad a.s., \tag{2.19}$$

where $\mathcal{I}_n = [\xi_p - n^{-3/8}\log n,\ \xi_p + n^{-3/8}\log n]$.

Similar results were obtained by Sen (1972) for ϕ-mixing processes and by Babu and Singh (1978) for α-mixing processes with exponentially decaying coefficients. Lemma 2.5 relaxes the weak dependence condition by exploiting the exponential inequality of Lemma 2.3.

Proof: The proof is based on Lemma 2.3, Bonferroni's inequality, and Borel-Cantelli lemma. Let

$$\eta_{r,n} = \xi_p + r\epsilon_n, \quad r = 0, \pm 1, \cdots, \pm d_n, \quad \epsilon_n = n^{-1/2}(\log n)^{-2}, \quad d_n = \lceil n^{1/8}(\log n)^3 \rceil .$$

We define

$$\Delta_{r,n} = F_n(\eta_{r,n}) - F(\eta_{r,n}) - F_n(\hat{\xi}_n) + p, \quad g(x) = F_n(x) - F(x) - F_n(\xi_p) + p .$$

Then for all $x \in [\eta_{r,n}, \eta_{r+1,n}]$,

$$\begin{aligned} g(x) &\leq F_n(\eta_{r+1,n}) - F(\eta_{r,n}) - F_n(\xi_p) + p \\ &\leq \Delta_{r+1,n} + [F(\eta_{r+1,n}) - F(\eta_{r,n})] \\ &\leq \Delta_{r+1,n} + d_2\epsilon_n . \end{aligned}$$

Likewise, $g(x) \geq \Delta_{r,n} - d_2\epsilon_n$.

Thus,

$$\Delta_n = \sup_{x \in \mathcal{I}_n} |F_n(x) - F(x) - F_n(\xi_p) + p| \leq \max_{|r| \leq d_n} \Delta_{r,n} + d_2\epsilon_n .$$

Hence it is enough to show that

$$\max_{-d_n \leq r \leq d_n} |F_n(\eta_{r,n}) - F(\eta_{r,n}) - F_n(\xi_p) + p| \leq \epsilon_n \quad \text{a.s.} \tag{2.20}$$

For $r = \pm 1, \cdots, \pm d_n$, we define $V_j^{(r)} = [I(X_j \leq \eta_{r,n}) - I(X_j \leq \xi_p)]\text{sign}(r), \quad j = 1, \cdots, n$, where $\text{sign}(t) = 1$ or -1 according as r is positive or negative. Then $V_j^{(r)}$ follows a Bernoulli $(p_{r,n})$ distribution with $p_{r,n} = EV_j^{(r)} = |F(\eta_{r,n}) - F(\xi_p)|$. Further,

$$\begin{aligned} d_1 n^{-1/2}(\log n)^{-2} &\leq p_{r,n} \leq d_2 n^{-3/8} \log n \\ E|V_j^{(r)} - p_{r,n}|^k &= p_{r,n}(1-p_{r,n})[(1-p_{r,n})^{k-1} + p_{r,n}^{k-1}] \\ &\leq k! E(V_j^{(r)} - p_{r,n})^2 \wedge 2p_{r,n} , \end{aligned} \tag{2.21}$$

for any integer $k > 2$. We now apply Lemma 2.3 to the row sum of the triangular array $\{\tilde{V}_j^{(r)} : 1 \le j \le n\}_{n\ge1}$, where

$$\tilde{V}_j^{(r)} = V_j^{(r)} - p_{r,n} ,$$

with $c = 1$, $\epsilon = n^{-1/2}(\log n)^{-2}$, $m_2^2 = p_{r,n}(1-p_{r,n})$, and $m_k^k = p_{r,n}(1-p_{r,n})[(1-p_{r,n})^{k-1} + p_{r,n}^{k-1}]$. Choosing $q = \lceil n^{5/8}(\log n)^7 \rceil$ leads to

$$\begin{aligned} a_1 &= 2\frac{n}{q} + 2\left(1 + \frac{\epsilon^2}{25m_2^2 + 5c\epsilon}\right) \\ &\le 2n^{3/8}(\log n)^{-7} + 2\left(1 + \frac{n^{-1}(\log n)^{-4}}{25d_2n^{-1/2}(\log n)^{-2} + 5n^{-1/2}(\log n)^{-2}}\right) \\ &\le O(n^{3/8}(\log n)^{-7}) ; \end{aligned}$$

and

$$\begin{aligned} a_2(k) &= 11n\left(1 + \frac{5m_k^{\frac{k}{2k+1}}}{\epsilon}\right) \\ &\le 11n\left(1 + \frac{5[2p_{r,n}]^{\frac{1}{2k+1}}}{n^{-1/2}(\log n)^{-2}}\right) \\ &\le 11n\left(1 + \frac{5(d_2n^{-3/8}\log n)^{\frac{1}{2k+1}}}{n^{-1/2}(\log n)^{-2}}\right) \\ &\le O\left(n^{3/2}(\log n)^2 \cdot (n^{-3/8}\log n)^{\frac{1}{2k+1}}\right) ; \end{aligned}$$

Also

$$\frac{q\epsilon^2}{25m_2^2 + 5c\epsilon} \ge \frac{n^{5/8}(\log n)^7 n^{-1}(\log n)^{-4}}{50d_2n^{-3/8}\log n + 5n^{-1/2}(\log n)^{-2}} \ge C_1(\log n)^2 ,$$

for some $C_1 > 0$, if n is sufficiently large. In view of the inequalities above, by Lemma 2.3 we get

$$\begin{aligned} & P\left(\frac{1}{n}\Big|\sum_{j=1}^n \tilde{V}_{n,j}^{(r)}\Big| > n^{-1/2}(\log n)^{-2}\right) \\ \le\ & a_1 \exp\{-\frac{q\epsilon^2}{25m_2^2 + 5c\epsilon}\} + a_2(k)\alpha_n\left(\left\lfloor\frac{n}{q+1}\right\rfloor\right)^{\frac{2k}{2k+1}} \\ \le\ & O(n^{3/8}(\log n)^{-7}) \cdot \exp\{-C_1(\log n)^2\} + O\left(n^{3/2}(\log n)^2 \cdot (n^{-3/8}\log n)^{\frac{1}{2k+1}}\right) \\ & \cdot C\left(n^{3/8}(\log n)^{-7}\right)^{-\beta\cdot\frac{2k}{2k+1}} \\ =\ & O\left(n^{\frac{3}{2}-\frac{3\beta}{8}}(\log n)^{7\beta+2} \cdot (n^{-3/8+3\beta/8}(\log n)^{7\beta})^{\frac{1}{2k+1}}\right) . \end{aligned}$$

Hence, by Bonferroni's inequality,

$$
\begin{aligned}
& P\Big(\max_{-d_n \le r \le d_n} |F_n(\eta_{r,n}) - F(\eta_{r,n}) - F_n(\xi_p) + p| > n^{-1/2}(\log n)^{-2}\Big) \\
= \;& P\Big(\max_{-b_n \le r \le b_n} \frac{1}{n}\Big|\sum_{j=1}^{n} \tilde{V}_{n,j}^{(r)}\Big| > n^{-1/2}(\log n)^{-2}\Big) \\
\le \;& 2d_n P\Big(\frac{1}{n}\Big|\sum_{j=1}^{n} \tilde{V}_{n,j}^{(r)}\Big| > n^{-1/2}(\log n)^{-2}\Big) \\
\le \;& 2n^{1/8}(\log n)^3 \cdot O\left(n^{\frac{3}{2}-\frac{3\beta}{8}}(\log n)^{7\beta+2} \cdot (n^{-3/8+3\beta/8}(\log n)^{7\beta})^{\frac{1}{2k+1}}\right). \qquad (2.22)
\end{aligned}
$$

Then we have, for $\beta > 7.5$ and k sufficiently large,

$$
\sum_{n=1}^{\infty} P\left(\max_{-d_n \le r \le d_n} |F_n(\eta_{r,n}) - F(\eta_{r,n}) - F_n(\xi_p) + p| > n^{-1/2}(\log n)^{-2}\right) < \infty .
$$

So Borel-Cantelli lemma yields (2.20). This completes the proof of Lemma 2.5. □

Lemma 2.6 *Let $\{W_n\}_{n\ge1}$ be a sequence of random variables that converge in distribution to a random variable W with a continuous distribution function $G(x)$. Then*

$$
\sup_{x\in\mathbb{R}} |G_n(x) - G(x)| = o(1)\,,
$$

where $G_n(x)$ is the distribution function of W_n, $n = 1, 2, \cdots$.

Proof. For any positive integer m, we set

$$
x_{k,m} = G^{-1}(k/m), \quad k = 1, \cdots, m-1\,,
$$

$$
x_{0,m} = -\infty, \quad x_{m,m} = +\infty\,.
$$

Then, by the continuity of G and the definition of convergence in distribution, for $k = 1, \cdots, m-1$,

$$
|G_n(x_{k,m}) - G(x_{k,m})| \to 0\,. \qquad (2.23)
$$

Note also that (2.23) holds trivially for the cases, $k = 0$ and $k = m$.

For $x \in (x_{k-1,m}, x_{k,m})$,

$$
G_n(x) - G(x) \le G_n(x_{k,m}) - G(x) \le G_n(x_{k,m}) - G(x_{k,m}) + 1/m\,,
$$

similarly,

$$G_n(x) - G(x) \ge G_n(x_{k-1,m}) - G(x) \ge G_n(x_{k-1,m}) - G(x_{k-1,m}) - 1/m\,.$$

Hence, for $x \in (x_{k-1,m}, x_{k,m})$, $k = 1, \cdots, m$, we have

$$|G_n(x) - G(x)| \le \max_{0 \le k \le m} |G_n(x_{k,m}) - G(x_{k,m})| + 1/m\,. \tag{2.24}$$

Lemma 2.6 follows from (2.23) and (2.24). □

Proof of Theorem 2.2: By Lemma 2.6 and Theorem 2.1, it is enough to show that for every $x \in \mathbb{R}$,

$$|P_*(\sqrt{n}(\xi_n^* - \tilde{\xi}_n) \le x) - \Phi(xf(\xi_p)/\sigma_\infty)| = o(1), \quad \text{a.s.} \tag{2.25}$$

For $1 \le i \le N$ and $x \in \mathbb{R}$, let $U_i(x) = \frac{1}{\ell}\sum_{j=0}^{\ell-1} I(X_{i+j} \le x)$, and let $Y_i(x)$ and $U_i^*(x)$ be defined as before, that is,

$$Y_i(x) = I(X_i \le x), \quad U_i^*(x) = \frac{1}{\ell}\sum_{j=1}^{\ell} I(X^*_{(i-1)\ell+j} \le x), \quad i = 1, \cdots, n, \quad i = 1, \cdots, b\,.$$

Write

$$\hat{x}_n = \tilde{\xi}_n + x/\sqrt{n}\,.$$

Then we have

$$\begin{aligned} P_*(F_n^*(\tilde{\xi}_n + x/\sqrt{n}) \ge p) &= P_*\left(F_n^*(\hat{x}_n) - E_*F_n^*(\hat{x}_n) \ge p - E_*F_n^*(\hat{x}_n)\right) \\ &= P_*\left(\frac{1}{b}\sum_{i=1}^{b}(U_i^*(\hat{x}_n) - E_*U_1^*(\hat{x}_n)) \ge p - \tilde{F}_n(\hat{x}_n)\right)\,. \end{aligned}$$

Then by the Berry-Esseen theorem (Feller (1966))

$$\left|P_*(F_n^*(\hat{x}_n) \ge p) - \Phi\left(\frac{\sqrt{b}(\tilde{F}_n(\hat{x}_n) - p)}{\sqrt{Var_*U_1^*(\hat{x}_n)}}\right)\right| \le \frac{3E_*|U_1^*(\hat{x}_n) - E_*U_1^*(\hat{x}_n)|^3}{\sqrt{b}(Var_*U_1^*(\hat{x}_n))^{3/2}}\,.$$

Similarly,

$$\left|P_*(F_n^*(\hat{x}_n) > p) - \Phi\left(\frac{\sqrt{b}(\tilde{F}_n(\hat{x}_n) - p)}{\sqrt{Var_*U_1^*(\hat{x}_n)}}\right)\right| \le \frac{3E_*|U_1^*(\hat{x}_n) - E_*U_1^*(\hat{x}_n)|^3}{\sqrt{b}(Var_*U_1^*(\hat{x}_n))^{3/2}}\,.$$

Since $P_*(F_n^*(\hat{x}_n) > p) \le P_*(\sqrt{n}(\xi_n^* - \tilde{\xi}_n) \le x) \le P_*(F_n^*(\hat{x}_n) \ge p)$,

$$\left| P_*(\sqrt{n}(\xi_n^* - \tilde{\xi}_n) \le x) - \Phi\left(\frac{\sqrt{b}(\tilde{F}_n(\hat{x}_n) - p)}{\sqrt{Var_* U_1^*(\hat{x}_n)}}\right)\right| \le \frac{3E_*|U_1^*(\hat{x}_n) - E_*U_1^*(\hat{x}_n)|^3}{\sqrt{b}(Var_* U_1^*(\hat{x}_n))^{3/2}}.$$

Therefore (2.25) will follow if we can show that

$$\frac{\sqrt{b}(\tilde{F}_n(\hat{x}_n) - p)}{\sqrt{Var_* U_1^*(\hat{x}_n)}} \to x f(\xi_p)/\sigma_\infty \quad \text{a.s.}$$

and

$$\frac{E|U_1^*(\hat{x}_n) - EU_1^*(\hat{x}_n)|^3}{\sqrt{b}(Var_* U_1^*(\hat{x}_n))^{3/2}} \to 0 \quad \text{a.s.}$$

Hence, it is enough to show the following three relations.

(i) $\sqrt{n}(\tilde{F}_n(\hat{x}_n) - p) \to x f(\xi_p)$ as $n \to \infty$ a.s.

(ii) $\frac{n}{b} Var_* U_1^*(\hat{x}_n) \to \sigma_\infty^2$ as $n \to \infty$ a.s.

(iii) $\frac{n\sqrt{n}}{b^2} E_*|U_1^*(\hat{x}_n) - E_*U_1^*(\hat{x}_n)|^3 \to 0$ as $n \to \infty$ a.s.

STEP I. Proof of (i): Note that (i) trivially holds for $x = 0$. Hence suppose that $x \ne 0$. Choose n sufficiently large so that $|x| \le \log n$. Also note that for $\beta > 9.5$, Lemma 2.4 (iii) implies that

$$|\tilde{\xi}_n - \xi_p| = o(n^{-3/8}\log n), \quad |\hat{x}_n - \xi_p| = o(n^{-3/8}\log n), \quad \text{a.s.}$$

Thus, Lemma 2.5 yields

$$|F_n(\hat{x}_n) - F_n(\tilde{\xi}_n) - F(\hat{x}_n) + F(\tilde{\xi}_n)| = O(n^{-1/2}(\log n)^{-2}) \quad \text{a.s.},$$

which, together-with Lemma 2.4, leads to

$$\begin{aligned}
\tilde{F}_n(\hat{x}_n) - p &= [\tilde{F}_n(\hat{x}_n) - \tilde{F}_n(\tilde{\xi}_n)] + [\tilde{F}_n(\tilde{\xi}_n) - p] \\
&= [F_n(\hat{x}_n) - F_n(\tilde{\xi}_n) + O(\ell/n)] + O(\ell/n + n^{-1/2}(\log n)^{-2}) \\
&= [F_n(\hat{x}_n) - F_n(\tilde{\xi}_n) - F(\hat{x}_n) + F(\tilde{\xi}_n)] + [F(\hat{x}_n) - F(\tilde{\xi}_n)] \\
&\quad + O(\ell/n + n^{-1/2}(\log n)^{-2}) \\
&= O(n^{-1/2}(\log n)^{-2}) + [F(\hat{x}_n) - F(\tilde{\xi}_n)] + O(\ell/n + n^{-1/2}(\log n)^{-2}) \\
&= f(\hat{\epsilon}_n)x/\sqrt{n} + o(n^{-1/2}) \quad \text{a.s.} \qquad (2.26)
\end{aligned}$$

where $\hat{\epsilon}_n \in (\tilde{\xi}_n \wedge \hat{x}_n, \tilde{\xi}_n \vee \hat{x}_n)$. Hence, by part (iii) of Lemma 2.4 and the assumption that $f(x)$ is continuous on $\mathcal{N}_p$ (a neighborhood of ξ_p), $\sqrt{n}(\tilde{F}_n(\hat{x}_n) - p) = xf(\xi_p) + o(1)$ a.s. Thus, (i) is proved.

STEP II. Proof of (ii): By definition,

$$Var_*(U_1^*(\hat{x}_n)) = \frac{1}{N}\sum_{i=1}^{N} U_i^2(\hat{x}_n) - \tilde{F}_n^{\,2}(\hat{x}_n)\,. \tag{2.27}$$

Let $\delta_0 \equiv \frac{5}{2+4\beta}$. Note that $\beta > 9.5$ implies $\delta_0 < \frac{1}{8}$. Given $\ell = O(n^{\frac{1}{2}-\eta})$ for some $\eta \in (\delta_0, \frac{1}{2})$, we fix a $\delta_1 \in (\delta_0, \eta \wedge \frac{1}{8})$ and define two sequences of real numbers $\{x_{n,1}\}_{n\geq 1}$ and $\{x_{n,2}\}_{n\geq 1}$ by

$$x_{n,1} = \xi_p - n^{-\frac{1}{2}+\delta_1}, \quad x_{n,2} = \xi_p + n^{-\frac{1}{2}+\delta_1}, \quad n \geq 1\,. \tag{2.28}$$

Note that by Lemma 2.4 (iii),

$$\begin{aligned} \hat{x}_n &= \tilde{\xi}_n + \frac{x}{\sqrt{n}} \\ &= \xi_p + O(n^{-\frac{1}{2}+\delta}\log n) \end{aligned} \tag{2.29}$$

for every $\delta \in (\delta_0, \frac{1}{8})$. Hence, by (2.28) and (2.29), there exists a set $A \in \mathcal{F}$ with $P(A) = 1$ such that for every $\omega \in A$, there exists an integer $n_\omega \in \mathbb{N}$ such that for all $n \geq n_\omega$,

$$x_{n,1} < \hat{x}_n(\omega) < x_{n,2}\,, \tag{2.30}$$

which, together with (2.27), implies that for all $n \geq n_\omega$,

$$Var_*(U_1^*(\hat{x}_n)) \leq \Big(N^{-1}\sum_{i=1}^{N} U_i^2(x_{n,2}) - \tilde{F}_n^{\,2}(x_{n,1})\Big)$$

and

$$Var_*(U_1^*(\hat{x}_n)) \geq \Big(N^{-1}\sum_{i=1}^{N} U_i^2(x_{n,1}) - \tilde{F}_n^{\,2}(x_{n,2})\Big)\,.$$

Since $n/b\ell \to 1$ as $n \to \infty$, it is enough to show that for $\{x_n\} = \{x_{n,1}\}, \{x_{n,2}\}$,

$$\ell \cdot \Big(N^{-1}\sum_{i=1}^{N} U_i^2(x_n) - \tilde{F}_n^{\,2}(x_n)\Big) \to \sigma_\infty^2 \quad \text{as} \quad n \to \infty \quad \text{a.s.} \tag{2.31}$$

$$\ell \cdot \Big(\tilde{F}_n^{\,2}(x_{n,2}) - \tilde{F}_n^{\,2}(x_{n,1})\Big) = o(1), \quad \text{as} \quad n \to \infty \quad \text{a.s.} \tag{2.32}$$

First, we consider (2.32). Note that by Lemma 2.4, Lemma 2.5, equation (2.28), and by the smoothness of F in a neighborhood of ξ_p,

$$\begin{aligned}
& \ell|\tilde{F}_n(x_{n,2}) - \tilde{F}_n(x_{n,1})| \\
\leq\ & \ell|F_n(x_{n,2}) - F_n(x_{n,1})| + 2\ell \sup_x |F_n(x) - \tilde{F}_n(x)| \\
\leq\ & \ell|[F_n(x_{n,2}) - F_n(\xi_p)] - [F_n(x_{n,1}) - F_n(\xi_p)]| + O(\ell^2/n) \\
\leq\ & \ell|[F_n(x_{n,2}) - F_n(\xi_p)] - [F(x_{n,2}) - F(\xi_p)]| \\
& \quad + \ell|[F_n(x_{n,1}) - F_n(\xi_p)] - [F(x_{n,1}) - F(\xi_p)]| \\
& \quad + \ell|F(x_{n,2}) - F(x_{n,1})| + O(\ell^2/n) \\
\leq\ & 2\ell \cdot O(n^{-1/2}(\log n)^{-2}) + C\ell|x_{n,2} - x_{n,1}| + O(\ell^2/n) \\
=\ & O(\ell n^{-\frac{1}{2}}(\log n)^{-2}) + O(\ell n^{-\frac{1}{2}+\delta_1}) + O(\ell^2/n) \\
=\ & o(1) \quad \text{a.s.,}
\end{aligned}$$

since $\ell = O(n^{\frac{1}{2}-\eta}) = o(n^{\frac{1}{2}-\delta_1})$. Hence, (2.32) follows.

Next consider (2.31). It is easy to verify that

$$\begin{aligned}
& \ell \cdot \left(N^{-1}\sum_{i=1}^{N} U_i^2(x_n) \; - \; \tilde{F}_n^{\,2}(x_n)\right) \\
=\ & \ell N^{-1}\sum_{i=1}^{N}[U_i(x_n) - F(x_n)]^2 \; - \; \ell[\tilde{F}_n(x_n) - F(x_n)]^2 \,. \qquad (2.33)
\end{aligned}$$

By Lemma 2.4 (i),

$$\ell^{1/2}|\tilde{F}_n(x_n) - F(x_n)| = O(\ell^{3/2}/n) = o(1) \quad \text{a.s.}$$

Also, note that by (2.7) and (2.28),

$$\ell E[U_1(x_n) - F(x_n)]^2 \;\; \to \;\; \sigma_\infty^2 \quad \text{as} \quad n \to \infty\,. \qquad (2.34)$$

Hence, to prove (ii), it remains to show that

$$N^{-1}\sum_{i=1}^{N} \tilde{W}_{ni} - E\tilde{W}_{ni} = o(1) \quad \text{a.s.,} \qquad (2.35)$$

where $\tilde{W}_{ni} = \ell[U_i(x_n) - F(x_n)]^2$. To prove (2.35), we now apply Lemma 2.2 with $W_{nj} = \tilde{W}_{nj} - E\tilde{W}_{nj}$. It is easy to check that for this choice of W_{nj}'s, $w_{n,\infty} \leq \ell$ and by (2.34), there exists a constant $C > 0$ such that

$$\begin{aligned} w_{n,2}^2 &= E(\ell[U_1(x_n) - F(x_n)]^2)^2 \\ &\le \ell E(l[U_1(x_n) - F(x_n)]^2) \\ &\le C\ell . \end{aligned}$$

Hence, with $d_n = n^{1/2}$ and $q = 2$, by Lemma 2.2 , for any $\epsilon > 0$, we have

$$\begin{aligned} & P\Big(|N^{-1}\sum_{i=1}^{N} \tilde{W}_{ni} - E\tilde{W}_{ni}| > \epsilon\Big) \\ \le\ & C_1 \exp\Big(-\frac{C_2[n/d_n]^2\epsilon^2}{C\ell + [n/d_n]^{1/2}C\ell\epsilon}\Big) + C_1[n/d_n]\Big\{1 \vee (C(\epsilon)\ell)^{1/2}\Big\}^{\frac{2}{5}} \cdot [\alpha(d_n/2)]^{\frac{4}{5}} \\ \le\ & C_1 \exp\Big(-C(\epsilon)[n/d_n]^{3/2}\ell^{-1}\Big) + C(\epsilon)[n/d_n]\ell^{\frac{1}{5}} d_n^{-\frac{4\beta}{5}} \\ =\ & C_1 \exp\Big(-C(\epsilon)n^{3/4}\ell^{-1}\Big) + C(\epsilon)n^{\frac{1}{2}-\frac{2\beta}{5}}\ell^{\frac{1}{5}}, \end{aligned}$$

which is summable over n, as $\ell^{\frac{1}{5}} = O(n^{\frac{1}{10}})$ and $\beta > 9.5$. This proves (2.35) and hence, completes the proof of STEP II.

STEP III. Proof of (iii): Observe that, by STEP II,

$$\begin{aligned} \frac{n\sqrt{n}}{b^2}E_*|U_1^*(\hat{x}_n) - E_*U_1^*(\hat{x}_n)|^3 &\le \frac{2n\sqrt{n}}{b^2}E_*|U_1^*(\hat{x}_n) - E_*U_1^*(\hat{x}_n)|^2 \\ &= 2\frac{\sqrt{n}}{b}\frac{n}{b}Var_*U_1^*(\hat{x}_n) \\ &= 2\frac{\ell}{\sqrt{n}}\sigma_\infty^2(1 + o(1)) \\ &\to 0 \quad \text{almost surely}. \end{aligned}$$

Hence, STEP III is proved, and so the proof of Theorem 2.2 is complete. □

2.4 Simulation Study

For the simulation study, we here focus on investigating the behavior of the MBB estimators of the sampling distribution of the scaled and centered sample quantile $Z_n = \sqrt{n}(\hat{\xi}_n - \xi_p)$. For simplicity, we consider the following three models:

(i) AR(1): $Y_t = 0.4Y_{t-1} + \epsilon_t$;

(ii) ARMA(1,1): $Y_t - 0.4Y_{t-1} = \epsilon_t + 0.3\epsilon_{t-1}$;

(iii) MA(1): $Y_t = \epsilon_t + 0.3\epsilon_{t-1}$.

In all three models above, we assume that $\{\epsilon_t\}$ are independent $N(0,1)$ random variables. For each model, we considered four different values of the sample size n, $n = 80, 140, 300, 500$. A brief description of the simulation procedure goes as follows.

Step 1: For a particular model, generate 20,000 samples of size n and compute the values of

$$G_n(x) = P(\sqrt{n}(\hat{\xi}_n - \xi_p) \le x)\,.$$

Step 2: Generate 500 data sets of size n and for the m-th data set, compute the conditional quantile $\tilde{\xi}_{n,m}$ and the 500 replicates of the MBB version of $\hat{\xi}_n$, denoted as

$$\xi^*_{n,m}(1), \cdots, \xi^*_{n,m}(500),$$

and then evaluate the corresponding estimate of $G_n(x)$:

$$\begin{aligned} \hat{G}_{n,m}(x) &= \frac{1}{500}\sum_{j=1}^{500} I(\sqrt{n}(\xi^*_{n,m}(j) - \tilde{\xi}_{n,m}) \le x) \\ &\approx P_*(\sqrt{n}(\xi^*_n - \tilde{\xi}_n) \le x)\,, \end{aligned}$$

for $m = 1, \cdots, 500$.

Step 3: Find the bias and variance of the MBB estimator:

$$\begin{aligned} Bias(\hat{G}_n(x)) &= \frac{1}{500}\sum_{m=1}^{500}[\hat{G}_{n,m}(x) - G_n(x)] \\ Var(\hat{G}_n(x)) &= \frac{1}{499}\sum_{m=1}^{500}[\hat{G}_{n,m}(x) - \overline{\hat{G}_n(x)}]^2 \\ SD(\hat{G}_n(x)) &= \sqrt{Var(\hat{G}_n(x))} \\ MSE(\hat{G}_n(x)) &= [Bias(\hat{G}_n(x))]^2 + Var(\hat{G}_n(x) \\ RMSE(\hat{G}_n(x)) &= MSE(\hat{G}_n(x))/(G_n(x))^2 \end{aligned}$$

where

$$\overline{\hat{G}_n(x)} = \frac{1}{500}\sum_{m=1}^{500} \hat{G}_{n,m}(x)\,.$$

For AR(1) model and ARMA(1,1) model, we did simulations with block length varying from 1-15 for all four sample sizes: $80, 140, 300$ and 500. It turned out that MBB estimation of the sample quantile of the MA(1) model is very different from the other two models in that much larger block lengths are needed to get good estimations. We simulated MA(1) model using 15 block lengths between 3-17 for sample sizes 80 and 140. The block lengths used for sample sizes 300 and 500 are between 20-34 and between 30-44, respectively.

Numerical results are presented in the form of tables and graphs below. As described in the simulation procedure, for each choice of the sample size n of a given model, the true value of $G_n(x) = P(\sqrt{n}(\hat{\xi}_n - \xi_p) \leq x)$ in the case of the median, is simulated from $20,000$ samples. The MBB distribution function estimates of $G_n(x)$, with a number of block lengths are calculated from 500 replicates. For simplicity, our simulation study is focused the cases $x = 0$ and $x = 1$. The $SD(\cdot)$, $MSE(\cdot)$, $RMSE(\cdot)$, and the histograms for MBB estimators of $G_n(0)$ and $G_n(1)$ are all based on the same 500 data sets generated from the given model (cf. Models (i)-(iii)). Tables 2.1-2.6 show the simulation results of the MBB approximations of $G_n(x)$ at $x = 0$ and $x = 1$, with the block length that results in the smallest MSE among a range of values. For example, in AR (1) model (i) estimation, when the sample size is chosen to be $n = 80$, the MBB approximation of $G_n(0)$ with block length $\ell = 7$ gives the smallest MSE among the results simulated with block length from 1 to 15 (cf. Figure 2.3), and the bias, SD, MSE, RMSE are -0.0397, 0.0332, 0.0027, 0.5078, 0.0053, respectively. The corresponding histogram of the 500 MBB estimates is given in the top left of Figure 2.1. The rest of the tables and histograms are presented in the same manner. Figures 2.3, 2.4 show respectively, how the MSEs of the MBB distribution function estimators of $G_n(0)$ and $G_n(1)$ for the AR(1) model vary with different combinations of of sample sizes and block lengths.

