CARATHÉODORY'S THEOREM AND MODULI OF LOCAL CONNECTIVITY

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Abstract. We give a constructive proof of the Carathéodory Theorem by means of the concept of a modulus of local connectivity and the extremal distance of the separating curves of an annulus.

1. Introduction

The goal of this paper is to give a new proof of the Carathéodory Theorem which states that if D is a Jordan domain, and if ϕ is a conformal map of D onto the unit disk, then ϕ extends to a homeomorphism of $\overline{D}$ with the closed unit disk (see e.g. [4] and [5]). This proof has a feature which appears to be new in that for each $\zeta \in \partial D$ it explicitly constructs a δ for each ϵ when proving the existence of $\lim_{z \to \zeta} \phi(z)$. Furthermore, a closed form expression for δ in terms of ϵ and ζ is obtained. Such expressions are potentially useful when estimating error in numerical computations. The proof also makes two seemingly new connections. First, we construct δ from ϵ by means of a *modulus of local connectivity* for the boundary of D. Roughly speaking, this is a function that predicts how close two boundary points must be in order to connect them with a small arc that is included in the boundary. Second, the proof constructs an upper bound on $|\phi(z) - \phi(\zeta)|$ from the extremal distance of the separating curves of an annulus.

The paper is organized as follows. Section 2 covers background material. Section 3 states the main ideas of the proof. Sections 4 and 5 deal with topological preliminaries. Our estimates are proven in Section 6 and Section 7 completes the proof.

2. Background

Let $\mathbb{N}$ denote the set of non-negative integers.

When $\mathcal{A}$ is an annulus with inner radius r and outer radius R, let

$$\lambda(\mathcal{A}) = \frac{2\pi}{\log(R/r)}.$$

$\lambda(\mathcal{A})$ is the extremal length of the family of separating curves of $\mathcal{A}$; see e.g. [3].

When X, Y, and Z are subsets of the plane, we say that X *separates* Y from Z if Y and Z are included in distinct connected components of $\mathbb{C} - X$. In the case where $Y = \{p\}$, we say that X separates p from Z. In the case where $Y = \{p\}$ and $Z = \{q\}$ we say that X separates p from q.

A topological space is *locally* connected if it has a basis of open connected sets. By the Hahn-Mazurkiewicz Theorem, every curve is locally connected; see e.g.

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Section 3-5 of [6]. Suppose X is a compact and connected metric space. Then, X is locally connected if and only if it is *uniformly locally arcwise connected*. This means that for every $\epsilon > 0$, there is a $\delta > 0$ so that whenever $p, q \in X$ and $0 < d(p,q) < \delta$, X includes an arc from p to q whose diameter is smaller than ϵ (although its length may be infinite); again, see Section 3-5 of [6]. Accordingly, we define a *modulus of local connectivity* for a metric space X to be a function $f : \mathbb{N} \to \mathbb{N}$ so that whenever $p, q \in X$ and $0 < d(p,q) \leq 2^{-f(k)}$, X includes an arc from p to q whose diameter is smaller than 2^{-k}. Thus, a metric space is uniformly locally arcwise connected if and only if it has a modulus of local connectivity, and a metric space that is compact and connected is locally connected if and only if it has a modulus of local connectivity. Note that if f is a modulus of local connectivity, then $\lim_{k\to\infty} f(k) = \infty$. In addition, if a metric space has a modulus of local connectivity, then it has a modulus of local connectivity that is increasing.

Moduli of local connectivity originated in the adaptation of local connectivity properties to the setting of theoretical computer science in [1] and [2]. Computational connections between moduli of local connectivity and boundary extensions of conformal maps are made in [7]. Here, we attempt to show that this notion may be useful in more traditional mathematical settings.

3. Outline of the proof

We first observe the following which is proven in Section 4.

Theorem 3.1. *If ζ_0 is a boundary point of a simply connected Jordan domain D, then for every $r > 0$, ζ_0 is a boundary point of exactly one connected component of $D_r(\zeta_0) \cap D$.*

Suppose ζ_0 is a boundary point of a simply connected Jordan domain D. In light of Theorem 3.1, when $r > 0$ we let $C(D; \zeta_0, r)$ denote the connected component of $D_r(\zeta_0) \cap D$ whose boundary contains ζ_0. Suppose ϕ is a conformal map of D onto the unit disk. The fundamental strategy of the proof is to bound the diameter of $\phi[C(D; \zeta_0, r)]$. To do so, we first construct an upper bound on the diameter of $\phi[C]$ where C is a connected component of $D_r(\zeta) \cap D$ for some point ζ in the complement of D. Namely, in Section 6 we prove the following.

