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Strong turbulence theory with propagator method

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TABLE OF CONTENTS

NOMENCLATURE

Roman Letters

A	Averaging operator
B	Magnetic field
c	Velocity of light
D	Diffusion coefficient or diffusional term
D_{T_p}	Diffusion coefficient in spatial space
D_{T_v}	Diffusion coefficient in velocity space
dA	An element of area
$d\ell$	An element of length
E	Electrical field
F	An arbitrary function
f	Distribution function
j	Electrical current
k	Wave number
L	A linear operator
l_o	Mean scale size of turbulence
m	Mass of the particle
n	Density of the particle
n_o	Density of a uniform background of neutralizing charges
$\bar{n}$	Density of average particles
$\bar{n}_o$	Density of average particles at the center
P	Pressure
q	Charge of the particle
R_i	Force per unit mass on the particle

r A radial distance along the minor radius of the torus

r_o Minor radius of the torus

T Temperature of the particle

t Time

t_c Randomization time

t_i Momentum exchange time

t_o Initial time

U Time propagator

U_A Weinstock's propagator

U_G Time propagator in the presence of effective collision frequency

x Spatial coordinate in general

v Velocity of the particle

Greek Letters

α A parameter which is a function of time and velocity

ϵ A small parameter in the expansion of distribution function

$\phi(v_i)$ A perturbation to distribution function

$\phi(x_i, v_i, t)$ A general function in moment equations

ν Collision frequency

τ Time

ω Wave frequency or unit propagation vector for time

ψ Pressure tensor

Subscripts

coll Collisional term

i The coordinate in i-direction

j The coordinate in j-direction

k The coordinate in k-direction

o Initial condition

r Radial direction

θ Poloidal direction

φ Toroidal direction

z The coordinate in z-direction

Notations

< > Ensemble average

— Average

' Fluctuating

$\perp$ Perpendicular

$\parallel$ Parallel

^ Effective

I. INTRODUCTION

The thermonuclear reactors are basically grouped into two general categories as magnetic and inertial confinement devices. In magnetic confinement devices deuterium and/or tritium ions are heated to the temperature of thermonuclear reactions by different methods such as ohmic heating, magnetic pumping, ion cyclotron resonance heating, etc. In inertial confinement devices solid pellets of fuel are bombarded by a laser beam.

The magnetic confinement devices are divided into several classes such as mirror machines, θ-pinches, astrons and toroidal devices depending on the geometry and the strength of the magnetic field.

The problems of gas dynamics in these devices naturally depend on the type of the reactor. Some of these problems are well understood while some remain unsolved. For instance the dynamics of mirror machines is well understood while many problems of toroidal devices still need be solved.

Today the most sophisticated magnetic confinement devices are toroidal ones. The current experiments are mostly conducted on a special group of toroids so-called Tokamaks which utilize double magnetic field in the confinement.

In order to reach the physical conditions of real thermonuclear reactions it is necessary to meet the so-called Lawson criterion [1] which says that the confinement time multiplied by the number density of the particles should be more than 10^{14} ions/sec cm^{-3}. In toroidal devices the particle density is about the order of 10^{14} per cubic centimeter. Therefore the confinement time of at least one second should be reached. The closest value to the Lawson criterion is achieved in Tokamaks among the other thermonuclear devices, though the confinement time is still in the order of milliseconds.

In Tokamak devices the hot plasma is contained by the poloidal magnetic field B_θ of a current circulating in it in axial z- or φ-direction. In addition a very strong magnetic field B_φ is applied to suppress the main magnetohydrodynamic instabilities.

One of the main problems of Tokamaks is that the particle and the energy losses from the plasma column are much larger than predictable through classical loss mechanisms such as the usual drifts and collisional diffusion of particles across the magnetic field, without turbulence. In order to account for these losses "neoclassical" theory was developed and studied by several authors [2-7]. According to this theory the presence of local magnetic mirrors in toroidal devices which are coupled to the inevitable inhomogeneity of the toroidal magnetic field results in the appearance of a group of particles called trapped particles. The longitudinal velocities of trapped particles are smaller than the transverse ones. The trapped particles are reflected from the intensified field zones and oscillate along the line of force. In the absence of drift, the projection of the trajectory represents an arc of the circle. When the trajectories of all particles are considered, their region of oscillation looks like the shape of a banana and this region is often called "banana region."

The difference in the trajectories of trapped and transit particles gives rise to enhanced diffusion and thermal conductivity. This effect was used to predict high losses in Tokamaks but had not been very successful. Then some attempts [8, 9] were made to introduce the effect of turbulent mixing. These modifications in neoclassical theory resulted in improvements to a certain extent but still not sufficient to explain the losses.

An important point about Tokamaks is that the magnetic field applied to suppress the main hydrodynamic instabilities is extremely high. For instance in Tokamak T-3a device the magnetic field ranges from 17 to 38 kOe and the plasma temperature is from 0.5 to 2 keV. Therefore the physical conditions are very severe. In addition to this, the magnetic field configurations are not very homogeneous. Therefore the behavior of the particles will be very nonlinear and gives rise to strong turbulence. In fact, the Texas-Tokamak device is designed to achieve the thermonuclear reaction temperature through electron turbulent heating. But there has not yet been any fruitful attempt to examine the particle behavior under turbulent mixing in Tokamaks. This is probably due to lack of a good strong turbulence theory about plasmas. However, some theoretical research was carried out in the past on strong turbulence for Vlasov plasmas which are subject to only an electrical field. The strong turbulence theory for plasmas utilizes a new method called time propagator technique. Thus it became possible to include all nonlinearities of turbulence not only in coordinate space but also in velocity space.

The strong turbulence theory based on the time propagator method can be used to interpret the high particle and energy losses in Tokamaks. In this method all linearizations which were heavily used in the past are avoided. Hence the nonlinear phenomena which are basically responsible for strong turbulence can be described and included in the strong turbulence equations which involve the time propagator.

II. LITERATURE REVIEW

In order to evaluate the dynamics of gases at the temperature of thermonuclear reactions the Boltzmann transport equation of microscopic level is the starting equation. Macroscopic manipulations usually give a rough picture of the real phenomenon.

The definition of turbulence at microscopic level is the excitation of many collective degrees of freedom in plasma. Therefore it is essentially different from hydrodynamic turbulence, because in hydrodynamics, turbulence represents the mutually interacting eddies in the system. In plasmas there are also many complicated oscillations besides the eddies. Depending on the degree of the freedom excited one can discuss the character of the excitation and the resulting oscillation.

Another important difference between fluid and plasma turbulence is that in order to predict most of the properties of fluid turbulence it is not necessary to go to microscopic level. When the motion gets to be sufficiently microscopic it is generally believed that turbulent energy is converted into heat energy. But in plasma turbulence such a conversion cannot be assumed, because one can have turbulence in collisionless plasma which can show many complicated features of the velocity distribution function that are not contained in the hydrodynamic equations. In other words, an effective microscopic motion of complicated particle trajectories takes place while there is also macroscopic motion. Therefore one must consider the deviation due to the turbulence of particle orbits from their unperturbed values.

As a result of the oscillations the particle diffusion coefficient deduced from collision theory may not be valid. Experiments have shown that in the presence of a magnetic field the particle diffusion coefficient across the field lines is much larger than that calculated from the collision theory. When the oscillations are of considerable importance, the diffusion coefficient is usually calculated from well-known Bohm's empirical relation,

$$D \sim \frac{T}{B} \; \ldots\ldots \qquad (1)$$

where D is the diffusion coefficient for the ionized gas, T the temperature, and B the magnetic field strength.

Dupree [10] predicts two resulting effects of the random microscopic scattering; (i) the energy of each particle changes and this results in wave growth since the total energy in a closed system must be conserved, (ii) the basic wave growth mechanism and thus the dispersion properties of the plasma medium are affected by the randomization of the spatial position of the particle.

In order to solve the Boltzmann equation one needs know an explicit expression for the distribution function $\ll f(x, v, t) \gg$. In many cases where weak turbulence approximation is sufficient, f can be written as,

$$f = f^{(o)} + \phi(v_i) \ldots\ldots \qquad (2)$$

where $f^{(o)}$ is an isotropic distribution which need not be Maxwellian and $\phi(v_i)$ is a small perturbation which causes f to be anisotropic. Equation 2 can be written in a power series expansion so-called Chapman-Enskog expansion as,

$$f = f^{(o)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \epsilon^3 f^{(3)} \ldots\ldots \quad (3)$$

where ϵ is a small parameter.

It is known that such an expansion limits one to those solutions which are determined by number density, mass average velocity and temperature.

The solution of the Boltzmann transport equation by using Chapman-Enskog expansion leads to a theory of flows which is linear in gradients. This can be seen from the solution such as given by Robinson and Bernstein [11]. Therefore for the cases of strong turbulence such a method would not be valid because of the nonlinear behavior of plasma. The linearization methods of the distribution function will bring in incomplete and sometimes wrong conclusions. For instance, turbulent heating, shock waves and particle trapping cannot be adequately described with weak turbulence approximation.

Particle trapping is caused by any perturbation in the orbit of the particle. Assuming a test particle with a wave number k_z and velocity v_z moving through a wave (k, ω), the particle feels a Doppler shifted frequency due to its relative motion to the wave. This Doppler shifted frequency can be expressed as $\omega - k_z v_z$. When $\omega - k_z v_z \rightarrow 0$, trapping becomes possible. When $\omega - k_z v_z \gg 0$, the situation describes the waves with higher frequencies. They grow and become unstable because of the increased energy content which is the source of instabilities and thus of turbulence. When $\omega - k_z v_z \ll 0$, the particles suppress the growth of the waves.

Since the collisions between particles are increased due to turbulent fluctuations, the resistivity, viscosity, heat conductivity and similar macroscopic properties are determined by the turbulent diffusion of the particles. If a strong turbulence takes place in a collisionless medium then the scattering of particles is carried out by waves instead of particle-particle collisions. The random motion of particles can be considered as a diffusion process with the mean square displacement proportional to time. The most random motion of particles is due to Coulomb scatterings which are made up of many small statistically independent perturbations. Because of these perturbations the phase relation between the wave and the resonant particles is destroyed in the time interval in which the particle position is randomized due to Coulomb scatterings. This means a certain time must pass for interaction. The time elapsed causes a broadening of resonances in the frequency domain. This phenomenon can be considered as caused by random Doppler shifts.

The broadening increases the range of the velocities of particles to interact and absorb energy from the wave, then the considered wave can be further stabilized.

Bugnolo [12] discussed that turbulent mixing theories restricted to positional space may be sufficient for an unmagnetized plasma, but in the presence of strong magnetic fields such as in plasma confinement devices it is necessary to consider turbulent diffusion also in velocity space. He suggested two turbulent-diffusion coefficients which are positional and velocity space diffusion coefficients,

$$D_{T_p} = l_o < v^2 >^{\frac{1}{2}} \Delta v \ldots.. \quad (4)$$

where l_o is the mean scale size of the turbulence for diffusion in positional space and

$$D_{T_v} = V < v^2 >^{\frac{1}{2}} \Delta v \ldots.. \quad (5)$$

for diffusion in velocity space; V denoting the characteristic turbulent velocity of electrons.

In another article [13] he used these coefficients together with the ensemble averaged distribution function to obtain a first-order estimate of the effects of strong turbulence on the confinement time of a plasma.

Dupree [14] did a pioneering work in the field of strong turbulence. His theory utilizes a statistical ensemble of perturbed particle orbits instead of unperturbed orbits usually employed in plasma perturbation theories. He studied the propagation of a few "test" waves coexisting with a set of random phase "background" waves. The effect of background waves was incorporated by using the perturbed trajectories of particles moving in the random phase background waves. This is achieved by introducing the statistical ensemble of trajectories. Hence a perturbation series can be obtained for the ensemble averaged distribution function $< f >$.

Dupree used Vlasov equation and averaged it as,

$$(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x}) < f > + \frac{q}{m} \frac{\partial}{\partial v} < E_f > = 0 \ldots.. \quad (6)$$

Now defining a propagation operator $U(t, t_o)$ as the solution to Vlasov equation,

$$(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x})U(t, t_o) + \frac{q}{m} \frac{\partial}{\partial v} U(t, t_o) = 0 \ldots\ldots \quad (7)$$

$$U(t_o, t_o) = 1 \ldots\ldots \quad (8)$$

one can relate $<f>$ to $U(t,t_o)$. If the following initial condition is imposed,

$$f(t_o) = \delta(r - r_o)\delta(v - v_o) \ldots\ldots \quad (9)$$

then it can be written that,

$$<U(t, t_o)> = \int dx_o \int dv_o <f(t)> \ldots\ldots \quad (10)$$

Under these conditions $<f(t)>$ becomes a Green's function. The perturbation of Eq. 7 results in,

$$(\frac{\partial}{\partial t} + v \frac{\partial}{\partial r})<f> - \frac{\partial}{\partial v} D(v) \frac{\partial}{\partial v} <f> = 0 \ldots\ldots \quad (11)$$

where,

$$D(v) = \frac{q^2}{m^2} \int_0^t d\tau \sum_k E_k(t)E_{-k}(\tau)e^{ikr}<U(t, \tau)>e^{-ikr} \ldots\ldots \quad (12)$$

$D(v)$ denotes the diffusion coefficient.