For all the MBB distribution estimators of the given models, as the sample size increases, the MSEs get closer to zero under all three models, which supports our theoretical findings.

Table 2.1 Distribution approximation of $G_n(0)$ for AR(1) Model (i) in the case of the median, based on 500 MBB replicates.

n	ℓ	Bias	SD	MSE	True $G_n(0)$	RMSE
80	7	-0.0397	0.0332	0.0027	0.5078	0.0105
140	1	-0.0290	0.0224	0.0014	0.5068	0.0055
300	13	-0.0195	0.0245	0.0010	0.5025	0.0040
500	9	-0.0159	0.0245	0.0008	0.5032	0.0032

Table 2.2 Distribution approximation of $G_n(0)$ for ARMA(1,1) Model (ii) in the case of the median, based on 500 MBB replicates.

n	ℓ	Bias	SD	MSE	True $G_n(0)$	RMSE
80	7	-0.0403	0.0361	0.0029	0.5022	0.0115
140	1	-0.0204	0.0245	0.0010	0.5010	0.0040
300	13	-0.0206	0.0245	0.0011	0.5006	0.0044
500	9	-0.0212	0.0245	0.0010	0.5042	0.0039

Table 2.3 Distribution approximation of $G_n(0)$ for MA(1) Model (iii) in the case of the median, based on 500 MBB replicates.

n	ℓ	Bias	SD	MSE	True $G_n(0)$	RMSE
80	7	-0.0471	0.0424	0.0040	0.4990	0.0161
140	9	-0.0430	0.0332	0.0029	0.5019	0.0115
300	31	-0.0292	0.0316	0.0018	0.4974	0.0073
500	37	-0.0235	0.0265	0.0013	0.4999	0.0052

Table 2.4 Distribution approximation of $G_n(1)$ for AR(1) Model (i) in the case of the median, based on 500 MBB replicates.

n	ℓ	Bias	SD	MSE	True $G_n(1)$	RMSE
80	6	-0.0114	0.1091	0.0120	0.7098	0.0169
140	5	-0.0133	0.0922	0.0087	0.7110	0.0122
300	8	-0.0075	0.0775	0.0061	0.7064	0.0086
500	5	-0.0021	0.0671	0.0045	0.7084	0.0064

Table 2.5 Distribution approximation of $G_n(1)$ for ARMA(1,1) Model (ii) in the case of the median, based on 500 MBB replicates.

n	ℓ	Bias	SD	MSE	True $G_n(1)$	RMSE
80	5	-0.0310	0.1269	0.0170	0.7659	0.0222
140	1	0.0101	0.0959	0.0093	0.7687	0.0121
300	2	-0.009	0.0854	0.0074	0.7690	0.0096
500	4	-0.0141	0.0742	0.0057	0.7658	0.0074

Table 2.6 Distribution approximation of $G_n(1)$ for MA(1) Model (iii) in the case of the median, based on 500 MBB replicates.

n	ℓ	Bias	SD	MSE	True $G_n(1)$	RMSE
80	14	-0.0246	0.1304	0.0176	0.8294	0.0212
140	11	-0.0337	0.1109	0.0135	0.8283	0.0163
300	22	-0.0221	0.1015	0.0107	0.8282	0.0129
500	36	-0.0179	0.0889	0.0082	0.8292	0.0099

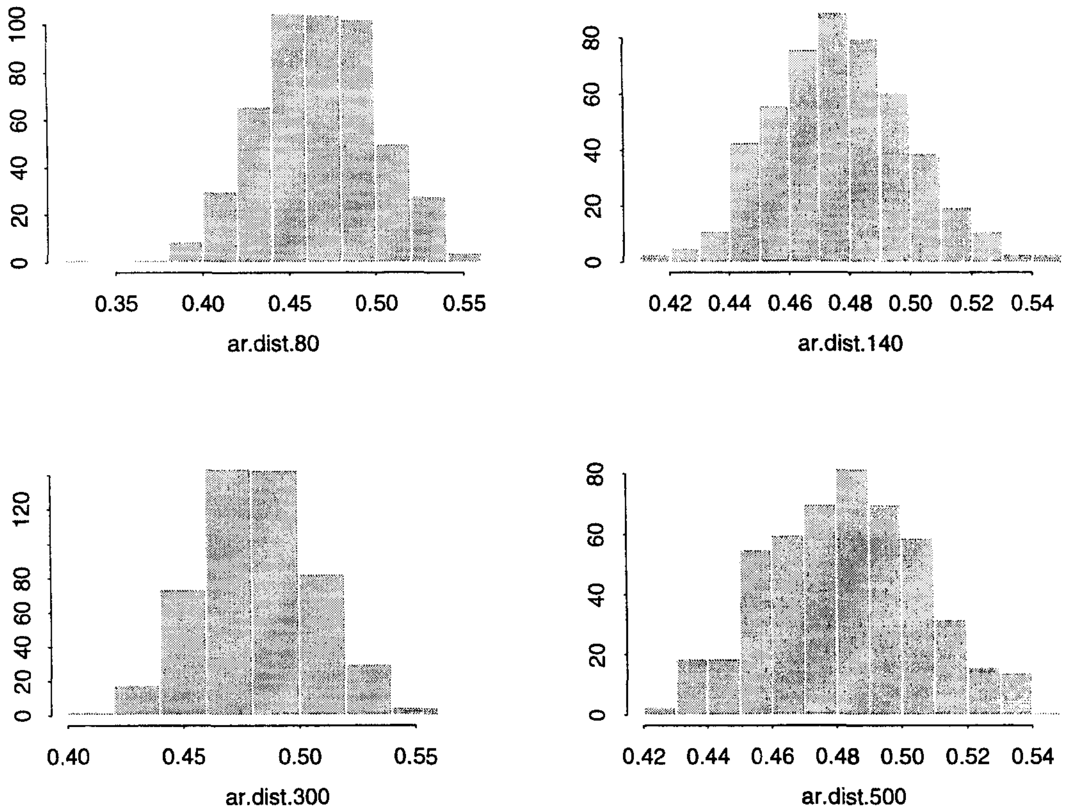

Figure 2.1 Histograms of distribution function estimates of $G_n(0)$ for AR(1) model (i) in the case of the median. All the histograms are based on 500 MBB replicates. They are from different combinations of sample sizes and block lengths: (1) $n = 80$, $\ell = 7$ (top left); (2) $n = 140$, $\ell = 1$ (top right); (3) $n = 300$, $\ell = 13$ (bottom left); (4) $n = 500$, $\ell = 9$ (bottom right).

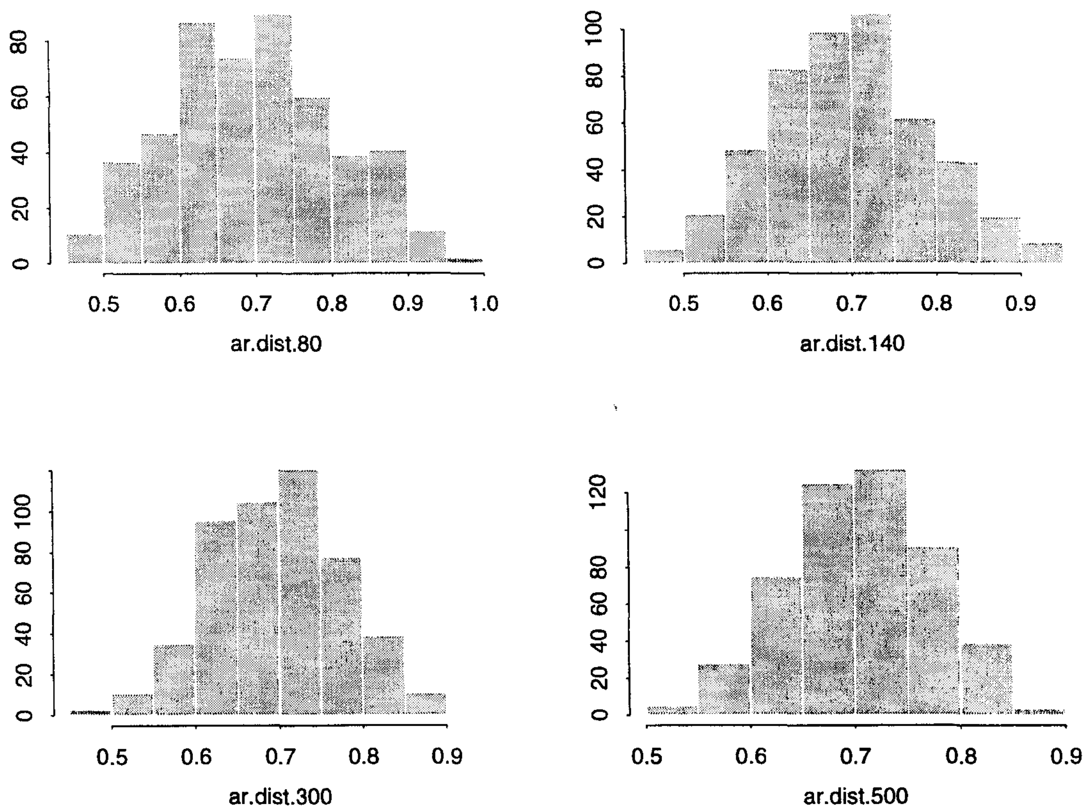

Figure 2.2 Histograms of distribution function estimates of $G_n(1)$ for AR(1) model (i) in the case of the median. All the histograms are based on 500 MBB replicates. They are from different combinations of sample sizes and block lengths: (1) $n = 80$, $\ell = 6$ (top left); (2) $n = 140$, $\ell = 5$ (top right); (3) $n = 300$, $\ell = 8$ (bottom left); (4) $n = 500$, $\ell = 5$ (bottom right).

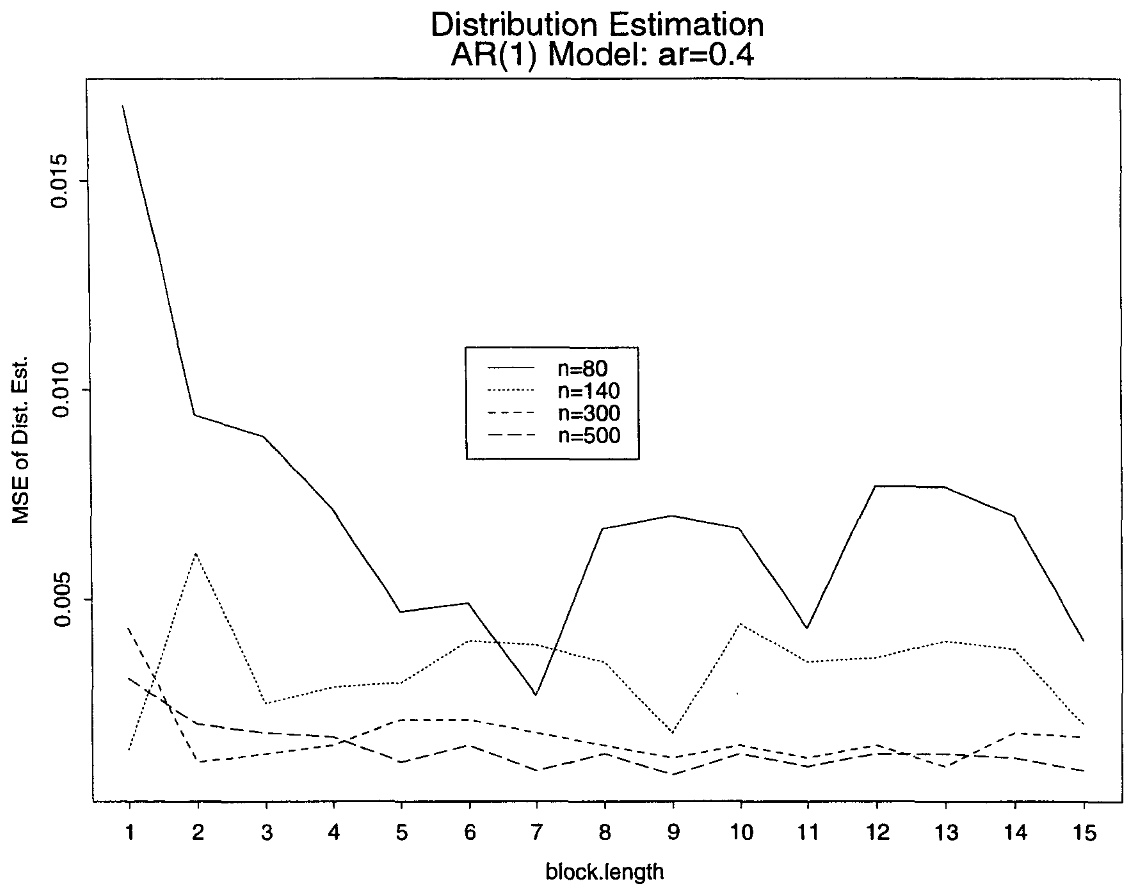

Figure 2.3 Block length impact on the MSE of the distribution function estimation of $G_n(0)$ for AR(1) model (i) in the case of the median, based on 500 MBB replicates. The solid and dotted lines are for sample size $n = 80$, and $n = 140$ respectively, while the dashed lines denote respectively the MBB approximation with sample size $n = 300$ and $n = 500$. In each case, the block length varies from 1 to 15.

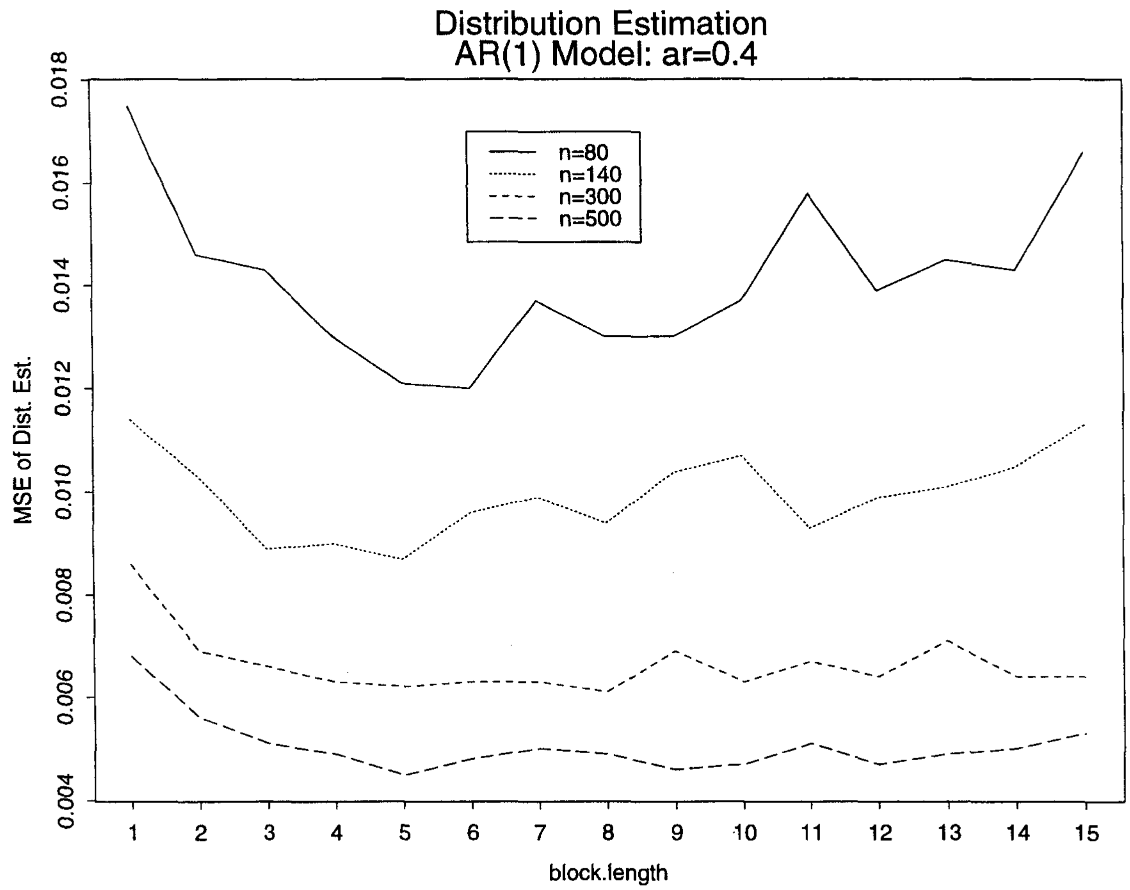

Figure 2.4 Block length impact on the MSE of the distribution function estimation of $G_n(1)$ for AR(1) model (i) in the case of the median, based on 500 MBB replicates. The solid and dotted lines are for sample size $n = 80$, and $n = 140$ respectively, while the dashed lines denote respectively the MBB approximation with sample size $n = 300$ and $n = 500$. In each case, the block length varies from 1 to 15.

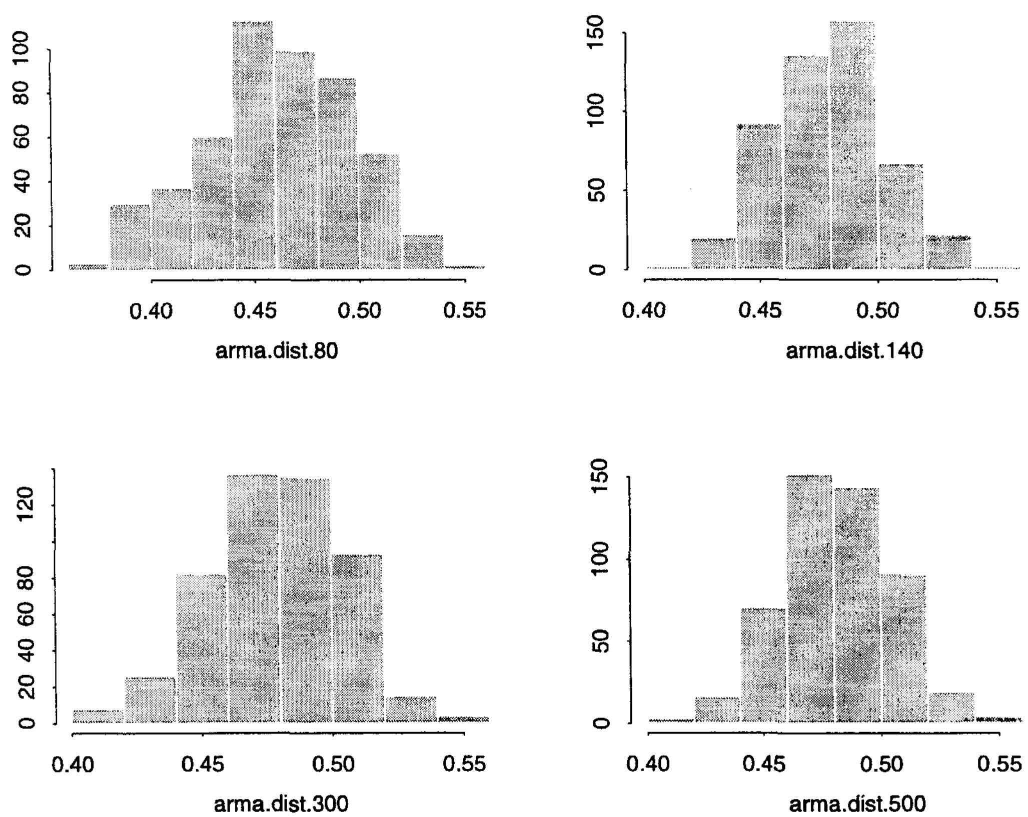

Figure 2.5 Histograms of distribution function estimates of $G_n(0)$ for ARMA(1,1) model (ii) in the case of the median. All the histograms are based on 500 MBB replicates. They are from different combinations of sample sizes and block lengths: (1) $n = 80$, $\ell = 7$ (top left); (2) $n = 140$, $\ell = 1$ (top right); (3) $n = 300$, $\ell = 13$ (bottom left); (4) $n = 500$, $\ell = 9$ (bottom right).

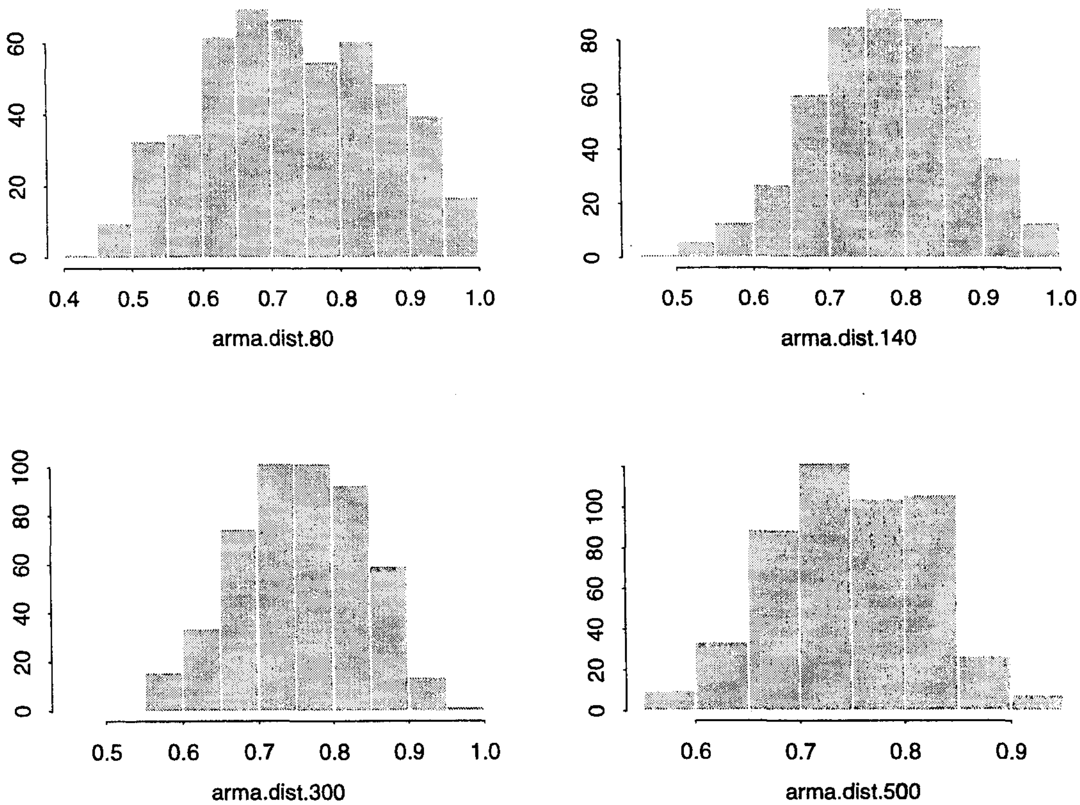

Figure 2.6 Histograms of distribution function estimates of $G_n(1)$ for ARMA(1,1) model (ii) in the case of the median. All the histograms are based on 500 MBB replicates. They are from different combinations of sample sizes and block lengths: (1) $n = 80$, $\ell = 5$ (top left); (2) $n = 140$, $\ell = 1$ (top right); (3) $n = 300$, $\ell = 2$ (bottom left); (4) $n = 500$, $\ell = 4$ (bottom right).

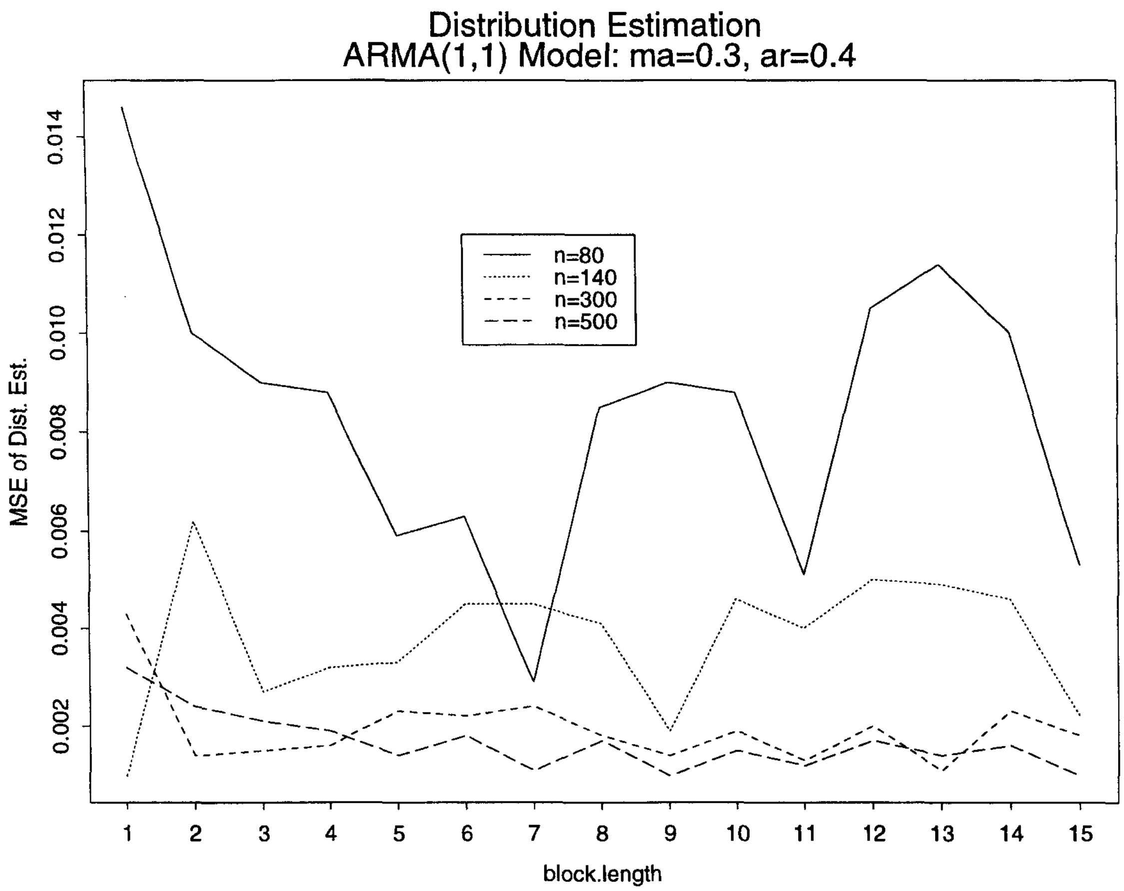

Figure 2.7 Block length impact on the MSE of the distribution function estimation of $G_n(0)$ for ARMA(1,1) model (ii) in the case of the median, based on 500 MBB replicates. The solid and dotted lines are for sample size $n = 80$, and $n = 140$ respectively, while the dashed lines denote respectively the MBB approximation with sample size $n = 300$ and $n = 500$. In each case, the block length varies from 1 to 15.

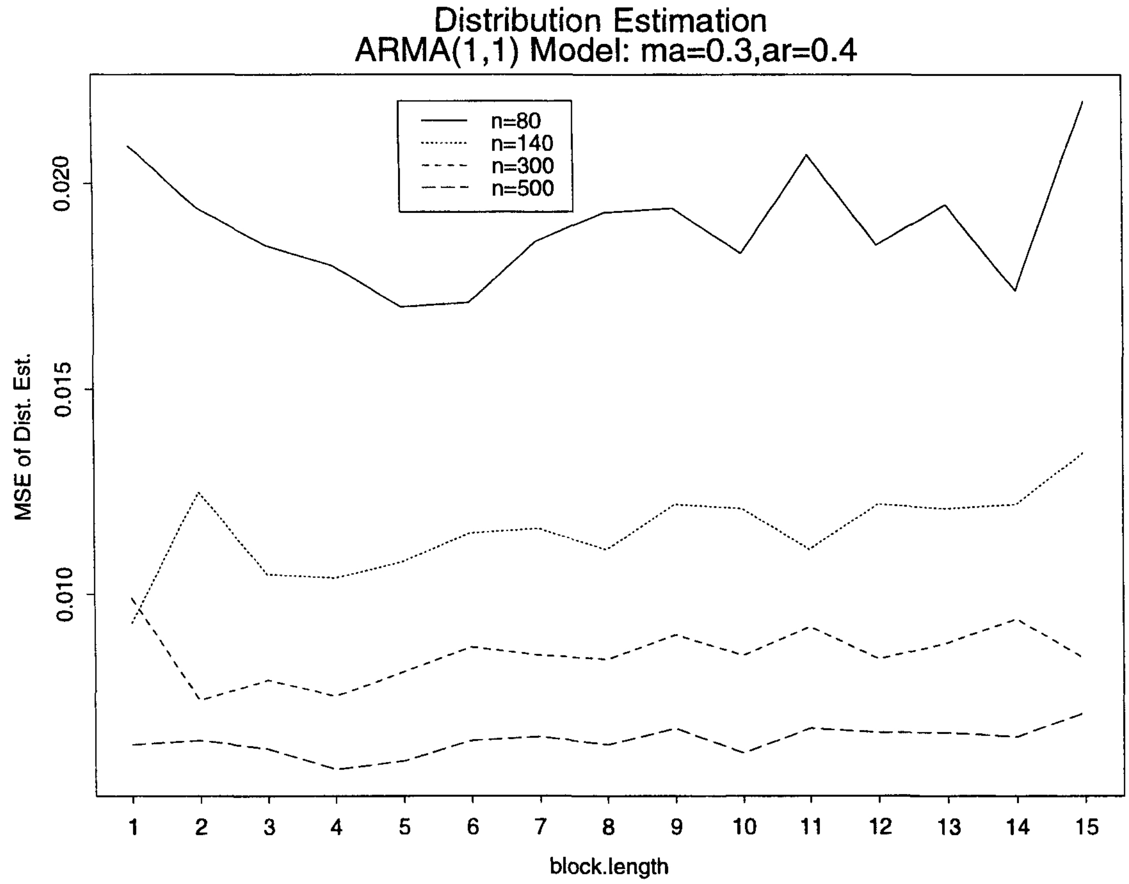

Figure 2.8 Block length impact on the MSE of the distribution function estimation of $G_n(1)$ for ARMA(1,1) model (ii) in the case of the median, based on 500 MBB replicates. The solid and dotted lines are for sample size $n = 80$, and $n = 140$ respectively, while the dashed lines denote respectively the MBB approximation with sample size $n = 300$ and $n = 500$. In each case, the block length varies from 1 to 15.

