Lonesum (0,1)-matrices and poly-Bernoulli numbers of negative index

by

Chad Roye Brewbaker

A thesis submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

Major: Computer Science

Program of Study Committee:
David Fernández-Baca, Co-major Professor
Ryan Martin, Co-major Professor
Andrew Miner
Sung-Yell Song
Kenneth Bryden

Iowa State University

Ames, Iowa

2005

Graduate College
Iowa State University

This is to certify that the master's thesis of

Chad Roye Brewbaker

has met the thesis requirements of Iowa State University

DEDICATION

I would like to dedicate this thesis to those who have helped me on my mathematical journey. My cohorts Brian Bossé, David Carlson, Mark Kalpakgian, Matt Poush, Shubhabrata Roy, Justin Reynolds, and Geoff Tims have always been there to coax me along. Also, my wonderful mentors Dan Alexander, Dan Ashlock, Maria Axenovitch, Luz DeAlba, John Hansen, Alexander Kleiner, Ken Kopecky, Ryan Martin, David Oakland, and Sung-Yell Song.

TABLE OF CONTENTS

ABSTRACT

This thesis shows that the number of (0,1)-matrices with n rows and k columns uniquely reconstructible from their row and column sums are the poly-Bernoulli numbers of negative index, $B_n^{(-k)}$. Two proofs of this main theorem are presented giving a combinatorial bijection between two poly-Bernoulli formula found in the literature. Next, some connections to Fermat are proved showing that for a positive integer n and prime number p

$$B_n^{(-p)} \equiv 2^n \pmod{p},$$

and that for all positive integers $\{x, y, z, n\}$ greater than two there exist no solutions to the equation:

$$B_x^{(-n)} + B_y^{(-n)} = B_z^{(-n)}.$$

In addition directed graphs with sum reconstructible adjacency matrices are surveyed, and enumerations of similar (0,1)-matrix sets are given as an appendix.

CHAPTER 1. General Introduction

This thesis concerns the set of binary matrices that can be described uniquely by their row and column sums.

Definition 1 (Lonesum matrix). *A Lonesum matrix is a binary matrix that can be uniquely reconstructed from its row and column sums.*

In chapter 1 we will give historical perspective, and define terms the reader may not be familiar with. Chapter 2 gives two proofs that the number of Lonesum matrices is equal to the poly-Bernoulli numbers of negative index, then points out some relations between Lonesum matrices and Fermat's last and little theorems. Our third chapter will explore directed graphs with Lonesum adjacency matrices. Chapter 4 counts similar classes of matrices using brute force computation.

1.1 Motivation

We are given the popular Milton Bradley game *Battleship*^{TM}. It consists of a rectangular array of coordinates, and a collection of vessels ranging from aircraft carriers to submarines.

How many configurations of vessels can we reconstruct given only the number of squares covered by a ship in each row and column? In general how many (0,1)-matrices can we reconstruct given their row and column sums?

1.2 Historical Perspective

The modern foundations of binary matrix reconstruction can be traced to Herb Ryser, Delbert Ray Fulkerson, and Richard Brualdi:[44],[11], [18],[23],[17],[24]. The first question we ask ourselves is: "What condition is needed for a (0,1)-matrix to be reconstructible from its row and column sum vectors?" Fortunately, we have the following theorem:

Definition 2 (interchange operation). *An interchange operation one of the following: replacing the sub-matrix* $\binom{01}{10}$ *with* $\binom{10}{01}$, *or replacing the sub-matrix* $\binom{10}{01}$ *with* $\binom{01}{10}$. *Note that row and column sum vectors stay unchanged after interchange operations are preformed.*

Theorem 1 (Ryser 1957[44]). *Any (0,1)-matrix with row sum vector R and column sum vector S can be transformed into any other (0,1)-matrix with row sum vector R and column sum vector S via interchange operations.*

This give rise to the notion of a Ryser class:

Definition 3 (Ryser Class). *A Ryser Class is the set of (0,1)-matrices with the same row sum vector, and the same column sum vector.[50]*

For example, the Ryser Class of $\binom{011}{100}$ is $\left\{\binom{011}{100}, \binom{101}{010}, \binom{110}{001}\right\}$

By Theorem 1, the Ryser classes can be represented as graphs where each vertex is a matrix in the Ryser class, and each edge is an interchange operation. The analysis of these graphs is an interesting area of research, but it is outside the scope of this thesis.

For this thesis we will be interested in Ryser classes of size one. By Theorem 1 we can deduce that a (0,1)-matrix will be uniquely determined by its row and column sums if and only if no interchange operation can be preformed on the matrix.

Definition 4 (Forbidden Minor). *A forbidden minor is a sub-matrix of the form* $\binom{01}{10}$ *or* $\binom{10}{01}$

1.3 Graph Reconstruction

Graph reconstruction is the study of how little, or what kinds of information are needed to construct a graph. Here are two major open problems:

Open Problem 1 (Vertex Reconstruction). *For an unlabeled simple graph with ≥ 3 vertexes can we reconstruct it uniquely given only the single vertex deleted subgraphs?*

Open Problem 2 (Edge Reconstruction). *For an unlabeled simple graph with ≥ 4 edges can we reconstruct it uniquely given only the single edge deleted subgraphs?*

The edge reconstruction problem has been proven true for graphs that contain only 2 vertex degrees [40], and graphs that have no two vertexes i, j such that $\deg(v_i)$=$\deg(v_j + 1)$.

1.4 Applications

There are four areas of interest that directly benefit from this thesis. First is the area of Discrete Tomography. Discrete Tomography deals with the reconstruction of discrete valued images from density information. For our purposes we view a (0,1)-matrix as an image formed by a 2D black and white pixel array. We are interested in how many images exist that we can uniquely reconstruct by using only the number of black pixels in every row and column.

Discrete Tomography is an active area of study. As we will prove later, the number of $n \times k$ (0,1)-matrices that are sum reconstructible is very small compared to all $2^{n\times k}$ of them. Research is currently being conducted into approximations for non-exact constructions, and creating higher resolution scans by taking many density profiles. Refer to [32],[10] for more details.

A second application is the simplification of number theory proofs. Instead of manipulating poly-Bernoulli numbers we will be able to do transformations on Lonesum matrices. As we will see in chapter 2 many proofs from the literature will be reduced from pages of manipulation to a few lines once we have shown that the number of Lonesum matrices of a given size is a poly-Bernoulli number of negative index.

A third area is graph compression and reconstruction. We can view square Lonesum matrices as adjacency matrices of directed graphs. A Lonesum adjacency matrix can be compressed by storing only the row and column sums. We get a compression ratio of $\frac{2n \log n}{\binom{n}{2}}$, hinting, as we prove in chapter 2, that there are not many of them compared to the set of all binary matrices.

The fourth area of interest is forbidden submatrix enumeration. Many combinatorial objects such as posets and labeled interval orders can be represented as digraphs that either avoid or force the inclusion of certain subgraphs. It is hoped that this research will aid the enumeration of combinatorial structures with larger forbidden subgraphs in their graph representation. A good survey article of this topic is chapter 10 of [57], and this thesis sheds light on Spinrad's open problem 10.1: "What is the number of I_2-free n by n matrices?" (The book defines I_2 as an induced 2 by 2 identity matrix.) Other applications of are perfect phylogenies[42][55], and electrical engineering[27].

