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ON THE STABILITY OF OSCILLATIONS IN A CLASS OF NONLINEAR
FEEDBACK SYSTEMS CONTAINING NUMERATOR DYNAMICS

Iowa State University

Ph.D. 1984

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On the stability in oscillations in
a class of nonlinear feedback systems
containing numerator dynamics

by

Gary Steven Krenz

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of the
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TABLE OF CONTENTS

INTRODUCTION	1
The Describing Function	3
Quasi-Static Stability Condition	5
EXPLANATION OF DISSERTATION FORMAT	8
REVIEW OF RELATED LITERATURE	9
Existence of Limit Cycles in the Unforced Case	12
Existence of Limit Cycles in the Forced Case	14
Stability of Unforced Oscillations	15
PART I. QUALITATIVE ANALYSIS OF OSCILLATION ^c IN NONLINEAR CONTROL SYSTEMS: A DESCRIBING FUNCTION APPROACH	20
ABSTRACT	21
I. INTRODUCTION	22
II. NOTATION AND STATEMENT OF THE MAIN RESULT	23
III. ANALYSIS OF THE FEEDBACK SYSTEM	29
IV. COMPARISON WITH THE QUASI-STATIC STABILITY ANALYSIS OF LIMIT CYCLES	44
V. EXAMPLES	46
VI. CONCLUDING REMARKS	52
VII. REFERENCES	61
PART II. STABILITY ANALYSIS OF ALMOST SINUSOIDAL PERIODIC OSCILLATIONS IN NONLINEAR CONTROL SYSTEMS SUBJECTED TO NONCONSTANT PERIODIC INPUT	62
ABSTRACT	63
I. INTRODUCTION	64
II. RELATED RESULTS	66
III. STATEMENT OF MAIN RESULT	71
IV. ANALYSIS OF THE FEEDBACK SYSTEM	78
V. EXAMPLES	93
VI. CONCLUDING REMARKS	101
VII. REFERENCES	110

CONCLUSION	112
REFERENCES	114
ACKNOWLEDGEMENTS	118
APPENDIX A: THE INTEGRAL MANIFOLD THEOREM	119
APPENDIX B	170

INTRODUCTION

The use of control systems dates back to ancient Greece. As early as 300 B.C., the Greeks had working water clocks which relied upon feedback in order to control the water volume flow rate. In [28], Mayr provides an excellent description of the Ktesibios water clock as well as other developments which led to the successful use of early feedback systems in hatcheries, mills and steam engines.

Having recognized that feedback within a system introduces some very desirable properties, engineers were confronted with the task of designing control systems which incorporate feedback with few analytical design tools at their disposal. Great strides in control design were made by Nyquist [36], Black [7], and Bode [9]. These works and others created the design tools now considered part of the classical linear control theory.

However, it soon became apparent that systems such as those containing relays, spool valves or dynamic vibration mounts had nonlinear characteristics which had to be considered in the system design process. Even linear systems were found to have only a limited range of linearity and exhibited a saturation behavior when physical variables assumed large values.

By 1950, a method was advanced to deal with nonlinear elements in feedback systems. This method was based upon harmonic balance (see [10]) and became known, in engineering literature, as the describing function method. (An extensive list of references that deal with the various

applications of describing functions prior to 1968 can be found in Gelb and Vander Velde [15].)

In this dissertation, we examine the use of the single-input sinusoidal describing function to predict the existence and stability of almost sinusoidal limit cycles in a class of periodically forced and unforced single loop nonlinear control systems (see figure 1).

In figure 1, the control system is linear if superposition is valid. That is, if inputs r_1 and r_2 result in outputs y_1 and y_2 , respectively, then the input $\alpha r_1 + \beta r_2$ yields $\alpha y_1 + \beta y_2$. If superposition fails, then the system is said to be nonlinear.

Since we will be using differential equations to realize the control systems, a limit cycle will mean a closed orbit in state-space (usual Euclidian space), such that no other closed orbits can be found arbitrarily close to it (c.f., [20]).

Moreover, the stability concepts used in Part I and Part II of this dissertation follow the standard definitions found in most texts on ordinary differential equations (see, e.g., [11], [13], [17], [18] or [33]). In particular, consider the equation

$$(D) \quad dy/dt = f(t, y) .$$

Let $\phi(t)$ be some solution of (D) for $0 \leq t < \infty$ and $\psi(t, t_0, y_0)$ be a solution of (D) for which $\psi(t_0, t_0, y_0) = y_0$.

The solution $\phi(t)$ is said to be stable if, for every $\epsilon > 0$ and every $t_0 > 0$, there exists $\delta > 0$ such that whenever

$|\phi(t_0) - y_0| < \delta$, the solution $\psi(t, t_0, y_0)$ exists for $t > t_0$ and satisfies $|\phi(t) - \psi(t, t_0, y_0)| < \varepsilon$, for $t > t_0$.

The solution $\phi(t)$ is said to be asymptotically stable if it is stable and if there exists $\delta > 0$ such that whenever $|\phi(t_0) - y_0| < \delta$, the solution $\psi(t, t_0, y_0)$ approaches the solution $\phi(t)$ as $t \rightarrow \infty$.

A T-periodic solution $\phi(t)$ is said to be orbitally stable if there is a $\delta > 0$ such that any solution $\psi(t, t_0, y_0)$, with $|\phi(t_0) - y_0| < \delta$, tends to the orbit $\{\phi(t): t_0 \leq t \leq t_0 + T\}$.

The solution $\phi(t)$ is said to be unstable in the sense of Lyapunov, if it is not stable.

The Describing Function

Originally, the single-input sinusoidal describing function, simply referred to as the describing function, was physically motivated.

The following heuristic derivation can be found in [15, pp. 49-52, 110-120]:

Assume that $n(\cdot)$ is an odd nonlinearity, i.e., $n(-y) = -n(y)$, and that $y = A \sin \omega t$. Then, the output of $n(A \sin \omega t)$ can be represented by the Fourier series

$$n(A \sin \omega t) = \sum_{n=1}^{\infty} A_n(A, \omega) \sin[n \omega t + \phi_n(A, \omega)] .$$

The describing function, denoted by $N(A, \omega)$, is

$$N(A, \omega) = \frac{\text{phasor representation of output component at frequency } \omega}{\text{phasor representation of input component at frequency } \omega}$$

$$= \frac{A_1(A, \omega) \exp(i\phi_1(A, \omega))}{A} .$$

Thus, the describing function $N(A, \omega)$ is an attempt to generalize the linear theory transfer function concept to a nonlinear setting. In particular, $N(A, \omega)$ is an attempt to represent the gain of the fundamental component of the limit cycle due to $n(\cdot)$ ignoring the higher harmonics.

When using the describing function in the unforced case ($r = 0$), the following assumptions are made:

- (1) The system is in a steady state limit cycle.
- (2) No subharmonics are generated by the nonlinearity in response to a sinusoidal input.
- (3) The system attenuates the higher harmonics such that y is almost entirely sinusoidal (the low-pass filter hypothesis).

In this case, $N(A, \omega)$ is used to describe the effect of the nonlinearity upon the limit cycle (see figure 2).

The application of linear theory to the quasi-linearized system, yields

$$[1 + G(i\omega)N(A,\omega)]A = 0 ,$$

or, since $A > 0$,

$$1 + G(i\omega)N(A,\omega) = 0 .$$

Solutions of this equation yield approximate amplitudes and frequencies of the closed loop limit cycles.

For $N(A,\omega) \neq 0$, the above "describing function equation" is equivalent to

$$G(i\omega) = \frac{-1}{N(A,\omega)} .$$

Thus, solutions to the describing function equation correspond to intersections of the curves $G(i\omega)$ and $-1/N(A,\omega)$ (see figure 3).

In figure 3, arrows indicate the direction of increasing A on the $-1/N(A,\omega)$ locus and increasing ω on the $G(i\omega)$ frequency locus.

Quasi-Static Stability Condition

Continuing with the heuristic derivation from [15, pp. 121-125], let $U(A,\omega)$ and $V(A,\omega)$, respectively, denote the real imaginary parts of $1 + G(i\omega)N(A,\omega)$ and assume that

$$U(A_0,\omega_0) + i V(A_0,\omega_0) = 1 + G(i\omega_0)N(A_0,\omega_0) = 0 .$$

We now consider small perturbations in limit cycle amplitude, rate of change of amplitude, and frequency by introducing the following changes in the above equation:

$$A_0 \longrightarrow A_0 + \Delta A$$

$$\omega_0 \longrightarrow \omega_0 + \Delta\omega + i\Delta\sigma .$$

The perturbation in the rate of change of amplitude has been associated with the frequency term; a technique which is motivated by the thinking of the limit cycle in the form $A_0 \exp(i\omega_0 t)$. Hence, we have

$$U(A_0 + \Delta A, \omega_0 + \Delta\omega + i\Delta\sigma) + i V(A_0 + \Delta A, \omega_0 + \Delta\omega + i\Delta\sigma) = 0 .$$

By definition, ΔA , $\Delta\omega$, and $\Delta\sigma$ are small. Thus, the Taylor series expansion of the perturbed describing function equation about the point (A_0, ω_0) , valid to first-order terms, is

$$\frac{\partial U}{\partial A} \Delta A + \frac{\partial U}{\partial \omega} (\Delta\omega + i\Delta\sigma) + i \frac{\partial V}{\partial A} \Delta A + i \frac{\partial V}{\partial \omega} (\Delta\omega + i\Delta\sigma) = 0 .$$

After separating this equation into its real and imaginary parts and eliminating $\Delta\omega$ from the new set of equations, we obtain

$$\left(\frac{\partial U}{\partial A} \frac{\partial V}{\partial \omega} - \frac{\partial U}{\partial \omega} \frac{\partial V}{\partial A} \right) \Delta A = \left[\left(\frac{\partial U}{\partial \omega} \right)^2 + \left(\frac{\partial V}{\partial \omega} \right)^2 \right] \Delta\sigma .$$

For a limit cycle to be orbitally stable, a positive increment ΔA must lead to a positive $\Delta \sigma$. Hence, for the proposed limit cycle to be orbitally stable, it is necessary that

$$\frac{\partial U}{\partial A} \frac{\partial V}{\partial \omega} - \frac{\partial U}{\partial \omega} \frac{\partial V}{\partial A} > 0 .$$

This condition is known as the Loeb criterion.

The stability of a limit cycle may be checked graphically under the same assumptions which apply to the analytic test and the argument is the same as that of linear system stability. That is, if the limit cycle amplitude perturbation is positive, we require a stable system configuration in which energy is dissipated until the amplitude decays to its unperturbed value. On the other hand, if the amplitude perturbation is negative, we require an unstable system configuration in which energy is absorbed until the amplitude grows to its unperturbed value. This behavior guarantees an orbitally stable limit cycle.

EXPLANATION OF DISSERTATION FORMAT

This dissertation contains two papers, written by the author, which have been submitted for publication. These papers are labeled Part I and Part II. Although Part I and Part II address two different problems, there is some duplication, particularly in their introductory material.

This dissertation consists of four distinct parts - the general introductory material preceding Part I; Part I; Part II; and the material following Part II. In each of these parts, equations and highlighted items such as theorems and figures are numbered consecutively, but separately from the other parts of the dissertation. In addition, the reference numbers in the introductory part and the part following Part II refer to the list of references at the end of this dissertation. However, the reference numbers in Part I and Part II refer to the separate reference lists contained in those two parts.

REVIEW OF RELATED LITERATURE

According to Atherton [1], the describing function method is the nonlinear technique most frequently used in industrial design. The advantages of the describing function method lie in its ease of use and its suitability for design purposes (see [12], [15] or [21]).

However, one disadvantage of the describing function method is that it's an approximate method and, as such, will occasionally yield inaccurate predictions. The second disadvantage arises from how, or why, the method is used. Commonly, the system designer is interested in whether or not an instability may occur in the system being designed. Under the assumption that the instability will manifest itself as a simple, almost sinusoidal, limit cycle, the describing function method is used. However, the describing function will give no indication whether or not a more complicated system instability could occur.

Most authors have dealt with the first problem - the question of accuracy of the describing function method and what constitutes sufficient filtering. However, before examining individual contributions, we will review some well-known manipulations (see [20], [32], [37], or [40]) which play an important role in much of the work on existence of limit cycles in nonlinear feedback systems.

Consider the Hammerstein equation

$$x = -Lx + r ,$$

where L is a linear operator mapping a real Banach space B into itself and N maps B into itself (see figure 4).

In particular, we are interested in the case where L is described by its frequency response transfer function, where N denotes composition, and B is a Hilbert space:

$$\begin{aligned}
 B &= \{x \in L^2[0, T] : \frac{1}{T} \int_0^T x^2(t) dt < \infty\} \\
 &= \{x \in L^2[0, T] : x(t) = \sum_{k=-\infty}^{\infty} c_k \exp(ik\omega t), \quad T = 2\pi/\omega, \\
 &\quad \sum_{k=-\infty}^{\infty} |c_k|^2 < \infty, \quad c_{-k} = \overline{c_k}, \\
 &\quad c_k = \frac{1}{T} \int_0^T x(t) \exp(-ik\omega t) dt\} ;
 \end{aligned}$$

$y = Nx$, for $x \in B$, is defined by $y(t) = n[x(t)]$, with $n: \mathbb{R} \rightarrow \mathbb{R}$, such that $y \in L^2[0, T]$;

$$\begin{aligned}
 y &= Lx, \text{ for } x \in B, \quad x(t) = \sum_{k=-\infty}^{\infty} c_k \exp(ik\omega t), \\
 &\text{is defined by } y(t) = \sum_{k=-\infty}^{\infty} G(ik\omega) c_k \exp(ik\omega t),
 \end{aligned}$$

where $\sup_k |G(ik\omega)| < \infty$ and $G(-ik\omega) = \overline{G(ik\omega)}$; and

$r \in B$.

Here, $L^2[0,T]$ denotes the usual Lebesgue square-integrable functions from $[0,T]$ into \mathbb{R} , where \mathbb{R} denotes the set of real numbers. We will use the complete orthonormal system consisting of sines and cosines for B , and we will use the complex exponentials to represent the sines and cosines.

By restricting $n(\cdot)$, the condition $Nx \in L^2[0,T]$ is easily satisfied (e.g., $n(\cdot)$ continuous and slope bounded). Moreover, for

any $x \in B$, with $x(t) = \sum_{k=-\infty}^{\infty} c_k \exp(ik\omega t)$, we have

$$\sum_{k=-\infty}^{\infty} |G(ik\omega)c_k|^2 \leq \sup_k |G(ik\omega)|^2 \sum_{k=-\infty}^{\infty} |c_k|^2.$$

Thus, we see $Lx \in L^2[0,T]$ by the Riesz-Fischer Theorem [20].

Let M be a nonempty subset of the integers such that, if $n \in M$, then $-n \in M$. We now define the projection P_M on B by

$$(P_M x)(t) = \sum_{k \in M} c_k \exp(ik\omega t),$$

for all $x(t) = \sum_{k=-\infty}^{\infty} c_k \exp(ik\omega t) \in B$.

Let P_M^* denote the complementary projection on B , i.e., $P_M^* = I - P_M$.

By letting $x_0 = P_M x$ and $x_1 = P_M^* x$, we may write the Hammerstein equation as the system

$$(P) \quad x_0 + P_M \operatorname{LN}(x_0 + x_1) = P_M r ,$$

$$(C) \quad x_1 + P_M^* \operatorname{LN}(x_0 + x_1) = P_M^* r .$$

Observe that if $x_1 = r = 0$ and $M = \{-1, 1\}$ in the first equation, we arrive at

$$1 + G(i\omega)N(A, \omega) = 0 .$$

Existence of Limit Cycles in the Unforced Case

Kudrewicz [26] considered the case where $M = \{-1, 1\}$ and $Nx = n(x, x')$, such that n satisfies the Lipschitz condition

$$|n(x, y) - n(w, z)| \leq \alpha |x - w| + \beta |y - z| .$$

In addition, $n(0, 0) = 0$. The linear operator L is required to satisfy some technical assumptions - assumptions which insure that $G(s)$ is a good filter at $\omega = \omega_0$. Using a contraction mapping argument on the complementary equation (C) over the space B_c ,

$$B_c = \{x \in L^2[0, T] : \int_0^T x(t) \exp(-i\omega t) dt = 0, \quad T = 2\pi/\omega\} ,$$

he obtained a unique fixed-point in B_c which depends continuously on ω and A . Substituting this fixed-point into the equation (P) over the

first Fourier subspace and using index theory, he obtained the existence of at least one periodic solution with amplitude near A_0 and frequency near ω_0 .

Similar results were obtained independently by Bergen and Franks [5]. Their results included backlash hysteresis as well as other important nonlinearities and placed fewer restrictions upon the linear operator. By requiring $n(\cdot)$ to be odd symmetric, they were able to perform the contraction mapping arguments on a subspace of L^2 :

$$H(\omega) = \{x \in L^2[0,T] : x(t) = \sum_{k \text{ odd}} c_k \exp(ik\omega t), \quad T = 2\pi/\omega\}.$$

Mees and Bergen [30] further simplified the argument presented in [5] by requiring $n(\cdot)$ to be single-valued and

$$\alpha(x - y) \leq n(x) - n(y) \leq \beta(x - y), \quad x > y.$$

Furthermore, they were able to give a simple graphical interpretation of their results in terms of the $G(i\omega)$ and $-1/N(A)$ loci.

The results of Mees and Bergen were generalized by Skar et al. [38] to include interconnected systems. Bergen et al. [6] showed that the results of [30] and [32] could be combined to handle more nonlinearities than originally discussed in [30].

Other related papers include Blackmore [8], Duncan and Johnson [14], Knyazev [23], Kou and Han [24], Swern [41] and an important early paper by Bass [4].

For higher order Galerkin approximations, where $M = \{-n, -n+1, \dots, n-1, n\}$, using similar fixed-point arguments, see Krasnoselskii [25] or Mees [29].

Existence of Limit Cycles in the Forced Case

Sandberg [37] examined the existence of a periodic solution to the forced functional equation

$$y = LN[y + r] .$$

(Observe that, without loss of generality, Sandberg analyzed systems with positive feedback.) Here, the nonlinear function $n(\cdot)$ is assumed to be real valued and, for some α, β ($\beta > 0$),

$$(\alpha + \beta)/2 = 1 \quad \text{and}$$

$$\alpha(x - y) \leq n(x) - n(y) \leq \beta(x - y), \quad \text{for } x > y .$$

Moreover, the condition he imposed upon the system resulted in a global contraction argument on $L^2[0, T]$, where $r \in L^2[0, T]$. Thus, he was able to assert the existence of a unique periodic response to an arbitrary periodic input of the same period. He also pointed out that if $n(\cdot)$ is odd symmetric, then the argument may be applied to the previously mentioned space $H(\omega)$ to obtain a similar fixed-point result.

In addition, he was able to give an upper bound on the mean square error between the actual periodic solution and the predicted response.

Holtzman [19] analyzed essentially the same problem as Sandberg. However, he used a vastly different, local approach. Instead of using an L^2 space and Fourier series methods, he worked with a space of continuous periodic functions. Hence, his error estimates resulted in sup norm bounds between the predicted solution x_0 and the actual solution, rather than the mean square error bound obtained by Sandberg. However, his method requires that the operator $F = LN$ have a Fréchet derivative in a neighborhood Ω of the predicted response x_0 and that $\|F'(x)\| = \alpha < 1$, for all $x \in \Omega$.

Neither Sandberg's nor Holtzman's results apply to systems containing important nonlinearities such as discontinuous relays. Miller and Michel [32], by using some very general results concerning Volterra equations and weak solutions, obtained existence results for a very general class of sinusoidally forced nonlinear systems. These results included discontinuous odd nonlinearities. Moreover, their method provides a sup norm bound on the error between the predicted solution and the actual solution.

Stability of Unforced Oscillations

Many of the early investigations concerning the stability of almost sinusoidal limit cycles examined the quasi-linear equation rather than the actual nonlinear equation.

Johnson and Leach [22], for example, showed that the Loeb criterion is an exact criterion for second order systems - assuming a "reasonably good describing function approximation".

Recently, Mees and Chua [31] used Hopf bifurcation to analyze the stability of limit cycles in nonlinear feedback systems. Their results allowed very general linear operators, but restricted the odd nonlinearities to those which have at least four continuous derivatives.

On the other hand, Balasubramanian [3] examined the stability of limit cycles in feedback systems which contain relays. His analysis used certain properties of the exact oscillatory solution, the fact that the output of a relay is easily calculated, and some well-known theorems on equations of first variation (see [33, Chapters 5, 6, and 8]).

Miller et al. [35] presented a stability analysis of nonlinear feedback systems using only elementary arguments. Their proof, however, required the linear part of the system to be given by $G(s) = 1/q(s)$, where $q(s)$ is a polynomial over the reals.

In Part I of this dissertation, we extend the result of Miller et al. to the case where $G(s) = p(s)/q(s)$, for p and q polynomials over the reals. To accomplish this, we will require a result concerning the existence and stability of an integral manifold for a system of differential equations. Thus, in Appendix A, we present a version of the integral manifold theorem that is suitable for our needs. The proof is a modification of those found in Hale [16,17,18], Bogoliubov and Mitropolsky [10].

In Part II of this dissertation, we give a new stability analysis for a class of periodically forced nonlinear feedback systems. That analysis requires a result which is an obvious simplification of the integral manifold theorem.

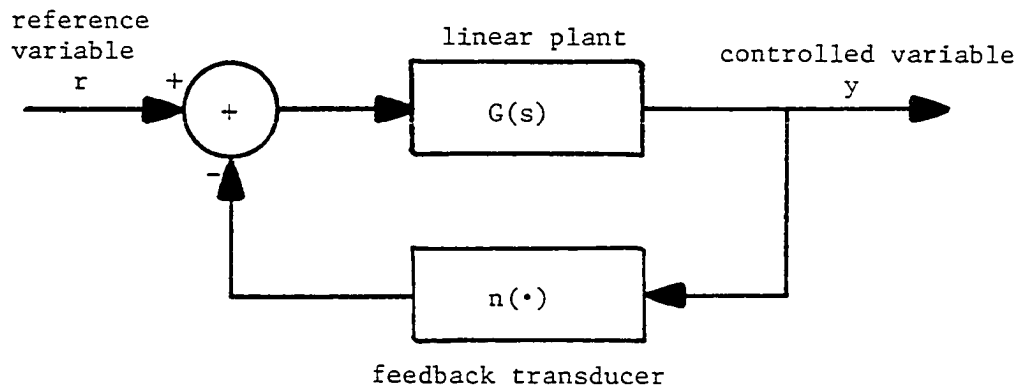


Figure 1. General control system block diagram

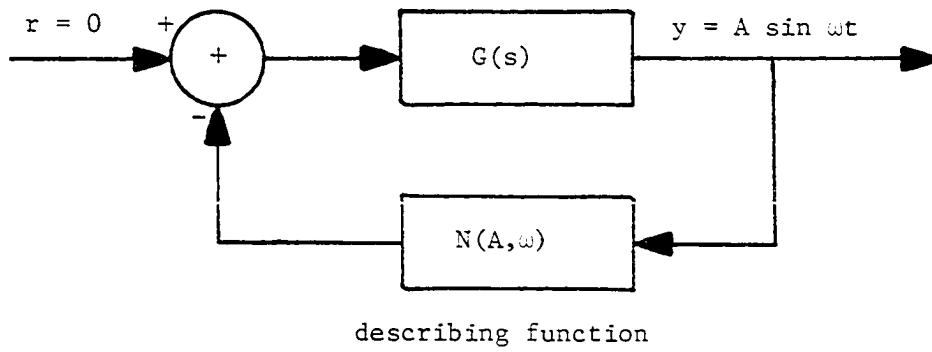


Figure 2. Quasi-linearized system

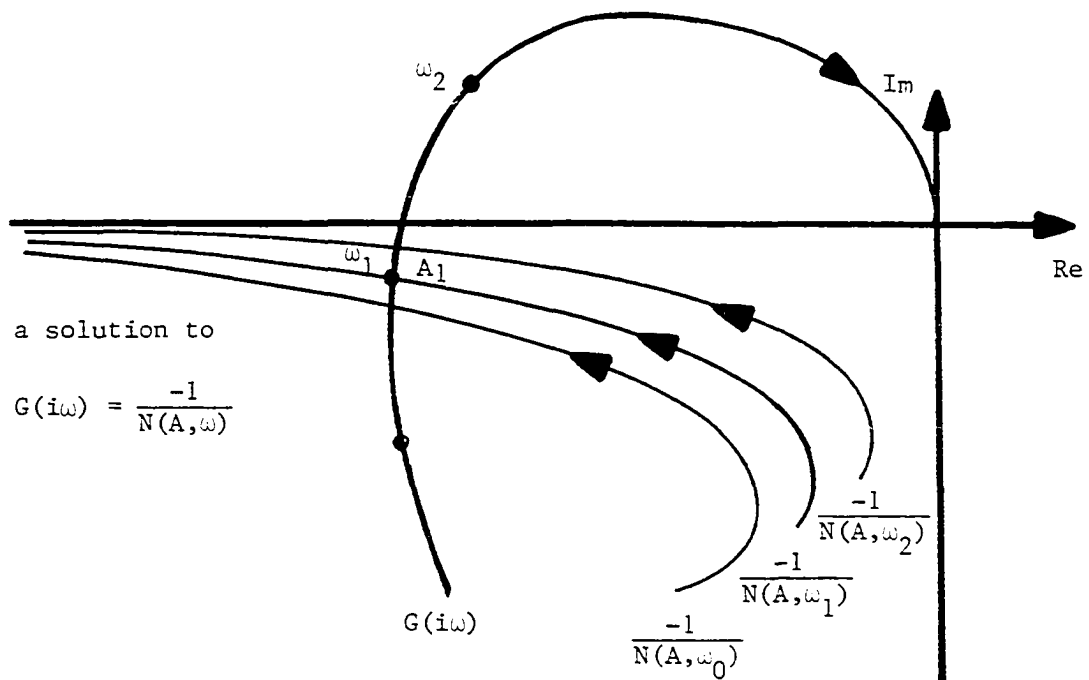


Figure 3. Polar plot of the graphical limit cycle determination.
 A_1 and ω_1 are the predicted limit cycle amplitude and frequency, respectively

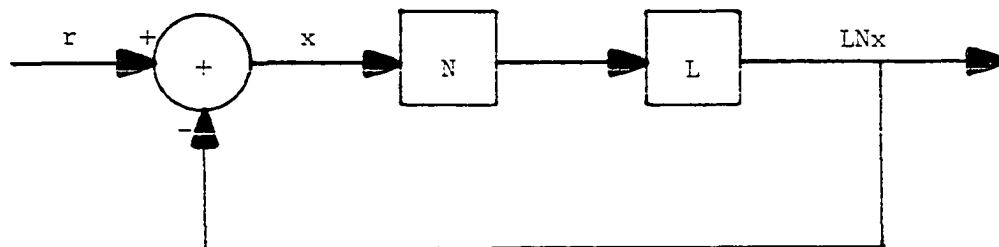


Figure 4. Block diagram of $x = -LNx + r$

PART I. QUALITATIVE ANALYSIS OF OSCILLATIONS IN
NONLINEAR CONTROL SYSTEMS: A DESCRIBING
FUNCTION APPROACH

ABSTRACT

We analyze the stability of oscillations in a wide class of nonlinear control systems which have their linear part given by the transfer function $G(s) = p(s)/q(s)$, where p and q are polynomials over the real numbers.

The analysis employs the classical single-input sinusoidal describing function, elementary control theory, and the theory of integral manifolds.

We demonstrate, by means of specific examples, how the present results can be used to obtain detailed information concerning the behavior of the solutions.

I. INTRODUCTION

In this paper, we extend some previous results [6] concerning the stability of limit cycles in certain nonlinear control systems. As in [6], we make use of several state-space coordinate transformations, averaging and a result on integral manifolds. Not only do our results justify the quasi-static stability analysis of limit cycles (Loeb's criterion) [2], they also offer rather detailed explanations of the behavior of solutions near the limit cycle.

We examine a class of control systems consisting of a linear part and a nonlinear part connected in a single loop feedback configuration. The linear part is given by a controllable and observable realization [3] of a real rational transfer function, $G(s)$, where the degree of the numerator of $G(s)$ is less than the degree of the denominator of $G(s)$. The nonlinearity is required to be an odd, continuous single-valued function with some additional piecewise differentiability properties. For this specific class of control systems, we give an easily computed stability criterion. We apply our result to specific examples.

This paper is divided into six sections, the first being an introduction to the problem. The second section explains some of the notation and gives the statement of our main result. The third section is devoted to the proof of our main result. In the fourth section, we investigate the relationship between our stability criterion and the quasi-static crossing condition. In the fifth section, we apply our main result to specific examples. We end the paper with some concluding remarks.

II. NOTATION AND STATEMENT OF THE MAIN RESULT

We will consider feedback systems of the form shown in figure 1, where $G(s)$ is a real rational function, i.e.,

$$G(s) = p(s)/q(s),$$

$$p(s) = \gamma_{J-1} s^{J-1} + \gamma_{J-2} s^{J-2} + \dots + \gamma_1 s + \gamma_0, \text{ and}$$

$$q(s) = s^J + \alpha_{J-1} s^{J-1} + \dots + \alpha_1 s + \alpha_0,$$

where $\alpha_k, \gamma_k \in \mathbb{R}$ for $k = 0, 1, 2, \dots, J-1$. Here, \mathbb{R} denotes the real line. We allow the leading coefficients of $p(s)$ to be zero, i.e., $0 \leq \deg p(s) \leq J-1$. Furthermore, we will assume that $p(s)$ and $q(s)$ have no common roots. We also assume that n is piecewise continuously differentiable and that $n''(y)$ exists and is uniformly continuous on all intervals, $\eta < y < \xi$, where $n'(y)$ exists and is continuous. For example, $n(y)$ could be the saturation function, given by

$$n(y) = \begin{cases} my & , \quad \text{if } |y| \leq d, \\ (\text{sgn } y)md & , \quad \text{if } |y| > d, \end{cases}$$

where m and d are positive constants.

Applying the describing function method [2] to the feedback system, we obtain the equation

$$(1) \quad 1 + G(i\omega) N(a) = 0,$$

where $N(a) = \frac{1}{\pi a} \int_0^{2\pi} n(a \cos \theta) \cos \theta d\theta$ is the sinusoidal-input describing function for the given nonlinearity $n(y)$. Suppose the pair (ω_0, a_0) solves (1), where $\omega_0 > 0$ and $a_0 > 0$. That is,

$$q(i\omega_0) + p(i\omega_0) N(a_0) = 0.$$

In addition, we assume a_0 is a value for which $n'(y)$ exists and $N(a_0) \neq 0$. For example, if $n(y)$ is the saturation function, then $a_0 \neq d$.

Since the feedback system in figure 1 has a natural canonical controllable and observable linear realization

$$(2) \quad \dot{x}_0 = Ax_0 + b_0 u,$$

$$(3) \quad y_0 = h_0^T x_0,$$

with transfer function $G(s) = f\{h_0^T e^{At} b_0\}$ (c.f. [3; Chapter 2], [7]), we will state our result for state-space solutions of (2). Here, A is the companion matrix for $q(s)$,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & \cdots & -\alpha_{J-1} \end{bmatrix},$$

$b_0 \in \mathbb{R}^J$, $b_0 = (0, 0, 0, \dots, 0, 1)^T$, $h_0^T = (\gamma_0, \gamma_1, \dots, \gamma_{J-1})$,
 $u = -n(y_0)$, T denotes the usual vector transpose and \mathcal{L} denotes the
 Laplace transform operator.

Definition: By an integral manifold for (2), we mean a surface, S , in
 real $(J+1)$ -dimensional space, such that if $(t_0, \xi_1, \dots, \xi_J) \in S$ and
 if $x_0(t)$ is any solution of (2) with initial condition
 $x_0(t_0) = (\xi_1, \dots, \xi_J)^T$, then $x_0(t)$ exists and $(t, x_0(t)^T) \in S$
 for all $t \in \mathbb{R}$.

For example, if $n(y) = N(a_0)y$, then (2) has an integral manifold

$$S_0 = \left\{ \left(t, \frac{a_0}{E} \sin(\omega_0 t + \tau), \frac{a_0 \omega_0}{E} \cos(\omega_0 t + \tau), \frac{-a_0 \omega_0^2}{E} \sin(\omega_0 t + \tau), \dots \right) : \right. \\
\left. -\infty < t < \infty, \quad 0 \leq \tau < 2\pi \right\},$$

where $E = |p(i\omega_0)|$. We will show that if two computable parameters,
 labelled $\hat{\beta}_1$ and $\hat{\beta}_2$, are sufficiently small, then (2) will have an
 integral manifold S_1 which approximates the surface S_0 . We will also
 state a stability criterion for S_1 .

First, however, we establish the following notation. Let

$$d(s) = q(s) + p(s) N(a_0) = d_0 s^J + d_1 s^{J-1} + \dots + d_{J-1} s + d_J,$$

$$\hat{\beta}_1 - i\hat{\beta}_2 = 2/(\omega_0 d'(i\omega_0)), \quad \text{and} \quad \Delta_1 + i\Delta_2 = (\hat{\beta}_1 - i\hat{\beta}_2) p(i\omega_0).$$

Clearly, $d_0=1$. We now define the Hurwitz determinants:

$$D_1 = d_1 \quad ,$$

$$D_2 = \begin{bmatrix} d_1 & d_3 \\ d_0 & d_2 \end{bmatrix} \quad ,$$

$$D_3 = \begin{bmatrix} d_1 & d_3 & d_5 \\ d_0 & d_2 & d_4 \\ 0 & d_1 & d_3 \end{bmatrix} \quad ,$$

and so on, where we let $d_j=0$ if $j > J$. Let D_0 be defined by

$$D_0 = \Delta_1 a_0 N'(a_0)/2.$$

We now state our main result.

Theorem 1: Suppose that system (2) - (3) satisfies

- (H-1) All d_j are real for $0 < j < J$ with $d_0=1$ and that $D_j \neq 0$, for $0 < j < J-2$.
- (H-2) The function n is an odd, continuous, piecewise continuously differentiable function. Moreover, on any interval $\eta < y < \xi$ where $n'(y)$ exists and is continuous, $n''(y)$ also exists and is uniformly continuous. The describing function for $n(y)$ will be denoted by $N(a)$.
- (H-3) There exists $\omega_0 > 0$ and $a_0 > 0$ such that $d(s)$ has simple roots $\pm i\omega_0$. The remaining roots of $d(s)$ must be noncritical. In addition, a_0 must be a point of continuity of $n'(y)$ and $N(a_0) \neq 0$.