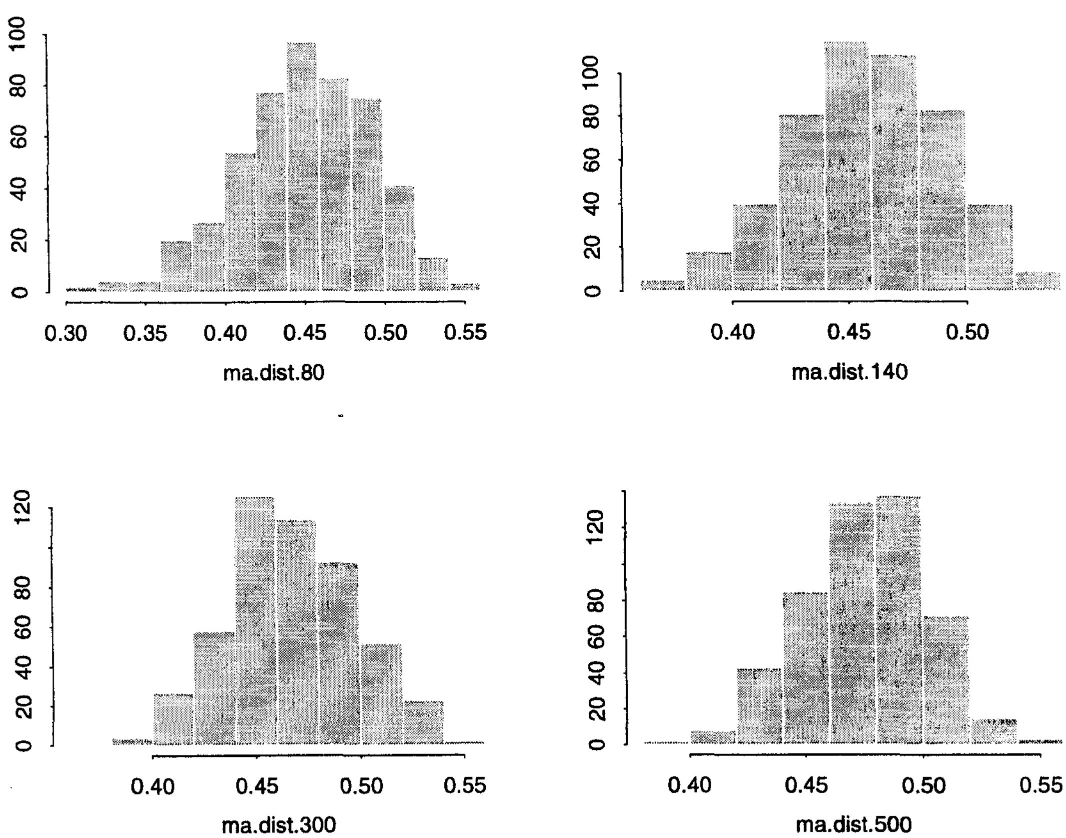

Figure 2.9 Histograms of distribution function estimates of $G_n(0)$ for MA(1) model (iii) in the case of the median. All the histograms are based on 500 MBB replicates. They are from different combinations of sample sizes and block lengths: (1) $n = 80$, $\ell = 7$ (top left); (2) $n = 140$, $\ell = 9$ (top right); (3) $n = 300$, $\ell = 31$ (bottom left); (4) $n = 500$, $\ell = 37$ (bottom right).

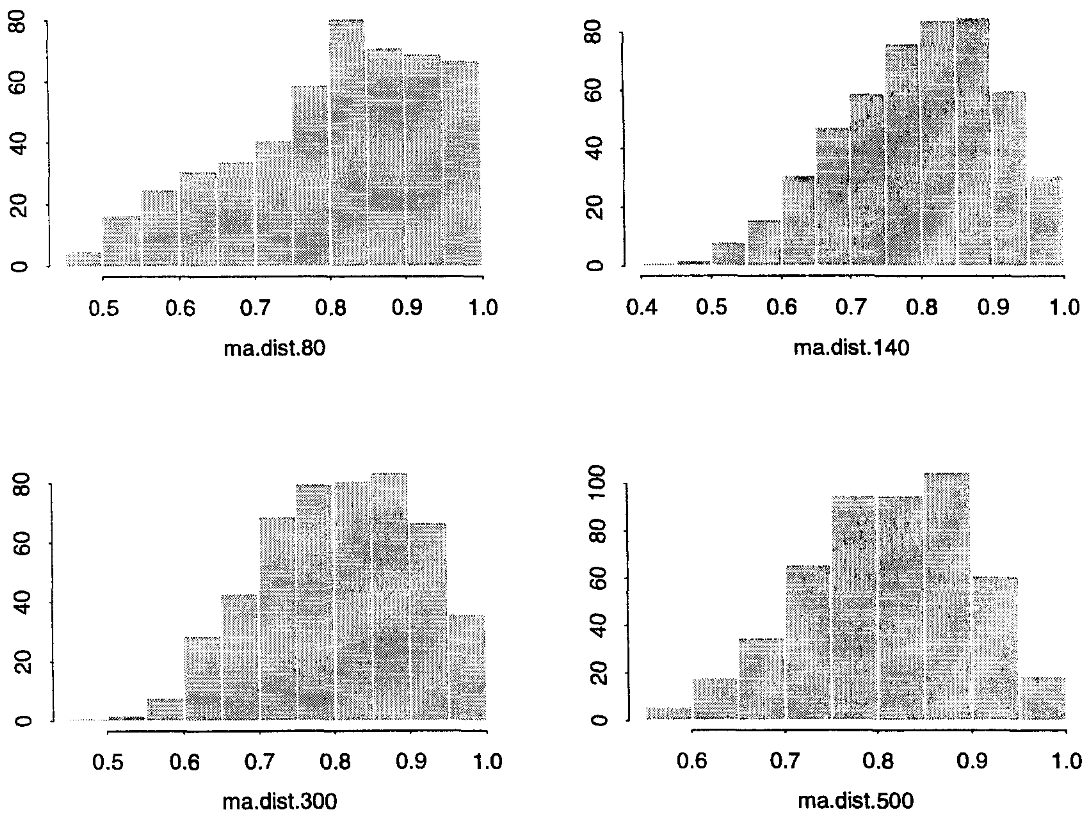

Figure 2.10 Histograms of distribution function estimates of $G_n(1)$ for MA(1) model (iii) in the case of the median. All the histograms are based on 500 MBB replicates. They are from different combinations of sample sizes and block lengths: (1) $n = 80$, $\ell = 14$ (top left); (2) $n = 140$, $\ell = 11$ (top right); (3) $n = 300$, $\ell = 22$ (bottom left); (4) $n = 500$, $\ell = 36$ (bottom right).

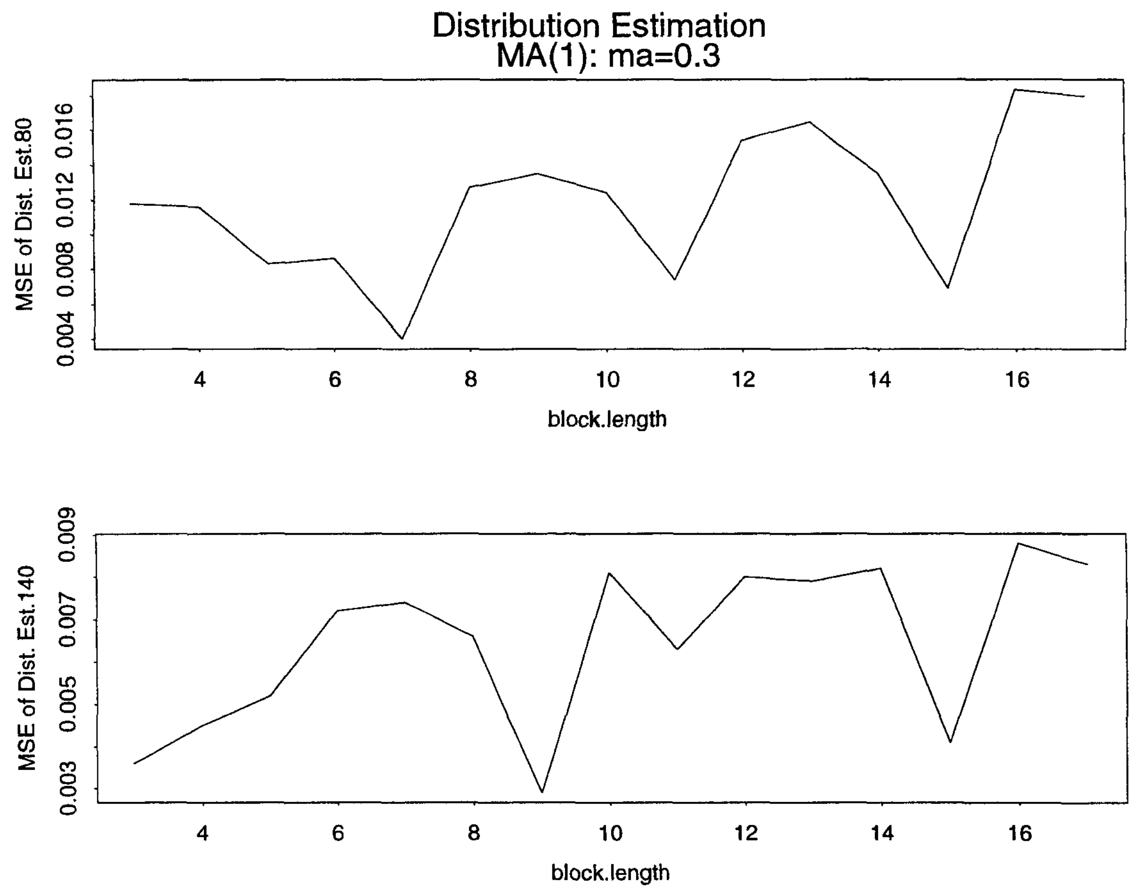

Figure 2.11 Block length impact on the MSE of the distribution function estimation of $G_n(0)$ for MA(1) model (iii) in the case of the median, based on 500 MBB replicates. The plot on the top is for sample size $n = 80$, while the one at the bottom denotes the MBB approximation for sample size $n = 140$. In both cases, block length varies from 3 to 17.

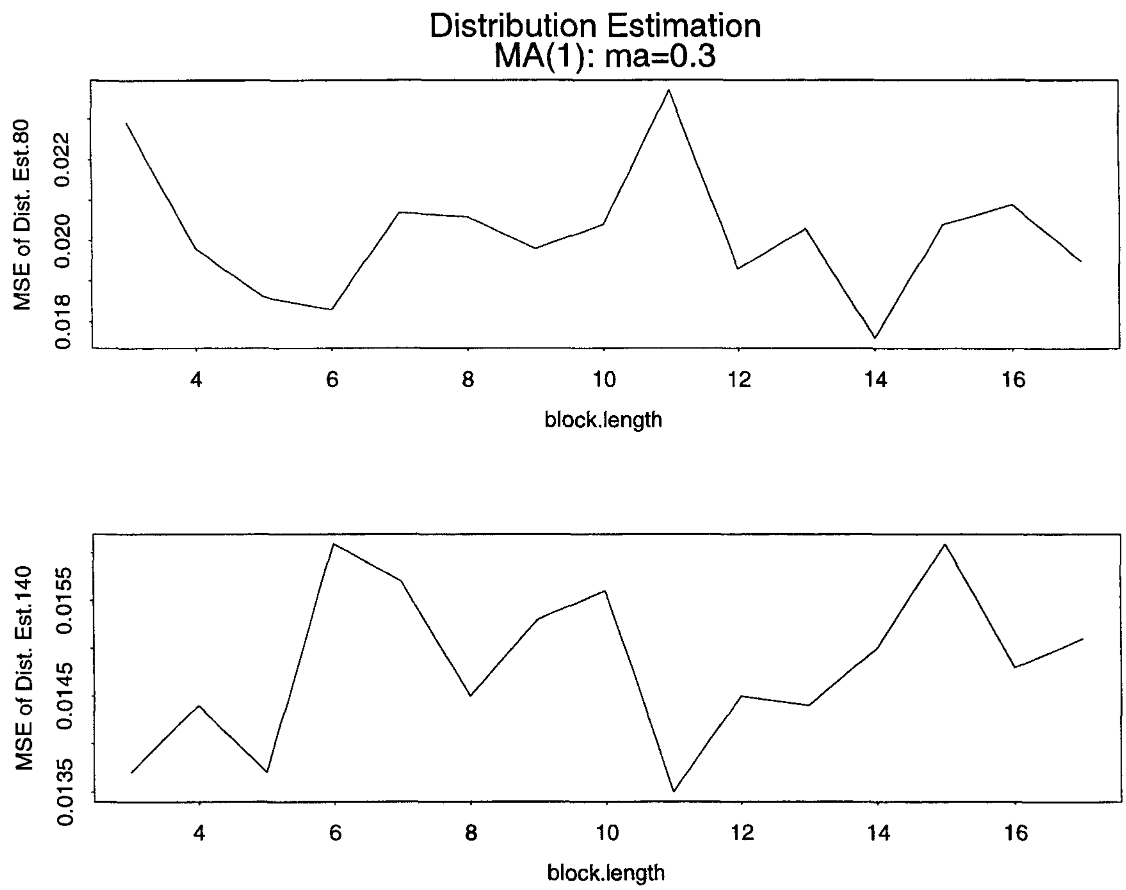

Figure 2.12 Block length impact on the MSE of the distribution function estimation of $G_n(1)$ for MA(1) model (iii) in the case of the median, based on 500 MBB replicates. The plot on the top is for sample size $n = 80$, while the one at the bottom denotes the MBB approximation for sample size $n = 140$. In both cases, block length varies from 3 to 17.

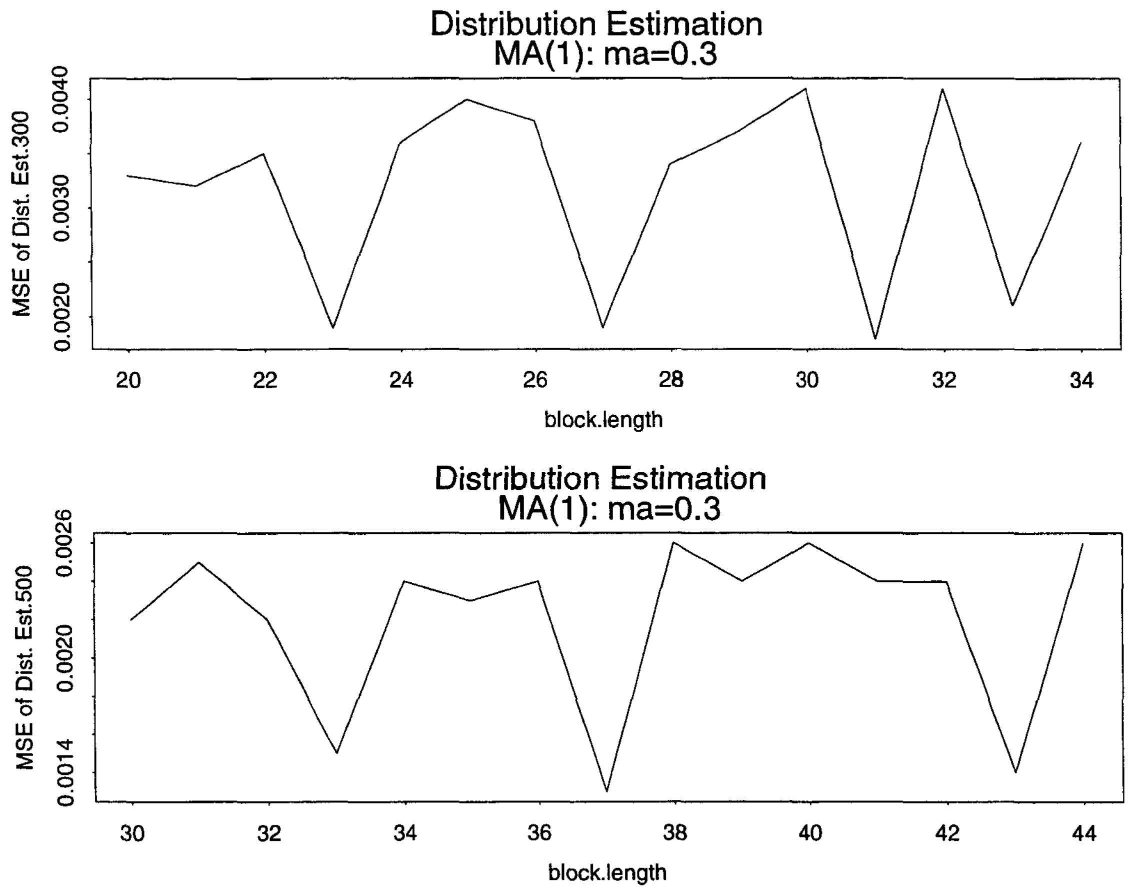

Figure 2.13 Block length impact on the MSE of the distribution function estimation of $G_n(0)$ for MA(1) model (iii) in the case of the median, based on 500 MBB replicates. The plot on the top is for sample size $n = 300$ with block length varying from 20 to 34, while the one at the bottom denotes the MBB approximation for sample size $n = 500$ with block length varying from 30 to 44.

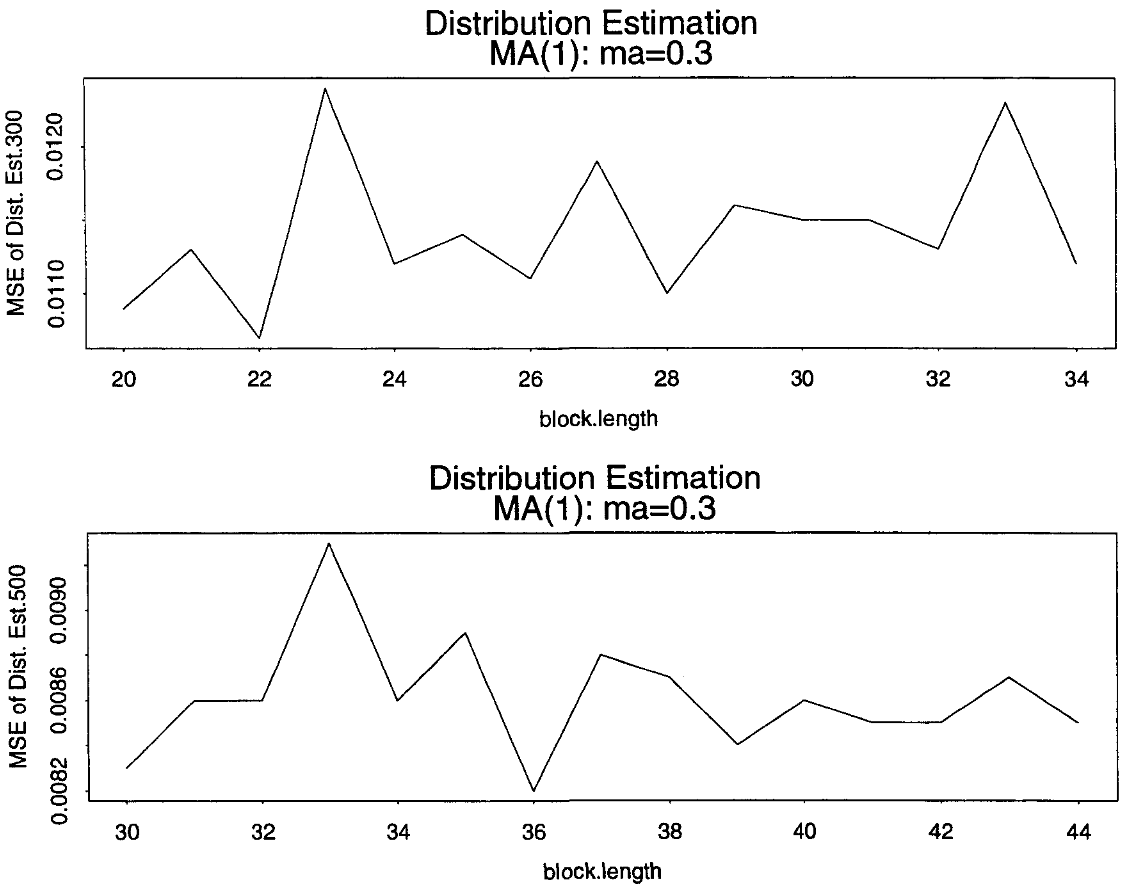

Figure 2.14 Block length impact on the MSE of the distribution function estimation of $G_n(1)$ for MA(1) model (iii) in the case of the median, based on 500 MBB replicates. The plot on the top is for sample size $n = 300$ with block length varying from 20 to 34, while the one at the bottom denotes the MBB approximation for sample size $n = 500$ with block length varying from 30 to 44.

CHAPTER 3 MBB VARIANCE ESTIMATION

3.1 Introduction

In this chapter, we investigate consistency properties of the moving block bootstrap (MBB) estimator of the asymptotic variances of the normalized sample quantiles based on weakly dependent data. In order to get critical values used for constructing confidence intervals or doing hypothesis testing, one needs to estimate the asymptotic variances. As pointed out earlier, traditional resampling methods, such as the Jackknife method does not provide a consistent estimator of the asymptotic variances of the sample quantiles. However, under mild weak dependence conditions and mild smoothness conditions on the one-dimensional marginal distribution function, we show that the moving block bootstrap variance estimator is strongly consistent.

As indicated in Section 1.1, for both i.i.d. and weakly dependent situations, the asymptotic variances of the normalized sample quantiles involve the value of the unknown (marginal) population density evaluated at the unknown population quantile. There have been a variety of density function estimation methods. Among those are kernel density estimation, histogram estimation, and spline estimation method. In the i.i.d. set up, a simple consistent estimator of $1/f(\xi_p)$ was proposed by Siddiqui (1960), and Bloch and Gastwirth (1968). It is defined as

$$S_{m,n} = (n/2m)(X_{n,r+m} - X_{n,r-m}),$$

where $r = r(n) = \lfloor np \rfloor + 1$, $m = m(n) \to \infty$. Here $X_{n,k}$ denotes the k-th order statistics. This variance estimator is based on the difference of two order statistics that are $2m$ apart. m is a smoothing parameter. Hall and Sheather (1988) studied the distribution of the sample quantile studentized by using the Siddiqui-Bloch-Gastwirth estimator of $1/f(\xi_p)$. Another alternative is bootstrap variance estimation. In fact, one of the most appealing characteristics of the bootstrap method is its ability and simplicity to produce variance estimation in some complicated problems by use of Monte Carlo simulation technique. In terms of theoretical development, Ghosh, Parr, Singh and Babu (1984) established the strong consistency result of Efron's bootstrap variance estimator of the sample quantile based on i.i.d. observations. Further, Hall and Martin (1988) found that the exact convergence rate of Efron's bootstrap quantile variance estimator is of order $O_p(n^{-1/4})$.

We note that, compared to the independent case, the asymptotic variance of $Z_n \equiv \sqrt{n}(\hat{\xi}_n - \xi_p)$ under dependence is more complicated, involving the infinite sum

$$\sum_{i \in \mathbb{Z}} Cov(I(X_1 \leq \xi_p), I(X_{i+1} \leq \xi_p))$$

of lag-covariances and the density function of X_1 at ξ_p. Even for a direct plug-in estimation of the asymptotic variance, it is evident that the user would have to employ different non-parametric functional estimation techniques for the numerator and the denominator of τ_∞^2 (cf. (1.3)), and specify smoothing parameters for each component (cf. Chen and Tang (2004)). In comparison, the MBB provides a unified way of approximating both parts of the asymptotic variance consistently using a single smoothing parameter, viz., the block-size variable ℓ, for a wide range of possible values of ℓ and under very weak smoothness conditions on F.

The layout of this chapter is as follows. In Section 3.2, we establish the consistency result of MBB estimators to the asymptotic variance of the scaled sample quantile. A small simulation study is presented in Section 3.3.

3.2 Consistency of the MBB Variance Estimator

The main result of this section shows that under some mild conditions, the MBB estimator $\hat{\tau}_n^2$ of the asymptotic variance of the centered and scaled p-th sample quantile is strongly consistent. Recall that $\hat{\tau}_n^2$ is the conditional variance of the MBB version of the centered and scaled p-th sample quantile based on block length ℓ, i.e.,

$$\hat{\tau}_n^2 = Var_*(Z_n^*),$$

where $Z_n^* = \sqrt{n}(\xi_n^* - \tilde{\xi}_n)$ is defined as in (1.6) of Chapter 1. Also, recall that $\alpha(n)$ denotes the strong mixing coefficient of the stationary process $\{X_n\}_{n\in\mathbb{Z}}$ as defined in Section 1.2.2. The main theorem of this chapter is stated as follows.

Theorem 3.1 *Assume that $\alpha(n) \leq Cn^{-\beta}$ for some $C \in (0,\infty)$ and $\beta > 9.5$, and that $\ell = O(n^{1/2-\eta})$ for some $\eta \in (5/(2+4\beta), 1/2)$. Also assume that for $0 < p < 1$, F has a positive and continuous derivative f in a neighborhood $\mathcal{N}_p$ of ξ_p with $f(\xi_p) > 0$, and*

$$E|X_1|^\alpha < \infty \quad \textit{for some} \quad \alpha > 0\,. \tag{3.1}$$

Then

$$\hat{\tau}_n^2 = \mathrm{Var}_*(\sqrt{n}(\xi_n^* - \tilde{\xi}_n)) \to \tau_\infty^2 \quad a.s.\,,$$

where

$$\tau_\infty^2 \equiv \Big[\sum_{i=-\infty}^{\infty} Cov\Big(I(X_1 \leq F^{-1}(p)), I(X_{i+1} \leq F^{-1}(p))\Big)\Big]/f^2(F^{-1}(p))\,.$$

Theorem 3.1 shows that under some mild conditions, the MBB estimator $\hat{\tau}_n^2$ of the asymptotic variance of the centered and scaled p-th sample quantile is strongly consistent. This is an extension of the important result of Ghosh, Parr, Singh and Babu (1984) on strong consistency of bootstrap variance estimator, from the i.i.d. set up to the weakly dependent case. Note that unlike the distribution function estimation problem treated in Chapter 2, for the consistency of the MBB variance estimator, we impose a mild moment condition, given by (3.1). It can be shown that (3.1) is a necessary condition for the validity of Theorem 3.1, i.e., the MBB variance estimator need not be consistent if (3.1) fails.

For proving Theorem 3.1, we need two lemmas that are standard facts from probability theory. We include them for the sake of completeness.

Lemma 3.1 *Let $\{W_n\}_{n\geq 1}$ be a sequence of random variables that converges in distribution to a random variable W_0. Let r be a positive integer. If $\sup\{E|W_n|^{r+\epsilon} : n \geq 1\} < \infty$ for some $\epsilon >$*

0 , *then for any* $1 \leq s \leq r$,

$$E|W_0|^s < \infty, \quad and \quad E|W_n|^s \to E|W_0|^s .$$

Proof: For $s \in [1, r]$, by Skorohod Embedding Theorem (cf. Billingsley (1995), Theorem 25.6), there exists a probability space $(\Omega, \mathcal{F}, P)$ and a sequence of random variables $\{V_{n,s}\}_{n \geq 0}$ defined on $(\Omega, \mathcal{F}, P)$ such that $V_{n,s}$ has the same distribution as W_n^s , for all $n \geq 0$ and $V_{n,s} \to V_{0,s}$ for each $\omega \in \Omega$. Then, we have

$$\begin{aligned}
\lim_{t\to\infty} \sup_{n\geq 1} \int_{|V_{n,s}|>t} |V_{n,s}| dP &= \lim_{t\to\infty} \sup_{n\geq 1} E|V_{n,s}| I(|V_{n,s}| > t) \\
&= \lim_{t\to\infty} \sup_{n\geq 1} E|W_n|^s I(|W_n|^s > t) \\
&\leq \lim_{t^{1/s}\to\infty} \sup_{n\geq 1} \left(E|W_n|^{r+\epsilon}\right)^{s/(r+\epsilon)} I(|W_n| > t^{1/s}) \\
&= \lim_{t^{1/s}\to\infty} \sup_{n\geq 1} \left(\int_{|W_n|>t^{1/s}} |W_n|^{r+\epsilon} dP \right)^{s/(r+\epsilon)} \\
&= 0 .
\end{aligned}$$

The last equality follows from the condition $\sup\{E|W_n|^{r+\epsilon} : n \geq 1\} < \infty$ and the monotone convergence theorem (cf. Billingsley (1995), Theorem 16.2). Thus, $\{V_{n,s}\}_{n\geq 1}$ is uniformly integrable and converges to random variable $V_{0,s}$. Then Theorem 16.14 of Billingsley (1995) implies that

$$E|V_{0,s}| < \infty, \quad \text{and} \quad E|V_{n,s}| \to E|V_{0,s}| .$$

Hence,

$$E|W_0|^s < \infty, \quad \text{and} \quad E|W_n|^s \to E|W_0|^s .$$

Lemma 3.1 is proved. □

Lemma 3.2 *Let* $\{X_i\}_{i\in\mathbb{Z}}$ *be a strictly stationary sequence with* $E|X_1|^\alpha < \infty$ *for some* $\alpha > 0$. *Then*

$$(|X_{(1)}| + |X_{(n)}|)/n^{1/\alpha} \to 0 \quad a.s.$$

where $X_{(1)} = \min\{X_1, \cdots, X_n\}, \quad X_{(n)} = \max\{X_1, \cdots, X_n\}$.

