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THE PROPAGATION AND GROWTH OF DISCONTINUITIES IN MAGNETOGASDYNAMICS IN THE PRESENCE OF FINITE CONDUCTIVITY.

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THE PROPAGATION AND GROWTH OF DISCONTINUITIES IN

MAGNETOGASDYNAMICS IN THE PRESENCE OF FINITE CONDUCTIVITY

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LIST OF SYMBOLS

a	velocity of sound
a_e	effective velocity of sound
A	Alfven velocity
$\underline{E}$	electric field
G	velocity of the wave-front normal to itself
h	component of the perturbed magnetic field in the y-direction
H	component of the magnetic field in the y-direction
H_i	component of the magnetic field in the x_i-direction
H_o	steady state component of the magnetic field in the x_i-direction
$\underline{j}$	current
k	Gaussian curvature
n_i	unit normal to surface
p	pressure
Q	energy
S	entropy
S_o	steady state entropy
t	time
T	temperature
T_o	steady state temperature
u	component of velocity in the x-direction
u_i	component of velocity in the x_i-direction
x	length along coordinate axis
$[z]$	jump in the quantity z $(= z_2 - z_1)$

I. GENERAL INTRODUCTION

The field of magnetogasdynamics is basically a study of the interaction of an electromagnetic field with a conducting compressible fluid. The mathematical model for such a study includes equations from gasdynamics and electromagnetic theory. By considering flows which have velocities much less than the speed of light, the relativistic effects can be neglected.

In most practical problems, we are interested more in the resultant effect due to the motion of a large number of particles, rather than in the motion of an individual particle in the fluid. Therefore we use a macroscopic analysis of the fluid rather than view it in terms of microscopic quantities. Hence the fundamental equations used in analyzing the dynamics of the conducting fluid are based on the conservation laws of mass, momentum, and energy together with Maxwell's equations. In this case, the electromagnetic forces are considered as well as the ordinary gasdynamical forces.

From a macroscopic viewpoint, there are two methods of attack in solving problems. If the conducting fluid is an ionized gas consisting of a mixture of species of ions, electrons, and neutral particles, the conservation laws can be written for each species. For other than the most simple problems, this involves a large number of equations to be solved. We can also choose to consider the fluid as a whole and use the gross quantities or the average quantities of all the species. Here the basic equations involved are the gasdynamic equations of conservation of mass, momentum, and energy with additional terms added to the energy and momentum equations to account for the electromagnetic field. Along with the gasdynamic equations, we use the Maxwell equations for electromagnetic fields and we usually employ Ohm's law as an approximation to more accurate analysis as an equation relating the gasdynamic equations and Maxwell's equations.

Along with the assumption that the velocity of the fluid is much smaller than the velocity of light, the additional assumptions will be made that the electric field is of the same order of magnitude as the induced electric field, and that problems of very high frequency are not considered. With these assumptions a number of authors (a recent development is found in Pai (1)) have shown that the displacement current can be neglected in Maxwell's equations, the excess electrical charge is negligible, and that the energy in the electric field is much smaller than that in the magnetic field. As a result, all the electromagnetic variables can be written in terms of the magnetic field. Thus we have reduced the problem to one involving the interaction of the magnetic field and the gasdynamical equations.

We are now interested in the wave motion in a magnetogasdynamic fluid. Much of the work done in wave motion considers the fluid to be perfectly conducting, that is, to have infinite electrical conductivity. With this assumption, the basic equations give the Lundquist (2) equations. This set of equations shares with the equations of gasdynamics the property of being a symmetric hyperbolic system of first order partial differential equations. Thus as in gasdynamics, disturbances propagate with finite speeds. Using the theory of characteristics, it has been shown that three types of wave motion can exist. There is a transverse wave called the Alfven wave and two longitudinal waves called the fast and slow waves. As a result of the nature of the Lundquist equations, much of the work in magnetogasdynamics involves the same techniques used in gasdynamics.

Friedrichs (2) studied the Lundquist equations in 1954 and brought out several of the analogies between these equations and the equations of gasdynamics. He showed that just as in the gasdynamic equations, the Lundquist equations possess real characteristics, Riemann invariants, and simple waves. He showed that three types of shock waves exist corresponding to the three types of wave motion, and he also showed the existence of contact discontinuities. In addition, a conducting fluid can possess switch on and switch off shocks, across which a component of the magnetic field is created or destroyed.

A weak shock or weak discontinuity is a surface across which there are discontinuities in the derivatives of certain variables. It has been shown (3) that these weak discontinuities occur along the characteristics of the flow. The growth of these weak discontinuities into a strong discontinuity or a shock was first obtained by Thomas (4). In gasdynamics the velocity of propagation of a weak discontinuity into a gas at rest is independent of the direction of the normal to the wave-front. In contrast, in magnetogasdynamics, the propagation is anisotropic, that is the velocity of the wave-front depends on the direction of the normal. This anisotropy introduces a number of complications in the integration of the equations. Lighthill (5) gave a general method of obtaining asmyptotic solutions of the linearized equations. Weitzner (6) gives a method of integrating these equations for all times in a two dimensional flow. Ludwig (7) also gives a method of obtaining the strength of the wave-front at singular points of the wave-front. The general equation of growth for an Alfven wave was obtained by Kaul (8), and for fast and slow waves was obtained by Nariboli (9).

All of the above studies are based on the assumption of infinite electrical conductivity. With finite conductivity, the picture changes. Its effect is dissipative. In gasdynamics, viscosity introduces dissipation, and it is well known that in the presence of viscosity (10), no shock can be formed. The influence of finite conductivity in magnetogasdynamics was studied by Ludford (11) in discussing the structure of stationary shock waves. He proved that when the normal component of the magnetic field is zero, an inviscid, finitely conducting gas admits of a shock across which the magnetic field is continuous while the velocity, density and pressure are discontinuous. Thus the shock is more of the gasdynamical nature. Pai shows the existence of real characteristics in a one dimensional flow with a perpendicular magnetic field.

The above discussion shows that the effect of finite conductivity in magnetogasdynamics is different from the effect of other dissipative parameters such as viscosity. Thomas (12) gave a general method of studying discontinuities in continuum theory. Using these compatibility conditions[1], one can study the propagation and growth of arbitrary

[1]Appendix.

discontinuities across moving surfaces, called singular surfaces. Using the compatibility relations, one can prove not only the existence (if they exist) of surfaces which admit discontinuities across them, but can also obtain the general equations of the propagation and growth of such discontinuities as the surface moves. Thomas has applied these to a number of gasdynamical (4) and plasticity (13) problems and obtained the growth of a sonic wave, the decay of a blast wave, the formation of Luderbands, etc. Truesdell (14) gives a general review of the historical development and the derivation of these equations.

In the present work, we first consider the one-dimensional flow of a fluid with finite conductivity, zero viscosity and heat conductivity, and the magnetic field perpendicular to the flow. We assume the quantities are all functions of the space variable x and the time t. When conductivity is infinite, the velocity of propagation is the "effective" speed of sound, that is, $a_e = (a^2 + A^2)^{\frac{1}{2}}$, where a is the sound velocity, and A is the Alfven velocity. Following this wave-front, we seek to study the growth of waves of finite amplitude along the lines that Lighthill (10) does for the gasdynamical case. We derive a "Burger's" type equation, which seems to be difficult to integrate.

Next we use the technique developed by Thomas (12) to study the equations of magnetogasdynamics in the presence of finite conductivity, but with zero viscosity and heat conduction. We prove the existence of a singular surface moving with the velocity of the gasdynamical speed of sound, and show that the discontinuities in density, velocity and pressure are stronger than those for the magnetic field. Equations are then obtained for the growth of the wave, and it is shown that the front may either terminate into a shock in a finite or infinite time, or may be damped out. The time for the formation of the shock is seen to depend on the direction of the normal to the wave-front.

In the second part, we study the decay of a shock wave along the same lines as Thomas. We obtain the differential equation for the velocity of the shock wave. However, it depends on the direction of the normal to the wave-front; so the integration has been done only for a perpendicular field.

In the last part, we apply the same technique to obtain the jump in vorticity and current across a shock wave. For completeness, we give the cases of both finite and infinite conductivity. Although the final results are not as elegant as for the gasdynamical case, the present technique is simpler and more straight forward than others.

II. THE EFFECT OF FINITE CONDUCTIVITY ON THE GROWTH OF WEAK DISCONTINUITIES IN MAGNETOGASDYNAMICS

A. Qualitative Discussion of the Effect of Finite Conductivity for Linear and Non-Linear Problems

It is well known that magnetogasdynamical equations belong, in the absence of all dissipative mechanisms, to a general class of hyperbolic equations, that is the symmetric hyperbolic equations. The gasdynamical equations also belong to the same class in absence of viscosity. Such a system of first order equations is known to have a number of common properties; existence of real characteristics, and corresponding Riemann invariants, and hence are expected to admit shock formation. But as discussed by Whitham (15), a shock is possible only in an ideal system; in all actual systems dissipative mechanisms cannot be neglected within the region where the gradients of quantities are large. Thus a general process of smoothing occurs due to these dissipative mechanisms. A study of such a problem is done by Whitham in full generality.

It is interesting to note that different dissipative mechanisms cannot be regarded to play an equally important role in the process of smoothing. In the study of the structure of shocks, Ludford and Pai have noted that the presence of viscosity admits of no sharp discontinuities in magnetogasdynamics, while the absence of viscosity, but the presence of finite electrical conductivity, admits of a shock under certain conditions. Ludford, in particular, notes that under certain conditions, this shock is one across which the magnetic field is continuous, while its derivative is discontinuous and the velocity and density are discontinuous.

This shock is then more of a gasdynamical nature. This feature distinguishes electrical conductivity from viscosity in magnetogasdynamics.

The study of wave phenomenon for the linearized problem of a finitely conducting medium appears to have been first made by Cole (16) for a perpendicular magnetic field. Tanuiti (17) also studied the case of an arbitrary magnetic field where the existence of real hyperbolic characteristics was shown. Nariboli and Hyayadhish (18) noted the complete identity in the mathematical sense of Cole's problem with that of thermo-elasticity and stressed the important point that the existence of real characteristics in a linear problem implies the admissibility of shock formation in a corresponding fully non-linear problem. It is precisely this feature we are going to study in greater detail. We first note a few general considerations.

Considering a simplified problem with a perpendicular magnetic field, the basic equations are,

$$\frac{\partial\rho}{\partial t} + u\,\frac{\partial\rho}{\partial x} + \rho\,\frac{\partial u}{\partial x} = 0 \qquad (2.1.1a)$$

$$\rho\,\left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x}\right) + \frac{\partial p}{\partial x} + \frac{1}{4\pi}\,H\,\frac{\partial H}{\partial x} = 0 \qquad (2.1.1b)$$

$$\frac{\partial H}{\partial t} + u\,\frac{\partial H}{\partial x} + H\,\frac{\partial u}{\partial x} = \frac{4\pi}{\sigma}\,\frac{\partial^2 H}{\partial x^2} \qquad (2.1.1c)$$

$$T\rho\,\left(\frac{\partial S}{\partial t} + u\,\frac{\partial S}{\partial x}\right) = \frac{1}{\sigma}\,\left(\frac{\partial H}{\partial x}\right)^2 \qquad (2.1.1d)$$

In the above, it is assumed that the viscosity and heat conduction are zero, ρ is the density, u is the velocity in the x-direction, H is the magnetic field in the y-direction, S is the entropy, T is the temperature, and that all quantities depend on the space variable x and the time t only. We employ e.m.u. units throughout. We also have the equation of state as $p = p(\rho,S)$, $T = T(\rho,S)$.

Linearizing these equations, by letting $\rho \to \rho_o + \rho$, $u \to u$, $H \to H_o + h$, $S \to S_o + S$, $T \to T_o + T$ and neglecting products and squares of (ρ,u,h,S,T) we get,

$$\frac{\partial \rho}{\partial t} + \rho_o \frac{\partial u}{\partial x} = 0 \tag{2.1.2a}$$

$$\rho_o \left(\frac{\partial u}{\partial t}\right) + a_o^2 \frac{\partial \rho}{\partial x} + A_o^2 \frac{\partial S}{\partial x} + \frac{1}{4\pi} H_o \frac{\partial h}{\partial x} = 0 \tag{2.1.2b}$$

$$\frac{\partial h}{\partial t} + H_o \frac{\partial u}{\partial x} = \frac{4\pi}{\sigma} \frac{\partial^2 h}{\partial x^2} \tag{2.1.2c}$$

$$\frac{dS}{dt} = 0 \tag{2.1.2d}$$

From Equation 2.1.2d, it follows that the entropy is constant. Thus we assume that $\frac{\partial S}{\partial x} = 0$ in Equation 2.1.2b, and eliminate the density from Equations 2.1.2a and 2.1.2b, to obtain finally,

$$\frac{\partial^2 u}{\partial t^2} - a_o^2 \frac{\partial^2 u}{\partial x^2} + \frac{H_o}{4\pi\rho_o} \frac{\partial^2 h}{\partial x \partial t} = 0 \tag{2.1.3a}$$

$$\frac{\partial h}{\partial t} + H_o \frac{\partial u}{\partial x} = \frac{4\pi}{\sigma} \frac{\partial^2 h}{\partial t^2} \tag{2.1.3b}$$

If we use the Laplace transform technique to solve these last equations with zero initial conditions, we obtain

$$\bar{u} = A_1 e^{-m_1 x} + B_1 e^{-m_2 x} \tag{2.1.4a}$$

$$\bar{h} = C_1 e^{-m_1 x} + D_1 e^{-m_2 x} \tag{2.1.4b}$$

where m_1 and m_2 are the positive roots of the equation

$$m^4 - m^2 \left[\frac{(1+r)s}{\epsilon} + \frac{s^2}{a_o^2} \right] + \frac{s^3}{\epsilon\, a_o^2} = 0$$

with $r = \frac{A_o^2}{a_o^2}$, $\epsilon = \frac{4\pi}{\sigma}$, where $A_o^2 = \frac{H_o^2}{4\pi\rho_o}$ is the Alfven velocity squared,

a_o is the sound velocity, A_1, B_1, C_1, and D_1 are arbitrary functions of s, and the barred quantities are the Laplace transforms of the corresponding variables. The other two roots are the negatives of m_1 and m_2, and are pertinent for flow in the negative x-direction.

The functions A_1, B_1, C_1, and D_1 are determined by the boundary conditions and cannot affect the nature of the solution. It is the arguments of the exponents, that is m_1 and m_2, which determine the nature of the propagation.

In the absence of the magnetic field, we obtain only one root given by s/a_o, leading to the gasdynamical wave.

If $\epsilon = 0$, that is, the conductivity is infinite, we obtain only one root, as s/a_e where $a_e^2 = A_o^2 + a_o^2$. Thus the nature of the solution remains the same as above; the ordinary speed of sound is merely replaced by the effective speed of sound.