If $\hat{\beta}_1$ and $\hat{\beta}_2$ are sufficiently small, then there is a small neighborhood N_0 of S_0 with the following properties:

- (i) An integral manifold S_1 of (2) lies inside N_0 .
- (ii) If $(t_0, n^T) = (t_0, n_1, \dots, n_J) \in N_0$, but $(t_0, n^T) \notin S_1$, then the solution of (2) satisfying $x_0(t_0) = n$ must leave N_0 in finite time. Hence, S_1 is the only integral manifold of (2) near S_0 .
- (iii) If $D_j > 0$ for $j=0, 1, \dots, J-2$ and $d_j > 0$ for $j=1, 2, \dots, J-1$, then S_1 is locally asymptotically stable.
- (iv) If one of the D_j , $0 \leq j \leq J-2$ or d_j , $1 \leq j \leq J-1$, is negative, then the integral manifold S_1 is unstable in the sense of Lyapunov.

Observe that conclusion (i) of the theorem states the existence of an integral manifold $S_1 \subseteq N_0$ near S_0 . This corresponds to a deformed torus fitted tightly around a predicted state-space limit cycle. In addition, N_0 will generally have a small diameter. Hence, a solution starting on S_1 will appear as if it is periodic, with radian frequency near ω_0 and amplitude near a_0 . Note that this result differs substantially from those which deal with the existence of observed limit cycles (for example, see the elegant results of [1] or [4]).

Finally, suppose $A_1 \in \mathbb{R}^J \times \mathbb{R}^J$, $b_1 \in \mathbb{R}^J$, $c \in \mathbb{R}^J$, $u \in \mathbb{R}$ and $u = -n(y)$ are such that the system

$$(2') \quad \dot{x}_1' = A_1 x_1 + b_1 u,$$

$$(3') \quad y = c^T x_1,$$

with transfer function $G(s)=p(s)/q(s)$, is controllable and observable. Then, by results in control theory (c.f. [3] or [7]), there is a nonsingular change of coordinates $x_1 = Px_0$, such that the system (2') - (3') is linearly equivalent to the system (2) - (3).

This yields the following corollary:

Corollary: Suppose that the system (2') - (3') satisfies hypotheses (H-1), (H-2) and (H-3) of Theorem 1. If $\hat{\beta}_1$ and $\hat{\beta}_2$ are sufficiently small, then there is a small neighborhood $PN_0 = \{(t, [Px]^T) : (t, x^T) \in N_0\}$ of $PS_0 = \{(t, [Px]^T) : (t, x^T) \in S_0\}$ with the following properties:

- (i) An integral manifold $PS_1 = \{(t, [Px]^T) : (t, x^T) \in S_1\}$ of (2') lies inside PN_0 .
- (ii) If $(t_0, \eta^T) \in PN_0$, but $(t_0, \eta^T) \notin PS_1$, then the solution of (2') satisfying $x_1(t_0) = \eta$ must leave PN_0 in finite time. Hence, PS_1 is the only integral manifold of (2') near PS_0 .
- (iii) If $D_j > 0$, for $j=0,1, \dots, J-2$ and $d_j > 0$, for $j=1,2, \dots, J-1$, then PS_1 is locally asymptotically stable.
- (iv) If one of the D_j , $0 \leq j \leq J-2$ or d_j , $1 \leq j \leq J-1$ is negative, then the integral manifold PS_1 is unstable in the sense of Lyapunov. ■

III. ANALYSIS OF THE FEEDBACK SYSTEM

Observe that the system (2) - (3), for the particular control $u = -n(y_0)$, is equivalent to the scalar equation

$$(4) \quad q(D)z + n(p(D)z) = 0,$$

where $D = \frac{d}{dt}$ and $x_0^T = (z, Dz, \dots, D^{J-1}z)$. Using the change of variables $\tau = \omega_0 t$, we see that (4) is equivalent to

$$(5) \quad d(\omega_0 \frac{d}{d\tau}) z = N(a_0)p(\omega_0 \frac{d}{d\tau})z - n(p(\omega_0 \frac{d}{d\tau})z),$$

where $d(s) = q(s) + p(s)N(a_0)$, as in Section II. Letting $\hat{p}(s) = d(\omega_0 s) \omega_0^{-J}$ and $n_1(y) = N(a_0)y - n(y)$, we can write (5) as

$$\hat{p}(\frac{d}{d\tau})z = n_1(p(\omega_0 \frac{d}{d\tau})z) \omega_0^{-J}.$$

In phase space, this becomes

$$(6) \quad v' = B_0 v + b_0 n_1(h_1^T v) \omega_0^{-J},$$

where

$$B_0 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \frac{-d_J}{\omega_0^J} & \frac{-d_{J-1}}{\omega_0^{J-1}} & \frac{-d_{J-2}}{\omega_0^{J-2}} & \dots & \frac{-d_1}{\omega_0} \end{bmatrix},$$

$$v^T = (z, \frac{dz}{d\tau}, \dots, \frac{d^{J-1}z}{d\tau^{J-1}})$$

and

$$h_1^T = (\gamma_0, \gamma_1 \omega_0, \gamma_2 \omega_0^2, \dots, \gamma_{J-1} \omega_0^{J-1}).$$

Having normalized time, we will simply replace τ by t and analyze (6).

Since B_0 is the companion matrix for $\hat{p}(s)$, we see that eigenvalues of B_0 are the roots of $\hat{p}(s)$. Thus, B_0 has two simple eigenvalues $s_1=i$ and $s_2=-i$. The remaining eigenvalues of B_0 are labelled s_3, s_4, \dots, s_J , with multiple eigenvalues repeated in the list as often as their multiplicities. Define $a(s) \in \mathbb{R}^J$ by

$$a(s) = (1, s, s^2, \dots, s^{J-1})^T.$$

Clearly, $a(i)$ is a [right] eigenvector of B_0 which corresponds to the eigenvalue $s_1=i$. Define $\xi_1, \xi_2 \in \mathbb{R}^J$ by

$$\xi_1 = \operatorname{Re} a(i) = (1, 0, -1, 0, 1, 0, \dots)^T$$

and

$$\xi_2 = \operatorname{Im} a(i) = (0, 1, 0, -1, 0, \dots)^T.$$

We now define vectors $\xi_3, \xi_4, \dots, \xi_J$. Suppose that s_k is a root of $p(s)$ with multiplicity $m_k > 1$, that is,

$$s_k = s_{k+1} = \dots = s_{k+m_k-1}.$$

Then, we define

$$\xi_k = \frac{1}{0!} a(s_k) = (1, s_k, s_k^2, \dots, s_k^{J-1})^T,$$

$$\begin{aligned} \xi_{k+1} &= \frac{1}{1!} a'(s_k) = (0, 1, 2s_k, \dots, (J-1)s_k^{J-2})^T, \\ &\vdots \end{aligned}$$

$$\xi_{k+m_k-1} = \frac{1}{(m_k-1)!} a^{(m_k-1)}(s_k).$$

Here, $\{\xi_k, \xi_{k+1}, \dots, \xi_{k+m_k-1}\}$ forms a chain of generalized eigenvectors of B_0 corresponding to the root s_k , i.e.,

$$(B_0 - s_k I) \xi_{k+j} = \xi_{k+j-1},$$

for $j = m_k-1, m_k-2, \dots, 1$ and $(B_0 - s_k I) \xi_k = 0$. Letting

$B = [\xi_1, \xi_2, \dots, \xi_J] \triangleq [b_{ij}]_{J \times J}$ and setting $B y = v$, we obtain

$$(7) \quad y' = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & & c \end{array} \right] y + B^{-1} b_0 n_1 (h_1^T B y) \omega_0^{-J}.$$

Using the notation

$$x_3 = (y_3, y_4, \dots, y_J)^T$$

and

$$x_6 = h_1^T [\xi_3, \xi_4, \dots, \xi_J] x_3,$$

we see that (7) is equivalent to

$$(8) \quad y' = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & & c \end{array} \right] y + \hat{\beta} n_1 (\operatorname{Re} p(i\omega_0) y_1 + \operatorname{Im} p(i\omega_0) y_2 + x_6),$$

where $\hat{\beta}$ denotes the vector $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2, \dots, \hat{\beta}_J)^T = \omega_0^{-J} B^{-1} b_0$.

Applying the Van der Pol transformation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \Phi(t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where

$$\Phi(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix},$$

to (8), we obtain

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \Phi(-t) \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} n_1 (\operatorname{Re} p(i\omega_0) [\cos t x_1 + \sin t x_2] +$$

$$\operatorname{Im} p(i\omega_0) [-\sin t x_1 + \cos t x_2] + x_6), \quad (9)$$

$$x_3' = Cx_3 + \begin{bmatrix} \hat{\beta}_3 \\ \vdots \\ \hat{\beta}_J \end{bmatrix} n_1 (\operatorname{Re} p(i\omega_0) [\cos t x_1 + \sin t x_2] +$$

$$\operatorname{Im} p(i\omega_0) [-\sin t x_1 + \cos t x_2] + x_6).$$

Observe that equation (9) has the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = f(t, x_1, x_2, x_6),$$

$$x_3' = Cx_3 + g(t, x_1, x_2, x_6),$$

where f and g are defined in the obvious manner.

Let $f_0(x_1, x_2) = \frac{1}{2\pi} \int_0^{2\pi} f(t, x_1, x_2, 0) dt$ and

$u(t, x_1, x_2) = \int_0^t \{f(s, x_1, x_2, 0) - f_0(x_1, x_2)\} ds$. Since u has the form considered in [5], we see that u is continuous, 2π -periodic in t , C^1 in (t, x_1, x_2) and u_{x_i} are Lipschitz continuous in (x_1, x_2) . These conditions on u will be needed later in order to use Theorem 2 of [5].

We now apply averaging to (9). This is accomplished by employing the following change of variables:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = w + u(t, w),$$

where

$$w = [w_1, w_2]^T.$$

Since

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = w' + u_t(t, w) + u_w(t, w)w' = f(t, x_1, x_2, x_6),$$

where $u_w(t, w) = \left[\frac{\partial u_i}{\partial x_j} \right]_{2 \times 2}$ is the Jacobian matrix of $u(t, w)$, and

since $u_t(t, w) = f(t, w, 0) - f_0(w)$, we see that

$$w' = f_0(w) + [I + u_w(t, w)]^{-1}$$

$$\{f(t, w + u(t, w), x_6) - f(t, w, 0) - u_w(t, w) f_0(w)\}.$$

Let $E = |p(i\omega_0)|$, $F = \arg p(i\omega_0)$, $w_1 = a \cos D$ and

$w_2 = a \sin D$. Then, we have

$$\begin{aligned} (10) \quad f_0(w) &= \frac{1}{2\pi} \int_0^{2\pi} \begin{bmatrix} \hat{\beta}_1 \cos t - \hat{\beta}_2 \sin t \\ \hat{\beta}_1 \sin t + \hat{\beta}_2 \cos t \end{bmatrix} \\ &\quad \cdot n_1(\operatorname{Re} p(i\omega_0)(\cos t w_1 + \sin t w_2) \\ &\quad + \operatorname{Im} p(i\omega_0)(-\sin t w_1 + \cos t w_2)) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \begin{bmatrix} \hat{\beta}_1 \cos t - \hat{\beta}_2 \sin t \\ \hat{\beta}_1 \sin t + \hat{\beta}_2 \cos t \end{bmatrix} \\ &\quad \cdot n_1(aE \cos(t + F - D)) dt. \end{aligned}$$

From the definition of N_1 , we have

$$\begin{aligned} \frac{e^{i[F-D]} N_1(aE) aE}{2} &= \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} n_1(aE \cos \theta) d\theta e^{i[F-D]} \\ &= \frac{1}{2\pi} \int_{D-F}^{2\pi+D-F} [\cos t - i \sin t] \\ &\quad \cdot n_1(aE \cos(t + F - D)) dt. \end{aligned}$$

After separating the real and imaginary parts and using periodicity of the integrands, we obtain

$$\begin{aligned} \frac{1}{2} aE N_1(aE) \cos(F-D) &= \frac{1}{2\pi} \int_0^{2\pi} \cos t n_1(aE \cos(t+F-D)) dt, \\ -\frac{1}{2} aE N_1(aE) \sin(F-D) &= \frac{1}{2\pi} \int_0^{2\pi} \sin t n_1(aE \cos(t+F-D)) dt. \end{aligned}$$

Applying the appropriate trigonometric identities yields

$$\begin{aligned} \frac{1}{2} N_1(aE) [\operatorname{Re} p(i\omega_0) w_1 + \operatorname{Im} p(i\omega_0) w_2] \\ (11) \quad = \frac{1}{2\pi} \int_0^{2\pi} \cos t n_1(aE \cos(t+F-D)) dt \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} N_1(aE) [\operatorname{Re} p(i\omega_0) w_2 - \operatorname{Im} p(i\omega_0) w_1] \\ (12) \quad = \frac{1}{2\pi} \int_0^{2\pi} \sin t n_1(aE \cos(t+F-D)) dt. \end{aligned}$$

From (10), (11), and (12), we obtain

$$(13) \quad f_0(w) = \frac{1}{2} N_1(aE) \begin{bmatrix} \hat{\beta}_1 [\operatorname{Re} p(i\omega_0) w_1 + \operatorname{Im} p(i\omega_0) w_2] \\ - \hat{\beta}_2 [\operatorname{Re} p(i\omega_0) w_2 - \operatorname{Im} p(i\omega_0) w_1] \\ \hat{\beta}_1 [\operatorname{Re} p(i\omega_0) w_2 - \operatorname{Im} p(i\omega_0) w_1] \\ + \hat{\beta}_2 [\operatorname{Re} p(i\omega_0) w_1 + \operatorname{Im} p(i\omega_0) w_2] \end{bmatrix}.$$

Recalling that $\Delta_1 + i\Delta_2 = (\hat{\beta}_1 - i\hat{\beta}_2) p(i\omega_0)$, we see that (13) is equivalent to

$$(14) \quad f_0(w) = 1/2 N_1(aE) \begin{bmatrix} \Delta_1 w_1 + \Delta_2 w_2 \\ -\Delta_2 w_1 + \Delta_1 w_2 \end{bmatrix}.$$

Let $f_1 = (f_{11}, f_{12})^T$ be the vector defined by

$$f_1(t, w, x_6) = [I + u_w(t, w)]^{-1} [f(t, w + u(t, w), x_6) - f(t, w, 0) - u_w(t, w)f_0(w)].$$

We see that (9) takes the form

$$(15) \quad w' = f_0(w) + f_1(t, w, x_6),$$

$$(16) \quad x_3' = Cx_3 + g(t, w + u(t, w), x_6).$$

The introduction of polar coordinates

$$w_1 = \left(r + \frac{a_0}{E}\right) \sin \theta,$$

$$w_2 = \left(r + \frac{a_0}{E}\right) \cos \theta,$$

transforms (15) into

$$(17) \quad r' = \frac{\Delta_1}{2} \left(r + \frac{a_0}{E} \right) N_1(Er + a_0) + \sin \theta f_{11}(t, w, x_6) \\ + \cos \theta f_{12}(t, w, x_6),$$

$$(18) \quad \theta' = \frac{\Delta_2}{2} N_1(Er + a_0) + [\cos \theta f_{11}(t, w, x_6) \\ - \sin \theta f_{12}(t, w, x_6)] / \left(r + \frac{a_0}{E} \right).$$

Setting $h(r) = \Delta_1 \left(r + \frac{a_0}{E} \right) N_1(Er + a_0)/2$, we see that

$$(i) \quad h(0) = 0, \quad \text{and}$$

$$(ii) \quad h'(0) = -\Delta_1 a_0 N'(a_0)/2 = -D_0.$$

If we assume either $h'(0) > 0$ or $h'(0) < 0$, then the function $h(r)$ has a simple zero at $r=0$. In addition, if we consider

$$(19) \quad r' = h(r),$$

we see that the trivial solution of (19) is asymptotically stable when $h'(0) < 0$, while the trivial solution of (19) is unstable if $h'(0) > 0$. This information is beneficial since (17) is a perturbation of (19), provided we show that f_{11} and f_{12} are, in some sense, small.

Two more transformations are applied to the x_3 coordinate. First, define $x_4 = B_1 x_3$, where B_1 is the submatrix of B given by

$$B_1 = \begin{bmatrix} b_{13} & b_{14} & \dots & b_{1J} \\ \vdots & & & \\ b_{J-2,3} & b_{J-2,4} & \dots & b_{J-2,J} \end{bmatrix},$$

while x_4 is a $J-2$ dimensional column vector,

$$x_4 = (x_{4,1}, \dots, x_{4,J-2})^T.$$

Observe that

$$\begin{aligned} x_6 &= h_1^T [\xi_3, \xi_4, \dots, \xi_J] x_3 \\ &= h_1^T [\xi_3, \xi_4, \dots, \xi_J] B_1^{-1} x_4. \end{aligned}$$

From (9), we see that x_4 satisfies

$$x_4' = [B_1 C B_1^{-1}] x_4 + B_1 \begin{bmatrix} \hat{\beta}_3 \\ \vdots \\ \hat{\beta}_J \end{bmatrix} n_1 (\operatorname{Re} p(i\omega_0) y_1 + \operatorname{Im} p(i\omega_0) y_2 + x_6).$$

The matrix $C_1 \triangleq B_1 C B_1^{-1}$ turns out to be the companion matrix of the $(J-2)$ -th degree polynomial,

$$\prod_{j=3}^J (s - s_j) = s^{J-2} + \hat{a}_{J-3} s^{J-3} + \dots + \hat{a}_1 s + \hat{a}_0.$$

This means C_1 has the form

$$C_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{J-3} \end{bmatrix}.$$

Moreover, since

$$b_0 = BB^{-1}b_0 = B \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_J \end{bmatrix} \omega_0^J,$$

we have

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \hat{\beta}_1 \hat{\xi}_1 + \hat{\beta}_2 \hat{\xi}_2 + B_1 \begin{bmatrix} \hat{\beta}_3 \\ \vdots \\ \hat{\beta}_J \end{bmatrix},$$

where $\hat{\xi}_1$ and $\hat{\xi}_2$ are, respectively, the first and second columns of B with their $J-1$ and J -th entries removed. Clearly,

$$B_1 \begin{bmatrix} \hat{\beta}_3 \\ \vdots \\ \hat{\beta}_J \end{bmatrix} = -[\hat{\beta}_1 \hat{\xi}_1 + \hat{\beta}_2 \hat{\xi}_2].$$

This gives us

$$(20) \quad \begin{aligned} \mathbf{x}'_4 = & C_1 \mathbf{x}_4 - [\hat{\beta}_1 \hat{\xi}_1 + \hat{\beta}_2 \hat{\xi}_2] n_1 (h_1^T [\xi_3, \dots, \xi_J] B_1^{-1} \mathbf{x}_4 + \\ & (Er + a_0) \sin(t + \theta + F) + E \cos(t + F) u_1(t, w) + E \sin(t + F) u_2(t, w)). \end{aligned}$$

Letting $\varepsilon = [\hat{\beta}_1^2 + \hat{\beta}_2^2]^{1/2}$ and setting $\mathbf{x}_4 = \varepsilon^{1/2} \mathbf{x}_5$ we have

$$(21) \quad \begin{aligned} \mathbf{x}'_5 = & C_1 \mathbf{x}_5 - \frac{1}{\sqrt{\varepsilon}} [\hat{\beta}_1 \hat{\xi}_1 + \hat{\beta}_2 \hat{\xi}_2] n_1 (\sqrt{\varepsilon} h_1^T [\xi_3, \dots, \xi_J] B_1^{-1} \mathbf{x}_5 + \\ & (Er + a_0) \sin(t + \theta + F) + E \cos(t + F) u_1(t, w) + E \sin(t + F) u_2(t, w)). \end{aligned}$$

The term $E \cos(t + F) u_1(t, w) + E \sin(t + F) u_2(t, w)$ is $\mathcal{O}(\varepsilon)$, with Lipschitz constants of order ε , for the variables r and θ (c.f. [5; section IV]). The definitions of ε and $f(t, x_1, x_2, x_6)$ imply that

$$\begin{aligned} f_1(t, w, x_6) &= [I + u_w(t, w)]^{-1} [f(t, w + u(t, w), x_6) - f(t, w, 0) - u_w(t, w) f_0(w)] \\ &= [f(t, w + u(t, w), x_6) - f(t, w, 0)] + \mathcal{O}(\varepsilon^2) \\ &= \begin{bmatrix} \hat{\beta}_1 \cos t - \hat{\beta}_2 \sin t \\ \hat{\beta}_1 \sin t + \hat{\beta}_2 \cos t \end{bmatrix} [n_1 (\sqrt{\varepsilon} h_1^T [\xi_3, \xi_4, \dots, \xi_J] B_1^{-1} \mathbf{x}_5 \\ &\quad + (Er + a_0) \sin(t + \theta + F) + \mathcal{O}(\varepsilon)) \\ &\quad - n_1 ((Er + a_0) \sin(t + \theta + F))] + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Hence, (17) and (18) take the form

$$(22) \quad r' = \frac{\Delta_1}{2} \left(r + \frac{a_0}{E} \right) N_1(Er + a_0) + [\hat{\beta}_1 \sin(t+\theta) + \hat{\beta}_2 \cos(t+\theta)] \cdot$$

$$[n_1(\sqrt{\epsilon} h_1^T [\xi_3, \dots, \xi_J] B_1^{-1} x_5 + (Er + a_0) \sin(t+\theta+F) + \mathcal{O}(\epsilon)) -$$

$$n_1((Er + a_0) \sin(t+\theta+F))] + \mathcal{O}(\epsilon^2),$$

$$(23) \quad \theta' = \frac{\Delta_2}{2} N_1(Er + a_0) + [\hat{\beta}_1 \cos(t+\theta) - \hat{\beta}_2 \sin(t+\theta)] \cdot$$

$$[n_1(\sqrt{\epsilon} h_1^T [\xi_3, \dots, \xi_J] B_1^{-1} x_5 + (Er + a_0) \sin(t+\theta+F) + \mathcal{O}(\epsilon^2))$$

$$- n_1((Er + a_0) \sin(t+\theta+F))] / \left(r + \frac{a_0}{E} \right) + \mathcal{O}(\epsilon^2).$$

The equations (21) - (23) satisfy the hypotheses of the invariant manifold theorem in [5]. Hence, there is an invariant manifold S_ϵ which, when ϵ is sufficiently small, is near the set $S = \{(t, \theta, 0, 0) : -\infty < t, \theta < \infty\}$. This integral manifold is locally asymptotically stable if the trivial solution of

$$(24) \quad r' = h(r), \quad x_5' = C_1 x_5$$

is asymptotically stable, i.e., if $D_j > 0$ for $j = 0, 1, \dots, J-2$. The integral manifold is unstable, in the sense of Lyapunov, if (24) is noncritical and unstable, i.e., if one of the D_j ,

$0 < j < J-2$ or d_j , $1 < j < J-1$, is negative. This implies that, for the original feedback system, there is an invariant integral manifold

S_1 in $R \times R^J$ space near

$$S_0 = \left\{ \left(t, \frac{a_0}{E} \sin(\omega_0 t + \tau), \frac{a_0}{E} \omega_0 \cos(\omega_0 t + \tau), \dots \right) : t \in R, 0 \leq \tau < 2\pi \right\}$$

which has the desired properties. ■

IV. COMPARISON WITH THE QUASI-STATIC STABILITY ANALYSIS OF LIMIT CYCLES

Let $G(i\omega) = q_1(\omega) + iq_2(\omega)$. We assume the Nyquist diagram for $G(s)$ indicates the noncritical roots of the describing function equation have negative real parts. This assumption is equivalent to requiring $D_k > 0$, for $k=1,2, \dots, J-2$. Then, graphically, the stability of the system depends on how the $G(i\omega)$ locus intersects the $\frac{-1}{N(a)}$ locus at the point $G(i\omega_0) = \frac{-1}{N(a_0)}$ (see figure 2 or [2]). This graphical stability condition can be expressed analytically by

$$N'(a_0)q_2'(\omega_0) < 0,$$

for $N(a)$ real and $N(a_0) \neq 0$. Observing that

$$\frac{d}{d\omega} G(i\omega) = iG'(i\omega),$$

we see that the graphical condition is equivalent to

$$N'(a_0) \operatorname{Re} G'(i\omega) < 0.$$

As stated in the theorem, we have asymptotic stability if

$$\Delta_1 N'(a_0) > 0.$$

But, since $\Delta_1 + i\Delta_2 = (\hat{\beta}_1 - i\hat{\beta}_2)p(i\omega_0) = \frac{-2}{\omega_0} \frac{1}{G'(i\omega_0)N^2(a_0)}$, we see that $\Delta_1 N'(a_0) > 0$ is equivalent to

$$\operatorname{Re} \overline{G'(i\omega_0)} N'(a_0) < 0.$$

That is, our analytic condition, $D_0 > 0$, is equivalent to the quasi-static crossing criterion for a stable system configuration.

V. EXAMPLES

We now give an example to which, when $\alpha > 3$ is sufficiently large, we may apply Theorem 1. Let $a_0 > 0$, $n(y)$ and $N(a_0)$ be fixed, but, for the moment, unknown. Set

$$\omega_0 = 1 ,$$

$$p(s) = (s+1)/\sqrt{2} ,$$

$$q(s) = s^4 + (\alpha+2)s^3 + (2\alpha+1)s^2 + \left(\alpha + 2 - \frac{N(a_0)}{\sqrt{2}} \right) s + 2\alpha - \frac{N(a_0)}{\sqrt{2}} .$$

Thus,

$$d(s) = d(\omega_0 s) = q(s) + p(s)N(a_0) = (s^2+1)(s+\alpha)(s+2),$$

$$E = 1 ,$$

$$F = \frac{\pi}{4} ,$$

$$\hat{\beta}_1 - i\hat{\beta}_2 = \frac{-(\alpha+2) - (2\alpha-1)i}{5(\alpha^2+1)} ,$$

$$5 \epsilon^2 = (\alpha^2+1)^{-1} ,$$

$$\Delta_1 + i\Delta_2 = \frac{(\alpha-3) + (-1-3\alpha)i}{5\sqrt{2}(\alpha^2+1)} = \epsilon \frac{(\alpha-3) + (-1-3\alpha)i}{\sqrt{10(\alpha^2+1)}} ,$$

$$x_6 = \sqrt{\frac{\varepsilon}{2}} x_5 ,$$

$$1 > \varepsilon^{-1} |\Delta_1| > \frac{1}{\sqrt{20}} ,$$

each component of e^{Ct} is bounded by $6e^{-2t}$ for all $t > 0$,

$$D_0 = \frac{\varepsilon(\alpha-3)}{2\sqrt{10(\alpha^2+1)}} a_0 N'(a_0) ,$$

$$D_1 > 0 ,$$

$$D_2 > 0 .$$

Moreover, since a_0 , $n(y)$, $N(a_0)$, and $p(i\omega_0)$ remain fixed as α increases, we see that $f_0(x_1, x_2)$ and $u(t, x_1, x_2)$ tend to 0 as $\alpha \rightarrow +\infty$. Thus, after choosing a nonlinear function, $n(y)$, and an $a_0 > 0$ which satisfy (H-2) and (H-3) of Theorem 1, we take $\alpha > 3$ sufficiently large in order to arrive at a sufficiently small ε .

We note that the saturation function, with $a_0 > d$ and the above choice of ω_0 , $p(s)$ and $q(s)$, will give rise to an unstable integral manifold. On the other hand, using the threshold nonlinearity, with $a_0 > \delta$, δ and m positive constants (see figure 5) and the above choice of ω_0 , $p(s)$ and $q(s)$, yields an asymptotically stable integral manifold. Also, observe that requiring α to be large implies that $G(s) = p(s)/q(s)$ satisfies the usual "low pass filtering hypothesis" associated with the describing function method [2].

Note that even in the above case, with a differentiable nonlinearity such as $n(y) = y^3$, obtaining an upper bound on admissible ϵ is a formidable task. Thus, although we have achieved our goal of providing a mathematical justification of the Loeb criterion, we see that further work is required in order to provide a simple analytical tool for the control engineer. However, it is evident that our analysis of the system provides further insights into the behavior of its solutions near the integral manifold. In particular, we present a rather striking example of how one can further analyze solutions via our method of proof.

Consider the feedback system with linear part

$$G(s) = \frac{9900s + 500000}{s(s^2 + 50s + 1)}$$

and nonlinearity, $n(y)$, given by

$$n(y) = \begin{cases} 0.02 y & , \quad \text{if } |y| \leq 2, \\ 0.04 \operatorname{sgn} y, & \text{if } |y| > 2. \end{cases}$$

From tabulations of describing functions such as those found in [2] we have

$$N(a) = \begin{cases} 0.02 & , \quad \text{if } 0 < a \leq 2, \\ 0.02 \frac{2}{\pi} \left[\arcsin \left(\frac{2}{a} \right) + \left(\frac{2}{a} \right) \sqrt{1 - \left(\frac{2}{a} \right)^2} \right] & , \quad \text{if } a > 2. \end{cases}$$

The describing function equation (1) gives us $\omega_0 = 10$ and $a_0 \approx 4.95082895$. Furthermore, since

$$d(s) = s^3 + 50s^2 + 100s + 5000 ,$$

we see that

$$\Delta_1 + i\Delta_2 \approx -0.1923 - i99.9615$$

and

$$\hat{\beta}_1 - i\hat{\beta}_2 \approx -3.8462 \times 10^{-5} - i1.923 \times 10^{-4}.$$

Since $D_i > 0$, for $i=0,1$, and since the $\hat{\beta}$'s are reasonably small in magnitude, we expect a stable manifold (also see figure 3). In addition, if we pick the initial conditions in such a way that the transient $x_6(t)$ is negligible, then

$$(25) \quad y_0(t) \approx (Er(t) + a_0) \cos(\omega_0 t + F - \theta(t)).$$

Ignoring the higher ordered terms in equations (22) and (23), we arrive at the following predictions:

1) The solution $y_0(t)$ will lock onto (what appears to be) a periodic orbit at a rate dictated roughly by $D_0\omega_0$, i.e.,

$$r(t) \approx r_0 e^{-D_0\omega_0 t} \approx r_0 e^{-0.009t}, \text{ where } r_0 \text{ is initial value of } r(t).$$

2a) If $r_0 > 0$, then $\theta'(t) \approx \frac{-\Delta_2}{2} EN'(a_0) r(t) \approx -47510r_0 e^{-0.009t} < 0$. Thus, $F - \theta(t)$ will tend upward to some phase shift. Hence, the oscillations of $y_0(t)$ will appear to speed up and

lock onto the frequency of the manifold.

2b) If $r_0 < 0$, then $\theta'(t) > 0$. We expect $F - \theta(t)$ to tend downward to some phase shift. Hence, the $y_0(t)$ oscillations will appear to slow down and lock onto the frequency of the manifold.

In case 2a) or 2b), the magnitude of the radial displacement from a_0 determines the magnitude of $\theta'(t)$. Recalling the role of $\theta(t)$ in (25), we see that a large change in $\theta(t)$ (i.e., a large $\theta'(t)$) will result in a drastic change in the oscillatory behavior of $y_0(t)$. To illustrate this point we numerically simulated the above feedback system. In figure 4, the effect of a large $r_0 > 0$ on the solution $y_0(t)$ can be easily observed, i.e., it can be seen that the oscillations speed up and eventually lock onto the frequency of the manifold as predicted. Similarly, numerical results for initial $r_0 < 0$ verified the predicted behavior, i.e., the oscillations slowed down and eventually locked onto the frequency of the manifold.

Our last example combines an unstable linear operator with a threshold nonlinearity to produce a stable integral manifold. Consider the feedback system with its linear part, $G(s)$, given by

$$G(s) = \frac{s + 24}{s^4 + 100s^3 + 2525s^2 - 100s + 100},$$

and its nonlinear part, $n(y)$, given by

$$n(y) = \begin{cases} 0 & , \quad \text{if } |y| < \delta, \\ m(y - (\text{sgn } y) \delta), & \text{if } |y| > \delta, \end{cases}$$

where $m = 2978.616155$ and $\delta = 0.5$ (see figure 5). For this nonlinearity, we have the corresponding describing function

$$N(a) = \begin{cases} 0 & , \text{ if } 0 < a < \delta, \\ m[1 - \frac{2}{\pi} (\arcsin(\frac{\delta}{a}) + (\frac{\delta}{a}) \sqrt{1 - (\frac{\delta}{a})^2})], & \text{ if } a > \delta. \end{cases}$$

The describing function equation (1) is solved for $a_0 = \omega_0 = 5$. Furthermore, since

$$d(s) = s^4 + 100s^3 + 2525s^2 + 2500s + 62500,$$

we see that

$$\hat{\beta}_1 - i\hat{\beta}_2 \approx -3.1369 \times 10^{-6} - i 1.5528 \times 10^{-5}$$

and

$$\Delta_1 + i\Delta_2 \approx 2.3527 \times 10^{-6} + i(-3.8835 \times 10^{-4}).$$

Observing that $D_i > 0$, for $i=0,1,2$, and since both $\hat{\beta}_1$ and $\hat{\beta}_2$ are small, we expect a stable integral manifold (also see figure 6). From the additional analysis of equations (22) and (23), we predict that the solution $y_0(t)$ will exhibit a "slow time" behavior (c.f. [6]). That is, we expect the amplitude of $y_0(t)$ to slowly approach a value near $a_0 = 5$. Furthermore, since $\theta'(t) \approx 0.36r(t)$, we expect very little deviation in the frequency of the $y_0(t)$ oscillations. Numerical simulation of this system confirms these predictions.

VI. CONCLUDING REMARKS

This paper shows that the describing function method for predicting the existence and stability of integral manifolds in a wide class of feedback systems is correct. Note that, as in [6], we have computable parameters $(\hat{\beta}_1 \text{ and } \hat{\beta}_2)$ which, when sufficiently small, guarantee the existence of an integral manifold. In addition, the integral manifold's stability is easily determined by computable parameters.

Finally, as pointed out in the examples, our analysis will indicate rather detailed behavior of the solution near the integral manifold.