Proof: The proof uses standard arguments as in Gosh, Parr, Singh, and Babu (1984). They showed the same result in the i.i.d. case. Note that (cf. Corollary 4.1.3, Chow and Teicher (1997)) for any random variable X and any $r \in (0, \infty)$,

$$\sum_{n=1}^{\infty} P(|X| \geq n^{1/r}) \leq E|X|^r \leq \sum_{n=0}^{\infty} P(|x| > n^{1/r}).$$

Then, for every $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_1| \geq \epsilon n^{1/\alpha}) \leq E|\epsilon^{-1} X_1|^{\alpha} < \infty,$$

since $E|X_1|^{\alpha} < \infty$. And, by stationarity,

$$\sum_{n=1}^{\infty} P(|X_n| \geq \epsilon n^{1/\alpha}) = \sum_{n=1}^{\infty} P(|X_1| \geq \epsilon n^{1/\alpha}) < \infty.$$

Thus, Borel-Cantelli lemma implies that, for any arbitrary $\epsilon > 0$, $|X_n| < \epsilon n^{1/\alpha}$ a.s., except for finitely many n's. Hence,

$$(|X_{(1)}| + |X_{(n)}|)/n^{1/\alpha} \to 0 \quad \text{a.s.},$$

which concludes the proof of Lemma 3.2. □

Proof of Theorem 3.1: By Theorem 2.1 and Theorem 2.2

$$\sqrt{n}(\xi_n^* - \tilde{\xi}_n) \to^d N(0, \sigma_\infty^2 / f^2(\xi_p)) \quad \text{a.s.}$$

Hence, by Lemma 3.1, it suffices to show that for some $\delta > 0$,

$$\sup_{n \geq 1} \{E_* |Z_n^*|^{2+\delta}\} < \infty \quad \text{a.s.}, \tag{3.2}$$

where $Z_n^* = \sqrt{n}(\xi_n^* - \tilde{\xi}_n)$. For $x > 0$, we define

$$\hat{H}_n^+(x) = P_*(Z_n^* > x), \quad \hat{H}_n^-(x) = P_*(Z_n^* < -x).$$

Then,

$$E_* |Z_n^*|^{2+\delta} = \int_0^{\infty} P_*(|Z_n^*|^{2+\delta} > x) \mathrm{d}x$$

$$
\begin{aligned}
&= \int_0^\infty P_*(|Z_n^*| > x^{1/(2+\delta)})\mathrm{d}x \\
&= (2+\delta)\int_0^\infty x^{1+\delta} P_*(|Z_n^*| > x)\mathrm{d}x \\
&= (2+\delta)\left[\int_0^\infty x^{1+\delta}\hat{H}_n^+(x)\mathrm{d}x + \int_0^\infty x^{1+\delta}\hat{H}_n^-(x)\mathrm{d}x\right].
\end{aligned}
$$

Hence (3.2) will follow from

$$
\int_0^\infty x^{1+\delta}\hat{H}_n^\pm(x)\mathrm{d}x < \infty \quad \text{a.s.} \tag{3.3}
$$

By Lemma 3.2 and part (iii) of Lemma 2.4, we get

$$
\begin{aligned}
|Z_n^*| &= |\sqrt{n}(\xi_n^* - \tilde{\xi}_n)| \\
&\le n^{1/2}|\xi_n^*| + n^{1/2}|\hat{\xi}_n| + n^{1/2}|\hat{\xi}_n - \tilde{\xi}_n| \\
&= 2n^{1/2}\cdot o(n^{1/\alpha}) + n^{1/2}\cdot o(1) \\
&= o(n^{1/2+1/\alpha}) \quad \text{a.s.},
\end{aligned}
$$

which implies that, almost surely,

$$
\hat{H}_n^\pm(x) = 0, \quad \text{for all} \quad x > n^{\frac{1}{2}+\frac{1}{\alpha}}, \quad \text{provided } n \text{ is large enough.} \tag{3.4}
$$

For $1 \le x < \infty$, we have

$$
\begin{aligned}
\hat{H}_n^+(x) &= P_*(\xi_n^* > \tilde{\xi}_n + x/\sqrt{n}) \\
&\le P_*(F_n^*(\hat{x}_n) < p) \\
&= P_*(F_n^*(\hat{x}_n) - \tilde{F}_n(\hat{x}_n) < p - \tilde{F}_n(\hat{x}_n)).
\end{aligned} \tag{3.5}
$$

By (2.26), for $1 \le x \le \log n$, n sufficiently large

$$
\begin{aligned}
p - \tilde{F}_n(\hat{x}_n) &= -\frac{x}{\sqrt{n}} f(\hat{\epsilon}_n) + o(n^{-1/2}) \\
&\le -d_1\frac{x}{\sqrt{n}} + o(n^{-1/2}) \\
&\le -A\frac{x}{\sqrt{n}} \quad \text{a.s.},
\end{aligned} \tag{3.6}
$$

where $0 < d_1 = \inf\{f(x) : x \in \mathcal{N}_p\}$, $0 < A < d_1$ and $\hat{\epsilon_n} \in (\tilde{\xi}_n \wedge \hat{x}_n, \tilde{\xi}_n \vee \hat{x}_n)$. Hence, for $0 < x \le \log n$, (3.5) and (3.6) yield that, for n large enough

$$
\begin{aligned}
\hat{H}_n^+(x) &\le P_*(F_n^*(\hat{x}_n) - \tilde{F}_n(\hat{x}_n) \le -Ax/\sqrt{n}) \\
&\le P_*(|F_n^*(\hat{x}_n) - \tilde{F}_n(\hat{x}_n)| \ge Ax/\sqrt{n}) \quad \text{a.s.}
\end{aligned} \tag{3.7}
$$

We now apply Corollary 1 of Fuk and Nagaev (1971) and then use Taylor's expansion to get an upper bound for $\hat{H}_n^+(x)$. Recall that

$$U_i^*(x) = \frac{1}{\ell}\sum_{j=1}^{\ell} I(X_{i+j}^* \le x), \quad i = 1, \cdots, b\,.$$

And for notational simplicity, we write $U_i^* \equiv U_i^*(\hat{x}_n), i = 1, \ldots, b$. By (ii) in the proof of Theorem 2.2,

$$\hat{\sigma}_n^2 = \frac{n}{b} Var_*(U_1^*) \to \sigma_\infty^2 \quad \text{as} \quad n \to \infty \quad \text{a.s.} \tag{3.8}$$

Hence, using Taylor's expansion of the function $g(y) = \log(1+y)$ around $y = 0$, we get, uniformly for $x \in [1, \log n]$, and sufficiently large n ,

$$\begin{aligned}
& P_*(|F_n^*(\hat{x}_n) - \tilde{F}_n(\hat{x}_n)| \ge Ax/\sqrt{n}) \\
= \;& P_*\left(\left|\sum_{i=1}^{b}(U_i^* - E_*U_i^*)\right| \ge bA\frac{x}{\sqrt{n}}\right) \\
\le \;& \exp\left\{\frac{Axb}{\sqrt{n}} - \left(\frac{Axb}{\sqrt{n}} + bVar_*(U_1^*)\right)\log\left(\frac{\frac{Axb}{\sqrt{n}}}{bVar_*(U_1^*)} + 1\right)\right\} \\
\le \;& C\exp\left\{\frac{Ax\sqrt{n}}{\ell} - \left(\frac{Ax\sqrt{n}}{\ell} + \frac{n\hat{\sigma}_n^2}{\ell^2}\right)\log\left(\frac{A\ell x}{\sqrt{n}\hat{\sigma}_n^2} + 1\right)\right\} \\
\le \;& C\exp\left\{-\frac{A^2x^2}{2\hat{\sigma}_n^2} + O\left(+\frac{\ell(\log n)^3}{\sqrt{n}}\right)\right\} \\
\le \;& C\exp\left\{-\frac{A^2x^2}{2\hat{\sigma}_n^2} + o(1)\right\} \\
\le \;& C\exp\left\{-\frac{A^2x^2}{4\sigma_\infty^2}\right\} \quad \text{a.s.},
\end{aligned} \tag{3.9}$$

where the last two steps follow from (3.8) and the condition $\ell = O(n^{1/2-\eta})$.

Next, let $a_n = 2\sigma_\infty\sqrt{(1/2 + 1/\alpha)(2+\delta)\log n}/A$, with A as in (3.6). Then, $a_n \le \log n$ for sufficiently large n. Define

$$I_1 = \sup_{n\ge 1}\int_1^{a_n} \hat{H}_n^+(x)x^{1+\delta}dx, \quad I_2 = \sup_{n\ge 1}\int_{a_n}^{n^{1/2+1/\alpha}} \hat{H}_n^+(x)x^{1+\delta}dx\,.$$

We have, by (3.4),

$$\sup_{n\ge 1}\int_0^\infty \hat{H}_n^+(x) \le 1/(2+\delta) + I_1 + I_2\,.$$

By (3.7) and (3.9),

$$I_1 = \sup_{n\geq 1} \int_1^{a_n} \hat{H}_n^+(x)x^{1+\delta}\mathrm{d}x \leq \sup_{n\geq 1} \int_1^{a_n} \exp\left\{-A^2x^2/(4\sigma_\infty^2)\right\}x^{1+\delta}\mathrm{d}x = O(1) \quad \text{a.s.}$$

Next, note that $\hat{H}_n^+(x)$ is nondecreasing in $x \in (0, \infty)$. Then

$$\begin{aligned}
I_2 &= \sup_{n\geq 1} \int_{a_n}^{n^{1/2+1/\alpha}} \hat{H}_n^+(x)x^{1+\delta}\mathrm{d}x \\
&\leq \sup_{n\geq 1} \int_{a_n}^{n^{1/2+1/\alpha}} \hat{H}_n^+(a_n)x^{1+\delta}\mathrm{d}x \\
&\leq \sup_{n\geq 1} \int_{a_n}^{n^{1/2+1/\alpha}} \exp\left\{-A^2a_n^2/(\sigma_\infty^2)\right\}x^{1+\delta}\mathrm{d}x \\
&= \sup_{n\geq 1} \int_{a_n}^{n^{1/2+1/\alpha}} \exp\left\{-A^2\left(\sigma_\infty\sqrt{(1/2+1/\alpha)(2+\delta)\log n}/A\right)^2/(\sigma_\infty^2)\right\}x^{1+\delta}\mathrm{d}x \\
&= \sup_{n\geq 1} \int_{a_n}^{n^{1/2+1/\alpha}} n^{-(1/2+1/\alpha)(2+\delta)}x^{1+\delta}\mathrm{d}x \\
&< C(\delta) \quad \text{a.s.}
\end{aligned}$$

Therefore

$$\sup_{n\geq 1} \int_0^{\infty} \hat{H}_n^+(x)x^{1+\delta}\mathrm{d}x < \infty \quad \text{a.s.}$$

Similarly, we may show that

$$\sup_{n\geq 1} \int_0^{\infty} \hat{H}_n^-(x)x^{1+\delta}\mathrm{d}(x) < \infty \quad \text{a.s.}$$

Thus (3.3) is proved. This completes the proof of Theorem 3.1. □

We conclude this section with several remarks.

Remark 3.1 *Note that in addition to the regularity conditions of Theorem 2.2, we require the moment condition (3.1) for the validity of Theorem 3.1. It can be shown that, as in the independent case, (strong) consistency of the MBB variance estimator does not hold without the moment condition (3.1).*

Remark 3.2 *It is worth noting that, unlike the distribution function estimation problem, the centering of the bootstrap sample quantile ξ_n^* at $\tilde{\xi}_n$ is of no importance. This is because the*

MBB variance estimator, being the conditional variance of Z_n^*, equals $n_1 Var_*(\xi_n^*)$, which does not involve the centering variable $\tilde{\xi}_n$.

Remark 3.3 *In this work, we focus on only one of many available block bootstrap methods, namely, the MBB method of Künsch (1989) and Liu and Singh (1992). It is not difficult to show that the conclusions of Theorem 3.1 and Theorem 2.2 also remain valid for the circular block bootstrap (CBB) method of Politis and Romano (1992) and the nonoverlapping block bootstrap (NBB) method of Carlstein (1986) under exactly the same sets of conditions on the marginal distribution function F and on the block size ℓ. Indeed, for the CBB method, analogs of Theorem 3.1 and Theorem 2.2 can be proved by using the main steps used in the proofs of Theorem 3.1 and Theorem 2.2 and using an estimate of the difference between the MBB and the CBB moments (cf. (5.10), pp. 129, Lahiri (2003)) involving certain indicator variables. The proofs for the NBB method are simpler (because of the absence of the extra dependence of the overlapping MBB blocks in the NBB blocks). In this case, Theorem 3.1 and Theorem 2.2 follow from straight-forward modifications of the arguments presented in Section 3.2 and Section 2.3. We omit the routine details.*

Remark 3.4 *We also mention that, like the sample mean, resampling a single data value at a time, as done in Efron (1979) for independent data, fails to provide a valid approximation for the sample quantiles under dependence. In view of the dependence of the asymptotic variance of Z_n on lag-covariances of arbitrarily high order (cf. (1.3)), it is clear that the MBB, the CBB, and the NBB methods with a bounded block size can not account for all such lag covariances and will, therefore, be inconsistent.*

3.3 Simulation Study

For the MBB variance estimation, we use the same models as in the MBB approximation to the distribution which was studied in Chapter 2. We also follow the similar simulation procedure as well.

Step 1: Select a model and generate 20,000 samples of size n and simulate the values of

$$\sigma_n^2 = n \cdot Var(\hat{\xi}_n)$$

Step 2: Generate 500 MBB data sets of size n and for the m-th data set, compute the 500 replicates of the MBB version of $\hat{\xi}_n$, denoted as

$$\xi^*_{n,m}(1), \cdots, \xi^*_{n,m}(500),$$

and then evaluate the MBB estimate of σ_n^2:

$$\begin{aligned} \hat{\sigma}^2_{n,m}(\ell) &= n \cdot [\frac{1}{499} \sum_{j=1}^{500} (\xi^*_{n,m}(j) - \overline{\xi^*_{n,m}})^2] \\ &\approx n \cdot Var_*(\xi^*_n), \quad m = 1, \cdots, 500, \end{aligned}$$

where

$$\overline{\xi^*_{n,m}} = \frac{1}{500} \sum_{m=1}^{500} \xi^*_{n,m}(j).$$

Step 3: Find the bias and variance of the MBB estimators:

$$\begin{aligned} Bias(\hat{\sigma}_n^2(\ell)) &= \frac{1}{500} \sum_{m=1}^{500} (\hat{\sigma}^2_{n,m}(\ell) - \sigma_n^2) \\ Var(\hat{\sigma}_n^2(\ell)) &= \frac{1}{499} \sum_{m=1}^{500} [\hat{\sigma}^2_{n,m}(\ell) - \overline{\hat{\sigma}_n^2}(\ell)]^2 \\ SD(\hat{\sigma}_n^2(\ell)) &= \sqrt{Var(\hat{\sigma}_n^2(\ell))} \\ MSE(\hat{\sigma}_n^2(\ell)) &= [Bias(\hat{\sigma}_n^2(\ell))]^2 + Var(\hat{\sigma}_n^2(\ell)) \\ RMSE(\hat{\sigma}_n^2(\ell)) &= MSE(\hat{\sigma}_n^2(\ell))/(\sigma_n^2)^2, \end{aligned}$$

where

$$\overline{\hat{\sigma}_n^2}(\ell) = \frac{1}{500} \sum_{m=1}^{500} \hat{\sigma}^2_{n,m}(\ell).$$

The simulation for the MBB variance estimation is based on the same 500 data sets used in the distribution function estimations treated in Chapter 1, and the results are here presented exactly in the same manner as that of the MBB distribution function estimation in Section 2.4. Again, as we may see from the tables and graphs below, the numerical results support the consistency result of the MBB variance estimator of the normalized sample quantile under weak dependence.

It is also worth mentioning that the optimal block lengths may differ for different estimation problems. There have been some studies about the choice of the optimal block length. See Lahiri et al. (2003), Hall et al. (1995), and Bühlmann and Künsch (1999). Here, we consider the effect of different block size on the MBB variance estimators for the three models described above. It appears that the optimal block size greatly depends on the model. For the AR(1) and the ARMA (1,1) models, a relatively smaller block size (between 1-6) tends to perform better, while for the MA(1) model, a choice of block length in the range 7-9 for $n = 80$ and $n = 140$ seems to have small MSE values.

Table 3.1 Variance estimation of σ_n^2 for AR(1) Model (i) in the case of the median, based on 500 MBB replicates.

n	ℓ	Bias	SD	MSE	True σ_n^2	RMSE
80	3	-0.6523	1.3011	2.1184	3.3911	0.1842
140	4	-0.5366	1.1794	1.6789	3.4533	0.1408
300	5	-0.3873	1.0467	1.2456	3.4800	0.1029
500	6	-0.3092	0.9449	0.9884	3.4813	0.0816

Table 3.2 Variance estimation of σ_n^2 for ARMA(1,1) Model (ii) in the case of the median, based on 500 MBB replicates.

n	ℓ	Bias	SD	MSE	True σ_n^2	RMSE
80	1	-0.1171	0.8170	0.6812	1.8954	0.1896
140	1	-0.1612	0.7187	0.5426	1.9064	0.1493
300	1	-0.2202	0.6223	0.4358	1.9331	0.1166
500	2	-0.1678	0.5833	0.3683	1.9354	0.0983

Table 3.3 Variance estimation of σ_n^2 for MA(1) Model (iii) in the case of the median, based on 500 MBB replicates.

n	ℓ	Bias	SD	MSE	True σ_n^2	RMSE
80	9	0.0291	0.6813	0.4652	1.0859	0.4284
140	7	0.0797	0.5688	0.3299	1.1061	0.2983
300	22	0.0224	0.5220	0.2730	1.0900	0.2505
500	36	-0.0016	0.4973	0.2473	1.0856	0.2278

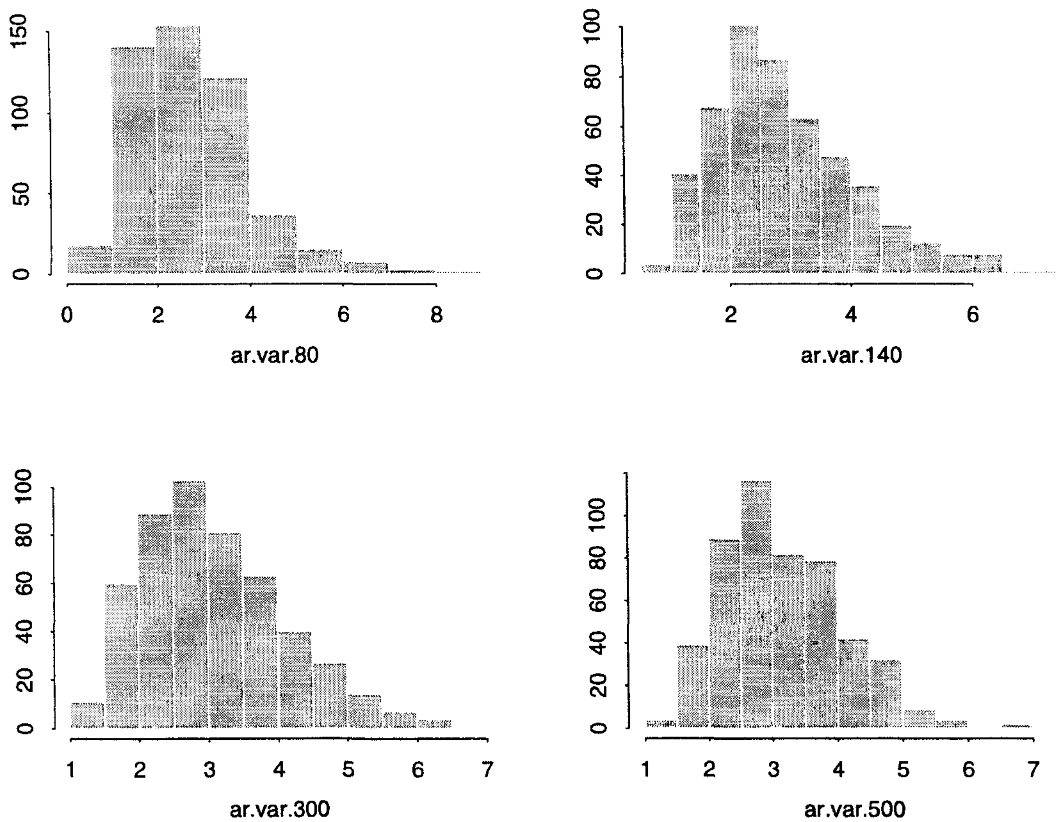

Figure 3.1 Histograms of the variance estimates of σ_n^2 for AR(1) model (i) in the case of the median. All the histograms are based on 500 MBB replicates. They are from different combinations of sample sizes and block lengths: (1) $n = 80$, $\ell = 3$ (top left); (2) $n = 140$, $\ell = 4$ (top right); (3) $n = 300$, $\ell = 5$ (bottom left); (4) $n = 500$, $\ell = 6$ (bottom right).

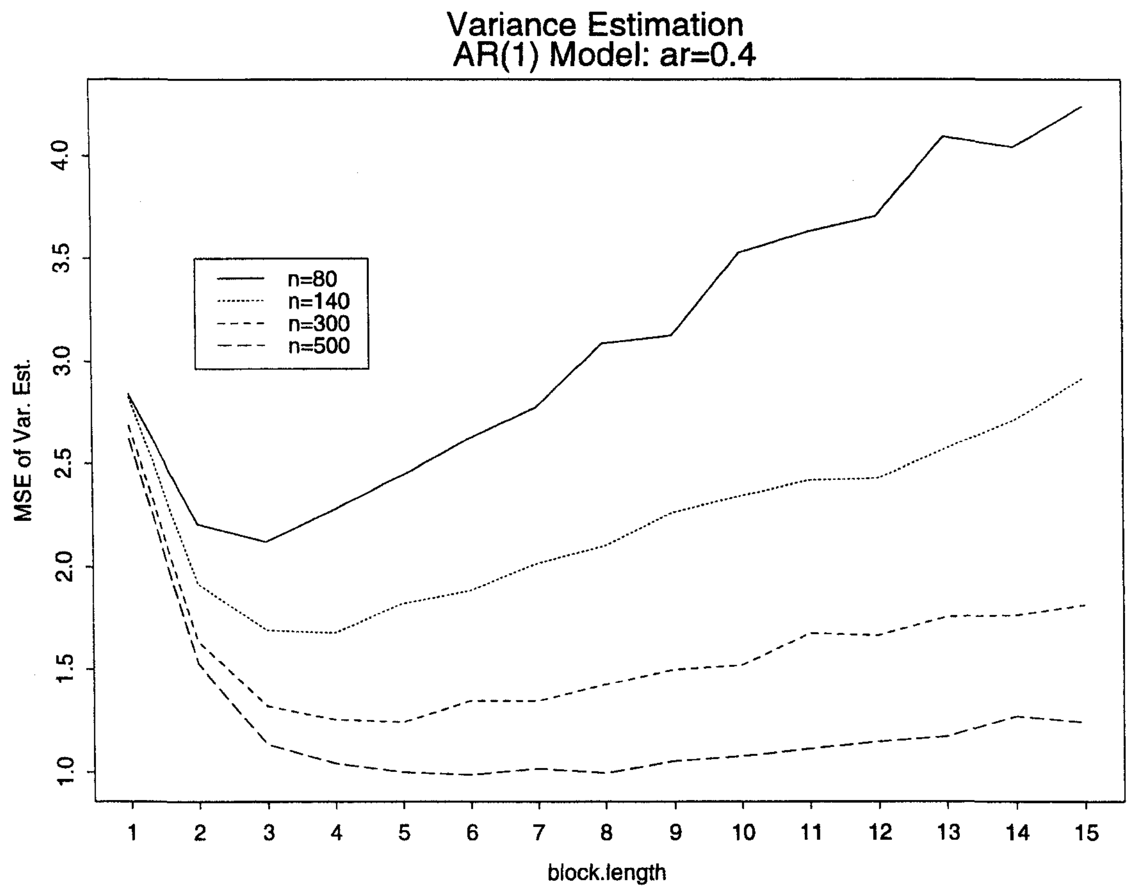

Figure 3.2 Block length impact on the MSE of the variance estimation of σ_n^2 for AR(1) model (i) in the case of the median, based on 500 MBB replicates. The solid and dotted lines are for sample size $n = 80$, and $n = 140$ respectively, while the dashed lines denote respectively the MBB approximation with sample size $n = 300$ and $n = 500$. In each case, the block length varies from 1 to 15.

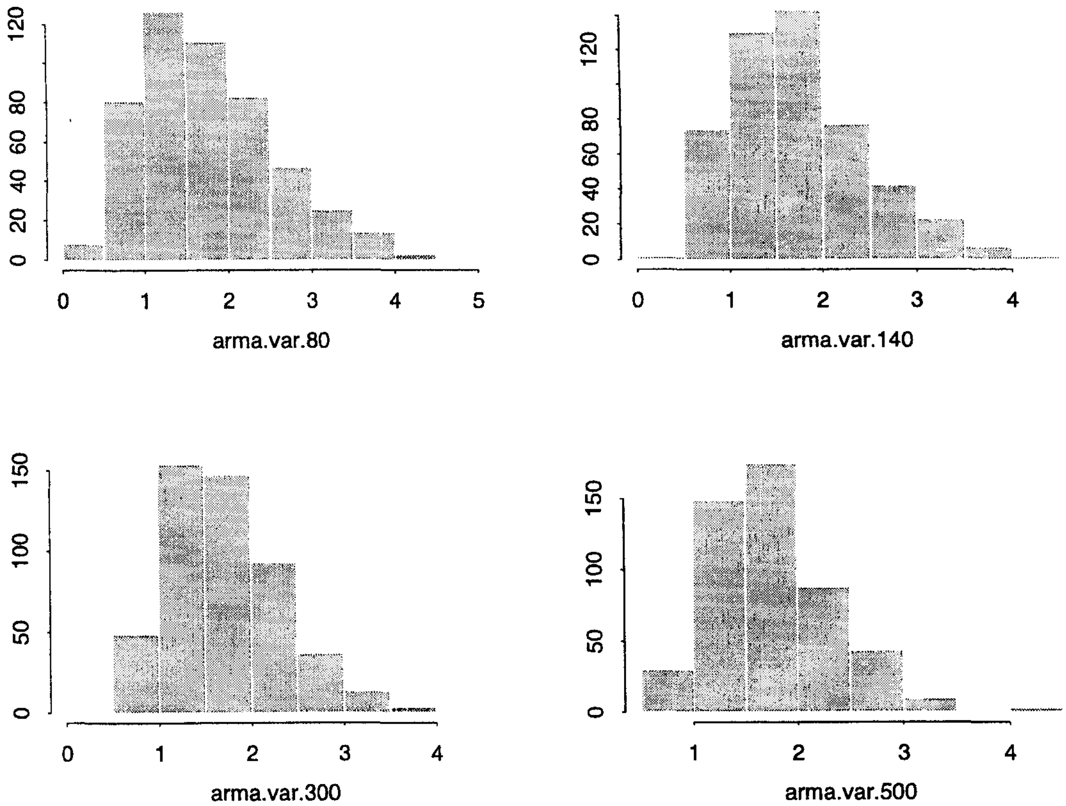

Figure 3.3 Histograms of the variance estimates of σ_n^2 for ARMA(1,1) model (ii) in the case of the median. All the histograms are based on 500 MBB replicates. They are from different combinations of sample sizes and block lengths: (1) $n = 80$, $\ell = 1$ (top left); (2) $n = 140$, $\ell = 1$ (top right); (3) $n = 300$, $\ell = 1$ (bottom left); (4) $n = 500$, $\ell = 2$ (bottom right).