Theorem 3.2. *Let ϕ be a conformal map of a domain D onto the unit disk. Suppose $\mathcal{A}$ is an annulus so that $\overline{\mathcal{A}}$ separates its center from $\phi[\overline{D_r(0)}]$ where $r \geq \sqrt{\pi\lambda(\mathcal{A})}$. Suppose C is a connected component of the points of D that are interior to the inner circle of $\mathcal{A}$. Then, the diameter of $\phi[C]$ is at most $\sqrt{l^2 + 4\pi\lambda(\mathcal{A})}$ where $l = 1 + \sqrt{r^2 - \pi\lambda(\mathcal{A})}$.*

Note that Theorem 3.2 applies to non-Jordan domains.

With Theorem 3.2 in hand, some basic calculations, which we perform in Section 6, lead us to the following.

Theorem 3.3. *Suppose ϕ is a conformal map of a Jordan domain D onto the unit disk. Let ζ_0 be a boundary point of D, and let $\epsilon > 0$. Then, the diameter of $\phi[C(D; \zeta_0, r_0)]$ is smaller than ϵ whenever r_0 is a positive number that is smaller than*

$$\sup_{0<l<\epsilon} \left(\exp\left(\frac{8\pi^2}{l^2 - \epsilon^2}\right) \min\left\{ |\zeta_0 - \phi^{-1}(w)| \ : \ |w| \leq \sqrt{(1-l)^2 + \frac{\epsilon^2 - l^2}{4}} \right\} \right). \tag{3.1}$$

When $0 < \epsilon < 1$ and $l = \frac{\epsilon}{2}$,

$$\frac{7}{16} < (1-l)^2 + \frac{\epsilon^2 - l^2}{4} < 1.$$

Thus, (3.1) is positive when $0 < \epsilon < 1$. In other words, for all sufficiently small $\epsilon > 0$, there *is* a positive number r_0 that is smaller than (3.1).

So, suppose ϕ is a conformal map of a Jordan domain D onto the unit disk. We use Theorem 3.3 to form an extension of ϕ to $\overline{D}$ as follows. Let ζ_0 be a boundary point of D. Note that $C(D; \zeta_0, r') \subseteq C(D; \zeta_0, r)$ when $0 < r' < r$. It follows from Theorem 3.3 that there is exactly one point in

$$\bigcap_{r>0} \overline{\phi[C(D; \zeta_0, r)]}.$$

We define this point to be $\phi(\zeta_0)$.

Our next goal is to show that this extension of ϕ is continuous. That is, $\lim_{z\to\zeta} \phi(z) = \phi(\zeta)$ whenever ζ is a boundary point of D. This is accomplished by showing that $z \in C(D; \zeta, r)$ whenever $z \in D$ is sufficiently close to ζ. This is where be begin to use moduli of local connectivity. Namely, in Section 4 we prove the following.

Theorem 3.4. *Suppose g is a modulus of local connectivity for a Jordan curve σ. Suppose D is an open disk whose boundary separates two points of σ. Suppose z_0 and ζ_0 are points so that $\zeta_0 \in \sigma \cap D$, $z_0 \in D - \sigma$, and $|z_0 - \zeta_0| < 2^{-g(k)}$ where $2^{-k} + 2^{-g(k)} \leq \max\{d(\zeta_0, \partial D), d(z_0, \partial D)\}$. Then, ζ_0 is a boundary point of the connected component of z_0 in $D - \sigma$.*

Theorem 3.4 was previously proven by means of the Carathéodory Theorem in [8]. We give another proof here with a few extra topological steps so as to avoid circular reasoning.

We then obtain the following form of the Carathéodory Theorem from Theorems 3.3 and Theorem 3.4.

Theorem 3.5. *Suppose ϕ is a conformal map of a Jordan domain D onto the unit disk. Let ζ_0 be a boundary point of D. Then, $\lim_{z\to\zeta_0} \phi(z) = \phi(\zeta_0)$. Furthermore, if g is a modulus of local connectivity for the boundary of D, then for each $\epsilon > 0$, $|\phi(z_0) - \phi(\zeta_0)| < \epsilon$ whenever z_0 is a point in D so that $|z_0 - \zeta_0| < 2^{-g(k)}$ and k is a non-negative integer so that $2^{-k} + 2^{-g(k)}$ is smaller than (3.1). Finally, the extension of ϕ to $\overline{D}$ is a homeomorphism of $\overline{D}$ with the closed unit disk.*

The proof of Theorem 3.5 is given in Section 7.