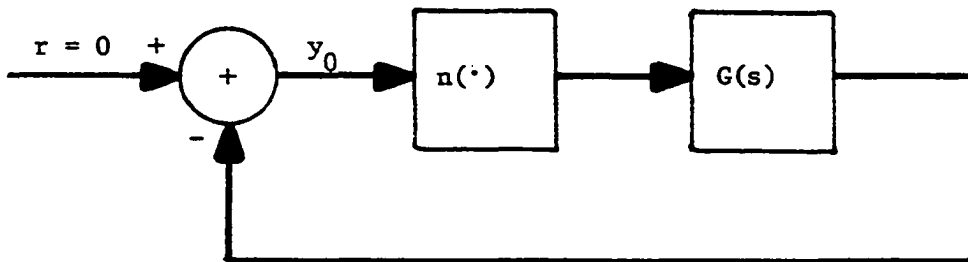


Figure 1. Feedback system configuration

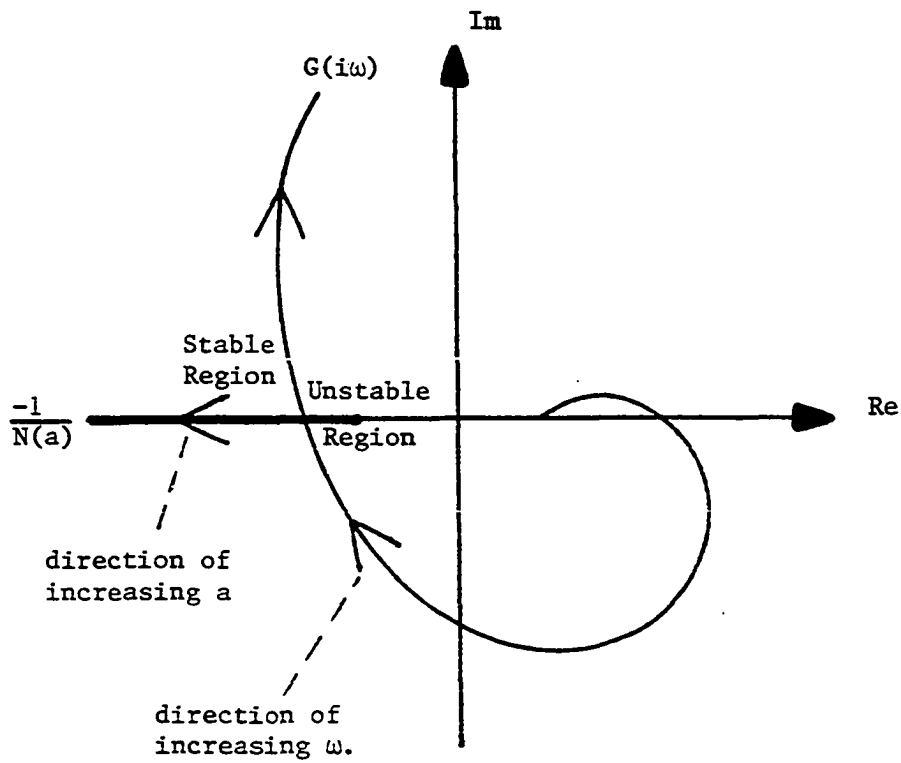


Figure 2. Graphical stability criterion for a stable limit cycle

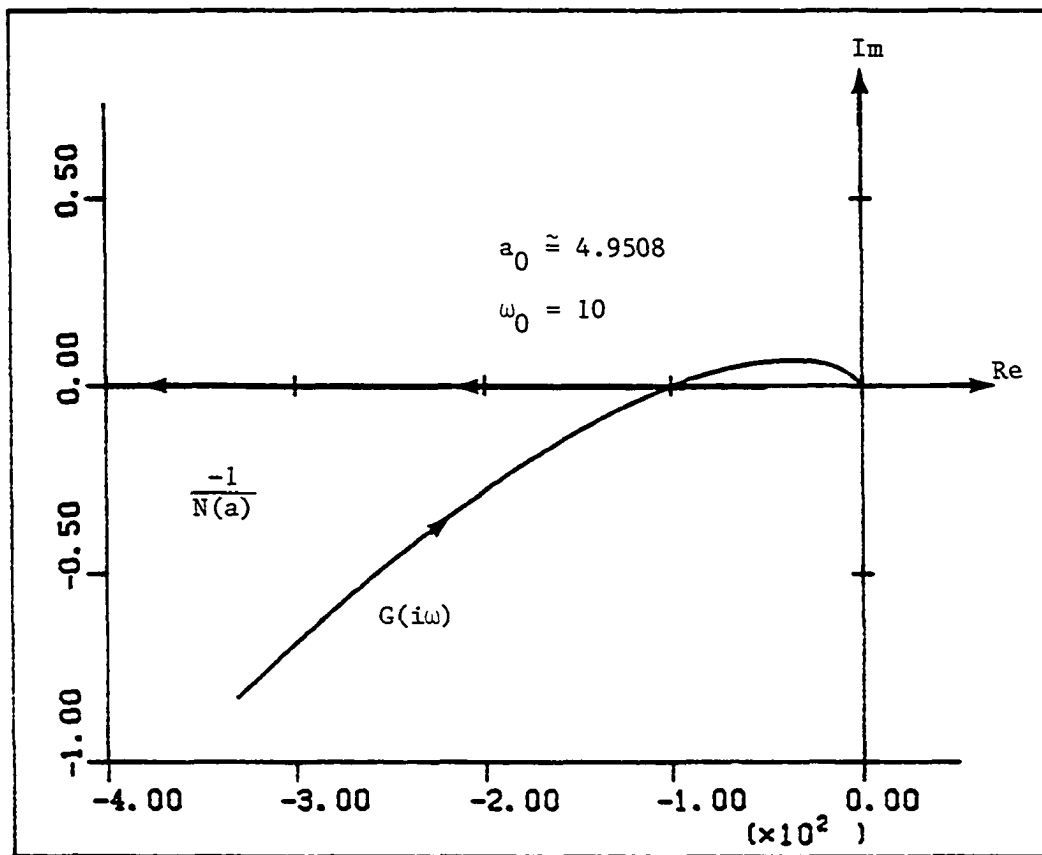


Figure 3. Polar Plot of Example 1

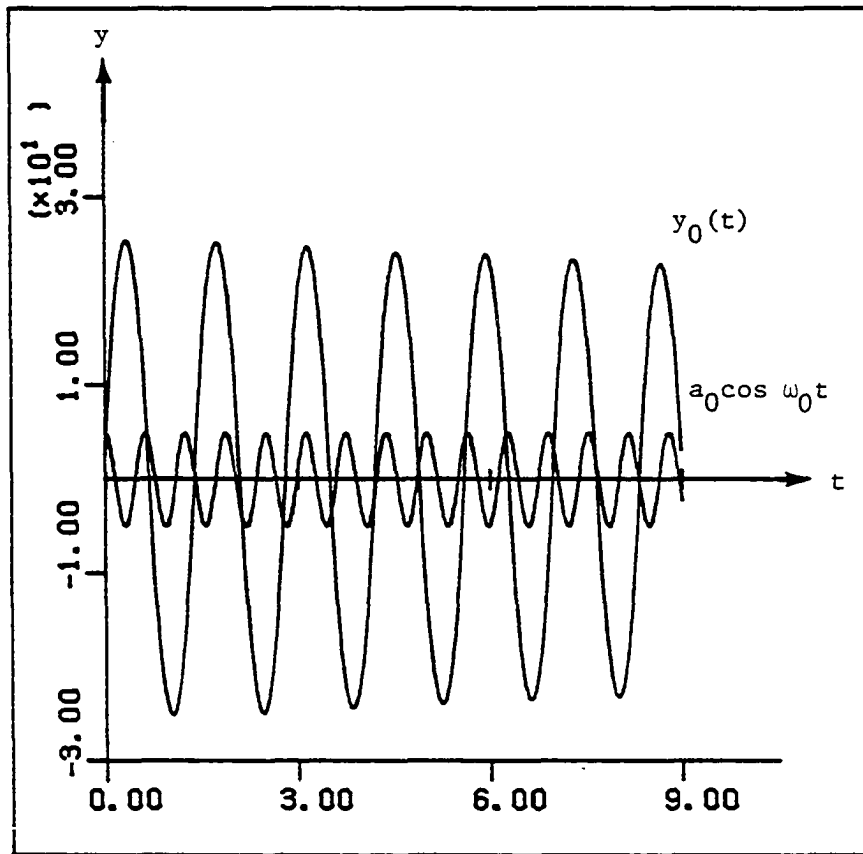


Figure 4a. The solution, $y_0(t)$, of Example 1 compared to $a_0 \cos \omega_0 t$, $0 \leq t \leq 9$ ($a_0 = 4.95$, $\omega_0 = 10$)

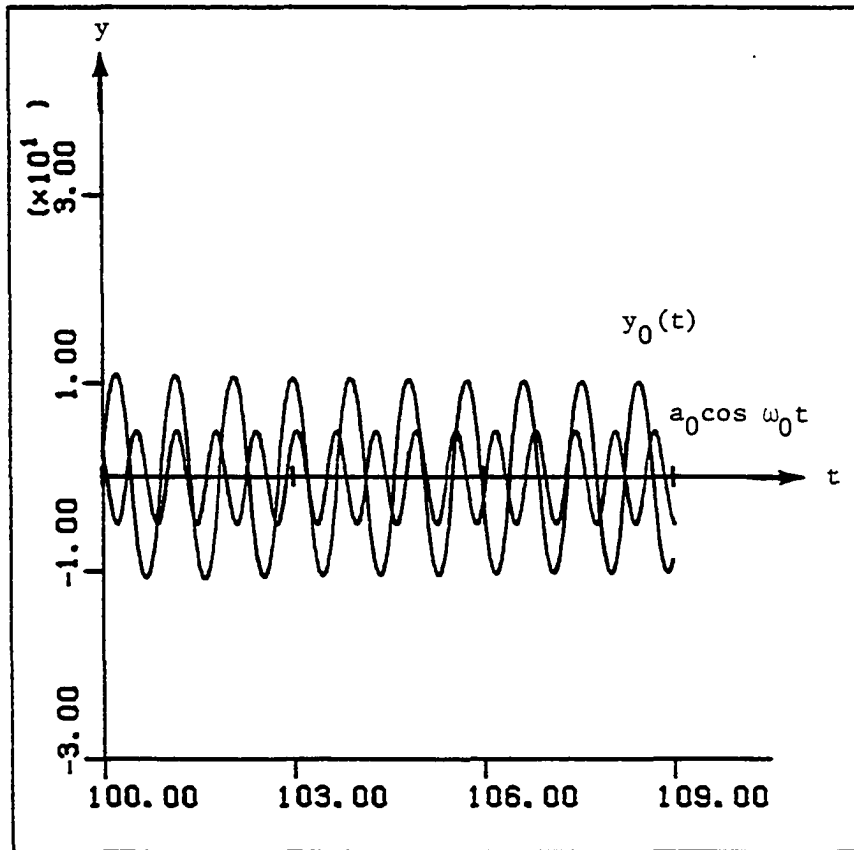


Figure 4b. The solution, $y_0(t)$, of Example 1 compared to $a_0 \cos \omega_0 t$, $100 \leq t \leq 109$

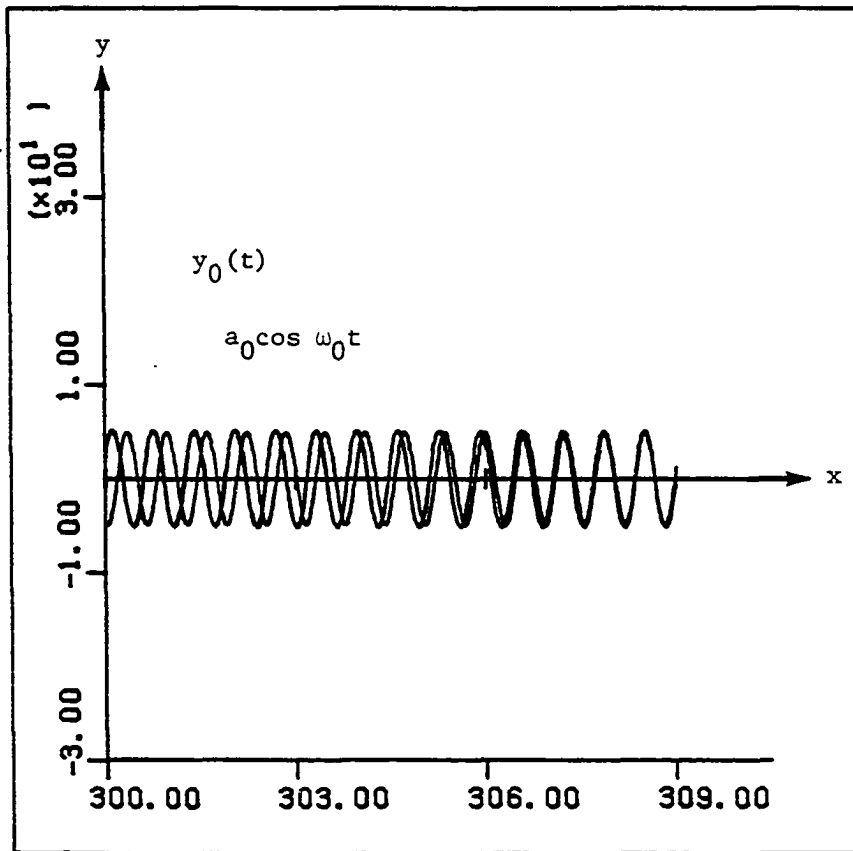


Figure 4c. The solution, $y_0(t)$, of Example 1
compared to $a_0 \cos \omega_0 t$, $300 \leq t \leq 309$

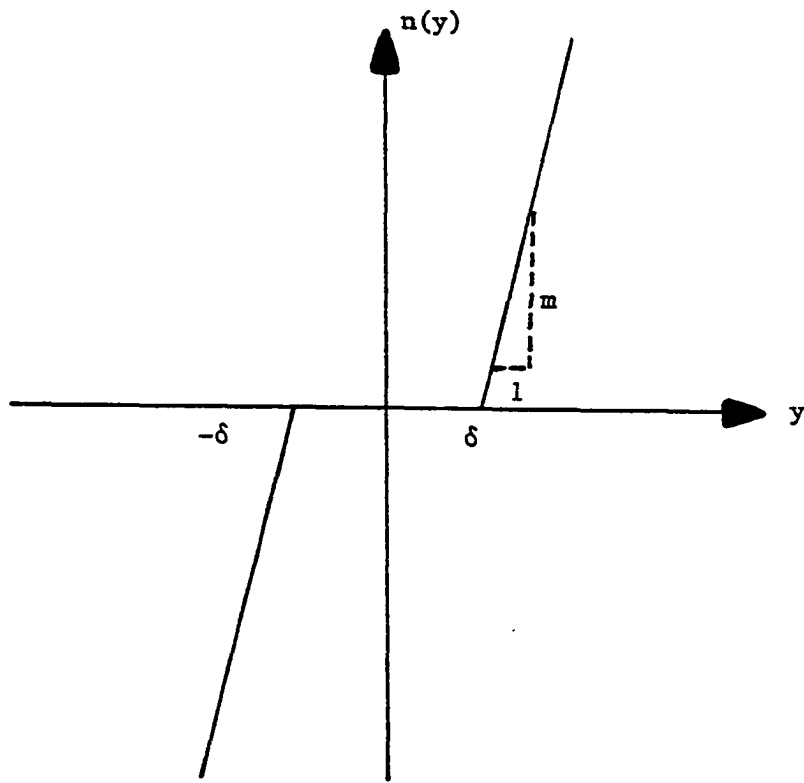


Figure 5. Threshold nonlinearity

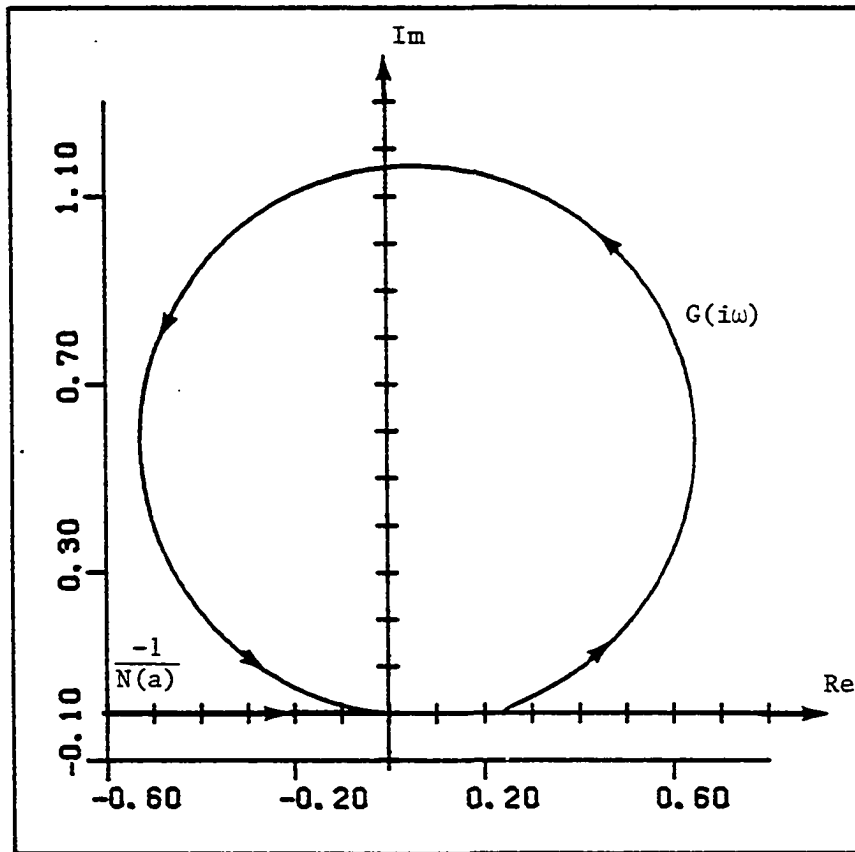


Figure 6a. Polar Plot of Example 2

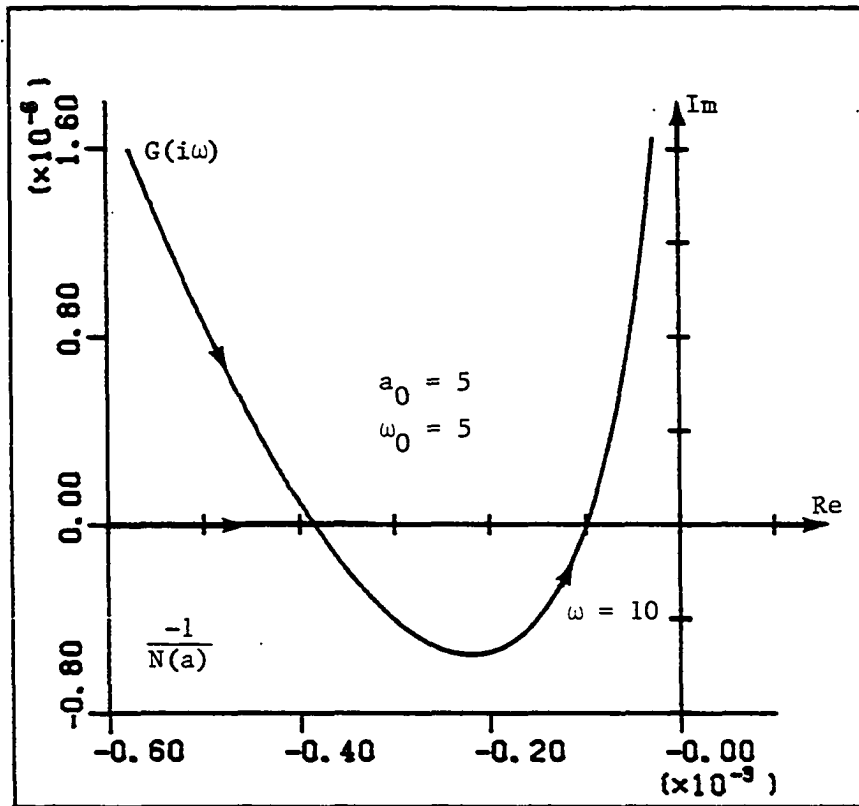


Figure 6b. Detailed view of the crossing

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**PART II. STABILITY ANALYSIS OF ALMOST SINUSOIDAL
PERIODIC OSCILLATIONS IN NONLINEAR CONTROL
SYSTEMS SUBJECTED TO NONCONSTANT PERIODIC
INPUT**

ABSTRACT

We investigate the existence, local uniqueness and local stability properties of almost sinusoidal periodic oscillations in a class of nonlinear control systems subjected to a nonconstant periodic input.

Provided two parameters are sufficiently small, a modified Routh-Hurwitz condition is given which determines the stability of the forced response. The analysis uses the classical single-input sinusoidal describing function to predict the amplitude and phase shift of the fundamental component of the forced response; a novel linearization of the forced problem; averaging; and a simple theorem concerning perturbed linear systems.

We present several systems which, in theory, satisfy our results. We also demonstrate, by means of a specific example, how the results could be used in practice.

I. INTRODUCTION

In this paper, we investigate the stability of periodic motions in a class of nonlinear control systems subjected to continuous, nonconstant periodic inputs. In particular, we use the classical single-input sinusoidal describing function method [6] to obtain the approximate amplitude, a_1 , and phase shift, a , of the system response. We then employ several state-space coordinate transformations, averaging and a result on perturbed linear systems in order to:

- (i) verify the existence and uniqueness of a periodic motion $x_p(t)$, near the approximate solution determined by a_1 and a ,
- (ii) analyze the stability properties of $x_p(t)$.

The class of control systems considered consists of a linear part and a nonlinear part connected in a single loop feedback configuration (see figure 1). The linear part is given by a controllable and observable realization [11] of a real rational transfer function, $G(s)$, where the degree of the numerator is less than the degree of the denominator of $G(s)$. The nonlinear part of the system is required to be an odd, continuous, single-valued function with some additional piecewise differentiability properties.

This paper is divided into six sections, the first being a brief overview of the paper. In the second section, we state some related results, and, for the reader's convenience, the statement of the above mentioned result on perturbed linear systems is given. The third section

explains some of the notation and gives the statement of our main result. In the fourth section, we present the proof of our main result. The fifth section is devoted to specific examples. We end the paper with some brief remarks.

II. RELATED RESULTS

There is extensive literature devoted to the theoretical justification of the describing function method as it is currently used in studying limit cycle behavior in nonlinear systems. In particular, the results of Bass [2], Bergen and Franks [3], Bergen et al. [4], Mees and Bergen [12], Skar et al. [18], and Swern [19] are concerned with the existence of self-sustained oscillations in systems subjected to zero forcing function. On the other hand, Holtzman [9], Miller and Michel [13], and Sandberg [17] used the describing function method to obtain sufficient conditions which guarantee the existence of periodic solutions of nonlinear control systems subjected to periodic forcing functions.

Sandberg's analysis is based on a global contraction mapping argument on the space of periodic functions which are square integrable over a period. His results require that the nonlinearity be Lipschitzian. With some additional restrictions, he is able to assert the existence of a unique periodic response to an arbitrary periodic input with the same period. Moreover, he is able to give an upper bound on the mean square error between the actual periodic system response and the predicted response. In addition, he gives a necessary condition for the occurrence of jump-resonance phenomena (see [6] or [10]) as well as conditions under which sub-harmonics and self-sustained oscillations cannot occur.

Holtzman, by requiring the local differentiability of the operator near the approximate solution, obtains a local existence result. As a consequence of this approach, he is able to give a uniform bound on the error between an actual solution and the approximate solution.

Miller and Michel, by applying results on the differential resolvent of Volterra equations and weak solutions, presented an existence result for sinusoidally forced nonlinear systems containing, for example, relays or hysteresis nonlinearities. Like Holtzman, a subspace of the continuous functions is used to obtain a uniform bound on the error between a solution and the describing function approximation.

The techniques employed in this paper are similar to those used by the present authors (see [15] or [16]) to study the stability of oscillations in nonlinear systems with zero forcing. However, in the current paper, the linearization of the problem must account for the effects of the fundamental component of the forcing. In addition, the role of the phase angle of the solution is drastically changed.

In Section IV, we will require a theorem on perturbed linear systems of the form

$$\begin{aligned} (E) \quad x' &= \varepsilon Ax + \varepsilon X(t, x, y, \varepsilon) , \\ y' &= By + \varepsilon^{1/2} Y(t, x, y, \varepsilon) , \end{aligned}$$

where

(G-1) X and Y are assumed to be defined and continuous
on a set

$$\Omega = \{(t, x, y, \varepsilon) \in \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^j \times \mathbb{R} : |x| < \tilde{\eta}, \\ |y| < \tilde{\eta}, 0 < \varepsilon \leq \varepsilon_0\},$$

for some $\tilde{\eta}, \varepsilon_0 > 0$,

(G-2) X and Y are 2π -periodic in t ,

(G-3) there exists a continuous, monotone increasing function

$\kappa(\cdot)$, with $\kappa(0) = 0$, such that

$$|X(t, x, y, \varepsilon)| < \kappa(\tilde{b}) + \kappa(\tilde{a})\tilde{a} \quad \text{for all } t \in \mathbb{R},$$

$$|x| < \tilde{a}, \quad |y| < \tilde{a}, \quad 0 < \varepsilon \leq \tilde{b},$$

(G-4) Y is Lipschitz in x and y , with Lipschitz constant M ,

(G-5) there exists a nonnegative step function, $L(t, v, w)$, which is 2π -periodic in t , such that, for $0 \leq t < 2\pi$,

$$L(t, v, w) = \sum_{n=1}^N c_n(v, w) \chi_{I_{n, v, w}}(t),$$

where

(a) $0 \leq c_n(v, w) \leq M_0 < \infty$, for $1 \leq n \leq N$, $0 \leq v < \tilde{\eta}$,
 $0 \leq w \leq \varepsilon_0$,

(b) $I_{n, v, w} = [a_{n, v, w}, b_{n, v, w}]$ is a subinterval of $[0, 2\pi]$,
for $1 \leq n \leq N$, and $I_{n_1, v, w} \cap I_{n_2, v, w} = \emptyset$, if $n_1 \neq n_2$,

- (c) $\chi_{I_{n,v,w}}$ is the characteristic function for $I_{n,v,w}$,
i.e.,

$$\chi_{I_{n,v,w}}(t) = \begin{cases} 1, & \text{for } t \in I_{n,v,w}, \\ 0, & \text{for } t \notin I_{n,v,w}, \end{cases}$$

- (d) $c_n(v,w) = (b_{n,v,w} - a_{n,v,w}) \rightarrow 0$, for all n , $1 \leq n \leq N$
as $(v,w) \rightarrow 0$,

and

$$|X(t, x_2, y_0, \varepsilon) - X(t, x_1, y_0, \varepsilon)| \leq L(t, \tilde{a}, \tilde{b}) |x_2 - x_1|,$$

$$|X(t, x_0, y_2, \varepsilon) - X(t, x_0, y_1, \varepsilon)| \leq L(t, \tilde{a}, \tilde{b}) |y_2 - y_1|,$$

for all $t \in \mathbb{R}$, $x_i \in \mathbb{R}^k$, $y_i \in \mathbb{R}^j$, with $|x_i| \leq \tilde{a} \leq \tilde{\eta}$, $|y_i| \leq \tilde{a} \leq \tilde{\eta}$
and $0 < \varepsilon \leq \tilde{b} \leq \varepsilon_0$. Here, \mathbb{R} denotes the set of real numbers.

Theorem 1: Suppose X and Y satisfy (G-1) through (G-5) and that A
and B are noncritical. Then, for each fixed ε , $0 < \varepsilon \leq \varepsilon_1$ (provided
 ε_1 is sufficiently small) there exists $C(\varepsilon)$ and $D(\varepsilon)$, with
 $\tilde{\eta} > C(\varepsilon) > D(\varepsilon) > 0$, so that within the region

$$\Omega_\varepsilon = \{(t, x, y) : (t, x, y, \varepsilon) \in \Omega, |x| \leq C, |y| \leq C\}$$

there is a unique 2π -periodic solution of (E), given by

$$S_\varepsilon = \{(t, f_1(t, \varepsilon), f_2(t, \varepsilon)) : t \in \mathbb{R}\},$$

where $f_1(t+2\pi, \varepsilon) = f_1(t, \varepsilon)$. The solution is unique in the sense that if a solution $(t, x(t), y(t)) \in \Omega_\varepsilon$, for all $t \in \mathbb{R}$, then $(t, x(t), y(t)) \in S_\varepsilon$, for all $t \in \mathbb{R}$.

In addition, if either A or B have an eigenvalue with a positive real part, then S_ε is unstable in the sense of Lyapunov. However, if both A and B are stable matrices, then S_ε is asymptotically stable.

The proof of Theorem 1 follows standard contraction mapping arguments, such as those found in [7], [8], [10] and [14].

III. STATEMENT OF MAIN RESULT

We will analyze feedback systems of the form displayed in figure 1, where $r_0 \in \mathbb{R}$ is continuous and $2\pi/\omega$ -periodic, with $\omega > 0$. Without loss of generality, we may assume r_0 has the form

$$r_0(t) = a_0 \sin \omega t + \psi(\omega t) ,$$

where

$$\int_0^{2\pi} \psi(t) e^{it} dt = 0 \quad \text{and} \quad a_0 > 0 .$$

The linear part of the feedback system is denoted by the transfer function $G(s)$. We assume $G(s)$ is a real rational function, i.e.,

$$G(s) = p(s)/q(s) ,$$

where
$$p(s) = \gamma_{J-1} s^{J-1} + \gamma_{J-2} s^{J-2} + \dots + \gamma_1 s + \gamma_0 ,$$

$$q(s) = s^J + \delta_{J-1} s^{J-1} + \dots + \delta_1 s + \delta_0 ,$$

$$\delta_k, \gamma_k \in \mathbb{R}, \quad 0 \leq k \leq J-1 .$$

Note that we will allow the leading coefficients of $p(s)$ to be zero, i.e., $0 \leq \deg p(s) \leq J-1$. In addition, we assume $p(s)$ and $q(s)$ have no common roots. The nonlinear part, $n(\cdot)$, must be an odd function which satisfies some additional smoothness requirements (see (H-2) of Theorem 2.)

Applying the describing function method [6] to the system in figure 1, we obtain

$$(1a) \quad [1 + G(i\omega)N(a)]a e^{-i\alpha_0} = G(i\omega)a_0 ,$$

where $N(a)$ is the sinusoidal-input describing function for the non-linear function $n(y)$. The term $e^{-i\alpha_0}$ corresponds to the phase shift required in order to balance the resulting fundamental components of the signals in the system.

Suppose there is a value $a = a_1 > 0$ for which (1a) holds, that is,

$$(1b) \quad |1 + G(i\omega)N(a_1)|a_1 = |G(i\omega)|a_0, \text{ and}$$

$$\alpha_0 = \arg([G(i\omega)]^{-1} + N(a_1)) .$$

We assume a_1 is a value for which $n'(y)$ exists. For example, if n is the saturation function

$$n(y) = \begin{cases} my & |y| \leq \delta , \\ m\delta \operatorname{sgn} y & |y| > \delta , \end{cases} \quad m, \delta > 0 \text{ and } \operatorname{sgn} y = \frac{y}{|y|} ,$$

then $a_1 \neq \delta$.

The assumption $p(s)$ and $q(s)$ have no common roots implies the feedback system in figure 1 has a natural phase space controllable and observable linear realization

$$(2) \quad \dot{x}_0 = Ax_0 + b_0 u ,$$

$$(3) \quad y_0 = h_0^T x_0 ,$$

with transfer function $G(s) = \mathcal{L}\{h_0^T e^{At} b_0\}$ [11]. Here, A is the companion matrix for $q(s)$,

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\delta_0 & -\delta_1 & -\delta_2 & & -\delta_{J-1} \end{bmatrix} ,$$

while b_0, h_0 are real J -tuples, with $b_0^T = (0, 0, \dots, 0, 1)$ and $h_0^T = (\gamma_0, \gamma_1, \dots, \gamma_{J-1})$. The control, u , will be $r_0(t) - n(y_0(t))$.

If we substitute the control $\tilde{u}(t) = a_0 \sin \omega t - N(a_1)y_0(t)$ for u in (2), we see that (2) has a periodic solution

$$\tilde{x}_0(t) = a_1 E^{-1}(\sin(\omega t - \alpha), \omega \cos(\omega t - \alpha), -\omega^2 \sin(\omega t - \alpha), \dots)^T$$

for all $t \in \mathbb{R}$,

where $\alpha = \alpha_0 + \arg(p(i\omega))$ and $E = |p(i\omega)|$. We will show that there exists a $2\pi/\omega$ -periodic solution $x_p(t)$ of the original system (2) near $\tilde{x}_0(t)$, provided two computable parameters are small. In addition, we will give a stability criterion for $x_p(t)$. Here, stability will mean local asymptotic stability, while instability will be in the sense of Lyapunov [14]. In order to state the stability results for the above phase space realization of the feedback system, we introduce the following notation:

Let $d(s)$ be given by

$$(4) \quad d(s) = q(s) + p(s)N(a_1) - \frac{a_0 E}{a_1} \left\{ \sin \alpha \frac{s}{\omega} + \cos \alpha \right\}$$

$$= d_0 s^J + d_1 s^{J-1} + \dots + d_{J-1} s + d_J ,$$

where α_0 and a_1 are given by equations (1a,b). Define $\hat{\beta}_1, \hat{\beta}_2$ by

$$\hat{\beta}_1 - i\hat{\beta}_2 = \frac{2}{\omega d'(i\omega)} .$$

For $k = 1, 2, \dots, J-2$, we define Hurwitz determinants D_k associated with $d(s)$ by

$$D_1 = d_1 , \quad D_2 = \det \begin{bmatrix} d_1 & d_3 \\ d_0 & d_2 \end{bmatrix} , \quad D_3 = \det \begin{bmatrix} d_1 & d_3 & d_5 \\ d_0 & d_2 & d_4 \\ 0 & d_1 & d_3 \end{bmatrix} ,$$

$$D_4 = \det \begin{bmatrix} d_1 & d_3 & d_5 & d_7 \\ d_0 & d_2 & d_4 & d_6 \\ 0 & d_1 & d_3 & d_5 \\ 0 & d_0 & d_2 & d_4 \end{bmatrix},$$

and so forth, where we take $d_j = 0$ if $j > J$. Define D_{J-1} and D_J by

$$D_{J-1} = 1 + a_1 N'(a_1) \operatorname{Re} \left\{ \frac{G(i\omega)}{1+G(i\omega)N(a_1)} \right\}$$

and

$$D_J = \operatorname{Re} \{ (\hat{\beta}_1 - i\hat{\beta}_2) (2[q(i\omega) + p(i\omega)N(a_1)] + a_1 N'(a_1)p(i\omega)) \}.$$

We now present our main result.