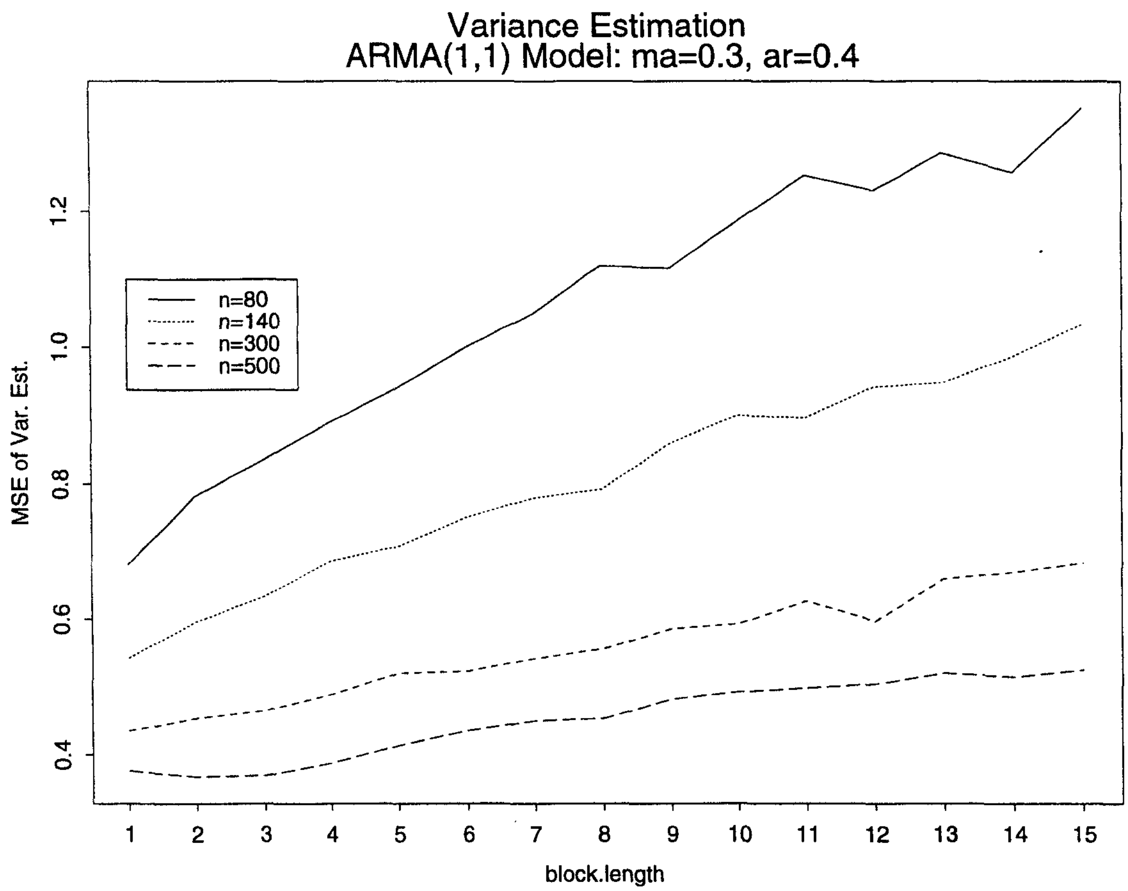

Figure 3.4 Block length impact on the MSE of the variance estimation of σ_n^2 for ARMA(1,1) model (ii) in the case of the median, based on 500 MBB replicates. The solid and dotted lines are for sample size $n = 80$, and $n = 140$ respectively, while the dashed lines denote respectively the MBB approximation with sample size $n = 300$ and $n = 500$. In each case, the block length varies from 1 to 15.

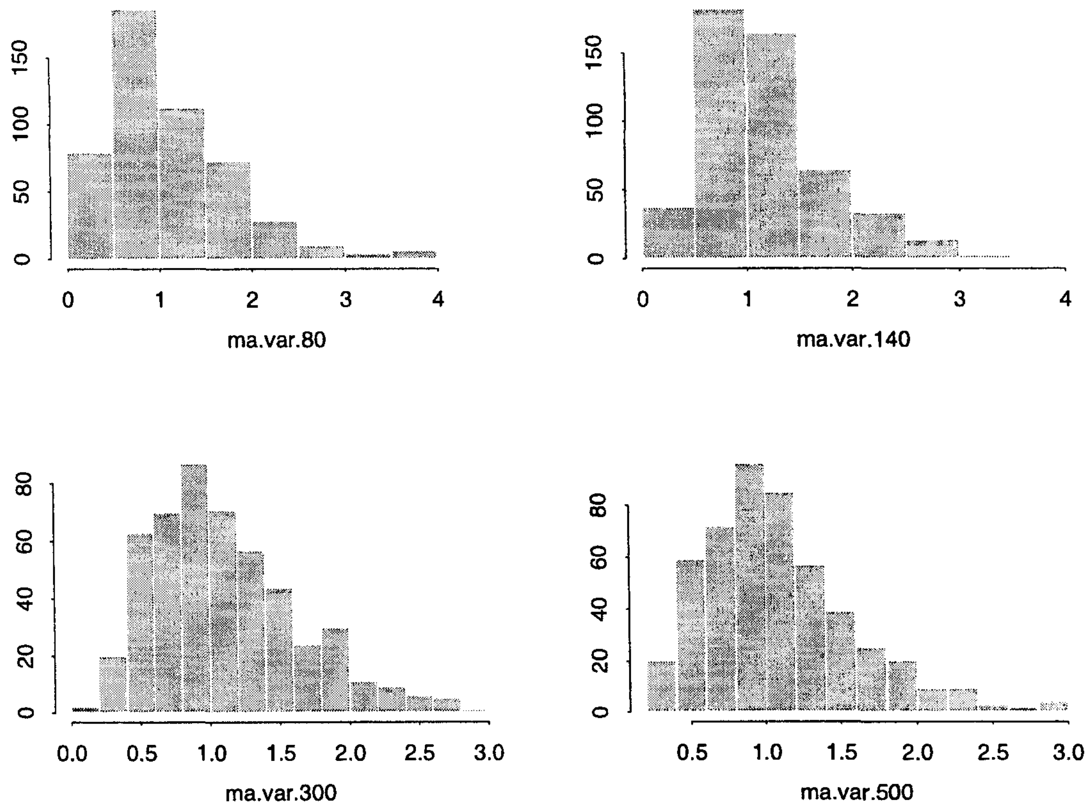

Figure 3.5 Histograms of the variance estimates of σ_n^2 for MA(1) model (iii) in the case of the median. All the histograms are based on 500 MBB replicates. They are from different combinations of sample sizes and block lengths: (1) $n = 80$, $\ell = 9$ (top left); (2) $n = 140$, $\ell = 7$ (top right); (3) $n = 300$, $\ell = 22$ (bottom left); (4) $n = 500$, $\ell = 36$ (bottom right).

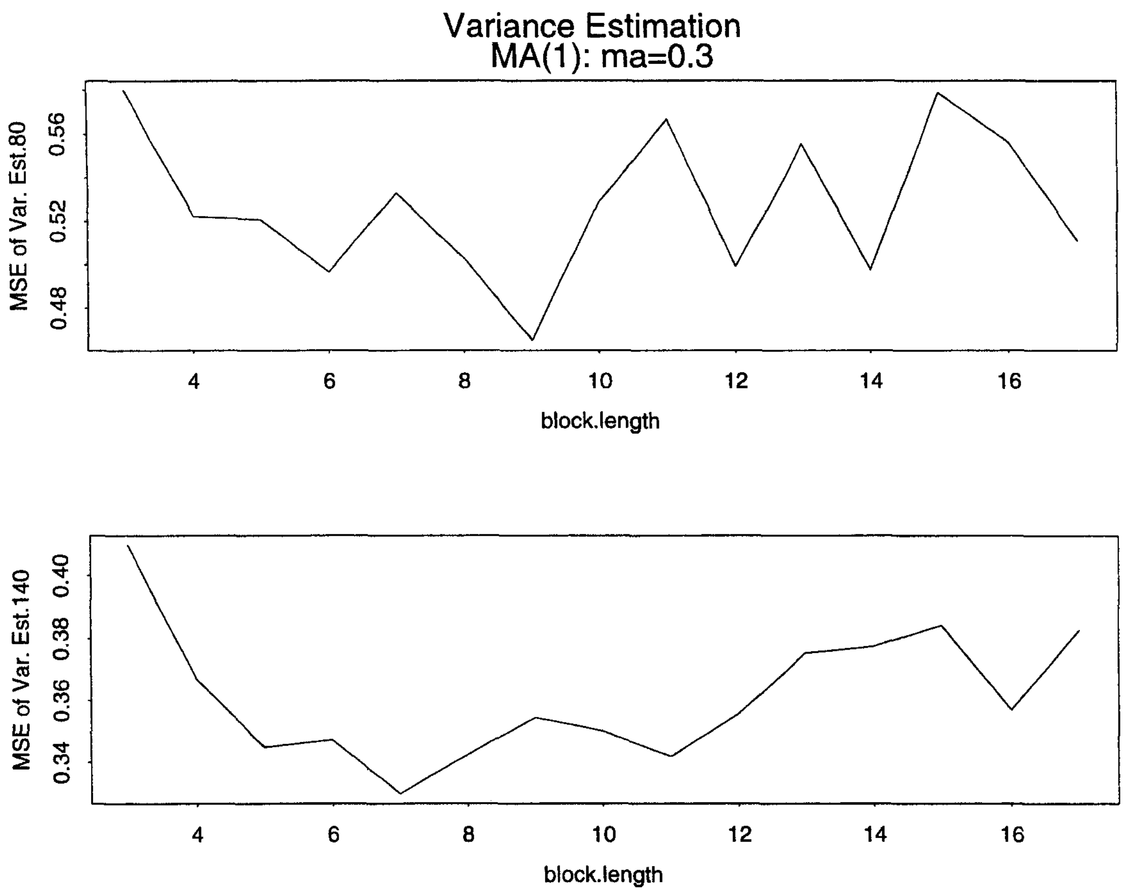

Figure 3.6 Block length impact on the MSE of the variance estimation of σ_n^2 for MA(1) model (iii) in the case of the median, based on 500 MBB replicates. The plot on the top is for sample size $n = 80$, while the one at the bottom denotes the MBB approximation for sample size $n = 140$. In both cases, block length varies from 3 to 17.

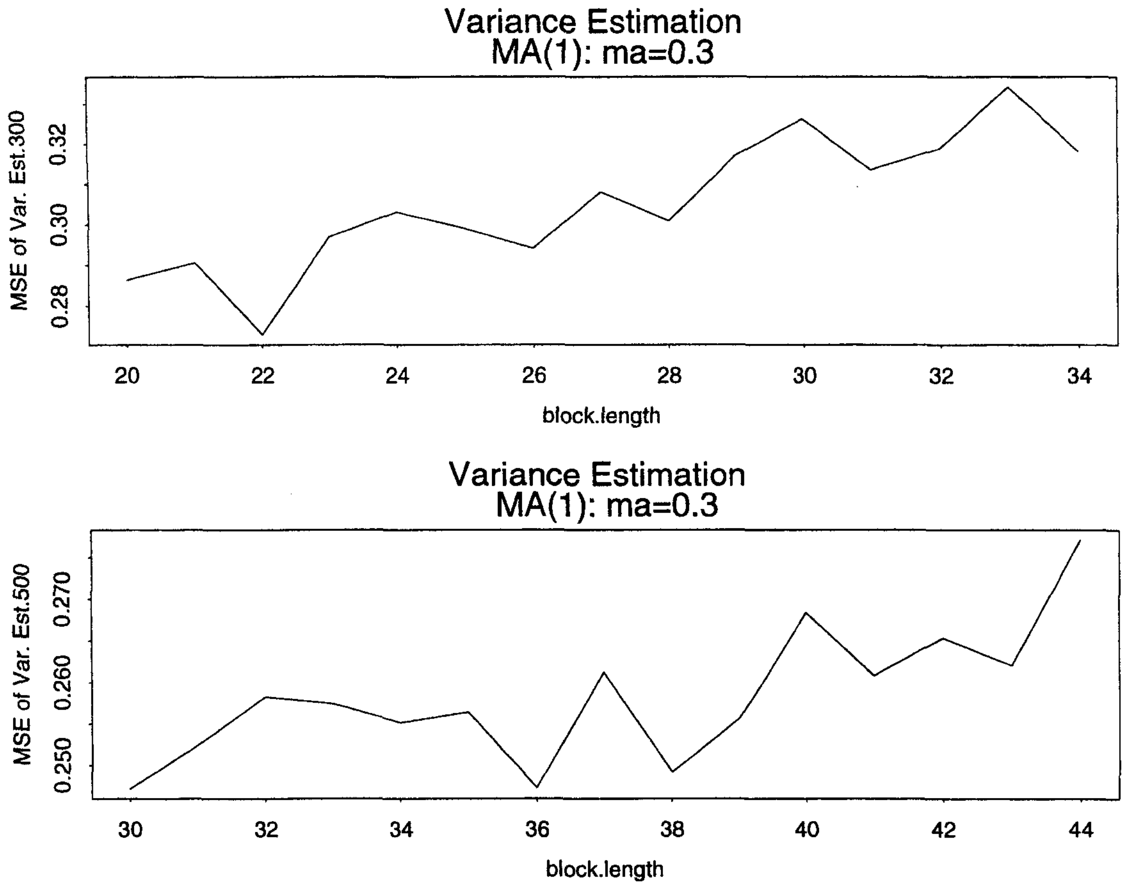

Figure 3.7 Block length impact on the MSE of the variance estimation of σ_n^2 for MA(1) model (iii) in the case of the median, based on 500 MBB replicates. The plot on the top is for sample size $n = 300$ with block length varying from 20 to 34, while the one at the bottom denotes the MBB approximation for sample size $n = 500$ with block length varying from 30 to 44.

CHAPTER 4 A BERRY-ESSEEN THEOREM

4.1 Introduction

This chapter proves a Berry-Esseen Theorem for sample quantile of strongly-mixing random variables under a polynomial mixing rate. When the process $\{X_i\}_{i\in\mathbb{Z}}$ is strongly mixing (see Section 4.2 for a definition) at a polynomial rate and F is differentiable at $F^{-1}(p)$ with a positive derivative $f(F^{-1}(p)) > 0$, it is known (cf. Theorem 2.1) that

$$\sqrt{n}(F_n^{-1}(p) - F^{-1}(p)) \to^d N(0, \tau_\infty^2) \tag{4.1}$$

as $n \to \infty$, where τ_∞^2 is defined as in (1.3).

The main result of this chapter (cf. Theorem 4.1) refines (4.1) by specifying the rate of normal approximation to the distribution of $\sqrt{n}(F_n^{-1}(p) - F^{-1}(p))$. More precisely, it is shown that under appropriate regularity conditions set forth in Section 4.2,

$$\sup_{x\in\mathbb{R}} |P(\sqrt{n}(F_n^{-1}(p) - F^{-1}(p)) \leq x) - \Phi(x/\tau_\infty)| = O(n^{-1/2}) \quad \text{as} \quad n \to \infty \tag{4.2}$$

Thus, as in the independence case, the rate of normal approximation is shown to be $O(n^{-1/2})$ as $n \to \infty$, and hence, the Berry-Esseen Theorem holds for the sample quantile for strongly mixing random variables under the conditions of Section 4.2. This is in marked contrast to the case of the sample mean of strongly mixing random variables, where a Berry-Esseen Theorem with the rate $O(n^{-1/2})$ of normal approximation is not available. The best possible rate for sums of strongly mixing random variables with an exponentially decaying mixing coefficient is only $O(n^{-1/2}(\log n)^c)$ for some suitable $c > 0$ (cf. Tikomirov(1980), Dasgupta(1989)). For processes satisfying stronger forms of dependence conditions like β-mixing or ϕ-mixing, (cf. Donkhan (1984)), a generalization of this result is recently obtained by Bentkus, Götze and Tikhomirov (1997) for U-statistics, but still with the rate $O(n^{-1/2}(\log n)^c)$ for some $c > 0$. Although validity of the Berry-Esseen Theorem with rate $O(n^{-1/2})$ for sample sums of strongly-mixing random variables remains unsolved, Theorem 4.1 below establishes the desired optimal rate $O(n^{-1/2})$ for the sample quantile, as in the independent case (cf. Reiss (1974)). The proof refines and adopts some of the arguments developed by Götze and Hipp (1983) and Lahiri (1993, 1996a) for deriving Edgeworth expansion for sums of strongly mixing random variables and also crucially exploits properties of the probability integral transform $F_n^{-1}(\cdot)$ of the empirical distribution function (edf) F_n that allows one to approximate the distribution function (df) of $F_n^{-1}(p)$ in terms of those of (normalized) sums of certain lattice random variables. The rest of this chapter is organized as follows. In Section 4.2, we state the conditions and the main result. Technical details are provided in Section 4.3.

4.2 Conditions and the Main Result

We establish the main result of this chapter under a general framework introduced in the seminal paper of Götze and Hipp (1983). Suppose that the random variables $\{X_i : i \in \mathbb{Z}\}$ are defined on a probability space $(\Omega, \mathcal{F}, P)$ and that $\{\mathcal{D}_i : i \in \mathbb{Z}\}$ is a collection of sub-σ-fields of $\mathcal{F}$. For $-\infty \leq a \leq b \leq \infty$, let $\mathcal{D}_a^b = \sigma\langle\{\mathcal{D}_i : i \in [a,b] \cap \mathbb{Z}\}\rangle$ denote the σ-field generated by $\{\mathcal{D}_i, a \leq i \leq b, i \in \mathbb{Z}\}$. We shall make use of the following conditions:

(C.1)

(i) F is differentiable in a neighborhood $\mathcal{N}_p$ of ξ_p with derivative $f(x)$ such that

$$0 < \inf\{f(x) : x \in \mathcal{N}_p\} \leq \sup\{f(x) : x \in \mathcal{N}_p\} < \infty\,,$$

where $\xi_p = F^{-1}(p)$.

(ii) $\sigma_\infty^2 = \sum_{i \in \mathbb{Z}} Cov(I(X_1 \leq \xi_p), I(X_{i+1} \leq \xi_p)) \in (0, \infty)$.

(C.2) There exist constants $d \in (0,1)$ and $\alpha_0 > 12$ such that, for all $n \geq 1$,

$$\begin{aligned} \alpha(n) &\equiv \sup\{|P(A \cap B) - P(A)P(B)| : A \in \mathcal{D}_{-\infty}^{i}, B \in \mathcal{D}_{i+n}^{\infty}, i \in \mathbb{Z}\} \\ &\leq d^{-1} n^{-\alpha_0}\,. \end{aligned} \tag{4.3}$$

(C.3) There exist constants $d \in (0,1)$ and $\beta_0 > 12$ such that $X^{\dagger}_{i,n}$ is $\mathcal{D}^{i+n}_{i-n}$-measurable, and for all $i \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$\beta(n) \equiv E|X_i - X^{\dagger}_{i,n}| \leq d^{-1} n^{-2\beta_0} \,. \tag{4.4}$$

(C.4) There exist constants $d \in (0,1)$ and $\gamma_0 > 12$ such that for all $m, n, r \in \mathbb{N}$ and $A \in \mathcal{D}^{r+m}_{r-m}$,

$$|P(A|\mathcal{D}_j : j \neq r) - P(A|X_j : 0 < |r-j| \leq m+n)| \leq d^{-1} n^{-\gamma_0} \,. \tag{4.5}$$

(C.5) There exist $d \in (0,1)$ and sub-σ-fields $\mathcal{C}_i, i \in \mathbb{Z}$, of $\mathcal{F}$ such that for every $i \in \mathbb{Z}$, $\sigma\langle \mathcal{D}_j : j \neq i \rangle \cup \sigma\langle \{X_j : j \neq i\} \rangle \subset \mathcal{C}_i$ and

$$P(G_i(\xi_p) = 1) \leq p - d \,, \tag{4.6}$$

where $G_i(y) = P(X_i \leq y|\mathcal{C}_i)$, $y \in \mathbb{R}$.

We now comment on the conditions. Condition (C.1) is a standard condition that is frequently used to ensure a nondegenerate normal distribution of the p-th sample quantile under dependence. In the independent case, (C.1)(i) is also almost necessary; see Bahadur (1966). Conditions (C.2)-(C.4) are similar to the conditions introduced in Götze and Hipp (1983) for deriving asymptotic expansion for sums of weakly dependent random vectors, where the right sides of (4.3)-(4.5) were assumed to be exponentially decaying functions of n. The reduction to the polynomial case here heavily borrows on the arguments of Lahiri (1996a) which proves Götze and Hipp's (1983) results under similar polynomial decay conditions. Condition (C.2) is a strong mixing condition in the auxiliary σ-fields $\mathcal{D}_j$'s which, together with condition (C.3), imposes an approximate strong-mixing structure to the given random variables $\{X_i\}_{i\in\mathbb{Z}}$. Condition (C.4) is an approximate Markov condition and in particular, it is satisfied, if $\{X_i\}_{i\in\mathbb{Z}}$ is an m-th order Markov process for a fixed $m \in \mathbb{N}$. Condition (C.5) is a regularity condition that says that the conditional distribution of X_i given $\{X_j : j \neq i\}$ and $\{\mathcal{D}_j : j \neq i\}$ has positive mass beyond ξ_p. It can be shown that (4.6) is approximately equivalent to requiring that for some $\epsilon \in (0,1)$,

$$P(I(X_i \leq \xi_p) = 1|\mathcal{C}_i) < p \tag{4.7}$$

on a set A_i with $P(A_i) > \epsilon$. Indeed, if $P(G_i(\xi_p) = 1) = p$, then it can be shown that the conditional distribution $\mathcal{L}(I(X_i \leq \xi_p)|\mathcal{C}_i)$ is degenerate at 1 on a set of probability p and it is degenerate at 0 on the complementary set of probability $q = 1 - p$. As a result, the conditional characteristic function of X_i given $\mathcal{C}_i$ becomes identically equal to 1 in absolute value on all of Ω, and the factorized conditional characteristic function of the scaled sum $S_n = n^{-1/2}\sum_{j=1}^{n}(I(X_i \leq \xi_p) - F(\xi_p))$ no longer provides a useful bound on the discrepancy between the distribution functions of S_n and $N(0, \sigma_\infty^2)$. However, once this degeneracy is ruled out at ξ_p by condition (C.5), it is possible to derive a suitably small upper bound on the conditional characteristic function of the scaled sums $n^{-1/2}\sum_{j=1}^{n}(I(X_i \leq y) - F(y))$ of the lattice random variables uniformly over y in a neighborhood of ξ_p (cf. Lemma 4.3 in Section 4.3). We exploit the arguments of Götze and Hipp (1983) and Lahiri (1993, 1996a) in conjunction with this bound to establish a uniform $O(n^{-1/2})$-order bound for the sums of such lattice variables. Note that in the independent case, if we take $\mathcal{D}_j = \sigma\langle X_j\rangle, j \in \mathbb{Z}$ and $\mathcal{C}_j = \sigma\langle\{X_i : i \neq j\}\rangle$, then $G_i(\xi_p) = P(X_i \leq \xi_p|\mathcal{C}_i) = P(X_1 \leq \xi_p) = p$ and hence condition (C.5) easily holds. Below we consider some important examples and verify conditions (C.2)-(C.5).

Example 4.1. Suppose $\{X_i\}_{i\in\mathbb{Z}}$ is m-dependent for some $m \in \mathbb{Z} \cup \{0\}$, i.e., $\sigma\langle\{X_i : i \leq k\}\rangle$ and $\sigma\langle\{X_i : i \geq k + m + 1\}\rangle$ are independent for all $k \in \mathbb{Z}$. Then, we take $\mathcal{D}_j = \sigma\langle X_j\rangle$ and $\mathcal{C}_j = \sigma\langle X_i : i \neq j\rangle$, $j \in \mathbb{Z}$. Then, it is easy to check that conditions (C.2)-(C.4) hold with $X_{i,m}^\dagger = X_i$ for all $i \in \mathbb{Z}$, $m \in \mathbb{N}$ and $\alpha_0, \beta_0, \gamma_0$ arbitrarily large. Furthermore, in this case, condition (C.5) reduces to

$$P(P(X_0 \leq \xi_p|X_i : 0 < |i| \leq m) = 1) < p\,. \tag{4.8}$$

We claim that (4.8) or equivalently (C.5) holds if there exists a set $A \in \mathcal{F}$ with $P(A) > 0$ and $\epsilon \in (0, 1/2)$, $a \leq \xi_p \leq b$ such that $G_0 \equiv$ the conditional distribution of X_0 given $\{X_i : 0 < |i| \leq$

$m\}$ puts at least ϵ mass on $(a-\epsilon, a]$ and on $(b, b+\epsilon]$ on the set A, i.e.,

$$G_0((a-\epsilon, a]) > \epsilon, \quad G_0((b, b+\epsilon]) > \epsilon, \quad \text{for all} \quad \omega \in A. \tag{4.9}$$

To see this, note that (writing G_0 also to denote the distribution function),

$$p = F(\xi_p) = EG_0(\xi_p) = P(G_0(\xi_p) = 1) + EG_0(\xi_p)I(0 < G_0(\xi_p) < 1). \tag{4.10}$$

Thus, $P(G_0(\xi_p) = 1) \leq p$.

If possible, suppose that (4.8) does not hold, i.e., $p = P(G_0(\xi_p) = 1)$, Then, by (4.10),

$$EG_0(\xi_p)I(0 < G_0(\xi_p) < 1) = 0,$$

which implies $P(G_0(\xi_p) \in (0,1)) = 0$. Consequently,

$$P(G_0(\xi_p) = 0) = 1 - [P(G_0(\xi_p) \in (0,1)) + P(G_0(\xi_p) = 1)] = 1 - p.$$

But then

$$\begin{aligned} P(A) &= P(A \cap \{G_0(\xi_p) = 0\}) + P(A \cap \{G_0(\xi_p) = 1\}) \\ &\leq P(\{G_0((a-\epsilon, a]) > \epsilon\} \cap \{G_0(\xi_p) = 0\}) \\ &\quad + P(\{G_0(b, b+\epsilon]) > \epsilon\} \cap \{G_0(\xi_p) = 1\}) \\ &= P(\phi) + P(\phi) = 0, \end{aligned}$$

which contradicts the fact that $P(A) > 0$. Hence, the claim is proved.

Example 4.2. Let $X_i = f(X_{0i}), i \in \mathbb{N}$, where f is a Borel measurable function and $\{X_{0i}\}_{i\in\mathbb{N}}$ is a stationary homogeneous Markov-process satisfying

$$|P(x; A) - P(y; A)| < 1 \tag{4.11}$$

for all $x, y \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$, the Borel σ-field on $\mathbb{R}$, where $P(\cdot;\cdot)$ denotes the transition probability function. Further suppose that $\mathcal{L}(X_{01}|X_{0i} : i \neq 1) = \mathcal{L}(X_{01}|X_{00})$. Then by (iii) on page 219 of Götze and Hipp (1983), conditions (C.2)-(C.4) hold with $\mathcal{D}_j = \sigma\langle X_{0j}\rangle$, and $X_{j,m}^{\dagger} = X_j$ for all $j \in \mathbb{Z}$ and with $\alpha_0, \beta_0, \gamma_0$ arbitrarily large positive real numbers. For condition (C.5), we take $\mathcal{C}_j = \sigma\langle\{X_{0i} : i \neq j\}\rangle, j \in \mathbb{Z}$. Let $A = f^{-1}((-\infty, \xi_p))$. Then,

$$G(\xi_p) = P(X_1 \leq \xi_p | X_{0i} : i \neq 1) = P(f(X_{01}) \leq \xi_p | X_{0i} : i \neq 1) = P(X_{01} \in A | X_{00})\,. \tag{4.12}$$

Next, we write $B_1 = \{\omega : P(X_{00}(\omega); A) = 1\}$, $B_2 = \{\omega : P(X_{00}(\omega); A) = 0\}$, and $B_3 = \{\omega : P(X_{00}(\omega); A) \in (0,1)\}$. Note that, by (4.12), $P(G(\xi_p) = 1) = P(B_1)$ and at least one of B_1, B_2 is empty because of (4.11). So condition (C.5) holds if $P(B_1) = 0$. Otherwise, if $P(B_2) = 0$, then we have the following:

$$P(B_3) = 1 - P(B_1) \geq 1 - p > 0\,,$$

implying

$$d = \int_{X_{00}(B_3)} P(x; A) P_0\{dx\} > 0\,,$$

but then

$$\begin{aligned} p &= P(X_1 \leq \xi_p) = P(X_{01} \in A) \\ &= \int_{\mathbb{R}} P(x; A) P_0\{dx\} \\ &= \int_{X_{00}(B_1)} P(x; A) P_0\{dx\} + \int_{X_{00}(B_3)} P(x; A) P_0\{dx\} \\ &= P(B_1) + d\,, \end{aligned}$$

where P_0 is the marginal distribution function of X_{00}. Hence condition (C.5) holds.

We conclude this section with presenting the main theorem of this chapter.

Theorem 4.1 *Under the conditions (C.1)-(C.5), we have*

$$\sup_{x \in \mathbb{R}} |P(\sqrt{n}(F_n^{-1}(p) - F^{-1}(p)) \leq x) - \Phi(x/\tau_\infty)| = O(n^{-1/2}) \quad n \to \infty\,.$$

4.3 Technical Details

In the proofs below, we write $C, C(\cdot)$ to denote generic constants with values in $(0, \infty)$ that may depend on the arguments (if any), but not on the variables, n, x, y. Also, unless otherwise mentioned, we take limits by letting $n \to \infty$. By the definition of the sample quantile, for any $y \in \mathbb{R}$,

$$P(F_n(y) > p) \leq P(F_n^{-1}(p) \leq y) \leq P(F_n(y) \geq p)\,. \tag{4.13}$$

Hence, we consider the sum of $\sum_{i=1}^{n} I(X_i \leq y)$ for y in a neighborhood of ξ_p and study the rate of convergence of the upper and lower bounds in (4.13). The first result gives an expansion for the log-characteristic function of a scaled sum of a general strong-mixing sequence $\{W_j\}_{j\in\mathbb{Z}}$ of random variables in a neighborhood of the origin.