Suppose ϕ, D, g, ζ_0 are as in Theorem 3.5. Without loss of generality suppose g is increasing. Thus $2^{-k} + 2^{-g(k)} \leq 2^{-k+1}$. Let $0 < \epsilon < 1$. We define a positive number $\delta(\zeta_0, \epsilon)$ so that $|\phi(z) - \phi(\zeta_0)| < \epsilon$ when $|z - z_0| < \delta(\zeta_0, \epsilon)$. Let:

$$\begin{aligned} k(\zeta_0, \epsilon) &= 2 - \Bigg\lfloor \sup_{0<l<\epsilon} \Bigg(\frac{8\pi^2}{l^2 - \epsilon^2} + \\ &\quad \min \left\{ \log|\zeta_0 - \phi^{-1}(w)| \ : \ |w| \leq \sqrt{(1-l)^2 + \frac{\epsilon^2 - l^2}{4}} \right\} \Bigg) \Bigg\rfloor \\ \delta(\zeta_0, \epsilon) &= 2^{-k(\zeta_0,\epsilon)} + 2^{-g(k(\zeta_0,\epsilon))} \end{aligned}$$

(Here, $\lfloor x \rfloor$ denotes the largest integer that is not larger than x.) Thus, by Theorem 3.5, $|\phi(z) - \phi(\zeta_0)| < \epsilon$ whenever $z \in D$ and $|z - \zeta_0| < \delta(\zeta_0, \epsilon)$.

4. Proofs of Theorems 3.1 and 3.4

Theorem 3.4 is used to prove Theorem 3.1. The proof of Theorem 3.4 is based on the following lemma and theorem.

Lemma 4.1. *Let D be a Jordan domain. Let α be a crosscut of D, and let γ_1, γ_2 be the subarcs of the boundary of D that join the endpoints of α. Then, the interior of $\gamma_1 \cup \alpha$ is one side of α, and the interior of $\gamma_2 \cup \alpha$ is the other side of α.*

Proof. Let U_j denote the interior of $\alpha \cup \gamma_j$. Choose a point p in $\alpha \cap D$. There is a positive number δ so that $D_\delta(p) \subseteq D$. Since p is a boundary point of U_j, $U_j \cap D_\delta(p)$ is non-empty. So, let $q_j \in U_j \cap D_\delta(p)$, and let D_j be the side of α that contains q_j.

We show that $U_j = D_j$. U_j is a connected subset of $D - \alpha$ that contains a point of D_j (namely q_j). So, $U_j \subseteq D_j$. On the other hand, D_j is a connected subset of $\mathbb{C} - (\gamma_j \cup \alpha)$ that contains a point of U_j. So, $D_j \subseteq U_j$.

$D_1 \neq D_2$ since $\partial D_1 \neq \partial D_2$. Thus, U_1 and U_2 are the two sides of α. □

Theorem 4.2. *Let D be an open disk, and let σ be a Jordan curve. Suppose the boundary of D separates two points of σ. Let C be a connected component of $D - \sigma$. Then, C is the interior of a Jordan curve. Furthermore, if p is a boundary point of C that also lies in D, then p lies on σ and the boundary of C includes the connected component of p in $D \cap \sigma$.*

Proof. Since $C \neq D$, the boundary of C contains a point of σ; let p denote such a point.

Since the boundary of D separates two points of σ, if G is a connected component of $D \cap \sigma$, then $\overline{G}$ is a crosscut of D.

Let E denote the connected component of p in $\sigma \cap D$. Since C is a connected subset of $D - E$, there is a side of E that includes C; let E^- denote this side, and let E^+ denote the other side. By Lemma 4.1, each of these sides is a Jordan domain. Again, since the boundary of D separates two points of σ, if G is a connected component of $\sigma \cap E^-$, then $\overline{G}$ is a crosscut of E^-.

We aim to show that the boundary of C is a Jordan curve which includes E. To this end, we construct an arc F so that $E \cup F$ is a Jordan curve whose interior is C. F will be a union of subarcs of σ and connected subsets of the boundary of D. To define these subarcs of σ, we define a partial ordering of the connected components of $\sigma \cap E^-$. Namely, when G_1, G_2 are connected components of $\sigma \cap E^-$, write $G_1 \prec G_2$ if G_2 is between G_1 and E; that is if E and G_1 lie in opposite sides of $\overline{G_2}$.

Since σ is locally connected, it follows that there is no increasing chain $G_1 \prec G_2 \prec G_3 \prec \ldots$. It then follows that if G_1 is a connected component of $\sigma \cap E^-$, then there is a $\preceq$-maximal component of $\sigma \cap E^-$, G, so that $G_1 \preceq G$.