1.5 The Stirling, Bernoulli, and poly-Bernoulli numbers

Definition 5 (Stirling Numbers of the Second Kind). *The Stirling numbers of the second kind are the number of ways to partition n elements into k non-empty subsets.*

We denote this the Stirling Numbers of the Second Kind by:

$$\left\{ {n \atop k} \right\} = \frac{(-1)^k}{k!} \sum_{l=0}^{k} (-1)^l \binom{k}{l} l^n$$

Table 1.1 Example: $\left\{ {4 \atop 2} \right\} = 7$

$\{1,2,3\} \bigcup \{4\}$	$\{1,2,4\} \bigcup \{3\}$
$\{1,3,4\} \bigcup \{2\}$	$\{2,3,4\} \bigcup \{1\}$
$\{1,2\} \bigcup \{3,4\}$	$\{1,3\} \bigcup \{2,4\}$
$\{1,4\} \bigcup \{2,3\}$	

Definition 6 (Bernoulli numbers). *The Bernoulli are denoted as B_n, and are the coefficients in the formula*

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

They have the recursive formula

$$B_0 = 1, B_n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} B_i.$$

: Combinatorially the Bernoulli numbers are an inclusion exclusion over the set of length n words, where the sum is taken over all words of length n with k distinct letters, and normalized by $k+1$.

Bernoulli numbers thus have the formula

$$B_n = \sum_{i=m}^{n} (-1)^{m+n} \frac{m! \left\{ {n \atop m} \right\}}{m+1}.$$

Table 1.2 combinatorial interpretation of the Bernoulli numbers

$B_0 =$	1	$\emptyset$
$B_1 =$	$\frac{1}{2}$	$\frac{\lvert\{a\}\rvert}{2}$
$B_2 =$	$\frac{1}{6}$	$-\frac{\lvert\{aa\}\rvert}{2} + \frac{\lvert\{ab,ba\}\rvert}{3}$
$B_3 =$	0	$\frac{\lvert\{aaa\}\rvert}{2} - \frac{\lvert\{aab,aba,baa,bab,bba\}\rvert}{3} + \frac{\lvert\{abc,acb,bac,bca,cab,cba\}\rvert}{4}$

An interesting relation for the Bernoulli numbers is that:

$$B_n = (-1)^{n+1} n\zeta(1-n)$$

Where $\zeta(x)$ is the famous Riemann Zeta Function

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{u^{x-1}}{e^u - 1} du$$

Where:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

For the more combinatorially inclined one can think of $\Gamma(x)$ as a continuous version of the factorial function, because $\Gamma(n+1) = n!$.

The following is a $1000000 USD prize problem from the Clay Mathematics Institute, http://www.claymath.org:

Open Problem 3 (Riemann Hypothesis). *Show that the only values of s over the complex numbers other than $-2, -4, -6, \ldots$ such that $\zeta(s) = 0$ have the form $s = \frac{1}{2} + xi$, where $i = \sqrt{-1}$.*

An extension to the Bernoulli numbers defined by Kaneko in [33]:

Definition 7 (poly-Bernoulli Numbers). *The poly-Bernoulli numbers are written as $B_n^{(k)}$ and*

$$\frac{Li_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^\infty B_n^{(k)} \frac{x^n}{n!}$$

where $Li_k(z)$ denotes the formal power series:

$$\sum_{m=1}^\infty \frac{z^m}{m^k}$$

Kaneko notes that when we set k equal to one we get back the Bernoulli numbers:

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_n^{(1)} \frac{x^n}{n!}$$

Fortunately, Kaneko also proves two combinatorial formulas.

$$B_n^{(-k)} = \sum_{m=0}^{n} (-1)^{n+m} m! \left\{ {n \atop m} \right\} (m+1)^k \tag{1.1}$$

$$B_n^{(-k)} = \sum_{j=0}^{min(n,k)} (j!)^2 \left\{ {n+1 \atop j+1} \right\} \left\{ {k+1 \atop j+1} \right\} \tag{1.2}$$

1.6 Poly-Bernoulli numbers in the literature

Here is a short survey of poly-Bernoulli number results in the literature.

Theorem 2 (Symmetry (Kaneko[33])). $B_n^{(-k)} = B_k^{(-n)}$

Theorem 3 (A theorem of Vandiver). *[33]*

$$B_i^{(1)} \equiv \sum_{m=1}^{p-2} (1 + \frac{1}{2} + ... + \frac{1}{m} (m+1)^i) \pmod p$$

Theorem 4 (A Clausen-von Staudt type theorem). *(Arakawa and Kaneko)[6] Let $k, p, n \in N$. Assume $k \geq 2$. Let p be a prime number satisfying $k + 2 \leq p \leq n + 1$.*

(i) If $n \equiv 0 \pmod{(p-1)}$, then $p^k B_n^{(k)}$ is a p-adic integer and satisfies

$$p^k B_n^{(k)} \equiv -1 \pmod{pZ_p}$$

(ii) If $n \not\equiv 0 \pmod{(p-1)}$, then $p^{k-1} B_n^{(k)}$ is a p-adic integer. It satisfies

$$p^{k-1}B_n^{(k)} \equiv \left\{ \begin{array}{ll} \frac{1}{p}\left\{ {n \atop p-1} \right\} - \frac{n}{2^k} \mod pZ_p & if n \equiv 1 \mod (p-1) \\ \frac{(-1)^{n-1}}{p}\left\{ {n \atop p-1} \right\} \pmod{pZ_p} & otherwise \end{array} \right\}$$

Also, in two articles [53],[52] Roberto Sanchez-Peregrino gives shorter proofs than Kaneko of equation 1.2.

Some other articles where poly-Bernoulli numbers are cited include:[51],[1],[34], [41], and [39].

CHAPTER 2. Poly-Bernoulli Numbers of Negative Index and Lonesum Matrices

In this chapter we shall prove that the cardinality of the set of n by k Lonesum matrices is the Poly-Bernoulli number $B_n^{(-k)}$.

Here is a recap of a few definitions from the last chapter:

Definition 8 (Forbidden Minor). *A forbidden minor is a sub-matrix of the form* $\binom{01}{10}$ *or* $\binom{10}{01}$

Definition 9 (Lonesum matrix). *A Lonesum matrix is a (0,1)-matrix uniquely determined by its ordered row and column sum vectors.*

Definition 10 (poly-Bernoulli Numbers). *The poly-Bernoulli numbers are written as* $B_n^{(k)}$ *and have the formula:*

$$B_n^{(k)} = \sum_{m=0}^{n} (-1)^{n+m} m! \left\{ {n \atop m} \right\} (m+1)^{-k}$$

2.1 The sieve formula

Hypothesis 1 (sieve formula). *The number of distinct $n \times k$ Lonesum matrices is*

$$B_n^{(-k)} = \sum_{m=0}^{n} (-1)^{n+m} m! \left\{ {n \atop m} \right\} (m+1)^{k}$$

.

To simplify the proof we will turn it into a word counting problem.

Definition 11 (column alphabet). *A column alphabet is a legal set of columns that can co-exist in a Lonesum (0,1)-matrix.*

Definition 12 (Forbidden word). *A forbidden word is an ordered list of {0,1} columns that when concatenated form a forbidden minor.*

For example, the three symbol word $\binom{010}{110}$ is allowed while the three symbol word $\binom{011}{110}$ is forbidden because it contains the forbidden minor $\binom{01}{10}$.