Theorem 2. Suppose that system (2)-(3) satisfies

- (H-1) All d_j are real for $0 \leq j \leq J$, with $d_0 = 1$ and that $D_j \neq 0$, for $1 \leq j \leq J$.
- (H-2) The function n is an odd, continuous, piecewise continuously differentiable function. Moreover, on any interval $y_a < y < y_b$, where $n'(y)$ exists and is continuous, $n''(y)$ also exists and is uniformly continuous. The describing function for $n(y)$ will be denoted by $N(a)$.
- (H-3) The polynomials $p(s)$ and $q(s)$ have no common factors.

(H-4) There exists $a_1 > 0$ satisfying (1b) and an associated α such that $d(s)$ has the simple roots $\pm i\omega$. In addition, a_1 must be a point of continuity of $n'(y)$.

If $\hat{\beta}_1$ and $\hat{\beta}_2$ are sufficiently small, then there is a (sup norm) neighborhood N_0 of $\{(t, \tilde{x}_0^T(t)) : t \in \mathbb{R}\}$ with the properties:

- (i) There exists a $2\pi/\omega$ -periodic solution $x_p(t)$ of (2) such that the graph of $x_p(t)$ lies in N_0 .
- (ii) If $(t_0, \eta^T) \in N_0$, but $x_p(t_0) \neq \eta$, then the solution of (2) satisfying $x_0(t_0) = \eta$ must leave N_0 in finite time. Hence, $x_p(t)$ is the only $2\pi/\omega$ -periodic solution of (2) near $\tilde{x}_0(t)$.
- (iii) If $D_j > 0$, for $j = 1, 2, \dots, J$, then $x_p(t)$ is stable.
- (iv) If $D_k < 0$, for some k , $1 \leq k \leq J$, then $x_p(t)$ is unstable. ■

Since $|x_p(t) - \tilde{x}_0(t)|$ will be small for all $t \in \mathbb{R}$, we have

$$\begin{aligned}
 y_p(t) &= h_0^T x_p(t) \\
 &\approx \frac{a_1}{E} [\gamma_0 \sin(\omega t - \alpha) + \gamma_1 \omega \cos(\omega t - \alpha) + \dots + \gamma_{J-1} \frac{d^{J-1}}{dt^{J-1}} \sin(\omega t - \alpha)] \\
 &= \frac{a_1}{E} \operatorname{Im}[p(i\omega)e^{i(\omega t - \alpha)}] \\
 &= a_1 \operatorname{Im}[\exp(i(\omega t - \alpha + \arg p(i\omega)))]
 \end{aligned}$$

$$= a_1 \sin(\omega t - \alpha_0).$$

Finally, suppose we have a controllable and observable system

$$(2') \quad x' = A_1 x + b_1 u,$$

$$(3') \quad y = c^T x,$$

with transfer function $G(s) = p(s)/q(s)$. Here, $A_1 \in R^{J \times J}$, $b_1, c \in R^J$ and $u \in R$, with $u = a_0 \sin \omega t - n(y)$. Then, by results from control theory [11], there is a nonsingular change of coordinates, $x = Px_0$, such that system (2')-(3') is linearly equivalent to system (2)-(3).

This gives rise to the following corollary.

Corollary. Suppose that system (2')-(3') satisfies hypotheses (H-1) through (H-4) of Theorem 2. If $\hat{\beta}_1$ and $\hat{\beta}_2$ are sufficiently small, then there is a neighborhood $PN_0 = \{(t, (Px)^T) : (t, x^T) \in N_0\}$ of $\tilde{Px}_0(t)$ with the properties:

- (i) There exists a $2\pi/\omega$ -periodic solution $Px_p(t)$ of (2'), such that the graph of $Px_p(t)$ lies in PN_0 .
- (ii) If $(t_0, \eta^T) \in PN_0$, but $Px_p(t_0) \neq \eta$, then the solution of (2') satisfying $x(t_0) = \eta$ must leave PN_0 in finite time. Hence, $Px_p(t)$ is only $2\pi/\omega$ -periodic solution of (2') near $\tilde{Px}_0(t)$.
- (iii) If $D_j > 0$, for all $j = 1, 2, \dots, J$, then $Px_p(t)$ is stable.

- (iv) If $D_j < 0$, for some j , $1 \leq j \leq J$, then $Px_p(t)$ is unstable. ■

IV. ANALYSIS OF THE FEEDBACK SYSTEM

Note that system (2) - (3), with control $u = r_0 - n(y_0)$, is equivalent to the scalar equation

$$(5) \quad q(D)z + n(p(D)z) = a_0 \sin \omega t + \psi(\omega t),$$

where $D = d/dt$ and $x_0^T = (z, Dz, \dots, D^{J-1}z)$. Via the time scaling $\tau = \omega t$, we see that (5) is equivalent to

$$(6) \quad q(\omega d/d\tau)z + n(p(\omega d/d\tau)z) = a_0 \sin \tau + \psi(\tau).$$

Define $\hat{d}(s)$, a monic J^{th} degree polynomial with real, constant coefficients, by

$$\hat{d}(s) = \omega^{-J} d(\omega s).$$

By the choice of a_1 and α , it is obvious that

$$\hat{d}(i) = \hat{d}(-i) = 0.$$

We now rewrite equation (6) to obtain

$$(7) \quad \hat{d}(d/d\tau)z = \omega^{-J} [N(a_1)p(\omega d/d\tau)z - \frac{Ea_0}{a_1} \{(\sin \alpha)dz/d\tau + (\cos \alpha)z\} - n(p(\omega d/d\tau)z) + a_0 \sin \tau + \psi(\tau)].$$

For the sake of notational convenience we replace τ by t and simply analyze equation (7). Let $\tilde{z} \in R^J$ be defined by

$$\tilde{z}^T = (z, Dz, D^2 z, \dots, D^{J-1} z).$$

We now rewrite (7) as the system

$$(8) \quad \begin{aligned} \tilde{z}' = B_0 \tilde{z} + \omega^{-J} b_0 \{ a_0 \sin t - \frac{Ea_0}{a_1} [(\sin \alpha) \tilde{z}_2 + \\ (\cos \alpha) \tilde{z}_1] + N(a_1) h^T \tilde{z} - n(h^T \tilde{z}) + \psi(t) \} , \end{aligned}$$

where B_0 is the companion form of $\hat{d}(s)$ and

$$h^T = (\gamma_0, \omega \gamma_1, \dots, \omega^{J-1} \gamma_{J-1}).$$

We will transform B_0 into a combination of its real and complex Jordan canonical forms. First, let $a(s)^T = (1, s, s^2, \dots, s^{J-1})$. Clearly, $a(i)$ is a right eigenvector of B_0 corresponding to the eigenvalue $\hat{s}_1 = i$ of B_0 . Let $\hat{s}_2 = -i$ and define $\xi_1, \xi_2 \in \mathbb{R}^J$ by

$$\xi_1 + i\xi_2 = a(i).$$

Next, we define vectors $\xi_3, \xi_4, \dots, \xi_J \in \mathbb{R}^J$. Suppose $\{\hat{s}_k\}_{k=3}^J$ are the remaining roots of $\hat{d}(s)$. Suppose that \hat{s}_k has multiplicity $m_k > 1$, that is,

$$\hat{s}_k = \hat{s}_{k+1} = \dots = \hat{s}_{k+m_k-1}.$$

We define ξ_{k+j} by

$$\xi_{k+j} = \frac{1}{j!} \left. \frac{d^j}{ds^j} a(s) \right|_{s=\hat{s}_k}, \quad \text{for } j = 0, 1, 2, \dots, m_k-1.$$

Observe that $\left\{ \xi_{k+j} \right\}_{j=0}^{m_k-1}$ forms a chain of generalized eigenvectors

of B_0 corresponding to the root \hat{s}_k , i.e.,

$$[B_0 - \hat{s}_k I] \xi_{k+j} = \xi_{k+j-1}, \quad \text{for } j = m_k-1, m_k-2, \dots, 1$$

and

$$[B_0 - \hat{s}_k I] \xi_k = 0.$$

Taking $B \in R^{J \times J}$,

$$B = [\xi_1, \xi_2, \xi_3, \dots, \xi_J] = [b_{ij}]_{J \times J}$$

and setting $Bv = \tilde{z}$, we transform (8) into

$$(9) \quad v' = \left[\begin{array}{c|c} 0 & 1 \\ -\frac{1}{c} & 0 \end{array} \right] v + B^{-1} b_0 \omega^{-J} [a_0 \sin t + \psi(t) - \frac{E a_0}{a_1} [(\cos \alpha) \tilde{z}_1 + (\sin \alpha) \tilde{z}_2] + N(a_1) h^T Bv - n(h^T Bv)].$$

Using the notation

$$\hat{\beta} = B^{-1} b_0 \omega^{-J},$$

$$n_1(y) = N(a_1)y - n(y),$$

$$x_3^T = (v_3, v_4, \dots, v_J),$$

$$\begin{bmatrix} x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & & h^T & & \end{bmatrix} [\xi_3, \xi_4, \dots, \xi_J] x_3$$

and letting

$$p_R + ip_I = p(i\omega) ,$$

we have

$$(10) \quad v' = \begin{bmatrix} 0 & 1 \\ -\frac{1}{c} & 0 \end{bmatrix} v + \hat{\beta} [a_0 \sin t - \frac{Ea_0}{a_1} [(\cos \alpha)(v_1 + x_6) \\ + (\sin \alpha)(v_2 + x_7)] + n_1(p_R v_1 + p_I v_2 + x_8) + \psi(t)] .$$

Applying the Van der Pol transformation,

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \phi(t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ,$$

where

$$\phi(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} ,$$

to equation (10) yields

$$(11) \quad \begin{cases} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = f(t, x_1, x_2, x_6, x_7, x_8) , \\ x_3' = Cx_3 + \begin{bmatrix} \hat{\beta}_3 \\ \vdots \\ \hat{\beta}_J \end{bmatrix} h(t, x_1, x_2, x_6, x_7, x_8) , \end{cases}$$

$$\begin{aligned} \text{where } h(t, x_1, x_2, x_6, x_7, x_8) = & a_0 \sin t + \psi(t) - \frac{Ea_0}{a_1} [(\cos \alpha)(\cos t x_1 + \\ & + \sin t x_2 + x_6) + (\sin \alpha)(-\sin t x_1 + \cos t x_2 + x_7)] + \\ & n_1(p_R(\cos t x_1 + \sin t x_2) + p_I(-\sin t x_1 + \cos t x_2) + x_8) \end{aligned}$$

$$\text{and } f(t, x_1, x_2, x_6, x_7, x_8) = \phi(-t) \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} h(t, x_1, x_2, x_6, x_7, x_8).$$

$$\text{Define } f_0(x_1, x_2) = \frac{1}{2\pi} \int_0^{2\pi} f(t, x_1, x_2, 0, 0, 0) dt \text{ and}$$

$$u(t, x_1, x_2) = \int_0^t [f(s, x_1, x_2, 0, 0, 0) - f_0(x_1, x_2)] ds .$$

Observe that $u(t, x_1, x_2)$ has the form considered in [15], with the addition of some terms linear in (x_1, x_2) . This implies that u is continuous, 2π periodic in t , C^1 in (t, x_1, x_2) and u_{x_i} are

Lipschitz continuous in (x_1, x_2) . These conditions on u will be needed later in order to apply Theorem 1.

The change of variables

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = w + u(t, w), \quad w^T = [w_1, w_2] ,$$

will be used to average equation (11). Since

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = w' + u_t(t, w) + u_w(t, w)w' = f(t, x_1, x_2, x_6, x_7, x_8) ,$$

where $u_w(t, w) = \left[\frac{\partial u_i}{\partial x_j} \right]_{2 \times 2}$, is the Jacobian matrix of $u(t, w)$ and

$$u_t(t, w) = f(t, w, 0, 0, 0) - f_0(w) ,$$

we see that (11) is equivalent to

$$(12) \quad \begin{cases} w' = f_0(w) + f_1(t, w, x_6, x_7, x_8) , \\ x_3' = Cx_3 + \begin{bmatrix} \hat{\beta}_3 \\ \vdots \\ \hat{\beta}_J \end{bmatrix} h(t, w+u(t, w), x_6, x_7, x_8) . \end{cases}$$

Here, $f_1 = (f_{11}, f_{12})^T$ is the vector function defined by

$$\begin{aligned}
f_1(t, w, x_6, x_7, x_8) &= [I + u_w(t, w)]^{-1} [f(t, w + u(t, w), x_6, x_7, x_8) \\
&\quad - f(t, w, 0, 0, 0) - u_w(t, w)f_0(w)] \\
&= [I + u_w(t, w)]^{-1} [\phi(-t) \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} \left\{ \frac{-Ea_0}{a_1} [(\cos t \, u_1(t, w) + \right. \\
&\quad \sin t \, u_2(t, w) + x_6) \cos \alpha + (-\sin t \, u_1(t, w) + \cos t \, u_2(t, w) + \\
&\quad x_7) \sin \alpha] + n_1(p_R((w_1 + u_1(t, w)) \cos t + (w_2 + u_2(t, w)) \sin t) \\
&\quad + p_I(-(w_1 + u_1(t, w)) \sin t + (w_2 + u_2(t, w)) \cos t) + x_8) \\
&\quad \left. - n_1(p_R(w_1 \cos t + w_2 \sin t) + p_I(-w_1 \sin t + w_2 \cos t)) \right\} \\
&\quad \left. - u_w(t, w)f_0(w) \right] .
\end{aligned}$$

We now compute $f_0(w)$. Let $w_1 = a \cos \gamma$, $w_2 = a \sin \gamma$ and $p(i\omega) = p_R + ip_I = E \exp(iF)$. Then,

$$\begin{aligned}
f_0(w) &= \frac{1}{2\pi} \int_0^{2\pi} f(t, w, 0, 0, 0) dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} \begin{bmatrix} \hat{\beta}_1 \cos t - \hat{\beta}_2 \sin t \\ \hat{\beta}_1 \sin t + \hat{\beta}_2 \cos t \end{bmatrix} \left\{ a_0 \sin t - \frac{Ea_0}{a_1} [(\cos t \, w_1 + \sin t \, w_2) \cos \alpha \right. \\
&\quad \left. + (-\sin t \, w_1 + \cos t \, w_2) \sin \alpha] + n_1(aE \cos(t+F-\gamma)) + \psi(t) \right\} dt
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left[\begin{aligned} &-\hat{\beta}_2 a_0 - \frac{E a_0}{a_1} [(\hat{\beta}_1 w_1 - \hat{\beta}_2 w_2) \cos \alpha + (\hat{\beta}_2 w_1 + \hat{\beta}_1 w_2) \sin \alpha] \\ &\hat{\beta}_1 a_0 - \frac{E a_0}{a_1} [(\hat{\beta}_2 w_1 + \hat{\beta}_1 w_2) \cos \alpha + (-\hat{\beta}_1 w_1 + \hat{\beta}_2 w_2) \sin \alpha] \end{aligned} \right] \\
&+ \frac{1}{2} a E N_1(a E) \left[\begin{aligned} &\hat{\beta}_1 \cos(F-\gamma) + \hat{\beta}_2 \sin(F-\gamma) \\ &-\hat{\beta}_1 \sin(F-\gamma) + \hat{\beta}_2 \cos(F-\gamma) \end{aligned} \right] \\
&= \frac{1}{2} \left[\begin{aligned} &-\hat{\beta}_2 a_0 - \frac{E a_0}{a_1} [(\hat{\beta}_1 w_1 - \hat{\beta}_2 w_2) \cos \alpha + (\hat{\beta}_2 w_1 + \hat{\beta}_1 w_2) \sin \alpha] \\ &\hat{\beta}_1 a_0 - \frac{E a_0}{a_1} [(\hat{\beta}_2 w_1 + \hat{\beta}_1 w_2) \cos \alpha + (-\hat{\beta}_1 w_1 + \hat{\beta}_2 w_2) \sin \alpha] \end{aligned} \right] \\
&+ \frac{1}{2} N_1(a E) \left[\begin{aligned} &\hat{\beta}_1 (p_R w_1 + p_I w_2) + \hat{\beta}_2 (p_I w_1 - p_R w_2) \\ &-\hat{\beta}_1 (p_I w_1 - p_R w_2) + \hat{\beta}_2 (p_R w_1 + p_I w_2) \end{aligned} \right] ,
\end{aligned}$$

where $a^2 = w_1^2 + w_2^2$ and $N_1(\cdot)$ is the describing function for $n_1(\cdot)$.

In order to analyze the radial and angular deviation from the predicted amplitude and phase shift, we now introduce the polar coordinates

$$w_1 = \left(r + \frac{a_1}{E}\right) \sin(\eta - \alpha) ,$$

$$w_2 = \left(r + \frac{a_1}{E}\right) \cos(\eta - \alpha) .$$

These coordinates transform (12) into

(13)

$$\begin{bmatrix} \eta \\ r \end{bmatrix}' = \frac{a_0}{2} \begin{bmatrix} \frac{-E}{a_1} [\hat{\beta}_1 \cos \alpha + \hat{\beta}_2 \sin \alpha] & \left[\frac{E}{a_1} \right]^2 (-\hat{\beta}_1 \sin \alpha + \hat{\beta}_2 \cos \alpha) - \frac{EN'(a_1)}{a_0} (\hat{\beta}_1 p_I - \hat{\beta}_2 p_R) \\ [\hat{\beta}_1 \sin \alpha - \hat{\beta}_2 \cos \alpha] & -\frac{E}{a_1} (\hat{\beta}_1 \cos \alpha + \hat{\beta}_2 \sin \alpha) - \frac{a_1 N'(a_1)}{a_0} (\hat{\beta}_1 p_R + \hat{\beta}_2 p_I) \end{bmatrix} \begin{bmatrix} \eta \\ r \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{\cos(\eta - \alpha)}{r + \frac{a_1}{E}} & \frac{-\sin(\eta - \alpha)}{r + \frac{a_1}{E}} \\ \sin(\eta - \alpha) & \cos(\eta - \alpha) \end{bmatrix} f_1(t, (r + \frac{a_1}{E}) \sin(\eta - \alpha), (r + \frac{a_1}{E}) \cos(\eta - \alpha),$$

$$x_6, x_7, x_8) + O(r^2 + \eta^2) + f_2(r),$$

$$(14) \quad x_3' = Cx_3 + \begin{bmatrix} \hat{\beta}_3 \\ \vdots \\ \hat{\beta}_J \end{bmatrix} h(t, w + u(t, w), x_6, x_7, x_8),$$

where

$$2f_2(r) = \begin{bmatrix} (N_1(Er + a_1) - N_1'(a_1)Er)(\hat{\beta}_1 p_I - \hat{\beta}_2 p_R) \\ (N_1(Er + a_1)(r + \frac{a_1}{E}) - N_1'(a_1)a_1 r)(\hat{\beta}_1 p_R + \hat{\beta}_2 p_I) \end{bmatrix}.$$

Let $M_R + iM_I = (\hat{\beta}_1 - i\hat{\beta}_2) \frac{a_0 E}{a_1} \exp(i\alpha)$ and

$Z_R + iZ_I = M_R + iM_I + (\hat{\beta}_1 - i\hat{\beta}_2) a_1 N'(a_1) p(i\omega)$. Then, equation (13) is equivalent to

$$(15) \quad \begin{bmatrix} \eta \\ r \end{bmatrix}' = \frac{1}{2} \begin{bmatrix} -M_R & -\frac{E}{a_1} Z_I \\ \frac{a_1}{E} M_I & -Z_R \end{bmatrix} \begin{bmatrix} \eta \\ r \end{bmatrix} + \text{perturbation terms.}$$

Observe that the matrix in (15) is stable if, and only if, D_{J-1} and D_J are positive. On the other hand, the matrix is unstable if, and only if, D_{J-1} or D_J is negative.

Before examining the hypotheses of Theorem 1, we must transform the x_3 coordinate. First, let $x_4 = B_1 x_3$, where B_1 is the submatrix of B given by

$$B_1 = \begin{bmatrix} b_{13} & b_{14} & \cdots & b_{1J} \\ b_{23} & b_{24} & \cdots & b_{2J} \\ \vdots & \vdots & & \vdots \\ b_{J-2,3} & b_{J-2,4} & \cdots & b_{J-2,J} \end{bmatrix}$$

and x_4 is a $J-2$ tuple with $x_4 = (x_{41}, x_{42}, \dots, x_{4,J-2})^T$. This implies

$$\begin{bmatrix} x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & & h^T & & \end{bmatrix} [\xi_3, \xi_4, \dots, \xi_J] B_1^{-1} x_4 .$$

From (14), we have

$$(16) \quad x_4' = B_1 C B_1^{-1} x_4 + B_1 \begin{bmatrix} \hat{\beta}_3 \\ \vdots \\ \hat{\beta}_J \end{bmatrix} h(t, w+u(t, w), x_6, x_7, x_8) .$$

Due to the structure of C and B_1 , $C_1 = B_1 C B_1^{-1}$ is the companion matrix of the $J-2$ degree polynomial

$$\prod_{j=3}^J (s - \hat{s}_j) .$$

Furthermore, since

$$b_0 = B B^{-1} b_0 = \hat{B} \hat{\beta}^J ,$$

we have

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \hat{\beta}_1 \hat{\xi}_1 + \hat{\beta}_2 \hat{\xi}_2 + B_1 \begin{bmatrix} \hat{\beta}_3 \\ \hat{\beta}_4 \\ \vdots \\ \hat{\beta}_J \end{bmatrix} ,$$

where $\hat{\xi}_1$ and $\hat{\xi}_2$ are, respectively, the first and second columns of B with their $J-1$ and J^{th} components removed. Defining

$\epsilon = [\hat{\beta}_1^2 + \hat{\beta}_2^2]^{1/2}$ and letting $x_4 = \epsilon^{1/2} x_5$, we see that (16) becomes

$$(17) \quad x_5' = C_1 x_5 - \epsilon^{-1/2} [\hat{\beta}_1 \hat{\xi}_1 + \hat{\beta}_2 \hat{\xi}_2] h(t, w + u(t, w), x_6, x_7, x_8) .$$

Observe that C_1 is stable if, and only if, $D_k > 0$, for $k = 1, 2, \dots, J-2$. Similarly, C_1 is unstable if $D_k < 0$, for some k , $1 < k < J-2$.

We note that in (13), with ϵ as a free parameter, $\epsilon^{-1}\{\text{perturbation terms}\}$ satisfies hypothesis (G-3) of Theorem 1. We now verify that the perturbation terms satisfy the Lipschitz function condition. However, due to the Lipschitz continuity of u_1 and u_{x_1} and the definition of f_2 , we need only examine the term involving n_1 , i.e.,

$$\begin{aligned} & n_1(p_R((w_1 + u_1(t, w))\cos t + (w_2 + u_2(t, w))\sin t) + p_I(-(w_1 + u_1(t, w))\sin t \\ & \quad + (w_2 + u_2(t, w))\cos t) + x_8) - n_1(p_R(w_1 \cos t + w_2 \sin t) \\ & \quad + p_I(-w_1 \sin t + w_2 \cos t)) \\ & = n_1((Er + a_1)\sin(t + \eta + F - \alpha) + x_8 + \mathcal{O}(\epsilon)) - n_1((Er + a_1)\sin(t + \eta + F - \alpha)) , \end{aligned}$$

where x_8 is now $\mathcal{O}(\sqrt{\epsilon} x_5)$ and the term $\mathcal{O}(\epsilon)$ has Lipschitz constants of order $\mathcal{O}(\epsilon)$. In particular, we must examine the Lipschitz continuity of

$$n_1((Er+a_1)\sin \theta + \delta) - n_1((Er+a_1)\sin \theta)$$

with respect to r (and n), for $0 < \theta < 2\pi$, $E|r| < \delta_1$, $|\delta| < \delta_1$, where δ_1 is a small positive number.

For any $\delta_2 > 0$, there is a $\delta_1 > 0$, such that the values of $y = a_1 \sin \theta + Er \sin \theta + \delta$, $0 < \theta < 2\pi$, cannot be within δ_2 of a point where $n'(y)$ does not exist, except on a set $S(\delta_1)$ of intervals, $S(\delta_1) \subseteq [0, 2\pi]$. Clearly, $S(\delta_1)$ can be chosen such that the measure of $S(\delta_1)$ tends to zero as δ_1 tends to zero. For any θ , $0 < \theta < 2\pi$, we have

$$\begin{aligned} D &= |\{n_1((Er_1+a_1)\sin \theta + \delta) - n_1((Er_1+a_1)\sin \theta)\} \\ &\quad - \{n_1((Er_2+a_1)\sin \theta + \delta) - n_1((Er_2+a_1)\sin \theta)\}| \\ &< 2EL|\sin \theta| |r_1 - r_2| \\ &< 2EL|r_1 - r_2|, \text{ where } L \equiv \max|n'(y)| + |N(a_1)|. \end{aligned}$$

For $\theta \in [0, 2\pi] - S(\delta_1)$, we note that $y = (Er+a_1)\sin \theta + \delta \in H$, $H = \{z: 0 < |z| < a_1 + 2\delta_1, |z-p| > \delta_2, \text{ for any point } p \text{ where } n' \text{ does not exist}\}$. Moreover, $n'(y)$ is uniformly continuous on components of H . Thus, there exists a continuous, increasing function σ , such that $\sigma(0) = 0$ and

$$|n'_1(y_1) - n'_1(y_2)| < \sigma(|y_1 - y_2|),$$

provided y_1 and y_2 lie in the same component of H . So, if $S(\delta_1)$ is properly chosen, we see that

$$\begin{aligned} D &\leq |n_1'((Er_3+a_1)\sin \theta + \delta) - n_1'((Er_4+a_1)\sin \theta)| E|r_1-r_2| \\ &\leq \sigma(|E(r_3-r_4)\sin \theta + \delta|)|r_1-r_2|E \\ &\leq \sigma(3\delta_1)|r_1-r_2|E, \end{aligned}$$

for $\theta \in [0, 2\pi] - S(\delta_1)$, where r_3 and r_4 are points which lie between r_1 and r_2 .

Hence, taking

$$L(t+\eta, \delta_1) = \begin{cases} 2LE, & \text{for } t+\eta+F-\alpha \in S(\delta_1), \\ \sigma(3\delta_1)E, & \text{for } t+\eta+F-\alpha \in [0, 2\pi] - S(\delta_1), \end{cases}$$

we see that

$$n_1((Er+a_1)\sin(t+\eta+F-\alpha) + \delta_1) - n_1((Er+a_1)\sin(t+\eta+F-\alpha))$$

is Lipschitz continuous in r , for $E|r| < \delta_1$, $t+\eta+F-\alpha \in [0, 2\pi]$, with Lipschitz function $L(t+\eta, \delta_1)$. In much the same manner, we can obtain a Lipschitz function for the variable η . So, provided ε is sufficiently small, we may apply Theorem 1 to obtain a unique periodic solution for equations (13) and (17) which is near $r = \eta = 0$, $x_5 = 0$, given by

$$\eta = f_1(t),$$

$$r = f_2(t),$$

$$x_5 = f_3(t) ,$$

with $|f_i(t)|$ uniformly small and $f_i(t+2\pi) = f_i(t)$ for $i = 1, 2, 3$.

That is, we have a 2π -periodic solution which is stable, if all $D_i > 0$, $i = 1, 2, \dots, J$. On the other hand, this periodic solution is unstable, if one of the D_i is negative. Combining this information with the Lipschitz continuity of u , we see that the periodic solution (in the variable v) is stable, if $D_k > 0$, $k = 1, 2, \dots, J$ and is unstable, if $D_k < 0$, for some k , $1 \leq k \leq J$. Since B is invertible, stability or instability in v is equivalent to that in \tilde{z} . After rescaling time, we arrive at the desired results. ■

V. EXAMPLES

Let

$$\omega = 1 ;$$

$$\alpha_0 = \frac{\pi}{4} ;$$

$a_1 > 0$ be a fixed, but arbitrary, real number;

$$\psi(t) \equiv 0 ;$$

$n(y)$ be any nonlinear function satisfying (H-2) and (H-3) of Theorem 2, such that its describing function, $N(a)$, satisfies the property $N'(a_1) > 0$;

$$p(s) = s ;$$

$$q(s) = s^4 + (k+1)s^3 + (k+1)s^2 + (k+2-N(a_1))s + (k-1),$$

where k is a parameter, with $k > 1$.

In addition, assume $n'(a_1)$ exists.

As required for controllability and observability, we see $p(s)$ and $q(s)$ have no common root. Moreover, evaluation at $s = i$ yields

$$p(i) = i ,$$

$$q(i) = -1 + (1-N(a_1))i ,$$

$$G(i\omega) = G(i) = p(i)/q(i) = i[-1 + (1-N(a_1))i]^{-1}.$$

Thus,

$$[1 + G(i\omega)N(a_1)]a_1 e^{-i\alpha_0} = G(i\omega)a_1\sqrt{2} .$$

So, if $a_0 = a_1\sqrt{2}$, we have a solution to the describing function equation (1).

Using the above parameters, we obtain

$$\alpha = \alpha_0 + \arg p(i\omega) = \frac{3\pi}{4} ,$$

$$\begin{aligned} d(s) &= q(s) + p(s)N(a_1) - \frac{a_0|p(i\omega)|}{a_1} \left\{ \sin \alpha \frac{s}{\omega} + \cos \alpha \right\} \\ &= s^4 + (k+1)s^3 + (k+1)s^2 + (k+1)s + k \\ &= (s^2+1)(s+1)(s+k) , \end{aligned}$$

$$\hat{\beta}_1 - i\hat{\beta}_2 = \frac{2}{\omega d'(i\omega)} = \frac{-(k+1) - (k-1)i}{2(k^2+1)} ,$$

$$\begin{aligned} D_3 &= 1 + a_1 N'(s_1) \operatorname{Re} \left\{ \frac{G(i\omega)}{1+G(i\omega)N(a_1)} \right\} \\ &= 1 + a_1 N'(a_1)/2 > 0 , \end{aligned}$$

$$\begin{aligned} D_4 &= \operatorname{Re} \{ (\hat{\beta}_1 - i\hat{\beta}_2) (2[q(i\omega) + p(i\omega)N(a_1)] + a_1 N'(a_1)p(i\omega)) \} \\ &= \frac{4k + (k-1)a_1 N'(a_1)}{2(k^2+1)} > 0 . \end{aligned}$$

Furthermore, since the roots of $d(s)$ are $\pm i$, -1 and $-k$, we see that it is unnecessary to check D_1 and D_2 .

Next, we compute $M_R + iM_I$, $Z_R + iZ_I$ and ε . Observe that

$$\begin{aligned} M_R + iM_I &= (\hat{\beta}_1 - i\hat{\beta}_2) \frac{a_0 |p(i\omega)|}{a_1} \exp(i\alpha) \\ &= \frac{k - li}{k^2 + 1}, \\ Z_R + iZ_I &= (\hat{\beta}_1 - i\hat{\beta}_2) \left(\frac{a_0}{a_1} \exp(i\alpha_0) + a_1 N'(a_1) \right) p(i\omega) \\ &= \frac{k - li}{k^2 + 1} + \frac{[(k-1) - (k+1)i] a_1 N'(a_1)}{2(k^2 + 1)} \end{aligned}$$

and

$$\varepsilon = (\hat{\beta}_1^2 + \hat{\beta}_2^2)^{1/2} = [2(k^2 + 1)]^{-1/2}.$$

In order to apply Theorem 1, we must have bounds on e^{At} and e^{Ct} , where

$$A = \begin{bmatrix} \frac{-k}{\sqrt{2(k^2+1)}} & \frac{2 + (k+1)a_1 N'(a_1)}{2a_1 \sqrt{2(k^2+1)}} \\ \frac{-a_1}{\sqrt{2(k^2+1)}} & \frac{-(2k+(k-1)a_1 N'(a_1))}{2\sqrt{2(k^2+1)}} \end{bmatrix} = [a_{ij}]_{2 \times 2}$$

and

$$C = \begin{bmatrix} 0 & 1 \\ -1 & -(k+1) \end{bmatrix}.$$

Observe that, as $k \rightarrow +\infty$, the eigenvalues of A tend to

$\frac{-1}{\sqrt{2}}$ and $\frac{-1}{\sqrt{2}} - \frac{a_1 N'(a_1)}{2\sqrt{2}}$. Thus, for all k sufficiently large,

$$e^{At} = \begin{bmatrix} 1 & 1 \\ v_1 & v_2 \end{bmatrix} e^{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} t} \begin{bmatrix} v_2 & -1 \\ -v_1 & 1 \end{bmatrix} \frac{a_{12}}{\lambda_2 - \lambda_1},$$

where λ_1 and λ_2 , $\lambda_1 \approx \frac{-1}{\sqrt{2}}$, $\lambda_2 \approx \frac{-1}{\sqrt{2}} - \frac{a_1 N'(a_1)}{2\sqrt{2}} \equiv \frac{-D_3}{\sqrt{2}}$, are the distinct eigenvalues of A , with $|\lambda_2 - \lambda_1| > \tilde{m} > 0$, \tilde{m} independent of

k . Here,

$$v_i = \frac{\lambda_i - a_{11}}{a_{12}}$$

is chosen so that $(1, v_i)^T$ is an eigenvector of A corresponding to the eigenvalue λ_i . We also note that

$$a_{12} \rightarrow \frac{N'(a_1)}{2\sqrt{2}},$$

$$v_1 \rightarrow 0$$

and

$$v_2 \rightarrow -a_1,$$

as $k \rightarrow +\infty$. A straightforward computation yields

$$e^{Ct} = (1-k)^{-1} \begin{bmatrix} e^{-kt} - ke^{-t} & e^{-kt} - e^{-t} \\ ke^{-t} - ke^{-kt} & e^{-t} - ke^{-kt} \end{bmatrix}.$$

Hence, there exist positive constants, m_∞ and $\lambda_\infty > 0$, independent of k , such that, for all $t > 0$ and any k sufficiently large,

$$|e^{At}| < m_\infty e^{-\lambda_\infty t} \quad \text{and} \quad |e^{Ct}| < m_\infty e^{-\lambda_\infty t}.$$

Also, due to the fact that $\deg p(s) < \deg q(s) - 3$, we have the following simple representation of $[x_6, x_7, x_8]^T$:

$$\begin{aligned} \begin{bmatrix} x_6 \\ x_7 \\ x_8 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ h^T \end{bmatrix} [\xi_3 \quad \xi_4] B_1^{-1} \varepsilon^{1/2} x_5 \\ &= \varepsilon^{1/2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 \\ 1 & k^2 \\ -1 & -k^3 \end{bmatrix} B_1^{-1} x_5 \\ &= \varepsilon^{1/2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} x_5. \end{aligned}$$

Moreover, since $a_0, a_1, \omega, p(i\omega)$ and α are fixed independent of the parameter k , $f_0(x_1, x_2)$ and $u(t, x_1, x_2)$ depend on k only through $\hat{\beta}_1$ and $\hat{\beta}_2$. Thus, we may apply Theorem 2 to obtain an asymptotically stable periodic solution, provided, of course, ε is sufficiently small, that is, provided k is sufficiently large.