As it is seen from Eq. 11 and 12, the solution of ensemble averaged distribution function is in terms of the time propagator $U(t, t_o)$.

These general treatments were applied by Dupree [15] to drift-wave turbulence and enhanced diffusion.

Flynn [16] reported that his experimental data for particle diffusion in turbulent electric fields were consistent with Dupree's prediction.

Dum and Dupree [17] used the propagator technique to show that the broadening of wave-particle resonances in turbulent electric field may determine the saturation level of high-frequency instabilities. Dum [18] used this technique also to compute the conductivity tensor of a turbulent plasma.

Manheimer [19] used Dupree's basic equations for nonlinear stabilization.

Orszag and Kraichnan [20] introduced integro-differential equations for strong turbulence based on Dupree's formulation. They argued for the superiority of their equations that correspond to the exact solution of dynamic equations. However since their equations are in closed form, they are difficult to manipulate and apply to physical problems.

Weinstock [21] developed a more general formulation of strong turbulence starting with Dupree's approach and introduced a new propagator $U_A(t, t_o)$ which also includes an averaging operator "A" and is a solution to the following equation,

$$\frac{\partial U_A(t, t_o)}{\partial t} = - (1 - A)\left(v \frac{\partial}{\partial r} + \frac{q}{m} E(r, t) \frac{\partial}{\partial v}\right) U_A(t, t_o) \ldots\ldots \quad (13)$$

Weinstock's equations reduce to Dupree's equations in the lowest order limit.

By means of the averaging operator introduced by Weinstock, the Vlasov equation can be written separately for both fluctuating and nonfluctuating particles in the case of weak coupling.

Birmingham and Bornatici [22] simplified Weinstock's equations and showed that there was a straightforward relationship among Weinstock's propagator, the Vlasov propagator and the ensemble averaged Vlasov propagator used by Dupree.

Benford and Thomson [23] argued that the diffusion coefficients derived by Dupree and Weinstock were not a strong function of velocity so that their results could be obtained by assuming the particles to diffuse in phase space in a Markovian way under the action of the fluctuating turbulent fields. A similar argument was made by Cook and Sanderson [24] about Dupree's diffusion theory that could be invalid for some general cases such as non-Markovian diffusion.

According to Peyraud and Coste [25] the validity of Dupree-Weinstock equations depend on the condition of $t_c \ll t_i$, where t_c is the randomization time of the field and t_i characterizes the momentum exchange time between field and particles.

Dupree-Weinstock theory is an important step in predicting the strong turbulence effects in a large variety of cases.

III. OBJECTIVES

The objectives of this research are to develop a strong turbulence theory for very general cases starting with Boltzmann equation and then apply it to find out the mechanism of particle and energy losses in Tokamak thermonuclear devices.

IV. GENERAL FORMULATION

The starting equation is the Boltzmann equation,

$$\left(\frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x} + R_i \frac{\partial}{\partial v}\right) f(x, v, t) = \left(\frac{\partial f}{\partial t}\right)_{coll.} \ldots\ldots \qquad (14)$$

where R_i is the force per unit mass on a particle in the gas. Neglecting the gravitational force, it can be written as Lorentz force,

$$R_i = \frac{q}{m_i} (E + v_i \times B) \ldots\ldots \qquad (15)$$

where q = electrical charge of the particle,

m_i = mass of the particle,

v_i = velocity of the particle,

E = electrical field,

B = magnetic field.

In Eq. 14 $f(x, v, t)$ refers to velocity distribution function and $(\partial f/\partial t)_{coll.}$ is collision term.

For self-consistency f is required to satisfy the Poisson equation.

$$\nabla \cdot E(x, t) = 4\pi q \int [f(x, v, t) - n_o \delta(v)] dv \ldots\ldots \qquad (16)$$

where n_o = density of a uniform background of neutralizing charge.

Now a linear operator "L" as defined below can be introduced to simplify mathematical manipulations,

$$L_{(t)} = v_i \frac{\partial}{\partial x} + R_i \frac{\partial}{\partial v} \ldots\ldots \qquad (17)$$

The Boltzmann equation can be written now as,

$$\left(\frac{\partial}{\partial t} + L_{(t)}\right) f = \left(\frac{\partial f}{\partial t}\right)_{coll.} \ldots\ldots \qquad (18)$$

Another useful operator is Weinstock's averaging operator "A" which gives the statistical ensemble average of everything it operates on. For an arbitrary function F it gives,

$$AF = \langle F \rangle = \bar{F} \ldots\ldots \quad (19)$$

The distribution function f can thus be divided into an average and a fluctuating part as,

$$Af = \langle f \rangle = \bar{f} \ldots\ldots \quad (20)$$

since

$$f = \bar{f} + f' \ldots\ldots \quad (21)$$

f' can be written as,

$$f' = f - Af = (1 - A)f \ldots\ldots \quad (22)$$

Now operating on Eq. 18 by A we obtain,

$$A\,\frac{\partial f}{\partial t} = \frac{\partial \bar{f}}{\partial t} = A\left(\frac{\partial f}{\partial t}\right)_{coll.} - ALf \ldots\ldots \quad (23)$$

By subtracting Eq. 23 from Eq. 18,

$$(1 - A)\,\frac{\partial f}{\partial t} = \frac{\partial f'}{\partial t} = (1 - A)\left(\frac{\partial f}{\partial t}\right)_{coll.} - (1 - A)Lf \ldots\ldots \quad (24)$$

Thus Eq. 18 is separated into average and fluctuating parts as Eq. 23 and Eq. 24.

Equation 24 can now be solved for $f'(t)$ in terms of $\bar{f}(t)$ and $f'(t_o)$, the latter being the initial condition for $f'(t)$. After this step Eq. 23 can be solved for $\bar{f}(t)$ in terms of $f'(t_o)$.

Let us rewrite Eq. 24 by first substituting Eq. 21,

$$\frac{\partial f'}{\partial t} = -\ (1 - A)L\bar{f} - (1 - A)Lf' + (1 - A)(\frac{\partial f}{\partial t})_{coll.}\ldots\ldots \quad (25)$$

$(1 - A)(\partial f/\partial t)_{coll.}$ is the collision term for the fluctuating particles. Therefore it can be expressed in terms of collision frequency as,

$$(1 - A)(\frac{\partial f}{\partial t})_{coll.} = \nu' f'\ldots\ldots \quad (26)$$

then Eq. 25 becomes,

$$\frac{\partial f'}{\partial t} = -\ (1 - A)L\bar{f} - (1 - A)Lf' + \nu' f'\ldots\ldots \quad (27)$$

Similarly it can be written that,

$$A(\frac{\partial f}{\partial t})_{coll.} = \overline{\nu f}\ldots\ldots \quad (28)$$

then Eq. 23 can be rewritten using Eq. 21 and 28,

$$\frac{\partial \bar{f}}{\partial t} = -\ AL\bar{f} - ALf' + \overline{\nu f}\ldots\ldots \quad (29)$$

The operator L can be separated into its average and fluctuating parts as,

$$L = AL + (1 - A)L = \bar{L} + L'\ldots\ldots \quad (30)$$

Using this relation Eq. 27 and Eq. 29 can be put into the following forms,

$$\frac{\partial f'}{\partial t} = -\ L'\bar{f} - \bar{L}f' - (1 - A)L'f' + \nu' f'\ldots\ldots \quad (31)$$

$$\frac{\partial \bar{f}}{\partial t} = -\ \overline{L\bar{f}} - AL'f' + \overline{\nu f}\ldots\ldots \quad (32)$$

These two equations were obtained in Appendix A as Eq. A-3 and Eq. A-5, respectively.

In order to find a solution for f' or $\bar{f}$ a time propagator $U(\tau, t)$ is employed. It is defined as the solution for the averaged differential operator,

$$\left(\frac{\partial}{\partial t} + \bar{L} - \bar{\nu}\right)U(\tau, t) = 0 \ldots\ldots \quad (33)$$

$$U(\tau, \tau) = U(t, t) = U(t_o, t_o) = 1 \ldots\ldots \quad (34)$$

Since $U(\tau, t)$ is a time propagator, the following general relation holds for an arbitrary function $F(t)$,

$$U(\tau, t)F(t) = F(\tau) \ldots\ldots \quad (35)$$

Now operating on Eq. 31 by $U(\tau, t)$,

$$U(\tau, t)\,\frac{\partial f'(t)}{\partial t} = -\,U(\tau, t)L'(t)\bar{f}(t) - U(\tau, t)\bar{L}(t)f'(t)$$

$$-\,(1 - A)U(\tau, t)L'(t)f'(t) + U(\tau, t)\nu' f'(t) \ldots\ldots \quad (36)$$

The integration of this equation is given in Appendix B as Eq. B-10 which is,

$$f'(\tau) = U(\tau, t_o)f'(t_o) - \int dt\,U(\tau, t)L'(t)\bar{f}(t)$$
$$-\int dt\,U(\tau, t)[(1 - A)L'f' - (\bar{\nu} + \nu')f'(t)] \ldots\ldots \quad (37)$$

f' appears on both sides of this equation. A solution through iteration can be achieved; by doing this the iteration series is assumed to converge. But there is not any criterion to require the convergence. However it is possible to define an "effective collision frequency" that sets an upper bound for convergence. Equation 33 can now be rewritten by adding to it the effective collision frequency "$\hat{\nu}$" such that;

$$\left(\frac{\partial}{\partial t} + \bar{L} - \bar{\nu}\right)U_G(\tau, t) = \hat{\nu}U_G(\tau, t)\ldots.. \tag{38}$$

where $U_G(\tau, t)$ is a new time propagator. Using the same procedure given in Appendix B, Eq. 37 takes the form of,

$$f'(\tau) = U_G(\tau, t)f'(t_o) - \int dtU_G(\tau, t)L'(t)\bar{f}(t)$$
$$- \int dtU_G(\tau, t)[(1 - A)L'f' - (\bar{\nu} + \nu')f'(t) - \hat{\nu}f'(t)]\ldots.. \tag{39}$$

An obvious way to cut off the iteration is to define,

$$\hat{\nu}f' = (1 - A)L'f' - (\bar{\nu} + \nu')f'(t)\ldots.. \tag{40}$$

Thus Eq. 38 becomes,

$$\left(\frac{\partial}{\partial t} + \bar{L} - (1 - A)L'\right)U_G(\tau, t) = - U_G(\tau, t)\nu'\ldots.. \tag{41}$$

Instead of defining an effective collision frequency, a special condition existing in Eq. 37 can be utilized. By requiring,

$$(\bar{\nu} + \nu')f'(t) = (1 - A)L'f'\ldots.. \tag{42}$$

the iteration can be still cut off.

Under this condition Eq. 37 becomes,

$$f'(\tau) = U(\tau, t)f'(t_o) - \int dtU(\tau, t)L'(t)\bar{f}(t)\ldots.. \tag{43}$$

Now substituting this equation into Eq. 32 it is obtained that,

$$\frac{\partial \bar{f}(\tau)}{\partial \tau} = -\bar{L}(\tau)\bar{f}(\tau) - AL'(\tau)U(\tau, t)f'(t_o)$$

$$+ \int dt AL'(\tau)U(\tau, t)L'(t)\bar{f}(t) + \overline{\nu f} \ldots.. \qquad (44)$$

The solution of this equation gives the distribution function for average particles.

In order to obtain a complete set of equations the electrical field and the distribution function must be related to each other with Poisson's relation as given by Eq. 16. For the average and the fluctuating distribution functions Eq. 16 can be written as,

$$\nabla \cdot \bar{E}(x, t) = 4\pi q \int [\bar{f}(x, v, t) - \bar{n}_o \delta(v)] d^3v \ldots.. \qquad (45)$$

$$\nabla \cdot E'(x, t) = 4\pi q \int f'(x, v, t) d^3v \ldots.. \qquad (46)$$

Equations 43, 44, 45, and 46 make a complete set for the four quantities f', $\bar{f}$, $\bar{E}$ and E'. These four equations are the basic strong turbulence equations.