Definition 13 (symbol weight). *Symbol weight is the number of 1's in a {0,1} column.*

Lemma 1 (Symbol Weight). *: No two symbols of the same* weight *occur in the same word of our language, unless they are the same symbol.*

Proof. Let S_1 and S_2 be symbols in our alphabet, $S_1 \neq S_2$, and weight(S_1) = weight(S_2). Since $S_1 \neq S_2$ there must be a row where they have different values. Scanning both symbols from top to bottom find the first row where their weights differ and call it X. At this point the running total of one symbol weight is bigger than the other, so scan down to the row where their weights are equal again and call that row Y. Thus, we have $\binom{X(0,1)}{Y(1,0)}$ or $\binom{X(1,0)}{Y(0,1)}$. Since both are forbidden minors the lemma holds. We can't have two symbols of the same weight occur in the same word of our language unless they are the same symbol. □

Observation 1 (Row Weight). *By a similar argument no two rows have the same weight unless they are identical.*

Lemma 2 (Swap). *Permuting rows or columns does not change membership in the class of Lonesum matrices.*

Proof. Note that the forbidden minors are mirror images. Swapping either the rows or columns of one forbidden minor will give you the other. Thus, any permutation of rows or columns will still yield the same number of forbidden minors. □

By Lemma 1 every column alphabet must consist of symbols with unique weight. For convenience we will throw out the all zero and all 1 symbol until later. If we order the symbols of a size $m-1$ alphabet in decreasing order of their weight and view the alphabet as a word every row will sum to an integer between zero and m. Thus, we have m distinct equivalence classes on the row weights. Partitioning the rows into m distinct equivalence classes, and assigning equivalence classes values of a permutation of the integers 0 to m, the number of alphabets is $(m)!\left\{{n \atop m}\right\}$.

Alphabets containing $m-1$ symbols will be a subset of the alphabets containing m symbols so we must do an inclusion exclusion to avoid over-counting.

Thus summing over all size $m-1$ alphabets not containing the all 1's symbol or the all zeros symbol, writing all possible length k words over these alphabets along with the all 1's symbol and all zeros symbol, and doing inclusion exclusion to avoid over counting yields the formula

$$\sum_{m=1}^{n}(-1)^{n+m}m!\left\{{n \atop m}\right\}(m+1)^k$$

.

Thus, we have proven the following theorem.

Theorem 5 (Sieve formula). *The number of distinct $n \times k$ Lonesum matrices is*

$$B_n^{(-k)} = \sum_{m=0}^{n}(-1)^{n+m}m!\left\{{n \atop m}\right\}(m+1)^k$$

.

We can now use transformations on sets of Lonesum matrices to simplify proofs about the poly-Bernoulli Numbers of negative index. For instance, take this identity proved by Kaneko[33] using a few pages of algebra. It is now a one line proof given Theorem 5.

Corollary 1 (Inversion). *For any $n, k \geq 0$ we have $B_n^{(-k)} = B_k^{(-n)}$*

Proof. Transpose the set of $n \times k$ Lonesum Matrices. Since $\binom{01}{10}$ and $\binom{10}{01}$ are duals under this transformation no forbidden matrices are added or deleted from our set. $\square$

2.2 The closed formula

Recent work has went into finding cleaner proofs for a closed equation of the Poly-Bernoulli numbers [53][52]:

$$\sum_{m=1}^{Min(n,k)} m!\left\{{n+1 \atop m+1}\right\}m!\left\{{k+1 \atop m+1}\right\}$$

Here we will give a combinatorial proof involving Lonesum matrices.

Hypothesis 2 (closed equation).

$$B_n^{(-k)} = \sum_{m=1}^{Min(n,k)} m!\left\{ {n+1 \atop m+1} \right\} m!\left\{ {k+1 \atop m+1} \right\}$$

and more specifically the number of Lonesum $n \times k$ (0,1)-matrices with exactly m distinct nonzero rows is $m!\left\{ {n+1 \atop m+1} \right\} m!\left\{ {k+1 \atop m+1} \right\}$.

We want to show that the number of Lonesum $n \times k$ (0,1)-matrices is equal to the new equation, and that m parameterizes the number of distinct nonzero rows.

Lemma 3 (Distinct row maps). *If a Lonesum matrix has m distinct rows out of n then there are $m!\left\{ {n \atop m} \right\}$ ways to map a set of m distinct nonzero rows if all rows are nonzero, and there are $(m+1)!\left\{ {n \atop m+1} \right\}$ ways to map a set of m distinct nonzero rows along with at least one copy of the zero row.*

Proof. This follows from having an onto function from an n set(matrix rows) to an m set(distinct row maps) along with an onto function from a n set(matrix rows) to an $m+1$(distinct row maps and the zero row) set. □

Now we have the question of given all possible mappings of n rows, how many sets of rows do we have to map?

Lemma 4 (Distinct m row sets). *The number of distinct m row sets is*

$$m!\left\{ {k \atop m} \right\} + (m+1)!\left\{ {k \atop m+1} \right\}$$

Proof. By the row weight lemma earlier in the chapter we know that rows have to obey a linear order, with row i less than row j when all 1 entries of row i are also contained in row j and row j has more 1 entries than row i.

Use this ordering to sort our nonzero row sets from highest weight to lowest weight. Notice that we have m rows and each of them is nonzero. From the existence of a linear order at least one column must have m 1's. Instead of writing out the whole row set we could represent it as a string of k numbers from zero to m representing column sums where at least one of the column sums is m. For example:

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}$$

could be written as $\{3, 2, 1\}$.

Thus, all row sets can be written as either an onto function from the k set of columns to the set of integers 1 to m, or as an onto function from the k set columns to the set of integers 0 to m in the case we get an all zero column

$$\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}$$

Thus, $m!\left\{{k \atop m}\right\}$ will count the sets without a zero column, and $(m+1)!\left\{{k \atop (m+1)}\right\}$ will count row sets with at least one zero column. Thus the number of legal distinct nonzero m row sets is $m!\left\{{k \atop m}\right\}+(m+1)!\left\{{k \atop m+1}\right\}$. □

Using the previous two lemmas we have:

$$(m!\left\{{n \atop m}\right\}+(m+1)!\left\{{n \atop (m+1)}\right\})\cdot(m!\left\{{k \atop m}\right\}+(m+1)!\left\{{k \atop (m+1)}\right\})$$

Lets do a little cleaning to get the form we want. Factor out the $m!$ and we get:

$$m!(\left\{{n \atop m}\right\}+(m+1)\left\{{n \atop (m+1)}\right\})\cdot m!(\left\{{k \atop m}\right\}+(m+1)\left\{{k \atop (m+1)}\right\})$$

Now use the substitution $\left\{{a \atop b}\right\}+(b+1)\left\{{a \atop b+1}\right\}=\left\{{a+1 \atop b+1}\right\}$.

$$m!\left\{{n+1 \atop m+1}\right\}m!\left\{{k+1 \atop m+1}\right\}$$

Now we have a formula for the number of Lonesum (0,1)-matrices with exactly m distinct nonzero rows.

Since the the number of set partitions where the number of partitions is bigger than the set is zero we only need to sum m up to $Min(n,k)$. For the set of all $n\times k$ Lonesum matrices we then get:

$$B_n^{(-k)}=\sum_{m=1}^{Min(n,k)} m!\left\{{n+1 \atop m+1}\right\}m!\left\{{k+1 \atop m+1}\right\}$$

Thus we have proved the following theorem.

Theorem 6 (closed equation).

$$B_n^{(-k)} = \sum_{m=1}^{Min(n,k)} m!\left\{ {n+1 \atop m+1} \right\} m!\left\{ {k+1 \atop m+1} \right\}$$

and more specifically the number of Lonesum $n \times k$ (0,1)-matrices with exactly m distinct nonzero rows is $m!\left\{ {n+1 \atop m+1} \right\} m!\left\{ {k+1 \atop m+1} \right\}$.