In the above example, admissible choices of $n(y)$ include

- (i) $n(y) = y^p$, where p is any odd integer greater than 1,
- (ii) $n(y) = |y|y^p$, where p is any odd positive integer,
- (iii) any threshold nonlinearity, with $a_1 > \delta$ (see figure 2).

Furthermore, the condition $N'(a_1) > 0$ is not essential in the above example. In particular, if $n(y)$ is an ideal saturation function (see figure 3) and $a_1 > \delta$, then

$$a_1 N'(a_1) = \frac{-4m\delta}{\pi a_1} \sqrt{1 - \left(\frac{\delta}{a_1}\right)^2}.$$

Thus, provided $N'(a_1) \neq 0$ and $D_3 > 0$, the above computations hold without the requirement $N'(a_1) > 0$. Hence, for the above choice of $\omega, \alpha_0, p(s)$ and $q(s)$ and for any ideal saturation nonlinearity, by taking a_1 and k sufficiently large, we will obtain an asymptotically stable periodic solution.

Although we have presented an existence, uniqueness and exact stability analysis based upon the describing function method, we concede that the task of checking the "sufficiently small" hypothesis is

formidable. Thus, in actual practice, we urge control engineers to use the above theory as they currently use the describing method:

when the theory predicts that the system has the desired response properties, verify by simulation.

The following example illustrates this approach.

Suppose $G(s)$ and $n(y)$ are given by

$$G(s) = \frac{s+1}{s(s+2)(s+3)}$$

and

$$n(y) = \begin{cases} 4y, & \text{for } |y| \leq 1, \\ 4|y|/y, & \text{for } |y| > 1. \end{cases}$$

The function, $n(y)$, is an ideal saturation function (see figure 3, with $m = 4$, $\delta = 1$). Observe that the describing function equation (1) is satisfied when

$$a_0 \approx 257.6772484 ,$$

$$a_1 \approx 2.4754145 \quad (N(a_1) = 2) ,$$

$$\omega = 10 ,$$

$$\alpha_0 \approx 2.745118723 \text{ (radians)}.$$

In addition, we have

$$d(s) = (s^2 + 100)(s + 5) ,$$

$$\alpha \approx -2.06693891 + 2k\pi , \quad k \text{ any integer},$$

$$\hat{\beta}_1 - i\hat{\beta}_2 = \frac{-2 - 1i}{2500} ,$$

$$D_1 > 0 ,$$

$$D_2 > 0 ,$$

and $D_3 > 0 .$

Since $\hat{\beta}_1$ and $\hat{\beta}_2$ are "small", we expect that system (2)-(3) has a locally asymptotically stable periodic solution near

$$\begin{aligned} \tilde{x}_0(t) &= (\tilde{x}_{01}(t), \tilde{x}_{02}(t), \tilde{x}_{03}(t))^T \\ &= \frac{a_1}{\sqrt{101}} (\sin(10t - \alpha), 10 \cos(10t - \alpha), -100 \sin(10t - \alpha))^T . \end{aligned}$$

In order to numerically substantiate this conjecture, we first simulated the system using the initial conditions $x_0(0) = \tilde{x}_0(0)$. In figure 4, the first component of the resulting solution, $x_0(t)$, is plotted superimposed on the plot of $\tilde{x}_{01}(t)$. Next, we conducted various simulations using initial conditions near $\tilde{x}_0(t)$. In figure 5, one such simulation is displayed. The plot shows the first component of the solution, $\tilde{x}_{01}(t)$.

The numerical evidence appears to support the existence of a locally asymptotically stable period solution near $\tilde{x}_0(t)$.

VI. CONCLUDING REMARKS

We have presented an exact stability analysis for nonlinear systems subjected to continuous, nonconstant periodic inputs. More precisely, if a system of the form (2')-(3') satisfies hypotheses (H-1) through (H-4), and provided $\epsilon = |\hat{\beta}_1 - i\hat{\beta}_2| = \left| \frac{2}{\omega d'(i\omega)} \right|$ is sufficiently small, then

(i) the existence of a $2\pi/\omega$ periodic state-space solution is guaranteed,

(ii) the $2\pi/\omega$ periodic solution is unique in the sense that it is the only solution which remains (for all $t \in \mathbb{R}$) in a particular neighborhood of the "approximate solution" predicted by the describing function technique,

(iii) the local stability (asymptotic stability or instability) of the state-space solution is easily obtained from the linearization of the problem.

The stability of the linearization is checked by a modified Hurwitz criterion.

In conclusion, we point out that, although we have used a describing function approximation, our results are for an actual periodic solution of the nonlinear system. That is, we have analyzed the the actual system (either (2)-(3) or (2')-(3')) and its actual response; not an approximate system or an approximate system response.

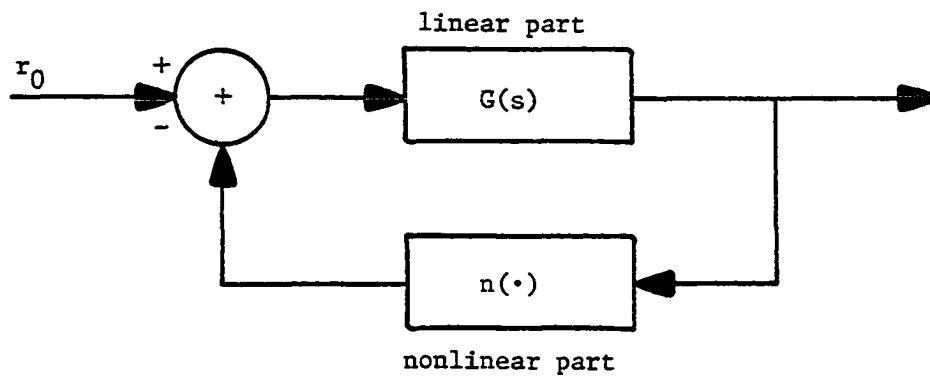


Figure 1. Block diagram of the system

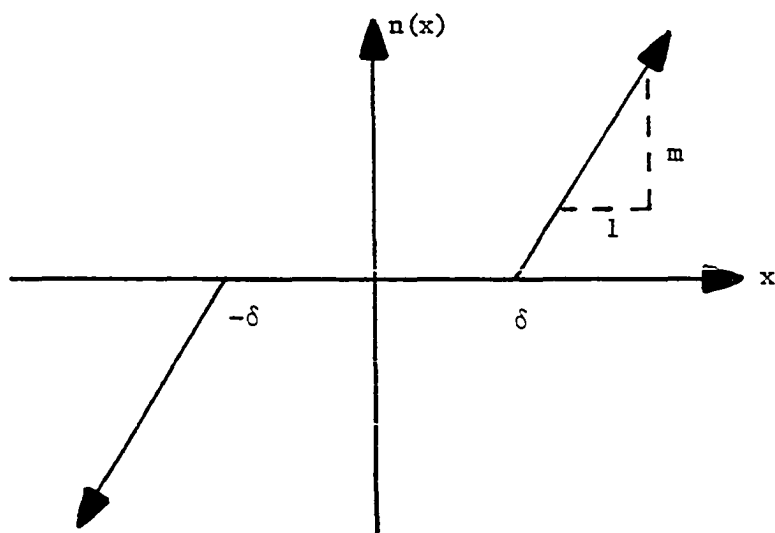


Figure 2. Threshold nonlinearity

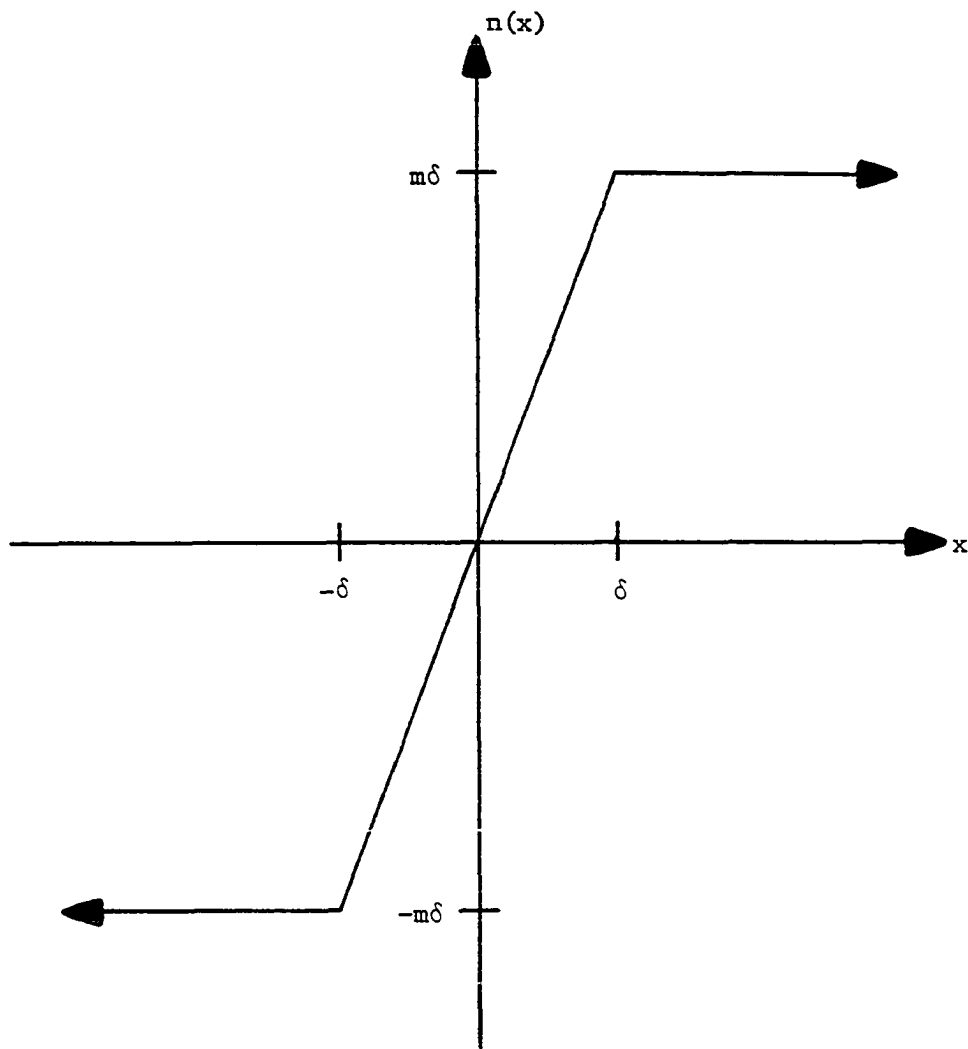


Figure 3. Saturation function

$$n(x) = \begin{cases} mx, & \text{for } |x| \leq \delta, \\ m\delta|x|/x, & \text{for } |x| > \delta \end{cases}$$

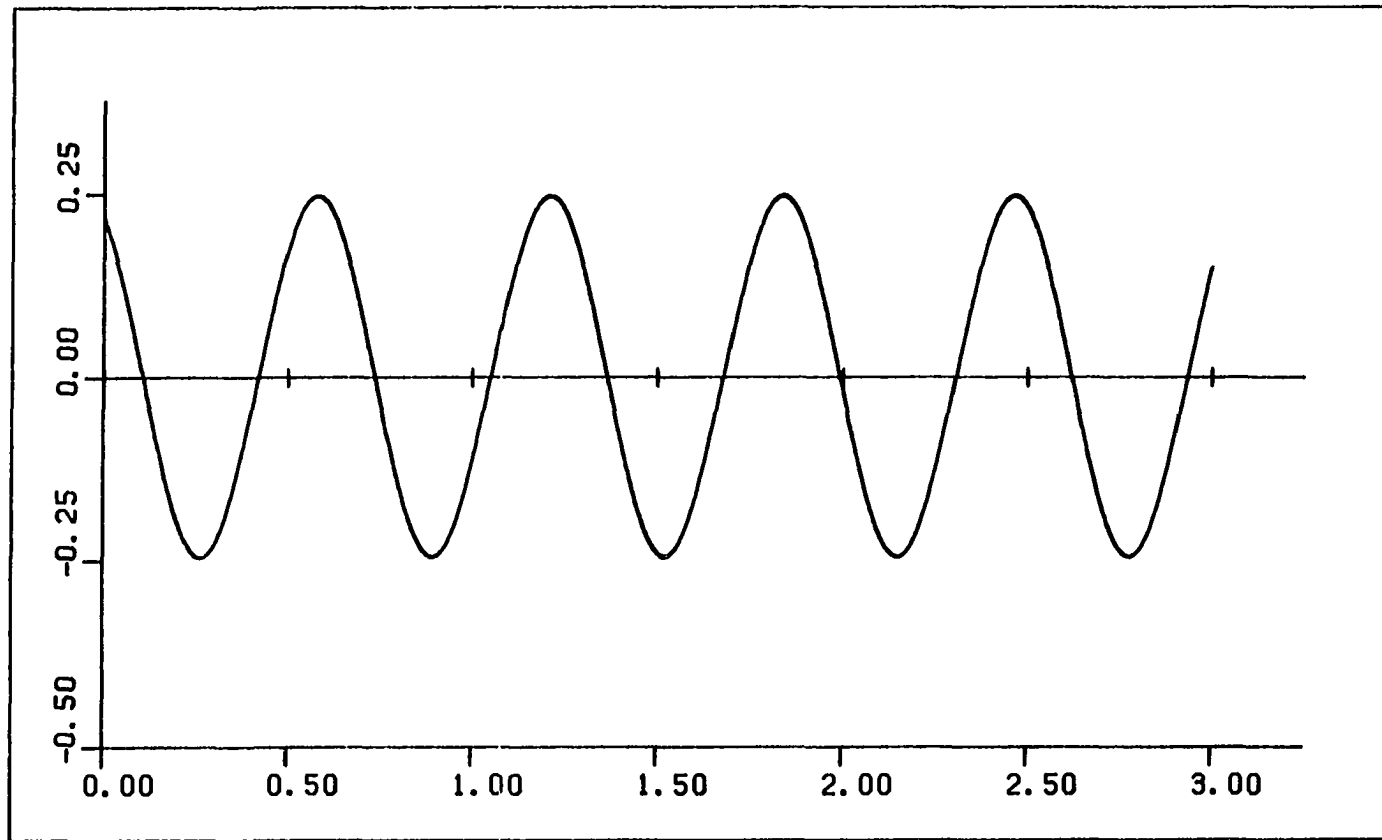


Figure 4. The graph of $x_{01}(t)$ superimposed upon the graph of

$$\tilde{x}_{01}(t) = \frac{a_1}{E} \sin(\omega t - \alpha)$$

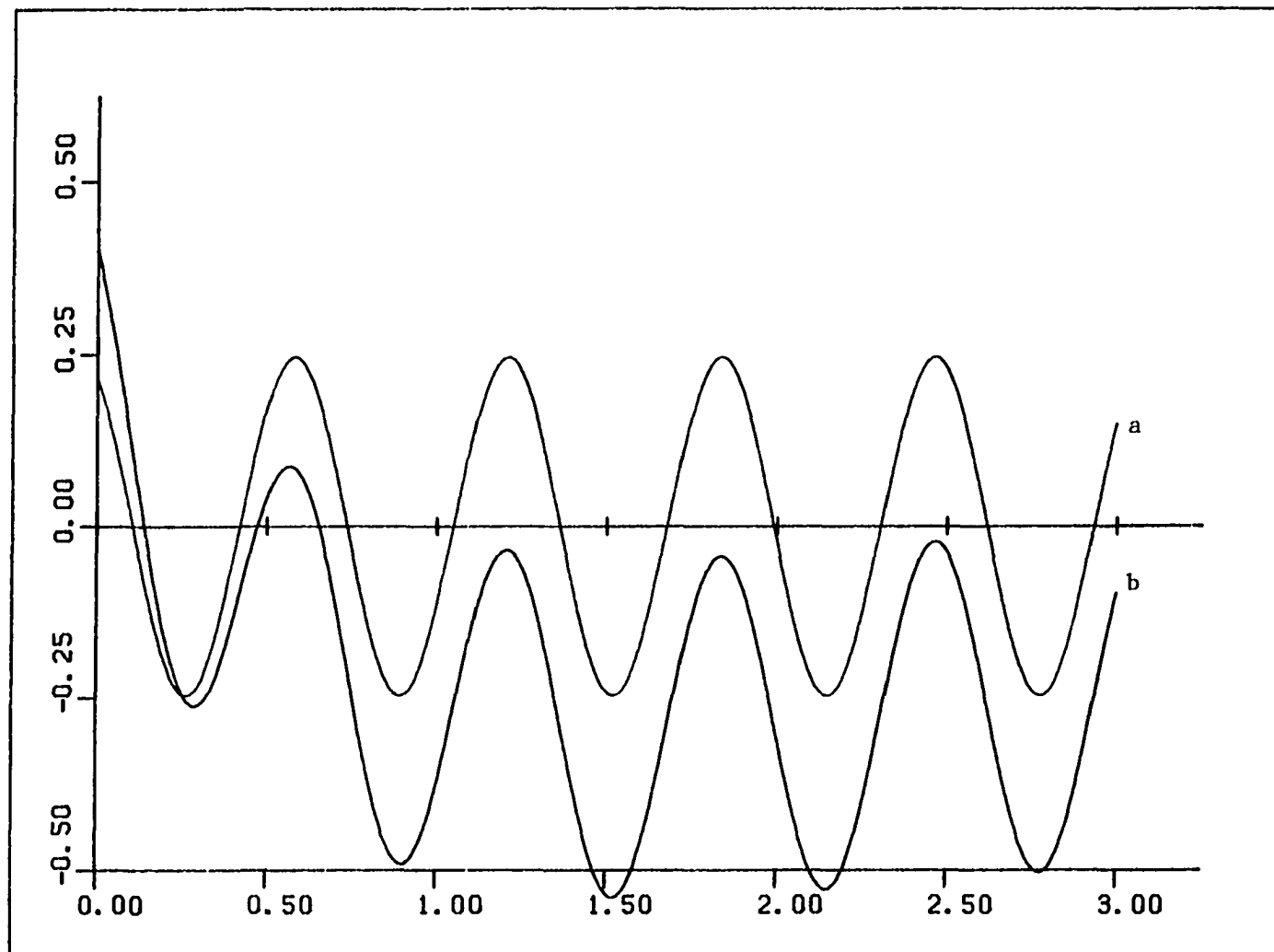


Figure 5. The graph of $\tilde{x}_{01}(t) = \frac{a_1}{E} \sin(\omega t - \alpha)$ [curve a] superimposed upon the numerical solution of $x_{01}(t)$ [curve b]

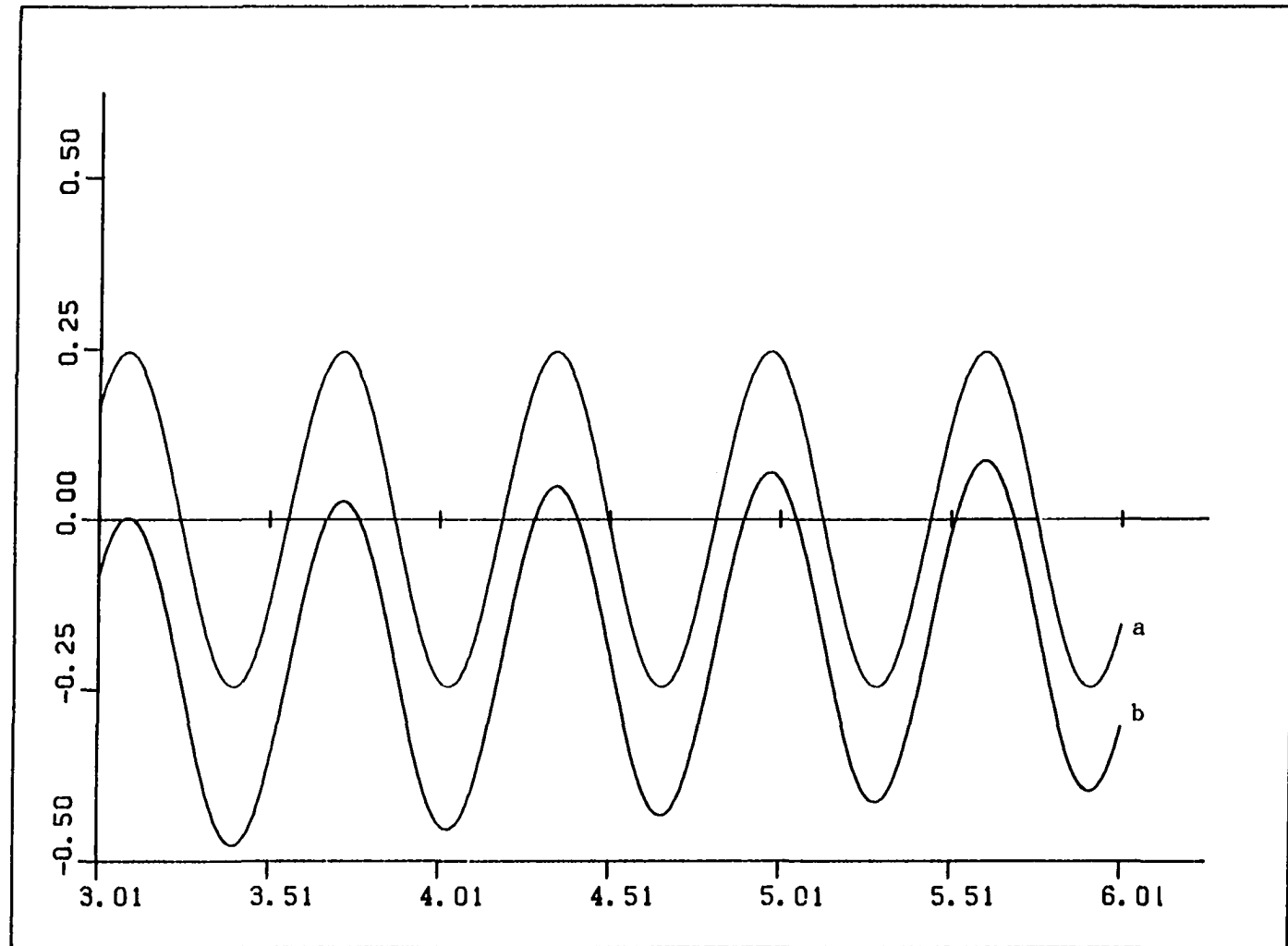


Figure 5 (continued)

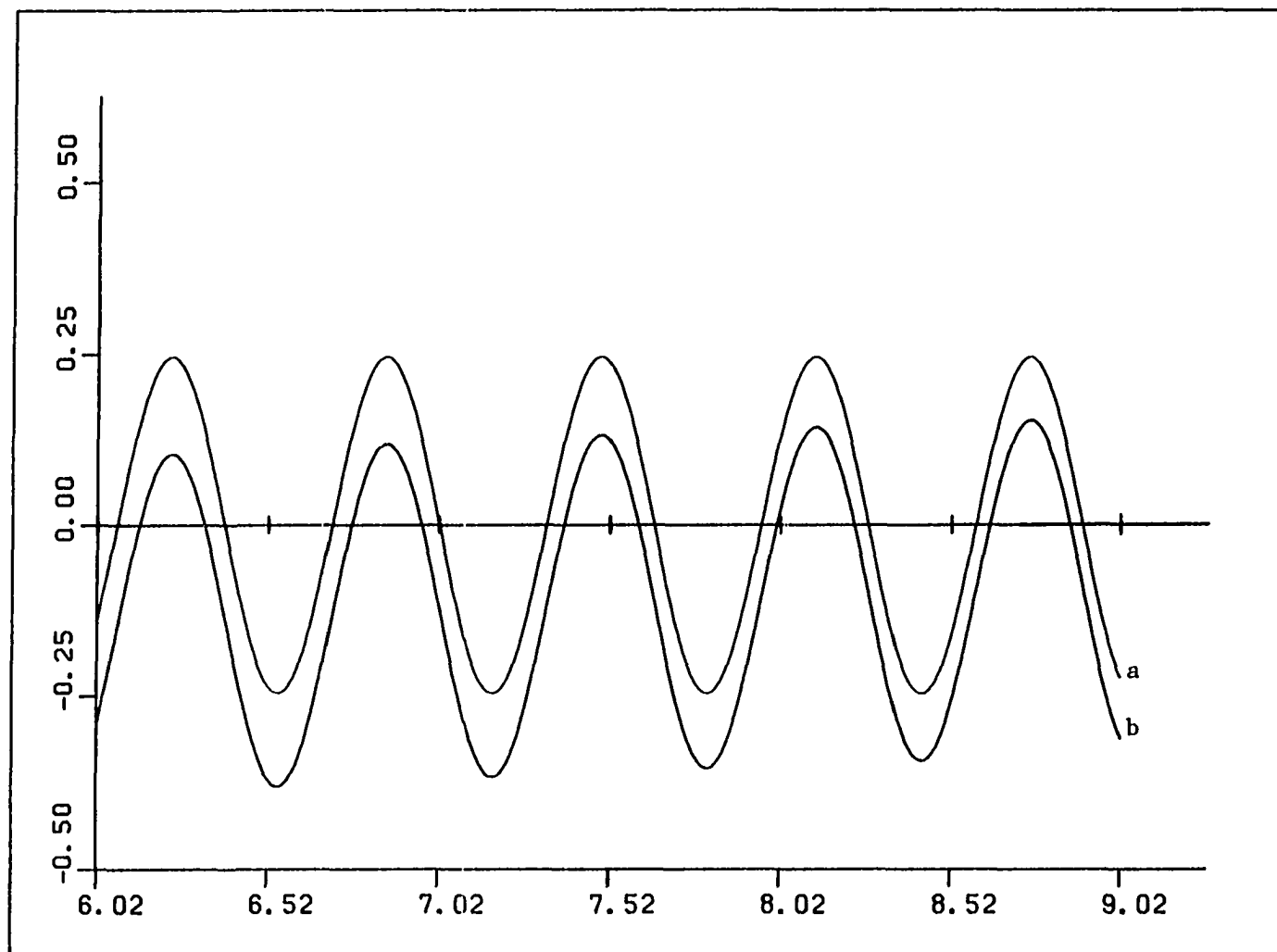


Figure 5 (continued)

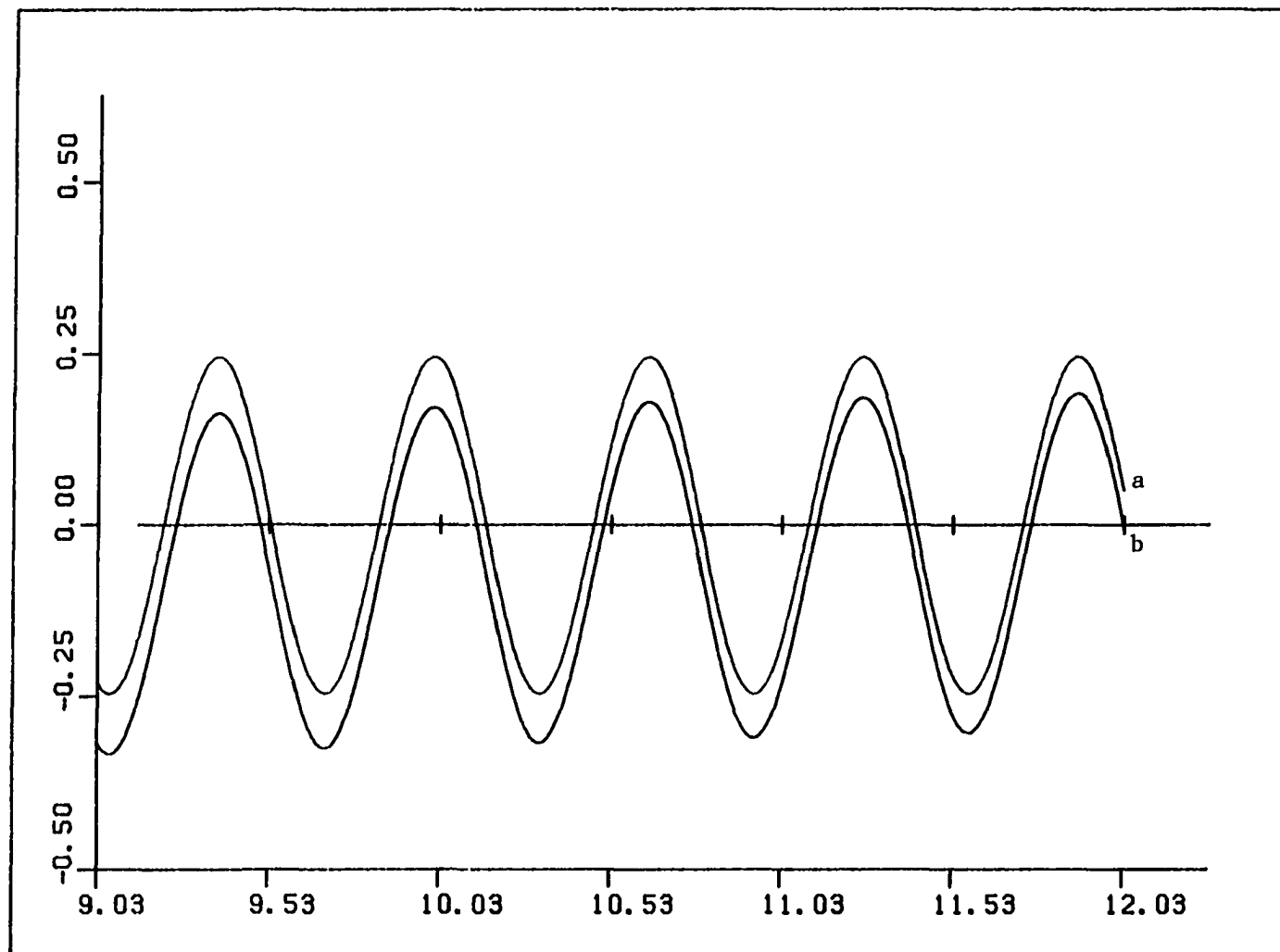


Figure 5 (continued)

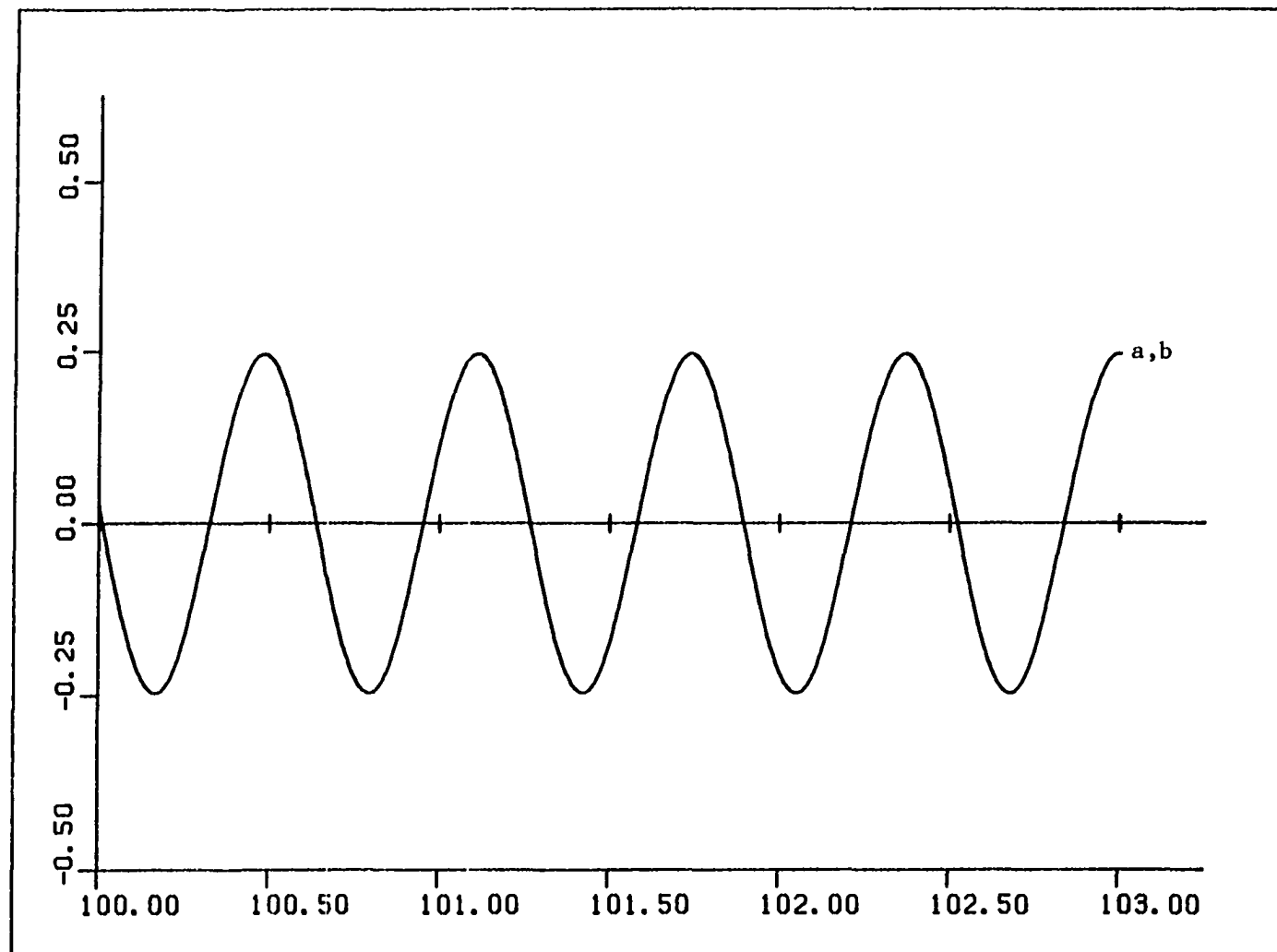


Figure 5 (continued)

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CONCLUSION

In Part I of this dissertation, we showed that the Loeb criterion is correct for the systems analyzed, provided that the quantities $\hat{\beta}_1$ and $\hat{\beta}_2$ are sufficiently small. In fact, the Loeb criterion inequality was shown to be equivalent to requiring the linearized polar equation for the fundamental amplitude to be asymptotically stable. This is consistent with the initial motivation of the stability criterion (see [15]). Moreover, we explicitly stated the requirement that the remaining roots of the linearized problem have negative real parts in order to have locally asymptotically stable oscillations. This requirement is often overlooked, even though it is implied by the heuristic motivation of the stability criterion.