In the general case, for large s, we obtain $m_1 = s/a_o$, $m_2 = (s/\epsilon)^{\frac{1}{2}}$. Now as discussed by Nariboli and Nyayadhish (18), and Erdelyi (19), the point $s \rightarrow \infty$ remains a so called saddle point of the problem for all ϵ, however small. Since a saddle point gives the major contribution to the integral, the wave nature described by the above values dominates for all times. m_1 gives the sound wave. m_2 can be compared to the case of viscosity in the gasdynamical case. Here $m = s/(a_o^2 + \upsilon s)^{\frac{1}{2}}$, υ being the kinematic coefficient of viscosity. The expansion of this for s infinite is exactly similar to that for m_2.

Pai has obtained the characteristics $\psi(x,t) =$ constant as given by

$$\rho^2 T_S \upsilon_H \psi_x^2 (\psi_t + u \psi_x) \cdot [\psi_t + (u + a_e)\psi_x] \cdot [\psi_t + (u - a_e)\psi_x] = 0$$

which gives parabolic and hyperbolic characteristics, corresponding exactly to m_2 and m_1 respectively. We thus expect the wave-front corresponding to m_1 to develop into a shock. This is exactly the result we prove in section C. As for the root m_2, this is the wave-front travelling with the effective speed of sound, but which is now diffused. The effect of finite conductivity on this wave can be seen to be exactly similar to that of viscosity in ordinary gasdynamics, the kinematic viscosity υ just replacing ϵ.

Lighthill (10) has studied the effect of finite viscosity on waves of finite amplitude. It is reasonable to assume that the wavefront, travelling with velocity a_e is damped just as the sound wave in the case of viscosity. With this idea in mind, we have obtained a non-linear equation of the Burger's type in section B. However, the equation turns out to be more complicated and not easily amenable to integration, as in Lighthill's case. The theory of singular surfaces or the method of characteristics does not admit a sharp discontinuity across this front. Whether the non-linearity of this type overcomes the damping to lead to a shock remains open, however, and although we feel a shock cannot develop across this front, we have been unable to prove it.

B. The "Burger's" Equation for the Wave-Front Travelling with the Effective Speed of Sound

We now wish to study the effect of finite conductivity on the wave-front travelling with the velocity a_e relative to the fluid. Consider the one-dimensional flow of a fluid with finite electrical conductivity, but which is inviscid and non heat-conducting. The additional assumptions are made that the flow is parallel to the x-axis, the magnetic field H is planar and perpendicular to the velocity, and that all quantities are functions of the space coordinate x and the time t only. The basic equations are then given by Equations 2.1.1.

Following Lighthill and neglecting squared and higher order terms, Equation 2.1.1 implies that the entropy S can be considered constant across the wavefront. For a perfectly conducting fluid, Equations 2.1.1a and 2.1.1c are equivalent and imply that $H=k\rho$. Also under this condition, it can be shown that the characteristics for the Equations 2.1.1b and 2.1.1c are

$$\frac{dx}{dt} = u \pm a_e \tag{2.2.1}$$

where $a_e^2 = a^2 + A^2$ is called the effective speed of sound, $a^2 = dp/d\rho$ is the ordinary speed of sound and $A^2 = \frac{H^2}{4\pi\rho}$ is the Alfven velocity.

The Riemann invariants for this system of equations are:

$$2r = u + \omega = \text{constant along } \frac{dx}{dt} = u + a_e \tag{2.2.2a}$$

$$2s = u - \omega = \text{constant along } dx/dt = u - a_e \tag{2.2.2b}$$

where

$$\omega = \int_0^\rho \frac{a_e(\rho)}{\rho} \, d\rho$$

Since the entropy is considered to be constant, the pressure is assumed to be a function of the density only. For an ideal gas $p = k\rho^\gamma$, so that

$$a_e = (k\gamma\rho^{\gamma-1} + \frac{1}{4\pi} k^2\rho)^{\frac{1}{2}}$$

From the paper by Mitchner (20), it would seem reasonable to assume a_e is proportional to some power of ρ, that is $a_e = D\rho^n$ where D is a proportionality constant and n is some given number. Using this value for a_e, it follows that

$$\omega = D/n\ \rho^n \tag{2.2.3a}$$

$$a_e = D(H/K)^n = n\omega \tag{2.2.3b}$$

Now consider Equations 2.1.1b and 2.1.1c. Using $H=K\rho$, and $p=p(\rho)$, these equations become,

$$\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} + H\frac{\partial u}{\partial x} = 4\pi/\sigma\ \frac{\partial^2 H}{\partial x^2}$$

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + a_e^2\frac{\partial \rho}{\partial x} = 0 \tag{2.2.4}$$

Then using $\frac{\partial H}{\partial x} = \frac{H}{a_e}\frac{\partial \omega}{\partial x}$ and $\frac{\partial H}{\partial t} = \frac{H}{a_e}\frac{\partial \omega}{\partial t}$, these reduce further to,

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + a_e\frac{\partial u}{\partial x} = \frac{4\pi a_e}{H\sigma}\ \frac{\partial^2 H}{\partial x^2} = \frac{4\pi}{\sigma}\left[\frac{\partial^2 \omega}{\partial x^2} + \frac{(a_e - Ha_e{}')}{a_e^2}\left(\frac{\partial \omega}{\partial x}\right)^2\right]$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + a_e\frac{\partial \omega}{\partial x} = 0 \tag{2.2.5}$$

Substituting from Equations 2.2.3 into these equations, we obtain

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + a_e\frac{\partial u}{\partial x} = \frac{4\pi}{\sigma}\left[\frac{\partial^2 \omega}{\partial x^2} + \left(\frac{1-n}{n\omega}\right)\left(\frac{\partial \omega}{\partial x}\right)^2\right] \tag{2.2.6a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + a_e\frac{\partial \omega}{\partial x} = 0 \tag{2.2.6b}$$

Add and subtract these equations and use the Riemann invariants 2.2.2 to further obtain,

$$\frac{\partial r}{\partial t} + (u + a_e)\frac{\partial r}{\partial x} = \frac{2\pi}{\sigma}\left[\frac{\partial^2 \omega}{\partial x^2} + \frac{1-n}{n\omega}\left(\frac{\partial \omega}{\partial x}\right)^2\right] \tag{2.2.7a}$$

$$\frac{\partial s}{\partial t} + (u - a_e)\frac{\partial s}{\partial x} = -\frac{2\pi}{\sigma}\left[\frac{\partial^2 \omega}{\partial x^2} + \frac{1-n}{n\omega}\left(\frac{\partial \omega}{\partial x}\right)^2\right] \tag{2.2.7b}$$

Following Lighthill and considering a disturbance initially limited to $x < 0$, all the backward leaning characteristics start from initial positions with $x > 0$, and therefore $s = s_o$ which is its value in the undisturbed fluid ahead of the wave. Thus Equations 2.2.7 reduce to the one equation

$$\frac{\partial r}{\partial t} + (u + a_e) \frac{\partial r}{\partial x} = \frac{2\pi}{\sigma} \left[\frac{\partial^2 r}{\partial x^2} + \frac{1-n}{n\omega} \quad (\frac{\partial r}{\partial \omega})^2 \right] \tag{2.2.8}$$

Now $u + a_e = r + s_o + n\omega = r + s_o + n\,(r - s_o) = (n+1)\,r + (1-n)s_o$ and $u_o + a_{eo} = (n+1)\,r_o + (1-n)\,s_o$.

Therefore Equation 2.2.8 becomes

$$\frac{\partial r}{\partial t} + ((1+n)r + (1-n)s_o\,) \frac{\partial r}{\partial x} = \frac{2\pi}{\sigma} \left[\frac{\partial^2 r}{\partial x^2} + \frac{1-n}{n} \, \frac{1}{r-s_o} \, (\frac{\partial r}{\partial x})^2 \right] \tag{2.2.9}$$

Let $z = (1+n)(r-r_o) = u+a_e - (u_o + a_{eo})$ and substitute into Equation 2.2.9, to obtain

$$\frac{\partial r}{\partial t} + (z + u_o + a_{eo}) \frac{\partial r}{\partial x} = \frac{2\pi}{\sigma} \left[\frac{\partial^2 r}{\partial x^2} + \frac{1-n}{n} \, \frac{1}{r-s_o} \, (\frac{\partial r}{\partial x})^2 \right] \tag{2.2.10}$$

Making the transformation $X = x-(u_o + a_{eo})\,t$, this becomes

$$\frac{\partial r}{\partial t} + z \frac{\partial r}{\partial x} = \frac{2\pi}{\sigma} \left[\frac{\partial^2 r}{\partial x^2} + \frac{1-n}{n} \, \frac{1}{r-s_o} \, (\frac{\partial r}{\partial x})^2 \right]$$

and transforming this to z, we get

$$\frac{\partial z}{\partial t} + z \frac{\partial z}{\partial x} = \frac{2\pi}{\sigma} \left[\frac{\partial^2 z}{\partial x^2} + \frac{1-n}{n(n+1)(\frac{z}{n+1} + r_o - s_o)} \, (\frac{\partial z}{\partial x})^2 \right] \tag{2.2.11}$$

Let $z = f(y)$, and find f such that the coefficient of the terms $(\frac{\partial y}{\partial x})^2$ vanish, that is, find $f(y)$ which satisfies the equation,

$$f''(y) + \frac{1-n}{n(f(y) + (n+1)(r_o - s_o)} [f'(y)]^2 = 0 \tag{2.2.12}$$

Thus $f(y) = (\frac{Cy+E}{n})^n - (n+1)(r_o - s_o)$ where C and E are constants of integration.

Under the transformation $z = f\,(y)$ as obtained, Equation 2.2.11 becomes

$$\frac{\partial y}{\partial t} + \quad (\frac{Cy + E}{n})^n - (n+1)(r_o - s_o) \quad \frac{\partial y}{\partial x} = \frac{2\pi}{\sigma} \frac{\partial^2 y}{\partial x^2} \tag{2.2.13}$$

which is a Burger's type equation;

$$\frac{\partial y}{\partial t} + g(y)\,\frac{\partial y}{\partial x} = \tfrac{1}{2}\,\delta\,\frac{\partial^2 y}{\partial x^2} \tag{2.2.14}$$

Some general results have been shown for an equation of this type. In particular, Lighthill has shown that for $g(y) = y$, the equation can be reduced to the familiar diffusion equation $\frac{\partial \psi}{\partial t} = \frac{1}{2}\,\delta\,\frac{\partial^2 \psi}{\partial x^2}$ and thus no shock will form in this case.

Lighthill implies that the $g(y)$ term in this equation acts to cause a shock wave to form, whereas the $\frac{\partial^2 y}{\partial x^2}$ is a dissipative term and tends to smooth the flow so as to eliminate a shock. There may or may not be a function $g(y)$ such that a shock will form, and since we are unable to integrate Equation 2.2.13, we cannot say whether a shock wave will form in our case or not. Therefore, although we do not feel the characteristic we are following will develop into a shock wave, we have been unable to prove it.

C. Propagation of a Weak Discontinuity into a Shock

In the last section, we derived the equation for a wave of finite amplitude, limiting our attention to the wave-front travelling with the effective speed of sound. The difficulties of integration did not lead to any final conclusion, although they do indicate the wave-front is diffused. In the present section, we study the general three-dimensional problem of magnetogasdynamics with finite electrical conductivity and arbitrary magnetic field. Using the compatability conditions, and the assumptions stated below, we first prove that there exists a singular surface with the velocity of propagation equal to the usual gasdynamical sound velocity. This verifies our earlier inference. Proceeding by the use of the second order compatability conditions, we obtain the equations of growth of a discontinuity. The velocity of propagation of the wavefront is now independent of the normal to the surface. It is therefore possible to integrate the equation of growth for an arbitrary initial waveform.

The result brings out a number of interesting results. First, it shows the possibility of the formation of a shock; it also brings out the dependence of the time of formation of the shock on the angle between the normal to the surface and the magnetic field, and on the initial curvature of the wave-front.

We are now going to consider the flow of an inviscid, non heat-conducting fluid with finite electrical conductivity. If we use the equation of state $p = p\,(\rho,S)$, $T = T(\rho,S)$, the equations for such a flow become,

$$\frac{\partial\rho}{\partial t} + \rho_{,i}\, u_i + \rho u_{i,i} = 0 \tag{2.3.1a}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\, u_{i,j}\right) = -\left(p\rho\, \rho_{,i} + p_S\, S_{,i}\right) + \frac{1}{4\pi}\left(H_{i,k}H_k - H_{k,i}H_k\right) \tag{2.3.1b}$$

$$\rho T\left(\frac{\partial S}{\partial t} + u_i\, S_{,i}\right) = \frac{1}{\sigma}\left(H_{k,m}\, H_{k,m} - H_{k,m}\, H_{m,k}\right) \tag{2.3.1c}$$

$$\frac{\partial H_i}{\partial t} = u_{i,k}\, H_k - u_{k,k}\, H_i - u_k\, H_{i,k} + \frac{4\pi}{\sigma}\, H_{i,kk} \tag{2.3.1d}$$

Assuming the velocity, density, entropy, and magnetic field are continuous across the shock, but that the first derivatives of the velocity and density are discontinuous, we obtain

$$[u_i] = [\rho] = [S] = [H_i] = 0$$

$$[u_{i,j}]n_i = \lambda_i \qquad [S_{,i}] = 0$$
$$[\rho_{,i}]n_i = \xi \qquad [H_{i,j}] = 0$$
$$[\frac{\partial u_i}{\partial t}] = - \lambda_i G \qquad [\frac{\partial S}{\partial t}] = 0 \tag{2.3.2}$$
$$[\frac{\partial \rho}{\partial t}] = - \xi G \qquad [\frac{\partial H_i}{\partial t}] = 0$$
$$[u_{i,jk}]\, n_j n_k = \bar{\lambda}_i \qquad [S_{,ij}]\, n_i n_j = \bar{\beta}$$
$$[\rho_{,ij}]\, n_i n_j = \bar{\xi} \qquad [H_{i,jk}]n_j n_k = \bar{\gamma}_i$$

where G is the velocity of the wave-front normal to itself, and the square brackets denote jumps across the surface.