Corollary 2 (Rank). *The number of $n \times k$ rank m Lonesum (0,1)-matrices is*

$$m!\left\{ {n+1 \atop m+1} \right\} m!\left\{ {k+1 \atop m+1} \right\}$$

Proof. Since they contain m distinct nonzero rows, and by the row weight lemma they must be linearly independent, the matrix has rank m. □

2.3 Connections to Fermat's last theorem

Before closing the chapter we would like to state one of our favorite open problems on language counting:

Open Problem 4 (Combinatorial FLT). *Give a COMBINATORIAL proof of the following: Given three alphabets $\sum_x, \sum_y, \sum_z$ with $|\sum_x| = x, |\sum_y| = y, |\sum_z| = z$ show that there is no bijection from $\sum_x^n \bigcup \sum_y^n$ to $\sum_z^n$ for $n \geq 3$, and every alphabet containing at least one letter.*

As you have noticed this is Fermat's last theorem stated as a counting problem. Andrew Wiles proved this theorem in the mid 1990's, but relied on heavy number theory not known to Fermat. A simple combinatorial proof using mathematics plausibly known to Fermat would be interesting indeed.

Ernst Kummer in the 1800's attempted to attack Fermat's last theorem with the Bernoulli numbers. His approach showed that for a prime number p, if p could not divide $B_p^{(1)}, ..., B_0^{(1)}$ then p could not be an exponent satisfying Fermat's last theorem.

There is also a related open problem[30]:

Open Problem 5 (Regular primes). *A prime p is called regular if it does not divide any of $B_p^{-1}, ..., B_0^{-1}$. Prove that there are an infinite number of regular primes.*

Note that we can encode the equation y^n as the set of $y \times n$ (0,1)-matrices, where scanning from top to bottom in each column there are zero or more ones followed by all zeros. Thus we can think of y^n as the set of $y \times n$ Lonesum (0,1)-matrices with unordered rows.

Theorem 7 (Ordered FLT). *There exist no integers x,y,z,n greater than 2 such that*

$$B_x^{(-n)} + B_y^{(-n)} = B_z^{(-n)}.$$

Proof. As above view the poly-Bernoulli numbers as the set of Lonesum (0,1)-matrices. For convenience let $x \leq y$. By the pigeonhole principle we also have y ¡ z.

The assumption we wish to prove false is that we can create a bijection from the height x and height y Lonesum matrix sets to the height z Lonsum matrix set.

Encode $B_x^{(-n)}$ into the set $B_z^{(-n)}$ by padding extra rows of zeros at the bottom until the height is $y+1$. Encode $B_y^{(-n)}$ into $B_z^{(-n)}$ by letting them be the matrices in B_z^n with at least one nonzero value in the bottom row.

We can set the bottom row to all ones and exactly copy over our matrices from $B_y^{(-n)}$ to the top of the matrix. The problem is that we also have the set of matrices where the last row is replaced with a nonzero row with at least one zero that does not induce a forbidden minor. For instance the all 1's $y \times n$ matrix induces this. Thus we obtain the contradiction that the set $B_z^{(-n)}$ is too large if it contains one more row than the y set, yet the z set have at least one more row to fit the x set. □

Furthermore we can derive a cohort to Fermat's little theorem stating $a^p \equiv a \pmod p$. We will start with the main lemma of [3].

Lemma 5. *Let S be a finite set, p be a prime number, and $f(x) : S \to S$ be a function that has the property $f^p(x) = x$ for all elements $x \in S$. Then $|S| \equiv |F| \mod p$ where F is the number of fixed points.*

One could view the function f as a directed graph on elements in S consisting of disjoint length p cycles and loops.

Theorem 8 (congruence). *Given a prime number p, and a positive integer n,*

$$B_n^{(-p)} \equiv 2^n \pmod p$$

Proof. Let S the set of $n \times p$ Lonesum (0,1)-matrices. Let f be the function that rotates the columns of a matrix once to the left. The matrices that are fixed points in this function are those consisting of all 1's and all zero rows. Their are 2^n such matrices, so let $F = 2^n$. From the above lemma

$$|S| \equiv |F| \pmod p$$

.

Substituting for S and F we get

$$B_n^{(-p)} \equiv 2^n \pmod p.$$

□

CHAPTER 3. Related Graphs

The following is intended to be a small survey of graphs with Lonesum adjacency matrices. Almost, if not all, of the results presented here can be found in the literature.

3.1 Labeled Directed Graphs (with loops)

First we will look at n by n Lonesum matrices with no restriction.

Corollary 3 (Labeled Directed Graphs). *The number of n by n labeled directed graphs uniquely reconstructible from their in and out degrees is $B_n^{(-n)}$*

Proof. By viewing the square (0,1)-matrix as an adjacency matrix the row and column sum vectors correspond to in and out degrees for each vertex. Thus, n by n Lonesum matrices are equivalent to labeled directed graphs uniquely reconstructible from their in and out degree vectors. From the previous chapter we know that the number of these matrices is $B_n^{(-n)}$. □

3.2 Labeled Directed Graphs (without loops)

How about labeled directed graphs without loops? This question was asked by D. Ray Fulkerson in his 1960 paper,"Zero-one matrices with zero trace"[18]

In this case we restrict ourselves to square Lonesum matrices that have 0 down the main diagonal: $\forall_i a_{i,i} = 0$. Instead of enumerating them outright we will show that they are equivalent to labeled interval orders on n elements. In Sloane's database of integer sequences[56] this is sequence number A079144.

Definition 14 (Interval Order). *An interval order is the combinatorial object where a, b, c, d are distinct elements in a set I with two properties:*

(i)Irreflexive: $a < a$ is not valid.

(ii)Relativity:$\{a < b, c < d\}$ implies at least one of $\{a < d, c < b\}$

Proposition 1 (Labeled Interval Orders on n Elements). *Square Lonesum Matrices with 0 on the main diagonal are the set of labeled interval orders on n elements*

Proof. View the matrix as $a_{i,j} = 1 \Leftrightarrow i < j$.

(i) is satisfied because $a_{i,i} = 0$, thus we never have $a < a$

(ii) is satisfied because we have no forbidden minors.

Take the intervals a, b, e, f. The forbidden minor:

$$\begin{pmatrix} < & e & f \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix}$$

shows that $a < e$, $b < f$. It violates condition (ii) because $a < f$ or $b < e$ must exist. □

Observation 2 (Labeled Interval Order Enumeration). *With the paper by Chen[13], and another by Zagier[65] we can naively construct a closed formula:*

$$I_n = \frac{1}{24^n}\sum_{k=0}^{n}\binom{n}{k}\cdot(-1)^{k+1}6^{2k+1}\frac{1}{12}\sum_{m=0}^{2k+1}((-1)^m m!\left\{\begin{matrix} n+1 \\ m+1 \end{matrix}\right\}a_{0,m}),$$

with $a_{0,m} = (-1)^{[m/4]} * 2^{-[m/2]} * 1 - \delta_{1,m+1}$,

and $\delta_{1,i} = 1$ if 4 divides i, and zero otherwise.

3.3 Simple Graphs

The set of Lonesum matrices that correspond the adjacency matrices of simple graphs is trivial. If $a_{i,j} = 1$ then $a_{j,i} = 1$ by symmetry, and we get the forbidden minor:

$$\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

Thus, the only matrix in this class is the all zero n by n matrix; the empty graph.

3.4 Simple Graphs (with loops)

We now allow every vertex in our graph to have a loop. This is the set of Lonesum n by n matrices with $a_{i,j} = a_{j,i}$.