Lemma 4.1 *For each $n \geq 1$, let $f_n : \mathbb{R} \to [-1, 1]$ be a Borel measurable function such that $Ef_n(X_i) = 0$, $E|f_n(X_i) - f_n(X_{i,k}^{\dagger})| \leq Ck^{-\beta_0}$ for all $i \in \mathbb{Z}$, $k \in \mathbb{N}$ and*

$$n^{-1}Var(\sum_{i=1}^{n} f_n(X_i)) = 1\,. \tag{4.14}$$

Then, for any $\epsilon \in (0, 1/4)$

$$\begin{aligned} &\sup_{t\in A_n} \Big|\log E\exp(\iota t S_n) - \sum_{r=2}^{5} \frac{(\iota t)^r}{r!}\chi_{r,n}\Big| \\ \leq\ & C(\epsilon)n^{-1/2-6\epsilon}(1 + \sup_{t\in A_n} |\theta_{1n}(t)|^6) \\ &+C(\epsilon)(\sup_{t\in A_n} |H_n(t)|^{-6}) \cdot n^{2\epsilon(\alpha_0\vee\beta_0)} \cdot \{n^{-1/2-\alpha_0/4} + n^{-1/2-\beta_0/4}\} \end{aligned}$$

for all $n \geq 1$, where $A_n = \{t \in R : |t| \leq (\log n)^{1/2}(\log\log(n+1))^{1/4}\}$ and where, with $W_{ni} \equiv f_n(X_i), i \in \mathbb{Z}, n \geq 1, S_n = n^{-1/2}\sum_{i=1}^{n} W_{ni}, H_n(t) = E\exp(\iota t S_n), t \in \mathbb{R}, \chi_{r,n}$ is the r-th cumulant of S_n, and $\theta_{1n}(t)$ is as defined in (4.19)below.

Proof: For any random variables $V_1, \cdots, V_p$, $p \in \mathbb{N}$, set

$$\mathcal{K}_t(V_1, \cdots, V_p) = \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_p} \log E\exp(\iota t S_n + x_1 V_1 + \cdots x_p V_p)\big|_{x_1=\cdots=x_p=0}\,. \tag{4.15}$$

Then, using Taylor's expansion of the cumulant generating function $\log E\exp(\iota t S_n)$ around $t = 0$, we get

$$\left|\log E\exp(\iota t S_n) - \sum_{r=2}^{5} \frac{(\iota t)^r}{r!}\chi_{r,n}\right| \le \sum_{k=0}^{n-1}\sum^{(k)} |\mathcal{K}_{\eta t}(V_{j1}, \cdots, V_{j6})|, \tag{4.16}$$

for any $t \in \mathbb{R}$ with $|E\exp(\iota t S_n)| > 0$, where $\eta = \eta(t) \in [0,1]$, $V_j = tW_{nj}/\sqrt{n}$ and for a given k, the summation $\sum^{(k)}$ extends over $j_1, \cdots, j_6$ with maximal gap k. Note that by Lemma 3.1 of Lahiri (1996a) (with $c_n = 1, t = 0$), for any $a_1, \cdots, a_r \in \mathbb{R}$, with $|a_j| \le 1, r \ge 2$,

$$\begin{aligned} |\mathcal{K}_0(a_1 S_n, \cdots, a_r S_n)| &\le C(r) n^{-(r-2)/2} \sum_{k=0}^{n-1} k^{r-1}[\alpha(k/3) + \beta_w(k/3)] \\ &\le C(r) n^{-(r-2)/2}, \end{aligned} \tag{4.17}$$

provided $\alpha_0 > r, \beta_0 > r$, where $\beta_w(k) = k^{-\beta_0}, k \in \mathbb{N}$.

Fix $\epsilon \in (0, 1/4)$ and let $a_n = n^{1/4-\epsilon}$. Then by (4.16) above and by Lemma 3.2 of Lahiri (1996a), (with $c_n = 1$), as in the proof of his Lemma 3.6 (cf. (3.9)), we have, for $t \in A_n$,

$$\begin{aligned} &\sum_{k=0}^{a_n}\sum^{(k)} |\mathcal{K}_{\eta t}(V_{j1}, \cdots, V_{j6})| \\ \le\ & \sum_{k=0}^{a_n} n \cdot (k+1)^5 \cdot C \cdot n^{-3} \cdot (1 + |t|^6)\{(1 + \theta_{1n}(\eta t))^6 + (1 + \theta_{2n}(\eta t))^6\} \\ \le\ & C(\epsilon) a_n^6 n^{-2} \cdot (1 + |t|^6)[1 + |\theta_{1n}(t)|^6 + |H_n(t)|^{-6}\{n^{-3\alpha_0/4} + n \cdot n^{-3\beta_0/4}\}], \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} \theta_{1n}(t) &= |H_n(t)|^{-1} \max\{|E\exp(S_I^{(l)})| : 1 \le l \le L, |I| \le 4, I \subset \{1, \cdots, n\}\} \\ \theta_{2n}(t) &= |H_n(t)|^{-1}(L2^L\{\alpha(m) + n\beta_w(m)\} + \zeta_t^L(m)) \\ \zeta_t(m) &= C|t|(n^{-1}m)^{1/2}, \quad \text{(correcting for a typographical error in Lahiri (1996a))} \\ m &= n^{3/4+\epsilon} \quad \text{and} \quad L = \log\log n\,. \end{aligned} \tag{4.19}$$

Here, for $I \subset \{1, \cdots, n\}, l \ge 0$, $S_I^{(l)} \equiv Ln^{-1/2}t\sum^{*(l)} W_{nj}$, with $\sum^{*(l)}$ ranging over all $j \in \{1, \cdots, n\}$ such that $|j - i| > lm$ for all $i \in I$. Next using Lemma 3.3 of Lahiri (1996a) with $K = L, m = 3Kn^{-\epsilon}$ and $c_n = 1$, for each $k \in (a_n, n)$, as in the proof of (3.10), page 217 of

Lahiri (1996a) (again correcting for the typo, $(1+\|t\|^{r/2})$, with $(1+\|t\|^r)n^{-r/2}$), we get

$$\begin{aligned}
&\sum_{k=a_n+1}^{n-1} \sum^{(k)} |\mathcal{K}_{\eta t}(V_{j1},\cdots,V_{j4})| \\
\leq\ & C\cdot n^{-3}(1+|t|^6)|H_n(\eta t)|^{-6}\cdot L2^L \\
&\cdot\left\{\sum_{k=a_n+1}^{n} n(k+1)^5[\alpha(kn^{-\epsilon})+n\beta_w(kn^{-\epsilon})+\zeta_t(3kn^{-\epsilon})^L\right\} \\
\leq\ & C(\epsilon)(1+|t|^6)|H_n(\eta t)|^{-6}\cdot n^{-2}\cdot L2^L\cdot\{n^{\epsilon\alpha_0}a_n^{(6-\alpha_0)}+n\cdot n^{\epsilon\beta_0}\cdot a_n^{(6-\beta_0)}+n^{-\epsilon L/4}\} \\
\leq\ & C(\epsilon)\cdot|H_n(\eta t)|^{-6}\cdot n^{2\epsilon(\alpha_0\vee\beta_0)}\cdot n^{-1/2}\cdot\{n^{-\alpha_0/4}+n\cdot n^{-\beta_0/4}\} \qquad (4.20)
\end{aligned}$$

for all $t\in A_n$. Hence, the lemma follows from (4.16), (4.18)-(4.20). $\square$

Remark 4.1 *It is possible to obtain a bound on the difference between $E\exp(\iota t S_n)$ and its s-th order Taylor's expansion $\sum_{r=2}^{s}\frac{(\iota t)^r}{r!}\chi_{r,n}$ for an integer $s\geq 3$ by suitably modifying the arguments in the proof of Lemma 3.1. It can be shown that for a small $\delta>0$, a $O(n^{-1/2-\delta})$ bound on the difference is assured, if the strong-mixing exponent α_0 satisfy*

$$\alpha_0 > s+4+9/(s-2)\,. \qquad (4.21)$$

By minimizing the right side of (4.21), we get $s=5$, which explains the reason behind considering the 5-th order Taylor's expansion in Lemma 4.1.

Lemma 4.2 *Let W_{nj}'s and S_n be as in Lemma 4.1. Then,*

(i) For any $a\in(0,1/2)$, there exists a $C_0=C_0(\alpha_0,\beta_0,\gamma_0,a)$ such that for all $n\geq C_0$,

$$\begin{aligned}
|H_n(t)| \ \leq\ & C_0\exp\left(\frac{-t^2}{2}[1-C_0(\log n)^{-2}]\right) \\
& + C_0 n^{1-a}\{n^{-a\alpha_0}+n^{-a\beta_0}+n^{-a\gamma_0}\}(\log n)^{C_0}
\end{aligned}$$

uniformly in $|t|\leq n^{(1-a)/2}(\log n)$.

(ii) There exist $\epsilon_0 \in (0,1)$ and $C_1 = C_1(\alpha_0, \beta_0, \gamma_0) \in (0, \infty)$ such that for all $n \geq C_1$,

$$|H_n(t)| \geq \epsilon_0 \exp(-t^2/2) - C_1(\log n)^{C_1}[n^{-\alpha_0/2} + n^{1/4} \cdot n^{-\beta_0/2} + n^{-\gamma_0/2}]$$

for all $|t| \leq \epsilon_0 \log n$.

Proof: Let $m = m_1(\log n)^{-2}$ and $m_1 = n^a(\log n)^{-6}$. Let $l, j_1, \cdots, j_l$ be the integers defined on page 218 of Lahiri (1996a)with $I = \{1, \cdots, n\}$ and $I_1 = \{m_1 + 1, \cdots, n - m_1\}$. Also, let

$$\begin{aligned}
\Gamma_k &= \prod\{\exp(\iota t W_{nj}/\sqrt{n}) : j \in I, |j - j_k| \leq m_1\}, \quad k = 1, \cdots, l \\
B &= \prod\{\exp(\iota t W_{nj}/\sqrt{n}) : j \in I, |j - j_k| > m_1\}, \quad \text{for all} \quad k = 1, \cdots, l\},
\end{aligned}$$

where $I = \{1, \cdots, n\}$.

Then, the arguments leading to (3.11) of Lahiri (1996a) (with $c_n = 1, R = 1$) yields

$$\begin{aligned}
&|H_n(t)| = \left|E(\prod_{k=1}^{l} \Gamma_k)B\right| \\
\leq \; & C\prod_{k=1}^{l} E|E(\Gamma_k|\mathcal{D}_j : j \neq j_k)| + C[l\alpha(m) + l\gamma(m) + \beta_w(m)\{m + n^{1/2}|t|\}] \qquad (4.22)
\end{aligned}$$

for all $n \geq 1, t \in \mathbb{R}$.

Next note that by (4.14) and the stationarity of X_i's,

$$\begin{aligned}
&\left|m_1^{-1} Var(\sum_{j=1}^{m_1} W_{nj}) - 1\right| \\
\leq \; & 2\sum_{j=m_1+1}^{n-1} |EW_{n_1}W_{n(j+1)}| + \frac{4}{m_1}\sum_{j=1}^{n} j|EW_{n1}W_{n(j+1)}| \\
\leq \; & C\left[\sum_{j=m_1+1}^{n} \{\alpha(j/3) + \beta_w(j/3)\} + \frac{1}{m}\sum_{j=1}^{\infty} j\{\alpha(j/3) + \beta_w(j/3)\}\right] \\
\leq \; & C(\alpha_0, \beta_0)[m_1^{-\alpha_0+1} + m_1^{-\beta_0+1} + m_1^{-1}] \qquad (4.23)
\end{aligned}$$

Hence, by (3.12) and the arguments following it on page 219 of Lahiri (1996a), and by (4.22) and (4.23) above, it follows that for all $n \geq C(\alpha_0, \beta_0)$,

$$\begin{aligned}
&\prod_{k=1}^{l} |E(\Gamma_k|\mathcal{D}_j : j \neq j_k)| \\
\leq \; & C\exp\left(-\frac{t^2}{2}\{n^{-1}2m_1 l(1 - C(\alpha_0, \beta_0)m_1^{-1}) - Cn^{-3/2}l|t|m_1^{3/2}\}\right) \\
\leq \; & C\exp\left(-\frac{t^2}{2}[1 - C(a, \alpha_0, \beta_0)(\log n)^{-2}]\right), \qquad (4.24)
\end{aligned}$$

for all $|t| \leq n^{(1-a)/2}(\log n)$, where in the second inequality, we make use of the fact

$$2m_1 l = n[1 - O(n^{-1}m_1 + m_1^{-1}m)] \quad \text{as} \quad n \to \infty\,. \tag{4.25}$$

Hence, part(i) of the lemma follows from (4.22) and (4.24). Part(ii) can be proved by retracing the arguments on pages 221-222 of Lahiri (1996a), with $c_n = 1$. □

For the next lemma, let $\mathcal{C}$ be a sub-σ-algebra of $\mathcal{F}$, $G(y;\cdot) = P(X_1 \leq y|\mathcal{C})$, $A_1(y) = \{\omega : G(y;\omega) = 1\}$, and $A_2(y) = \{\omega : 0 < G(y;\cdot) < 1\}, y \in \mathbb{R}$. Also, let $g(y) = P(\{\omega : G(y;\omega) = 1\}) = P(A_1(y)), y \in \mathbb{R}$. Let $\Psi_a(t) = (ae^{\iota t} + 1 - a)$ denote the characteristic function of a random variable Y with $P(Y = 0) = 1 - a, P(Y = 1) = a, a \in (0,1)$.

Lemma 4.3 *If $g(\xi_p) < p$, then there exists $\delta, \epsilon \in (0,1)$ such that for all $t \in \mathbb{R}$,*

$$\sup_{|y-\xi_p|\leq\delta} E|E\{\exp(\iota t I(X_1 \leq y)|\mathcal{C}\}| \leq 1 - (1 - |\Psi_\epsilon(t)|)\delta\,.$$

Proof: By definition, for all $y \in \mathbb{R}$

$$\begin{aligned} F(y) &= P(X_1 \leq y) = E\{P(X_1 \leq y|\mathcal{C})\} \\ &= \int_{A_1(y)\cup A_2(y)} G(y;\cdot)dP \\ &= \int_{A_2(y)} G(y;\cdot)dP + g(y) \end{aligned} \tag{4.26}$$

Note that $A_1(y_1) \subset A_1(y_2)$ for all $y_1 < y_2$ and that $G(\cdot;\omega)$ is a valid distribution function for each $\omega \in \Omega$. Hence it follows that

(i) $g(\cdot)$ is nondecreasing, and

(ii) $g(\cdot)$ is right continuous on $\mathbb{R}$.

Indeed, for any sequence $y_n \downarrow y \in \mathbb{R}$, $\{\omega : G(y;\omega) = 1\} \subset \cap_{n\geq 1}\{\omega : G(y_n;\omega) = 1\}$ by the monotonicity of $G(y;\cdot)$ in y and the reverse inclusion follows from the right-continuity of $G(y;\omega)$ in y for each ω, proving (ii). Since $F(\xi_p) = p$, by (4.26),

$$g(\xi_p) \leq p\,. \tag{4.27}$$

First suppose that $g(\xi_p) < p$. Then by the right continuity of $g(\cdot)$, there exists a $\delta_0 > 0$ such that

$$g(\xi_p + \delta_0) < p - \delta_0 \,. \tag{4.28}$$

By (i), this implies that $g(y) < p - \delta_0$ for all $y < \xi_p + \delta_0$. Since F is continuous at ξ_p, there exists a $0 < \delta_1 \le \delta_0$ such that

$$F(\xi_p - \delta_1) > p - \delta_0/2 \,. \tag{4.29}$$

Next, write $A_3(y;\epsilon) = \{\omega : \epsilon < G(y;\omega) < 1-\epsilon\}, A_4 = \{\omega : G(y;\omega) \ge 1-\epsilon\}, \epsilon \in (0,1), y \in \mathbb{R}$. Note that $A_4(y;\epsilon) \downarrow A_1(y)$ as $\epsilon \downarrow 0$ and $A_4(y;\epsilon) \subset A_4(y+h;\epsilon)$ for all $y \in \mathbb{R}, h > 0, \epsilon \in (0,1)$. In particular, for any $y \in \mathbb{R}$,

$$\lim_{\epsilon \to 0} \int_{A_4(y;\epsilon)} G(y;\cdot)dP = \int_{A_1(y)} G(y;\cdot)dP = P(A_1(y)) = g(y) \tag{4.30}$$

Hence, by (4.26), and (4.28)-(4.30), there exists $0 < \epsilon < \delta_0/8$ such that for all $y \in (\xi_p - \delta_1, \xi_p + \delta_1)$,

$$\begin{aligned}
P(A_3(y;\epsilon)) &\ge \int_{A_3(y;\epsilon)} G(y;\cdot)dP \\
&= \int G(y;\cdot)dP - \int G(y;\cdot)I(G(y;\cdot) \le \epsilon)dP \\
&\quad - \int G(y;\cdot)I(G(y;\cdot) \ge 1-\epsilon)dP \\
&\ge F(y) - \epsilon - \int G(\xi_p + \delta_1;\cdot)I(G(\xi_p+\delta_1;\cdot) \ge 1-\epsilon)dP \\
&\ge F(\xi_p - \delta_1) - \epsilon - [g(\xi_p + \delta_1) + \epsilon] \\
&\ge [p - \delta_0/2] - 2\epsilon - [p - \delta_0] \\
&= \delta_0/4 \,.
\end{aligned} \tag{4.31}$$

Next, writing $\Psi_\epsilon(t) = |\epsilon e^{\iota t} + 1 - \epsilon|, t \in \mathbb{R}$, and $G(y) = G(y;\cdot)$ for notational simplicity, by (4.31), for $y \in (\xi_p - \delta_1, \xi_p + \delta_1)$, we have

$$\begin{aligned}
&E|E(\exp(\iota t I(X_1 \le y))|\mathcal{C})| \\
&= E|G(y)e^{\iota t} + (1 - G(y))|
\end{aligned}$$

$$
\begin{aligned}
&= E|1-4G(y)(1-G(y))\sin^2(t/2)|^{1/2} \\
&\leq P(A_3^c(y;\epsilon)) + EI_{A_3(y;\epsilon)} \cdot |1-4G(y)(1-G(y))\sin^2(t/2)|^{1/2} \\
&\leq P(A_3^c(y;\epsilon)) + |1-\epsilon(1-\epsilon)\sin^2(t/2)|^{1/2} P(A_3(y;\epsilon)) \\
&= 1-(1-|\Psi_\epsilon(t)|)P(A_3(y;\epsilon)) \\
&\leq 1-(1-|\Psi_\epsilon(t)|)\delta_0/4\,. \qquad (4.32)
\end{aligned}
$$

Lemma 4.3 is proved. □

Lemma 4.4 *Suppose that condition (C.2) holds and that $\{Y_j\}_{j\geq 1}$ be a sequence of zero-mean real-valued random variables such that*

(i) Y_j is $\mathcal{D}_{j-k}^{j+k}$- measurable for all $j \geq 1$ with $1/k + k/n = o(1)$ as $n \to \infty$

(ii) there exists a constant $c > 0$ such that

$$E|Y_j|^l \leq c^{l-2} l! EY_j^2 < \infty; \quad j = 1, \cdots, n; l = 3, 4, \cdots$$

Then for each $n \geq 2$, each integer $q \in [1, \frac{n}{2k+1}]$, each $\epsilon > 0$ and each $s \geq 3$,

$$P\left(\left|\sum_{j=1}^n Y_j\right| > n\epsilon\right) \leq a_1 \exp\left\{-\frac{q\epsilon^2}{25m_2^2 + 5c\epsilon}\right\} + a_2(s)\alpha_n\left(\left\lfloor\frac{n}{q+1} - 2k\right\rfloor\right)^{\frac{2s}{2s+1}},$$

where

$$a_1 = 2\frac{n}{q} + 2\left(1 + \frac{\epsilon^2}{25m_2^2 + 5c\epsilon}\right), \quad \textit{with} \quad m_2^2 = \max_{1\leq j\leq n} E|Y_j|^2\,,$$

$$a_2(s) = 11n\left(1 + \frac{5m_s^{\frac{s}{2s+1}}}{\epsilon}\right), \quad \textit{with} \quad m_s = \max_{1\leq j\leq n} \|Y_j\|_s\,.$$

Proof: Since the proof of this lemma follows the same line as in that of Lemma 2.3, we only give an outline. For details, see the proof of Lemma 2.3. We define the blocks of random variables as follows:

$$Z_j' = \sum_{i=1}^q Y_{(i-1)r+j}, \quad j = 1, \cdots, r, \quad R_n' = (Y_{qr+1} + \cdots + Y_n)I(qr < n)\,.$$

where $q \in [1, n/(2k+1)]$, $r = \lfloor n/(q+1) \rfloor$. Then, by equation (2.14),

$$P\left(|R_n'| > \frac{n\epsilon}{5}\right) \leq 2\left(1 + \frac{\epsilon^2}{5(5m_2^2 + c\epsilon)}\right) \exp\left(-\frac{q\epsilon^2}{5(5m_2^2 + c\epsilon)}\right). \tag{4.33}$$

And by Lemma 1.2 of Bosq (1998), there exist random variables $\{W_i\}_{i=1}^q$ such that $W_i =^d Y_{(i-1)r+j}$, $i = 1, \cdots, q$, and

$$P\left(|W_i - Y_{(i-1)r+j}| > \frac{2\epsilon}{5}\right) \leq 11\left(1 + \frac{5m_s}{\epsilon}\right)^{\frac{s}{2s+1}} \left(\alpha_n\left(\left\lfloor \frac{n}{q+1} - 2k \right\rfloor\right)\right)^{\frac{2s}{2s+1}}. \tag{4.34}$$

Hence, (4.33), (4.34), Bonferroni's inequality, and Bernstein's inequality lead to

$$\begin{aligned} P\left(|Z_j'| > \frac{4n\epsilon}{5r}\right) &\leq P\left(|\sum_{i=1}^q W_i| > \frac{2q\epsilon}{5}\right) + \sum_{i=1}^q P\left(|Y_{(i-1)r+j} - W_i| > \frac{2\epsilon}{5}\right) \\ &\leq 11q\left(1 + \frac{5m_s}{\epsilon}\right)^{\frac{s}{2s+1}} \left(\alpha_n\left(\left\lfloor \frac{n}{q+1} - 2k \right\rfloor\right)\right)^{\frac{2s}{2s+1}} \\ &\quad + 2\exp\left(-\frac{q\epsilon^2}{5(5m_2^2 + c\epsilon)}\right). \end{aligned} \tag{4.35}$$

Thus, by (4.33) and (4.35)

$$\begin{aligned} P\left(\left|\sum_{j=1}^n Y_j\right| > n\epsilon\right) &\leq \sum_{j=1}^r P\left(|Z_j'| > \frac{4n\epsilon}{5r}\right) + P\left(|R_n'| > \frac{n\epsilon}{5}\right) \\ &\leq r \cdot 11q\left(1 + \frac{5m_s}{\epsilon}\right)^{\frac{s}{2s+1}} \left(\alpha_n\left(\left\lfloor \frac{n}{q+1} - 2k \right\rfloor\right)\right)^{\frac{2s}{2s+1}} \\ &\quad + r \cdot 2\exp\left(-\frac{q\epsilon^2}{5(5m_2^2 + c\epsilon)}\right) \\ &\quad + 2\left(1 + \frac{\epsilon^2}{5(5m_2^2 + c\epsilon)}\right) \exp\left(-\frac{q\epsilon^2}{5(5m_2^2 + c\epsilon)}\right), \end{aligned}$$

which completes the proof of Lemma 4.4. □

Lemma 4.5 *Suppose that conditions (C.1)-(C.3) hold. Then for any* $\delta \in (5/(2+4\alpha_0), 1/2)$,

$$P(|\sqrt{n}(\hat{\xi}_n - \xi_p)| > n^\delta \log n) \leq Cn^{-1/2},$$

for some constant $C > 0$.

Proof: Let $\epsilon_n = n^{-1/2+\delta}\log n$, $x_n = \xi_p + \epsilon_n$. Then we have by (4.13)

$$\begin{aligned}
& P(\sqrt{n}(\hat{\xi}_n - \xi_p) > n^\delta \log n) \\
\leq\ & P(F_n(x_n) \leq p) \\
\leq\ & P\left(\frac{1}{n}\sum_{i=1}^n [I(X_i \leq x_n) - F(x_n)] \leq p - F(x_n)\right) \\
\leq\ & P\left(\frac{1}{n}\sum_{i=1}^n [I(X_i \leq x_n) - F(x_n)] \leq -A\epsilon_n\right) \\
\leq\ & P\left(|\sum_{i=1}^n [I(X_i \leq x_n) - F(x_n)]| \geq An\epsilon_n\right),
\end{aligned} \tag{4.36}$$

where $A = \sup\{f(x) : x \in \mathcal{N}_p\}$. It is easy to check that for any $i \in \mathbb{Z}$, any $x \in \mathbb{R}$ and any $\epsilon_0 > 0$,

$$\begin{aligned}
& E|I(X_i \leq x) - I(X_{i,k}^\dagger \leq x)| \\
\leq\ & P\left(X_i \leq x < X_{i,k}^\dagger\right) + P\left(X_i > x \geq X_{i,k}^\dagger\right) \\
\leq\ & P\left(\min(X_i, X_{i,k}^\dagger) \leq x \leq \max(X_i, X_{i,k}^\dagger)\right) \\
\leq\ & P(|X_i - x| < \epsilon_0) + P\left(|X_i - X_{i,k}^\dagger| \geq \epsilon_0\right) \\
\leq\ & [F(x+\epsilon_0) - F(x-\epsilon_0)] + \epsilon_0^{-1} E|X_i - X_{i,k}^\dagger|.
\end{aligned} \tag{4.37}$$

Hence, by conditions (C.1) and (C.3) for $\beta(n)$, there exists $\delta_0 > 0$ such that for all $|x - \xi_p| < \delta_0$, and $i \in \mathbb{Z}, k \in \mathbb{N}$, (with $\epsilon_0 = k^{-\beta_0}$ in (4.37)),

$$E|I(X_i \leq x) - I(X_{i,k}^\dagger \leq x)| \leq A \cdot 2\epsilon_0 + \epsilon_0^{-1} \cdot d^{-1} n^{-2\beta_0} = (2A + d^{-1})k^{-\beta_0}. \tag{4.38}$$

Thus, for $|x - \xi_p| < \delta_0$, $k = \lfloor n^{(1-\delta)/\beta_0} \rfloor$,

$$\begin{aligned}
& |\sum_{i-1}^n [E(I(X_{i,k}^\dagger \leq x) - F(x)]| \\
\leq\ & \sum_{i-1}^n |E(I(X_{i,k}^\dagger \leq x) - I(X_i \leq x)| \\
\leq\ & \sum_{i-1}^n E|I(X_{i,k}^\dagger \leq x) - I(X_i \leq x)| \\
\leq\ & n \cdot (2A + d^{-1})k^{-\beta_0} \\
\leq\ & An\epsilon_n/3,
\end{aligned} \tag{4.39}$$

and

$$\begin{aligned}
& P\left(\left|\sum_{i=1}^{n}[I(X_i \le x) - I(X_{i,k}^{\dagger} \le x)]\right| > \frac{An\epsilon_n}{3}\right) \\
\le\ & \frac{E\left|\sum_{i=1}^{n}[I(X_i \le x) - I(X_{i,k}^{\dagger} \le x)]\right|}{An\epsilon_n/3} \\
\le\ & \frac{\sum_{i=1}^{n} E|I(X_i \le x) - I(X_{i,k}^{\dagger} \le x)|}{An\epsilon_n/3} \\
\le\ & \frac{n \cdot (2A + d^{-1})k^{-\beta_0}}{An\epsilon_n/3} \\
\le\ & \frac{3(2A + d^{-1}) \cdot n^{-(1-\delta)}}{An^{-1/2+\delta}\log n} \\
\le\ & O(n^{-1/2})\,. \qquad (4.40)
\end{aligned}$$

Next, let $Y_i = I(X_{i,k}^{\dagger} \le x) - EI(X_{i,k}^{\dagger} \le x), i = 1, 2, \cdots$ with $1/k + k/n = o(1)$ as $n \to \infty$. Then it can be seen easily that $\{Y_i\}_{j \ge 1}$ satisfies the conditions of Lemma 4.4 (with $c = 1, m_2^2 = EI(X_{i,k}^{\dagger} \le x)(1 - EI(X_{i,k}^{\dagger} \le x))$. Applying Lemma 4.4 to the sequence $\{Y_i\}_{j \ge 1}$ with

$$k = \lfloor n^{(1-\delta)/\beta_0} \rfloor, \quad \epsilon = An^{-1/2+\delta}(\log n)/3, \quad q = \lfloor n^{1-2\delta}(\log\log n) + 1 \rfloor\,,$$

and exploiting the arguments in the proof of Lemma 2.4 (iii) will lead to

$$\begin{aligned}
& P\left(\left|\sum_{i=1}^{n} I(X_{i,k}^{\dagger} \le x) - EI(X_{i,k}^{\dagger} \le x)\right| > n \cdot An^{-1/2+\delta}(\log n)/3\right) \\
=\ & P\left(\left|\sum_{i=1}^{n} Y_i\right| > n \cdot An^{-1/2+\delta}(\log n)/3\right) \\
\le\ & O(n^{-1/2})\,. \qquad (4.41)
\end{aligned}$$

Note that there exist some $\delta_1 < \delta_0$ such that (4.37)-(4.41) hold uniformly for x satisfying $|x - \xi_p| < \delta_1$ since the bounds in those relations depend on x only through the moments of $I(X_{i,k}^{\dagger} \le x)$.