Diffusional Term:

The linear operator L was defined in Eq. 17. It can be separated into its average and fluctuating parts as follows,

$$L = \bar{L} + L' \ldots.. \qquad (47)$$

$$\bar{L} = v_i \frac{\partial}{\partial x_i} + \bar{R}_i \frac{\partial}{\partial v_i} \ldots.. \qquad (48)$$

$$L' = R'_i \frac{\partial}{\partial v_i} \ldots.. \qquad (49)$$

The third term on the right side of Eq. 44 can be put into a different form by making use of Eq. 49,

$$\int dt A L'(\tau) U(\tau, t) L'(t) \bar{f}(t) = \int dt < R_i'(\tau) \frac{\partial}{\partial v(\tau)} U(\tau, t) R_i'(t) \frac{\partial}{\partial v(t)} > \bar{f}(t)$$

$$= \frac{\partial}{\partial v(\tau)} \left(\int dt < R_i'(\tau) U(\tau, t) R_i'(t) > \right) \frac{\partial}{\partial v(t)} \bar{f}(t)$$

$$= \frac{\partial}{\partial v(\tau)} D \frac{\partial \bar{f}}{\partial v(t)} \ldots.. \quad (50)$$

where $$D = \int dt < R_i'(\tau) U(\tau, t) R_i'(t) > \ldots.. \quad (51)$$

"D" is a kind of diffusional operator.

Equation 51 can be expressed in terms of the electrical and the magnetic fields by using Eq. 15. For the average and fluctuating parts Eq. 15 can be written as,

$$\bar{R} = \frac{q}{m} (\bar{E} + v \times \bar{B}) \ldots.. \quad (52)$$

$$R' = \frac{q}{m} (E' + v \times B') \ldots.. \quad (53)$$

then

$$D = \frac{q^2}{m^2} \int dt < (E'(x, \tau) + v_i \times B'(x, \tau)) U(\tau, t) (E'(x, t) + v_i \times B(x, t)) >$$

$$\ldots.. (54)$$

A simplification can be made in the strong turbulence equations. R_i can be assumed to be statistically independent of $f'(x, v, t_o)$ for all t. Then the second term on the right side of Eq. 44 is now expressed as,

$$A L'(\tau) U(\tau, t) f'(t_o) = < L'(\tau) U(\tau, t) >< f'(t_o) >$$

$$= 0 \ldots.. \quad (55)$$

since

$$< f'(t_o) > = 0. \quad (56)$$

Thus using Eqs. 28, 50 and 55, Eq. 44 can be written in the following form,

$$\frac{\partial\bar{f}(\tau)}{\partial\tau} + \bar{L}(\tau)\bar{f}(\tau) - \left(\frac{\partial}{\partial v(\tau)} D \frac{\partial}{\partial v(t)}\right)\bar{f}(t) = A\left(\frac{\partial f}{\partial\tau}\right)_{coll.} \quad \ldots (57)$$

This is the transport equation for average particles in strong turbulence. The moment equations can now be derived starting with this equation.

V. GENERAL MOMENT EQUATIONS

Once the velocity distribution function is known, the mean value of any molecular property such as density, pressure, mean velocity, and temperature of the gas can be found out. However these quantities can be directly derived from the Boltzmann equation.

In the development of the strong turbulence equations in the previous section, the Boltzmann equation was divided into average and fluctuating parts. It is known that the fluctuating particles constitute only a small fraction of the total particles in many cases at least in Tokamaks because of the near Maxwellian distribution. Therefore the solution of Eq. 57 gives almost a complete picture of particle behavior for Tokamaks.

Equation 57 describes the time variation of the average velocity distribution function $\bar{f}$. Now a general moment equation that describes the average time variation of any function $\phi(x_j, v_j, t)$ which is a function of the position, velocity and time of the particle can be derived. Since it is known from the definition that the average value of ϕ is obtained by multiplying it by the distribution function and integrating over velocity space, similarly the time variation of the average value of ϕ is obtained by multiplying Eq. 57 by ϕ and integrating over velocity space. So starting with Eq. 57 it can be written that,

$$\int \phi[\frac{\partial \bar{f}}{\partial \tau} + \bar{L}(\tau)\bar{f}(\tau) - (\frac{\partial}{\partial v(\tau)} D \frac{\partial}{\partial v(t)})\bar{f}]\, dv = \int \phi A (\frac{\partial f}{\partial \tau})_{coll.} dv \ldots\ldots \quad (58)$$

Substituting Eq. 48 for $\bar{L}$, it is obtained that,

$$\int \phi[\frac{\partial \bar{f}}{\partial \tau} + v_i \frac{\partial \bar{f}}{\partial x_i} + \bar{R}_i \frac{\partial \bar{f}}{\partial v_i} - (\frac{\partial}{\partial v_i(\tau)} D \frac{\partial}{\partial v_i(t)})\bar{f}(t)]dv = \int \phi A (\frac{\partial f}{\partial \tau})_{coll.} dv \tag{59}$$

The evaluation of the first, second and third terms are given in many textbooks [26] and are presented in Appendix C, Appendix D and Appendix E for completeness. The collision term could be integrated using Landau's form of collision term [27] but there is no need for this evaluation.

The fourth term of Eq. 59 is evaluated as follows,

$$\int \phi (\frac{\partial}{\partial v(\tau)} D \frac{\partial}{\partial v(t)})\bar{f}(t)dv = \int \phi \frac{\partial}{\partial v(\tau)} (D \frac{\partial \bar{f}(t)}{\partial v(t)}) dv \ldots\ldots \tag{60}$$

$$= \int \phi \frac{\partial D}{\partial v(\tau)} \frac{\partial \bar{f}(t)}{\partial v(t)} dv + \int \phi D \frac{\partial^2 \bar{f}(t)}{\partial v(\tau) \partial v(t)} dv \ldots\ldots \tag{61}$$

The first term of Eq. 61 can be written as,

$$\int \phi \frac{\partial D}{\partial v(\tau)} \frac{\partial \bar{f}(t)}{\partial v(t)} dv = \int \phi \frac{\partial}{\partial v(t)} (\frac{\partial D}{\partial v(\tau)} \cdot \bar{f}) dv - \int \phi \bar{f} \frac{\partial^2 D}{\partial v(\tau) \partial v(t)} dv \ldots\ldots \tag{62}$$

The first term of this equation also can be written as,

$$\int \phi \frac{\partial}{\partial v(t)} (\frac{\partial D}{\partial v(\tau)} \bar{f}) dv = \int \frac{\partial}{\partial v(t)} (\phi \frac{\partial D}{\partial v(\tau)} \cdot \bar{f}) dv - \int \frac{\partial D}{\partial v(\tau)} \cdot \bar{f} \cdot \frac{\partial \phi}{\partial v(t)} dv \ldots\ldots \tag{63}$$

Now Eq. 54 can be written in a more explicit form,

$$D = \frac{q^2}{m^2} \int dt \langle E'(x, \tau)U(\tau, t)E'(x, t) + E'(x, \tau)U(\tau, t)(v \times B'(x, t)) + (v_i \times B'(x, \tau))U(\tau, t)E'(x, t) + (v_i \times B'(x, \tau))U(\tau, t)(v_i \times B'(x, t)) \rangle \ldots\ldots \tag{64}$$

As it is seen from this equation that the derivative of D with respect to v gives a linear equation in velocity which then appears only in the last term. Therefore this term does not create any difficulty when taking the integration in Eq. 63. The first term on the right of Eq. 63 can now be written as,

$$\int \frac{\partial}{\partial v(\tau)} \left(\phi \frac{\partial D}{\partial v(t)} \bar{f}\right) dv = \phi \frac{\partial D}{\partial v(t)} \bar{f}\Big]_{-\infty}^{+\infty} = 0 \ldots\ldots \qquad (65)$$

Since $\bar{f}$ is assumed to go to zero at infinity, the integration naturally yields to zero in Eq. 65.

Equation 63 now becomes,

$$\int \phi \frac{\partial}{\partial v(t)} \left(\frac{\partial D}{\partial v(\tau)} \bar{f}\right) dv = -\int \frac{\partial D}{\partial v(\tau)} \cdot \frac{\partial \phi}{\partial v(t)} \cdot \bar{f}\, dv \ldots\ldots \qquad (66)$$

Substitution of this equation into Eq. 62 gives,

$$\int \phi \frac{\partial D}{\partial v(\tau)} \frac{\partial \bar{f}}{\partial v(t)}\, dv = -\int \frac{\partial D}{\partial v(\tau)} \cdot \frac{\partial \phi}{\partial v(t)}\, \bar{f}\, dv - \int \phi \bar{f} \frac{\partial^2 D}{\partial v(\tau) \partial v(t)}\, dv \qquad (67)$$

Utilizing Eq. C-2 in Appendix C,

$$\int \phi \frac{\partial D}{\partial v(\tau)} \frac{\partial \bar{f}}{\partial v(t)}\, dv = -\bar{n} \left\langle \frac{\partial D}{\partial v(\tau)} \cdot \frac{\partial \phi}{\partial v(t)} \right\rangle - \bar{n} \left\langle \phi \frac{\partial^2 D}{\partial v(\tau) \partial v(t)} \right\rangle \qquad (68)$$

Hence the first term on the right of Eq. 61 has been evaluated. Now the second term can be written as,

$$\int \phi D \frac{\partial^2 \bar{f}(t)}{\partial v(\tau) \partial v(t)}\, dv = \int \frac{\partial^2}{\partial v(\tau) \partial v(t)} (\phi D \bar{f}) dv - \int \bar{f} \frac{\partial^2}{\partial v(\tau) \partial v(t)} (\phi D) dv \qquad (69)$$

The first term on the right can be expressed as,

$$\int \frac{\partial^2}{\partial v(\tau)\partial v(t)} (\phi D\bar{f})dv = \frac{\partial}{\partial v(\tau)} \int \frac{\partial}{\partial v(t)} (\phi D\bar{f})dv \ldots\ldots \tag{70}$$

$$= \frac{\partial}{\partial v(\tau)} \left\{ [\phi D\bar{f}]_{-\infty}^{\infty} \right\} = 0 \ldots\ldots \tag{71}$$

This conclusion is derived from Eq. 65.

The second term of Eq. 69 is written as,

$$\int \bar{f} \frac{\partial^2}{\partial v(\tau)\partial v(t)} (\phi D)dv = \int \bar{f}D \frac{\partial^2 \phi}{\partial v(\tau)\partial v(t)} dv + \int \bar{f}\phi \frac{\partial^2 D}{\partial v(\tau)\partial v(t)} dv$$
$$+ \int \bar{f} \frac{\partial \phi}{\partial v(\tau)} \frac{\partial D}{\partial v(t)} dv + \int \bar{f} \frac{\partial D}{\partial v(\tau)} \frac{\partial \phi}{\partial v(t)} \ldots\ldots \tag{72}$$

By adding Eq. 71 and Eq. 72, Eq. 69 becomes,

$$\int \phi D \frac{\partial^2 \bar{f}}{\partial v(\tau)\partial v(t)} dv = -\int \bar{f}D \frac{\partial^2 \phi}{\partial v(\tau)\partial v(t)} dv - \int \bar{f}\phi \frac{\partial^2 D}{\partial v(\tau)\partial v(t)} dv$$
$$- \int \bar{f} \frac{\partial \phi}{\partial v(\tau)} \cdot \frac{\partial D}{\partial v(t)} dv - \int \bar{f} \frac{\partial D}{\partial v(\tau)} \frac{\partial \phi}{\partial v(t)} \ldots\ldots \tag{73}$$

$$= -\bar{n} < D \frac{\partial^2 \phi}{\partial v(\tau)\partial v(t)} > - \bar{n} < \phi \frac{\partial^2 D}{\partial v(\tau)\partial v(t)} > - \bar{n} < \frac{\partial \phi}{\partial v(\tau)} \frac{\partial D}{\partial v(t)} >$$
$$- \bar{n} < \frac{\partial D}{\partial v(\tau)} \frac{\partial \phi}{\partial v(t)} > \ldots\ldots \tag{74}$$

Equation 61 is obtained by adding Eq. 68 and Eq. 74.

$$\int \phi \left(\frac{\partial}{\partial v(\tau)} D \frac{\partial}{\partial v(t)}\right)\bar{f}dv = - 2\bar{n} < \frac{\partial D}{\partial v(\tau)} \frac{\partial \phi}{\partial v(t)} > - \bar{n} < \frac{\partial \phi}{\partial v(\tau)} \frac{\partial D}{\partial v(t)} >$$
$$- \bar{n} < D \frac{\partial^2 \phi}{\partial v(\tau)\partial v(t)} > - 2\bar{n} < \phi \frac{\partial^2 D}{\partial v(\tau)\partial v(t)} > \tag{75}$$

Now the substitution of Eq. C-3 (Appendix C), Eq. D-4 (Appendix D), Eq. E-11 (Appendix E), and Eq. 75 into Eq. 59 gives,

$$\frac{\partial}{\partial\tau}(\bar{n}<\phi>) - \bar{n}<\frac{\partial\phi}{\partial\tau}> + \frac{\partial}{\partial x_i}(\bar{n}<v_i\phi>)$$

$$- \bar{n}<v_i \frac{\partial\phi}{\partial x_i}> - \bar{n}<\bar{R}_i \frac{\partial\phi}{\partial v_i(\tau)}> + 2\bar{n}<\frac{\partial D}{\partial v(\tau)} \frac{\partial\phi}{\partial v(t)}>$$

$$+ \bar{n}<\frac{\partial\phi}{\partial v(\tau)} \frac{\partial D}{\partial v(t)}> + \bar{n}<D \frac{\partial^2\phi}{\partial v(\tau)\partial v(t)} + 2\bar{n}<\phi \frac{\partial^2 D}{\partial v(\tau)\partial v(t)}>$$

$$= \int \phi A\left(\frac{\partial f}{\partial\tau}\right)_{coll.} dv \ldots\ldots \quad (76)$$

The moment equations are obtained by letting ϕ take the values proportional to increasing power of velocity. $\phi = 1$ gives the continuity equation (or so-called zeroth moment), $\phi = mv_j(\tau)$ gives the momentum equation (i.e. first moment) and $\phi = \frac{1}{2} mv_j(\tau)v_j(\tau)$ gives the energy equation (i.e. second moment).