List the forbidden minors and their implications. Assume a, b, c, d are distinct vertexes:

$$\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

(1) Every edge is connected to a looped vertex. If the edge ab exists, but a loop on a or b does not, then we get the above forbidden minor.

$$\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

(2) Every two loops are connected by an edge. Two loops not connected by an edge forms the above forbidden minor.

$$\begin{array}{cc} & \begin{array}{cc} a & c \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

(3)Every loop is distance ≤ 1 from every edge. From the above forbidden minor, given the loops a and c we must also include at least one of the edges ac or ab. Either edge would make the distance between a and bc at most one.

$$\begin{array}{c|cc} & a & c \\ \hline a & 0 & 1 \\ b & 1 & 0 \end{array}$$

(4)Any two edges sharing a non-loop vertex form a triangle. If a is a non-loop vertex, and ac, bc are edges in the graph, then edge bc must be added to avoid the above forbidden minor.

$$\begin{array}{c|cc} & c & d \\ \hline a & 1 & 0 \\ b & 0 & 1 \end{array}$$

(5)Every two non-adjacent edges are connected by an edge. Given the edges ac, bd we must insert either edge bc or edge ad to avoid the above forbidden minor.

The resulting graph consists of a completely connected and looped center, surrounded by unlooped leaves.

Theorem 9 (Labeled Lonesum Graphs). *The number of labeled graphs with a Lonesum adjacency matrix on n vertexes is:*

$$G(n) = \sum_{k=0}^{n} [\binom{n}{k} 2^{k \cdot (n-k)}]$$

Proof. Since the looped vertexes are totally connected make a choice for each set of k looped vertexes. Next we must decide how to connect the remaining $n-k$ vertexes to the looped vertexes. Each vertex has k possible edges to a looped vertex,or 2^kedge configurations. There are $n-k$ vertexes that must make this choice, thus $2^{k*(n-k)}$ total edge configurations for every k set of looped vertexes. $\square$

CHAPTER 4. Conclusion

In this chapter we shall investigate future areas of research.

4.1 Computational Results on forbidden sub-matrices

The obvious continuation of this thesis is to start a program for the enumeration of (0,1)-matrices that avoid a finite set of forbidden matrices. Even for the case of 2×2 forbidden matrices this is not an easy task. We have 2^{16} different sets to inspect. To begin this we have written a program in C that does a depth first search of the matrix space branching for every possible (0,1) matrix and bounding by making sure the branch did not induce one of our forbidden submatrices. To prevent overflow the GNU GMP library was used to get arbitrarily large integers. Also, since the problem is intractable by brute force all computations were bounded by a time limit, usually 5-10 seconds per table entry on a Pentium4 1.8ghz Linux machine. Some of these numbers grew rather large, so the output $\geq NUMBER$ means that the computation took too long to run and we produce a lower bound. These tables can be found in appendix A, and the source code is in appendix B.

4.2 Labeled Poset Enumeration

Open Problem 6 (Poset Enumeration [9]). *How many posets are there on n elements for n greater than 16?*

One very outstanding problem related to this thesis is the enumeration of labeled partially ordered sets. A partially ordered set on n elements can be viewed as an $n \times n$ (0,1) adjacency matrix A, where $A_{i,j} \bigwedge A_{j,k} \Rightarrow A_{i,k}$. Thus, the number of labeled sets on n elements is equivalent to the number of $n \times n$ (0,1)-matrices with 1 on the diagonal that avoid permutations of the forbidden induced subgraphs:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

It is our view that a program should be started to enumerate the number of labeled graphs that avoid small forbidden minors, and the pairwise intersection of the sets. Then, for combinatorial structures with an arbitrarily small finite set of forbidden minors we could use inclusion-exclusion to automatically come up with an enumerative formula for our new set of objects.

APPENDIX A. More forbidden (0,1) matricies

This appendix contains tables enumerating more classes of forbidden (0,1) matricies. The source code used can be found in the next appendix. Our program takes as input a list of forbidden (0,1) matricies and does a depth first search to count matricies that avoid them.

More stuff

We start by looking at the $\binom{16}{2}$ sets of forbiden 2×2 matrix pairs:

Table A.1 Poly-Bernoulli Numbers of Negative Index $B_n^{(-k)}$ Avoids $\binom{01}{10}, \binom{10}{01}$ OEIS A099594,A048163

f(n,k)	2	3	4	5	6	7
2	14	46	146	454	1394	4246
3	46	230	1066	4718	20266	85310
4	146	1066	6902	41506	237686	1315666
5	454	4718	41506	329462	$\geq$ 2240224	$\geq$ 1010101
6	697	20266	237686	2270762	$\geq$ 990099	$\geq$ 800080
7	2123	85310	1315666	$\geq$ 1450145	$\geq$ 800080	$\geq$ 660066

Table A.2 Avoids $\binom{10}{01}$

f(n,k)	2	3	4	5	6	7
2	15	54	189	648	2187	7290
3	54	330	1888	10304	54272	278016
4	189	1888	16927	140626	1103671	5202340
5	648	10304	140626	1725316	$\geq$ 4820482	$\geq$ 2370237
6	891	54272	1103671	$\geq$ 5490549	$\geq$ 2270227	$\geq$ 1890189
7	2916	278016	5772397	$\geq$ 3070307	$\geq$ 1880188	$\geq$ 1630163

Table A.3 Avoids $\binom{10}{11}$

f(n,k)	2	3	4	5	6	7
2	15	54	189	648	2187	7290
3	54	330	1888	10304	54272	278016
4	189	1888	16927	140626	1103671	5505552
5	648	10304	140626	1725320	$\geq$ 5000500	$\geq$ 2490249
6	891	54272	1103671	$\geq$ 5770577	$\geq$ 2330233	$\geq$ 2020202
7	2916	278016	6022604	$\geq$ 3050305	$\geq$ 1990199	$\geq$ 1710171

Table A.4 Avoids $\binom{11}{11}$

f(n,k)	2	3	4	5	6	7
2	15	54	189	648	2187	7290
3	54	334	1952	10944	59392	313856
4	189	1952	18521	165120	1401445	$\geq$ 6040604
5	270	10944	165120	2293896	$\geq$ 5470547	$\geq$ 2490249
6	891	59392	1401445	$\geq$ 5930593	$\geq$ 2550255	$\geq$ 2100210
7	2916	313856	$\geq$ 7200720	$\geq$ 2670267	$\geq$ 2080208	$\geq$ 1630163

Table A.5 Avoids $\binom{11}{11}$, $\binom{00}{00}$

f(n,k)	2	3	4	5	6	7
2	14	44	128	352	928	2368
3	44	156	408	720	720	0
4	128	408	840	720	720	0
5	352	720	720	0	0	0
6	928	720	720	0	0	0
7	2368	0	0	0	0	0

Table A.6 Avoids $\binom{11}{11}$, $\binom{10}{11}$

f(n,k)	2	3	4	5	6	7
2	14	46	146	454	1394	4246
3	46	230	1066	4718	20266	85310
4	146	1066	6902	41506	237686	1315666
5	454	4718	41506	329462	1625752	$\geq$ 1460146
6	486	20266	237686	1595749	$\geq$ 1500150	$\geq$ 750075
7	1458	85310	1315666	$\geq$ 1910191	$\geq$ 810081	$\geq$ 650065

Table A.7 Avoids $\binom{11}{11}$, $\binom{10}{01}$

f(n,k)	2	3	4	5	6	7
2	14	44	128	352	928	2368
3	44	192	726	2492	7976	24208
4	128	726	3384	13872	51888	181120
5	352	2492	13872	66134	282514	516060
6	928	7976	51888	282514	549981	509079
7	2368	24208	181120	596068	549083	$\geq$ 610061