Hence, we have by (4.39), (4.40) and (4.41)

$$\begin{aligned}
& P\left(\left|\sum_{i=1}^{n}[I(X_i \le x_n) - F(x_n)]\right| \ge An\epsilon_n\right) \\
\le\ & \sup_{|x-\xi_p| \le \delta_1} P\left(\left|\sum_{i=1}^{n}[I(X_{i,k}^{\dagger} \le x) - EI(X_{i,k}^{\dagger} \le x)]\right| \ge An\epsilon_n/3\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \sup_{|x-\xi_p|\le\delta_1} P\left(|\sum_{i=1}^{n}[I(X_i \le x) - I(X_{i,k}^{\dagger} \le x)]| \ge An\epsilon_n/3 \right) \\
&\le \; Cn^{-1/2}, \qquad (4.42)
\end{aligned}
$$

where C_1 is a positive constant. Thus, (4.36) and (4.42) lead to

$$P(\sqrt{n}(\hat{\xi}_n - \xi_p) > n^{\delta}\log n) \le C_1 n^{-1/2}.$$

Similarly,

$$P(\sqrt{n}(\hat{\xi}_n - \xi_p) < -n^{\delta}\log n) \le C_1 n^{-1/2}.$$

Thus, Lemma 4.5 follows. □

Proof of Theorem 4.1 By (3.1) and Lemma 4.5, it is enough to show that

$$\sup_{|y-\xi_p|\le n^{-1/2+\delta}\log n} |P(F_n(y) \ge p) - \Phi(y/\tau_\infty)| = O(n^{-1/2}) \quad \text{as} \quad n \to \infty. \tag{4.43}$$

For $x \in \mathbb{R}$, let $x_n = \xi_p + n^{-1/2+\delta}x$, $\sigma_n^2(x) = nVar(F_n(x))$ and $S_n(x) = \sqrt{n}(F_n(x) - F(x))/\sigma_n(x)$. By the smoothing inequality (cf. Lemma 2, page 538 of Feller (1971))

$$
\begin{aligned}
\Delta_n &\equiv \sup_{|x|\le n^{\delta}\log n} \left| P(F_n(x) \le p) - \Phi\left(\frac{\sqrt{n}(p - F(x_n))}{\sigma_n(x)}\right) \right| \\
&\le \sup_{|x|\le n^{\delta}\log n}\sup_{y\in\mathbb{R}} |P(S_n(x_n) \le y) - \Phi(y)| \\
&\le \sup_{|x|\le n^{\delta}\log n} \left[\frac{1}{\pi}\int_{-\kappa\sqrt{n}}^{\kappa\sqrt{n}} \frac{|E\exp(\iota t S_n(x_n)) - e^{-t^2/2}|}{|t|}dt + \frac{1}{24\sqrt{2\pi}}\frac{1}{\kappa\sqrt{n}} \right], \qquad (4.44)
\end{aligned}
$$

where $\kappa \in (0, \infty)$ is a constant (independent of x), to be specified later.

Let δ_* be such that

$$\liminf_{n\to\infty} \inf\{\sigma_n^2(\xi_p + x) : |x| \le \delta_*\} > \sigma_\infty^2/2, \tag{4.45}$$

and $0 < \delta_* < \delta_0$, here δ_0 is as in (4.38) in Lemma 4.5. Note that (4.45) holds, because $\sigma_\infty^2 > 0$, F is continuous at ξ_p and $\sup\{\sum_{j=n}^{\infty} |Cov(I(X_1 \le x), I(X_{j+1} \le x))| : x \in \mathbb{R}\} \le C\sum_{j=n}^{\infty}[\alpha(j/3) + \beta(j/3))] \to 0$ as $n \to \infty$. Let $\mathcal{N}_p = \{x : |x - \xi_p| \le \delta_*\}$, and let $\chi_{r,n}(x)$ denote the r-th cumulant of $S_n(x) \equiv \sqrt{n}(F_n(x) - F(x))/\sigma_n(x), x \in \mathcal{N}_p$. Then, we have by (4.38), for any $i \in \mathbb{Z}$, there exists $C > 0$ and $0 < \delta_{**} \leq \delta_*$ such that for all $|x - \xi_p| < \delta_{**}$, and $i \in \mathbb{Z}, k \in \mathbb{N}$,

$$E|I(X_i \leq x) - I(X_{i,k}^\dagger \leq x)| \leq Ck^{-\beta_0}. \tag{4.46}$$

Without loss of generality, let $\delta_{**} = \delta_*$. Also, let $W_{ni}(x) = [I(X_i \leq x) - F(x)]/\sigma_n(x), i \in \mathbb{Z}, n \geq 1, x \in \mathcal{N}_p$. Then, by (4.45), (4.46) and the condition (C.3), $\{W_{ni}(x) : i \in \mathbb{Z}\}_{n\geq 1}$ satisfies the conditions of Lemma 3.1 uniformly in $x \in \mathcal{N}_p$. By Lemma 4.1 and Lemma 4.2 (ii) above and the induction arguments used in the proof of Lemma 3.28 of Götze and Hipp (1983), it follows that there exists $\epsilon_1 = \epsilon_1(\alpha_0, \beta_0) \in (0, 1/4)$ such that for all $0 < \epsilon < \epsilon_1$,

$$\begin{aligned}
&\sup_{x\in\mathcal{N}_p}\sup_{t^2\leq\log n}\left|\log E\exp(\iota t S_n(x)) - \sum_{r=2}^{5}\frac{(\iota t)^r}{r!}\chi_{r,n}(x)\right| \\
\leq\ & C(\epsilon)[n^{-1/2-6\epsilon} + \{\epsilon_0\exp(-(\sqrt{\log n})^2/2)\}^{-6}n^{2\epsilon(\alpha_0\vee\beta_0)}\{n^{-1/2-\alpha_0/4} + n^{-1/2-\beta_0/4}\}] \\
\leq\ & C(\epsilon,\epsilon_0)n^{-1/2-C(\epsilon)},
\end{aligned} \tag{4.47}$$

for all $n \geq C_1$, where C_1 is as in Lemma 4.2(ii). Note also that $|e^x - 1| \leq |x|e^{|x|}$. Hence, for $|t|^2 \leq \log n$, by (4.17) and (4.47), there exist constants C_1 and C_2 such that

$$\begin{aligned}
&\left|E\exp(\iota t S_n(x)) - e^{-\frac{1}{2}t^2}\right| \\
\leq\ & \left|E\exp(\iota t S_n(x)) - \exp\left\{\sum_{r=2}^{5}(\iota t)^r(r!)^{-1}\chi_{r,n}(x)\right\}\right| + \left|\exp\left\{\sum_{r=2}^{5}(\iota t)^r(r!)^{-1}\chi_{r,n}(x)\right\} - e^{-\frac{1}{2}t^2}\right| \\
=\ & \exp\left\{\sum_{r=2}^{5}(\iota t)^r(r!)^{-1}\chi_{r,n}(x)\right\}\left|\exp\left\{\log E\exp(\iota t S_n(x)) - \sum_{r=2}^{5}(\iota t)^r(r!)^{-1}\chi_{r,n}(x)\right\} - 1\right| \\
& + e^{-\frac{1}{2}t^2}\left|\exp\left\{\sum_{r=3}^{5}(\iota t)^r(r!)^{-1}\chi_{r,n}(x)\right\} - 1\right| \\
\leq\ & C_0\cdot C(\epsilon,\epsilon_0)n^{-1/2-C(\epsilon)} + C_2e^{-\frac{1}{2}t^2}\cdot n^{-1/2}|t|^3.
\end{aligned} \tag{4.48}$$

Thus,

$$\begin{aligned}
&\sup_{x\in\mathcal{N}_p}\int_{t^2\leq\log n}|E\exp(\iota t S_n(x)) - e^{-t^2/2}||t|^{-1}dt \\
\leq\ & \sup_{x\in\mathcal{N}_p}\int_{t^2\leq\log n}\left\{C_0C(\epsilon,\epsilon_0)n^{-1/2-C(\epsilon)} + C_2e^{-\frac{1}{2}t^2}\cdot n^{-1/2}|t|^3\right\}|t|^{-1} \\
\leq\ & Cn^{-1/2},
\end{aligned}$$

for all $n \geq C_1$. On the other hand

$$\int_{t^2>\log n} e^{-t^2/2}dt = o(n^{-1/2}),$$

so it remains to show that

$$\sup_{x\in\mathcal{N}_p}\int_{(\log n)^{1/2}<|t|<\kappa n^{1/2}} |E\exp(\iota t S_n(x))||t|^{-1}dt = O(n^{-1/2}). \tag{4.49}$$

To this end, we split the set of t-values in (4.49) into the sets $B_{1n} = \{t \in \mathbb{R} : (\log n)^{1/2} \leq |t| \leq n^{7/16}\}$ and $B_{2n} = \{t \in \mathbb{R} : n^{7/16} < |t| < \kappa n^{1/2}\}$. Then, using Lemma 4.2(i) with $W_{nj} = W_{nj}(x), x \in \mathcal{N}_p$ with $a = 1/8$, we have

$$\begin{aligned}
&\sup_{x\in\mathcal{N}_p}\int_{B_{1n}} |E\exp tS_n(x))||t|^{-1}dt \\
\leq\ & 2C_0\int_{(\log n)^{1/2}}^{n^{7/16}} \exp(-t^2/2)\cdot\exp(C_0t^2(\log n)^{-2}/2)|t|^{-1}dt \\
&+2C_0\left(\int_{(\log n)^{1/2}}^{n^{7/16}} |t|^{-1}dt\right)(\log n)^{C_0}\cdot n^{-1/2-1/8} \\
\leq\ & 2C_0\left[\exp(C_0/2)\int_{(\log n)^{1/2}}^{\log n}\exp(-t^2/2)|t|^{-1}dt + \int_{\log n}^{n^{7/16}}\exp(-t^2/4)|t|^{-1}dt\right] \\
&+2C_0(\log n)^{C_0+1}n^{-1/2-1/8} \\
=\ & o(n^{-1/2}).
\end{aligned}$$

Next, note that $(1-u)^{1/2} \leq 1 - 1/2u$ for all $0 < u < 1$. Hence, there exists $\kappa_0 = \kappa_0(\delta, \epsilon) \in (0, \infty)$, depending on δ and ϵ of Lemma 4.3 such that for $|t| \leq \kappa_0$

$$\begin{aligned}
1-(1-|\Psi_\epsilon(t)|)\delta &= (1-\delta)+\delta(1-4\epsilon(1-\epsilon)\sin^2(t/2))^{1/2} \\
&\leq (1-\delta)+\delta(1-2\epsilon(1-\epsilon)\sin^2(t/2)) \\
&\leq 1-C(\epsilon,\delta)t^2.
\end{aligned} \tag{4.50}$$

Also, for a bounded random variable y and σ-fields $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{F}$, $E(Y|\mathcal{G}_1) = E\{E(Y|\mathcal{G}_2)|\mathcal{G}_1\}$ a.s. (P). Hence, setting $\kappa = \kappa_0$ in B_{2n}, and using (4.22) (with $a = 1/8$), we have

$$\begin{aligned}
&\sup_{x\in\mathcal{N}_p}\int_{B_{2n}} |E\exp(\iota tS_n(x))||t|^{-1}dt \\
\leq\ & \sup_{x\in\mathcal{N}_p}\int_{B_{2n}}\prod_{k=1}^{l} E|E(\Gamma_k(x)|\mathcal{D}_j : j\neq j_k)||t|^{-1}dt
\end{aligned}$$

$$
\begin{aligned}
& +C[n^{1-a}(n^{-a(\alpha_0\wedge\beta_0\wedge\gamma_0)})\cdot(\log n)^{C(\alpha_0,\beta_0,\gamma_0)}] \\
\leq\ & 2\log n\cdot\sup\left\{\prod_{k=1}^{l}E|E(\Gamma_k(x)|\mathcal{C})| : t\in B_{2n}, x\in\mathcal{N}_p\right\}+o(n^{-1/2}) \\
=\ & 2\log n\cdot\sup\{E|E(\exp(\iota tI(X_1\leq x)/\sqrt{n})|\mathcal{C})| : t\in B_{2n}, x\in\mathcal{N}_p\}^l+o(n^{-1/2}) \\
\leq\ & (2\log n)\cdot\sup_{t\in B_{2n}}\left\{1-C(\epsilon,\delta)\cdot t^2/n\right\}^l+o(n^{-1/2}) \\
=\ & O(\log n\cdot\exp(-C(\epsilon,\delta)\cdot n^{-1/8}\cdot l))+o(n^{-1/2}) \\
=\ & o(n^{-1/2}), \qquad (4.51)
\end{aligned}
$$

where $l = n/(2m_1)(1+o(1)) = n^{1-a}(\log n)^6(1+o(1))$ (with $a = 1/8$) and where the variables $\Gamma_k(x)$'s are defined as in the proof of Lemma 4.2 with $W_{jk} = W_{jk}(x), x \in \mathcal{N}_p$. Hence, (4.49) follows, and this completes the proof of the Theorem. □

CHAPTER 5 ACCURACY OF MBB APPROXIMATION

5.1 Introduction

This chapter gives a refined version of Theorem 2.2 in Chapter 2, and studies the rate of bootstrap approximations to the distribution of sample quantiles. The main results of Chapter 5 show that, under the assumption of exponentially decaying strongly mixing coefficients, the rate of convergence of the moving block bootstrap (MBB) approximations to the distributions of the centered and scaled sample quantiles is of order $O(n^{-1/4}(\log n)(\log\log n)^{1/4})$.

It is known that, in the i.i.d. set up, the accuracy of the bootstrap approximation to the unknown sampling distributions of many regular statistics such as smooth functions of sample means, is of order $o(n^{-1/2})$ (cf. Singh (1981), Babu (1986)). Thus, in these situations, the bootstrap works better than the classical normal approximation which has the accuracy of order $O(n^{-1/2})$. For the weakly dependent case, the second order property of the block bootstrap methods have been studied by several authors. See Lahiri (1991, 1996b), Götze and Künsch (1996). However, for approximations to the distributions of irregular statistics, such as sample quantiles, the bootstrap approximation is inferior to the classical normal approximation. Singh (1981) showed that the exact rate of convergence of the bootstrap approximation to the distribution of the sample quantile based on i.i.d. observations is $O(n^{-1/4}(\log\log n)^{1/2})$.

In the i.i.d. case, smoothing techniques have been introduced to improve the performance of the bootstrap approximation to the sample quantile. Hall, DiCiccio and Romano (1989) showed that, if the distribution function is sufficiently smooth, smoothing appropriately can improve the bootstrap estimator of the distribution of sample quantile to the order of $O(n^{-1/2+\epsilon})$, for any $\epsilon > 0$. For other works in this context, we refer to Falk and Reiss (1989), Falk and Janas (1990), and Janas (1993). However, as we may see from this chapter that, unlike the i.i.d. case, smoothing may not improve the performance of the MBB approximation to the distribution of the sample quantile based on weakly dependent observations. This is mainly because, in the weak dependence case, the rate of approximation for the denominator of the asymptotic variance (cf. (1.3)) of the sample quantile may not be improved, though that of the numerator can be improved as in the i.i.d. case. Details are provided in the following section. We next state the main theorem and prove some auxiliary results which will be used to investigate the convergence rate of the MBB approximation.

5.2 MBB Approximation Rate

Throughout this chapter, we assume that $\{X_i\}_{i\in\mathbb{Z}}$ is a sequence of α-mixing strictly stationary random variables with exponentially decaying coefficients, i.e., there exist constants $C>0$, $0<\rho<1$ such that $\alpha(n)\le C\rho^n$, for any integer n. As in the previous chapters, we use F and f to denote, respectively, the corresponding marginal distribution function and the marginal probability density function of the random variables. We also follow the notation that are used in the first four chapters. In particular, for $p\in(0,1)$, $\xi_p=F^{-1}(p)$, $\hat{\xi}_n=F_n^{-1}(p)$, $\xi_n^*=F_n^{*-1}(p)$, $\tilde{\xi}_n=\tilde{F_n}^{-1}(p)$ (cf. Section 1.1, 1.2), and for all $x\in\mathbb{R}$,

$$
\begin{aligned}
U_i(x) &\equiv \frac{1}{\ell}\sum_{j=1}^{\ell} I(X_{(i-1)\ell+j}\le x), \quad i=1,\cdots,b,\\
U_i^*(x) &\equiv \frac{1}{\ell}\sum_{j=1}^{\ell} I(X^*_{(i-1)\ell+j}\le x), \quad i=1,\cdots,b,\\
\sigma_n^2(x) &\equiv Var(\sqrt{n}F_n(x))\equiv\frac{1}{n}\sum_{j=-(n-1)}^{n-1} Cov(I(X_1\le x),I(X_{1+j}\le x)\,,\\
\sigma_\infty^2(x) &\equiv \sum_{j=-\infty}^{\infty} Cov(I(X_1\le x),I(X_{1+j}\le x)\,. \qquad (5.1)
\end{aligned}
$$

Again, P_*, Var_*, and E_* represent respectively the conditional probability, the conditional variance, and the conditional expectation, given $(X_1,\cdots,X_n)$. Here, we define $\hat{\sigma}_n^2(x)$ as the conditional variance of $\sqrt{\ell}U_1^*(x)$, i.e.,

$$
\hat{\sigma}_n^2(x)\equiv Var_*(\sqrt{\ell}U_1^*(x))\equiv\frac{1}{N}\sum_{i=1}^{N}(\sqrt{\ell}U_i(x))^2-\ell\tilde{F_n}^2(x)\,, \qquad (5.2)
$$

where $N=n-\ell+1$. Now, we are in a position to state the main theorem of this chapter.

Theorem 5.1 *Suppose that the MBB block length ℓ satisfies $\ell = o(n^{1/2})$ and that there exists a neighborhood of ξ_p, say $\mathcal{N}_p$, such that $\sigma_\infty^2(x)$ (cf. (5.1)) is positive and has a bounded derivative on $\mathcal{N}_p$. Suppose also that there exists $d \in (0,1)$ such that*

$$P(G_i(\xi_p) = 1) \le p - d\,,$$

where $G_i(y) = P(X_i \le y | X_j : j \ne i)$. And

$$0 < d_0 = \inf\{f(x) : x \in \mathcal{N}_p\} \le \sup\{f(x) : x \in \mathcal{N}_p\} = d_1 < \infty \tag{5.3}$$

$$0 < \inf\{f'(x) : x \in \mathcal{N}_p\} \le \sup\{f'(x) : x \in \mathcal{N}_p\}\} < \infty\,. \tag{5.4}$$

Then, under the assumption of α-mixing with exponential decay rate, we have

$$\sup_{x\in\mathbb{R}} |P_*(\sqrt{n}(\xi_n^* - \tilde{\xi}_n) \le x) - P(\sqrt{n}(\hat{\xi}_n - \xi_p) \le x)| = \delta_n \quad a.s.\,, \tag{5.5}$$

where

$$\delta_n = O(\ell^{-1} + \ell n^{-1/2}\log n + n^{-1/4}(\log n)(\log\log n)^{1/4})$$

Thus, a wide range of choice of block length ℓ will lead to $\delta_n = O(n^{-1/4}(\log n)(\log\log n)^{1/4})$, which is the optimal rate of convergence.

Theorem 5.1 indicates that, under suitable choice of the block length, the MBB approximation to the distribution of the normalized sample quantile of α-mixing sequence with exponentially decaying rate, has the accuracy of order $O(n^{-1/4}(\log n)(\log\log n)^{1/4})$. This conclusion is almost identical to the approximation result of the bootstrapped sample quantile based on i.i.d. observations (cf. Singh (1981)). Note that the first two terms of the right hand side of equation (5.5) both involve the block length ℓ. Here we explain briefly why this is so. Details are provided in the proofs given below. The first term ℓ^{-1} is due to the bias of $\hat{\sigma}_n^2(x)$ for estimating $\sigma_\infty^2(x)$. Note that by (5.13) and $\sup_{x\in\mathbb{R}}|F_n(x) - \tilde{F}_n(x)| = O(\ell^{1/2}n^{-1})$ (cf. Götze and Künsch (1996)), one may get

$$\begin{aligned}
E\hat{\sigma}_n^2(x) &= E(\sqrt{\ell}U_1(x))^2 - \ell E\tilde{F}_n^{\,2}(x) \\
&= [Var(\sqrt{\ell}U_1(x)) + \ell F^2(x)] - \ell E(F_n(x) + O(\ell^{1/2}n^{-1}))^2 \\
&= [\sigma_\ell^2(x) + \ell F^2(x)] - \ell[\sigma_n^2(x)/n + F^2(x) + O(\ell^{1/2}n^{-1})] \\
&= \sigma_\infty^2(x) + O(\ell^{-1} + \ell^{3/2}n^{-1})\,,
\end{aligned} \tag{5.6}$$

uniformly in $x \in \mathbb{R}$. The second term $\ell n^{-1/2} \log n$ is due to the random quantity $\hat{x}_n$ occurring inside of the conditional variance of $\sqrt{\ell} U_1^*(\cdot)$. Note that by Lemma 5.1 (ii) and Lemma 2.4 (iii), for $|x| \leq \log n$,

$$\hat{x}_n = \tilde{\xi}_n + x n^{-1/2} = \xi_p + O(\ell^{1/2} n^{-1} + n^{-1/2} \log n) \quad \text{a.s.} \tag{5.7}$$

Hence, we have, by (5.6), (5.7), and the smoothness condition on $\sigma_\infty^2(x)$,

$$\begin{aligned} \ell Var_* U_1^*(\hat{x}_n) &= \sigma_\infty^2 + O(\ell^{-1} + \ell^{3/2} n^{-1}) + \ell \cdot O(\ell^{1/2} n^{-1} + n^{-1/2} \log n) \\ &= \sigma_\infty^2 + O(\ell^{-1} + \ell n^{-1/2} \log n) \quad \text{a.s.} \end{aligned} \tag{5.8}$$

On the other hand, it can be shown as in (5.22) that, under the exponentially mixing condition,

$$\sqrt{n}(\tilde{F}_n(\tilde{\xi}_n + x n^{-1/2}) - p) = x f(\xi_p) + O(\ell^{1/2} n^{-1/2} + n^{-1/4} (\log n)(\log\log n)^{1/4}) \quad \text{a.s.} \tag{5.9}$$

So, for appropriate values of the block length ℓ, the accuracy of the MBB approximation to the distribution of sample quantile is dominated by equations (5.8) and (5.9) , since by Lemma 5.5, (5.23) and (5.28)

$$\begin{aligned} & \sup_{x \in \mathbb{R}} |P_*(\sqrt{n}(\xi_n^* - \tilde{\xi}_n) \leq x) - P(\sqrt{n}(\hat{\xi}_n - \xi_p) \leq x)| \\ = \quad & \sup_{|x| \leq \log n} |P_*(\sqrt{n}(\xi_n^* - \tilde{\xi}_n) \leq x) - P(\sqrt{n}(\hat{\xi}_n - \xi_p) \leq x)| \\ \leq \quad & \sup_{|x| \leq \log n} \left| \Phi\left(\frac{x f(\xi_p)}{\sigma_\infty}\right) - \Phi\left(\frac{\sqrt{b}(\tilde{F}_n(\hat{x}_n) - p)}{\sqrt{Var_* U_1^*(\hat{x}_n)}}\right) \right| + O(\ell n^{-1/2}) \quad \text{a.s.} \end{aligned}$$

Note that the above term $O(\ell n^{-1/2})$ is from (5.27). It can be shapened to be of order $O(\ell^{-1/2} n^{1/2})$ by using the similar arguments as in Lemma 5.4. Here, this term is dampened by the term $O(\ell n^{-1/2} \log n)$ in (5.8).

As in the i.i.d. set up, if the marginal distribution function F is sufficiently smooth, kernel smoothing can improve the right hand side of equation (5.9) to be of order $O(\ell^{1/2} n^{-1/2} + n^{-1/2+\epsilon}(\log n))$, for arbitrary $\epsilon > 0$. However, smoothing can not reduce the order of the right hand side of equation (5.8).

For proving the main theorem, we first present the following lemmas. The first three lemmas give some asymptotic properties about the deviations of the empirical distribution function and the sample quantile. These are extensions of the results of Babu and Singh (1978).

Lemma 5.1 *Suppose that (5.3) holds. Then, almost surely*

(i) $\sup_{x\in\mathcal{N}_p} |F_n(x) - F(x)| = O(n^{-1/2}(\log\log n)^{1/2})$;

(ii) $\sup_{0\le t\le 1} |F_n^{-1}(t) - F^{-1}(t)| = O(n^{-1/2}(\log\log n)^{1/2})$;

(iii) $\sup_{t\in\{F(x):x\in\mathcal{N}_p\}} |F_n(F_n^{-1}(t)) - t| = O(n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4})$.

Poof: Note that under condition (5.3), F is strictly increasing on $\mathcal{N}_p$. Then, $F_n(x) = F_n(F^{-1}(t))$ with $t = F(x)$. Hence

$$\begin{aligned}
\sup_{x\in\mathcal{N}_p} |F_n(x) - F(x)| &= \sup_{t\in\{F(x):x\in\mathcal{N}_p\}} |F_n(F^{-1}(t)) - t| \\
&= \sup_{t\in\{F(x):x\in\mathcal{N}_p\}} |E_n(t) - t| \\
&\le \sup_{0\le t\le 1} |E_n(t) - t| \\
&\le cn^{-1/2}(\log\log n)^{1/2} ,
\end{aligned}$$

where the last inequality follows from Lemma 3.4 of Babu and Singh (1978). c is a constant. Here we use E_n, E_n^{-1} to denote, respectively, the empirical distribution function and the sample quantile function of the uniform distribution defined on $(0,1)$. So Lemma 5.1 (i) is proved. For the proof of 5.1 (ii), see Lemma 4.2 of Babu and Singh (1978).

We now prove (iii). Note that

$$\begin{aligned}
\sup_{t\in\{F(x):x\in\mathcal{N}_p\}} |F_n(F_n^{-1}(t)) - t| &= \sup_{t\in\{F(x):x\in\mathcal{N}_p\}} |F_n(F^{-1}(E_n^{-1}(t))) - t| \\
&= \sup_{t\in\{F(x):x\in\mathcal{N}_p\}} |E_n(E_n^{-1}(t)) - t| \\
&\le \sup_{0\le t\le 1} |E_n(E_n^{-1}(t)) - t| \\
&= O(n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}) .
\end{aligned}$$

Here, we used the arguments on pages 538-539 of Babu and Singh (1978). $\square$

Lemma 5.2 *Under the conditions of Lemma 5.1, we have*

$$\sup_{|x-\xi_p|\le 2n^{-1/2}\log n} |F_n(x) - F_n(\xi_p) - F(x) + p| = O(n^{-3/4}(\log n)(\log\log n)^{1/4}) \quad a.s.$$

Proof: By modifying the arguments on page 539 of Babu and Singh (1978) and applying their Lemma 3.3 (an exponential type of inequality) with

$$b_n = n^{-3/4}(\log n)(\log\log n)^{1/4}, \quad b = 2d_1 n^{-1/2}\log n, \quad D = 2d_1 n b_n\,,$$

where $d_1 = \max\{f(x) : x \in \mathcal{N}_p\}$, we have

$$\sup_{|t-p|\le 2d_1 n^{-1/2}\log n} |E_n(t) - E_n(p) - t + p| = O(n^{-3/4}(\log n)(\log\log n)^{1/4})\,,$$

which implies

$$\begin{aligned}
& \sup_{|x-\xi_p|\le 2n^{-1/2}\log n} |F_n(x) - F_n(\xi_p) - F(x) + p| \\
= & \sup_{|x-\xi_p|\le 2n^{-1/2}\log n} |F_n(F^{-1}(F(x))) - F_n(F^{-1}(F(\xi_p))) - F(x) + p| \\
= & \sup_{|x-\xi_p|\le 2n^{-1/2}\log n} |E_n(F(x)) - E_n(F(\xi_p)) - F(x) + p| \\
= & \sup_{|F^{-1}(t)-F^{-1}(\xi_p)|\le 2n^{-1/2}\log n} |E_n(t) - E_n(p) - t + p| \\
\le & \sup_{|t-p|\le 2d_1 n^{-1/2}\log n} |E_n(t) - E_n(p) - t + p| \\
= & \; O(n^{-3/4}(\log n)(\log\log n)^{1/4})\,.
\end{aligned}$$

We complete the proof of Lemma 5.2. □

As we may see that, the above three lemmas and the lemma below are actually refinements of Lemma 2.4 and Lemma 2.5 in Chapter 2 by imposing stronger conditions on the dependence structure of the stationary process.