We now define F. Let $F' = \partial E^- \cap \partial D$. Thus, $E \cup F' = \partial E^-$. Let $\mathcal{M}$ denote the set of all $\preceq$-maximal components of $\sigma \cap E^-$. For each $G \in \mathcal{M}$, let λ_G be the subarc of F' that joins the endpoints of $\overline{G}$. Let F be formed by removing each λ_G from F' and replacing it with $\overline{G}$.

Thus, F is an arc that joins the endpoints of E and that contains no other points of E. Let $J = E \cup F$. Then, J is a Jordan curve. We show that C is the interior of J. Note that since $J \subseteq \overline{E^-}$, E^- includes the interior of J.

When $G \in \mathcal{M}$, let G^+ be the side of $\overline{G}$ that includes E (when $\overline{G}$ is viewed as a crosscut of D rather than E^-), and let G^- denote the other side. The rest of the proof revolves around the following four claims.

(1) For each $G \in \mathcal{M}$, the exterior of J includes G^-.
(2) The interior of J includes $\bigcap_{G \in \mathcal{M}} G^+ \cap E^-$.
(3) For each $G \in \mathcal{M}$, G^+ includes C.
(4) The interior of J contains no point of σ.

Claims (2) and (3) together imply that the interior of J includes C. Claim (1) will be used to prove (4). Claim (4) shows that the interior of J is included in a connected component of $D - \sigma$ which then must be C.

We begin by proving (1). Let $p' \in G^-$. Let $z_0 \in \mathbb{C} - \overline{D}$. Thus, z_0 is exterior to J since $J \subseteq \overline{D}$. We construct an arc from p' to z_0 that contains no point of J. Let $q \in \lambda_G - \overline{G}$. By Lemma 4.1, G^- is the interior of $G \cup \lambda_G$. So, there is an arc σ_1 from p' to q so that $\sigma' \cap \partial G^- = \{q\}$. There is an arc σ_2 from q to z_0 so that $\sigma_2 \cap \partial D = \{q\}$. Thus, $\sigma_1 \cup \sigma_2$ is an arc from p' to z_0 that contains no point of J. Thus, p' is exterior to J for every $p' \in G^-$.

We now prove (2). Suppose $p_0 \in E^-$ belongs to G^+ for every $G \in \mathcal{M}$. By way of contradiction, suppose p_0 is exterior to J. Again, let $z_0 \in \mathbb{C} - \overline{D}$. Thus, the exterior of J includes an arc from p_0 to z_0; let α denote such an arc. By examination of cases, α cannot cross the boundary of D at any boundary point of E^-. So, it must do so at a boundary point of E^+. But, this entails that α crosses E which it does not since J includes E. This is a contradiction, and so p_0 is interior to J.

Next, we prove (3). Let $G \in \mathcal{M}$. Since σ is locally connected, and since $p \in E$, there is a positive number δ so that $D_\delta(p)$ contains no point of any connected component of $\sigma \cap E^-$. However, this disk must contain a point of C, p'. So, $[p', p]$ contains a point of E but no point of G. Hence, $p' \in G^+$. Since C is a connected subset of $D - G$, $C \subseteq G^+$.

Finally, we prove (4). By way of contradiction, suppose p' is a point on σ that is interior to J. As noted above, E^- includes the interior of J. So, $p' \in \sigma \cap E^-$. Let G_1 be the connected component of p' in $\sigma \cap E^-$. Let G be a $\preceq$-maximal component of $\sigma \cap E^-$ so that $G_1 \preceq G$. Since p' is interior to J, and since J includes G, $p' \notin G$. So, $G_1 \prec G$. This means that $G_1 \subseteq G^-$. By (1), p' is exterior to J- a contradiction. So, the interior of J contains no point of σ.

By the remarks after (4), C is the interior of J and the proof is complete. □

Proof of Theorem 3.4. Let C be the connected component of z_0 in $D - \sigma$. Let $l = [z_0, \zeta_0]$. Let z_1 be the point in $l \cap \sigma$ that is closest to z_0. Thus, $z_1 \in \partial C$. Since $|z_1 - \zeta_0| < 2^{-g(k)}$, σ contains an arc from z_1 to ζ_0 whose diameter is smaller than 2^{-k}; call this arc σ_1.

We claim that D includes σ_1. For, let $q \in \sigma_1$. It follows that

$$\max\{|q - z_0|, |q - \zeta_0|\} < 2^{-k} + 2^{-g(k)}.$$

Since $2^{-k} + 2^{-g(k)} \leq \max\{d(\zeta_0, \partial D), d(z_0, \partial D)\}$, it follows that $q \in D$.