It is also assumed that quantities in front of the shock wave are constant in space and time and that the velocity is zero in front of the wave. The compatibility conditions[1] with the relations 2.3.2 become,

$$[\frac{\partial^2 u_i}{\partial x_j \partial t}] = (-G\, \bar{\lambda}_i + \frac{\delta \lambda_i}{\delta t})\, n_j - G\, g^{\alpha\beta}\, \lambda_{i,\alpha}\, x_{j,\beta}$$
$$[u_{i,jk}] = \bar{\lambda}_i\, n_j n_k + g^{\alpha\beta}\, \lambda_{i,\alpha}\, (n_j x_{k,\beta} + n_k x_{j,\beta}) - \lambda_i\, g^{\alpha\beta}\, g^{\sigma\tau}\, b_{\alpha\sigma}\, x_{j,\beta}\, x_{k,\tau} \tag{2.3.3}$$
$$[\frac{\partial^2 u_i}{\partial t^2}] = G^2 \bar{\lambda}_i - 2G\, \frac{\delta \lambda_i}{\delta t}$$
$$[\rho_{,ij}] = \bar{\xi}\, n_i n_j + g^{\alpha\beta}\, \xi_{,\alpha}\, (n_i x_{j,\beta} + n_j x_{i,\beta}) - \xi\, g^{\alpha\beta}\, g^{\sigma\tau}\, b_{\alpha\sigma}\, x_{i,\beta}\, x_{j,\tau}$$
$$[\frac{\partial^2 \rho}{\partial x_i \partial t}] = (-G\bar{\xi} + \frac{\delta \xi}{\delta t})\, n_i - G\, g^{\alpha\beta}\, \xi_{,\alpha}\, x_{i,\beta}$$

---

[1] Appendix.

$$[\frac{\partial^2 \rho}{\partial t^2}] = G^2 \bar{\xi} - 2G \frac{\delta \xi}{\delta t}$$

$$[S_{,ij}] = \bar{\beta} n_i n_j \qquad [\frac{\partial^2 S}{\partial x_i \partial t}] = - G \bar{\beta} n_i$$

$$[\frac{\partial^2 S}{\partial t^2}] = G^2 \bar{\beta} \qquad [H_{i,jk}] = \bar{\gamma}_i n_j n_k$$

$$[\frac{\partial^2 H_i}{\partial x_j \partial t}] = - G \bar{\gamma}_i n_j, \qquad [\frac{\partial^2 H_i}{\partial t^2}] = G^2 \bar{\gamma}_i$$

Now applying these compatability relations to the equations 2.3.1, we obtain,

$$- \xi G + \rho \lambda_i n_i = 0 \tag{2.3.4a}$$

$$-\rho G\lambda_i + p\rho \xi n_i = 0 \tag{2.3.4b}$$

$$H_k \lambda_i n_k - H_i \lambda_k n_k + \frac{4\pi}{\sigma} \bar{\gamma}_i n_k n_k = 0 \tag{2.3.4c}$$

or

$$- \xi G + \rho \lambda_n = 0 \tag{2.3.5a}$$

$$-\rho G\lambda_i = - p\rho \xi n_i \tag{2.3.5b}$$

$$H_n \lambda_i - H_i \lambda_n - \frac{4\pi}{\sigma} \bar{\gamma}_i \tag{2.3.5c}$$

Multiply Equation 2.3.5b by n_i and sum over the repeated indices to obtain, $-\rho G \lambda_n = - p\rho\xi$ and then substitute this into Equation 2.3.5a to obtain $-\xi G + \frac{p\rho\xi}{G} = 0$.

Therefore, $\xi G^2 = p\rho\xi = a^2 \xi$ where $a^2 = p\rho$ and thus $G = \pm a$.

Thus the velocity of the wavefront is constant; so the moving wavefront forms a system of parallel surfaces. The successive positions of the surface can be obtained by erecting equal normals to the initial surface. The geometry of the wave-front does not change with time.

With this last result, Equations 2.3.5 become,

$$-a\xi + \rho \lambda_n = 0 \tag{2.3.6a}$$

$$\rho\lambda_i = a\xi\, n_i \tag{2.3.6b}$$

$$H_i\lambda_n - H_n\lambda_i = \frac{4\pi}{\sigma}\bar{\gamma}_i \tag{2.3.6c}$$

Now differentiate Equations 2.3.1 with respect to x_j to obtain,

$$\frac{\partial^2\rho}{\partial x_j\partial t} + \rho_{,ij}\, u_i + \rho_{,i}\, u_{i,j} + \rho_{,j}\, u_{i,i} + \rho u_{i,ij} = 0 \tag{2.3.7a}$$

$$\rho\left(\frac{\partial^2 u_i}{\partial x_j\partial t} + u_{k,j}\, u_{i,k} + u_k u_{i,kj}\right) + \rho_{,j}\left(\frac{\partial u_i}{\partial t} + u_k u_{i,k}\right)$$

$$= -[p_{\rho\rho}\,\rho_{,j}\,\rho_{,i} + p_{\rho S}\,(S_{,j}\,\rho_{,i} + \rho_{,j}\,S_{,i}) + p_{SS}\,S_{,j}\,S_{,i}$$

$$+ p_\rho\,\rho_{,ij} + p_S\,S_{,ij}] + \frac{1}{4\pi}[H_{i,kj}\,H_k + H_{i,k}\,H_{k,j} - H_{k,ij}\,H_k$$

$$- H_{k,i}\,H_{k,j}] \tag{2.3.7b}$$

$$\rho_{,j}\,T\left(\frac{\partial S}{\partial t} + u_i\,S_{,i}\right) + \rho\,(T_\rho\,\rho_{,j} + T_S\,S_{,j})\left(\frac{\partial S}{\partial t} + u_i\,S_{,i}\right)$$

$$+ \rho T\left(\frac{\partial^2 S}{\partial x_j\partial t} + u_{i,j}\,S_{,i} + u_i\,S_{,ij}\right)$$

$$= \frac{1}{\sigma}\,[H_{k,mj}\,H_{k,m} + H_{k,m}\,H_{k,mj} - H_{k,mj}\,H_{m,k} - H_{k,m}\,H_{m,kj}] \tag{2.3.7c}$$

Applying the compatability conditions 2.3.3 to these equations they become,

$$(-G\bar{\xi} + \frac{\delta\xi}{\delta t})n_j - G\,g^{\alpha\beta}\,\xi_{,\alpha}\,x_{j,\beta} + \xi n_i\lambda_i n_i + \rho\,[\bar{\lambda}_i n_i n_j$$

$$+ g^{\alpha\beta}\,\lambda_{i,\alpha}\,(n_i x_{j,\beta} + n_j\,x_{i,\beta}) - \lambda_i\,g^{\alpha\beta}\,g^{\sigma\tau}\,b_{\alpha\sigma}\,x_{i,\beta}\,x_{j,\tau}] = 0 \tag{2.3.8a}$$

$$\rho\,[(-G\bar{\lambda}_i + \frac{\delta\lambda_i}{\delta t})\,n_j - G\,g^{\alpha\beta}\,\lambda_{i,\alpha}\,x_{j,\beta}] + \rho\,[\lambda_k\,n_j\lambda_i n_k] - \xi n_j\lambda_i G$$

$$= -p_{\rho\rho}\,\bar{\xi}\,n_i n_j - p_\rho\,[\bar{\xi}\,n_i n_j + g^{\alpha\beta}\,\xi_{,\alpha}\,(n_i x_{j,\beta} + n_j x_{i,\beta})$$

$$- \xi\,g^{\alpha\beta}\,g^{\sigma\tau}\,b_{\alpha\sigma}\,x_{i,\beta}\,x_{j,\tau}] - p_S\,\bar{\beta}\,n_i n_j + \frac{1}{4\pi}\,H_k\,\bar{\gamma}_i\,n_j n_k - \frac{1}{4\pi}\,H_k\bar{\gamma}_k\,n_k n_j \tag{2.3.8b}$$

$$- \rho TG \, \bar{\beta} \, n_j = 0 \tag{2.3.8c}$$

Since ρ, T, and G are not zero, then $\bar{\beta} = [S_{,ij}] = 0$.

Multiplying Equations 2.3.8a and 2.3.8b by n_j, summing over the repeated indices, and using this last result, we obtain

$$- G \, \bar{\xi} + \frac{\delta \xi}{\delta t} + 2\xi \, \lambda_n + \rho \, \bar{\lambda}_n + \rho \, g^{\alpha\beta} \lambda_{i,\alpha} \, x_{i,\beta} = 0 \tag{2.3.9a}$$

$$\rho \, (-G \, \bar{\lambda}_i + \frac{\delta \lambda_i}{\delta t}) + \rho \, \lambda_n \lambda_i - \xi \, G \lambda_i = - p_{\rho\rho} \, S^2 \, n_i - p_\rho \, \bar{\xi} \, n_i$$

$$- p_\rho \, g^{\alpha\beta} \, \xi_{,\alpha} \, x_{i,\beta} + \frac{1}{4\pi} H_n \, \bar{\gamma}_i - \frac{1}{4\pi} H_k \, \bar{\gamma}_k \, n_i \tag{2.3.9b}$$

Equation 2.3.4b implies that $\lambda_i = \lambda n_i$. Using this condition, Equations 2.3.6 become,

$$\lambda = \frac{a\xi}{\rho} \tag{2.3.10a}$$

$$\frac{4\pi}{\sigma} \, \bar{\gamma}_i = \lambda \, (H_i - H_n \, n_i)$$

and Equations 2.3.9 become,

$$-G\bar{\xi} + \frac{\delta \xi}{\delta t} + 2\xi\lambda + \rho\bar{\lambda} \, n + \rho \, g^{\alpha\beta} \, (\lambda n_i)_{,\alpha} \, x_{i,\beta} = 0 \tag{2.3.11a}$$

$$\rho \, (-G\bar{\lambda}_i + \frac{\delta(\lambda n_i)}{\delta t}) + \rho \, \lambda^2 \, n_i - \xi \, G\lambda n_i = - p_{\rho\rho} \, \xi^2 \, n_i$$

$$- p_\rho \, \bar{\xi} \, n_i - p_\rho \, g^{\alpha\beta} \, \xi_{,\alpha} \, x_{i,\beta} + \frac{1}{4\pi} H_n \, \bar{\gamma}_i - \frac{1}{4\pi} H_k \, \bar{\gamma}_k \, n_i \tag{2.3.11b}$$

Multiplying Equation 2.3.11b by n_i and summing over the repeated indices, we obtain

$$\frac{\delta \xi}{\delta t} = G \, \bar{\xi} - 2\xi \, \lambda - \rho\bar{\lambda}n + 2\rho\lambda \, \Omega \tag{2.3.12a}$$

$$-\rho \, G \, \bar{\lambda}n + \rho \, n_i \, (\frac{\delta \lambda}{\delta t} + \lambda \, \frac{\delta n_i}{\delta t}) + \rho\lambda^2 - \xi G\lambda = - p_{\rho\rho} \, \xi^2 - p_\rho \, \bar{\xi}$$

$$+ \frac{1}{4\pi} H_n \, \bar{\gamma}_n - \frac{1}{4\pi} H_k \, \bar{\gamma}_k \tag{2.3.12b}$$

or

$$\frac{\delta\xi}{\delta t} = G\ \bar{\xi} - 2\xi\lambda + 2\rho\lambda\Omega - \rho\ \bar{\lambda}_n \tag{2.3.13a}$$

$$\rho\ \frac{\delta\lambda}{\delta t} - \rho\ G\ \bar{\lambda}_n + \rho\ \lambda^2 - \xi G\lambda = -p_{\rho\rho}\ \xi^2 - p_\rho\ \bar{\xi} + \frac{1}{4\ \pi}\ H_n\ \bar{\gamma}_n - \frac{1}{4\pi}\ H_k\bar{\gamma}_k,$$

where Ω is the mean curvature of the surface.

Differentiating Equation 2.3.10a, we obtain $\frac{\delta\lambda}{\delta t} = \frac{a}{\rho}\frac{\delta\xi}{\delta t}$. Using this, and the facts that $G = a$ and $p_\rho = a^2$, Equations 2.3.13 become,

$$\frac{\delta\xi}{\delta t} = a\bar{\xi} - 2\xi\lambda + 2\rho\Omega\ \lambda - \rho\ \bar{\lambda}_n \tag{2.3.14a}$$

$$a\ \frac{\delta\xi}{\delta t} - \rho\ a\bar{\lambda}_n = -\ p_{\rho\rho}\ \bar{\xi} - a^2\ \bar{\xi} - \frac{\sigma\lambda}{16\pi^2}\ (H^2 - H_n^2) \tag{2.3.14b}$$

Multiplying Equation 2.3.14a by a and adding the two equations, we obtain finally,

$$a\ \frac{d\xi}{dt} + \xi^2\ \left(\frac{2a}{\rho} + \frac{p_{\rho\rho}}{2a}\right) + \xi\ \left(\frac{\sigma}{32\pi^2\rho}\ H_\alpha^2 - a\Omega\right) = 0\ , \tag{2.3.15}$$

where $H_\alpha = H \sin\theta$, θ denoting the angle between the magnetic field and the normal to the wave-front.

This is the differential equation for the variation of ξ along the normal trajectories of the family of sonic wave surfaces, as these surfaces, as these surfaces are propagated into a uniform gas at rest.

Let n be the distance along the normal trajectory to the initial wavefront. Then since $G = a$ is the velocity of the wavefront, we have

$$\frac{d\xi}{dt} = \frac{d\xi}{dn}\frac{dn}{dt} = G\ \frac{d\xi}{dn} = a\ \frac{d\xi}{dn} \tag{2.3.16}$$

Now let

$$A = \frac{2}{\rho} + \frac{p_{\rho\rho}}{2a^2} \tag{2.3.17a}$$

$$B = \frac{\sigma\ H_\alpha^2}{32\pi^2\ \rho a} \tag{2.3.17b}$$

Therefore Equation 2.3.15 becomes,

$$\frac{d\xi}{dn} + A\xi^2 + (B - \Omega)\,\xi = 0 \tag{2.3.18}$$

We will now discuss the cases corresponding to different initial waveforms.

1. Plane wave

For a plane wave, $\Omega = 0$. Therefore Equation 2.3.18 becomes,

$$\frac{d\xi}{dn} + A\,\xi^2 + B\xi = 0 \tag{2.3.19}$$

Integrating this equation, we have, if $B \neq 0$,

$$\xi = \frac{1}{e^{Bn}\left(\frac{1}{\xi_o} + \frac{A}{B}\right) - \frac{A}{B}} \tag{2.3.20}$$

where ξ_o is the value of ξ at the initial wavefront. A shock is said to form in this case when the denominator of the expression for ξ goes to zero, or equivalently $\xi \quad \infty$. That is,

$$e^{Bn_1}\left(\frac{1}{\xi_o} + \frac{A}{B}\right) - \frac{A}{B} = 0 \tag{2.3.21}$$

Solving for n_1 we obtain,

$$n_1 = -\frac{1}{B}\ln\left(1 + \frac{B}{A\xi_o}\right) \tag{2.3.22}$$

It can be seen by use of Weyl's condition, that both A and B are greater than zero, so we have two cases to consider.

$\xi_o > 0$

This case corresponds to an expansion wave and no shock will form.

$\xi_o < 0$

This is a compressive case and $n_1 > 0$ and is finite if $0 < \frac{B}{A\xi_o} + 1 < 1$.

Letting $\xi_o = - \bar{\xi}_o$ we get the restriction on the strength of the magnetic field allowable for a shock to form:

$$H_\alpha^2 < \frac{32\pi^2 \ \rho \ a \ \bar{\xi}_o}{\sigma} \left(\frac{2}{\rho} + \frac{p_{\rho\rho}}{2a^2}\right) \tag{2.3.23}$$

which is seen to depend on the speed of sound and the density ahead of the shock.

2. *Spherical waves*

In this case $\Omega = - \frac{1}{R}$, where the negative sign is used since the normal is chosen in the direction of the propagation of the wave. We also can replace n by R in this case, and Equation 2.3.18 becomes,

$$\frac{d\xi}{dR} + A\xi^2 + (B + \frac{1}{R})\xi = 0 \tag{2.3.24}$$

Integrating this, we have

$$\xi = \frac{e^{-BR}/R}{e^{-BR_o}/R_o\xi_o + A\displaystyle\int_{R_o}^{R} \frac{e^{-BR}}{R} \, dR} \tag{2.3.25}$$

where R_o and ξ_o are the values of R and ξ at the initial wave-front.