Table A.8 Avoids $\binom{11}{11}$, $\binom{10}{00}$

f(n,k)	2	3	4	5	6	7	8	9	10
2	14	44	128	352	928	2368	5888	14336	34304
3	44	177	582	1693	4584	11847	29634	72345	173292
4	128	582	1992	5860	15882	40924	101922	247668	94270
5	352	1693	5860	17255	46662	119853	131437	72612	65016
6	928	4584	15882	46662	125848	131139	81833	63180	55882
7	2368	11847	40924	119853	211147	121609	72423	63957	46974
8	5888	29634	101922	271451	131838	92425	73396	65150	38516
9	14336	72345	247668	242629	133187	93960	65150	57190	40973
10	34304	173292	384299	205030	125889	86978	78520	50974	45304

Table A.9 This is from [43] Theorem 2 and avoids the matrix $(110)^T, (101)^T$

f(n,k)	2	3	4	5	6	7	8
2	16	64	256	1024	4096	16384	65536
3	63	490	3773	28812	218491	1647086	12353145
4	117	3552	50413	698898	7961086	$\geq$ 3540354	$\geq$ 2450245
5	432	24425	614664	$\geq$ 5110511	$\geq$ 1830183	$\geq$ 1310131	$\geq$ 1150115
6	1566	160218	$\geq$ 4570457	$\geq$ 1340134	$\geq$ 1020102	$\geq$ 790079	$\geq$ 720072
7	5589	1008647	$\geq$ 2300230	$\geq$ 880088	$\geq$ 690069	$\geq$ 550055	$\geq$ 500050
8	19683	$\geq$ 3400340	$\geq$ 920092	$\geq$ 640064	$\geq$ 500050	$\geq$ 410041	$\geq$ 360036

Table A.10 Also from [43] and avoids $\binom{01}{10}$, $\binom{01}{11}$, $\binom{11}{01}(101)^T$

f(n,k)	2	3	4	5	6	7	8	9	10
2	13	38	104	272	688	1696	4096	9728	22784
3	33	129	450	1452	4424	12896	36288	99200	264704
4	71	356	1531	5927	21237	71682	230672	≥ 600060	≥ 320032
5	60	851	4415	20138	83538	322023	≥ 410041	≥ 240024	≥ 220022
6	97	1828	11257	59690	283375	≥ 380038	≥ 210021	≥ 180018	≥ 150015
7	147	3613	26069	158985	≥ 430043	≥ 190019	≥ 150015	≥ 130013	≥ 100010
8	212	6679	55855	388315	≥ 210021	≥ 140014	≥ 110011	≥ 90009	≥ 80008
9	294	11686	112216	≥ 350035	≥ 130013	≥ 110011	≥ 90009	≥ 70007	≥ 50005
10	395	19526	213544	≥ 220022	≥ 110011	≥ 90009	≥ 70007	≥ 50005	≥ 50005

Table A.11 Avoids $\binom{01}{10}$, $\binom{10}{01}$, (11), from [27]

f(n,k)	2	3	4	5	6	7	8	9	10
2	7	10	13	16	19	22	25	28	31
3	15	22	29	36	43	50	57	64	71
4	31	46	61	76	91	106	121	136	151
5	63	94	125	156	187	218	249	280	311
6	127	190	253	316	379	442	505	568	631
7	255	382	509	636	763	890	1017	1144	1271
8	511	766	1021	1276	1531	1786	2041	2296	2551
9	1023	1534	2045	2556	3067	3578	4089	4600	5111
10	2047	3070	4093	5116	6139	7162	8185	9208	10231

Table A.12 Another application is the threshold functions for the Bipartite Turan Property [26]. In this problem we are looking at the function $z(n,t)$ which is the maximum number of 1's in an $n \times n$ matrix such that no size t submatrix is all 1's. For this problem our code was slightly modified in six lines to count the largest number of 1's encountered. Thus a set of matrices can be input and the program will count the largest number of 1's present in matrices of each size avoiding the forbidden submatrices. This avoids the 3×3 all 1's submatrix

f(n,k)	2	3	4	5
2	16			
3	64	511		
4	256	4067	63935	
5	1024	32242	993206	$\geq$ 2340234
6	4096	254506	$\geq$ 2640264	
7	16384	2000033	$\geq$ 1490149	
8	65536	$\geq$ 3120312		
9	262144	$\geq$ 2270227		
10	1048576	$\geq$ 1690169		

APPENDIX B. Source Code

This appendix contains the source code used.