Since $\sigma_1 \subseteq D$, ζ_0 belongs to the connected component of z_1 in $D \cap \sigma$. So, by Theorem 4.2, ζ_0 is a boundary point of C since z_1 is. □

Proof of Theorem 3.1. Without loss of generality, suppose $D_r(\zeta_0)$ does not include D. Let J denote the boundary of D. It follows that $\partial D_r(\zeta_0)$ separates two points of J.

It follows from Theorem 3.4 that ζ_0 is a boundary point of at least one connected component of $D_r(\zeta_0) - J$. We now show it is a boundary point of exactly two such components. Let E be the connected component of ζ_0 in $D_r(\zeta_0) \cap J$. Thus, as noted in the proof of Theorem 4.2, $\overline{E}$ is a crosscut of $D_r(\zeta_0)$. If C is a connected component of $D_r(\zeta_0) - J$, and if ζ_0 is a boundary point of C, then exactly one side of E includes C. By the proof of Theorem 3.1, if C is a connected component of $D_r(\zeta_0) - J$, then the side of E that includes C completely determines the boundary of C. Thus, ζ_0 is a boundary point of exactly two connected components of $D - J$; one for each side of E.

So, let C_1, C_2 denote the two connected components of $D_r(\zeta_0) - J$ whose boundaries contain ζ_0. Each of these components is a connected subset of $\mathbb{C} - J$. So each is either included in the interior of J or in the exterior of J. Since there are points of the interior and exterior of J that are arbitrarily close to ζ_0, it follows from Theorem 3.4 that one of these components is included in the interior of J and one is included in the exterior of J. Suppose C_1 is included in the interior of J; that is, $D \supseteq C_1$.

Let $p \in C_1$, and let U be the connected component of p in $D \cap D_r(\zeta_0)$. We show that $U = C_1$. Since C_1 is a connected subset of $D \cap D_r(\zeta_0)$ that contains p, $C_1 \subseteq U$. Since U is a connected subset of $D_r(\zeta_0) - J$ that contains p, $U \subseteq C_1$. This completes the proof of the theorem. □

5. Preliminaries to proof of Theorem 3.2: polar separations

Definition 5.1. Let $\mathcal{A}$ be an annulus, and let Ω be an open subset of $\mathcal{A}$. A *polar separation* of the boundary of Ω is a pair of disjoint sets (E, F) so that whenever C is an intermediate circle of $\mathcal{A}$, there is a connected component of $C \cap \Omega$ whose boundary contains a point of E and a point of F.

Our goal in this section is to prove the following.

Theorem 5.2. *Let $\mathcal{A}$ be an annulus, and let D be a simply connected Jordan domain. Suppose that $\mathcal{A}$ separates two boundary points of D, and let γ_1 and γ_2 be the subarcs of the boundary of D that join these points. Then, $(\gamma_1 \cap \mathcal{A}, \gamma_2 \cap \mathcal{A})$ is a polar separation of the boundary of $D \cap \mathcal{A}$.*

Our proof of Theorem 5.2 is based on the following lemma.

Lemma 5.3. *Let C be a circle, and let D be a simply connected Jordan domain. Suppose C separates two boundary points of D. Then, there is a connected component of $C \cap D$ whose boundary hits both subarcs of the boundary of D that join these two boundary points of D.*

Proof. Let p be a boundary point of D that is exterior to C, and let q be a boundary point of D that is interior to C.

Let γ_1, γ_2 denote the subarcs of the boundary of D that join p and q. Let α be a crosscut of D so that $\alpha \cap C$ consists of a single point; label this point p'. Let D_j denote the interior of $\alpha \cup \gamma_j$. By Lemma 4.1, D_1 and D_2 are the sides of α.

Now, for each $j \in \{1, 2\}$, we construct a point q_j in $C \cap D_j$ so that p' is a boundary point of the connected component of q_j in $C \cap D_j$. Since D is open, there is a positive number δ so that $D_\delta(p') \subseteq D$. Let $C' = C \cap D_\delta(p')$. Thus, C' is a subarc of C. Let $q \in C' - \{p'\}$. Then, $q \notin \alpha$ since $C \cap \alpha = \{p'\}$. So, $q \in D_1 \cup D_2$. Without loss of generality, suppose $q \in D_1$. Relabel q as q_1. Let q_2 be a point of C'

so that p' is between q_1 and q_2 on C'. Again, $q_2 \in D_1 \cup D_2$. Since D_1 is the interior of a Jordan curve, and since the subarc of C' from q_1 to q_2 crosses the boundary of D_1 exactly once, $q_2 \notin D_1$. So, $q_2 \in D_2$.