Here again we say a shock forms if the denominator of this quantity goes to zero, that is,

$$A \int_{R_o}^{R} \frac{e^{-BR}}{R} \, dR = - \frac{e^{-BR_o}}{R_o\xi_o} \tag{2.3.26}$$

If $\xi_o > 0$, no shock will form, whereas if $\xi_o < 0$, there is a positive value R_1 for which this equation is satisfied. It is interesting to note that since $B = B(\theta)$, the wave front does not develop into a shock simultaneously at all points of the surface. To see this, let R_1

be the root of this equation for $\theta = 0$ and R the root for any θ. Then it can be written as

$$A \int_{R}^{R_1} \frac{1-e^{-Bu}}{u} \, du = \frac{1}{R_o \xi_o} (e^{-BR_o} - 1)$$

$$< 0,$$

implying $R_1 < R$.

Thus, when the magnetic field is parallel to the normal to the surface, the magnetic field has no effect, as is well known. While, at points where the normal to the front is inclinded to the magnetic field, the field retards the formation of the shock. This retardation is maximum for $\theta = \frac{\pi}{2}$.

3. General mean curvature

In the general case (4), $\Omega = \dfrac{\Omega_o - k_o n}{1 - 2\Omega_o n + x_o n^2}$ where n is the distance along the normal to the wave surface as before, and k_o and Ω_o are the Gaussian and mean curvatures of the initial surface.

The differential Equation 2.3.18 can now be written in the form,

$$\frac{d}{dn} \left(\frac{e^{\int (\Omega (n) - B) dn}}{\xi} \right) = A \, e^{\int (\Omega - B) \, dn} \qquad (2.3.27)$$

But $\int (\Omega - B) dn = - \ln \phi(n) - Bn$ where $\phi(n) = \sqrt{1 - 2\Omega_o n + k_o n^2}$.

Integrating Equation 2.3.17 form 0 to n, we obtain,

$$\frac{1}{\xi} = e^{Bn} \phi(n) \left[\frac{1}{\xi_o} + A \int_o^n \frac{e^{-Bn}}{\phi(n)} \, dn \right]$$

and thus,

$$\xi = \frac{e^{-Bn}}{\phi(n)\left[\frac{1}{\xi_o} + A\int_o^n \frac{e^{-Bn}}{\phi(n)}\,dn\right]} \tag{2.3.28}$$

Let

$$J(n) = \frac{1}{\xi_o} + A\int_o^n \frac{e^{-Bn}}{\phi(n)}\,dn$$

Equation 2.3.28 then becomes,

$$\xi = \frac{e^{-Bn}}{\phi(n)\ J(n)} \tag{2.3.29}$$

It is interesting to note, that as in the last two examples, the magnetic field tends to retard the growth of the shock wave, when one is formed, so that all points of the surface do not develop into a shock simultaneously.

Now, following Thomas, we consider two cases:

Case 1. $K_o \geq 0, \Omega_o < 0$ Here $\phi(n) > 0$ for $n \geq 0$

a) $\xi_o > 0$ Under these assumptions, there is no value of n for which $J(n)$ is zero. Thus no shock wave will form.

b) $\xi_o < 0$ Here there is a positive value n_1 for which $J(n_1) = 0$, implying the existence of a strong shock.

Case 2. $K_o \leq 0, \Omega_o > 0$

For this case, there exists a value n_1 for which $\phi(n_1) = 0$, with $\phi(n) > 0$ for $0 \leq n < n_1$.

a) $\xi_o > 0$ With this assumption, $J(n) < 0$ for $0 \leq n \leq n_1$, and $J(n_1)$ is finite. Therefore a shock wave exists since $\phi(n_1) = 0$.

b) $\xi_o < 0$ If $J(n_1) \geq 0$, a shock forms as $n \rightarrow n_1$. If $J(n_1) < 0$, there exists a value $n_2 < n_1$ for which $J(n_2) = 0$. Thus in this case a shock develops as $n \rightarrow n_2$.

Thus a shock develops in all cases for this particular geometry of the initial wave-front.

Following Thomas, similar results could be obtained for other initial wave-fronts.

We finally note that $\xi_o < 0$ implies that the discontinuity is compressive. Thus, a compressive weak wave can develop into a shock in a finite time or an infinite time, or may be completely damped out. It depends on ξ_o, the strength of the initial discontinuity.

III. BLAST WAVES IN MAGNETOGASDYNAMICS WITH FINITE CONDUCTIVITY

In this section, we assume a medium with finite conductivity and zero viscosity and heat conduction. Helliwell and Pack (21) obtain shock relations for this case. However, they assume the velocity of propagation of the shock is zero. We rederive the jump conditions for a propagating shock wave. If in these relations, we further assume the magnetic field is continuous across the shock wave, we obtain relations anälogous to gasdynamics, with an extra condition for the jumps in the derivatives of the magnetic field across the shock.

Courant and Friedrichs (3) have studied the decay of a plane shock wave. The methods used in treating the plane wave differ from the techniques used by other authors for the spherical wave. Thomas (22) discusses the problem again by the use of the compatability conditions. The shock relations leave the velocity of propagation indeterminate. One needs an additional assumption to make it determinate. Thomas makes an energy hypothesis and integrates the equation for the velocity of propagation of the shock.

Following the last author, we obtain the differential equation for the velocity of propagation of the wave-front in the fluid we are considering. Since the resulting differential equation depends on the angle between the magnetic field and the normal to the surface, its integration in the general case cannot be achieved. However, we integrate the equation for the case of a cylindrical wave-front with a perpendicular magnetic field.

The flow of a fluid with the above assumptions is governed by the following set of equations,

$$\frac{\partial \rho}{\partial t} + \rho_{,i} u_i + \rho u_{i,i} = 0 \tag{3.1a}$$

$$\rho \frac{\partial u_i}{\partial t} + \rho u_k \, u_{i,k} + p_{,i} = \frac{1}{4\pi} H_k H_{i,k} - \frac{1}{4\pi} H_k, \, H_{k,i} \tag{3.1b}$$

$$\frac{\partial H_i}{\partial t} = u_{i,k} \, H_k - u_{k,k} \, H_i - u_k H_{i,k} + \frac{4\pi}{\sigma} H_{i,kk} \tag{3.1c}$$

where the equation of state for the gas that has been used is $p = k \, \rho^{\gamma}$, γ being the gas constant.

If we assume the magnetic field is continuous across the shock wave, shock conditions derived by Kanwall (24) for the equation of continuity and the equation of motion hold and reduce to the ordinary gasdynamical jump conditions. That is

$$\rho_1 \, (u_{1n} - G) = \rho_2 \, (u_{2n} - G) \tag{3.2a}$$

$$\rho_1 \, (u_{1n} - G) \, [u_i] = - \, [p] \, n_i \tag{3.2b}$$

Also from the equation $H_{i,i} = 0$, we can obtain

$$[H_n] = 0 \tag{3.2c}$$

Since Kanwall considers the infinite conductivity case, we cannot use his jump conditions for the energy equation, and Helliwell and Pack consider a stationary wave. Therefore, since we wish to consider a moving front, we must derive the energy jump condition.

The energy equation with finite conductivity is

$$\frac{dp}{dt} = \rho \frac{dh}{dt} - \frac{j^2}{\sigma} \tag{3.3}$$

where $j = \nabla \times \underline{H}$. Multiplying the equation of motion by the vector $\underline{u}$

we obtain,

$$\rho \underline{u} \cdot \frac{d\underline{u}}{dt} = - \underline{u} \cdot \nabla p + \frac{1}{4\pi} \underline{u} \cdot (\underline{j} \times \underline{H}) \qquad (3.4)$$

Adding this equation to 3.3 and using the equation of continuity to give

$$\frac{dp}{dt} = \rho \frac{d}{dt} \left(\frac{p}{\rho}\right) - p \quad \nabla \cdot \underline{u}$$

we obtain,

$$\rho \frac{dh}{dt} + \frac{1}{2} \rho \frac{du^2}{dt} - \rho \frac{d}{dt} \left(\frac{p}{\rho}\right) + p \quad \nabla \cdot \underline{u} + \underline{u} \cdot \nabla p - \frac{j^2}{\sigma}$$

$$+ \frac{1}{4\pi} \underline{j} \cdot (\underline{u} \times \underline{H}) = 0 \qquad (3.5)$$

Using the generalized Ohm's law and Maxwell's equations, we get

$$\frac{1}{4\pi} \underline{j} \cdot (\underline{u} \times \underline{H}) - \frac{j^2}{\sigma} = - \underline{j} \cdot \underline{E} = - \underline{E} \cdot (\nabla \times \underline{H}) = \nabla \cdot (\underline{E} \times \underline{H})$$

$$- \underline{H} \cdot (\nabla \times \underline{E})$$

But, we have

$$\nabla \times \underline{E} = - \frac{1}{4\pi} \frac{\partial H}{\partial t} \quad .$$

Therefore integrating Equation 3.5 over an arbitrary volume, and using the above results, we obtain

$$\int_V \rho \frac{d}{dt} \left[h + \frac{1}{2} u^2 - \frac{p}{\rho}\right] dv + \int_V (p \nabla \cdot \underline{u} + \underline{u} \cdot \nabla p + \nabla \cdot (\underline{E} \times \underline{H})$$

$$+ \frac{1}{8\pi} \frac{\partial H^2}{\partial t}) \; dv = 0 \qquad (3.6)$$

Using the relation $\frac{d}{dt} \int_V \rho \, f dv = \int_V \rho \frac{df}{dt} dv$ and Gauss' theorem, this can be reduced to

$$\frac{d}{dt} \int_V \rho \left[h + \frac{1}{2} u^2 - \frac{p}{\rho}\right] dv + \int_S (p\underline{u}) \cdot \underline{n} \, dS + \int_S (\underline{E} \times \underline{H}) \cdot nS$$

$$+ \int_V \frac{1}{8\pi} \frac{\partial H^2}{\partial t} \; dv = 0 \qquad (3.7)$$

Applying the equation[1]

$$\frac{d}{dt}\int_V f dV = \int_V \frac{\partial f}{\partial t}\, dV + \int_S f u_n\, d\sigma + \int_\Sigma (f_1 - f_2) G\, d\sigma$$

to this equation and letting V approach zero in such a way that in the limit it passes into a finite part Σ_o of the shock surface Σ. Then the volume integral in the above is of higher order than the surface integrals and can be neglected. We must also consider that

$$\int_{S_1} f\, u_n\, d\sigma \rightarrow -\int_{\Sigma_o} f_1 u_{1n}\, d\sigma$$

$$\int_{S_2} f\, u_n\, d\sigma \rightarrow \int_{\Sigma_o} f_2\, u_{2n}\, d\sigma$$

where S_1 and S_2 are the parts of the surface S on sides 1 and 2 of the shock respectively.

Under these conditions, Equation 3.7 becomes

$$\int_{\Sigma_o} \Big[\rho_2\, (u_{2n} - G)(h + \tfrac{1}{2}\, u^2 - \tfrac{p}{\rho})_2 - \rho_1\, (u_{1n} - G)(h + \tfrac{1}{2} u^2 - \tfrac{p}{\rho})_1$$

$$+ p_2\, u_{2n} - p_1\, u_{1n} + (\underline{E} \times \underline{H}) \cdot n|_2 - (\underline{E} \times \underline{H}) \cdot \underline{n}|_1 \Big]\, d\sigma = 0 \tag{3.8}$$

Since this integral is independent of the extent of the surface Σ_o, we obtain the jump condition desired,

$$[\rho\, (u_n - G)(h + \tfrac{1}{2}\, u^2 - \tfrac{p}{\rho})] + [p\, u_n] + [(\underline{E} \times \underline{H}) \cdot \underline{n}] = 0 \tag{3.9}$$

---

[1]Appendix.

Helliwell and Pack derive another jump condition from Maxwell's equation

$$\nabla \times \underline{E} = - \frac{1}{4\pi} \frac{\partial \underline{H}}{\partial t} \quad .$$

It is,

$$[\underline{n} \times \underline{E}] = 0$$

Since we are assuming the magnetic field to be continuous across the shock, it is seen that this condition is the same as $[(\underline{E} \times \underline{H}) \cdot \underline{n}] = 0$ in Equation 3.9. Thus Equation 3.9 reduces to

$$\rho_1 \, (u_{1n} - G) \; [h + \tfrac{1}{2} u^2 - \tfrac{p}{\rho}] + [pu_n] = 0 \qquad (3.11)$$

where we have also used Equation 3.2a.

Now considering the velocity in front of the shock to be zero, and using the notation $C_1^2 = \frac{\gamma p_1}{\rho_1}$, we can reduce Equations 3.2a, 3.2b and 3.11 to

$$[\rho] = \frac{2\rho_1 \, (G^2 - C_1^2)}{(\gamma+1)G^2 + 2C_1^2} \qquad (3.12a)$$

$$[p] = \frac{2\rho_1 \, (G^2 - C_1^2)}{\gamma+1} \qquad (3.12b)$$

$$[u_i] = \frac{2 \, (G^2 - C_1^2)}{(\gamma+1)G} \; n_i \qquad (3.12c)$$

Using the following notation,

$$[u_i] = \lambda_i, \; [\rho] = \xi, \; [p] = \beta \; , \; [H_i] = 0$$

$$[u_{i,j}] \, n_j = \bar{\lambda}_i, \quad [\rho_{,i}]n_i = \bar{\xi}, \quad [p_{,i}]n_i = \bar{\beta} \; , \; [H_{i,j}]n_j = \bar{\alpha}_i \qquad (3.13)$$

the compatability relations become,

$$[u_{i,j}] = \bar{\lambda}_i \, n_j + g^{\alpha\beta} \quad \lambda_{i,\alpha} \, x_{j,\beta}$$

$$[\rho_{,i}] = \bar{\xi} \, n_i + g^{\alpha\beta} \, \xi_{,\alpha} \, x_{i,\beta}$$

$$[p_{,i}] = \bar{\beta} n_i + g^{\alpha\beta} \beta_{,\alpha} \; x_{i,\beta}$$

$$[H_{i,j}] = \bar{\alpha}_i \; n_j$$

$$[\frac{\partial u_i}{\partial t}] = - \bar{\lambda}_i \; G + \frac{\delta \lambda_i}{\delta t}$$

$$[\frac{\partial \rho}{\partial t}] = - \bar{\xi} \; G + \frac{\delta \xi}{\delta t}$$

$$[\frac{\partial p}{\partial t}] = - \bar{\beta} \; G + \frac{\delta \beta}{\delta t}$$

$$[\frac{\partial H_i}{\delta t}] = - \; \bar{\alpha}_i \; G \tag{3.14}$$

Applying these compatability relations to Equations 3.1 with $u_{(\,)i} = 0$, this gives,

$$- \bar{\xi} \; G + \frac{\delta \xi}{\delta t} + \lambda_i \; (\bar{\xi} \; n_i + g^{\alpha\beta} \xi_{,\alpha} \; x_{i,\beta}) + (\rho_1 + \xi)(\bar{\lambda}_i n_i + g^{\alpha\beta} \lambda_{i,\alpha} x_{i,\beta}) = 0 \tag{3.15a}$$

$$\xi \; (-\bar{\lambda}_i G + \frac{\delta \lambda_i}{\delta t}) + (\rho_1 + \xi)\lambda_k \; (\bar{\lambda}_i \; n_k + g^{\alpha\beta} \lambda_{i,\alpha} x_{k,\beta}) + \bar{\beta} n_i + g^{\alpha\beta} \beta_{,\alpha} x_{i,\beta}$$
$$= \frac{1}{4\pi} H_{1k} \; \bar{\alpha}_i \; n_k - \frac{1}{4\pi} \; H_{1k} \; \bar{\alpha}_k \; n_i \tag{3.15b}$$

$$\frac{1}{\gamma-1} \; (-\bar{\beta} \; G + \frac{\delta \beta}{\delta t}) + \frac{1}{\gamma-1} \lambda_i \; (\bar{\beta} n_i + g^{\alpha\beta} \beta_{,\alpha} \; x_{i,\beta} + (\beta + p_1)(\bar{\lambda}_i n_i +$$
$$g^{\alpha\beta} \lambda_{i,\alpha} \; x_{i,\beta}) = \frac{1}{\sigma} \bar{\alpha}_k \; n_j \; \bar{\alpha}_k n_j - \frac{1}{\sigma} \bar{\alpha}_k n_j \; \bar{\alpha}_j \; n_k \tag{3.15c}$$