Forbiden (0,1) matrix enumeration code

```
/*
Input is a list of forbidden matrix minors.

3 3 //Size of minor
0 0 1  //The minor itself (Only needs to be sepeated by whitespace,
1 0 1  //but newlines are pretty ;)
1 1 1
//Either another minor or EOF.


The program then enumerates a table of size 1,1....n,n zero one matricies
that avoid the set of forbidden matrix minors.


We are lazy so forbidden minors are stored in a list.

There is an option to also include all permutations of forbiden minors.
There is also an option to ignore all minors that involve diagonal elements
```

```
  Optimization can be done to eliminate bigger forbidden minors that have
  smaller forbidden minors as part of them.

  Also, we can make sure that the forbidden masks are not repeated.

  Another great optimization would be to create minimal circuts of that could
detect forbidden subgraphs of that size.


  The program DFS generates row by row the whole matrix.
  All minors possible minors with the new element in the lower left hand
corner are checked.

*/


#include <stdlib.h>
#include <stdio.h>
#include <gmp.h>
#include <time.h>

#define RowMin 2
#define RowMax 10
#define ColMin 2
#define ColMax 10
```

```
#define ForbMatrixSizeMax 3
#define MaxForbMats 30
#define TEX_PRINT 1
#define SEARCH_TIME 5



struct forbMat{
  int rows;
  int cols;
  int arr[ForbMatrixSizeMax][ForbMatrixSizeMax];
};

struct forbMat forbMats[MaxForbMats];
int forbNum,TOO_LONG;
unsigned long count;
mpz_t bigCount;
int chosenX[RowMax];
int chosenY[ColMax];
short wrkMat[20][20];


time_t START_TIME,END_TIME;

int matrixCollision( int index)
{
  int i,j,k;
  for(i=0;i<forbMats[index].rows; i++){
```

```
    for(j=0;j<forbMats[index].cols;j++){
      if(wrkMat[chosenX[i]][chosenY[j]]!=forbMats[index].arr[i][j])
return 0;
    }
  }
  return 1;
}

/*Check forbMats[index] to see if it violates the matrix with the last row
fixed to rmax, and the last col fixed to colmax

Return 1 if there is a conflict, return 0 if they don't conflict
*/
int dfsCheck(int usedX, int usedY, int index, int rmax, int colmax)
{
  int i,j,k;
  if(usedX < forbMats[index].rows-1){
    if(usedX==0)
      i=0;
    else
      i=chosenX[usedX-1]+1;
    for(i; i<rmax; i++){
      chosenX[usedX]=i;
      if(dfsCheck(usedX+1,usedY,index,rmax,colmax))
return 1;
    }/*Choose all copies of this row greater than the last chosen*/
  }
  else{
```

```
    chosenX[forbMats[index].rows-1]=rmax;
    if(usedY < forbMats[index].cols-1){
      if(usedY==0)
i=0;
      else
i=chosenY[usedY-1]+1;
      for(;i<colmax; i++){
chosenY[usedY]=i;
if(dfsCheck(usedX,usedY+1,index,rmax,colmax))
  return 1;
      }
    }
    else
      return matrixCollision(index);
  }
  return 0;
}

dfsSearch(int row, int col, int rmax, int colmax)
{
  int i,j,k,bad;


  if(TOO_LONG)
    return;

  bad=0;
  for(i=0;i<forbNum;i++){
```

```
  /*Check to see if the forbiden matrix is applicable*/
  if(row<forbMats[i].rows-1 || col< forbMats[i].cols-1)
    continue;
  /*Check for conflicts and return if an conflict is found*/
  chosenX[forbMats[i].rows-1]=row;
  chosenY[forbMats[i].cols-1]=col;
  if(dfsCheck(0,0,i,row,col))
    return;
}


/*Set zero and 1*/
col++;
if(col<colmax){
  wrkMat[row][col]=1;
  dfsSearch(row,col,rmax,colmax);
  wrkMat[row][col]=0;
  dfsSearch(row,col,rmax,colmax);
}
else{
  col=0;row++;
  if(row<rmax){
    wrkMat[row][col]=1;
    dfsSearch(row,col,rmax,colmax);
    wrkMat[row][col]=0;
    dfsSearch(row,col,rmax,colmax);
  }
  else{
```

```
      /*We have found another legal matrix*/
      count++;
      if(count > 10000){
mpz_add_ui(bigCount,bigCount,count);
count=0;
time(&END_TIME);
if(((int)difftime(END_TIME,START_TIME))> SEARCH_TIME)
  TOO_LONG=1;
      }


      /*  for(i=0;i<colmax;i++){ */
      /*  for(j=0;j<rmax;j++) */
      /*    printf("%d",wrkMat[i][j]); */
      /*  printf("\n"); */
      /*   } */
    /*  printf("\n"); */
    }
  }
}



int main()
{
  int i,j,k,rows,cols;
  //Read in forbidden Matricies.
  forbNum=0;
  mpz_init(bigCount);
  while(1){
```

```
    if(EOF==scanf("%d",&forbMats[forbNum].rows))
      break;
    scanf("%d",&forbMats[forbNum].cols);
    for(i=0;i<forbMats[forbNum].rows;i++){
      for(j=0;j<forbMats[forbNum].cols;j++){
scanf("%d",&forbMats[forbNum].arr[i][j]);
      }
    }
    forbNum++;
  }
  if(TEX_PRINT)
    {
  printf("\nThe matricies you gave me were:\n");
  for(i=0;i<forbNum;i++){
    gmp_printf("\\[\\left(\\begin{array}{");
    for(j=0;j<forbMats[i].cols;j++)
      gmp_printf("c");
    gmp_printf("}\n");
    for(j=0;j<forbMats[i].rows;j++){
      for(k=0;k<forbMats[i].cols-1;k++)
gmp_printf("%d %",forbMats[i].arr[j][k]);
      gmp_printf("%d \\\\\n",forbMats[i].arr[j][forbMats[i].cols-1]);
    }
    gmp_printf("\\end{array}\\right)\\]\n");
  }


    }
  if(TEX_PRINT){
```

```
    gmp_printf("\\begin{table}\n\\begin{tabular}{|");
    for(i=ColMin;i<=ColMax;i++){
      gmp_printf("c|");
    }
    gmp_printf("c|}\\hline\n");
    gmp_printf("f(n,k) ");
    for(i=ColMin;i<=ColMax;i++){
      gmp_printf("& %d",i);
    }
    gmp_printf("\\\\");
}
  for(i=RowMin;i<=RowMax;i++){
    if(TEX_PRINT){
      gmp_printf("\n %d",i);
    }
    else
      gmp_printf("\n");
    for(j=RowMin;j<=RowMax;j++){
      count=0;
      mpz_set_ui(bigCount,0);
      wrkMat[0][0]=0;
      dfsSearch(0,0,i,j);
      wrkMat[0][0]=1;
      time(&START_TIME);
      TOO_LONG=0;
      dfsSearch(0,0,i,j);
      mpz_add_ui(bigCount,bigCount,count);
      if(TEX_PRINT){
```

```
if(TOO_LONG)
  gmp_printf(" & $\\geq%Zd$",bigCount);
else
  gmp_printf(" & %Zd",bigCount);
      }
      else
{
  if(TOO_LONG)
    gmp_printf(">%Zd \t",bigCount);
  else
    gmp_printf("%Zd \t",bigCount);
}
    }
    if(TEX_PRINT)
      gmp_printf(" \\\\");
  }
  if(TEX_PRINT)
    gmp_printf("\n\\end{tabular}\n\\end{table}\n");
  mpz_clear(bigCount);
  return 0;
}
```

Bibliography

[1] Adelberg, A. (2002). Kummer congruences for universal Bernoulli numbers and related congruences for poly-bernoulli numbers. *International Mathematical Journal*, 1(1):53–63.

[2] Agarwaka, R. and Fernández-Baca, D. (1994). A polynomial-time algorithm for the perfect phylogeny problem when the number of character states is fixed. *SIAM Journal on Computing*, 23(6):1216–1224.

[3] Anderson, P. G., Benjamin, A. T., and Rouse, J. A. (2005). Combinatorial proofs of Fermat's, Lucas's, and Wilson's theorems. *American Mathematical Monthly*, 112(3):266–268.

[4] Anstee, R., Ferguson, R., and Sali, A. (2001). Small forbidden configurations ii. *Electronic Journal of Combinatorics*, 8(R4):24pp.

[5] Anstee, R., Griggs, J., and Sali, A. (1997). Small forbidden configurations. *Graphs and Combinatorics*, 13:97–118.

[6] Arakawa, T. and Kaneko, M. (1999). On poly-Bernoulli numbers. Online Paper.

[7] Ashlock, D. (2003). On sets of 0-1 matrices with equal marginal sums. lecture notes.

[8] Bona, M. (2002). *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory*. World Scientific, New Jersey.

[9] Brinkman, G. and McKay, B. D. (2002). Posets on up to 16 points. *Order*, 19:147–149.

[10] Brualdi, Mehrmann, and Schnider (2001). Special issue on discrete tomography. *Linear Algebra and Its Applications*, 339.

[11] Brualdi, R. A. (1980). Matrices of zeros and ones with fixed row and column sums. *Linear Algebra and Its Applications*, 33:159–231.

[12] Brualdi, R. A. and Ryser, H. J. (1991). *Combinatorial Matrix Theory*. Cambridge Press.

[13] Chen, K.-W. (2001). Algorithms for Bernoulli numbers and Euler numbers. *Journal of Integer Sequences*, 4.

[14] Conway, J. H. and Guy, R. K. (1996). *The Book of Numbers*. Springer, New York.

[15] Erickson, M. J. (1996). *Introduction to Combinatorics*. Wiley-Interscience.

[16] Fishburn, P. C. (1985). *Interval Orders and Interval Graphs*. John Wiley Sons Inc.

[17] Fulkerson, D. R. (1960a). Traces, term ranks, widths and heights. *IBM journal of Research and Development*, 4:455–459.

[18] Fulkerson, D. R. (1960b). Zero-one matricies with zero trace. *Pacific Journal of Mathematics*, 10(3):831–836.

[19] Fulkerson, D. R. (1964). The maximum number of disjoint permutations contained in a matrix of zeros and ones. *Canadian Journal of Mathematics*, 16:729–735.

[20] Fulkerson, D. R. and Gross, O. (1964). Incidence matricies with the consecutive 1's property. *Bulliten of the American Mathematical Society*, 70:681–684.