It is seen that since v_j, x_i and τ are independent variables, then $\partial\phi/\partial\tau$, and $\partial\phi/\partial x_i$ are both zero. $\partial\phi/\partial v(t)$ also is zero since ϕ is a function of $v(\tau)$ and not of $v(t)$. Therefore, Eq. 76 can now be written as,

$$\frac{\partial}{\partial\tau}(\bar{n}<\phi>) + \frac{\partial}{\partial x_i}(\bar{n}<v_i\phi>) - \bar{n}<\bar{R}_i \frac{\partial\phi}{\partial v_i(\tau)}> + \bar{n}<\frac{\partial\phi}{\partial v(\tau)} \frac{\partial D}{\partial v(t)}>$$

$$+ 2\bar{n}<\phi \frac{\partial^2 D}{\partial v(\tau)\partial v(t)}> = \int \phi A\left(\frac{\partial f}{\partial\tau}\right)_{coll.} dv \ldots\ldots \quad (77)$$

A. Continuity Equation

By letting $\phi = 1$ in Eq. 77, the continuity equation is obtained,

$$\frac{\partial\bar{n}}{\partial\tau} + \frac{\partial}{\partial x_i}(\overline{nv_i}) + 2\bar{n}<\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}> = \int A\left(\frac{\partial f}{\partial\tau}\right)_{coll.} dv \ldots\ldots \quad (78)$$

since $<v_i> = \bar{v}_i$.

The collision term represents the rate at which particles gained or lost due to collisions resulting in ionization, attachment and recombination. If these processes do not take place then the continuity equation can be written without the collision term,

$$\frac{\partial\bar{n}}{\partial\tau} + \frac{\partial}{\partial x_i}(\overline{nv_i}) + 2\bar{n}<\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}>= 0..... \tag{79}$$

B. Momentum Equation

The momentum equation is obtained by letting $\phi = mv_j(\tau)$ in Eq. 77.

$$\frac{\partial}{\partial\tau}(\bar{n}<mv_j>) + \frac{\partial}{\partial x_i}(\bar{n}<v_i mv_j>) - \bar{n}<\bar{R}_i \frac{\partial(mv_j(\tau))}{\partial v_i(\tau)}>$$
$$+ \bar{n}<\frac{\partial(mv_j(\tau))}{\partial v_i(\tau)} \cdot \frac{\partial D}{\partial v_i(t)}>+ 2\bar{n}<mv_j \frac{\partial^2 D}{\partial v_i(\tau)\partial v_i(t)}>$$
$$= \int mv_j A(\frac{\partial f(\tau)}{\partial\tau})_{coll.} dv..... \tag{80}$$

This equation can now be further simplified. First, the velocity can be separated into average and fluctuating parts as,

$$v = \bar{v} + v'..... \tag{81}$$

then,

$$<v_i v_j> =<(\bar{v}_i + v_i')(\bar{v}_j + v_j')>$$
$$=<\bar{v}_i\bar{v}_j>+<\bar{v}_i v_j'>+<v_i'\bar{v}_j>+<v_i'v_j'>$$
$$= \bar{v}_i\bar{v}_j + \bar{v}_i<v_j'>+<v_i'>\bar{v}_j +<v_i'v_j'>..... \tag{82}$$

since $<v'> = 0$, Eq. 82 can be written as,

$$<v_i v_j> = \bar{v}_i\bar{v}_j +<v_i'v_j'>..... \tag{83}$$

Another simplification is,

$$\frac{\partial(mv_j)}{\partial v_i} = m\delta_{ij} = m \ldots\ldots \qquad (84)$$

Now Eq. 81 is substituted into the fifth and the collisional terms of Eq. 80. Then Eq. 83 is substituted into the second term; and Eq. 84 into the third and fourth terms of Eq. 80.

$$\frac{\partial}{\partial\tau}(\bar{n}m\bar{v}_j) + \frac{\partial}{\partial x_i}(\bar{n}m\bar{v}_i\bar{v}_j) + \frac{\partial}{\partial x_i}(\bar{n}m<v_i'v_j'>) - m\bar{n}<\bar{R}_i>$$
$$+ m\bar{n}<\frac{\partial D}{\partial v_i(t)}> + 2\bar{n}<m\bar{v}_j \frac{\partial^2 D}{\partial v_i(\tau)\partial v_i(t)}> + 2\bar{n}<mv_j' \frac{\partial^2 D}{\partial v_i(\tau)\partial v_i(t)}>$$
$$= \int m\bar{v}_j A(\frac{\partial f(\tau)}{\partial\tau})_{coll.} dv + \int mv_j' A(\frac{\partial f(\tau)}{\partial\tau})_{coll.} dv \ldots\ldots \qquad (85)$$

The first and the second terms can be differentiated as,

$$\frac{\partial}{\partial\tau}(\bar{n}m\bar{v}_j) + \frac{\partial}{\partial x_i}(\bar{\bar{n}}m\bar{v}_i\bar{v}_j) = m\bar{v}_j\frac{\partial\bar{n}}{\partial\tau} + m\bar{n}\frac{\partial\bar{v}_j}{\partial\tau}$$
$$+ m\bar{v}_j\frac{\partial}{\partial x_i}(\overline{nv_i}) + m\bar{v}_i\frac{\partial}{\partial x_i}(\overline{nv_j}) \ldots\ldots \qquad (86)$$

After substituting Eq. 86 into Eq. 85 and rearranging,

$$m\bar{v}_j\,[\frac{\partial\bar{n}}{\partial\tau} + \frac{\partial}{\partial x_i}(\overline{nv_i}) + 2\bar{n}<\frac{\partial^2 D}{\partial v_i(\tau)\partial v_i(t)}> - \int A(\frac{\partial f(\tau)}{\partial\tau})_{coll.} dv]$$
$$+ m\bar{n}\frac{\partial\bar{v}_j}{\partial\tau} + m\bar{v}_i\frac{\partial}{\partial x_i}(\overline{nv_j}) + m\frac{\partial}{\partial x_i}(\bar{n}<v_j'v_i'>)$$
$$- m\bar{n}\bar{R}_i + m\bar{n}<\frac{\partial D}{\partial v_i(t)}> + 2m\bar{n}<v_j'\frac{\partial^2 D}{\partial v_i(\tau)\partial v_i(t)}>$$
$$= \int mv_j' A(\frac{\partial f}{\partial\tau})_{coll.} dv \ldots\ldots \qquad (87)$$

It is seen that the first four terms in the bracket are equal to the continuity equation i.e. Eq. 78. So the remaining terms are,

$$m\bar{n}\frac{\partial\bar{v}_j}{\partial\tau} + m\bar{v}_i\frac{\partial}{\partial x_i}(\overline{nv_j}) + m\frac{\partial}{\partial x_i}(\bar{n}<v_i'v_j'>) - m\bar{n}<\bar{R}_i>$$
$$+ m\bar{n}<\frac{\partial D}{\partial v_i(t)}> + 2m\bar{n}<v_j'\frac{\partial^2 D}{\partial v_i(\tau)\partial v_i(t)}>$$
$$= \int mv_j'A(\frac{\partial f}{\partial\tau})_{coll.}dv \ldots\ldots \quad (88)$$

From the definition the hydrostatic pressure can be written as,

$$P = \frac{1}{3}\,nm<v_i'v_i'> \ldots\ldots \quad (89)$$

then the pressure tensor ψ_{ij} is defined as,

$$\bar{\psi}_{ij} = \bar{n}m<v_i'v_j'> \ldots\ldots \quad (90)$$

The relation between P and ψ is,

$$P = \frac{1}{3}\,\psi_{ii} \ldots\ldots \quad (91)$$

Now substituting Eq. 90 into Eq. 88,

$$m\bar{n}\frac{\partial\bar{v}_j}{\partial\tau} + m\bar{v}_i\frac{\partial}{\partial x_i}(\overline{nv_j}) + \frac{\partial}{\partial x_i}<\bar{\psi}_{ij}> - m\bar{n}<\bar{R}_i>$$
$$+ m\bar{n}<\frac{\partial D}{\partial v_i(t)}> + 2m\bar{n}<v_j'\frac{\partial^2 D}{\partial v_i(\tau)\partial v_i(t)}>$$
$$= \int mv_j'A(\frac{\partial f}{\partial\tau})_{coll.}dv \ldots\ldots \quad (92)$$

Equation 90 shows that ψ has the dimension of an energy density and is in fact a measure of the thermal motions of the fluid. When all the particles have the same velocity ψ becomes zero since $v' = 0$.

If ψ is a diagonal unit tensor then the kinetic pressure is said to be scalar, or isotropic as given in Eq. 91.

C. Energy Equation

The energy equation is obtained by letting $\phi = \frac{1}{2} mv_j(\tau)v_j(\tau)$ in Eq. 77. But in order to obtain a general kinetic pressure expression different subscripts should be used as $\phi = \frac{1}{2} mv_j(\tau)v_k(\tau)$.

$$\frac{\partial}{\partial\tau} (\bar{n}<\frac{1}{2} mv_j v_k>) + \frac{\partial}{\partial x_i} (\bar{n}<v_i \frac{1}{2} mv_j v_k>) - \bar{n}<\bar{R}_i \frac{\partial(\frac{1}{2} mv_j(\tau)v_k(\tau))}{\partial v_i(\tau)}>$$
$$+ \bar{n}<\frac{\partial(\frac{1}{2} mv_j(\tau)v_k(\tau))}{\partial v(\tau)} \cdot \frac{\partial D}{\partial v(t)}>$$
$$+ 2\bar{n}<\frac{1}{2} mv_j v_k \frac{\partial^2 D}{\partial v(\tau)\partial v(t)}> = \int \frac{1}{2} mv_j v_k A(\frac{\partial f}{\partial\tau})_{coll.} dv \ldots\ldots \tag{93}$$

From Eq. 83, one can write,

$$<v_j v_k> = \bar{v}_j \bar{v}_k + <v_j' v_k'> \ldots\ldots \tag{94}$$

Using Eq. 81, $<v_i v_j v_k>$ can be written as,

$$<v_i v_j v_k> = <(\bar{v}_i + v_i')(\bar{v}_j + v_j')(\bar{v}_k + v_k')> \ldots\ldots \tag{95}$$

$$= \bar{v}_i \bar{v}_j \bar{v}_k + \bar{v}_i \bar{v}_j <v_k'> + \bar{v}_i \bar{v}_k <v_j'>$$
$$+ \bar{v}_i <v_j' v_k'> + <v_i'> \bar{v}_j \bar{v}_k + \bar{v}_j <v_i' v_k'>$$
$$+ <v_i' v_j'> \bar{v}_k + <v_i' v_j' v_k'> \ldots\ldots \tag{96}$$

The second, the third, and the fifth terms are zero since $<v'> = 0$. The remaining terms give,

$$<v_i v_j v_k> = \bar{v}_i \bar{v}_j \bar{v}_k + \bar{v}_i <v_j' v_k'> + \bar{v}_j <v_i' v_k'>$$
$$+ \bar{v}_k <v_i' v_j'> + <v_i' v_j' v_k'> \ldots\ldots \tag{97}$$

The derivative of the kinetic pressure with respect to velocity can be taken as,

$$\frac{\partial}{\partial v_i}\left(\frac{1}{2}mv_jv_k\right)=\frac{1}{2}mv_k\delta_{ij}+\frac{1}{2}mv_j\delta_{ik}$$

$$=\frac{1}{2}mv_k+\frac{1}{2}mv_j\ldots\ldots \tag{98}$$

Substituting Eq. 81 into this equation,

$$\frac{\partial}{\partial v_i}\left(\frac{1}{2}mv_jv_k\right)=\frac{1}{2}m\bar{v}_k+\frac{1}{2}mv'_k+\frac{1}{2}m\bar{v}_j+\frac{1}{2}mv'_j\ldots\ldots \tag{99}$$