Let E_j denote the connected component of q_j in $C \cap D_j$. By construction, p' is a boundary point of E_j. So, the other endpoint of E_j must be in γ_j since $C \cap \alpha = \{p'\}$. Set $E = E_1 \cup E_2$. Thus, E is a connected component of $C \cap D$. One endpoint of E belongs to γ_1, and the other belongs to γ_2. This proves the lemma. □

Proof of Theorem 5.2. By assumption, $\mathcal{A}$ separates two boundary points of D. One of these points is interior to the inner circle of $\mathcal{A}$, and the other is exterior to the outer circle of $\mathcal{A}$. Let p denote a point that is exterior to the outer circle of $\mathcal{A}$, and let q denote a point that is interior to the inner circle of $\mathcal{A}$.

Let C be an intermediate circle of $\mathcal{A}$. Then, p is exterior to C and q is interior to C. So, by Lemma 5.3, there is a connected component of $C \cap D$ so that one of its endpoints lies on γ_1 and the other lies on γ_2. Thus, $(\gamma_1 \cap \mathcal{A}, \gamma_2 \cap \mathcal{A})$ is a polar separation of the boundary of $D \cap \mathcal{A}$. □

6. Proof of Theorems 3.2 and 3.3

When $X, Y \subseteq \mathbb{C}$, let $d_{\inf}(X, Y)$ denote the infimum of $|z - w|$ as z ranges over all points of X and w ranges over all points of Y.

The proof of the following is essentially the same as the proof of Lemma 4.1 of [7] which is a standard length-area argument.

Lemma 6.1. *Let $\mathcal{A}$ be an annulus, and let Ω be an open subset of $\mathcal{A}$. Suppose (E, F) is a polar separation of the boundary of Ω. Then,*

$$\lambda(\mathcal{A}) \geq \sup_{\phi} \frac{d_{\inf}(\phi[E], \phi[F])^2}{\operatorname{Area}(\phi[\Omega])}$$

where ϕ ranges over all maps that are conformal on a neighborhood of $\overline{\Omega}$.

Proof of Theorem 3.2. Note that $r < 1$ since C is non-empty.

We begin by constructing a rectangle R as follows. Let z_0 be any point of $\phi[C]$. Choose m, l_0 so that $l_0 > l$, $m > \sqrt{\pi\lambda(\mathcal{A})}$, and $(1 - l_0)^2 + m^2 < (1 - l)^2 + \pi\lambda(\mathcal{A})$. Since $r^2 = (1 - l)^2 + \pi\lambda(\mathcal{A})$, z is exterior to the outer circle of $\mathcal{A}$ whenever $|\phi(z)| \leq \sqrt{(1 - l_0)^2 + m^2}$. Let:

$$\begin{aligned} \nu_1 &= \frac{z_0}{|z_0|}(1 - l_0 + mi) \\ \nu_2 &= \frac{z_0}{|z_0|}(1 - l_0 - mi) \end{aligned}$$

Thus, the radius $[0, z_0/|z_0|]$ is a perpendicular bisector of the line segment $[\nu_1, \nu_2]$. The midpoint of $[\nu_1, \nu_2]$ is $(1 - l_0)z_0/|z_0|$, and the length of $[\nu_1, \nu_2]$ is $2m$. Let:

$$\begin{aligned} \nu_3 &= \frac{z_0}{|z_0|}(1 + mi) \\ \nu_4 &= \frac{z_0}{|z_0|}(1 - mi) \end{aligned}$$

Thus, the line segment $[\nu_3, \nu_4]$ is perpendicular to the radius $[0, z_0/|z_0|]$. Furthermore, the length of this segment is $2m$ and its midpoint is $z_0/|z_0|$.

Let R be the open rectangle whose vertices are ν_1, ν_2, ν_3, and ν_4. That is, R is the interior of $[\nu_1, \nu_3] \cup [\nu_3, \nu_4] \cup [\nu_4, \nu_2] \cup [\nu_2, \nu_1]$.

Note that the diameter of R is $\sqrt{l_0^2 + 4m^2}$. Also, the diameter of R approaches $\sqrt{l^2 + 4\pi\lambda(\mathcal{A})}$ as $(l_0, m) \to (l, \sqrt{\pi\lambda(\mathcal{A})})$. It thus suffices to show that $\phi[C] \subseteq R$.