[21] Fulkerson, D. R. and Gross, O. A. (1965). Incidence matricies and interval graphs. *Pacific Journal of Mathematics*, 15:835–855.

[22] Fulkerson, D. R., Hoffman, A. J., and McAndrew, M. H. (1965). Some properties of graphs with multiple edges. *Canadian Journal of Mathematics*, 17:166–177.

[23] Fulkerson, D. R. and Ryser, H. J. (1961). Widths and heights of (0,1)-matrices. *Canadian Journal of Mathematics*, 13:239–255.

[24] Fulkerson, D. R. and Ryser, H. J. (1962). Multiplicities and minimal widths for (0,1)-matrices. *Canadian Journal of Mathematics*, 14:498–508.

[25] Fulkerson, D. R. and Ryser, H. J. (1963). Width sequences for special classes of (0,1) matricies. *Canadian Journal of Mathematics*, 15:371–396.

[26] Godbole, A. P., Lamorte, B., and Sandquist, E. J. (1997). Threshold functions for the bipartite Turan property. *Electronic Journal of Combinatorics*, R18.

[27] Golin, M. J., Yong, X., Zhang, Y., and Sheng, L. (2004). New upper and lower bounds on the channel capacity of read/write isolated memory. *Discrete Applied Mathematics*, 140:35 48.

[28] Graham, R. L., Knuth, D. E., and Patashnik, O. (1994). *Concrete Mathematics: A Foundation for Computer Science*. Addison-Wesley, 2nd edition.

[29] Greenough, T. L. (1976). *Representation and Enumeration of Interval Orders and Semiorders*. PhD dissertation, Darmouth College.

[30] Guy, R. K. (1994). *Unsolved Problems in Number Theory 2nd Ed.* Springer-Verlag, New York.

[31] Harary, F. and Palmer, E. M. (1973). *Graphical Enumeration*. Academic Press.

[32] Herman, G. T. and Kuba, A. (1999). *Discrete Tomography: Foundations, Algorithms, and Applications*. World Scientific Publishing Co.

[33] Kaneko, M. (1997). Poly-Bernoulli numbers. *Journal de Théorie des Nombres de Bordeaux*, 9:221–228.

[34] Kaneko, M. (1999). Multiple zeta values, poly-Bernoulli numbers, and related zeta functions. *Nagoya Math Journal*. 153:189 201.

[35] Kaneko, M. (2000). The Akiyama-Tanigawa algorithm for Bernoulli numbers. *Journal of Integer Sequences*, 3(00.2.9).

[36] Lauri, J. and Scapellato, R. (2003). *Topics in Graph Automorphisms and Reconstruction*. Cambridge Press.

[37] Lin, I.-J. and West, D. B. (1996). Interval digraphs that are indefference digraphs. Technical report, University of Illinois Urbana-Champaign.

[38] Marcus, A. and Tardos, G. (2004). Excluded permutation matrices and the Stanley-Wilf conjecture. *Journal of Combinatorial Theory Series A*, 107:153–160.

[39] Minh, H. N. (2003). Finite polyzetas, poly-Bernoulli numbers, indentities of polyzetas and noncommutative rational power series. *Proceedings of WORDS03*.

[40] Myrvold, W. J., Ellingham, M. N., and Hoffman, D. G. (1987). Bidegreed graphs are edge reconstructible. *Journal of Graph Theory*, 11(3):281–302.

[41] Ohno, Y. (1998). Multiple zeta values and a zeta function related to poly-bernoulli number. In *Proc. of 3rd. Sympos. on Number Theory at Tsudajyuku Univ*, pages 98–107.

[42] Pe'er, I., R., S., and Sharan, R. (2002a). On the generality of phylogenies from incomplete directed characters. In *LNCS*, volume 2368, pages 258–367. 8th SWAT.

[43] Pe'er, I., Shamir, R., and Sharan, R. (2002b). Incomplete directed perfect phylogeny. Technical report, Tel-Aviv University.

[44] Ryser, H. J. (1957). Combinatorial properties of matrices of zeros and ones. *Canadian Journal of Mathematics*, 9:371–377.

[45] Ryser, H. J. (1960a). Matricies of zeros and ones. *Bulliten of the American Mathematical Society*, 16:442–464.

[46] Ryser, H. J. (1960b). Traces of matricies of zeros and ones. *Canadian Journal of Math*, 12:463–476.

[47] Ryser, H. J. (1969). Permanents and systems of distinct representatives. *Combinatoral Mathematics and its Applications:Proc. Conf. Chapel Hill, N.C., 1967*, 3:55–70.

[48] Ryser, H. J. (1970). A generalization of the matrix equation $a^2 = j$. *Linear Algebra and its Applications*, 3:451–460.

[49] Ryser, H. J. and Jurkat, W. (1966). Matrix factorizations of determinants and permanents. *Journal of Algebra*, 3:1–27.

[50] Sachkov, V. and Tarakanov, V. (2002). *Combinatorics of Nonnegative Matrices*. American Mathematical Society.

[51] Sanchez-Peregrino, R. (2000). The lucas congruence for Stirling numbers of the second kind. *Acta Arithmatica*, 94(1):41–52.

[52] Sanchez-Peregrino, R. (2002a). Closed formula for poly-Bernoulli numbers. *Fibonacci Quarterly*, 40(4):362–364.

[53] Sanchez-Peregrino, R. (2002b). A note on a closed formula for poly-bernoulli numbers. *American Mathematical Monthly*, 109(8):755–758.

[54] Schröder, B. (2002). *Ordered Sets*. Birkhauser.

[55] Sharan, R. (2002). *Graph Modification Problems and their Applications to Genomic Research*. PhD dissertation, Tel-Aviv University.

[56] Sloane, N. (2005). The on-line encyclopedia of integer sequences.

[57] Spinrad, J. (2003). *Efficient Graph Representations*. American Mathematical Society.

[58] Stanley, R. P. (2000). *Enumerative Combinatorics Vol. 1*. Cambridge Press.

[59] Szmidt, J., Urbanowicz, J., and Zagier, D. (1995). Congruences among generalized Bernoulli numbers. *Acta Arithmatica*, 71:273–278.

[60] Trotter, W. (1992). *Combinatorics and Partially Ordered Sets: Dimension Theory*. Johns Hopkins University Press.

[61] vanLint, J. and Wilson, R. M. (2001). *A Course in Combinatorics*. Cambridge University Press, 2nd edition.

[62] Wernicke, S., Alber, J., Gramm, J., Gou, J., and Niedermeier, R. (2004). Avoiding forbiden submatricies by row deletions. Technical report, Universitat Tubingen.

[63] West, D. B. (1996). Short proofs for interval digraphs. Technical report, University of Illinois Urbanna-Champaign.

[64] Zagier, D. (1998). A modified Bernoulli number. *Niew Arch. Wisk*, 4(16):63–72.

[65] Zagier, D. (2001). Vassiliev invariants and a strange identity related to the Dedekind eta-function. *Topology*, 40:941–960.

ACKNOWLEDGEMENTS

I would like to take this opportunity to express my thanks to those who helped me with various aspects of conducting research and the writing of this thesis. First of all I would like to thank Dr. Dan Ashlock for introducing me to the matrix reconstruction problem. Also, thanks to Dr. Ashlock's wife Wendy for introducing him to the problem while reading an issue of Games magazine.

I would also like to thank Dr. Ryan Martin for his months of insightful editorial criticism, and the rest of my committee.

Lastly, I would like to thank the IMA (Institute for Mathematics and its Applications) for putting on a wonderful summer program in combinatorics at Georgia Tech in the summer of 2003.