Lemma 5.3 *Under the conditions of Lemma 5.1, we have*

(i) $|\tilde{F}_n(\tilde{\xi}_n) - p| \le O(\ell^{1/2}n^{-1} + n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4})$ *a.s.*

(ii) $|\tilde{\xi}_n - \xi_p| \le O(\ell^{1/2}n^{-1} + n^{-1/2}(\log\log n)^{1/2})$ *a.s.*

Proof: We may refine the result of Lemma 2.4 (i) (cf. Götze and Künsch (1996)) as

$$\sup_{x\in\mathbb{R}} |F_n(x) - \tilde{F}_n(x)| = O(\ell^{1/2}n^{-1}) \quad \text{a.s.} \tag{5.10}$$

Then, there exists a constant $C_1 > 0$ such that

$$\sup_{x\in\mathbb{R}} |F_n(x) - \tilde{F}_n(x)| \le C_1\ell^{1/2}n^{-1} \quad \text{a.s.}$$

Thus, by the definition of quantile function, we have

$$F_n^{-1}(p - C_1\ell^{1/2}n^{-1}) \le \tilde{F_n}^{-1}(p) \le F_n^{-1}(p + C_1\ell^{1/2}n^{-1}) \quad \text{a.s.} \tag{5.11}$$

So (5.10) and (5.11) lead to

$$\begin{aligned} & F_n(F_n^{-1}(p - C_1\ell^{1/2}n^{-1})) - C_1\ell^{1/2}n^{-1} \\ \le\ & \tilde{F}_n(\tilde{F_n}^{-1}(p)) \\ \le\ & F_n(F_n^{-1}(p + C_1\ell^{1/2}n^{-1})) + C_1\ell^{1/2}n^{-1} \quad \text{a.s.,} \end{aligned}$$

which, together with Lemma 5.1 (iii), gives

$$\begin{aligned} & |\tilde{F}_n(\tilde{\xi}_n) - p| \\ \le\ & C_1\ell^{1/2}n^{-1} + |F_n(F_n^{-1}(p - C_1\ell^{1/2}n^{-1})) - p| \vee |F_n(F_n^{-1}(p + C_1\ell^{1/2}n^{-1})) - p| \\ \le\ & 2C_1\ell^{1/2}n^{-1} + O(n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}) \\ =\ & O(\ell^{1/2}n^{-1} + n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}) \quad \text{a.s.} \end{aligned}$$

Here $a \vee b = \max\{a, b\}$. So (i) is proved.

Note that, by condition (5.3), the mean value theorem, and Lemma 5.1,

$$\begin{aligned} |F_n^{-1}(p \pm C_1\ell^{1/2}n^{-1}) - \xi_p| &\le |F_n^{-1}(p \pm C_1\ell^{1/2}n^{-1})) - F^{-1}(p \pm C_1\ell^{1/2}n^{-1})| \\ &\quad + |F^{-1}(p \pm C_1\ell^{1/2}n^{-1})) - F^{-1}(p)| \\ &\le O(n^{-1/2}(\log\log n)^{1/2}) + \frac{1}{d_0}C_1\ell^{1/2}n^{-1} \\ &= O(l^{1/2}n^{-1} + n^{-1/2}(\log\log n)^{1/2}). \end{aligned} \tag{5.12}$$

Hence, Lemma 5.3 (ii) follows from (5.11) and (5.12). This concludes the proof of Lemma 5.3. □

In Chapter 2, we showed that $\ell Var_* U_1^*(\hat{x}_n) \to \sigma_\infty^2(\xi_p) = \sigma_\infty^2$. The next lemma gives a refinement of this result.

Lemma 5.4 *Under the conditions of Theorem 4.1, if the block length* $\ell = o(n^{1/2})$, *then we have*

$$\ell Var_* U_1^*(\hat{x}_n) = \sigma_\infty^2 + O(\ell^{-1} + \ell n^{-1/2} \log n)\,.$$

Proof: Let $\sigma_n^2(x)$ and $\sigma_\infty^2(x)$ be defined as in (5.1). Then

$$\begin{aligned}
\sigma_n^2(x) &= \sum_{j=-(n-1)}^{n-1} (1-|j|/n) Cov(I(X_1 \le x), I(X_{1+j} \le x)) \\
&= \sum_{j=-\infty}^{\infty} Cov(I(X_1 \le x), I(X_{1+j} \le x)) - \sum_{|j|=n}^{\infty} Cov(I(X_1 \le x), I(X_{1+j} \le x)) \\
&\quad -\frac{1}{n} \sum_{j=-(n-1)}^{n-1} |j| Cov(I(X_1 \le x), I(X_{1+j} \le x)) \\
&= \sigma_\infty^2(x) + O(n^{-1})\,,
\end{aligned} \tag{5.13}$$

since, by Billingsley's inequality

$$\left| \sum_{|j|=n}^{\infty} Cov(I(X_1 \le x), I(X_{1+j} \le x)) \right| \le \sum_{|j|=n}^{\infty} 4\alpha(j) = O(n^{-m})\,,$$

for any $m \ge 1$, and

$$\sum_{j=-(n-1)}^{n-1} |j| Cov(I(X_1 \le x), I(X_{1+j} \le x)) < \infty\,.$$

We next exploit the similar arguments as used in Chapter 2. Let

$$x_{n,1} = \xi_p - 2n^{-1/2} \log n, \quad x_{n,2} = \xi_p + 2n^{-1/2} \log n\,.$$

By Lemma 5.1 and Lemma 5.3, there exist constants $C_3, C_4 > 0$ such that

$$|\hat{\xi}_n - \xi_p| \le C_3 n^{-1/2} (\log\log n)^{1/2}, \quad |\tilde{\xi}_n - \xi_p| \le C_3 n^{-1/2} (\log\log n)^{1/2} \quad \text{a.s.}$$

Then, there exists a set $A \in \mathcal{F}$ with $P(A) = 1$, such that, for any $\omega \in A$, there exists a positive integer n_ω and for all $n > n_\omega$, $|x| \le \log n$, $|\hat{x}_n(w) - \xi_p| \le 2n^{-1/2}\log n$. Recall that $\hat{x}_n = \tilde{\xi}_n + xn^{-1/2}$. Then, we have

$$x_{n,1} \le \hat{x}_n(\omega) \le x_{n,2} \quad \text{for} \quad n > n_\omega, \quad |x| \le \log n \quad \text{a.s.}$$

Thus,

$$\begin{aligned} Var_* U_1^*(\hat{x}_n) &= \frac{1}{N}\sum_{i=1}^{N} U_i^2(\hat{x}_n) - \tilde{F_n}^2(\hat{x}_n) \\ &\le \frac{1}{N}\sum_{i=1}^{N} U_i^2(x_{n,2}) - \tilde{F_n}^2(x_{n,1}) \\ &= \frac{1}{N}\sum_{i=1}^{N} [U_i(x_{n,2}) - F(x_{n,2})]^2 + 2F(x_{n,2})\tilde{F}_n(x_{n,2}) \\ &\quad -F^2(x_{n,2}) - \tilde{F_n}^2(x_{n,1}) \quad \text{a.s.} \end{aligned} \tag{5.14}$$

And

$$\begin{aligned} Var_* U_1^*(\hat{x}_n) &\ge \frac{1}{N}\sum_{i=1}^{N} U_i^2(x_{n,1}) - \tilde{F_n}^2(x_{n,2}) \\ &= \frac{1}{N}\sum_{i=1}^{N} [U_i(x_{n,1}) - F(x_{n,1})]^2 + 2F(x_{n,1})\tilde{F}_n(x_{n,1}) \\ &\quad -F^2(x_{n,1}) - \tilde{F_n}^2(x_{n,2}) \quad \text{a.s.} \end{aligned} \tag{5.15}$$

By Lemma 5.1 and Lemma 5.2, we have

$$\begin{aligned} & |2F(x_{n,2})\tilde{F}_n(x_{n,2}) - F^2(x_{n,2}) - \tilde{F_n}^2(x_{n,1})| \\ \le\ & |\tilde{F_n}^2(x_{n,2}) - \tilde{F_n}^2(x_{n,1})| + (\tilde{F}_n(x_{n,2}) - F(x_{n,2}))^2 \\ \le\ & 2|\tilde{F}_n(x_{n,2}) - \tilde{F}_n(x_{n,1})| + (\tilde{F}_n(x_{n,2}) - F(x_{n,2}))^2 \\ =\ & 2|F_n(x_{n,2}) - F_n(x_{n,1})| + O(\ell^{1/2}n^{-1}) \quad \text{a.s.} \end{aligned}$$

And

$$|F_n(x_{n,2}) - F_n(x_{n,1})| = |[F_n(x_{n,2}) - F_n(\xi_p)] - [F_n(x_{n,1}) - F_n(\xi_p)]|$$

$$\begin{aligned}
&\leq |F_n(x_{n,2}) - F_n(\xi_p) - F(x_{n,2}) + F(\xi_p)| \\
&\quad +|F_n(x_{n,1}) - F_n(\xi_p) - F(x_{n,1}) + F(\xi_p)| \\
&\quad +|F(x_{n,2}) - F(x_{n,1})| \\
&\leq 2 \cdot O(n^{-3/4}(\log n)(\log\log n)^{1/4}) + d_1|x_{n,2} - x_{n,1}| \\
&= O(n^{-1/2}\log n) \quad \text{a.s.}
\end{aligned}$$

Thus,

$$\ell|2F(x_{n,2})\tilde{F_n}(x_{n,2}) - F^2(x_{n,2}) - \tilde{F_n}^2(x_{n,1})| = O(\ell n^{-1/2}\log n) \quad \text{a.s.} \tag{5.16}$$

Likewise,

$$\ell|2F(x_{n,1})\tilde{F_n}(x_{n,1}) - F^2(x_{n,1}) - \tilde{F_n}^2(x_{n,2})| = O(\ell n^{-1/2}\log n) \quad \text{a.s.} \tag{5.17}$$

We now evaluate

$$\ell N^{-1}\sum_{i=1}^{N}[U_i(x_{n,j}) - F(x_{n,j})]^2, \quad j = 1, 2\,.$$

Let

$$W_{n,i} = \ell[U_i(x_n) - F(x_n)]^2, \quad i = 1, \cdots, N, \quad \text{and} \quad \{x_n\} = \{x_{n,1}\}, \{x_{n,2}\}\,,$$

then,

$$\ell N^{-1}\sum_{i=1}^{N}[U_i(x_{n,j}) - F(x_{n,j})]^2 = N^{-1}\sum_{i=1}^{N}W_{n,i}\,. \tag{5.18}$$

By (5.13) and the condition on $\sigma_\infty^2(x)$, we get

$$\begin{aligned}
EW_{n,i} &= E\ell[U_i(x_{n,j}) - F(x_{n,j})]^2 \\
&= \sigma_\infty^2(x_n) + O(\ell^{-1}) \\
&= \sigma_\infty^2(\xi_p) + O(n^{-1/2}\log n) + O(\ell^{-1}) \\
&= \sigma_\infty^2 + O(n^{-1/2}\log n + \ell^{-1})\,.
\end{aligned} \tag{5.19}$$

Note that $w_{n,\infty} = \|W_{n,1}\|_\infty \leq \ell$, and

$$\begin{aligned}
w_{n,2}^2 &= EW_{n,1}^2 = E(\ell[U_i(x_{n,j}) - F(x_{n,j})]^2)^2 \\
&\leq \ell E(\ell[U_i(x_{n,j}) - F(x_{n,j})]^2) \\
&\leq C_5\ell, \quad \text{for some} \quad C_5 > 0\,,
\end{aligned}$$

On the other hand, (5.19) indicates that, there exists some positive constant C_6 such that

$$w_{n,2}^2 \geq |EW_{n,1}|^2 \geq C_6 .$$

We next apply Lemma 2.2 to triangular array $\{W_{n,i}\}$ with

$$a = 1, \quad q = 2, \quad d_n = (\log n)^2, \quad \epsilon_n = \ell^{1/2} n^{-1} (\log n)^3 ,$$

and get

$$\begin{aligned}
& P\left(\left| N^{-1} \sum_{i=1}^{N} W_{n,i} - EW_{n,i} \right| > \epsilon_n \right) \\
\leq \; & C^* \exp\left\{ -\frac{C_1^* [n/d_n]^2 \epsilon_n^2}{w_{n,2}^2 + [n/d_n]^{1/2} w_{n,\infty} \epsilon_n} \right\} \\
& + C^* [n/d_n] \cdot \max\{1, \frac{w_{n,2}}{\epsilon_n}\}^{2/5} [\alpha(d_n/2)]^{4/5} \\
\leq \; & C^* \exp\left\{ -\frac{C_1^* n^2 (\log n)^{-4} \ell n^{-2} (\log n)^6}{C_5 \ell + n^{1/2} (\log n)^{-1} \cdot \ell \cdot \ell^{1/2} n^{-1} (\log n)^3} \right\} \\
& + C^* n (\log n)^{-2} \cdot (C_5^{1/2} \ell^{1/2} \ell^{-1/2} n (\log n)^{-3})^{2/5} [C \rho^{-\frac{(\log n)^2}{2}}]^{4/5} \\
= \; & C^* \exp\left\{ -\frac{C_1^* (\log n)^2}{C_5 + \ell^{1/2} n^{-1/2} (\log n)^2} \right\} + C^{4/5} C^* C_5^{1/5} n^{7/5} (\log n)^{-16/5} \rho^{-\frac{2(\log n)^2}{5}} \\
\leq \; & O(n^{-m}), \quad \forall m > 1 .
\end{aligned}$$

Then, Borel-Cantelli lemma implies

$$\left| N^{-1} \sum_{i=1}^{N} W_{n,i} - EW_{n,i} \right| \leq \epsilon_n = \ell^{1/2} n^{-1} (\log n)^3 \quad \text{a.s.},$$

which together with (5.19) leads to

$$\begin{aligned}
N^{-1} \sum_{i=1}^{N} W_{n,i} &= EW_{n,1} + O(\ell^{1/2} n^{-1} (\log n)^3) \\
&= \sigma_\infty^2 + O(\ell^{-1} + n^{-1/2} \log n) \quad \text{a.s.}
\end{aligned} \tag{5.20}$$

Hence, by (5.14)-(5.18) and (5.20), we have

$$\begin{aligned}
\ell Var_* U_1^*(\hat{x}_n) &= \sigma_\infty^2 + O(\ell^{-1} + n^{-1/2} \log n) + O(\ell n^{-1/2} \log n) \\
&= \sigma_\infty^2 + O(\ell^{-1} + \ell n^{-1/2} \log n) \quad \text{a.s.}
\end{aligned}$$

Lemma 5.4 is proved. □

The last lemma below investigates the tail behavior of the MBB estimator of the sample quantile.

Lemma 5.5 *Under the conditions of Theorem 4.1, for any $m \geq 1$, we have*

$$P_*(\sqrt{n}|\xi_n^* - \tilde{\xi}_n)| > \log n) = O(n^{-m}) \quad a.s.$$

Proof: Let $\hat{y}_n = \tilde{\xi}_n + n^{-1/2}\log n$, then

$$\begin{aligned} P_*(\sqrt{n}(\xi_n^* - \tilde{\xi}_n) > \log n) &= P_*(\xi_n^* > \tilde{\xi}_n + n^{-1/2}\log n) \\ &\leq P_*(F_n^*(\tilde{\xi}_n + n^{-1/2}\log n) \leq p) \\ &= P_*(F_n^*(\hat{y}_n) - \tilde{F}_n(\hat{y}_n) \leq p - \tilde{F}_n(\hat{y}_n)) \\ &= P_*\left(\frac{1}{b}\sum_{i=1}^{b}[U_i^*(\hat{y}_n) - E_*(U_i^*(\hat{y}_n))] \leq p - \tilde{F}_n(\hat{y}_n)\right). \quad (5.21) \end{aligned}$$

Recall that

$$\hat{x}_n = \tilde{\xi}_n + xn^{-1/2}, \quad \sup_{x\in\mathbb{R}}|\tilde{F}_n(x) - F_n(x)| = O(\ell^{1/2}n^{-1}),$$

which together with Lemma 5.1, Lemma 5.2, and Lemma 5.3 leads to

$$\begin{aligned} \tilde{F}_n(\hat{x}_n) - p &= [\tilde{F}_n(\hat{x}_n) - \tilde{F}_n(\tilde{\xi}_n)] + [\tilde{F}_n(\tilde{\xi}_n) - p] \\ &= [F_n(\hat{x}_n) - F_n(\tilde{\xi}_n) + O(\ell^{1/2}n^{-1})] \\ &\quad +O(\ell^{1/2}n^{-1} + n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}) \\ &= [F_n(\hat{x}_n) - F(\hat{x}_n) - F_n(\tilde{\xi}_n) + F(\tilde{\xi}_n)] + [F(\hat{x}_n) - F(\tilde{\xi}_n)] \\ &\quad +O(\ell^{1/2}n^{-1} + n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}) \\ &= 2\cdot O(n^{-3/4}(\log n)(\log\log n)^{1/4}) + (f(\xi_p) + O(n^{-1/2}\log n))\cdot xn^{-1/2} \\ &\quad +O(\ell^{1/2}n^{-1} + n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}) \\ &= xf(\xi_p)n^{-1/2} + O(\ell^{1/2}n^{-1} + n^{-3/4}(\log n)(\log\log n)^{1/4}) \quad \text{a.s.} \qquad (5.22) \end{aligned}$$

By equation (5.22) and the condition of Theorem 5.1, $\ell = o(n^{1/2})$, there exists $C_0 > 0$ such that

$$\tilde{F}_n(\hat{y}_n) - p \leq C_0 n^{-1/2}\log n, \quad \text{a.s.}$$

for $|x| \le \log n$. So, we have, by (5.21), Lemma 5.4, and Bernstein's inequality

$$\begin{aligned}
P_*(\sqrt{n}(\xi_n^* - \tilde{\xi}_n) > \log n) &\le P_*\left(\sum_{i=1}^{b}[U_i^*(\hat{y}_n) - E_*(U_i^*(\hat{y}_n))] \le -C_0 b n^{-1/2} \log n\right) \\
&\le P_*\left(\left|\sum_{i=1}^{b}[\ell^{1/2}U_i^*(\hat{y}_n) - E_*(\ell^{1/2}U_i^*(\hat{y}_n))]\right| \ge C_0 \ell^{1/2} b n^{-1/2} \log n\right) \\
&= P_*\left(\left|\sum_{i=1}^{b}[\ell^{1/2}U_i^*(\hat{y}_n) - E_*(\ell^{1/2}U_i^*(\hat{y}_n))]\right| \ge C_0 b^{1/2} \log n\right) \\
&\le 2\exp\left\{-\frac{[C_0 b^{1/2}\log n]^2}{4\sum_{i=1}^{b} \ell E[U_i^*(\hat{y}_n) - \tilde{F}_n(\hat{y}_n)]^2 + 2C_0 b^{1/2} \log n}\right\} \\
&= 2\exp\left\{-\frac{C_0^2 b(\log n)^2}{4b\ell Var_* U_1^*(\hat{y}_n) + 2C_0 b^{1/2}\log n}\right\} \\
&= 2\exp\left\{-\frac{C_0^2 b(\log n)^2}{4b(\sigma_\infty^2 + o(1)) + 2C_0 b^{1/2}\log n}\right\} \\
&= 2\exp\left\{-\frac{C_0^2(\log n)^2}{4(\sigma_\infty^2 + o(1)) + 2C_0 b^{-1/2}\log n}\right\} \\
&\le O(n^{-m}), \quad \forall m > 1, \quad \text{a.s.}
\end{aligned}$$

We finish the proof of Lemma 5.5. $\square$

Proof of Theorem 5.1 By Lemma 5.1 and Lemma 5.5, it suffices to show that

$$\sup_{|x|\le \log n} |P_*(\sqrt{n}(\xi_n^* - \tilde{\xi}_n) \le x) - P(\sqrt{n}(\hat{\xi}_n - \xi_p) \le x)| = O(\ell^{-1} + \ell n^{-1/2}\log n) \quad \text{a.s.}$$

We see, from the proof of Theorem 2.2 that

$$\left|P_*(F_n^*(\hat{x}_n) > p) - \Phi\left(\frac{\sqrt{b}(\tilde{F}_n(\hat{x}_n) - p)}{\sqrt{Var_* U_1^*(\hat{x}_n)}}\right)\right| \le \frac{3E_*|U_1^*(\hat{x}_n) - E_*U_1^*(\hat{x}_n)|^3}{\sqrt{b}(Var_* U_1^*(\hat{x}_n))^{3/2}},$$

and by the Berry-Esseen Theorem for sample quantile (cf. Theorem 4.1 of Chapter 4)

$$\sup_{x\in\mathbb{R}}\left|P(\sqrt{n}(\hat{\xi}_n - \xi_p) \le x) - \Phi\left(\frac{xf(\xi_p)}{\sigma_\infty}\right)\right| \le O(n^{-1/2}).$$

Therefore,

$$\sup_{|x|\le \log n} |P_*(\sqrt{n}(\xi_n^* - \tilde{\xi}_n) \le x) - P(\sqrt{n}(\hat{\xi}_n - \xi_p) \le x)|$$

$$\leq \sup_{|x|\leq \log n} \left| \Phi\left(\frac{xf(\xi_p)}{\sigma_\infty}\right) - \Phi\left(\frac{\sqrt{b}(\tilde{F}_n(\hat{x}_n) - p)}{\sqrt{Var_* U_1^*(\hat{x}_n)}}\right) \right|$$

$$+ \sup_{|x|\leq \log n} \frac{3E_*|U_1^*(\hat{x}_n) - E_* U_1^*(\hat{x}_n)|^3}{\sqrt{b}(Var_* U_1^*(\hat{x}_n))^{3/2}} + O(n^{-1/2}). \tag{5.23}$$

We have, by equation (5.22),

$$\sqrt{n}(\tilde{F}_n(\hat{x}_n) - p) = xf(\xi_p) + O(\ell^{1/2} n^{-1/2} + n^{-1/4}(\log n)(\log\log n)^{1/4}) \quad \text{a.s.} \tag{5.24}$$

By Lemma 5.4

$$\frac{1}{\sqrt{\ell Var_* U_1^*(\hat{x}_n)}} = \frac{1}{\sigma_\infty} + O(\ell^{-1} + \ell n^{-1/2} \log n) \quad \text{a.s.} \tag{5.25}$$

So, (5.24) and (5.25) imply

$$\frac{\sqrt{b}(\tilde{F}_n(\hat{x}_n) - p)}{\sqrt{Var_* U_1^*(\hat{x}_n)}} = \frac{xf(\xi_p)}{\sigma_\infty} + O(\ell^{-1} + \ell n^{-1/2} \log n + n^{-1/4}(\log n)(\log\log n)^{1/4}) \quad \text{a.s.} \tag{5.26}$$

Note also that

$$\frac{n\sqrt{n}}{b^2} E_*|U_1^*(\hat{x}_n) - E_* U_1^*(\hat{x}_n)|^3 \leq \frac{\ell^{-1} n\sqrt{n}}{b^2}[\ell E_*|U_1^*(\hat{x}_n) - E_* U_1^*(\hat{x}_n)|^2] = O(\ell n^{-1/2}). \tag{5.27}$$

Thus

$$\frac{E_*|U_1^*(\hat{x}_n) - E_* U_1^*(\hat{x}_n)|^3}{\sqrt{b}(Var_* U_1^*(\hat{x}_n))^{3/2}} = \frac{\frac{n\sqrt{n}}{b^2} \cdot E_*|U_1^*(\hat{x}_n) - E_* U_1^*(\hat{x}_n)|^3}{(\ell Var_* U_1^*(\hat{x}_n))^{3/2}} \leq O(\ell n^{-1/2}). \tag{5.28}$$

Hence, we have, by (5.23), (5.26), and (5.28),

$$\begin{aligned} &\sup_{|x|\leq \log n} |P_*(\sqrt{n}(\xi_n^* - \tilde{\xi}_n) \leq x) - P(\sqrt{n}(\hat{\xi}_n - \xi_p) \leq x)| \\ \leq\ & O(\ell^{-1} + \ell n^{-1/2} \log n + n^{-1/4}(\log n)(\log\log n)^{1/4}) \\ &+ O(\ell n^{-1/2}) + O(n^{-1/2}) \\ =\ & O(\ell^{-1} + \ell n^{-1/2} \log n + n^{-1/4}(\log n)(\log\log n)^{1/4}), \end{aligned}$$

and $\ell = \tilde{C} n^{1/4}(\log n)^{-1/2}$, $\tilde{C} > 0$, gives the optimal rate $O(n^{-1/4}(\log n)(\log\log n)^{1/4})$. We complete the proof of Theorem 5.1. □

Note that we may sharpen the right hand side of (5.28) to be of order $O(\ell^{1/2}n^{-1/2})$, by borrowing on the same arguments as used in Lemma 5.4. However, the result from (5.28) is sufficient for handling our problem here.

CHAPTER 6 CONCLUSION

In this dissertation, we first establish the strong consistency results for the moving block bootstrap (MBB) approximations to the distributions and variances of sample quantiles based on weakly dependent observations. The consistency result of the MBB distribution function estimation is studied in Chapter 2. Theorem 2.2 of this chapter indicates that, under the assumption of α-mixing with polynomially decaying rate, if the one-dimensional (marginal) distribution function F has a positive derivative in a neighborhood of $\xi_p = F^{-1}(p), \quad p \in (0,1)$, the MBB method with the block length in a wide range of possible values provides a valid approximation to the distribution of the centered and scaled sample quantile $Z_n = \sqrt{n}(\hat{\xi}_n - \xi_p)$. The consistency of the MBB distribution function estimator is also supported by the numerical results from a small simulation study presented in Section 2.4. In the independence case, validity of Efron's (1979) bootstrap for the sample quantile was established by Bickel and Freedman (1981) and Singh (1981). Thus Chapter 2 extends this basic consistency result to the case of weakly dependent random variables for the MBB method.

In Chapter 3, we investigate the validity of the MBB estimation of the asymptotic variance of the normalized sample quantile. It is proved that, under the same set of conditions as required for the valid MBB distribution function approximation, if the random variables satisfy a fairly non-restrictive moment condition, the MBB variance estimator of the normalized sample quantile is strongly consistent (cf. Theorem 3.1). This is an extension of the consistency result of Ghosh, Parr, Singh and Babu (1984) from the i.i.d. set up to the situation where the random variables are weakly dependent. We note that the asymptotic variance of Z_n under weak dependence involves both an infinite sum of lag-covariances and the one-dimensional density function evaluated at the unknown quantile ξ_p (cf. (1.3)). The MBB resampling procedure captures both the effect of the dependence structure of the α-mixing stationary process and the effect of the nonlinear nature of the sample quantile, and thus, provides a unified way of approximating both parts using only one smoothing parameter, say, the block length ℓ.

In Chapter 4, we present a Berry-Esseen Theorem for sample quantiles based on α-mixing stationary random variables with polynomially decaying rate. It is pointed out that the classical normal approximation to the distribution of the sample quantile under weak dependence has accuracy of order $O(n^{-1/2})$.

The accuracy of the MBB distribution function estimation to the sample quantile under weak dependence is investigated in Chapter 5. We show that the rate of convergence of the MBB distribution function estimation of $G_n(x) = P(\sqrt{n}(\hat{\xi}_n - \xi_p) \leq x)$ is of order $O(n^{-1/4}(\log n)(\log\log n)^{1/4})$. This slow convergence rate is not unexpected considering that, in the i.i.d. case, the exact rate of convergence of Efron's (1979) bootstrapped sample quantile distribution estimation is of order $O(n^{-1/4}(\log\log n)^{1/2})$ (cf. Singh (1981)). However, unlike Efron's i.i.d. bootstrap approximation, smoothing may not improve the performance of the MBB distribution function estimation of sample quantiles based on weakly dependent observations. Thus, in terms of distribution function estimation of sample quantiles under dependence, the classical large sample approximation method outperforms the MBB method.

In the end, we discuss some possible future investigations along the line of this dissertation. As indicated in Section 2.1 of Chapter 2, the Bahadur representation allows one to express sample quantiles in terms of empirical functions which are more easily handled with. This idea was initiated by Bahadur (1966) for i.i.d. random variables. Sen (1972) and Babu and Singh (1978) established, respectively, the Bahadur representations for sample quantiles of ϕ-mixing random variables and the Bahadur representations of α-mixing random variables under exponential decay rate. Based on our knowledge, the most recent result for a Bahadur representation is due to Yoshihara (1995). He proposed a Bahadur representation for sample quantiles of bounded random variables under the structure of α-mixing with polynomial decay rate. It is of interest to extend Yoshihara's result to situations where the random variables are not necessarily bounded.

It is known that, in several respects studentizing before bootstrapping improves the accuracy of bootstrap distribution approximations (cf. Hall (1992)). Indeed, in the i.i.d. set up, Janas (1993) showed that studentizing sample quantiles by means of a kernel density estimate can improve the rate of the bootstrap approximation significantly. In weak dependence case, due to the complexity of the variance structure, the procedure of studentizing a sample quantile will be more complicated. The impact of studentizing on the MBB distribution approximation of sample quantiles based on weakly dependent data remains unknown.

It follows from Theorem 3.1 in Chapter 3 that, the MBB variance estimator of sample quantiles is strongly consistent. In the i.i.d. case, Hall and Martin (1988) showed that the exact rate of bootstrap variance estimator of sample quantiles is of order $O_p(n^{-1/4})$. Under dependence, the performance of the MBB variance estimator depends crucially on the smoothing parameter, say, the block length ℓ. Further investigation of the convergence rate of the MBB variance estimator for sample quantiles under weak dependence will be desirable.

It is known in the literature, the MBB approximations are very sensitive to the choice of block lengths. There have been several studies about the choice of block lengths for the MBB approximation. Among those are Hall, Horowitz and Jing (1995), Bühlmann and Künsch (1999), and Lahiri, Furukawa and Lee (2003). The optimal choice of block lengths for the MBB sample quantile estimations also needs further investigation.

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