The fifth term of Eq. 93 can be written in a more explicit form using Eq. 82 in the nonaveraged form,

$$\left<\frac{1}{2}mv_jv_k\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}\right>=\left<\frac{1}{2}m\bar{v}_j\bar{v}_k\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}\right>$$
$$+\left<\frac{1}{2}m\bar{v}_jv'_k\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}\right>+\left<\frac{1}{2}mv'_j\bar{v}_k\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}\right>$$
$$+\left<\frac{1}{2}mv'_jv'_k\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}\right>\ldots\ldots \tag{100}$$

Now substituting Eqs. 94, 97, 99 and 100 into Eq. 93, it is obtained that,

$$\frac{\partial}{\partial\tau}\left(\bar{n}\frac{1}{2}m\bar{v}_j\bar{v}_k\right)+\frac{\partial}{\partial\tau}\left(\bar{n}\left<\frac{1}{2}mv'_jv'_k\right>\right)+\frac{\partial}{\partial x_i}\left(\bar{n}\frac{1}{2}m\bar{v}_i\bar{v}_j\bar{v}_k\right)$$
$$+\frac{\partial}{\partial x_i}\left(\bar{n}\frac{1}{2}m\bar{v}_i\left<v'_jv'_k\right>\right)+\frac{\partial}{\partial x_i}\left(\bar{n}\frac{1}{2}m\bar{v}_j\left<v'_iv'_k\right>\right)$$
$$+\frac{\partial}{\partial x_i}\left(\bar{n}\frac{1}{2}m\bar{v}_k\left<v'_iv'_j\right>\right)+\frac{\partial}{\partial x_i}\left(\bar{n}\frac{1}{2}m\left<v'_iv'_jv'_k\right>\right)$$
$$-\bar{n}\left<\bar{R}_i\frac{1}{2}m\bar{v}_k\right>-\bar{n}\left<\bar{R}_i\frac{1}{2}mv'_k\right>-\bar{n}\left<\bar{R}_i\frac{1}{2}m\bar{v}_j\right>-\bar{n}\left<\bar{R}_i\frac{1}{2}mv'_j\right>$$
$$+\bar{n}\left<\frac{1}{2}m\bar{v}_k\frac{\partial D}{\partial v_i(t)}\right>+\bar{n}\left<\frac{1}{2}mv'_k\frac{\partial D}{\partial v_i(t)}\right>+\bar{n}\left<\frac{1}{2}m\bar{v}_j\frac{\partial D}{\partial v_i(t)}\right>$$
$$+\bar{n}\left<\frac{1}{2}mv'_j\frac{\partial D}{\partial v_i(t)}\right>+2\bar{n}\left<\frac{1}{2}m\bar{v}_j\bar{v}_k\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}\right>$$
$$+2\bar{n}\left<\frac{1}{2}m\bar{v}_jv'_k\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}\right>+2\bar{n}\left<\frac{1}{2}mv'_j\bar{v}_k\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}\right>$$

$$+ 2\bar{n}<\frac{1}{2}\, mv_j'v_k' \frac{\partial^2 D}{\partial v(\tau)\partial v(t)}>$$

$$= \int \frac{1}{2}\, m\bar{v}_j\bar{v}_k A(\frac{\partial f}{\partial \tau})_{coll.} dv + \int \frac{1}{2}\, m\bar{v}_j v_k' A(\frac{\partial f}{\partial \tau})_{coll.} dv$$

$$+ \int \frac{1}{2}\, mv_j'\bar{v}_k A(\frac{\partial f}{\partial \tau})_{coll.} dv + \int \frac{1}{2}\, mv_j'v_k' A(\frac{\partial f}{\partial \tau})_{coll.} dv \cdots\cdots \tag{101}$$

Now some simplifications can be carried out on this equation. The first, third, and sixth terms can be written as,

$$\frac{\partial}{\partial \tau} (\bar{n}\, \frac{1}{2}\, m\bar{v}_j\bar{v}_k) + \frac{\partial}{\partial x_i} (\bar{n}\, \frac{1}{2}\, m\bar{v}_i\bar{v}_j\bar{v}_k) + \frac{\partial}{\partial x_i} (\bar{n}\, \frac{1}{2}\, m\bar{v}_k <v_i'v_j'>)$$

$$= \frac{1}{2}\, m\bar{v}_j\bar{v}_k \frac{\partial \bar{n}}{\partial \tau} + \bar{n}\, \frac{1}{2}\, m\bar{v}_k \frac{\partial \bar{v}_j}{\partial \tau} + \bar{n}\, \frac{1}{2}\, m\bar{v}_j \frac{\partial \bar{v}_k}{\partial \tau}$$

$$+ \frac{1}{2}\, m\bar{v}_j\bar{v}_k \frac{\partial}{\partial x_i} (\overline{nv_i}) + \frac{1}{2}\, m\bar{v}_i\bar{v}_k \frac{\partial}{\partial x_i} (\overline{nv_j})$$

$$+ \frac{1}{2}\, m\bar{v}_i\bar{v}_j \frac{\partial}{\partial x_i} (\overline{nv_k}) + \frac{1}{2}\, m\bar{v}_k \frac{\partial}{\partial x_i} (\bar{n}<v_i'v_j'>)$$

$$+ \frac{1}{2}\, m\bar{n}<v_i'v_j'> \frac{\partial \bar{v}_k}{\partial x_i} \cdots\cdots \tag{102}$$

Now substituting this equation into Eq. 101 and doing some rearrangements,

$$\frac{1}{2}\, m\bar{v}_j\bar{v}_k \,[\frac{\partial \bar{n}}{\partial \tau} + \frac{\partial}{\partial x_i} (\overline{nv_i}) + 2\bar{n}<\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}> - \int A(\frac{\partial f}{\partial \tau})_{coll.} dv]$$

$$+ \frac{1}{2}\, \bar{v}_k [\bar{n}m \frac{\partial \bar{v}_j}{\partial \tau} + m\bar{v}_i \frac{\partial}{\partial x_i} (\overline{nv_j}) + m \frac{\partial}{\partial x_i} (\bar{n}<v_i'v_j'>)$$

$$- m\bar{n}<\bar{R}_i> + m\bar{n}<\frac{\partial D}{\partial v_i(t)}> + 2m\bar{n}<v_j' \frac{\partial^2 D}{\partial v(\tau)\partial v(t)}>$$

$$- \int v_j' A(\frac{\partial f}{\partial \tau})_{coll.} dv] + \frac{\partial}{\partial \tau} (\bar{n}<\frac{1}{2}\, mv_j'v_k'>)$$

$$+ \bar{n}\, \frac{1}{2}\, m\bar{v}_j \frac{\partial \bar{v}_k}{\partial \tau} + \frac{1}{2}\, m\bar{v}_i\bar{v}_j \frac{\partial}{\partial x_i} (\overline{nv_k}) + \frac{1}{2}\, m\bar{n}<v_i'v_j'> \frac{\partial \bar{v}_k}{\partial x_i}$$

$$+ \frac{\partial}{\partial x_i} (\bar{n} \frac{1}{2} m\bar{v}_i <v_j'v_k'>) + \frac{\partial}{\partial x_i} (\bar{n} \frac{1}{2} m\bar{v}_j <v_i'v_k'>)$$

$$+ \frac{\partial}{\partial x_i} (\bar{n} \frac{1}{2} m <v_i'v_j'v_k'>) - \bar{n}<\bar{R}_i \frac{1}{2} mv_k'> - \bar{n}<\bar{R}_i \frac{1}{2} m\bar{v}_j>$$

$$- \bar{n}<\bar{R}_i \frac{1}{2} mv_j'> + \bar{n}<\frac{1}{2} mv_k' \frac{\partial D}{\partial v_i(t)}> + \bar{n}<\frac{1}{2} m\bar{v}_j \frac{\partial D}{\partial v_i(t)}>$$

$$+ \bar{n}<\frac{1}{2} mv_j' \frac{\partial D}{\partial v_i(t)}> + 2\bar{n}<\frac{1}{2} m\bar{v}_j v_k' \frac{\partial^2 D}{\partial v(\tau)\partial v(t)}>$$

$$+ 2\bar{n}<\frac{1}{2} mv_j'v_k' \frac{\partial^2 D}{\partial v(\tau)\partial v(t)}>$$

$$= \int \frac{1}{2} m\bar{v}_j v_k' A(\frac{\partial f}{\partial \tau})_{coll.} dv + \int \frac{1}{2} mv_j'v_k' A(\frac{\partial f}{\partial \tau})_{coll.} dv \ldots\ldots \quad (103)$$

It is seen that the first bracket in Eq. 103 is identical to Eq. 78 and gives the continuity equation. The second bracket is the same as Eq. 88 and gives the momentum equation. Now the remaining terms of Eq. 103 give,

$$\frac{\partial}{\partial \tau} (\bar{n}<\frac{1}{2} mv_j'v_k'>) + \bar{n} \frac{1}{2} m\bar{v}_j \frac{\partial \bar{v}_k}{\partial \tau} + \frac{1}{2} m\bar{v}_i\bar{v}_j \frac{\partial}{\partial x_i} (\overline{nv_k})$$

$$+ \frac{1}{2} m\bar{n}<v_i'v_j'> \frac{\partial \bar{v}_k}{\partial x_i} + \frac{\partial}{\partial x_i} (\bar{n} \frac{1}{2} m\bar{v}_i <v_j'v_k'>)$$

$$+ \frac{\partial}{\partial x_i} (\bar{n} \frac{1}{2} m\bar{v}_j <v_i'v_k'>) + \frac{\partial}{\partial x_i} (\bar{n} \frac{1}{2} m <v_i'v_j'v_k'>$$

$$- \bar{n}<\bar{R}_i \frac{1}{2} mv_k'> - \bar{n}<\bar{R}_i \frac{1}{2} m\bar{v}_j> - \bar{n}<\bar{R}_i \frac{1}{2} mv_j'>$$

$$+ \bar{n}<\frac{1}{2} mv_k' \frac{\partial D}{\partial v_i(t)}> + \bar{n}<\frac{1}{2} m\bar{v}_j \frac{\partial D}{\partial v_i(t)}> + \bar{n}<\frac{1}{2} mv_j' \frac{\partial D}{\partial v_i(t)}>$$

$$+ 2\bar{n}<\frac{1}{2} m\bar{v}_j v_k' \frac{\partial^2 D}{\partial v(\tau)\partial v(t)}> + 2\bar{n}<\frac{1}{2} mv_j'v_k' \frac{\partial^2 D}{\partial v(\tau)\partial v(t)}>$$

$$= \int \frac{1}{2} m\bar{v}_j v_k' A(\frac{\partial f}{\partial \tau})_{coll.} dv + \int \frac{1}{2} mv_j'v_k' A(\frac{\partial f}{\partial \tau})_{coll.} dv \ldots\ldots \quad (104)$$

This equation can be further simplified; the fifth term can be written as,

$$\frac{\partial}{\partial x_i}(\bar{n}\,\frac{1}{2}\,m\bar{v}_j<v_i'v_k'>) = \frac{1}{2}\,m\bar{v}_j\,\frac{\partial}{\partial x_i}(\bar{n}<v_i'v_k'>)$$
$$+\frac{1}{2}\,m\bar{n}<v_i'v_k'>\,\frac{\partial}{\partial x_i}(\bar{v}_j)\ldots\ldots \qquad (105)$$

Substituting this equation into Eq. 104 and rearranging some of the terms yield to,

$$\frac{1}{2}\,\bar{v}_j[m\bar{n}\,\frac{\partial\bar{v}_k}{\partial\tau} + m\bar{v}_i\,\frac{\partial}{\partial x_i}(\overline{nv_k}) + m\,\frac{\partial}{\partial x_i}(\bar{n}<v_i'v_k'>)$$
$$- m\bar{n}<\bar{R}_i> + m\bar{n}<\frac{\partial D}{\partial v_i(\tau)}> + 2m\bar{n}<v_k'\,\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}>$$
$$- \int v_k'A(\frac{\partial f}{\partial\tau})_{coll.}dv] + \frac{\partial}{\partial\tau}(\bar{n}<\frac{1}{2}\,mv_j'v_k'>) + \frac{1}{2}\,m\bar{n}<v_i'v_k'>\,\frac{\partial\bar{v}_j}{\partial x_i}$$
$$+\frac{1}{2}\,m\bar{n}<v_i'v_j'>\,\frac{\partial\bar{v}_k}{\partial x_i} + \frac{\partial}{\partial x_i}(\bar{n}\,\frac{1}{2}\,m\bar{v}_i<v_j'v_k'>)$$
$$+\frac{\partial}{\partial x_i}(\bar{n}\,\frac{1}{2}\,m<v_i'v_j'v_k'>) - \bar{n}<\bar{R}_i\,\frac{1}{2}\,mv_k'> - \bar{n}<\bar{R}_i\,\frac{1}{2}\,mv_j'>$$
$$+\bar{n}<\frac{1}{2}\,mv_k'\,\frac{\partial D}{\partial v_i(t)}> + \bar{n}<\frac{1}{2}\,mv_j'\,\frac{\partial D}{\partial v_i(t)}>$$
$$+2\bar{n}<\frac{1}{2}\,mv_j'v_k'\,\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}> = \int\frac{1}{2}\,mv_j'v_k'A(\frac{\partial f}{\partial\tau})_{coll.}dv\ldots\ldots$$
$$(106)$$