We claim that it suffices to show that $\phi[C]$ contains no boundary point of R. For, since $\phi^{-1}(z_0)$ is interior to the outer circle of $\mathcal{A}$, the modulus of z_0 is larger than $\sqrt{(1-l_0)^2 + m^2}$ which is larger than $l - l_0$. This implies that $z_0 \in R$. Since R contains at least one point of $\phi[C]$, namely z_0, and since $\phi[C]$ is connected, it suffices to show that $\phi[C]$ contains no boundary point of R.

Since $[\nu_3, \nu_4]$ contains no point of the unit disk, it contains no point of $\phi[C]$. By construction, $|\nu_1| = |\nu_2| = \sqrt{(1-l_0)^2 + m^2}$. Thus, $|z| \leq \sqrt{(1-l_0)^2 + m^2}$ whenever $z \in [\nu_1, \nu_2]$. It follows from what has been observed about l_0 and m that $[\nu_1, \nu_2]$ contains no point of $\phi[C]$. So, it suffices to show that $[\nu_1, \nu_3] \cup [\nu_4, \nu_2]$ contains no point of $\phi[C]$.

Let us begin by showing that $[\nu_1, \nu_3]$ contains no point of $\phi[C]$. By way of contradiction, suppose otherwise. In order to obtain a contradiction, we construct a Jordan curve J so that $\mathcal{A}$ separates two points of J as follows. Let z_1 be a point of $\phi[C]$ that belongs to $[\nu_1, \nu_3]$. Thus, by what has just been observed, $z_1 \neq \nu_1$. Let σ_0 be the pre-image of ϕ on $[\nu_1, 0]$. Let σ_1' be the pre-image of ϕ on $[\nu_1, z_1]$. Let σ_3' be the pre-image of ϕ on $[0, z_0]$. Since C is connected, it includes an arc from $\phi^{-1}(z_1)$ to $\phi^{-1}(z_0)$; label this arc σ_2'. Let w_1 be the first point on σ_1' that belongs to σ_2'. Let w_2 be the first point on σ_3' that belongs to σ_2'. Let σ_1 be the subarc of σ_1' from $\phi^{-1}(\nu_1)$ to w_1, and let σ_3 be the subarc of σ_3' from w_2 to $\phi^{-1}(0)$. Let σ_2 be the subarc of σ_2' from w_1 to w_2. Let $J = \sigma_0 \cup \sigma_1 \cup \sigma_2 \cup \sigma_3$. Thus, J is a Jordan curve. By construction, $\mathcal{A}$ separates two points of J.

Let D' denote the interior of J. Let $\Omega = D' \cap \mathcal{A}$. Let $E = \sigma_1 \cap \mathcal{A}$, and let $F = \sigma_3 \cap \mathcal{A}$. We claim that (E, F) is a polar separation of the boundary of Ω. For, let $p = \phi^{-1}(\nu_1)$, and let $q = w_1$ (where w_1 is as in the construction of J). Thus, p is exterior to the outer circle of $\mathcal{A}$. Since $q \in C$, q is interior to the inner circle of $\mathcal{A}$. Let $\gamma_1 = \sigma_1$, and let $\gamma_2 = \sigma_2 \cup \sigma_3 \cup \sigma_0$. Therefore, γ_1, γ_2 are the subarcs of the boundary of D' that join p and q. So, by Theorem 5.2, $(\gamma_1 \cap \mathcal{A}, \gamma_2 \cap \mathcal{A})$ is a polar separation of the boundary of Ω. Since σ_0 is the pre-image of ϕ on $[\nu_1, 0]$, σ_0 contains no point of $\overline{\mathcal{A}}$. Since $\sigma_2 \subseteq C$, σ_2 contains no point of $\overline{\mathcal{A}}$. Thus, $E = \gamma_1 \cap \mathcal{A}$, and $F = \gamma_2 \cap \mathcal{A}$. Hence, (E, F) is a polar separation of the boundary of Ω.

By construction, $d_{\inf}(\phi[E], \phi[F]) = m$. So, by Lemma 6.1, the area of $\phi[\Omega]$ is at least as large as

$$m^2 \lambda(\mathcal{A})^{-1} > \pi.$$

This is impossible since the unit disk includes $\phi[\Omega]$. Thus, $[\nu_1, \nu_3]$ contains no point of $\phi[C]$.