These equations become

$$\bar{\xi} \; (\lambda_n - G) + \bar{\lambda}_n \; (\rho_1 + \xi) = A \tag{3.16a}$$

$$\bar{\lambda}_i \; ((\rho_1 + \xi)\lambda_n - \xi G) + \bar{\beta} n_i = f_i \tag{3.16b}$$

$$\bar{\beta} \; (\lambda_n - G) + (\gamma-1)(\beta + p_1)\bar{\lambda}_n = B \tag{3.16c}$$

where

$$A = - \frac{\delta \xi}{\delta t} - g^{\alpha\beta} \xi_{,\alpha} \lambda_{,\beta} - (\rho_1 + \xi) \; g^{\alpha\beta} \; \lambda_{i,\alpha} \; x_{i,\beta} \tag{3.17a}$$

$$f_i = -\xi \frac{\delta\lambda_i}{\delta t} - (\rho_1 + \xi) g^{\alpha\beta} \lambda_{i,\alpha} \lambda_\beta - g^{\alpha\beta} \beta_{,\alpha} x_{i,\beta}$$

$$+ \frac{1}{4\pi} H_{1n} \bar{\alpha}_i - \frac{1}{4\pi} H_{1k} \bar{\alpha}_k n_i \qquad (3.17b)$$

$$B = -\frac{\delta\beta}{\delta t} - g^{\alpha\beta} \beta_{,\alpha} \lambda_\beta - (\gamma-1)(\beta + p_1) g^{\alpha\beta} \lambda_{i,\alpha} x_{i,\beta}$$

$$+ \frac{\gamma-1}{\sigma} \bar{\alpha}_k \bar{\alpha}_k - \frac{\gamma-1}{\sigma} \bar{\alpha}_n^2 \qquad (3.17c)$$

Now consider Equation 3.10. In subscript notation this becomes

$$[\frac{1}{\sigma} (n_j H_{j,i} - n_j H_{i,j}) - \frac{1}{4\pi} n_j u_i H_j + \frac{1}{4\pi} n_j u_j H_i] = 0 \qquad (3.18)$$

where Ohm's law has been used to eliminate $\underline{E}$.

Using Equation 3.2c this gives,

$$\bar{\alpha}_i = \frac{\sigma}{4\pi} H_{,i} \lambda_n - \frac{\sigma}{4\pi} H_{1n} \lambda_i \qquad (3.19)$$

From Equation 3.12c we obtain

$$\lambda_n = \frac{2(G^2 - C_1^2)}{(\gamma+1)G} \qquad (3.20a)$$

$$\lambda_\beta = 0 \qquad (3.20b)$$

Therefore we find that;

$$\rho_1 + \xi = \frac{\rho_1 (\gamma+1) G^2}{(\gamma-1)G^2 + 2C_1^2} \qquad (3.21a)$$

$$p_1 + \beta = \frac{\rho_1}{\gamma+1} (2G^2 - C_1^2 + \frac{p_1}{\rho_1}) \qquad (3.21b)$$

$$\lambda_n - G = \frac{(1-\gamma) G^2 - 2C_1^2}{(\gamma+1)G} \qquad (3.21c)$$

$$\rho_1 \lambda_n + \xi (\lambda_n - G) = 0 \qquad (3.21d)$$

$$A = -\frac{\delta\xi}{\delta t} + (\rho_1 + \xi) g^{\alpha\beta} \lambda_n b_{\alpha\beta} \qquad (3.21e)$$

$$f_n = -\xi \frac{\delta\lambda_n}{\delta t} - \frac{1}{4\pi} H_{1k} \bar{\alpha}_k \tag{3.21f}$$

$$B = -\frac{\delta\beta}{\delta t} + (\gamma-1)(\beta + p_1)g^{\alpha\beta}\, b_{\alpha\beta}\, \lambda_n + \frac{\gamma-1}{\sigma}\bar{\alpha}^2 \tag{3.21g}$$

where we have used,

$$\lambda_{i,\alpha}\, x_{i,\beta} = -\lambda_n\, b_{\alpha\beta}$$

$$\frac{\delta\lambda_i}{\delta t} n_i = \frac{\delta\lambda_n}{\delta t} \tag{3.22}$$

With these results, we obtain

$$\bar{\lambda}_n = \frac{B - (\lambda_n - G)\, f_n}{(\gamma-1)(\beta + p_1)} \tag{3.23a}$$

$$\bar{\beta} = f_n \tag{3.23b}$$

$$\xi = \frac{A}{\lambda_n - G} - \frac{(\rho_1 + \xi)[B - (\lambda_n - G)\, f_n]}{(\lambda_n - G)(\gamma-1)(\beta + p_1)} \tag{3.23c}$$

Differentiating Equations 3.12 and substituting into the quantities for A, f_n, and B, we get,

$$A = -\frac{4\,(\gamma+1)\,\rho_1\, C_1^2\, G}{(2C_1^2 + (\gamma-1)\, G^2)^2}\frac{dG}{dt} + \frac{4\,\rho_1 \Omega\, G\,(G^2 - C_1^2)}{(\gamma-1)\, G^2 + 2C_1^2} \tag{3.24a}$$

$$f_n = \frac{-4\rho_1\,(G^2 - C_1^2)}{(\gamma+1)((\gamma-1)G^2 + 2C_1^2)}\left(1 + \frac{C_1^2}{G^2}\right)\frac{dG}{dt} - \frac{\sigma(H_1^2 - H_{1n}^2)(G^2 - C_1^2)}{8\pi^2\,(\gamma+1)G} \tag{3.24b}$$

$$B = -\frac{4\rho_1 G}{\gamma+1}\frac{dG}{dt} + \frac{4\rho_1\Omega\,(\gamma-1)(G^2 - C_1^2)}{(\gamma+1)^2 G}\left(2G^2 - C_1^2 + \frac{p_1}{\rho_1}\right) + \frac{(\gamma-1)\sigma\,(H_1^2 - H_{1n}^2)(G^2-C_1^2)^2}{4\pi^2\,(\gamma+1)^2\, G^2} \tag{3.24c}$$

Therefore $\bar{\lambda}_n$, $\bar{\beta}$, and $\bar{\xi}$ become respectively

$$\bar{\lambda}_n = \frac{\rho_1}{(\gamma-1)(2G^2 - C_1^2 + \frac{p_1}{\rho_1})}\left[-4\rho_1 G\frac{dG}{dt} + \frac{4\rho_1\Omega(\gamma-1)(G^2 - C_1^2)}{(\gamma+1)G}(2G^2\right.$$

$$- C_1^2 + \frac{p_1}{\rho_1}) + \frac{(\gamma-1)\sigma\,(H_1^2 - H_{1n}^2)(G^2 - C_1^2)}{4\pi^2\,(\gamma+1)\,G^2} + \frac{(1-\gamma)G^2 - 2C_1^2)}{G}$$

$$\frac{4\rho_1\,(G^2 - C_1^2)}{(\gamma+1)[(\gamma-1)G^2 + 2c_1^2]}\,(1 + \frac{C_1^2}{G^2})\,\frac{dG}{dt} + \frac{\sigma(H_1^2 - H_{1n}^2)(G^2 - C_1^2)}{8\pi^2\,(\gamma+1)G}\Bigg] \quad (3.25a)$$

$$\bar{\beta} = \frac{-4\rho_1\,(G^2 - C_1^2)}{(\gamma+1)[(\gamma-1)G^2 + 2C_1^2]}\,(1 + \frac{C_1^2}{G^2})\frac{dG}{dt} - \frac{\sigma(H_1^2 - H_{1n}^2)(G^2 - C_1^2)}{8\pi^2\,(\gamma+1)G} \quad (3.25b)$$

$$\bar{\xi} = \frac{(\gamma+1)G}{(1-\gamma)G^2 - 2C_1^2}\left[- \frac{4(\gamma+1)\,\rho_1\,C_1^2\,G}{(2C_1^2 + (\gamma-1)G^2)^2}\,\frac{dG}{dt} + \frac{4\rho_1\quad G(G^2 - C_1^2)}{(\gamma-1)G^2 + 2C_1^2}\right]$$

$$+ \frac{\rho_1\,(\gamma+1)^2\,G^3}{((\gamma-1)G^2 + 2C_1^2)^2}\,\bar{\lambda}_n \quad (3.25c)$$

Thus it can be seen that the jumps in the derivatives of the velocity, density and pressure across the shock wave are known in terms of the quantities in front of the shock, G and derivatives of G. We need an additional assumption before we can completely specify $\bar{\lambda}_n$, $\bar{\xi}$, and $\bar{\beta}$.

Following Thomas, we state an energy hypothesis from which we are able to obtain G. Let ΔQ be the energy in a differential shell element of the shock surface. Then the energy hypotehsis says that the energy ΔQ is a) proportional to the total energy Q released by the explosion, b) proportional to the volume ΔV of the differential shell element, and c) inversely proportional to the volume V(t) enclosed by the shock wave.

This then becomes,

$\Delta Q = \frac{\alpha\, Q\Delta V}{V(t)}$ where the proportionality constant α will depend on the gas considered.

Now the energy in the shell will consist of the energy ΔQ and the energy contributed by the undisturbed gas. This latter energy is $E_1\rho_2\Delta V$. Also in the case of Joule heating, an additional term must be added which is,

$$\frac{j^2}{\sigma}\rho_2\Delta V$$

Therefore the total energy in the shell is equal to

$$\frac{\alpha Q\Delta V}{V} + E_1\rho_2\Delta V + \frac{j^2}{\sigma}\rho_2\Delta V \tag{3.26}$$

The energy in the shell can also be expressed by

$$E_2\rho_2\Delta V \tag{3.27}$$

Equate these, and simplify to obtain,

$$[E] = \frac{\alpha Q}{\rho_2 V} + \frac{j^2}{\sigma} \tag{3.28}$$

where

$$[E] = E_2 - E_1$$

The Equation 3.11 implies that,

$$[E] = \frac{p_2\, u_{2n}}{\rho_1 G} \tag{3.29}$$

Equating Equations 3.28 and 3.29, we obtain

$$\frac{p_2\, u_{2n}}{\rho_1 \mathbf{G}} = \frac{\alpha Q}{\rho_2 V} + \frac{j^2}{\sigma} \tag{3.30}$$

Since

$$\frac{j^2}{\sigma} = \frac{1}{\sigma}\left(H_{2k,j}\, H_{2k,j} - H_{2k,j}\, H_{2j,k}\right)$$

$$= \frac{1}{\sigma}\left(\bar{\alpha}^2 - \bar{\alpha}_n^2\right)$$

we find that,

$$\frac{j^2}{\sigma} = \frac{1}{\sigma}\bar{\alpha}^2 \tag{3.31}$$

where we have used the facts that $H_{1i,j} = 0$ and $\bar{\alpha}_n = 0$.

From Equations 3.19 and 3.20a, we find that this can be written further as,

$$\frac{j^2}{\sigma} = \frac{\sigma\,(H_1^2 - H_{1n}^2)(G^2 - C_1^2)^2}{4\pi^2\,(\gamma+1)^2\,G^2} \tag{3.32}$$

Substituting this into Equation 3.30, we see that,

$$\frac{p_2 u_{2n}}{\rho_1 G} = \frac{\alpha Q}{\rho_2 V} + \frac{\sigma(H_1^2 - H_{1n}^2)(G^2 - C_1^2)^2}{4\pi^2\,(\gamma+1)^2\,G^2} \tag{3.33}$$

Solving Equations 3.12 for u_{2n}, p_2, and ρ_2, we obtain

$$u_{2n} = \frac{2(G^2 - C_1^2)}{(\gamma + 1)G} \tag{3.34a}$$

$$p_2 = \frac{2\rho_1}{\gamma+1}\,(G^2 - \frac{\gamma-1}{2\gamma}\,C_1^2) \tag{3.34b}$$

$$\rho_2 = \frac{(\gamma+1)\;\rho_1 G^2}{2C_1^2 + (\gamma-1)G^2} \tag{3.34c}$$

Substituting these expressions into Equation 3.33 and rearranging, we obtain the following quadratic in $G^2 - C_1^2$;

$$(G^2 - C_1^2)^2 \left[\frac{4}{\gamma+1} - \frac{\sigma(H_1^2 - H_{1n}^2)}{4\pi^2\,(\gamma+1)}\right] + (G^2 - C_1^2)(\frac{2C_1^2}{\gamma} - \frac{\alpha Q\,(\gamma-1)}{\rho_1 V}) - \frac{\alpha Q\,(\gamma+1)C_1^2}{\rho_1 V} = 0 \tag{3.35}$$

Let

$$A = \frac{16\pi^2 - \sigma H_\alpha^2}{4\pi^2}$$

Then Equation 3.35 becomes,

$$(G^2 - C_1^2)^2\;(\frac{A}{\gamma+1}) + (G^2 - C_1^2)(\frac{2C_1^2}{\gamma} - \frac{\alpha Q(\gamma-1)}{\rho_1 V}) - \frac{\alpha Q(\gamma+1)C_1^2}{\rho_1 V} \tag{3.36}$$

Solving this equation for $G^2 - C_1^2$, we obtain,

$$G^2 - C_1^2 = \frac{\alpha Q\ (\gamma^2 - 1)}{2\rho_1\ VA} - \frac{(\gamma+1)C_1^2}{\gamma A}$$

$$+ \frac{1}{2}\left[\frac{4\ (\frac{1}{2}+1)^2 C_1^4}{\gamma^2\ A^2} - \frac{4(\gamma+1)C_1^2\ \alpha Q(\gamma^2-1)}{\gamma\rho_1 VA} + \frac{\alpha^2 Q^2(\gamma^2-1)^2}{\rho_1^2\ V^2\ A^2} + \frac{4\alpha Q(\gamma+1)^2 C_1^2}{\rho_1 VA}\right]^{\frac{1}{2}} \tag{3.37}$$

To determine the constant of proportionality α, write Equation 3.28 in the form

$$\alpha = \frac{E_2\rho_2 V}{Q} - \frac{E_1\rho_2 V}{Q} - \frac{\rho_2 V\ j^2}{Q\sigma} \tag{3.38}$$