It is seen that the terms in the bracket give the momentum equation i.e. Eq. 88. The remaining terms yield the following equation,

$$\frac{\partial}{\partial\tau}(\bar{n}<\frac{1}{2}\,mv_j'v_k'>) + \frac{1}{2}\,m\bar{n}<v_i'v_k'>\,\frac{\partial\bar{v}_j}{\partial x_i} + \frac{1}{2}\,m\bar{n}<v_i'v_j'>\,\frac{\partial\bar{v}_k}{\partial x_i}$$
$$+\frac{\partial}{\partial x_i}(\bar{n}\,\frac{1}{2}\,m\bar{v}_i<v_j'v_k'>) + \frac{\partial}{\partial x_i}(\bar{n}\,\frac{1}{2}\,m<v_i'v_j'v_k'>)$$
$$- \bar{n}<\bar{R}_i\,\frac{1}{2}\,mv_k'> - \bar{n}<\bar{R}_i\,\frac{1}{2}\,mv_j'> + \bar{n}<\frac{1}{2}\,mv_k'\,\frac{\partial D}{\partial v_i(t)}>$$

$$+ \bar{n}<\frac{1}{2}\, mv'_j \frac{\partial D}{\partial v_i(t)}> + 2\bar{n}<\frac{1}{2}\, mv'_j v'_k \frac{\partial^2 D}{\partial v(\tau)\partial v(t)}>$$

$$= \int \frac{1}{2}\, mv'_j v'_k A(\frac{\partial f}{\partial \tau})_{coll.} dv \ldots\ldots \qquad (107)$$

Now this equation can be expressed in a different notation. First, defining a new term Q_i so-called heat flux vector as,

$$\bar{Q}_i = \bar{n}m<v'_i v'_j v'_k> \ldots\ldots \qquad (108)$$

and also expressing the fourth term of Eq. 107 as,

$$\frac{\partial}{\partial x_i}\left(\bar{n}\frac{1}{2}m\bar{v}_i<v'_j v'_k>\right) = \frac{1}{2}\, m\bar{n}<v'_j v'_k> \frac{\partial \bar{v}_i}{\partial x_i} + \frac{1}{2}\,\bar{v}_i \frac{\partial}{\partial x_i}\left(\bar{n}m<v'_j v'_k>\right)$$

(109)

then Eq. 107 can be put into the following form by using Eqs. 108, 109 and 90,

$$\frac{\partial}{\partial \tau}\bar{\psi}_{jk} + \bar{\psi}_{ik}\frac{\partial \bar{v}_j}{\partial x_i} + \bar{\psi}_{ij}\frac{\partial \bar{v}_k}{\partial x_i} + \bar{\psi}_{jk}\frac{\partial \bar{v}_i}{\partial x_i} + \bar{v}_i \frac{\partial}{\partial x_i}\bar{\psi}_{jk}$$

$$+ \frac{\partial \bar{Q}}{\partial x_i} - m\bar{n}<\bar{R}_i v'_k> - m\bar{n}<\bar{R}_i v'_j>$$

$$+ m\bar{n}<v'_k \frac{\partial D}{\partial v}> + m\bar{n}<v'_j \frac{\partial D}{\partial v}>$$

$$+ 2<\bar{\psi}_{jk}\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}> = 2\int \frac{1}{2}\, mv'_j v'_k A(\frac{\partial f}{\partial \tau})_{coll.} dv \ldots\ldots$$

(110)

This tensorial expression can be written in a somewhat simpler form by making use of Eq. 78 which can be written as,

$$\frac{\partial \bar{n}}{\partial \tau} + \bar{v}_i \frac{\partial \bar{n}}{\partial x_i} + \bar{n}\frac{\partial \bar{v}_i}{\partial x_i} + 2\bar{n}<\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}> = \int A(\frac{\partial f}{\partial \tau})_{coll.} dv \ldots\ldots$$

(111)

$$\frac{\partial \bar{v}_i}{\partial x_i} = \frac{1}{\bar{n}} [- \frac{\partial \bar{n}}{\partial \tau} - \bar{v}_i \frac{\partial \bar{n}}{\partial x_i} - 2\bar{n} < \frac{\partial^2 D}{\partial v(\tau)\partial v(t)} > + \int A (\frac{\partial f}{\partial \tau})_{coll.} dv] \ldots\ldots \quad (112)$$

The first two terms of Eq. 112 can be expressed in total time derivative as,

$$\frac{\partial}{\partial \tau} + v \cdot \frac{\partial}{\partial x} = \frac{d}{d\tau} \ldots\ldots \quad (113)$$

First this equation is substituted into Eq. 112 which is then substituted into Eq. 110 for the fourth term. After some rearrangements it is obtained that,

$$\begin{aligned}
&(\frac{\partial}{\partial \tau} + \bar{v}_i \frac{\partial}{\partial x_i})\bar{\psi}_{jk} - \frac{\bar{\psi}_{jk}}{\bar{n}} \frac{d\bar{n}}{d\tau} + \bar{\psi}_{ik} \frac{\partial \bar{v}_j}{\partial x_i} + \bar{\psi}_{ij} \frac{\partial \bar{v}_k}{\partial x_i} \\
&+ \frac{\partial \bar{Q}}{\partial x_i} - m\bar{n} < \bar{R}_i v_k' > - m\bar{n} < \bar{R}_i v_j' > + m\bar{n} < v_k' \frac{\partial D}{\partial v} > \\
&+ m\bar{n} < v_j' \frac{\partial D}{\partial v} > + 2 < \bar{\psi}_{jk} \frac{\partial^2 D}{\partial v(\tau)\partial v(t)} > \\
&- 2\bar{\psi}_{jk} < \frac{\partial^2 D}{\partial v(\tau)\partial v(t)} > = 2 \int \frac{1}{2} m v_j' v_k' A (\frac{\partial f}{\partial \tau})_{coll.} dv \\
&- \frac{\bar{\psi}_{jk}}{\bar{n}} \int A (\frac{\partial f}{\partial \tau})_{coll.} dv \ldots\ldots
\end{aligned} \quad (114)$$

Equation 113 can also be substituted for the first two terms in the bracket in the above equation; then a simplification can be carried out including the third term as follows,

$$(\frac{\partial}{\partial \tau} + \bar{v}_i \frac{\partial}{\partial x_i})\bar{\psi}_{jk} - \frac{\bar{\psi}_{jk}}{\bar{n}} \frac{d\bar{n}}{d\tau} = \frac{d\bar{\psi}_{jk}}{d\tau} - \frac{\bar{\psi}_{jk}}{\bar{n}} \frac{d\bar{n}}{d\tau} \ldots\ldots \quad (115)$$

$$= \bar{n}[\frac{1}{\bar{n}} \frac{d\bar{\psi}_{jk}}{d\tau} - \frac{\bar{\psi}_{jk}}{\bar{n}^2} \frac{d\bar{n}}{d\tau}]$$

$$= \bar{n} \frac{d}{d\tau} (\frac{\bar{\psi}_{jk}}{\bar{n}}) \ldots\ldots \quad (116)$$

Now substitution of Eq. 116 into Eq. 114 results in,

$$\frac{d}{d\tau}\left(\frac{\overline{\psi}_{jk}}{\overline{n}}\right) + \frac{\overline{\psi}_{ik}}{\overline{n}} \cdot \frac{\partial \overline{v}_j}{\partial x_i} + \frac{\overline{\psi}_{ij}}{\overline{n}} \frac{\partial \overline{v}_k}{\partial x_i} + \frac{1}{\overline{n}} \frac{\partial \overline{Q}}{\partial x_i} - m\langle \overline{R}_i v_k' \rangle$$

$$- m\langle \overline{R}_i v_j' \rangle + m\langle v_k' \frac{\partial D}{\partial v} \rangle + m\langle v_j' \frac{\partial D}{\partial v} \rangle$$

$$+ \frac{2}{\overline{n}} \langle \overline{\psi}_{jk} \frac{\partial^2 D}{\partial v(\tau)\partial v(t)} \rangle - \frac{2\overline{\psi}_{jk}}{\overline{n}} \langle \frac{\partial^2 D}{\partial v(\tau)\partial v(t)} \rangle$$

$$= \frac{1}{\overline{n}} \int m v_j' v_k' A\left(\frac{\partial f}{\partial \tau}\right)_{coll.} dv - \frac{\overline{\psi}_{jk}}{\overline{n}^2} \int A\left(\frac{\partial f}{\partial \tau}\right)_{coll.} dv \ldots\ldots \quad (117)$$

The trace of this tensorial equation gives the energy or heat equation.

VI. PARTICLE DISTRIBUTION IN TOKAMAKS

The distribution of particles in a Tokamak device can be found from continuity equation. Since the temperature of the Tokamak plasma is very high (200-400 eV) the fuel atoms can be assumed to be fully ionized. Thus the collision term makes no contribution under this condition. Therefore the starting equation will be Eq. 79 rather than Eq. 78. Equation 79 can now be written as,

$$\frac{\partial \bar{n}}{\partial \tau} + \nabla(\overline{nv}) + 2\bar{n}\left\langle \frac{\partial^2 D}{\partial v(\tau)\partial v(t)} \right\rangle = 0 \ldots\ldots \tag{118}$$

where ∇ is substituted for $\partial/\partial x$ which represents the spatial coordinates in general.

Equation 118 can be written as,

$$\frac{\partial \bar{n}}{\partial \tau} + \bar{v}\nabla\bar{n} + \bar{n}\nabla\bar{v} + 2\bar{n}\left\langle \frac{\partial^2 D}{\partial v(\tau)\partial v(t)} \right\rangle = 0 \ldots\ldots \tag{119}$$

The second term can be dropped since the velocity is independent of coordinates. Then expressing the first term in cylindrical coordinates, it is obtained that,

$$\frac{\partial \bar{n}}{\partial \tau} + \bar{v}\,\frac{1}{r}\frac{\partial}{\partial r}\,(r\bar{n}) + \bar{v}\,\frac{1}{r}\frac{\partial \bar{n}}{\partial \theta} + \bar{v}\,\frac{\partial \bar{n}}{\partial \varphi} + 2\bar{n}\left\langle \frac{\partial^2 D}{\partial v(\tau)\partial v(t)} \right\rangle = 0 \ldots\ldots \tag{120}$$

In order to make further simplifications the general features of a Tokamak must be taken into consideration. Figure 1 shows a typical Tokamak device. The plasma column inside the torus is shown in Fig. 2. Under the effect of poloidal and toroidal magnetic fields its motion becomes helical.