In the forced case (Part II of this dissertation), we showed that the describing function method can be used to predict the existence and stability of a periodic response. More precisely, the describing function method yields a predicted amplitude and phase shift. We then showed that, in a sup norm neighborhood of the predicted sinusoidal response, there is one and only one periodic solution, i.e., local uniqueness.

In Parts I and II, the exact stability type is easily checked by a modified Routh-Hurwitz test. In both Parts I and II, the Routh-Hurwitz test depends upon the linear part's transfer function, the nonlinearities' describing function, and upon parameters from the describing function equation solution. We do not require exact information concerning the solutions. Of course, the price for using approximate

information is the "sufficiently small" assumption. This is a curse typically associated with qualitative stability analysis. Although, in theory, the "sufficiently small" can be quantified, in practice, it is extremely difficult. Furthermore, since all estimates are absolute and fail to account for the oscillatory nature of the solutions, even if we could obtain a quantified "sufficiently small", we would expect the result to be very conservative.

As observed in Part II, stability in the forced case is extremely complex. The linearized problem clearly shows the coupling of the amplitude deviations and the phase angle deviations. However, this is to be expected since we are examining the stability properties of a single periodic response rather than the stability properties of a surface of solutions, i.e., the integral manifold.

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APPENDIX A: THE INTEGRAL MANIFOLD THEOREM

Consider a coupled system of ordinary differential equations of the form

$$(E) \quad \begin{cases} \theta' = \varepsilon \theta(t, \theta, x, y, \varepsilon) \\ x' = \varepsilon A x + \varepsilon X(t, \theta, x, y, \varepsilon) \\ y' = B y + \varepsilon^{1/2} Y(t, \theta, x, y, \varepsilon) \end{cases},$$

defined on a set $\Omega = \{(t, \theta, x, y, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^j \times \mathbb{R} : |x| < \tilde{\eta}, |y| < \tilde{\eta}, 0 < \varepsilon < \varepsilon_0\}$, with θ, X and Y continuous on Ω . We assume that A and B are noncritical and, without loss of generality, that $A = \text{diag}(A_+, A_-)$, $B = \text{diag}(B_+, B_-)$, where the eigenvalues of the matrices A_+, B_+ have positive real parts and the eigenvalues of A_-, B_- have negative real parts. Let θ, X and Y be 2π -periodic in both t and θ . Let Y be Lipschitz continuous in (θ, x, y) , with Lipschitz constant $M > 0$. Let X satisfy the condition

$$|X(t, \theta, x, y, \varepsilon)| < \kappa(\tilde{b}) + \tilde{a} \kappa(\tilde{a}),$$

for all $(t, \theta, x, y, \varepsilon) \in \Omega$ with $|x| < \tilde{a}$, $|y| < \tilde{a}$ and $0 < \varepsilon < \tilde{b}$, where $\kappa(\cdot)$ is a continuous, monotone increasing function with $\kappa(0) = 0$.

Moreover, we assume there is a nonnegative function, $L(u, v, w)$, which is 2π -periodic in u , such that, for $u \in [0, 2\pi]$

$$(1) \quad L(u, v, w) = \sum_{n=1}^N c_n(v, w) \chi_{I_{n, v, w}}(u), \quad \text{where}$$

$$(a) \quad 0 < c_n(v, w) < M_0 < \infty \quad \text{for } 1 \leq n \leq N, \quad 0 < v < \tilde{\eta}, \quad 0 < w < \varepsilon_0,$$

$$(b) \quad I_{n, v, w} \text{ is an interval in } [0, 2\pi], \quad \text{for } 1 \leq n \leq N \text{ and}$$

$$I_{n_1, v, w} \cap I_{n_2, v, w} = \emptyset \quad \text{if } n_1 \neq n_2,$$

$$(c) \quad \chi_{I_{n, v, w}}(u) \text{ is the characteristic function for } I_{n, v, w}, \text{ i.e.,}$$

$$\chi_{I_{n, v, w}}(u) = \begin{cases} 1, & \text{for } u \in I_{n, v, w}, \\ 0, & \text{for } u \notin I_{n, v, w}, \end{cases}$$

$$(d) \quad c_n(v, w) m[I_{n, v, w}] \rightarrow 0 \quad \text{for all } n, \quad 1 \leq n \leq N, \quad \text{as } (v, w) \rightarrow 0$$

(here, $m[I]$ denotes the Lebesgue measure of the set I),

and both X and θ satisfy the following, nonstandard, Lipschitz conditions:

$$(2) \quad \begin{aligned} & |\theta(t, \theta_2, x, y, \varepsilon) - \theta(t, \theta_1, x, y, \varepsilon)| \leq L(t + \theta_1, \tilde{a}, \tilde{b}) |\theta_2 - \theta_1|, \\ & |X(t, \theta_2, x, y, \varepsilon) - X(t, \theta_1, x, y, \varepsilon)| \leq L(t + \theta_1, \tilde{a}, \tilde{b}) |\theta_2 - \theta_1|, \\ & |\theta(t, \theta, x_2, y, \varepsilon) - \theta(t, \theta, x_1, y, \varepsilon)| \leq L(t + \theta, \tilde{a}, \tilde{b}) |x_2 - x_1|, \\ & |X(t, \theta, x_2, y, \varepsilon) - X(t, \theta, x_1, y, \varepsilon)| \leq L(t + \theta, \tilde{a}, \tilde{b}) |x_2 - x_1|, \\ & |\theta(t, \theta, x, y_2, \varepsilon) - \theta(t, \theta, x, y_1, \varepsilon)| \leq L(t + \theta, \tilde{a}, \tilde{b}) |y_2 - y_1|, \\ & |X(t, \theta, x, y_2, \varepsilon) - X(t, \theta, x, y_1, \varepsilon)| \leq L(t + \theta, \tilde{a}, \tilde{b}) |y_2 - y_1|, \end{aligned}$$

for all $t, \theta, \theta_1, \theta_2 \in \mathbb{R}$, $x, x_1, x_2 \in \mathbb{R}^k$, $y, y_1, y_2 \in \mathbb{R}^j$, with $|x| < \tilde{a}$, $|x_i| < \tilde{a}$, $|y| < \tilde{a}$, $|y_i| < \tilde{a}$ and $0 < \varepsilon < \tilde{b}$.

We will need the concept of an integral manifold of the system (E).

Definition: A surface S_ε in (t, θ, x, y) space is an **integral manifold** of system (E) for a fixed ε , if, for any point $(t_0, \theta_0, x_0, y_0, \varepsilon) \in \Omega$ such that $(t_0, \theta_0, x_0, y_0) \in S_\varepsilon$, the solution of (E) passing through (θ_0, x_0, y_0) at time t_0 satisfies $(t, \theta(t), x(t), y(t)) \in S_\varepsilon$, for all $t \in \mathbb{R}$.

For example, if we set $\theta \equiv 0$, $X \equiv 0$, $Y \equiv 0$ and $\varepsilon > 0$, then there is an integral manifold

$$S = \{(t, \theta, 0, 0) : -\infty < t, \theta < \infty\}$$

for the unperturbed system (E). Clearly, when both A and B are stable matrices, solutions of the unperturbed linear system will tend to S as $t \rightarrow +\infty$. However, if either A or B has an eigenvalue with a positive real part, then S is unstable (with respect to the unperturbed linear system) in the sense of Lyapunov [33]. From the assumption on the magnitude of X and since $\varepsilon^{1/2} Y$ tends to zero as $\varepsilon \rightarrow 0^+$, we expect, for all sufficiently small $\varepsilon > 0$, an integral manifold, S_ε , for the perturbed system (E). We also expect S_ε to be near S and that S_ε will inherit the stability properties of S .

The above discussion gives rise to the theorem:

Theorem 1: Suppose that θ , X and Y satisfy the periodicity, continuity, Lipschitz and norm conditions stated above. Assume that L satisfies (1) and suppose A and B are noncritical. Then, there exist $\varepsilon_1 > 0$, $C(\varepsilon) > D(\varepsilon) > 0$ and $\Delta(\varepsilon) > 0$, such that in the region

$$\Omega_\varepsilon = \{(t, \theta, x, y) : (t, \theta, x, y, \varepsilon) \in \Omega, \quad |x| < C, \quad |y| < C, \quad 0 < \varepsilon < \varepsilon_1\},$$

there is an integral manifold S_ε of (E) , given by

$$(3) \quad S_\varepsilon = \{(t, \theta, f_1(t, \theta, \varepsilon), f_2(t, \theta, \varepsilon)) : t, \theta \in \mathbb{R}\},$$

where $|f_i(t, \theta, \varepsilon) - f_i(t, \bar{\theta}, \varepsilon)| < \Delta(\varepsilon)|\theta - \bar{\theta}|$, for all $t, \theta, \bar{\theta} \in \mathbb{R}$,

$$\|f_i\| < D(\varepsilon),$$

$$f_i(t+2\pi, \theta, \varepsilon) = f_i(t, \theta+2\pi, \varepsilon) = f_i(t, \theta, \varepsilon), \quad \text{for all } t, \theta \in \mathbb{R},$$

and

$$f_i \in C(\mathbb{R}^2 \times \{\varepsilon\}).$$

The integral manifold S_ε is unique in the sense that if a solution $(t, \theta(t), x(t), y(t)) \in \Omega_\varepsilon$, for all $t \in \mathbb{R}$, then $(t, \theta(t), x(t), y(t)) \in S_\varepsilon$, for all $t \in \mathbb{R}$.

In addition, if either A or B have an eigenvalue with a positive real part, then S_ε is unstable in the sense of Lyapunov. However, if both A and B are stable matrices, then, provided $|\rho - f_1(t_0, \theta_0, \varepsilon)|$ and $|\xi - f_2(t_0, \theta_0, \varepsilon)|$ are sufficiently small, the solution of (E) which passes through (θ_0, ρ, ξ) at time t_0 will tend exponentially to some solution on S_ε as $t \rightarrow +\infty$. ■

We will present the proof of Theorem 1 as a sequence of lemmas. In these lemmas we will adopt the following notation:

$$(4) \quad J(t) = - \begin{bmatrix} -A_+ t & 0 \\ e & 0 \\ 0 & 0 \end{bmatrix}, \quad K(t) = - \begin{bmatrix} -B_+ t & 0 \\ e & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{for } t > 0,$$

$$J(t) = \begin{bmatrix} 0 & 0 \\ 0 & -A_- t \\ 0 & e \end{bmatrix}, \quad K(t) = \begin{bmatrix} 0 & 0 \\ 0 & -B_- t \\ 0 & e \end{bmatrix}, \quad \text{for } t < 0$$

and $J(-0) - J(+0) = I$, $K(-0) - K(+0) = I$. It is obvious that there are constants $\alpha > 0$, $\beta > 0$, such that $|J(t)| < \beta e^{-\alpha|t|}$ and $|K(t)| < \beta e^{-\alpha|t|}$, for all $t \in \mathbb{R}$.

By $\psi(t, \tau, \theta, f, \varepsilon)$, we will mean the solution of

$$\frac{d\psi}{dt} = \varepsilon \theta(t, \psi, f_1(t, \psi, \varepsilon), f_2(t, \psi, \varepsilon), \varepsilon),$$

$$\psi(\tau) = \theta,$$

where $f = (f_1, f_2)$. The existence and uniqueness of $\psi(t, \tau, \theta, f, \varepsilon)$ will be obvious from later restrictions on the functions. Moreover, since ε will be a fixed positive number, we will usually drop the explicit statement of dependence of the functions upon ε .

Let $\|\cdot\|$ denote the supremum of the norm of a function over its domain (with ε fixed). For example, $\|X\| = \sup |X(t, \theta, x, y, \varepsilon)|$, where the supremum is over (t, θ, x, y) with $t, \theta \in \mathbb{R}$, $|x| < \tilde{\eta}$ and $|y| < \tilde{\eta}$.

The first lemma gives the existence and "uniqueness" of an integral manifold.

Lemma 1. Suppose that θ , X and Y satisfy the above continuity, periodicity, Lipschitz and norm conditions. Assume there exists a $C(\varepsilon) > 0$, $D(\varepsilon) > 0$ and $\Delta(\varepsilon) > 0$, with $D < C < \tilde{\eta}$, such that

$$(5) \quad 2\alpha^{-1}\beta \max\{\kappa(\varepsilon) + C\kappa(C), \varepsilon^{1/2} \|Y\|\} < D,$$

$$(6) \quad \max_{t \in \mathbb{R}} \int_{-\infty}^{\infty} \varepsilon \beta e^{-\varepsilon \alpha |s-t|} |L(s+\psi(s, t, \theta, f), C, \varepsilon)|$$

$$\cdot \exp(\varepsilon(1+2\Delta)) \left| \int_t^s L(u+\psi(u, t, \theta, f), C, \varepsilon) du \right| ds < \frac{\Delta}{1+2\Delta}$$

and

$$(7) \quad \max_{t \in \mathbb{R}} \int_{-\infty}^{\infty} \varepsilon^{1/2} \beta M \exp(-\alpha |s-t| + \varepsilon(1+2\Delta)) \left| \int_t^s L(u+\psi(u, t, \theta, f), C, \varepsilon) du \right| ds$$

$$< \frac{\Delta}{1+2\Delta}$$

hold for all $f = (f_1, f_2)$, with $\|f_i\| < C$, such that $\psi(u, t, \theta, f)$ exists and is unique for all $u \in R$, $(t, \theta) \in \text{domain } f$. Then, there exists a $f = (f_1, f_2) \in C(R^2)$, such that (3) is an integral manifold of (E), with $\|f_i\| < D$, $|f_i(t, \theta) - f_i(t, \bar{\theta})| < \Delta|\theta - \bar{\theta}|$, for $i = 1, 2$, $t, \theta, \bar{\theta} \in R$ and $f(t+2\pi, \theta) = f(t, \theta+2\pi) = f(t, \theta)$. Moreover, if $(t, \theta(t), x(t), y(t))$ is a solution of (E), such that $|x(t)| < C$, $|y(t)| < C$, for all $t \in R$, then $(t, \theta(t), x(t), y(t)) \in S_\varepsilon$, for all $t \in R$.

Proof. Let $F(D(\varepsilon), \Delta(\varepsilon))$ be the space of continuous functions given by

$$F(D, \Delta) = \{f \in C(R^2) : \|f\| < D, \quad f(t+2\pi, \theta) = f(t, \theta) = f(t, \theta+2\pi),$$

$$\text{for all } t, \theta \in R, \quad f: R^2 \rightarrow R^k \times R^j, \text{ with}$$

$$|f_i(t, \theta) - f_i(t, \bar{\theta})| < \Delta|\theta - \bar{\theta}|, \quad i = 1, 2\},$$

where $\|f\| = \max\{\|f_1\|, \|f_2\|\}$. Clearly, $F(D, \Delta)$ is a complete metric space under the metric induced by $\|\cdot\|$.

We now define an operator T , on $F(D, \Delta)$, by

$$Tf(t, \theta) = \begin{bmatrix} T_1 f(t, \theta) \\ T_2 f(t, \theta) \end{bmatrix},$$

where

$$T_1 f(t, \theta) = \int_{-\infty}^{\infty} J(\varepsilon(s-t)) \varepsilon X(s, \psi(s, t, \theta, f), f_1(s, \psi(s, t, \theta, f))),$$

$$f_2(s, \psi(s, t, \theta, f)) ds ,$$

$$T_2 f(t, \theta) = \int_{-\infty}^{\infty} K(s-t) \varepsilon^{1/2} Y(s, \psi(s, t, \theta, f), f_1(s, \psi(s, t, \theta, f))),$$

$$f_2(s, \psi(s, t, \theta, f)) ds .$$

We now derive some estimates which will be used later to show $T: F(D, \Delta) \rightarrow F(D, \Delta)$, such that T is a contraction map. Since, for $f \in F(D, \Delta)$,

$$|\psi(s, t, \theta, f) - \psi(s, t, \bar{\theta}, f)|$$

$$< |\theta - \bar{\theta}| + \left| \int_t^s \varepsilon |\theta(u, \psi(u, t, \theta, f), f_1(u, \psi(u, t, \theta, f))),$$

$$f_2(u, \psi(u, t, \theta, f))$$

$$- \theta(u, \psi(u, t, \bar{\theta}, f), f_1(u, \psi(u, t, \bar{\theta}, f))),$$

$$f_2(u, \psi(u, t, \bar{\theta}, f)) |du|$$

$$< |\theta - \bar{\theta}| + \left| \int_t^s \varepsilon L(u, \psi(u, t, \theta, f), C) (1+2\Delta) \right.$$

$$\cdot |\psi(u, t, \theta, f) - \psi(u, t, \bar{\theta}, f)| du | ,$$

by Gronwall's inequality [33], we have

$$(8) \quad |\psi(s, t, \theta, f) - \psi(s, t, \bar{\theta}, f)| \leq |\theta - \bar{\theta}| \exp\left(\left|\int_t^s \varepsilon L(u, \psi(u, t, \theta, f), C)(1+2\Delta) du\right|\right)$$

Similarly, for $f, g \in F(D, \Delta)$,

$$\begin{aligned} & |\psi(s, t, \theta, f) - \psi(s, t, \theta, g)| \\ & \leq \varepsilon \left| \int_t^s \left| \theta(u, \psi(u, t, \theta, f), f_1(u, \psi(u, t, \theta, f)), f_2(u, \psi(u, t, \theta, f))) \right. \right. \\ & \quad \left. \left. - \theta(u, \psi(u, t, \theta, g), g_1(u, \psi(u, t, \theta, g)), g_2(u, \psi(u, t, \theta, g))) \right| du \right| \\ & \leq \varepsilon \left| \int_t^s \left\{ \left| \theta(u, \psi(u, t, \theta, f), f_1(u, \psi(u, t, \theta, f)), f_2(u, \psi(u, t, \theta, f))) \right. \right. \right. \\ & \quad \left. \left. - \theta(u, \psi(u, t, \theta, g), f_1(u, \psi(u, t, \theta, f)), f_2(u, \psi(u, t, \theta, f))) \right| \right. \\ & \quad + \left| \theta(u, \psi(u, t, \theta, g), f_1(u, \psi(u, t, \theta, f)), f_2(u, \psi(u, t, \theta, f))) \right. \\ & \quad \left. - \theta(u, \psi(u, t, \theta, g), f_1(u, \psi(u, t, \theta, g)), f_2(u, \psi(u, t, \theta, f))) \right| \\ & \quad + \left| \theta(u, \psi(u, t, \theta, g), f_1(u, \psi(u, t, \theta, g)), f_2(u, \psi(u, t, \theta, f))) \right. \\ & \quad \left. - \theta(u, \psi(u, t, \theta, g), f_1(u, \psi(u, t, \theta, g)), f_2(u, \psi(u, t, \theta, g))) \right| \\ & \quad \left. + \left| \theta(u, \psi(u, t, \theta, g), f_1(u, \psi(u, t, \theta, g)), f_2(u, \psi(u, t, \theta, g))) \right. \right. \\ & \quad \left. \left. - \theta(u, \psi(u, t, \theta, g), g_1(u, \psi(u, t, \theta, g)), g_2(u, \psi(u, t, \theta, g))) \right| \right\} du \right| \end{aligned}$$

$$\leq \varepsilon \left| \int_t^s L(u+\psi(u,t,\theta,g),C)(1+2\Delta) \left\{ |\psi(u,t,\theta,f) - \psi(u,t,\theta,g)| \right. \right. \\ \left. \left. + \frac{\|f_1 - g_1\| + \|f_2 - g_2\|}{1+2\Delta} \right\} du \right| .$$

Adding $2 \|f-g\| / (1+2\Delta)$ to both sides, yields

$$\left\{ |\psi(s,t,\theta,f) - \psi(s,t,\theta,g)| + 2 \|f-g\| / (1+2\Delta) \right\} \\ \leq 2 \|f-g\| / (1+2\Delta) + \left| \int_t^s \varepsilon(1+2\Delta)L(u+\psi(u,t,\theta,g),C) \right. \\ \left. \cdot \left\{ |\psi(u,t,\theta,f) - \psi(u,t,\theta,g)| + 2 \|f-g\| / (1+2\Delta) \right\} du \right| .$$

By applying Gronwall's inequality, we have

$$\left\{ |\psi(s,t,\theta,f) - \psi(s,t,\theta,g)| + 2 \|f-g\| / (1+2\Delta) \right\} \\ \leq \left\{ 2 \|f-g\| / (1+2\Delta) \right\} \exp \left(\left| \int_t^s \varepsilon(1+2\Delta)L(u+\psi(u,t,\theta,g),C) du \right| \right) ,$$

which implies

$$(9) \quad |\psi(s,t,\theta,f) - \psi(s,t,\theta,g)|$$

$$\leq 2 \|f-g\| \left\{ \exp \left(\left| \int_t^s \varepsilon(1+2\Delta)L(u+\psi(u,t,\theta,g),C) du \right| \right) - 1 \right\} / (1+2\Delta) .$$

For $h \in \mathbb{R}$ and $f \in F(D, \Delta)$, we have

$$\begin{aligned}
 & |\psi(s, t+h, \theta, f) - \psi(s, t, \theta, f)| \\
 & \leq \varepsilon \left| \int_{t+h}^s \theta(u, \psi(u, t+h, \theta, f), f_1(u, \psi(u, t+h, \theta, f)), f_2(u, \psi(u, t+h, \theta, f))) du \right. \\
 & \quad \left. - \int_t^s \theta(u, \psi(u, t, \theta, f), f_1(u, \psi(u, t, \theta, f)), f_2(u, \psi(u, t, \theta, f))) du \right| \\
 & \leq \varepsilon \left| \int_{t+h}^t \theta(u, \psi(u, t+h, \theta, f), f_1(u, \psi(u, t+h, \theta, f)), f_2(u, \psi(u, t+h, \theta, f))) du \right| \\
 & \quad + \varepsilon \left| \int_t^s L(u, \psi(u, t, \theta, f), C)(1+2\Delta) |\psi(u, t+h, \theta, f) - \psi(u, t, \theta, f)| du \right|.
 \end{aligned}$$

Thus, by Gronwall's inequality, we have

$$\begin{aligned}
 (10) \quad & |\psi(s, t+h, \theta, f) - \psi(s, t, \theta, f)| \\
 & \leq \varepsilon \left| \int_t^{t+h} \theta(u, \psi(u, t+h, \theta, f), f_1(u, \psi(u, t+h, \theta, f)), f_2(u, \psi(u, t+h, \theta, f))) du \right| \\
 & \quad \cdot \exp(\varepsilon(1+2\Delta) \left| \int_t^s L(u, \psi(u, t, \theta, f), C) du \right|).
 \end{aligned}$$

First, we will show that for $f \in F(D, \Delta)$, $|||Tf||| < D$. Since

$$\begin{aligned}
 |T_1 f(t, \theta)| &< \int_{-\infty}^{\infty} \epsilon \beta e^{-\alpha \epsilon |s-t|} \\
 &\quad \cdot |X(s, \psi(s, t, \theta, f), f_1(s, \psi(s, t, \theta, f)), f_2(s, \psi(s, t, \theta, f)))| ds \\
 &< 2 \alpha^{-1} \beta (\kappa(\epsilon) + C\kappa(C)),
 \end{aligned}$$

$$\begin{aligned}
 |T_2 f(t, \theta)| &< \int_{-\infty}^{\infty} \epsilon^{1/2} \beta e^{-\alpha \epsilon |s-t|} \\
 &\quad \cdot |Y(s, \psi(s, t, \theta, f), f_1(s, \psi(s, t, \theta, f)), f_2(s, \psi(s, t, \theta, f)))| ds \\
 &< 2 \alpha^{-1} \beta \epsilon^{1/2} \|Y\|,
 \end{aligned}$$

and (5) hold, we have

$$|||Tf||| < D.$$

From the uniqueness of solutions, periodicity of θ , and the periodicity of $f \in F(D, \Delta)$, we have

$$\psi(t+2\pi+z, t+2\pi, \theta, f) = \psi(t+z, t, \theta, f)$$

and

$$\psi(t+z, t, \theta+2\pi, f) = \psi(t+z, t, \theta, f) + 2\pi.$$

Hence, after the change of variables $z = s - t$ in the definition of Tf , it is obvious that Tf is 2π -periodic in both t and θ .

Next, we show that Tf is Lipschitz in θ with Lipschitz constant Δ . For $f \in F(D, \Delta)$, we observe

$$\begin{aligned}
 & |T_1 f(t, \theta) - T_1 f(t, \bar{\theta})| \\
 & \leq \int_{-\infty}^{\infty} \beta e^{-\varepsilon \alpha |s-t|} \varepsilon |X(s, \psi(s, t, \theta, f), f_1(s, \psi(s, t, \theta, f)), f_2(s, \psi(s, t, \theta, f))) \\
 & \quad - X(s, \psi(s, t, \bar{\theta}, f), f_1(s, \psi(s, t, \bar{\theta}, f)), f_2(s, \psi(s, t, \bar{\theta}, f)))| ds \\
 & \leq \int_{-\infty}^{\infty} \varepsilon \beta e^{-\varepsilon \alpha |s-t|} L(s + \psi(s, t, \theta, f), C)(1+2\Delta) |\psi(s, t, \theta, f) - \psi(s, t, \bar{\theta}, f)| ds \\
 & \leq \int_{-\infty}^{\infty} \varepsilon \beta e^{-\varepsilon \alpha |s-t|} L(s + \psi(s, t, \theta, f), C)(1+2\Delta) |\theta - \bar{\theta}| \\
 & \quad \cdot \exp\left(\varepsilon(1+2\Delta) \left| \int_t^s L(u + \psi(u, t, \theta, f), C) du \right| \right) ds \\
 & \leq \Delta |\theta - \bar{\theta}|,
 \end{aligned}$$

since (8) and (6) hold. Similarly, from (7), we have

$$\begin{aligned}
& |\mathbf{T}_2 f(t, \theta) - \mathbf{T}_2 f(t, \bar{\theta})| \\
& \leq \int_{-\infty}^{\infty} \varepsilon^{1/2} \beta e^{-\alpha|s-t|} |Y(s, \psi(s, t, \theta, f), f_1(s, \psi(s, t, \theta, f)), f_2(s, \psi(s, t, \theta, f))) \\
& \quad - Y(s, \psi(s, t, \bar{\theta}, f), f_1(s, \psi(s, t, \bar{\theta}, f)), f_2(s, \psi(s, t, \bar{\theta}, f)))| ds \\
& \leq \int_{-\infty}^{\infty} \varepsilon^{1/2} \beta e^{-\alpha|s-t|} M(1+2\Delta) |\psi(s, t, \theta, f) - \psi(s, t, \bar{\theta}, f)| ds \\
& \leq \int_{-\infty}^{\infty} \varepsilon^{1/2} \beta e^{-\alpha|s-t|} M(1+2\Delta) |\theta - \bar{\theta}| \exp\left(\varepsilon(1+2\Delta) \left| \int_t^s L(u, \psi(u, t, \theta, f), C) du \right| \right) ds \\
& \leq \Delta |\theta - \bar{\theta}| .
\end{aligned}$$

Using (9), for $f, g \in \mathbf{F}(D, \Delta)$, we have

$$\begin{aligned}
& |\mathbf{T}_1 f(t, \theta) - \mathbf{T}_1 g(t, \theta)| \\
& \leq \int_{-\infty}^{\infty} \varepsilon \beta e^{-\varepsilon \alpha|s-t|} |X(s, \psi(s, t, \theta, f), f_1(s, \psi(s, t, \theta, f)), f_2(s, \psi(s, t, \theta, f))) \\
& \quad - X(s, \psi(s, t, \theta, g), g_1(s, \psi(s, t, \theta, g)), g_2(s, \psi(s, t, \theta, g)))| ds \\
& \leq \int_{-\infty}^{\infty} \varepsilon \beta e^{-\varepsilon \alpha|s-t|} \{ |X(s, \psi(s, t, \theta, f), f_1(s, \psi(s, t, \theta, f)), f_2(s, \psi(s, t, \theta, f))) \\
& \quad - X(s, \psi(s, t, \theta, g), f_1(s, \psi(s, t, \theta, f)), f_2(s, \psi(s, t, \theta, f)))|
\end{aligned}$$

$$\begin{aligned}
& + |X(s, \psi(s, t, \theta, g), f_1(s, \psi(s, t, \theta, f)), f_2(s, \psi(s, t, \theta, f))) \\
& - X(s, \psi(s, t, \theta, g), f_1(s, \psi(s, t, \theta, g)), f_2(s, \psi(s, t, \theta, f)))| \\
& + |X(s, \psi(s, t, \theta, g), f_1(s, \psi(s, t, \theta, g)), f_2(s, \psi(s, t, \theta, f))) \\
& - X(s, \psi(s, t, \theta, g), f_1(s, \psi(s, t, \theta, g)), f_2(s, \psi(s, t, \theta, g)))| \\
& + |X(s, \psi(s, t, \theta, g), f_1(s, \psi(s, t, \theta, g)), f_2(s, \psi(s, t, \theta, g))) \\
& - X(s, \psi(s, t, \theta, g), g_1(s, \psi(s, t, \theta, g)), g_2(s, \psi(s, t, \theta, g)))| \} ds \\
\leq & \int_{-\infty}^{\infty} \varepsilon \beta e^{-\varepsilon \alpha |s-t|} L(s+\psi(s, t, \theta, g), C) \{ (1+2\Delta) |\psi(s, t, \theta, f) - \psi(s, t, \theta, g)| \\
& + |f_1(s, \psi(s, t, \theta, g)) - g_1(s, \psi(s, t, \theta, g))| \\
& + |f_2(s, \psi(s, t, \theta, g)) - g_2(s, \psi(s, t, \theta, g))| \} ds \\
\leq & \int_{-\infty}^{\infty} 2\varepsilon \beta e^{-\varepsilon \alpha |s-t|} L(s+\psi(s, t, \theta, g), C) |||f-g||| \\
& \cdot \exp\left(\varepsilon(1+2\Delta) \left| \int_t^s L(u+\psi(u, t, \theta, g), C) du \right| \right) ds .
\end{aligned}$$

Similarly,

$$\begin{aligned}
 & |T_2 f(t, \theta) - T_2 g(t, \theta)| \\
 & < \int_{-\infty}^{\infty} 2\epsilon^{1/2} M \beta e^{-\alpha|s-t|} \|f-g\| \exp(\epsilon(1+2\Delta) \left| \int_t^s L(u+\psi(u, t, \theta, g), C) du \right|) ds .
 \end{aligned}$$

Thus, since $\tilde{p} \equiv \frac{2\Delta}{1+2\Delta} < 1$, we have

$$(11) \quad \|Tf - Tg\| < \tilde{p} \|f-g\| .$$

The inequality in (11) implies T is distance contracting on $F(D, \Delta)$.

We now use (10) to show that Tf is continuous in (t, θ) . However, since

$$\begin{aligned}
 & |Tf(t+h, \theta+k) - Tf(t, \theta)| \\
 & < |Tf(t+h, \theta+k) - Tf(t+h, \theta)| + |Tf(t+h, \theta) - Tf(t, \theta)| \\
 & < \Delta|k| + |Tf(t+h, \theta) - Tf(t, \theta)| ,
 \end{aligned}$$

it suffices to show Tf is continuous in t . Consider

$$\begin{aligned}
(12) \quad & |T_1 f(t+h, \theta) - T_1 f(t, \theta)| \\
& = \left| \int_{-\infty}^{\infty} \{J(\varepsilon(s-t-h))\varepsilon X(s, \psi(s, t+h, \theta, f), f_1(s, \psi(s, t+h, \theta, f)), \right. \\
& \quad \left. f_2(s, \psi(s, t+h, \theta, f))) \right. \\
& \quad \left. - J(\varepsilon(s-t))\varepsilon X(s, \psi(s, t, \theta, f), f_1(s, \psi(s, t, \theta, f)), f_2(s, \psi(s, t, \theta, f)))\} ds \right|.
\end{aligned}$$

From (4), we see that (12) consists of two components. Using (10) and (6) on the first component of $T_1 f$, where $X = (X_1^T, X_2^T)^T \in \mathbb{R}^k$, we obtain

$$\begin{aligned}
& \left| \int_{t+h}^{\infty} e^{\varepsilon A_+(t+h-s)} \varepsilon X_1(s, \psi(s, t+h, \theta, f), f_1(s, \psi(s, t+h, \theta, f)), \right. \\
& \quad \left. f_2(s, \psi(s, t+h, \theta, f))) ds \right. \\
& \quad \left. - \int_t^{\infty} e^{\varepsilon A_+(t-s)} \varepsilon X_1(s, \psi(s, t, \theta, f), f_1(s, \psi(s, t, \theta, f)), \right. \\
& \quad \left. f_2(s, \psi(s, t, \theta, f))) ds \right| \\
& \leq \left| \int_{t+h}^{\infty} [e^{\varepsilon A_+(t+h-s)} - e^{\varepsilon A_+(t-s)}] \right. \\
& \quad \cdot \varepsilon X_1(s, \psi(s, t+h, \theta, f), f_1(s, \psi(s, t+h, \theta, f)), \\
& \quad \left. f_2(s, \psi(s, t+h, \theta, f))) ds \right| +
\end{aligned}$$

$$\begin{aligned}
& \left| \int_{t+h}^t e^{\varepsilon A_+ (t-s)} \varepsilon X_1(s, \psi(s, t+h, \theta, f), f_1(s, \psi(s, t+h, \theta, f))), \right. \\
& \quad \left. f_2(s, \psi(s, t+h, \theta, f)) \right) ds \Big| + \\
& \left| \int_t^\infty e^{\varepsilon A_+ (t-s)} \varepsilon \{ X_1(s, \psi(s, t+h, \theta, f), f_1(s, \psi(s, t+h, \theta, f))), \right. \\
& \quad \left. f_2(s, \psi(s, t+h, \theta, f)) \right) \\
& \quad - X_1(s, \psi(s, t, \theta, f), f_1(s, \psi(s, t, \theta, f))), \\
& \quad \left. f_2(s, \psi(s, t, \theta, f)) \right) \Big| ds \Big| \\
& \leq \left| e^{\varepsilon A_+ h} - I \right| \alpha^{-1} \beta e^{-\varepsilon \alpha h} \|X\| + \alpha^{-1} \beta \left| 1 - e^{-\varepsilon \alpha h} \right| \|X\| \\
& \quad + \int_t^\infty \beta e^{\varepsilon \alpha (t-s)} \varepsilon L(s, \psi(s, t, \theta, f), C) (1+2\Delta) \varepsilon \\
& \quad \cdot \left| \int_t^{t+h} \theta(u, \psi(u, t+h, \theta, f), f_1(u, \psi(u, t+h, \theta, f)), f_2(u, \psi(u, t+h, \theta, f))) du \right| \\
& \quad \cdot \exp(\varepsilon(1+2\Delta) \int_t^s L(u, \psi(u, t, \theta, f), C) du) \Big| ds \\
& \leq \left\{ \left| e^{\varepsilon A_+ h} - I \right| e^{-\varepsilon \alpha h} + \left| 1 - e^{-\varepsilon \alpha h} \right| \right\} \alpha^{-1} \beta \|X\| + \varepsilon \Delta \|\theta\| |h|.
\end{aligned}$$

Thus, the first component of $T_1 f$ is continuous in t . In a similar manner, we obtain continuity in the second component of $T_1 f$ and in both components of $T_2 f$.