Now consider this equation as $V \to 0$, that is,

$$\alpha = \lim_{V\to 0} \frac{E_2\rho_2 V}{Q} - \lim_{V\to 0} \frac{E_1\rho_2 V}{Q} - \lim_{V\to 0} \frac{\rho_2 V\ j^2}{Q\sigma} \tag{3.39}$$

It is seen from Equation 3.37 that $G \to \infty$ as $V \to 0$. Using this in Equation 3.34c, we see that,

$$\lim_{G\to\infty} \rho_2 = \frac{(\gamma+1)\rho_1}{\gamma-1} \tag{3.40}$$

Therefore ρ_2 is bounded and the last two limits in Equation 3.39 vanish. Thus, we have

$$\begin{aligned}\alpha &= \lim_{V\to 0} \frac{E_2\rho_2\ V}{Q} \\ &= \lim_{V\to 0} \left(\frac{E_2\rho_1 V}{Q}\right) \lim_{V\to 0} \left(\frac{\rho_2}{\rho_1}\right) \\ &= \frac{\gamma+1}{\gamma-1} \lim_{V\to 0} \left(\frac{E_2 V\ \rho_1}{Q}\right)\end{aligned} \tag{3.41}$$

If it is assumed that the energy released in the explosion is distributed uniformly throughout the volume at the first instant, this implies that,

$$\lim_{V \to 0} \frac{E_2 V\rho_1}{Q} = 1$$

Using this value of α, Equation 3.37 becomes,

$$(G^2 - C_1^2) = \frac{Q(\gamma+1)^2}{2\rho_1 VA} - \frac{(\gamma+1)\ C_1^2}{\gamma A}$$

$$+ \frac{1}{2} \left[\frac{4\ (\gamma+1)^2\ C_1^4}{\gamma^2\ A^2} - \frac{4\ (\gamma+1)^3\ C_1^2\ Q}{\gamma\rho_1\ VA^2} + \frac{(\gamma+1)^4\ Q^2}{\rho_1^2\ V^2\ A^2} + \frac{4Q(\gamma+1)^3\ C_1^2}{(\gamma+1)\rho_1 VA} \right]^{\frac{1}{2}} \tag{3.42}$$

From this equation, it can be seen that the velocity of propagation of the wave-front approaches the speed of sound in the undisturbed region as $V \to \infty$, that is $G \to c_1$.

Equation 3.42 can now be used to determine the jump in the pressure across the shock wave as $V \to \infty$. Using Equation 3.12b to eliminate $G^2 - c_1^2$, in Equation 3.42 we obtain,

$$[p] = \frac{Q(\gamma+1)}{VA} - \frac{2\rho_1\ C_1^2}{\gamma A}$$

$$+ \frac{\rho_1}{\gamma+1} \left[\frac{4(\gamma+1)^2\ C_1^4}{\gamma^2 A^2} - \frac{4(\gamma+1)^3\ C_1^2\ Q}{\gamma\ \rho_1\ VA^2} + \frac{(\gamma+1)^4 Q^2}{\rho_1^2\ V^2 A^2} + \frac{4Q(\gamma+1)^3\ C_1^2}{(\gamma-1)\rho_1\ VA} \right]^{\frac{1}{2}} \tag{3.43}$$

It can be seen from this relation that as $V \to \infty$, $[p] \to 0$, which implies that the shock decays as it travels.

We will now consider a cylindrical blast wave. Since the radius R of the cylinder is in the direction of the normal to the wave-front, we have $G = dR/dt$. Considering a unit length of the cylindrical wave, we see that $V = \pi R^2$. Therefore the differential equation for the determination of R as a function of the time t is,

$$\frac{dR}{dt} = \Bigg\{ C_1^2 + \frac{Q(\gamma+1)^2}{2\rho_1\ \pi R^2 A} - \frac{(\gamma+1)C_1^2}{\gamma A}$$

$$+\frac{1}{2}\left[\frac{4(\gamma-1)^2 C_1^4}{\gamma^2 A^2} - \frac{4(\gamma+1)^3 C_1^2 Q}{\gamma\rho_1 \pi R^2 A^2} + \frac{(\gamma+1)^4 Q^2}{\rho_1^2 \pi^2 R^4 A^2} + \frac{4Q(\gamma+1)^3 C_1^2}{(\gamma-1)\rho_1 \pi R^2 A}\right]^{\frac{1}{2}}\Bigg\}^{\frac{1}{2}} \tag{3.44}$$

Thus, with the addition of an energy hypothesis, we are able to determine G and make the problem determinate. The jumps in the derivatives of the velocity, density and pressure across the shock wave are completely known.

VI. JUMPS IN THE VORTICITY AND CURRENT ACROSS A MAGNETO-GASDYNAMIC SHOCK

In 1952, Truesdell (24) studied the jump in the vorticity across a stationary shock wave for a two-dimensional wave in gasdynamics. He showed that the relation is of a purely kinematic nature, in that he only used the equation of motion to obtain his result.

In 1957, Lighthill (25) considered the same problem in a three-dimensional flow, but he used the equation of energy to obtain his final relations.

In the same year, Hayes (26) obtained a result similar to Lighthill, without the use of the equation of energy. He considered the case of a moving shock wave.

Later, Kanwal (27) discusses the two-dimensional shock wave for magnetogasdynamics. He considers a stationary wave-front, and obtains both the jumps in vorticity and current across the shock wave.

All of these authors employ different techniques. In the following section, we show that the results can be deduced in a simple, elegant, and straightforward manner by use of the compatibility conditions.

In addition, for the sake of completeness, we discuss the complete three-dimensional case for a moving shock wave with finite and infinite electrical conductivity.

A. Finite Electrical Conductivity

In this section, we will find expressions for the jump in vorticity and the jump in current across a shock wave. We assume the magnetic field is continuous across the shock wave, but that the velocity, density, and pressure have finite discontinuities across the front.

The equation of motion for a finite conducting fluid is,

$$\rho \frac{\partial u_i}{\partial t} + \rho\, u_k\, u_{i,k} + p_{,i} + \frac{1}{4\pi} H_k\, H_{k,i} - \frac{1}{4\pi} H_{i,k} H_k = 0 \qquad (4.1.1)$$

The notation that will be used for the jumps in the quantities across the shock wave is given in Equation 3.13.

Considering the fluid in front of the shock to be uniform, we apply the compatibility conditions from Equation 3.14 to the Equation 4.1.1 to obtain,

$$(\rho_1 + \xi)(-\lambda_i\, G + \frac{\delta\lambda_i}{\delta t}) + (\xi\, \lambda_k + \rho_1\lambda_k + u_{1k}\xi + \rho_1 u_{1k})(\bar{\lambda}_i n_k + g^{\alpha\beta}\lambda_{i,\alpha} x_{k,\beta})$$

$$+ \bar{\beta}\, n_i + g^{\alpha\beta}\beta_{,\alpha}\, x_{i,\beta} = \frac{1}{4\pi} H_{1k}\, \bar{\alpha}_i\, n_k - \frac{1}{4\pi} H_{1k}\, \bar{\alpha}_k n_i \qquad (4.1.2)$$

Now if we let

$$A_k = \lambda_k + u_{1k}$$

and rearrange, this can be written as,

$$(\rho_1 + \xi)(A_n - G)\bar{\lambda}_i = - \bar{\beta} n_i - (\rho_1 + \xi)\frac{\delta\lambda_i}{\delta t} - (\rho_1 + \xi)A_\beta\, g^{\alpha\beta}\ \lambda_{i,\alpha}$$

$$- g^{\alpha\beta}\ \beta_{,\alpha}\, x_{i,\beta} + \frac{1}{4\pi} H_{1n}\, \bar{\alpha}_i - \frac{1}{4\pi} H_{1k}\, \bar{\alpha}_k n_i \qquad (4.1.3)$$

Using the compatibility conditions for the jump in vorticity, we obtain

$$[\omega] = \epsilon_{ijk}\, [u_{k,j}]$$

$$= \epsilon_{ijk}\, (\bar{\lambda}_k\, n_j + g^{\alpha\beta}\lambda_{k,\alpha}\, x_{j,\beta}) \qquad (4.1.4)$$

The jump conditions as given in Equations 3.2a and 3.2b imply that,

$\lambda_i = \lambda\, n_i$.

Therefore Equation 4.1.4 becomes,

$$[\omega_i] = \epsilon_{ijk}\,\bar{\lambda}_k n_j + \epsilon_{ijk}\, g^{\alpha\beta}\, x_{j,\beta}\lambda_{,\alpha} n_k$$

$$-\lambda\epsilon_{ijk}\, g^{\alpha\beta}\, x_{j,\beta}\, g^{\gamma\delta}\, b_{\alpha\gamma}\, x_{k,\delta} \qquad (4.1.5)$$

where we have used,

$$n_{k,\alpha} = -\, g^{\gamma\beta}\, b_{\alpha\gamma}\, x_{k,\beta}$$

But,

$$\lambda\epsilon_{ijk}\, g^{\alpha\beta} g^{\gamma\delta}\, b_{\alpha\gamma}\, x_{j,\beta}\, x_{k,\delta} = 0$$

Therefore, this is further reduced to,

$$[\omega_i] = \epsilon_{ijk}\,\bar{\lambda}_k n_j + \epsilon_{ijk}\, g^{\alpha\beta}\, x_{j,\beta}\, \lambda_{,\alpha}\, n_k \qquad (4.1.6)$$

Now multiply this equation by $(\rho_1 + \xi)(A_n - G)$ and substitute for $\bar{\lambda}_k$ from Equation 4.1.3 to obtain,

$$(\rho_1 + \xi)(A_n - G)[\omega_i] = (\rho_1 + \xi)(A_n - G)\epsilon_{ijk} g^{\alpha\beta}\, x_{j,\beta}\, \lambda_{,\alpha}\, n_k$$

$$+ \lambda(\rho_1 + \xi)\epsilon_{ijk}\, n_j g^{\alpha\beta}\, G_{,\alpha} x_{k,\beta}$$

$$+ \lambda(\rho_1 + \xi)\epsilon_{ijk}\, n_j\, A_\beta\, g^{\alpha\beta}\, g^{\gamma\delta} b_{\alpha\gamma}\, x_{k,\delta}$$

$$- \epsilon_{ijk} n_j g^{\alpha\beta} \beta_{,\alpha}\quad x_{k,\beta}$$

$$+ \frac{1}{4\pi}\; H_{1n}\; \epsilon_{ijk}\; n_j\;\; \bar{\alpha}_k \qquad (4.1.7)$$

where we have used the relations,

$$n_{k,\alpha} = -\, g^{\gamma\beta}\, b_{\alpha\gamma}\, x_{k,\beta}$$

$$\frac{\delta n_k}{\delta t} = -\, g^{\alpha\beta}\, G_{,\alpha}\, x_{k,\beta}$$

From Equation 3.2b, we can write

$$\rho_1 \, (u_{1n} - G) \, \lambda = - \beta \tag{4.1.8}$$

Now differentiate this with respect to tangential coordinate u^{α} to obtain,

$$\beta_{,\alpha} = \lambda \rho_1 \, G_{,\alpha} - \rho_1 \, (u_{1n} - G) \, \lambda_{,\alpha} - \rho_1 \, \lambda u_{,i} \, n_{i,\alpha}$$

$$= \lambda \rho_1 \, G_{,\alpha} + \rho_1 \, \lambda u_{1\beta} g^{\alpha\beta} b_{\alpha\gamma} - \rho_1 \, (u_{1n} - G) \, \lambda_{,\alpha} \tag{4.1.9}$$

From Equation 3.2a, we obtain

$$\rho_2 \lambda = - \, (u_{1n} - G) \xi$$

and since

$$\rho_1 + \xi = \rho_2$$

we obtain

$$(\rho_1 + \xi)(A_n - G) = \rho_2 \, (\lambda + u_{1n} - G)$$

$$= \rho_1 \, (u_{1n} - G) \tag{4.1.10}$$

Using Equations 4.1.9 and 4.1.10 in Equation 4.1.7 we obtain,

$$\rho_1 \, (u_{1n} - G)[\omega_i] = \lambda \xi \, \epsilon_{ijk} \, n_j \, g^{\alpha\beta} \, G_{,\alpha} \, x_{k,\beta}$$

$$- \lambda \rho_1 \, u_{,\delta} \, \epsilon_{ijk} \, n_j \, g^{\alpha\beta} \, x_{k,\beta} \, g^{\gamma\delta} b_{\alpha\gamma}$$

$$+ \lambda \, (\rho_1 + \xi) A_\beta \, \epsilon_{ijk} \, n_j \, g^{\alpha\beta} \, g^{\gamma\delta} \, b_{\alpha\gamma} \, x_{k,\delta}$$

$$+ \frac{\sigma}{16\pi^2} \, H_{1n} \, \epsilon_{ijk} \, n_j \, H_{1k} \lambda_n \tag{4.1.11}$$

where we have used Equation 3.19 for $\bar{\alpha}_k$.

Therefore we can write,

$$[\omega_i] = \epsilon_{ijk} \quad - \frac{\delta^2}{1+\delta} \, n_j \, g^{\alpha\beta} \, G_{,\alpha} \, x_{k,\beta}$$

$$+ \frac{\delta}{1+\delta} u_1^\alpha n_j x_{k,\beta} b_\alpha^\beta - \delta A^\alpha n_j x_{k,\beta} b_\alpha^\beta$$

$$- \frac{\sigma\delta}{\rho_1(1+\delta)} H_{1n} n_j H_{1k}) \tag{4.1.12}$$

where

$$\lambda = - \frac{(u_{1n} - G)\delta}{1+\delta}$$

has been used, and

$$\delta = \frac{\rho_2 - \rho_1}{\rho_1}$$

is the shock strength.

Using the compatibility conditions for the jump in current, we obtain

$$[J_i] = \frac{1}{4\pi} \epsilon_{ijk} [H_{k,j}]$$

$$= \frac{1}{4\pi} \epsilon_{ijk} \bar{\alpha}_k n_j \tag{4.1.13}$$

Now using Equation 3.19 and the fact that $\lambda_k = \lambda_{nk}$, this becomes

$$[J_i] = \frac{\sigma}{16\pi^2} \epsilon_{ijk} H_{1k} \lambda_{nj} \tag{4.1.14}$$

Substituting for λ in terms of the shock strength, δ, we finally obtain

$$[J_i] = - \frac{\sigma(u_{1n} - G)\delta}{16\pi^2 (1+\delta)} \epsilon_{ijk} H_{1k} n_j \tag{4.1.15}$$

Therefore we have been able to find the jumps in vorticity and current across the shock wave in terms of the shock strength δ and the quantities ahead of the shock. We also see from the above, that the jumps in vorticity and current can be obtained in the case of finite electrical conductivity by use of only the equation of motion and the shock relations. Thus the result is purely a kinematical one.

B. Infinite Electrical Conductivity

For completeness in demonstrating the use of the compatibility relations for determining the jumps in the vorticity and velocity across a shock wave, we now consider the case of infinite electrical conductivity.