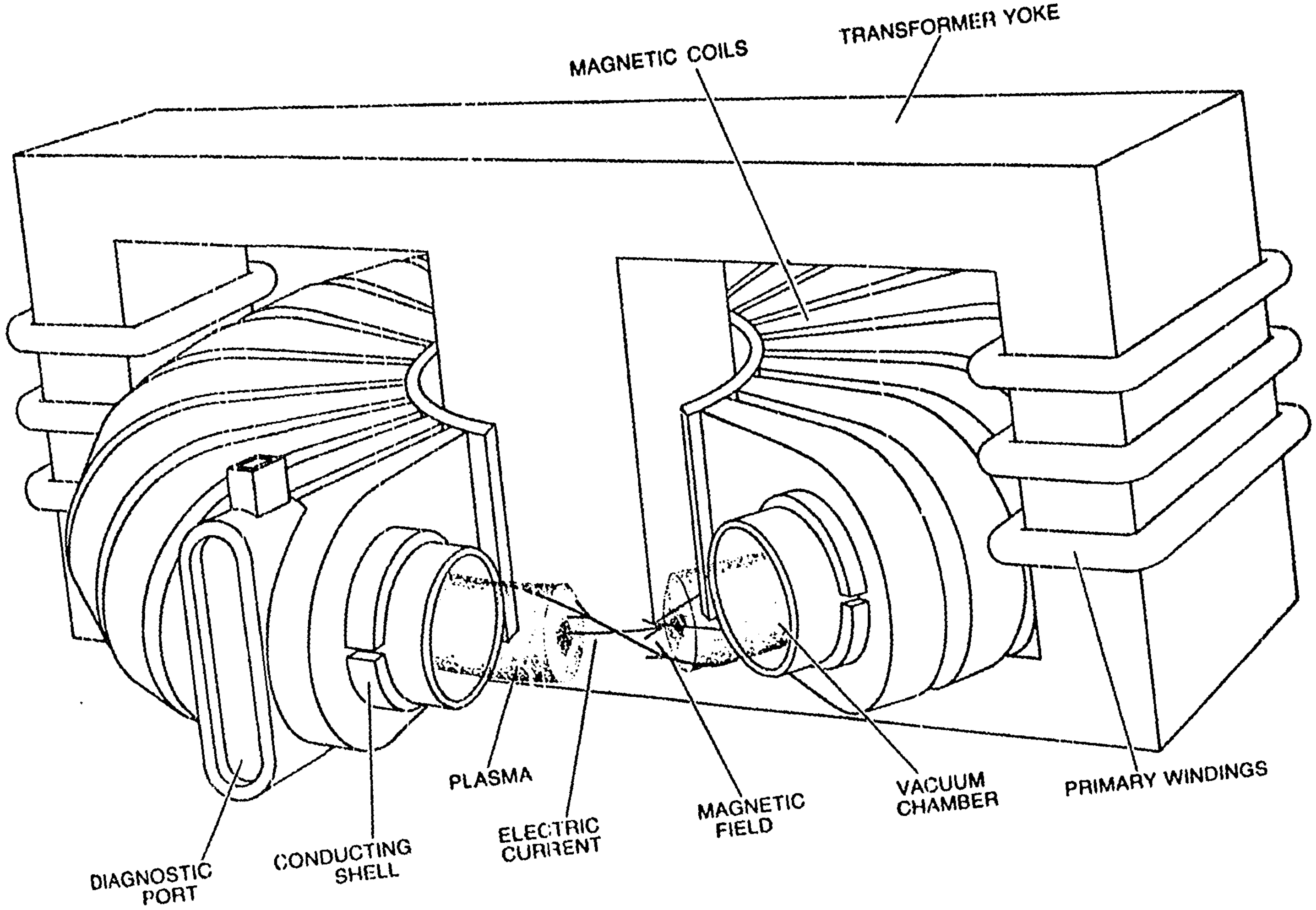

Fig. 1. A typical Tokamak device

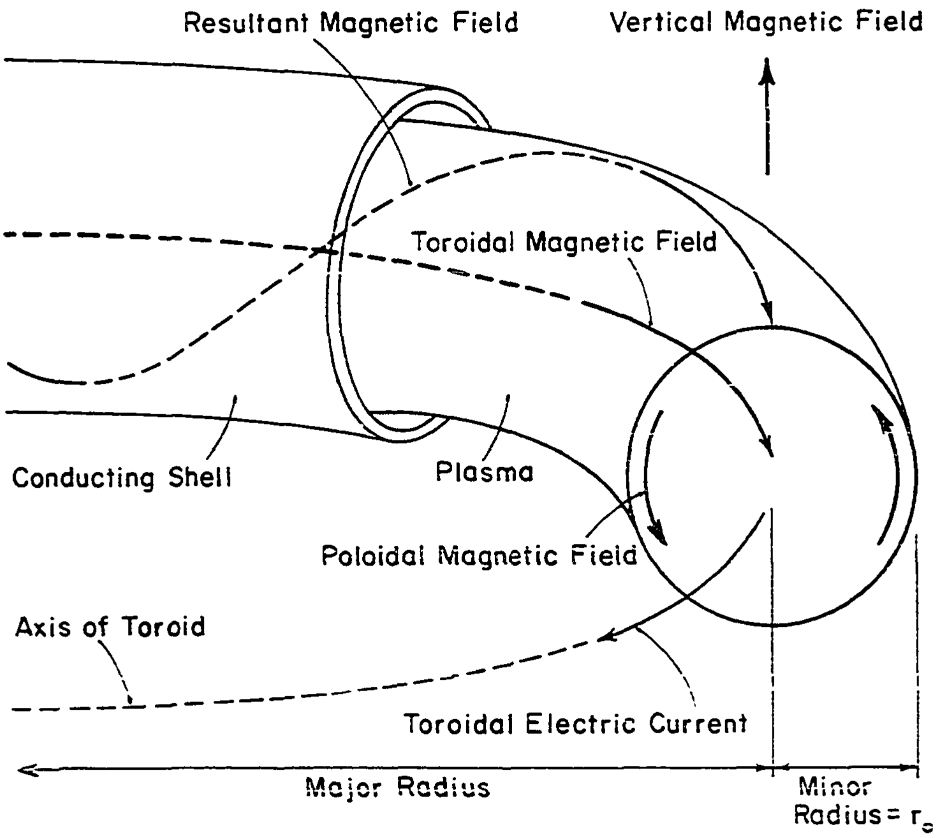

Fig. 2. The plasma column inside the torus

A typical Tokamak device consists of two major parts; one is a toroidal vacuum chamber surrounded by a set of coils to provide a strong magnetic field and the other is an iron-core transformer which provides the poloidal magnetic field B_θ. A varying current in the primary of the transformer induces an electric field in the torus, which in turn drives the current of the plasma column which actually is the secondary of the transformer.

The toroidal magnetic field B_φ, produced by the current through the coils of vacuum chamber compresses the particles toward the center of the torus.

The toroidal magnetic field is much larger than the poloidal magnetic field which is induced by the current of the plasma column, that is $B_\varphi \gg B_\theta$.

The plasma current runs only as long as the poloidal magnetic flux is changing. But the maximum magnetic flux is limited by the cross section of the iron-core and also by the maximum magnetic field it can handle before saturation. Therefore the experiment becomes time limited.

Because of the compression of the particles by B_φ in r-direction, the particle density changes along the minor radius r. There is no real net force in θ-direction and the plasma column can be assumed to be homogeneous in θ-direction so that the third term in Eq. 120 can be dropped. The movement of the plasma column in θ-direction results because of the combined effect of perpendicular and parallel forces to φ-direction. Even though the particles are derived in the φ-direction there will not be any gradient for particle density because of the uniformity of the driving force all along the φ-direction. Therefore the fourth term of Eq. 120 also can be dropped. Now it can be written as,

$$\frac{\partial \bar{n}}{\partial \tau} + \frac{\bar{v}}{r}\frac{\partial}{\partial r}(r\bar{n}) + 2\bar{n}<\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}>= 0\ldots\ldots \tag{121}$$

This equation can be put in a more explicit form by using Eq. 64,

$$\frac{\partial \bar{n}}{\partial \tau} + \frac{\bar{v}}{r}\bar{n} + \bar{v}\frac{\partial \bar{n}}{\partial r} + 2\bar{n}<\frac{\partial^2}{\partial v(\tau)\partial v(t)}\int dt\, \frac{q^2}{m^2}\Big[\sum_k <E'_k(r, \tau)U(\tau, t)E'_{-k}(r, t)$$

$$+ E'_k(r, \tau)U(\tau, t)(v \times B'(r, t))_{-k}$$

$$+ (v \times B'(r, \tau))_k U(\tau, t)E'_{-k}(r, t)$$

$$+ (v \times B'(r, \tau))_k U(\tau, t)(v \times B'(r, t))_{-k}>\Big]> = 0\ldots\ldots \tag{122}$$

Now $\bar{n}(r, t)$ can be written in a Fourier analyzed form,

$$\bar{n}(r, \tau) = \sum \bar{n}(r)e^{-i\omega\tau}\ldots\ldots \tag{123}$$

then,

$$\frac{\partial \bar{n}(r, \tau)}{\partial \tau} = -\, i\omega\bar{n}(r, \tau)\ldots\ldots \tag{124}$$

Now substituting Eq. 124 into Eq. 122 and expressing all terms in an integration form it is obtained that,

$$\int\frac{d\bar{n}}{\bar{n}} = \int\frac{i\omega}{\bar{v}}\,dr - \int\frac{dr}{r} - \frac{2}{\bar{v}}<\frac{\partial^2}{\partial v(\tau)\partial v(t)}$$

$$\Big[\frac{q^2}{m^2}\iint\sum_k \Big[<E'_k(r, \tau)U(\tau, t)E'_{-k}(r, t)$$

$$+ E'_k(r, \tau)U(\tau, t)(v \times B'(r, t))_{-k}$$

$$+ (v \times B'(r, \tau))_k U(\tau, t) E'_{-k}(r, t)$$

$$+ (v \times B'(r, \tau))_k U(\tau, t)(v \times B'(r, t))_{-k} > dtdr\Big]\Big] > \ldots\ldots \quad (125)$$

In order to carry out the integration E' and B' must be explicitly expressed in terms of r. This is also achieved by writing each quantity in Fourier analyzed form, such that,

$$E'_{-k}(r, t) = \sum_k E'(t) e^{-ikr} \ldots\ldots \quad (126)$$

$$B_{-k}(r, t) = \sum_k B'(t) e^{-ikr} \ldots\ldots \quad (127)$$

Equation 125 can now be written as,

$$\int \frac{d\bar{n}}{\bar{n}} = \int \frac{i\omega}{\bar{v}} dr - \int \frac{dr}{r} - \frac{2}{\bar{v}} \frac{q^2}{m^2} < \frac{\partial^2}{\partial v(\tau) \partial v(t)}$$

$$\iint < \sum_k \Big[E'(\tau) e^{ikr} U(\tau, t) E'(t) e^{-ikr}$$

$$+ E'(\tau) e^{ikr} U(\tau, t)(v \times B'(t)) e^{-ikr}$$

$$+ (v \times B'(\tau)) e^{ikr} U(\tau, t) E'(t) e^{-ikr}$$

$$+ (v \times B'(\tau)) e^{ikr} U(\tau, t)(v \times B'(t)) e^{-ikr} \Big] > dtdr > \ldots\ldots \quad (128)$$

From the definition of the $U(\tau, t)$ propagator as given in Eq. 35, it can be written that,

$$U(\tau, t) E'(t) e^{-ikr} = E'(\tau) e^{-ikr_p} \ldots\ldots \quad (129)$$

$$U(\tau, t)(v \times B(t)) e^{-ikr} = (v \times B(\tau)) e^{-ikr_p} \ldots\ldots \quad (130)$$

where "r_p" is the perturbed trajectory of the particle as introduced by Dupree [14]. When the particle first has an unperturbed trajectory "r" at time "t", it then has its trajectory perturbed because of turbulence with a new value of "r_p" at time "τ". In fact the time propagator transforms (r, t) into (r_p, τ).

The problem now is to find out the value of r_p. This can be estimated by using Eq. 43 and Eq. 57. First substituting r' for f' in Eq. 43 yields,

$$r'(\tau) = r_p' = U(\tau, t)r'(0) - \int dt\ U(\tau, t)\cdot\frac{q}{m} R' \frac{\partial}{\partial v}\bar{r}(t)\ldots\ldots \tag{131}$$

since $r'(0) = 0$,

$$r_p' = -\frac{q}{m}\int dt\ U(\tau, t)(E'(r, t) + v \times B'(r, t))t\ldots\ldots \tag{132}$$

Now by setting $\bar{r}$ for $\bar{f}$ in Eq. 57 and using Eq. 49,

$$\frac{\partial\bar{r}(\tau)}{\partial\tau} + \bar{L}(\tau)\bar{r}(\tau) - \left(\frac{\partial}{\partial v(\tau)} D \frac{\partial}{\partial v(t)}\right)\bar{r}(t) = A\left(\frac{\partial r}{\partial\tau}\right)_{coll.}\ldots\ldots \tag{133}$$

Neglecting the collisional contribution and using Eq. 48, Eq. 133 can now be expressed as,

$$\frac{\partial\bar{r}(\tau)}{\partial\tau} + \bar{v}\frac{1}{\bar{r}(\tau)}\frac{\partial}{\partial\bar{r}(\tau)}(\bar{r}(\tau)\bar{r}(\tau)) + \bar{R}_i\frac{\partial\bar{r}(\tau)}{\partial v} - \left(\frac{\partial}{\partial v(\tau)} D \frac{\partial}{\partial v(t)}\right)\bar{r}(t) = 0 \ldots\ldots \tag{134}$$

$$\frac{\partial\bar{r}(\tau)}{\partial\tau} + 2\bar{v} + \bar{R}_i\tau - \frac{\partial D}{\partial v(\tau)}\frac{\partial\bar{r}(t)}{\partial v(t)} + D\frac{\partial^2\bar{r}(t)}{\partial v(\tau)\partial v(t)} = 0\ldots\ldots \tag{135}$$

$$d\bar{r}(\tau) = -2\bar{v}d\tau - \bar{R}_i\tau d\tau - \frac{\partial D}{\partial v(\tau)}\ t\ d\tau\ldots\ldots \tag{136}$$

The integration can be carried out with the boundary conditions of,

$\tau \to t \qquad r(\tau) \to r(t)$

$\tau \to \tau \qquad r(\tau) \to r(\tau)$

$$\bar{r}(\tau) = \bar{r}_p = \bar{r}(t) - 2\bar{v}(\tau - t) - \frac{\bar{R}_i}{2}(\tau^2 - t^2) - \int\frac{\partial D}{\partial v(\tau)}\ t\ d\tau\ldots\ldots \tag{137}$$

By adding Eq. 132 and 137,

$$r_p = \bar{r}_p + r'_p = \bar{r}(t) - 2\bar{v}(\tau - t) - \frac{\bar{R}_i}{2}(\tau^2 - t^2) - \int \frac{\partial D}{\partial v(\tau)} t \, d\tau$$

$$- \frac{q}{m}\int dt \; U(\tau, t)(E'(r, t) + v \times B'(r, t))t \ldots\ldots \quad (138)$$

Now Eq. 138 can be substituted into Eq. 129 and 130 which are then both substituted into Eq. 128 to obtain,

$$\int \frac{d\bar{n}}{\bar{n}} = \int \frac{i\omega}{\bar{v}} \, dr - \int \frac{dr}{r} - \frac{2}{\bar{v}} \frac{q^2}{m^2} < \frac{\partial^2}{\partial v(\tau)\partial v(t)} \iint < \sum_k \Big[E'(\tau)E'(\tau)$$

$$+ E'(\tau)(v \times B'(\tau)) + (v \times B'(\tau))E'(\tau)$$

$$+ (v \times B'(\tau))(v \times B'(\tau))\Big] e^{ikr} e^{-ikr_p} > dt \; dr > \ldots\ldots \quad (139)$$

Since r_p has been expressed in terms of velocity, the differentiation with respect to velocity can be carried out in the above equation.