Hence, we have shown T maps $F(D, \Delta)$ into $F(D, \Delta)$. Furthermore, we have shown that T is contractive. Thus, by the Banach fixed-point theorem [43], there exists a unique $f \in F(D, \Delta)$, such that $Tf = f$.

We claim that $S_\varepsilon = \{(t, \theta, x, y) : x = f_1(t, \theta), y = f_2(t, \theta), -\infty < t, \theta < \infty\}$ is an integral manifold for (E). Let $\theta(t, \tau, \eta)$, $x(t, \tau, f_1(\tau, \eta))$ and $y(t, \tau, f_2(\tau, \eta))$ denote the solution of (E), where

$$\theta(\tau, \tau, \eta) = \eta ,$$

$$x(\tau, \tau, f_1(\tau, \eta)) = f_1(\tau, \eta) ,$$

$$y(\tau, \tau, f_2(\tau, \eta)) = f_2(\tau, \eta) .$$

We now show

$$\theta(t, \tau, \eta) = \psi(t, \tau, \eta, f) ,$$

$$(13) \quad x(t, \tau, f_1(\tau, \eta)) = f_1(t, \psi(t, \tau, \eta, f)) ,$$

$$y(t, \tau, f_2(\tau, \eta)) = f_2(t, \psi(t, \tau, \eta, f)) .$$

First, we observe that the fixed-point relation yields

$$(14a) \quad f_1(t, \theta) = \int_{-\infty}^{\infty} J(\varepsilon(s-t)) \varepsilon X(s, \psi(s, t, \theta, f), f_1(s, \psi(s, t, \theta, f))),$$

$$f_2(s, \psi(s, t, \theta, f)) ds \quad \text{and}$$

$$(14b) \quad f_2(t, \theta) = \int_{-\infty}^{\infty} K(s-t) \varepsilon^{1/2} Y(s, \psi(s, t, \theta, f), f_1(s, \psi(s, t, \theta, f))), \\ f_2(s, \psi(s, t, \theta, f))) ds .$$

Substituting $\psi(t, \tau, \eta, f)$ in place of θ , in (14), and noting that

$$\psi(s, t, \psi(t, \tau, \eta, f), f) = \psi(s, \tau, \eta, f),$$

we have

$$(15a) \quad f_1(t, \psi(t, \tau, \eta, f)) \\ = \int_{-\infty}^{\infty} J(\varepsilon(s-t)) \varepsilon X(s, \psi(s, \tau, \eta, f), f_1(s, \psi(s, \tau, \eta, f))), \\ f_2(s, \psi(s, \tau, \eta, f))) ds \quad \text{and}$$

$$(15b) \quad f_2(t, \psi(t, \tau, \eta, f)) \\ = \int_{-\infty}^{\infty} K(s-t) \varepsilon^{1/2} Y(s, \psi(s, \tau, \eta, f), f_1(s, \psi(s, \tau, \eta, f))), \\ f_2(s, \psi(s, \tau, \eta, f))) ds .$$

Since (15) is equivalent to

$$\begin{aligned}
 & f_1(t, \psi(t, \tau, \eta, f)) \\
 &= - \int_t^\infty \begin{bmatrix} e^{\epsilon A_+(t-s)} & 0 \\ 0 & 0 \end{bmatrix} \epsilon X(s, \psi(s, \tau, \eta, f), f_1(s, \psi(s, \tau, \eta, f)), \\
 & \quad f_2(s, \psi(s, \tau, \eta, f))) ds \\
 &+ \int_{-\infty}^t \begin{bmatrix} 0 & 0 \\ 0 & e^{\epsilon A_-(t-s)} \end{bmatrix} \epsilon X(s, \psi(s, \tau, \eta, f), f_1(s, \psi(s, \tau, \eta, f)), \\
 & \quad f_2(s, \psi(s, \tau, \eta, f))) ds \quad \text{and}
 \end{aligned}$$

$$\begin{aligned}
 & f_2(t, \psi(t, \tau, \eta, f)) \\
 &= - \int_t^\infty \begin{bmatrix} e^{B_+(t-s)} & 0 \\ 0 & 0 \end{bmatrix} \epsilon^{1/2} Y(s, \psi(s, \tau, \eta, f), f_1(s, \psi(s, \tau, \eta, f)), \\
 & \quad f_2(s, \psi(s, \tau, \eta, f))) ds \\
 &+ \int_{-\infty}^t \begin{bmatrix} 0 & 0 \\ 0 & e^{B_-(t-s)} \end{bmatrix} \epsilon^{1/2} Y(s, \psi(s, \tau, \eta, f), f_1(s, \psi(s, \tau, \eta, f)), \\
 & \quad f_2(s, \psi(s, \tau, \eta, f))) ds ,
 \end{aligned}$$

we see that

$$\theta = \psi(t, \tau, \eta, f) ,$$

$$x = f_1(t, \psi(t, \tau, \eta, f)) ,$$

$$y = f_2(t, \psi(t, \tau, \eta, f)) ,$$

is a solution of (E), such that

$$\theta(\tau) = \eta ,$$

$$x(\tau) = f_1(\tau, \eta) ,$$

$$y(\tau) = f_2(\tau, \eta) .$$

By the uniqueness of solutions, we see (13) holds for all choices of t and η . Hence, f gives rise to an integral manifold S_ε , satisfying (3). Having shown existence of an integral manifold, we now verify that if $(t, \theta(t), x(t), y(t))$ is a solution of (E) with $|x(t)| \leq C$, $|y(t)| \leq C$, for all $t \in \mathbb{R}$, then $(t, \theta(t), x(t), y(t)) \in S_\varepsilon$, for all $t \in \mathbb{R}$.

Define $g(t, \theta) = (g_1(t, \theta), g_2(t, \theta))$ by

$$g_1(t, \theta) = x(t)$$

and

$$g_2(t, \theta) = y(t) ,$$

for $\theta = \theta(t)$, $t \in \mathbb{R}$. Clearly, g gives rise to an integral manifold for (E). We now derive a relation that g must satisfy. In fact, the following derivation can be thought of as the motivation behind the definition of the operator T .

From the variation of constants formula, we have

$$\begin{aligned}
 (16a) \quad g_1(t, \psi(t, t_0, \theta, g)) &= e^{\varepsilon A(t-t_0)} g_1(t_0, \theta) + \int_{t_0}^t e^{\varepsilon A(t-s)} \\
 &\quad \cdot \varepsilon X(s, \psi(s, t_0, \theta, g), g_1(s, \psi(s, t_0, \theta, g))), \\
 &\quad g_2(s, \psi(s, t_0, \theta, g)) ds \quad \text{and} \\
 (16b) \quad g_2(t, \psi(t, t_0, \theta, g)) &= e^{B(t-t_0)} g_2(t_0, \theta) + \int_{t_0}^t e^{B(t-s)} \\
 &\quad \cdot \varepsilon^{1/2} Y(s, \psi(s, t_0, \theta, g), g_1(s, \psi(s, t_0, \theta, g))), \\
 &\quad g_2(s, \psi(s, t_0, \theta, g)) ds .
 \end{aligned}$$

Since $A = \text{diag}(A_+, A_-)$, we see that (16a) can be split into two parts, g_{11} and g_{12} , given by

$$\begin{aligned}
 (17) \quad g_{11}(t, \psi(t, t_0, \theta, g)) &= e^{\varepsilon A_+(t-t_0)} g_{11}(t_0, \theta) + \int_{t_0}^t e^{\varepsilon A_+(t-s)} \\
 &\quad \cdot \varepsilon X_1(s, \psi(s, t_0, \theta, g), g_1(s, \psi(s, t_0, \theta, g))), \\
 &\quad g_2(s, \psi(s, t_0, \theta, g))) ds
 \end{aligned}$$

and

$$\begin{aligned}
 (18) \quad g_{12}(t, \psi(t, t_0, \theta, g)) &= e^{\varepsilon A_-(t-t_0)} g_{12}(t_0, \theta) + \int_{t_0}^t e^{\varepsilon A_-(t-s)} \\
 &\quad \cdot \varepsilon X_2(s, \psi(s, t_0, \theta, g), g_1(s, \psi(s, t_0, \theta, g))), \\
 &\quad g_2(s, \psi(s, t_0, \theta, g))) ds,
 \end{aligned}$$

where $X^T = (X_1^T, X_2^T)$. Equation (17) is equivalent to

$$\begin{aligned}
 (19) \quad e^{\varepsilon A_+(t_0-t)} g_{11}(t, \psi(t, t_0, \theta, g)) \\
 = g_{11}(t_0, \theta) + \int_{t_0}^t e^{\varepsilon A_+(t_0-s)} \varepsilon X_1(s, \psi(s, t_0, \theta, g), g_1(s, \psi(s, t_0, \theta, g))), \\
 g_2(s, \psi(s, t_0, \theta, g))) ds.
 \end{aligned}$$

Since g is bounded, letting $t \rightarrow +\infty$ in (19) implies

$$\begin{aligned}
 (20) \quad g_{11}(t_0, \theta) &= - \int_{t_0}^{+\infty} e^{\varepsilon A_+(t_0-s)} \varepsilon X_1(s, \psi(s, t_0, \theta, g), g_1(s, \psi(s, t_0, \theta, g))), \\
 &\quad g_2(s, \psi(s, t_0, \theta, g))) ds.
 \end{aligned}$$

Similary, equation (18) becomes

$$\begin{aligned}
 & e^{\epsilon A_-(t_0-t)} g_{12}(t, \psi(t, t_0, \theta, g)) \\
 &= g_{12}(t_0, \theta) + \int_{t_0}^t e^{\epsilon A_-(t_0-s)} \epsilon X_2(s, \psi(s, t_0, \theta, g), g_1(s, \psi(s, t_0, \theta, g))), \\
 & \quad g_2(s, \psi(s, t_0, \theta, g)) ds
 \end{aligned}$$

and letting $t \rightarrow -\infty$ yields

$$\begin{aligned}
 (21) \quad g_{12}(t_0, \theta) &= \int_{-\infty}^{t_0} e^{\epsilon A_-(t_0-s)} \epsilon X_2(s, \psi(s, t_0, \theta, g), g_1(s, \psi(s, t_0, \theta, g))), \\
 & \quad g_2(s, \psi(s, t_0, \theta, g)) ds .
 \end{aligned}$$

By combining (20) and (21), we obtain

$$\begin{aligned}
 (22) \quad g_1(t_0, \theta) &= \int_{-\infty}^{\infty} J(\epsilon(s-t_0)) \epsilon X(s, \psi(s, t_0, \theta, g), g_1(s, \psi(s, t_0, \theta, g))), \\
 & \quad g_2(s, \psi(s, t_0, \theta, g)) ds .
 \end{aligned}$$

In a similar manner, we have

$$\begin{aligned}
 (23) \quad g_2(t_0, \theta) &= \int_{-\infty}^{\infty} K(s-t_0) \epsilon^{1/2} Y(s, \psi(s, t_0, \theta, g), g_1(s, \psi(s, t_0, \theta, g))), \\
 & \quad g_2(s, \psi(s, t_0, \theta, g)) ds .
 \end{aligned}$$

Thus, by extending the definition of T , we can represent (22) and (23) by

$$g(t_0, \theta) = Tg(t_0, \theta) .$$

Observe that the derivations of (9) and (11) depend only upon the Lipschitz continuity of f (provided that $\|f_i - g_i\|$ for $i = 1, 2$ and $\|f - g\|$ are taken to mean supremum over the domain of g). Hence, from (11), we have

$$\|f - g\| < \tilde{p} \|f - g\| .$$

Thus, $f(t_0, \theta) = g(t_0, \theta)$ for all $(t_0, \theta) \in \text{domain } g$. ■

In Lemma 2 we examine the stability properties of the integral manifold. Observe that the proof of Lemma 2 implies that the stability properties of the equilibrium of the unperturbed system are inherited by the integral manifold.

Lemma 2. Suppose that system (E) has an integral manifold given by (3), with $\Delta < 1/2$. Assume there is a $\delta \equiv \delta(C, \epsilon) > 0$ so that $D + \delta < C < \tilde{\eta}$ and suppose

$$(24) \quad 3 \int_{t_0}^{\infty} \beta \epsilon e^{-\frac{\epsilon \alpha}{2} |s-t|} L(s + \psi(s, t_0, \eta, f) + h(s), C, \epsilon) ds < 1/2$$

and

$$(25) \quad 3\beta\epsilon^{1/2} M \left[\left(\alpha + \frac{\epsilon\alpha}{2} \right)^{-1} + \left(\alpha - \frac{\epsilon\alpha}{2} \right)^{-1} \right] < 1/2$$

hold for all $t > t_0$, $\eta \in \mathbb{R}$, $h: \mathbb{R} \rightarrow \mathbb{R}$, with $h \in C(\mathbb{R})$ and $\|h\| < \delta$.

In addition, assume that either $\text{rank } A_- > 1$ or $\text{rank } B_- > 1$. Then, there is a set S_M properly containing $\{(\theta, f_1(t_0, \theta), f_2(t_0, \theta)) : \theta \in \mathbb{R}\}$, such that solutions starting in S_M tend exponentially to some solution on the integral manifold. Moreover, if A and B are stable matrices, then

$$N = \bigcup_{\theta \in \mathbb{R}} \{(\eta, \rho, \xi) : \max\{|\eta - \theta|, |\rho - f_1(t_0, \theta)|, |\xi - f_2(t_0, \theta)|\} < \frac{1-2\Delta}{2(1+\Delta)} \delta\} \subseteq S_M.$$

Proof: We again employ a contraction mapping argument. However, this time we define an operator U , on a space of exponentially decaying functions, such that a fixed-point of U corresponds to the difference between known solutions on the integral manifold and unknown solutions off the integral manifold.

Suppose that $\text{rank } A_- = a$ and $\text{rank } B_- = b$. Let $\Omega(\frac{\delta}{2}) \subseteq \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^j$ be given by

$$\Omega(\frac{\delta}{2}) = \{(\eta, \rho, \xi) : \eta \in \mathbb{R}, \rho - f_1(t_0, \eta) \in \{0\} \times \mathbb{R}^a,$$

$$\xi - f_2(t_0, \eta) \in \{0\} \times \mathbb{R}^b \text{ with } \beta |J(-0)(\rho - f_1(t_0, \eta))| < \frac{\delta}{2}$$

$$\text{and } \beta |K(-0)(\xi - f_2(t_0, \eta))| < \frac{\delta}{2} \}.$$

Let G be the space of exponentially decaying functions defined by

$$G = \{ \phi : [t_0, \infty) \times \Omega(\frac{\delta}{2}) \rightarrow \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^j : \phi \in C([t_0, \infty) \times \Omega(\frac{\delta}{2})) \},$$

$$\sup_{\substack{t > t_0 \\ (\eta, \rho, \xi) \in \Omega(\delta/2)}} |\phi_i(t, \eta, \rho, \xi)| e^{\frac{\varepsilon \alpha}{2}(t-t_0)} < \delta,$$

$$\phi(t, \eta + 2\pi, \rho, \xi) = \phi(t, \eta, \rho, \xi) \quad \text{and}$$

$$\phi(t, \eta, f_1(t_0, \eta), f_2(t_0, \eta)) = 0, \quad \text{for all } t > t_0 \}.$$

Observe that G is complete under the metric

$$\|\phi - \tilde{\phi}\|_e = \max_{1 \leq i \leq 3} \sup_{\substack{t > t_0 \\ (\eta, \rho, \xi) \in \Omega(\delta/2)}} |\phi_i(t, \eta, \rho, \xi) - \tilde{\phi}_i(t, \eta, \rho, \xi)| e^{\frac{\varepsilon \alpha}{2}(t-t_0)}.$$

Define U on G by

$$U\phi = \begin{bmatrix} U_1 \phi \\ U_2 \phi \\ U_3 \phi \end{bmatrix}, \quad \text{for } \phi \in G,$$

where

$$\begin{aligned}
U_1 \phi(t, \eta, \rho, \xi) = & - \int_t^\infty \varepsilon [\theta(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta))) \\
& + \phi_2(s, \eta, \rho, \xi), f_2(s, \psi(s, t_0, \eta)) + \phi_3(s, \eta, \rho, \xi)) \\
& - \theta(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta)), \\
& f_2(s, \psi(s, t_0, \eta)))] ds,
\end{aligned}$$

$$\begin{aligned}
U_2 \phi(t, \eta, \rho, \xi) = & J(\varepsilon(t_0 - t)) [\rho - f_1(t_0, \eta)] + \int_{t_0}^\infty J(\varepsilon(s - t)) \varepsilon \\
& \{X(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta)) \\
& + \phi_2(s, \eta, \rho, \xi), f_2(s, \psi(s, t_0, \eta)) + \phi_3(s, \eta, \rho, \xi)) \\
& - X(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta)), \\
& f_2(s, \psi(s, t_0, \eta)))\} ds,
\end{aligned}$$

$$\begin{aligned}
U_3 \phi(t, \eta, \rho, \xi) = & K(t_0 - t) [\xi - f_2(t_0, \eta)] + \int_{t_0}^\infty K(s - t) \varepsilon^{1/2} \\
& \{Y(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta)) \\
& + \phi_2(s, \eta, \rho, \xi), f_2(s, \psi(s, t_0, \eta)) + \phi_3(s, \eta, \rho, \xi)) \\
& - Y(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta)), f_2(s, \psi(s, t_0, \eta)))\} ds,
\end{aligned}$$

for $(\eta, \rho, \xi) \in \Omega(\frac{\delta}{2})$, $t > t_0$, where $\psi(s, t_0, \eta)$ is defined to be the solution of

$$\frac{d}{ds} \psi = \varepsilon \theta(s, \psi, f_1(s, \psi), f_2(s, \psi)) ,$$

$$\psi(t_0) = \eta .$$

In order to apply the Banach fixed-point theorem, we must show that U maps G back into G and that U is a contraction mapping.

First, we examine the exponential boundedness condition required of functions residing the space G . Consider

$$\begin{aligned} |U_1 \phi(t, \eta, \rho, \xi)| e^{\frac{\varepsilon \alpha}{2} (t-t_0)} &< \varepsilon \int_t^\infty L(s+\psi(s, t_0, \eta), C) [|\phi_1(s, \eta, \rho, \xi)| + \\ &|\phi_2(s, \eta, \rho, \xi)| + |\phi_3(s, \eta, \rho, \xi)|] ds e^{\frac{\varepsilon \alpha}{2} (t-t_0)} \\ &< 3\delta \varepsilon \int_t^\infty L(s+\psi(s, t_0, \eta), C) e^{-\frac{\varepsilon \alpha}{2} (s-t)} ds , \\ |U_2 \phi(t, \eta, \rho, \xi)| e^{\frac{\varepsilon \alpha}{2} (t-t_0)} &< e^{-\frac{\varepsilon \alpha}{2} (t-t_0)} \frac{\delta}{2} + 3\delta \varepsilon \int_{t_0}^\infty \beta e^{-\varepsilon \alpha |s-t|} \\ &\cdot L(s+\psi(s, t_0, \eta), C) e^{-\frac{\varepsilon \alpha}{2} (s-t_0)} ds e^{\frac{\varepsilon \alpha}{2} (t-t_0)} \\ &< \frac{\delta}{2} + 3\delta \varepsilon \int_{t_0}^\infty e^{-\frac{\varepsilon \alpha}{2} |s-t|} L(s+\psi(s, t_0, \eta), C) ds \end{aligned}$$

and

$$\begin{aligned}
 |U_3 \phi(t, \eta, \rho, \xi)| e^{\frac{\varepsilon \alpha}{2} (t-t_0)} &\leq e^{(\frac{\varepsilon \alpha}{2} - \alpha)(t-t_0)} \frac{\delta}{2} \\
 &+ 3\beta \delta \varepsilon^{1/2} M \int_{t_0}^{\infty} e^{-\alpha|s-t|} e^{-\frac{\varepsilon \alpha}{2} (s-t_0)} ds e^{\frac{\varepsilon \alpha}{2} (t-t_0)} \\
 &\leq \frac{\delta}{2} + 3\beta \delta \varepsilon^{1/2} M [(\alpha + \frac{\varepsilon \alpha}{2})^{-1} + (\alpha - \frac{\varepsilon \alpha}{2})^{-1}] .
 \end{aligned}$$

From (24) and (25), we may conclude that

$$\sup_{\substack{t > t_0 \\ (\eta, \rho, \xi) \in \Omega(\frac{\delta}{2})}} |U_i \phi(t, \eta, \rho, \xi)| e^{\frac{\varepsilon \alpha}{2} (t-t_0)} < \delta, \text{ for } i = 1, 2, 3 .$$

Having shown $U_i \phi$ satisfies the exponential boundedness condition, we now verify that $U\phi$ is 2π -periodic in η . Recalling that $f(s, \eta+2\pi) = f(s, \eta)$, $\psi(s, t_0, \eta+2\pi) = \psi(s, t_0, \eta) + 2\pi$ and $\theta(s, \eta+2\pi, x, y) = \theta(s, \eta, x, y)$ for all $s, \eta \in \mathbb{R}$, we see that

$$\begin{aligned}
 U_1 \phi(t, \eta+2\pi, \rho, \xi) &= - \int_{t_0}^{\infty} \varepsilon [\theta(s, \psi(s, t_0, \eta)+2\pi, \phi_1(s, \eta+2\pi, \rho, \xi), \\
 &\quad f_1(s, \psi(s, t_0, \eta)+2\pi, \phi_2(s, \eta+2\pi, \rho, \xi), f_2(s, \psi(s, t_0, \eta)+2\pi, \phi_3(s, \eta+2\pi, \rho, \xi)) \\
 &\quad - \theta(s, \psi(s, t_0, \eta)+2\pi, f_1(s, \psi(s, t_0, \eta)+2\pi, f_2(s, \psi(s, t_0, \eta)+2\pi))] ds \\
 &= U_1 \phi(t, \eta, \rho, \xi) ,
 \end{aligned}$$

for any $\phi \in G$, $t > t_0$, $(\eta, \rho, \xi) \in \Omega(\frac{\delta}{2})$. Similar computations yield

$$U_i \phi(t, \eta + 2\pi, \rho, \xi) = U_i \phi(t, \eta, \rho, \xi),$$

for $i = 2, 3$, $\phi \in G$, $t > t_0$, $(\eta, \rho, \xi) \in \Omega(\frac{\delta}{2})$.

Next, we show that $U\phi$ is a continuous function from $[t_0, \infty) \times \Omega(\frac{\delta}{2})$ into $R \times R^k \times R^j$. Recalling the definition of $U\phi$, i.e.,

$$\begin{aligned} U_1 \phi(t, \eta, \rho, \xi) = & -\varepsilon \int_t^\infty [\theta(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), \\ & f_1(s, \psi(s, t_0, \eta) + \phi_2(s, \eta, \rho, \xi), f_2(s, \psi(s, t_0, \eta) + \phi_3(s, \eta, \rho, \xi)) \\ & - \theta(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta), f_2(s, \psi(s, t_0, \eta)))] ds, \end{aligned}$$

$$U_2 \phi(t, \eta, \rho, \xi) = \begin{bmatrix} 0 & 0 \\ 0 & e^{\varepsilon A_-(t-t_0)} \end{bmatrix} [\rho - f_1(t_0, \eta)] + \varepsilon \int_{t_0}^t \begin{bmatrix} 0 & 0 \\ 0 & e^{\varepsilon A_-(t-s)} \end{bmatrix}$$

$$\cdot [X(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta) + \phi_2(s, \eta, \rho, \xi),$$

$$f_2(s, \psi(s, t_0, \eta) + \phi_3(s, \eta, \rho, \xi))]$$

$$- X(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta)), f_2(s, \psi(s, t_0, \eta))) ds$$

$$- \varepsilon \int_t^\infty \begin{bmatrix} e^{\varepsilon A_+(t-s)} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\bullet [X(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta)) + \phi_2(s, \eta, \rho, \xi),$$

$$f_2(s, \psi(s, t_0, \eta)) + \phi_3(s, \eta, \rho, \xi))$$

$$- X(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta)), f_2(s, \psi(s, t_0, \eta))) ds ,$$

$$U_3 \phi(t, \eta, \rho, \xi) = \begin{bmatrix} 0 & 0 \\ 0 & e^{B_-(t-t_0)} \end{bmatrix} [\xi - f_2(t_0, \eta)] + \varepsilon^{1/2} \int_{t_0}^t \begin{bmatrix} 0 & 0 \\ 0 & e^{B_-(t-s)} \end{bmatrix}$$

$$\bullet [Y(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta)) + \phi_2(s, \eta, \rho, \xi),$$

$$f_2(s, \psi(s, t_0, \eta)) + \phi_3(s, \eta, \rho, \xi))$$

$$- Y(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta)), f_2(s, \psi(s, t_0, \eta))) ds$$

$$- \varepsilon^{1/2} \int_t^\infty \begin{bmatrix} e^{B_+(t-s)} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\bullet [Y(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta) + \phi_2(s, \eta, \rho, \xi),$$

$$f_2(s, \psi(s, t_0, \eta) + \phi_3(s, \eta, \rho, \xi))$$

$$- Y(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta)), f_2(s, \psi(s, t_0, \eta)))] ds ,$$

we see that each component of $U\phi$ is differentiable with respect to t .

Moreover, $\frac{\partial}{\partial t} U_i \phi$ is bounded for $i = 1, 2, 3$. Thus, $U_i \phi$ is Lipschitz continuous in t . Hence, to show that $U\phi \in C([t_0, \infty) \times \Omega(\frac{\delta}{2}))$, it suffices to show that $U\phi \in C(\Omega(\frac{\delta}{2}))$ for any fixed $t > t_0$.

Fix $t > t_0$ and $\gamma > 0$. There is a $T > t > t_0$, so that

$$|\varepsilon \int_T^\infty [\theta(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta) + \phi_2(s, \eta, \rho, \xi),$$

$$f_2(s, \psi(s, t_0, \eta) + \phi_3(s, \eta, \rho, \xi))$$

$$- \theta(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta)), f_2(s, \psi(s, t_0, \eta)))] ds|$$

$$< 3\delta \varepsilon \int_T^\infty L(s, \psi(s, t_0, \eta), C) e^{-\frac{\varepsilon \alpha}{2}(s-t_0)} ds < \frac{\gamma}{3} ,$$

$$|\varepsilon \int_T^\infty J(\varepsilon(s-t)) [X(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta) + \phi_2(s, \eta, \rho, \xi),$$

$$f_2(s, \psi(s, t_0, \eta) + \phi_3(s, \eta, \rho, \xi))$$

$$\begin{aligned}
& - X(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta)), f_2(s, \psi(s, t_0, \eta)))] ds | \\
& \leq 3\delta\epsilon\beta \int_T^\infty e^{-\epsilon\alpha(s-t)} L(s, \psi(s, t_0, \eta), C) e^{-\frac{\epsilon\alpha}{2}(s-t_0)} ds < \frac{\delta}{4}
\end{aligned}$$

and

$$\left| \epsilon \int_T^{1/2} K(s-t) [Y(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta)) + \phi_2(s, \eta, \rho, \xi),
\right.$$

$$\left. f_2(s, \psi(s, t_0, \eta)) + \phi_3(s, \eta, \rho, \xi) \right]$$

$$- Y(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta)), f_2(s, \psi(s, t_0, \eta)))] ds |$$

$$\leq 3\delta\epsilon \int_T^{1/2} e^{-\alpha(s-t) - \frac{\epsilon\alpha}{2}(s-t_0)} ds$$

$$= \frac{3\delta\epsilon}{\alpha + \frac{\epsilon\alpha}{2}} \exp\left(\alpha t + \frac{\epsilon\alpha}{2} t_0 - \left(\alpha + \frac{\epsilon\alpha}{2}\right)T\right) < \frac{\gamma}{4}.$$

From the continuity of ψ , ϕ , f , θ , X and Y , compactness, and provided

$|(\eta, \rho, \xi) - (\bar{\eta}, \bar{\rho}, \bar{\xi})|$ is sufficiently small, with (η, ρ, ξ) and

$(\bar{\eta}, \bar{\rho}, \bar{\xi}) \in \Omega(\frac{\delta}{2})$, we may assume

$$\epsilon | [\theta(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta)) + \phi_2(s, \eta, \rho, \xi),$$

$$f_2(s, \psi(s, t_0, \eta)) + \phi_3(s, \eta, \rho, \xi)]$$

$$- \theta(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta)), f_2(s, \psi(s, t_0, \eta)))]$$

$$- [\theta(s, \psi(s, t_0, \bar{n}) + \phi_1(s, \bar{n}, \bar{\rho}, \bar{\xi}), f_1(s, \psi(s, t_0, \bar{n}) + \phi_2(s, \bar{n}, \bar{\rho}, \bar{\xi}),$$

$$f_2(s, \psi(s, t_0, \bar{n}) + \phi_3(s, \bar{n}, \bar{\rho}, \bar{\xi}))$$

$$- \theta(s, \psi(s, t_0, \bar{n}), f_1(s, \psi(s, t_0, \bar{n})), f_2(s, \psi(s, t_0, \bar{n})))] |$$

$$< \frac{\gamma}{3(T-t_0)},$$

$$\beta | [\rho - f_1(t_0, n)] - [\bar{\rho} - f_1(t_0, \bar{n})] | < \frac{\gamma}{4},$$

$$\beta | [\xi - f_2(t_0, n)] - [\bar{\xi} - f_2(t_0, \bar{n})] | < \frac{\gamma}{4},$$

$$\beta \epsilon | [X(s, \psi(s, t_0, n) + \phi_1(s, n, \rho, \xi), f_1(s, \psi(s, t_0, n) + \phi_2(s, n, \rho, \xi),$$

$$f_2(s, \psi(s, t_0, n) + \phi_3(s, n, \rho, \xi))$$

$$- X(s, \psi(s, t_0, n), f_1(s, \psi(s, t_0, n)), f_2(s, \psi(s, t_0, n)))]$$

$$- [X(s, \psi(s, t_0, \bar{n}) + \phi_1(s, \bar{n}, \bar{\rho}, \bar{\xi}), f_1(s, \psi(s, t_0, \bar{n}) + \phi_2(s, \bar{n}, \bar{\rho}, \bar{\xi}),$$

$$f_2(s, \psi(s, t_0, \bar{n}) + \phi_3(s, \bar{n}, \bar{\rho}, \bar{\xi}))$$

$$- X(s, \psi(s, t_0, \bar{n}), f_1(s, \psi(s, t_0, \bar{n})), f_2(s, \psi(s, t_0, \bar{n})))] |$$

$$< \frac{\gamma}{4(T-t_0)}$$

and

$$\begin{aligned}
& \leq \frac{1}{2} \left| \left[Y(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta)) + \phi_2(s, \eta, \rho, \xi), \right. \right. \\
& \quad \left. \left. f_2(s, \psi(s, t_0, \eta)) + \phi_3(s, \eta, \rho, \xi)) \right] \right. \\
& \quad \left. - Y(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta)), f_2(s, \psi(s, t_0, \eta))) \right] \\
& \quad - \left[Y(s, \psi(s, t_0, \bar{\eta}) + \phi_1(s, \bar{\eta}, \bar{\rho}, \bar{\xi}), f_1(s, \psi(s, t_0, \bar{\eta})) + \phi_2(s, \bar{\eta}, \bar{\rho}, \bar{\xi}), \right. \\
& \quad \left. f_2(s, \psi(s, t_0, \bar{\eta})) + \phi_3(s, \bar{\eta}, \bar{\rho}, \bar{\xi})) \right] \\
& \quad \left. - Y(s, \psi(s, t_0, \bar{\eta}), f_1(s, \psi(s, t_0, \bar{\eta})), f_2(s, \psi(s, t_0, \bar{\eta}))) \right] \Big| \\
& \leq \frac{\gamma}{4(T-t_0)},
\end{aligned}$$

for all $s \in [t_0, T]$. These inequalities imply

$$\begin{aligned}
& |U_1 \phi(t, \eta, \rho, \xi) - U_1 \phi(t, \bar{\eta}, \bar{\rho}, \bar{\xi})| \\
& \leq \varepsilon \left| \int_{t_0}^T \left\{ \left[\theta(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta)) + \phi_2(s, \eta, \rho, \xi), \right. \right. \right. \\
& \quad \left. \left. f_2(s, \psi(s, t_0, \eta)) + \phi_3(s, \eta, \rho, \xi)) \right] \right. \\
& \quad \left. - \theta(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta)), f_2(s, \psi(s, t_0, \eta))) \right] \\
& \quad - \left[\theta(s, \psi(s, t_0, \bar{\eta}) + \phi_1(s, \bar{\eta}, \bar{\rho}, \bar{\xi}), f_1(s, \psi(s, t_0, \bar{\eta})) + \phi_2(s, \bar{\eta}, \bar{\rho}, \bar{\xi}), \right. \\
& \quad \left. f_2(s, \psi(s, t_0, \bar{\eta})) + \phi_3(s, \bar{\eta}, \bar{\rho}, \bar{\xi})) \right] \\
& \quad \left. - \theta(s, \psi(s, t_0, \bar{\eta}), f_1(s, \psi(s, t_0, \bar{\eta})), f_2(s, \psi(s, t_0, \bar{\eta}))) \right] \Big\} ds \Big| +
\end{aligned}$$

$$\begin{aligned}
& \left| \varepsilon \int_T^\infty \{ [\theta(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta)) + \phi_2(s, \eta, \rho, \xi), \right. \\
& \quad \left. f_2(s, \psi(s, t_0, \eta) + \phi_3(s, \eta, \rho, \xi)) \right. \\
& \quad \left. - \theta(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta)), f_2(s, \psi(s, t_0, \eta))) \right] \\
& \quad - [\theta(s, \psi(s, t_0, \bar{\eta}) + \phi_1(s, \bar{\eta}, \bar{\rho}, \bar{\xi}), f_1(s, \psi(s, t_0, \bar{\eta})) + \phi_2(s, \bar{\eta}, \bar{\rho}, \bar{\xi}), \\
& \quad \left. f_2(s, \psi(s, t_0, \bar{\eta}) + \phi_3(s, \bar{\eta}, \bar{\rho}, \bar{\xi})) \right. \\
& \quad \left. - \theta(s, \psi(s, t_0, \bar{\eta}), f_1(s, \psi(s, t_0, \bar{\eta})), f_2(s, \psi(s, t_0, \bar{\eta}))) \right] \} ds \mid \\
& < \frac{\gamma}{3} + \frac{2\gamma}{3} = \gamma, \\
& \quad | \mathfrak{U}_2 \phi(t, \eta, \rho, \xi) - \mathfrak{U}_2 \phi(t, \bar{\eta}, \bar{\rho}, \bar{\xi}) | \\
& < \beta e^{-\varepsilon \alpha(t-t_0)} | [\rho - f_1(t_0, \eta)] - [\bar{\rho} - f_1(t_0, \bar{\eta})] | + \\
& \quad \varepsilon \left| \int_t^T J(\varepsilon(s-t)) \{ [X(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta)) + \phi_2(s, \eta, \rho, \xi), \right. \\
& \quad \left. f_2(s, \psi(s, t_0, \eta) + \phi_3(s, \eta, \rho, \xi)) \right. \\
& \quad \left. - X(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta)), f_2(s, \psi(s, t_0, \eta))) \right] \\
& \quad - [X(s, \psi(s, t_0, \bar{\eta}) + \phi_1(s, \bar{\eta}, \bar{\rho}, \bar{\xi}), f_1(s, \psi(s, t_0, \bar{\eta})) + \phi_2(s, \bar{\eta}, \bar{\rho}, \bar{\xi}), \\
& \quad \left. f_2(s, \psi(s, t_0, \bar{\eta}) + \phi_3(s, \bar{\eta}, \bar{\rho}, \bar{\xi})) \right. \\
& \quad \left. - X(s, \psi(s, t_0, \bar{\eta}), f_1(s, \psi(s, t_0, \bar{\eta})), f_2(s, \psi(s, t_0, \bar{\eta}))) \right] \} ds \mid +
\end{aligned}$$

$$\begin{aligned}
& \varepsilon \left| \int_T^\infty J(\varepsilon(s-t)) \{ [X(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta)) + \phi_2(s, \eta, \rho, \xi), \right. \\
& \quad \left. f_2(s, \psi(s, t_0, \eta)) + \phi_3(s, \eta, \rho, \xi)) \right. \\
& \quad - X(s, \psi(s, t_0, \eta), f_1(s, \psi(s, t_0, \eta)), f_2(s, \psi(s, t_0, \eta)))] \\
& \quad - [X(s, \psi(s, t_0, \bar{\eta}) + \phi_1(s, \bar{\eta}, \bar{\rho}, \bar{\xi}), f_1(s, \psi(s, t_0, \bar{\eta})) + \phi_2(s, \bar{\eta}, \bar{\rho}, \bar{\xi}), \\
& \quad \left. f_2(s, \psi(s, t_0, \bar{\eta})) + \phi_3(s, \bar{\eta}, \bar{\rho}, \bar{\xi})) \right. \\
& \quad \left. - X(s, \psi(s, t_0, \bar{\eta}), f_1(s, \psi(s, t_0, \bar{\eta})), f_2(s, \psi(s, t_0, \bar{\eta})))] \} ds \right| \\
& < \frac{\gamma}{4} + \frac{\gamma}{4} + \frac{2\gamma}{4} = \gamma
\end{aligned}$$

and a similar computation yields

$$|U_3\phi(t, \eta, \rho, \xi) - U_3\phi(t, \bar{\eta}, \bar{\rho}, \bar{\xi})| < \gamma.$$

Thus, $U\phi \in C(\Omega(\frac{\delta}{2}))$, for any fixed $t > t_0$.