By similar reasoning, $[\nu_4, \nu_2]$ contains no point of $\phi[C]$. Thus, $\phi[C] \subseteq R$, and the theorem is proven. $\square$

Proof of Theorem 3.3. Suppose r_0 is a positive number that is smaller than (3.1). We begin by defining an annulus $\mathcal{A}$ as follows. Choose l so that $0 < l < \epsilon$ and so that

$$r_0 < \exp\left(\frac{8\pi^2}{l^2 - \epsilon^2}\right) \min\left\{ |\zeta_0 - \phi^{-1}(w)| \ : \ |w| \leq \sqrt{(1-l)^2 + \frac{\epsilon^2 - l^2}{4}} \right\}.$$

There is a positive number r_1 so that

$$r_1 < \min\left\{|\zeta_0 - \phi^{-1}(w)| \ : \ |w| \leq \sqrt{(1-l)^2 + \frac{1}{4}(\epsilon^2 - l^2)}\right\}$$

and so that

$$r_0 < \exp\left(\frac{8\pi^2}{l^2 - \epsilon^2}\right) r_1.$$

Since $l < \epsilon$, $r_0 < r_1$. So, define $\mathcal{A}$ to be the annulus whose center is ζ_0, whose outer radius is r_1, and whose inner radius is r_0.

We now show that the diameter of $\phi[C(D; \zeta_0, r_0)]$ is smaller than ϵ. First, note that $\pi\lambda(\mathcal{A}) < (\epsilon^2 - l^2)/4$. Set $r = \sqrt{(l-1)^2 + \pi\lambda(\mathcal{A})}$. Then, $\mathcal{A}$, r, and l satisfy the hypotheses of Theorem 3.2. By Theorem 3.2, the diameter of $\phi[C(D; \zeta_0, r_0)]$ is at most

$$\sqrt{l^2 + 4\pi\lambda(\mathcal{A})}.$$

We have

$$\begin{aligned} l^2 + 4\pi\lambda(\mathcal{A}) &= l^2 + \frac{8\pi^2}{\log(r_1/r_0)} \\ &< l^2 + \epsilon^2 - l^2 = \epsilon^2. \end{aligned}$$

Thus, the diameter of $\phi[C(D; \zeta_0, r_0)]$ is smaller than ϵ. □

7. Proof of the Carathéodory Theorem

We now conclude with the proof of Theorem 3.5. Set $r_0 = 2^{-k} + 2^{-g(k)}$. By Theorem 3.4, $z_0 \in C(D; \zeta_0, r_0)$. By Theorem 3.3, $|\phi(z_0) - \phi(\zeta_0)| < \epsilon$. Thus, $\lim_{z \to \zeta_0} \phi(z) = \phi(\zeta_0)$.

We now show that this extension of ϕ is injective. It suffices to show that $\phi(\zeta_0) \neq \phi(\zeta_1)$ whenever ζ_0 and ζ_1 are distinct boundary points of D. By way of contradiction, suppose $\phi(\zeta_0) = \phi(\zeta_1)$. Let $p = \phi(\zeta_0)$.

We construct a Jordan curve σ as follows. Let α be a crosscut of D that joins ζ_0 and ζ_1. Thus, $\phi[\alpha]$ is a Jordan curve that contains no unimodular point other than p. Let $\sigma = \phi[\alpha]$.

We now construct an annulus $\mathcal{A}$ that separates two points of σ. Choose a positive number R so that $R < \max\{|z - p| \ : \ z \in \sigma\}$. Choose another positive number r so that $r < R$. Let $\mathcal{A}$ be the annulus whose center is p, whose inner radius is r, and whose outer radius is R. By the choice of R, there is a point $q \in \sigma$ that is exterior to the outer circle of $\mathcal{A}$. Let γ_1 and γ_2 be the subarcs of σ that join p and q. Let $E = \gamma_1 \cap \mathcal{A}$, and let $F = \gamma_2 \cap \mathcal{A}$. Finally, let $\Omega = \mathcal{A} \cap \mathbb{D}$ (where $\mathbb{D}$ is the unit disk). Then, by Theorem 5.2, (E, F) is a polar separation of the boundary of Ω. Now, as $r \to 0^+$, $\lambda(\mathcal{A}) \to 0$. However, by the choice of R, $d_{\inf}(E, F)$ is bounded away from 0 as $r \to 0^+$. Thus, by Lemma 6.1, Area$(\phi^{-1}[\Omega]) \to \infty$ as $r \to 0^+$. Since $\phi^{-1}[\Omega] \subseteq D$, this is a contradiction. Thus, $\phi(\zeta_0) \neq \phi(\zeta_1)$.

Finally, we show that this extension of ϕ is surjective. Let ζ be a point on the unit circle. It follows from the Balzano-Weirstrauss Theorem that there is a boundary point of D, ζ_1, so that $\zeta_1 \in \overline{\{\phi^{-1}(r\zeta) \ : \ 0 < r < 1\}}$. Thus, $\phi(\zeta_1) = \zeta$ by the continuity of ϕ.

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