The equation of motion and the magnetic field equation for this case are

$$\rho \frac{du_i}{dt} + p_{,i} + \frac{1}{4\pi} H_k H_{k,i} - \frac{1}{4\pi} H_{i,j} H_j = 0 \qquad (4.2.1a)$$

$$\frac{\partial H_i}{\partial t} - u_{i,j} H_j + H_{i,j} u_j + H_i u_{k,k} = 0 \qquad (4.2.b)$$

We assume there are jumps in the density, pressure, velocity, and magnetic field across the shock wave. The notation to be used is then

$$[u_i] = \lambda_i \qquad [u_{i,j}]n_j = \bar{\lambda}_i$$

$$[\rho] = \xi \qquad (\rho_{,i}]n_i = \bar{\xi}$$

$$[p] = \beta \qquad [p_{,i}]n_i = \bar{\beta}$$

$$[H_i] = \alpha_i \qquad [H_{i,j}]n_j = \bar{\alpha}_i \qquad (4.2.2)$$

With this notation, the compatibility conditions become,

$$[u_{i,j}] = \bar{\lambda}_i n_j + g^{\alpha\beta} \lambda_{i,\alpha} x_{j,\beta}$$

$$[\rho_{,i}] = \bar{\xi}\, n_i + g^{\alpha\beta}\, \xi_{,\alpha} x_{i,\beta}$$

$$[p_{,i}] = \bar{\beta} n_i + g^{\alpha\beta} \beta_{,\alpha}\ x_{i,\beta}$$

$$[H_{i,j}] = \bar{\alpha}_i n_j + g^{\alpha\beta}\, \alpha_{i,\alpha} x_{j,\beta}$$

$$[\frac{\partial u_i}{\partial t}] = - \bar{\lambda}_i\, G + \frac{\delta \lambda_i}{\delta t}$$

$$[\frac{\partial \rho}{\partial t}] = - \bar{\xi}\ G + \frac{\delta \xi}{\delta t}$$

$$[\frac{\partial p}{\partial t}] = - \bar{\beta}\ G + \frac{\delta \beta}{\delta t}$$

$$[\frac{\partial H_i}{\partial t}] = - \bar{\alpha}_i\ G + \frac{\delta \alpha_i}{\delta t} \qquad (4.2.3)$$

Applying these compatibility conditions to Equations 4.2.1a and 4.2.1b, we obtain

$$(\rho_1 + \xi)(-\bar{\lambda}_i G + \frac{\delta \lambda_i}{\delta t}) + (\rho_1 + \xi)(u_{,j} + \lambda_j)(\bar{\lambda}_i n_j + g^{\alpha\beta}\ \lambda_{i,\alpha} x_{j,\beta})$$

$$+ \bar{\beta}\ n_i + g^{\alpha\beta}\beta_{,\alpha}\ x_{i,\beta} + \frac{1}{4\pi}\ (H_{1k} + \alpha_k)(\bar{\alpha}_k n_i + g^{\alpha\beta}\alpha_{k,\alpha}\ x_{i,\beta})$$

$$- \frac{1}{4\pi}\ (H_{,j} + \alpha_j)(\bar{\alpha}_i n_j + g^{\alpha\beta}\alpha_{i,\alpha}\ x_{j,\beta}) = 0 \qquad (4.2.4a)$$

$$-\bar{\alpha}_i\ G + \frac{\delta \alpha_i}{\delta t} - (H_{,j} + \alpha_j)(\bar{\lambda}_i n_j + g^{\alpha\beta}\ \lambda_{i,\alpha}\ x_{j,\beta})$$

$$+ (u_{,j} + \lambda_j)(\bar{\alpha}_i n_j + g^{\alpha\beta}\ \alpha_{i,\beta}\ x_{j,\beta})$$

$$+ (H_{,i} + \alpha_i)(\lambda_k n_k + g^{\alpha\beta}\lambda_{k,\alpha}\ x_{k,\beta}) = 0 \qquad (4.24b)$$

We can now write these as,

$$(\rho_1 + \xi)(u_{1n} + \lambda_n - G)\ \bar{\lambda}_i + \frac{1}{4\pi}\ (H_{1k} + \alpha_k)\bar{\alpha}_k\ n_i$$

$$- \frac{1}{4\pi}\ (H_{1n} + \alpha_n)\ \bar{\alpha}_i + \bar{\beta} n_i = g_i \qquad (4.2.5a)$$

$$(u_{1n} + \lambda_n - G)\ \bar{\alpha}_i + (H_{,i} + \alpha_i)\bar{\lambda}_n - (H_{1n} + \alpha_n)\ \bar{\lambda}_n = f_i \qquad (4.2.5b)$$

where we have used the notation that,

$$g_i = - (\rho_1 + \xi)\frac{\delta \lambda_i}{\delta t} - (\rho_1 + \xi)(u_1 + \lambda_\beta)g^{\alpha\beta}\ \lambda_{i,\alpha}$$

$$- g^{\alpha\beta}\beta_{,\alpha}\, x_{i,\beta} - \frac{1}{4\pi}\,(H_{1k} + \alpha_k)g^{\alpha\beta}\,\alpha_{k,\alpha}\, x_{i,\beta}$$

$$+ \frac{1}{4\pi}\,(H_{1\beta} + \alpha_\beta)\; g^{\alpha\beta}\alpha_{i,\alpha} \tag{4.2.6a}$$

$$f_i = - \frac{\delta\alpha_i}{\delta t} + (H_{1\beta} + \alpha_\beta)\; g^{\alpha\beta}\quad{}_{i,\alpha} - (u_{1\beta} + \quad{}_\beta)g^{\alpha\beta}\,\alpha_{i,\alpha}$$

$$- (H_{1i} + \alpha_i)g^{\alpha\beta}\,\lambda_{k,\alpha}x_{k,\beta} \tag{4.2.6b}$$

Resolving these into the normal and tangential components, we obtain

$$(u_{1n} + \lambda_n - G)\quad{}_n = f_n \tag{4.2.7a}$$

$$(u_{1n} + \lambda_n - G)\bar{\alpha}_\beta + (H_{1\beta} + \alpha_\beta)\bar{\lambda}_n - (H_{1n} + \alpha_n)\bar{\lambda}_\beta = f_\beta \tag{4.2.7b}$$

$$(\rho_1 + \xi)(u_{1n} + \lambda_n - G)\bar{\lambda}_n + \frac{1}{4\pi}\,(H_{1\gamma} + \alpha_\gamma)\bar{\alpha}^\gamma + \bar{\beta} = g_n \tag{4.2.7c}$$

$$(\rho_1 + \Gamma)(u_{1n} + \lambda_n - G)\bar{\lambda}_\beta - \frac{1}{4\pi}\,(H_{1n} + \alpha_n)\bar{\alpha}_\beta = g_\beta \tag{4.2.7d}$$

First eliminating $\bar{\alpha}_\beta$ from Equations 4.2.7b and 4.2.7d, and then eliminating $\bar{\lambda}_\beta$ we obtain,

$$((\rho_1 + \xi)(u_{1n} + \lambda_n - G)^2 - \frac{1}{4\pi}\,(H_{1n} + \alpha_n)^2)\;\bar{\lambda}_\beta + \frac{1}{4\pi}(H_{1n}+\alpha_n)(H_{1\beta}+\alpha_\beta)\;\bar{\lambda}_n$$

$$= (u_{1n} + \lambda_n - G)g_\beta + \frac{1}{4\pi}\,(H_{1n} + \alpha_n)\; f_\beta \tag{4.2.8a}$$

$$((\rho_1 + \xi)(u_{1n} + \lambda_n - G)^2 - \frac{1}{4\pi}\,(H_{1n} + \alpha_n)^2)\bar{\alpha}_\beta + (\rho_1+\xi)(u_{1n}+\lambda_n-G)(H_{1\beta}+\alpha_\beta)\bar{\lambda}_n$$

$$= (H_{1n} + \alpha_n)\; g_\beta + (\rho_1 + \xi)(u_{1n} + \lambda_n - G)f_\beta \tag{4.2.8b}$$

These can now be written in the form

$$L\,\bar{\lambda}_\beta + M_\beta\,\bar{\lambda}_n = h_\beta \tag{4.2.9a}$$

$$L\;\bar{\alpha}_\beta + N_\beta\,\bar{\lambda}_n = e_\beta \tag{4.2.9b}$$

where we have used the notation,

$$L = (\rho_1 + \xi)(u_{1n} + \lambda_n - G)^2 - \frac{1}{4\pi} (H_{1n} + \alpha_n)^2 \tag{4.2.10a}$$

$$M_\beta = (H_{1n} + \alpha_n)(H_{1\beta} + \alpha_\beta) \tag{4.2.10b}$$

$$N_\beta = (\rho_1 + \xi)(u_{1n} + \lambda_n - G)(H_{1\beta} + \alpha_\beta) \tag{4.2.10c}$$

$$h_\beta = (u_{1n} + \lambda_n - G)g_\beta + \frac{1}{4\pi} (H_{1n} + \alpha_n)f_\beta \tag{4.2.10d}$$

$$e_\beta = (H_{1n} + \alpha_n)g_\beta + (\rho_1 + \xi)(u_{1n} + \lambda_n - G)f_\beta \tag{4.2.10e}$$

We now substitute for $\bar{\alpha}_\beta$ from Equation 4.2.9b into Equation 4.2.7c to obtain,

$$_n\left((\rho_1 + \xi)(u_{1n} + \lambda_n - G) - \frac{N_\beta}{4\pi L} (H_1^\gamma + \alpha^\gamma)\right) = g_n - \frac{e_\beta}{4\pi L} (H_1^\gamma + \alpha^\gamma) - \bar{\beta} \tag{4.2.11}$$

Now let

$$p = (\rho_1 + \xi)(u_{1n} + \lambda_n - G) - \frac{N_\beta}{4\pi L} (H_1^\gamma + \alpha^\gamma) \tag{4.2.12a}$$

$$Q = g_n - \frac{e_\beta}{4\pi L} (H_1^\gamma + \alpha^\gamma) \tag{4.2.12b}$$

and this equation becomes,

$$\rho\, \bar{\lambda}_n = Q - \bar{\beta} \tag{4.2.13}$$

Therefore we have found $\bar{\lambda}_n$, $\bar{\lambda}_\beta$, $\bar{\alpha}_n$, $\bar{\alpha}_\beta$ in terms of known quantities and the unknown quantity $\bar{\beta}$. That is,

$$\bar{\lambda}_n = \frac{Q - \bar{\beta}}{p} \tag{4.2.14a}$$

$$\bar{\lambda}_\beta = \frac{M_\beta}{Lp} \bar{\beta} + \frac{h_\beta}{L} - \frac{M_\beta Q}{Lp} \tag{4.2.14b}$$

$$\bar{\alpha}_n = \frac{f_n}{(u_{1n} + \lambda_n - G)} \tag{4.2.14c}$$

$$\bar{\alpha}_\beta = \frac{N_\beta}{Lp} \bar{\beta} + \frac{e_\beta}{L} - \frac{N_\beta Q}{Lp} \tag{4.2.14d}$$

Now using the compatibility conditions in the relations for the jumps in the vorticity and current, we obtain

$$[\omega_i] = \epsilon_{ijk}\,[u_{k,j}]$$

$$= \epsilon_{ijk}\,(\bar{\lambda}_k n_j + g^{\alpha\beta}\,\lambda_{k,\alpha} x_{j,\beta})$$

$$= \epsilon_{ijk}\,n_j\,(\bar{\lambda}_n\,n_k + \bar{\lambda}^{\alpha}\,x_{k,\alpha}) + \epsilon_{ijk}\,g^{\alpha\beta}\,\lambda_{k,\alpha} x_{j,\beta}$$

$$= \epsilon_{ijk}\,\bar{\lambda}^{\beta}\,n_j\,x_{k,\beta} + \epsilon_{ijk}\,g^{\alpha\beta}\,\lambda_{k,\alpha} x_{j,\beta} \tag{4.2.15a}$$

$$[J_i] = \frac{1}{4\pi}\,\epsilon_{ijk}\,[H_{k,j}]$$

$$= \frac{1}{4\pi}\,\epsilon_{ijk}\,(\bar{\alpha}_k\,n_j + g^{\alpha\beta}\,\alpha_{k,\alpha}\,x_{j,\beta})$$

$$= \frac{1}{4\pi}\,\epsilon_{ijk}\,\bar{\alpha}^{\beta}\,n_j\,x_{k,\beta} + \frac{1}{4\pi}\,\epsilon_{ijk}\,g^{\alpha\beta}\,\alpha_{k,\alpha}\,x_{j,\beta} \tag{4.2.15b}$$

Thus it is seen that the jumps in the vorticity and current depend only on the quantities $\bar{\lambda}_\beta$ and $\bar{\alpha}_\beta$ respectively, and do not depend on the normal components $\bar{\lambda}_n$ and $\bar{\alpha}_n$.

Since $\bar{\lambda}_\beta$ and $\bar{\alpha}_n$ depend on the unknown quantity $\bar{\beta}$, we must use another basic equation besides the equation of motion and the magnetic field equation to make the problem determinate. This means it is no longer of a purely kinematic nature.

Therefore we use the equation of energy as our other equation, that is,

$$\rho\,\frac{dh}{dt} = \frac{dp}{dt} \tag{4.2.16}$$

where h is enthalpy.

Using the equation of state,

$$h = \frac{\gamma p}{(\gamma-1)\rho}$$

Equation 4.2.16 becomes,

$$\frac{dp}{dt} + \gamma\, p\, u_{i,i} = 0 \qquad (4.2.17)$$

where we have used the equation of motion to simplify the equation.

Applying the compatibility equations to this equation we obtain,

$$-G\bar{\beta} + \frac{\delta\beta}{\delta t} + (u_{1i} + \lambda_i)(\bar{\beta}\, n_i + g^{\alpha\beta}\beta_{,\alpha}\, x_{i,\beta})$$

$$+ \gamma(p_1 + \beta)(\bar{\lambda}_i\, n_i + g^{\alpha\beta}\lambda_{i,\alpha}\, x_{i,\beta}) = 0 \qquad (4.2.18)$$

Letting,

$$d = -\frac{\delta\beta}{\delta t} - (u_{1\beta} + \lambda_\beta)\, g^{\alpha\beta}\, \beta_{,\alpha} - \gamma(p_1 + \beta)g^{\alpha\beta}\, \lambda_{i,\alpha}\, x_{i,\beta}, \qquad (4.2.19)$$

this can be written as,

$$(u_{1n} + \lambda_n - G)\bar{\beta} + \gamma\,(p_1 + \beta)\,\bar{\lambda}_n = d \qquad (4.2.20)$$

With this last equation, the quantities $\bar{\lambda}_\beta$ and $\bar{\alpha}_\beta$ can now be written in terms of known quantities. That is,

$$\bar{\lambda}_\beta = \frac{M_\beta}{Lp}\left(\frac{d}{R} - \frac{\gamma(p_1 + \beta)Q}{RP}\right) + \frac{h_\beta}{L} - \frac{M_\beta Q}{Lp} \qquad (4.2.21a)$$

$$\bar{\alpha}_\beta = \frac{N_\beta}{Lp}\left(\frac{d}{R} - \frac{\gamma(p_1 + \beta)Q}{RP}\right) + \frac{e_\beta}{L} - \frac{N_\beta Q}{Lp} \qquad (4.2.21b)$$

where

$$R = u_{1n} + \lambda_n - G - \frac{\gamma(p_1 + \beta)}{P} \qquad (4.2.22)$$

Thus the above results lead to the calculation of the jumps in the vorticity and the current across a shock wave. We see that in this case, the energy equation is needed.