From the definition of $U\phi(t, \eta, \rho, \xi)$, it is obvious that

$U\phi(t, \eta, f_1(t_0, \eta), f_2(t_0, \eta)) = 0$, for all $\phi \in G$. Hence, we see that

$U: G \rightarrow G$. We now show that U is a contraction mapping. Observe that

(24) implies

$$\begin{aligned}
& |\mathbf{U}_1 \phi(t, \eta, \rho, \xi) - \mathbf{U}_1 \tilde{\phi}(t, \eta, \rho, \xi)| e^{\frac{\varepsilon \alpha}{2} (t - t_0)} \\
& \leq e^{\frac{\varepsilon \alpha}{2} (t - t_0)} \varepsilon \int_t^\infty \mathbf{L}(s + \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), C) \\
& \quad \cdot \left\{ \sum_{i=1}^3 |\phi_i(s, \eta, \rho, \xi) - \tilde{\phi}_i(s, \eta, \rho, \xi)| \right\} ds \\
& \leq 3\varepsilon \int_t^\infty \mathbf{L}(s + \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), C) e^{-\frac{\varepsilon \alpha}{2} (s - t)} ds \|\phi - \tilde{\phi}\|_e \\
& \leq 1/2 \|\phi - \tilde{\phi}\|_e
\end{aligned}$$

and

$$\begin{aligned}
& |\mathbf{U}_2 \phi(t, \eta, \rho, \xi) - \mathbf{U}_2 \tilde{\phi}(t, \eta, \rho, \xi)| e^{\frac{\varepsilon \alpha}{2} (t - t_0)} \\
& \leq e^{\frac{\varepsilon \alpha}{2} (t - t_0)} \left| \int_{t_0}^\infty \varepsilon \mathbf{J}(\varepsilon(s - t)) \right. \\
& \quad \cdot [\mathbf{X}(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta) + \phi_2(s, \eta, \rho, \xi), \\
& \quad f_2(s, \psi(s, t_0, \eta) + \phi_3(s, \eta, \rho, \xi)) \\
& \quad - \mathbf{X}(s, \psi(s, t_0, \eta) + \tilde{\phi}_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta) + \tilde{\phi}_2(s, \eta, \rho, \xi), \\
& \quad f_2(s, \psi(s, t_0, \eta) + \tilde{\phi}_3(s, \eta, \rho, \xi))] ds \left| \right. \\
& \leq 3\beta\varepsilon \int_{t_0}^\infty \mathbf{L}(s + \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), C) e^{-\frac{\varepsilon \alpha}{2} |s - t|} ds \|\phi - \tilde{\phi}\|_e \\
& \leq 1/2 \|\phi - \tilde{\phi}\|_e, \quad \text{for } \phi, \tilde{\phi} \in G.
\end{aligned}$$

Similarly, using (25), we obtain

$$\begin{aligned}
& |U_3\phi(t, \eta, \rho, \xi) - U_3\tilde{\phi}(t, \eta, \rho, \xi)| e^{\frac{\varepsilon\alpha}{2}(t-t_0)} \\
& \leq e^{\frac{\varepsilon\alpha}{2}(t-t_0)} \frac{1}{\varepsilon} \int_{t_0}^{\infty} K(s-t) \\
& \quad \cdot [Y(s, \psi(s, t_0, \eta) + \phi_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta)) + \phi_2(s, \eta, \rho, \xi), \\
& \quad f_2(s, \psi(s, t_0, \eta)) + \phi_3(s, \eta, \rho, \xi)) \\
& \quad - Y(s, \psi(s, t_0, \eta) + \tilde{\phi}_1(s, \eta, \rho, \xi), f_1(s, \psi(s, t_0, \eta)) + \tilde{\phi}_2(s, \eta, \rho, \xi), \\
& \quad f_2(s, \psi(s, t_0, \eta)) + \tilde{\phi}_3(s, \eta, \rho, \xi))] ds | \\
& \leq 3\beta\varepsilon \int_{t_0}^{\infty} e^{-\alpha|s-t| - \frac{\varepsilon\alpha}{2}(s-t)} ds \|\phi - \tilde{\phi}\|_e \\
& \leq 3\beta\varepsilon \int_{t_0}^{\infty} e^{-\alpha|s-t| - \frac{\varepsilon\alpha}{2}(s-t)} ds \left[\left(\alpha - \frac{\varepsilon\alpha}{2}\right)^{-1} + \left(\alpha + \frac{\varepsilon\alpha}{2}\right)^{-1} \right] \|\phi - \tilde{\phi}\|_e \\
& \leq \frac{1}{2} \|\phi - \tilde{\phi}\|_e, \quad \text{for } \phi, \tilde{\phi} \in G.
\end{aligned}$$

This implies $\|U\phi - U\tilde{\phi}\|_e \leq \frac{1}{2} \|\phi - \tilde{\phi}\|_e$, for any $\phi, \tilde{\phi} \in G$, that is, U is a contraction map on G . Applying the Banach fixed-point theorem to U produces a fixed-point $\phi \in G$, such that

$$\phi(t, \eta, \rho, \xi) = U\phi(t, \eta, \rho, \xi), \quad \text{for all } t > t_0 \quad \text{and}$$

$$(\eta, \rho, \xi) \in \Omega\left(\frac{\delta}{2}\right).$$

From the above fixed-point and the known solutions of (E) on the integral manifold, we will construct solutions of (E) which tend exponentially to the integral manifold. For $(\eta^*, \rho^*, \xi^*) \in \Omega\left(\frac{\delta}{2}\right)$, let

$$\theta(t, t_0, \eta, \rho, \xi) = \psi(t, t_0, \eta^*) + \phi_1(t, \eta^*, \rho^*, \xi^*),$$

$$x(t, t_0, \eta, \rho, \xi) = f_1(t, \psi(t, t_0, \eta^*)) + \phi_2(t, \eta^*, \rho^*, \xi^*),$$

$$y(t, t_0, \eta, \rho, \xi) = f_2(t, \psi(t, t_0, \eta^*)) + \phi_3(t, \eta^*, \rho^*, \xi^*),$$

where

$$\theta(t_0, t_0, \eta, \rho, \xi) = \eta = \eta^* + \phi_1(t_0, \eta^*, \rho^*, \xi^*),$$

$$x(t_0, t_0, \eta, \rho, \xi) = \rho = \rho^* + \int_{t_0}^{\infty} J(\varepsilon(s - t_0))$$

$$\cdot \varepsilon X(s, \psi(s, t_0, \eta^*) + \phi_1(s, \eta^*, \rho^*, \xi^*),$$

$$f_1(s, \psi(s, t_0, \eta^*)) + \phi_2(s, \eta^*, \rho^*, \xi^*),$$

$$f_2(s, \psi(s, t_0, \eta^*)) + \phi_3(s, \eta^*, \rho^*, \xi^*)) ds,$$

$$\begin{aligned}
y(t_0, t_0, \eta, \rho, \xi) = \xi = \xi^* + \int_{t_0}^{\infty} K(s-t_0) \\
\cdot \varepsilon^{1/2} Y(s, \psi(s, t_0, \eta^*) + \phi_1(s, \eta^*, \rho^*, \xi^*), \\
f_1(s, \psi(s, t_0, \eta^*)) + \phi_2(s, \eta^*, \rho^*, \xi^*), \\
f_2(s, \psi(s, t_0, \eta^*)) + \phi_3(s, \eta^*, \rho^*, \xi^*)) ds.
\end{aligned}$$

The fixed-point relation implies

$$\begin{aligned}
(26a) \quad \phi_1'(t, \eta^*, \rho^*, \xi^*) = \varepsilon \theta(t, \theta(t, t_0, \eta, \rho, \xi), x(t, t_0, \eta, \rho, \xi), \\
y(t, t_0, \eta, \rho, \xi)) -
\end{aligned}$$

$$\varepsilon \theta(t, \psi(t, t_0, \eta^*), f_1(t, \psi(t, t_0, \eta^*)),$$

$$f_2(t, \psi(t, t_0, \eta^*))),$$

$$(26b) \quad \phi_2'(t, \eta^*, \rho^*, \xi^*) = \varepsilon A \phi_2(t, \eta^*, \rho^*, \xi^*) + \varepsilon \left\{ \begin{bmatrix} \bar{0} & \bar{0} \\ 0 & \bar{I} \end{bmatrix} + \begin{bmatrix} \bar{I} & \bar{0} \\ 0 & 0 \end{bmatrix} \right\}$$

$$\cdot [X(t, \theta(t, t_0, \eta, \rho, \xi), x(t, t_0, \eta, \rho, \xi),$$

$$y(t, t_0, \eta, \rho, \xi))$$

$$- X(t, \psi(t, t_0, \eta^*), f_1(t, \psi(t, t_0, \eta^*)),$$

$$f_2(t, \psi(t, t_0, \eta^*)))],$$

$$\begin{aligned}
 (26c) \quad \phi_3'(t, \eta^*, \rho^*, \xi^*) &= B\phi_3(t, \eta^*, \rho^*, \xi^*) + \varepsilon^{1/2} \left\{ \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right\} \\
 &\cdot [Y(t, \theta(t, t_0, \eta, \rho, \xi), x(t, t_0, \eta, \rho, \xi), \\
 &\quad y(t, t_0, \eta, \rho, \xi)) \\
 &\quad - Y(t, \psi(t, t_0, \eta^*), f_1(t, \psi(t, t_0, \eta^*)), \\
 &\quad f_2(t, \psi(t, t_0, \eta^*)))].
 \end{aligned}$$

Since

$$\begin{aligned}
 \theta &= \psi(t, t_0, \eta^*), \\
 x &= f_1(t, \psi(t, t_0, \eta^*)), \\
 y &= f_2(t, \psi(t, t_0, \eta^*)),
 \end{aligned}$$

solves (E), we see that (26) implies

$$\begin{aligned}
 \theta &= \theta(t, t_0, \eta, \rho, \xi), \\
 x &= x(t, t_0, \eta, \rho, \xi), \\
 y &= y(t, t_0, \eta, \rho, \xi),
 \end{aligned}$$

also solves (E). Moreover, by its construction, the solution

$(\theta(t, t_0, \eta, \rho, \xi), x(t, t_0, \eta, \rho, \xi), y(t, t_0, \eta, \rho, \xi))$ tends exponentially to some solution on the integral manifold. Taking $S_M = \{(\theta(t_0, t_0, \eta, \rho, \xi), x(t_0, t_0, \eta, \rho, \xi), y(t_0, t_0, \eta, \rho, \xi)) : (\eta^*, \rho^*, \xi^*) \in \Omega(\frac{\delta}{2})\}$ yields the first result. We now show that S_M contains N when both A and B are stable. Let $(\eta, \rho, \xi) \in N$. Then, there is an $\theta \in \mathbb{R}$ such that

$$|\eta - \theta| < \frac{1 - 2\Delta}{2(1+\Delta)} \delta ,$$

$$|\rho - f_1(t_0, \theta)| < \frac{1 - 2\Delta}{2(1+\Delta)} \delta ,$$

$$|\xi - f_2(t_0, \theta)| < \frac{1 - 2\Delta}{2(1+\Delta)} \delta .$$

For any $\eta^* \in [\eta - \delta, \eta + \delta]$, we have

$$\begin{aligned} & |\rho - f_1(t_0, \eta^*)| \\ & \leq |\rho - f_1(t_0, \theta)| + |f_1(t_0, \theta) - f_1(t_0, \eta)| + |f_1(t_0, \eta) - f_1(t_0, \eta^*)| \\ & \leq \frac{1 - 2\Delta}{2(1+\Delta)} \delta + \Delta \frac{1 - 2\Delta}{2(1+\Delta)} \delta + \Delta \delta \\ & = \frac{\delta}{2} \end{aligned}$$

and, similarly,

$$|\xi - f_2(t_0, \eta^*)| < \frac{\delta}{2} .$$

Thus, $(\eta^*, \rho, \xi) \in \Omega(\frac{\delta}{2})$ for all $\eta^* \in [\eta - \delta, \eta + \delta]$. This implies $\eta^* + \phi_1(t_0, \eta^*, \rho, \xi)$ is defined for all $\eta^* \in [\eta - \delta, \eta + \delta]$. Furthermore,

$$\eta - \delta + \phi_1(t_0, \eta - \delta, \rho, \xi) \leq \eta \leq \eta + \delta + \phi_1(t_0, \eta + \delta, \rho, \xi),$$

since $|\phi_1(t_0, \eta^*, \rho, \xi)| \leq \delta$ for all $(\eta^*, \rho, \xi) \in \Omega(\frac{\delta}{2})$. Hence, by connectedness, there is at least one $\eta^* \in [\eta - \delta, \eta + \delta]$, such that $\eta^* + \phi_1(t_0, \eta^*, \rho, \xi) = \eta$. Hence, $(\eta, \rho, \xi) \in S_M$. ■

In order to prove Theorem 1 we need only show that, for sufficiently small C and ε , the hypothesis of Lemmas 1 and 2 are satisfied.

Proof of Theorem 1. First, we restrict $\varepsilon > 0$ so that $\varepsilon \|\theta\| < 1$. We also fix $\Delta \in \mathbb{R}$ so that $0 < \Delta < 1/2$. Now, for any $C \in \mathbb{R}$, $0 < C \leq \tilde{\eta}$ and $f \in C(\mathbb{R}^2)$ with $f_1: \mathbb{R}^2 \rightarrow \mathbb{R}^k$, $f_2: \mathbb{R}^2 \rightarrow \mathbb{R}^j$ and $\|f_i\| \leq C$, we have

$$u \equiv s + \psi(s, t, \eta, f)$$

implies

$$\frac{du}{ds} = 1 + \frac{d\psi}{ds}(s, t, \eta, f) > 0.$$

Thus, a simple change of variables and periodicity yields

$$\begin{aligned}
\int_{\gamma}^{\gamma+2\pi} L(s+\psi(s,t,\eta,f), C, \varepsilon) ds &\leq (1-\varepsilon\|\theta\|)^{-1} \int_{\gamma+\psi(\gamma,t,\eta,f)}^{\gamma+2\pi+\psi(\gamma+2\pi,t,\eta,f)} L(u, C, \varepsilon) du \\
&\leq (1-\varepsilon\|\theta\|)^{-1} \int_{\gamma+\psi(\gamma,t,\eta,f)}^{\gamma+2\pi+\psi(\gamma,t,\eta,f)+2\pi\varepsilon\|\theta\|} L(u, C, \varepsilon) du \\
&\leq 2(1-\varepsilon\|\theta\|)^{-1} \|L(\cdot, C, \varepsilon)\|_1,
\end{aligned}$$

where $\|L(\cdot, C, \varepsilon)\|_1$ denotes $\int_0^{2\pi} |L(t, C, \varepsilon)| dt$.

Furthermore, for $\alpha_0 < \beta_0$, there is a $k_0 > 0$ such that

$$2k_0\pi < \beta_0 - \alpha_0 < 2(k_0+1)\pi \quad \text{and}$$

$$\begin{aligned}
&\exp[\varepsilon(1+2\Delta) \int_{\alpha_0}^{\beta_0} L(s+\psi(s,t,\eta,f), C, \varepsilon) ds] \\
&\leq \exp[2\varepsilon(\int_{\alpha_0}^{\alpha_0+2k_0\pi} L(s+\psi(s,t,\eta,f), C, \varepsilon) ds \\
&\quad + \int_{\alpha_0+2k_0\pi}^{\beta_0} L(s+\psi(s,t,\eta,f), C, \varepsilon) ds)] \\
&\leq \exp[4k_0\varepsilon(1-\varepsilon\|\theta\|)^{-1} \|L(\cdot, C, \varepsilon)\|_1] \exp[4\varepsilon(1-\varepsilon\|\theta\|)^{-1} \|L(\cdot, C, \varepsilon)\|_1] \\
&\leq \exp[2\varepsilon(\beta_0 - \alpha_0)(\pi - \varepsilon\pi\|\theta\|)^{-1} \|L(\cdot, C, \varepsilon)\|_1] \\
&\quad \cdot \exp[4\varepsilon(1-\varepsilon\|\theta\|)^{-1} \|L(\cdot, C, \varepsilon)\|_1]
\end{aligned}$$

Recalling equation (1), we see that, for C and ε sufficiently

small, $\tau_0 \equiv \alpha - 2(\pi - \varepsilon\pi\|\theta\|)^{-1} \|L(\cdot, C, \varepsilon)\|_1 > 0$. Let

$\tau_1 \equiv \exp[4\varepsilon(1 - \varepsilon\|\theta\|)^{-1} \|L(\cdot, C, \varepsilon)\|_1]$. Then,

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \varepsilon \beta e^{-\varepsilon \alpha |s-t|} L(s+\psi(s, t, \theta, f), C, \varepsilon) \\
 & \quad \cdot \exp[\varepsilon(1+2\Delta) \left| \int_t^s L(u+\psi(u, t, \theta, f), C, \varepsilon) du \right|] ds \\
 & \leq \int_{-\infty}^{\infty} \varepsilon \beta e^{-\varepsilon \tau_0 |s-t|} L(s+\psi(s, t, \theta, f), C, \varepsilon) ds \tau_1 \\
 & = \sum_{n=-\infty}^{\infty} \int_{t+2n\pi}^{t+2(n+1)\pi} \varepsilon \beta e^{-\varepsilon \tau_0 |s-t|} L(s+\psi(s, t, \theta, f), C, \varepsilon) ds \tau_1 \\
 & \leq 4\varepsilon \beta (1 - \varepsilon\|\theta\|)^{-1} \tau_1 \sum_{n=0}^{\infty} [e^{-\varepsilon \tau_0 2\pi n}] \|L(\cdot, C, \varepsilon)\|_1 \\
 & = \frac{\varepsilon}{(1 - e^{-2\pi \varepsilon \tau_0})(1 - \varepsilon\|\theta\|)} 4\beta \tau_1 \|L(\cdot, C, \varepsilon)\|_1
 \end{aligned}$$

Now, as $(C, \varepsilon) \rightarrow 0$ we have

$$\begin{aligned}
 \frac{\varepsilon}{(1 - e^{-2\pi \varepsilon \tau_0})(1 - \varepsilon\|\theta\|)} & \rightarrow \frac{1}{2\pi \alpha}, \\
 4\beta \tau_1 & \rightarrow 4\beta,
 \end{aligned}$$

and

$$\|L(\cdot, C, \varepsilon)\|_1 \rightarrow 0.$$

Hence, for all C and ε sufficiently small, with $\varepsilon > 0$, (6) is true for any $t, \theta \in \mathbb{R}$ and $f \in C(\mathbb{R}^2)$, where $\|f_i\| < C$ for $i = 1, 2$.

Similarly, by perhaps further restricting the size of C and ε , we see that (7) holds for all $t, \theta \in \mathbb{R}$ and $f \in C(\mathbb{R}^2)$, where $\|f_i\| < C$ for $i = 1, 2$. Also, observe that, regardless of the size of $C > 0$ or $D > 0$, equation (25) is attainable by taking ε sufficiently small.

Next, we show equation (24) holds for all sufficiently small C and ε .

With this goal in mind we derive two inequalities. For $h \in C(\mathbb{R})$,

$\|h\| < \delta$ and for any $\gamma \in \mathbb{R}$, we have

$$\begin{aligned} & \int_{\gamma}^{\gamma+2\pi} |L(u+h(u), v, w) - L(u, v, w)| du \\ & \leq \sum_{n=1}^N c_n(v, w) \int_{\delta}^{2\pi-\delta} |\chi_{I_{n,v,w}}(u+h(u+\gamma)) - \chi_{I_{n,v,w}}(u)| du + 4M_0\delta \\ & \leq \sum_{n=1}^N c_n(v, w) 2\delta + 4M_0\delta \\ & \leq 2M_0(N+2)\delta. \end{aligned}$$

Next, using change of variables $u = s + \psi(s, t_0, n, f)$, letting

$\tilde{h}(u) = h(s)$, and $\gamma = t + 2n\pi + \psi(t+2n\pi, t_0, n, f)$, we obtain

$$\begin{aligned}
& \int_{t+2n\pi}^{t+2(n+1)\pi} L(s+\psi(s, t_0, \eta, f) + h(s), C, \varepsilon) ds \\
& \leq \int_{\gamma}^{\gamma+(\varepsilon\|\theta\|+1)2\pi} L(u+\tilde{h}(u), C, \varepsilon) du (1-\varepsilon\|\theta\|)^{-1} \\
& \leq \int_{\gamma}^{\gamma+4\pi} L(u+\tilde{h}(u), C, \varepsilon) du (1-\varepsilon\|\theta\|)^{-1} \\
& \leq (4M_0(N+2)\delta + 2\|L(\cdot, C, \varepsilon)\|_1)(1-\varepsilon\|\theta\|)^{-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& 3 \int_{t_0}^{\infty} \beta \varepsilon e^{-\frac{\varepsilon \alpha}{2} |s-t|} L(s+\psi(s, t_0, \eta, f)+h(s), C, \varepsilon) ds \\
& \leq 3 \int_{-\infty}^{\infty} \beta \varepsilon e^{-\frac{\varepsilon \alpha}{2} |s-t|} L(s+\psi(s, t_0, \eta, f)+h(s), C, \varepsilon) ds \\
& \leq \frac{6\beta\varepsilon}{1-e^{-\varepsilon\alpha\pi}} \max_n \int_{t+2n\pi}^{t+2(n+1)\pi} L(s+\psi(s, t_0, \eta, f)+h(s), C, \varepsilon) ds \\
& \leq \frac{6\beta\varepsilon}{(1-e^{-\varepsilon\alpha\pi})(1-\varepsilon\|\theta\|)} (4M_0(N+2)\delta + 2\|L(\cdot, C, \varepsilon)\|_1) \rightarrow 0,
\end{aligned}$$

provided $(C, \varepsilon) \rightarrow 0$ and $\delta \rightarrow 0$. Hence, for all C and ε sufficiently small, with δ sufficiently small, we may assume (24) holds independent of $t_0, t, \eta \in R$, for all $h \in C(R)$ with $\|h\| \leq \delta$. Moreover, by perhaps further restricting C and ε , with δ sufficiently small, we may assume

$$22\alpha^{-1}\beta\epsilon^{1/2}\|Y\| \leq C - \delta \quad \text{and} \quad 2\alpha^{-1}\beta\kappa(\epsilon) \leq [1-2\alpha^{-1}\beta\kappa(c)]C - \delta ,$$

i.e., equation (5) holds for all ϵ sufficiently small. Thus, for C and ϵ sufficiently small (and for any smaller $\epsilon > 0$) with $D = C - \delta > 0$, we may apply Lemma 1 and 2 to obtain the desired results. ■

APPENDIX B

Consider a perturbed system of linear equations of the form

$$(E) \quad \begin{aligned} x' &= \varepsilon Ax + \varepsilon X(t, x, y, \varepsilon) , \\ y' &= By + \varepsilon^{1/2} Y(t, x, y, \varepsilon) , \end{aligned}$$

where

(G-1) X and Y are assumed to be defined and continuous on a set

$$\Omega = \{(t, x, y, \varepsilon) \in \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^j \times \mathbb{R} : |x| < \tilde{\eta}, \\ |y| < \tilde{\eta}, 0 < \varepsilon < \varepsilon_0\} ,$$

for some $\tilde{\eta}, \varepsilon_0 > 0$,

(G-2) X and Y are 2π -periodic in t ,

(G-3) there exists a continuous, monotone increasing

function $\kappa(\cdot)$, with $\kappa(0) = 0$, such that

$$|X(t, x, y, \varepsilon)| < \kappa(\tilde{b}) + \kappa(\tilde{a})\tilde{a} \quad \text{for all } t \in \mathbb{R},$$

$$|x| < \tilde{a}, \quad |y| < \tilde{a}, \quad 0 < \varepsilon < \tilde{b},$$

(G-4) Y is Lipschitz in x and y , with Lipschitz constant M ,

- (G-5) there exists a nonnegative step function, $L(t, v, w)$,
which is 2π -periodic in t , such that, for $0 \leq t \leq 2\pi$,

$$L(t, v, w) = \sum_{n=1}^N c_n(v, w) \chi_{I_{n, v, w}}(t),$$

where

- (a) $0 \leq c_n(v, w) \leq M_0 < \infty$, for $1 \leq n \leq N$, $0 \leq v \leq \tilde{\eta}$,
 $0 \leq w \leq \varepsilon_0$,
- (b) $I_{n, v, w} = [a_{n, v, w}, b_{n, v, w}]$ is a subinterval of $[0, 2\pi]$,
for $1 \leq n \leq N$, and $I_{n_1, v, w} \cap I_{n_2, v, w} = \emptyset$, if $n_1 \neq n_2$,
- (c) $\chi_{I_{n, v, w}}$ is the characteristic function for $I_{n, v, w}$,
i.e.,

$$\chi_{I_{n, v, w}}(t) = \begin{cases} 1, & \text{for } t \in I_{n, v, w}, \\ 0, & \text{for } t \notin I_{n, v, w}, \end{cases}$$

- (d) $c_n(v, w) \cdot (b_{n, v, w} - a_{n, v, w}) \rightarrow 0$, for all n , $1 \leq n \leq N$
as $(v, w) \rightarrow 0$,

and

$$|X(t, x_2, y_0, \varepsilon) - X(t, x_1, y_0, \varepsilon)| \leq L(t, \tilde{a}, \tilde{b}) |x_2 - x_1|,$$

$$|X(t, x_0, y_2, \varepsilon) - X(t, x_0, y_1, \varepsilon)| \leq L(t, \tilde{a}, \tilde{b}) |y_2 - y_1|,$$

for all $t \in \mathbb{R}$, $x_i \in \mathbb{R}^k$, $y_i \in \mathbb{R}^j$, with $|x_i| < \tilde{a} < \tilde{\eta}$, $|y_i| < \tilde{a} < \tilde{\eta}$ and $0 < \varepsilon < \tilde{b} < \varepsilon_0$. Here, \mathbb{R} denotes the set of real numbers.

Theorem 2: Suppose X and Y satisfy (G-1) through (G-5) and that A and B are noncritical. Then, for each fixed ε , $0 < \varepsilon < \varepsilon_1$ (provided ε_1 is sufficiently small) there exists $C(\varepsilon)$ and $D(\varepsilon)$, with $\tilde{\eta} > C(\varepsilon) > D(\varepsilon) > 0$, so that within the region

$$\Omega_\varepsilon = \{(t, x, y) : (t, x, y, \varepsilon) \in \Omega, |x| < C, |y| < C\}$$

there is a unique 2π -periodic solution of (E), given by

$$S_\varepsilon = \{(t, f_1(t, \varepsilon), f_2(t, \varepsilon)) : t \in \mathbb{R}\},$$

where $f_i(t+2\pi, \varepsilon) = f_i(t, \varepsilon)$. The solution is unique in the sense that if a solution $(t, x(t), y(t)) \in \Omega_\varepsilon$, for all $t \in \mathbb{R}$, then $(t, x(t), y(t)) \in S_\varepsilon$, for all $t \in \mathbb{R}$.

In addition, if either A or B have an eigenvalue with a positive real part, then S_ε is unstable in the sense of Lyapunov. However, if both A and B are stable matrices, then S_ε is asymptotically stable.

Proof. The proof of the theorem is an obvious simplification of the integral manifold theorem. For example, in lemma 1, $F(D, \Delta)$ becomes

$$F(D) = \{f \in C(\mathbb{R}) : |||f||| < D, f(t+2\pi) = f(t)\};$$

equation (6) is replaced with

$$\int_{-\infty}^{\infty} \epsilon \beta e^{-\epsilon \alpha |s-t|} L(s, C, \epsilon) ds < 1/2 ;$$

while (7) becomes

$$2 \epsilon^{1/2} M \beta / \alpha < 1/2 .$$

In addition, all estimates involving ψ and the θ Lipschitz arguments may be eliminated.

Similarly, we modify the definition of $\Omega(\delta/2)$, G and U in lemma 2:

$$\Omega(\delta/2) = \{(\rho, \xi) : \rho - f_1(t_0) \in \{0\} \times \mathbb{R}^a, \quad \xi - f_2(t_0) \in \{0\} \times \mathbb{R}^b,$$

$$\beta |J(-0)(\rho - f_1(t_0))| < \frac{\delta}{2}, \quad \beta |K(-0)(\xi - f_2(t_0))| < \frac{\delta}{2} \} ,$$

$$G = \{ \phi : [t_0, \infty) \times \Omega(\delta/2) \rightarrow \mathbb{R}^k \times \mathbb{R}^j : \phi \text{ is continuous},$$

$$\sup_{\substack{t > t_0 \\ (\rho, \xi) \in \Omega(\delta/2)}} |\phi_1(t, \rho, \xi)| e^{\frac{\epsilon \alpha}{2} (t-t_0)} < \delta \quad \text{and} \quad \phi(t, f_1(t_0), f_2(t_0)) = 0$$

for all $t > t_0$ } ,

and

$$U\phi = \begin{bmatrix} U_2\phi \\ U_3\phi \end{bmatrix} .$$

Of course, (24) and (25) can be replaced with

$$2 \int_{t_0}^{\infty} \beta \epsilon e^{-\frac{\epsilon \alpha}{2} |s-t|} L(s, C, \epsilon) ds < \frac{1}{2}$$

and

$$2\beta e^{1/2} M \left[\left(\alpha + \frac{\epsilon \alpha}{2} \right)^{-1} + \left(\alpha - \frac{\epsilon \alpha}{2} \right)^{-1} \right] < \frac{1}{2} ,$$

respectively. Finally, the definition of N must be changed to

$$N = \{ (\rho, \xi) : \max\{ |\rho - f_1(t_0)|, |\xi - f_2(t_0)| \} < \frac{\delta}{2} \} . \quad \blacksquare$$