The quantities like λ_i obtained from the shock relations can be substituted and simplified further, but since the derivatives of λ_i are involved in the final expressions, the simplifications do not lead very far.

V. CONCLUSION

In this work, we have studied magnetogasdynamical problems in the presence of finite conductivity. Although resistivity is a dissipative parameter, its nature is distinct from that of other similar parameters. It is interesting to make a comprehensive study of the relative importance of such parameters. The distinctive feature of conductivity in magnetogasdynamics is that it does admit of certain singular surfaces. The singular surfaces which we have defined are the ones which admit of discontinuities in certain variables. We have not only asserted the existence of such surfaces, but have also studied how the discontinuities grow.

In a real gas, all the dissipative parameters are called into play. But as we have shown, these act more predominately at different wavefronts. The question remains open as to which discontinuity is generated for an assigned set of initial conditions. There is the fast wave, the slow wave, the Alfven wave and the sound wave. These waves are all possible in a given medium. It is possible, although we do not feel it has been proven, that there do exist certain initial conditions due to which only one type of wave is generated.

As is well known, a shock wave is a mathematical discontinuity which is introduced for convenience. Such an ideal discontinuity surface cannot exist in a real gas. The non-linearity of the equations has the effect of continually changing the waveform, to cause it to become steeper and steeper. On the contrary, the dissipative mechanisms continually assert themselves to smooth out the profile. There exists a competition between these two mechanisms. With the increase in the gradients of the quantities, the effect of the dissipative mechanisms increases. Thus it is possible that there exists a limit beyond which the gradients cannot increase and therefore do not allow a shock to form. Even so, it is well known that the region in which these dissipative mechanisms ultimately dominate, is of very small thickness; so in actuality, we always have a shock layer instead of a shock surface. It is also well known, that in these regions the picture of a gas as a continuum ceases to remain valid (10). Since the thickness of the layer is small, the usual procedure is to substitute a shock for the layer and modify it by the use of the asymptotic theory. The latter studies are known as the studies in the structure of shock waves.

For all real gases, we thus find that the combination of the studies of discontinuity surfaces with an asymptotic study of thin layers provides a fairly good picture of the flow. There do remain other interesting questions. It would be more realistic to take the conductivity as variable, instead of constant as we have done. The theory of singular surfaces can then also be applied. Such studies are being made for stationary shocks (28, 29, 30). Since the temperature changes across a shock wave may be large, a variable conductivity would describe the actual situation more correctly. In fact, in these papers, the studies are made under the assumptions of zero conductivity ahead of the shock and infinite conductivity behind the shock. These are called ionization fronts by the authors.

The theory of singular surfaces enables us to study these problems for moving wave profiles. This study would be more general than the studies done thus far.

VI. APPENDIX

Throughout the work, we assume a number of results taken from differential geometry, the geometry of a moving surface, and the compatibility conditions which were obtained from the stated references. Thus we note below some of the results needed in the work.

A. Results from Differential Geometry for a Moving Surface

Consider a moving surface $\sum(t)$ represented by,

$$x_i = \phi_i\,(u^1, u^2, t) \qquad i = 1, 2, 3 \tag{A.1}$$

where the u^1 and u^2 are curvilinear coordinates of the surface and the x_i are the orthogonal cartesian coordinates referred to a fixed coordinate system. We assume throughout that the ϕ_i possess the necessary differentiability and continuity properties required. We shall use the summation convention from tensor analysis, where we must distinguish between covariant and contravariant indices when the indices represent the curvilinear coordinates u^α, while for x_i we do not make this distinction. Latin indices range over 1, 2, 3 and Greek indices will range over 1, 2.

From any book on differential geometry (32), we can write the coefficients of the first fundamental form for the surface as

$$g_{\alpha\beta} = \phi_{i,\alpha}\,\phi_{i,\beta} \tag{A.2}$$

where the comma denotes partial differentiation.

Since,

$$x_{i,\alpha} \qquad \alpha = 1, 2 \tag{A.3}$$

are vectors lying in the tangent plane, we define n_i to be a unit vector normal to the surface, and thus we have the relations,

$$n_i \, n_i = 1 \tag{A.4a}$$

$$x_{i,\alpha} \, n_i = 0 \tag{A.4b}$$

It can be seen that any vector can then be decomposed into its normal and tangential components, that is,

$$A_i = A_n \, n_i + A^{\alpha} \, x_{i,\alpha} \tag{A.5}$$

where A_i is any arbitrary vector.

Also from the theory of surfaces, we have the relations,

$$x_{i,\alpha\beta} = b_{\alpha\beta} n_i \tag{A.6a}$$

$$n_{i,\alpha} = - g^{\beta\gamma} \, b_{\beta\alpha} \, x_{i,\gamma} \tag{A.6b}$$

where the $x_{i,\alpha\beta}$ are the components of the second covariant derivative of the quantities x_i and the $b_{\alpha\beta}$ are the components of the second fundamental form of the surface.

In addition, we have,

$$2\Omega = g^{\alpha\beta} \, b_{\alpha\beta} \tag{A.7}$$

where Ω is the mean curvature of the surface and is given in terms of the curvature as,

$$\Omega = \tfrac{1}{2} \, (k_1 + k_2) \tag{A.8}$$

where k_1 and k_2 are curvatures in the u^1 and u^2 directions respectively.

Lane (32) shows that for parallel surfaces,

$$\Omega = \frac{\Omega_o - k_o n}{1 - 2\Omega_o n + k_o n^2} \tag{A.9}$$

where Ω_o is the mean curvature of the first surface, k_o is the Gaussian curvature of the first surface, where in general,

$$k = k_1 \, k_2 \tag{A.10}$$

and n is the distance along the normal to the first surface.

We have also made use of the fact that,

$$\frac{\delta n_i}{\delta t} = - g^{\alpha\beta} G_{,\alpha} x_{i,\beta} \tag{A.11}$$

which Thomas (11) has proved.

B. Compatibility Relations

A singular surface (t) is one across which there are jumps in a function Z or its derivatives. We consider the jumps or the discontinuities in a function Z and its derivatives across a moving surface. The function Z could be pressure, density, entropy, or the components of velocity and magnetic field.

We define the discontinuity in Z as

$$[Z] = Z_2 - Z_1 \tag{A.12}$$

where the subscripts 1 and 2 refer to the sides 1 and 2 of the surface. We assume the normal is pointing from side 2 to side 1, and that the side 1 is ahead of the moving surface. A similar notation is used for the derivatives of Z.

We use the notation,

$$[Z] = A \quad , \; [Z_{,i}]n_i = B \quad , \; [Z_{,ij}]\, n_i n_j = C \tag{A.13}$$

Then Thomas (11) has obtained several compatibility conditions. The geometrical conditions of compatibility of the first order are,

$$[Z_{,i}] = Bn_i + g^{\alpha\beta} A_{,\alpha} x_{i,\beta} \tag{A.14a}$$

The kinematical condition of compatibility of the first order is,

$$[\frac{\partial Z}{\partial t}] = - BG + \frac{\delta A}{\delta t} \tag{A.14b}$$

The geometrical conditions of compatibility of the second order are,

$$[Z_{,ij}] = Cn_i n_j + g^{\alpha\beta}(B_{,\alpha} + g^{\sigma\tau}\, b_{\alpha\sigma} A_{,\tau})(n_i\, x_{j,\beta} + n_j\, x_{i,\beta})$$

$$+ g^{\alpha\beta}\, g^{\sigma\tau}\, (A_{,\alpha\sigma} - Bb_{\alpha\sigma})\, x_{i,\beta}\, x_{j,\tau} \qquad \text{(A.14c)}$$

And finally we have the kinematical conditions of compatibility of the second order, which are

$$[\frac{\partial^2 Z}{\partial x_i \partial t}] = (-CG + \frac{\delta B}{\delta t} - g^{\alpha\beta}\, A_{,\alpha}\, x_{k,\beta}\, \frac{\delta n_k}{\delta t})n_i + g^{\alpha\beta}\, A^1_{,\alpha}\, x_{i,\beta} \qquad \text{(A.14d)}$$

$$[\frac{\partial^2 Z}{\partial t^2}] = CG^2 - G\, \frac{\delta B}{\delta t} + G\, g^{\alpha\beta}\, A_{,\alpha}\, x_{i,\beta}\, \frac{\delta n_i}{\delta t} + \frac{\delta A^1}{\delta t} \qquad \text{(A.14e)}$$

where

$$A^1 = [\frac{\partial Z}{\partial t}]$$

These are the compatibility conditions which we have used throughout the work where the A, B, and C are replaced by the appropriate quantities depending on which property of the fluid we are considering.

Another relation we have used frequently is,

$$[PQ] = [P][Q] + P_1\, [Q] + Q_1\, [P] \qquad \text{(A.15)}$$

which follows immediately from the definition of a jump across the surface.

C. Fundamental Conservation Equations and Shock Conditions for Gases

Equations in continuum mechanics can always be written as conservation laws. When written in this form, they can be rearranged and combined in an integral form instead of as the usual system of first order partial differential equations. The integral forms are more fundamental since they are valid even across surfaces of discontinuity. Using Reynolds transport theorem, one can always obtain the differential forms from the integral forms. In addition, the integral forms can be used to obtain the shock relations. The differential forms are not valid in this region, where the derivatives have no meaning.

In order to make the work self-contained, we indicate below the proof of the Reynolds transport theorem, and explain how the integral relations can be used to obtain the differential forms and the shock relations.

D. Reynolds Transport Theorem

This theorem states that

$$\frac{d}{dt}\int_{V(t)} f\, dV = \int_{V(t)} \left(\frac{df}{dt} + f\, u_{i,i}\right) dV \tag{A.16}$$

where f is an arbitrary differential function of x_i and t.

Proof:

Let

$$x_i = x_i\,(X_j,\, t) \qquad i,j = 1,\, 2,\, 3$$

where X_j is a coordinate system moving with the fluid, and x_i is a fixed system.

Let,

$$J = \frac{\partial(x_1,\, x_2,\, x_3)}{\partial(X_1,\, X_2,\, X_3)}$$

Then we have,

$$\frac{d}{dt}\int_{V(t)} f(x_i,\, t)\, dV = \frac{d}{dt}\int_{V_o} f(x_i\,(X_j,t),\, t)\, J\, dV_o$$

$$= \int_{V_o} \left(\frac{df}{dt} J + f\,\frac{dJ}{dt}\right) dV_o$$

$$= \int_{V_o} \left(\frac{df}{dt} + f\, u_{i,i}\right) J dV_o$$

$$= \int_V \left(\frac{df}{dt} + f\, u_{i,i}\right) dV$$

where we have used the fact that,

$$\frac{dJ}{dt} = J\, u_{i,i}$$

Thus we have obtained the desired result.

Expanding the material derivative and applying Gauss' Theorem, we can further write this as,

$$\frac{d}{dt} \int_V f\, dV = \int_V \frac{\partial f}{\partial t}\, dV + \int_S f\, G\, dS \qquad (A.17)$$

where G is the component of the velocity of the surface S along the outward normal to S.

We will now give the derivation of the differential form of the equation of continuity from integral form;

$$\frac{d}{dt} \int_V \rho\, dV = 0 \qquad (A.18)$$

Applying Equation A.16 to this equation, with $f = \rho$, we obtain

$$\int_V \left(\frac{d\rho}{dt} + \rho\, u_{i,i}\right) dV = 0 \qquad (A.19)$$

Since V is arbitrary, it follows that,

$$\frac{d\rho}{dt} + \rho\, u_{i,i} = 0 \qquad (A.20)$$

which is the familiar differential form of the equation of continuity. Similarly, the differential forms can be derived from the integral forms for the conservation of momentum and the conservation of energy.

Now to use the integral forms to obtain the shock relations, we first consider V to be a moving volume in the fluid which is divided by the moving surface Σ into two volumes, V_1 and V_2. Let S be the surface of V, and let S_1 and S_2 denote the portions of S which form part of the boundary of the volumes V_1 and V_2 respectively. The remaining part of the boundaries of V_1 and V_2 will be furnished by the surface Σ .

Now,

$$\frac{d}{dt}\int_V f\, dV = \frac{d}{dt}\int_{V_1} f\, dV + \frac{d}{dt}\int_{V_2} f\, dV \tag{A.21}$$

Applying Equation A.17 to this equation, we obtain,

$$\frac{d}{dt}\int_{V_1} f dV = \int_{V_1} \frac{\partial f}{\partial t}\, dV + \int_{S_1} f u_n\, dS + \int_{\Sigma} f_1\, G dS$$

$$\frac{d}{dt}\int_{V_2} f\, dV = \int_{V_2} \frac{\partial f}{\partial t}\, dV + \int_{S_2} f u_n\, dS - \int_{\Sigma} f_2 G\, dS$$

where u_n is the unit vector normal to the surface S everywhere. It is to be noted, that if the unit normal vector to the surface Σ is assumed positive when it points from side 2 to side 1 across Σ, the normal velocity of the surface, G, is positive when Σ is taken as part of the boundary of V_1 and negative when it is taken as part of the boundary of V_2.

Thus substituting these results into Equation A.21, we obtain,

$$\frac{d}{dt}\int_V f\, dV = \int_V \frac{\partial f}{\partial t}\, dV + \int_S f u_n dS + \int_{\Sigma} (f_1 - f_2) G\, dS \tag{A.22}$$

Using the equation of continuity, we now give an example of the derivation of the shock condition. Using Equation A.22 on the integral form of the equation of continuity as given by Equation A.18, we obtain

$$\int_V \frac{\partial \rho}{\partial t}\, dV + \int_{S_1} \rho u_n dS + \int_{S_2} \rho u_n\, dS + \int_{\Sigma} (\rho_1 - \rho_2) G\, dS = 0 \tag{A.23}$$

where ρ_1 and ρ_2 denote the values of the density on the sides 1 and 2 of Σ. Now let the volume V approach zero at a fixed time t, such that in passing to the limit, it becomes a finite part Σ_o of the surface Σ. Then the volume integrals are of a higher order than the surface integrals, and can be neglected. Also,

$$\int_{S_1} \rho u_n \, dS \rightarrow - \int_{\Sigma_o} \rho_1 u_{1n} \, dS$$

$$\int_{S_2} \rho u_n \, dS \rightarrow \int_{\Sigma_o} \rho_2 u_{2n} \, dS$$

where u_{1n} and u_{2n} are the normal components of the fluid velocity on sides 1 and 2 of the surface respectively.

Therefore we obtain,

$$\int_{\Sigma_o} [\rho_1 (u_{1n} - G) - \rho_2 (u_{2n} - G)] \, dS = 0$$

Since the Σ_o is arbitrary, we finally obtain,

$$\rho_1 (u_{1n} - G) = \rho_2 (u_{2n} - G) \tag{A.24}$$

which is the desired shock relation.

Similar results can be obtained from the other conservation laws.

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