The integration of the right side of Eq. 139 is complicated. So it can be written as follows in a semi-closed form,

$$\frac{\bar{n}}{\bar{n}_o} = \frac{r_o}{r} \exp\Big[\int \frac{i\omega}{\bar{v}} \, dr - \frac{2}{\bar{v}} \frac{q^2}{m^2} < \frac{\partial^2}{\partial v(\tau)\partial v(t)} \iint < \sum_k \Big[E'(\tau)E'(\tau)$$

$$+ E'(\tau)(v \times B'(\tau)) + (v \times B'(\tau))E'(\tau)$$

$$+ (v \times B'(\tau))(v \times B'(\tau))\Big] e^{ikr} e^{-ikr_p} > dt \, dr > \Big] \ldots\ldots \quad (140)$$

where (r, r_o) and $(\bar{n}, \bar{n}_o)$ are boundary conditions. This is the time dependent solution of Eq. 122. A steady-state solution of Eq. 122, can now be obtained by dropping the first term.

Now a slight modification in the propagator $U(\tau, t)$ can be made. It can also be written as,

$$U(\tau, t) = U(\tau - t, 0) \ldots\ldots \tag{141}$$

Using this equation and considering the steady-state condition, Eq. 122 is written as,

$$\bar{v}\frac{\partial\bar{n}}{\partial r} + \frac{\overline{vn}}{r} + 2\bar{n} < \frac{\partial^2}{\partial v^2}\int\left[\frac{q^2}{m^2}\int < \sum_k \left[E'_k(r, \tau)U(\tau - t, 0)E'_{-k}(r, t)\right.\right.$$
$$+ E'_k(r, \tau)U(\tau - t, 0)(v \times B'(r, t))_{-k}$$
$$+ (v \times B'(r, \tau))_k U(\tau - t, 0)E'_{-k}(r, t)$$
$$\left.\left. + (v \times B'(r, \tau))_k U(\tau - t, 0)(v \times B'(r, t))_{-k}\right] > \right] dt > \ldots\ldots \tag{142}$$

Since τ is an arbitrary time, it can be set equal to zero, so that,

$$\bar{v}\frac{\partial\bar{n}}{\partial r} + \frac{\overline{vn}}{r} + 2\bar{n} < \frac{\partial^2}{\partial v^2}\int\left[\frac{q^2}{m^2}\int < \sum_k \left[E'_k(r, 0)U(- t, 0)E'_{-k}(r, t)\right.\right.$$
$$+ E'_k(r, 0)U(- t, 0)(v \times B'(r, t))_{-k}$$
$$+ (v \times B'(r, 0))_k U(- t, 0)E'_{-k}(r, t)$$
$$\left.\left. + (v \times B'(r, 0))_k U(- t, 0)(v \times B'(r, t))_{-k}\right] > \right] dt > \ldots\ldots \tag{143}$$

From Eq. 35, it is concluded that when $U(- t, 0)$ operated on an arbitrary function $F(t, 0)$, it gives,

$$U(- t, 0)F(t, 0) = F(- t, 0) = F(- t) \ldots\ldots \tag{144}$$

Using this fact, the operator $U(- t, 0)$ can be removed by operating on the quantities on the right of it in Eq. 143.

A further simplification in Eq. 143 can be carried out by taking the derivatives of the diffusional terms twice with respect to velocity. When this is done it is seen that only the last term survives. Therefore a new diffusional term $D_{B'B'}$ can now be defined as,

$$D_{B'B'} = \frac{\partial^2}{\partial v_i^2}\left[\frac{q^2}{m^2}\int dt < (v \times B'(r, 0))(v \times B'(r, -t)) >\right] \ldots\ldots \tag{145}$$

which was evaluated in Appendix F to give,

$$D_{B'B'} = 2\,\frac{q^2}{m^2}\int dt < B_j'^2 + B_k'^2 > \ldots\ldots \tag{146}$$

In cylindrical coordinates it can be written as,

$$D_{B'B'} = 2\,\frac{q^2}{m^2}\int dt < B_\theta'^2 + B_\varphi'^2 > \ldots\ldots \tag{147}$$

or in general,

$$D_{B'B'} = 2\,\frac{q^2}{m^2}\int dt < \sum_{m=j,k} B_m^2 > \ldots\ldots \tag{148}$$

Equation 143 now becomes,

$$\bar{v}\,\frac{\partial \bar{n}}{\partial r} + \frac{\overline{vn}}{r} - 2\bar{n} < D_{B'B'} > = 0 \ldots\ldots \tag{149}$$

In order to integrate this equation, $D_{B'B'}$ must be first expressed explicitly in terms of r. It was found out in Appendix G that B' changes inversely with radius r. From Eq. G-5 in Appendix G, B' can be written as,

$$B'_{(t)} = \frac{Cst}{r} \ldots\ldots \tag{150}$$

or using the boundary conditions it can be expressed as

$$B'_{(t)} = \frac{r_o B'_o(t)}{r} \ldots\ldots \qquad (151)$$

where B'_o is the value of B' at r_o which is the minor radius of the torus. (See Fig. 2, page 40).

Substituting Eq. 151 into Eq. 148 and then the resulting equation into Eq. 149,

$$\bar{v}\,\frac{\partial \bar{n}}{\partial r} + \frac{\overline{vn}}{r} - 2\bar{n}\left(2\,\frac{q^2}{m^2}\int dt < B'_o B'_o(-t) >\right)\frac{r_o^2}{r^2} = 0 \ldots\ldots \qquad (152)$$

Now defining a new quantity α such as,

$$\alpha = -\frac{4r_o^2}{\bar{v}}\,\frac{q^2}{m^2}\int dt < B'_o B'_o(-\,t) > \ldots\ldots \qquad (153)$$

Equation 152 can now be written as,

$$\frac{d\bar{n}}{\bar{n}} = -\frac{dr}{r} + \alpha\,\frac{dr}{r^2} \ldots\ldots \qquad (154)$$

The integration can be carried out with the boundary conditions of,

as $\quad r = r_o \qquad \bar{n} = \bar{n}_o$

$\quad\;\; r = r \qquad\; \bar{n} = \bar{n}$

The integration is carried out from the wall to the center of the torus in order to be physically consistent. The result is,

$$\frac{\bar{n}}{\bar{n}_o} = \frac{r_o}{r}\exp\left[-\,\alpha\left(\frac{1}{r} - \frac{1}{r_o}\right)\right] \ldots\ldots \qquad (155)$$

The profile of the number density of electrons can now be seen by plotting this equation as $\bar{n}/\bar{n}_o$ versus the distance from the center of the torus.

As it is seen from Eq. 153 α is a function of time. Therefore it takes different values depending on the time of the discharge stages.

Although $\overline{n}/\overline{n}_o$ in Eq. 155 represents the average particles, it can be generalized to represent both average and fluctuating particles. It is seen from Eq. 43 that $f'(\tau)$ is a function of $\overline{f}(t)$ only if one considers the fact that $f'(t_o) = 0$ at the beginning i.e. $t_o = 0$. In fact, it is also possible to neglect the fluctuating particles, because they are only a fraction of the total particles.

Anashin, et al. [28] published a complete set of experimental results on laser and microwave probing of plasma in the Tokamak T-3a device. Their results for the electron distribution were reproduced in Figs. 3, 4 and 5, in Section VIII, each corresponding to a different discharge stage. Also for different values of α the values of $\overline{n}/\overline{n}_o$ were found from Eq. 155 and plotted in the same figures. The numerical values of $\overline{n}/\overline{n}_o$ with change of r were tabulated in Table 1 in Section VIII.

A very good agreement between the theoretical predictions and the experimental results can be noticed in Figs. 3, 4 and 5.

VII. ENERGY EQUATIONS FOR TOKAMAKS

Equation 117 can now be applied to the physical conditions of Tokamaks to obtain an energy equation. Defining S_{ijk} to be,

$$S_{ijk} = \frac{1}{\bar{n}}\frac{\partial \bar{Q}}{\partial x_i} - m\langle \bar{R}_i v_k' \rangle - m\langle \bar{R}_i v_j' \rangle + m\langle v_k' \frac{\partial D}{\partial v}\rangle + m\langle v_j' \frac{\partial D}{\partial v}\rangle + \frac{2}{\bar{n}}\langle \bar{\psi}_{jk} \frac{\partial^2 D}{\partial v(\tau)\partial v(t)}\rangle - \frac{2\bar{\psi}_{jk}}{\bar{n}}\langle \frac{\partial^2 D}{\partial v(\tau)\partial v(t)}\rangle - \frac{1}{\bar{n}}\int m v_j' v_k' A\left(\frac{\partial f}{\partial \tau}\right)_{coll.} dv \ldots\ldots \quad (156)$$

then Eq. 117 is written as,

$$\frac{d}{d\tau}\left(\frac{\bar{\psi}_{jk}}{\bar{n}}\right) + \frac{\bar{\psi}_{ik}}{\bar{n}}\frac{\partial \bar{v}_j}{\partial x_i} + \frac{\bar{\psi}_{ij}}{\bar{n}}\frac{\partial \bar{v}_k}{\partial x_i} + S_{ijk} = 0 \ldots\ldots \quad (157)$$

where the last collisional term was dropped because it can be taken zero when integrated over velocity space as explained before.

The temperature T which denotes the energy of the particle, is defined as,

$$T = \frac{1}{3} m\langle v_i' v_i' \rangle = \frac{1}{3} m\langle v_j' v_j' \rangle = \frac{1}{3} m\langle v_k' v_k' \rangle \ldots\ldots \quad (158)$$

Using this definition T can be written in terms of ψ as,

$$T = \frac{1}{3\bar{n}}\langle \psi_{ii} \rangle = \frac{1}{3\bar{n}}\langle \psi_{jj} \rangle = \frac{1}{3\bar{n}}\langle \psi_{kk} \rangle \ldots\ldots \quad (159)$$

Now Eq. 157 can be written in its x, y, and z components for j = k in order to obtain energy equation.

For j = k, Eq. 157 becomes,

$$\frac{d}{d\tau}\left(\frac{\bar{\psi}_{jj}}{\bar{n}}\right) + \frac{2\bar{\psi}_{ij}}{\bar{n}}\frac{\partial \bar{v}_j}{\partial x_i} + S_{ijj} = 0 \ldots\ldots \quad (160)$$

The x, y, and z components of this equation are given in Appendix H.

In cylindrical coordinates we can write them as follows,

$$\frac{d}{d\tau}\left(\frac{\overline{\psi}_{rr}}{\overline{n}}\right) + \frac{2}{\overline{n}}\left(\frac{\overline{\psi}_{rr}}{r} \cdot \frac{\partial(r\overline{v}_r)}{\partial r} + \frac{\overline{\psi}_{r\theta}}{r}\frac{\partial \overline{v}_r}{\partial \theta} + \overline{\psi}_{\varphi r}\frac{\partial \overline{v}_\varphi}{\partial \varphi}\right)$$

$$+ S_{rrr} + S_{\theta rr} + S_{\varphi rr} = 0\ldots\ldots \quad (161)$$

$$\frac{d}{d\tau}\left(\frac{\overline{\psi}_{\theta\theta}}{\overline{n}}\right) + \frac{2}{\overline{n}}\left(\frac{\overline{\psi}_{r\theta}}{r} \cdot \frac{\partial(r\overline{v}_\theta)}{\partial r} + \frac{\overline{\psi}_{\theta\theta}}{r} \cdot \frac{\partial \overline{v}_\theta}{\partial \theta} + \overline{\psi}_{\varphi\theta}\frac{\partial \overline{v}_\theta}{\partial \varphi}\right)$$

$$+ S_{r\theta\theta} + S_{\theta\theta\theta} + S_{\varphi\theta\theta} = 0\ldots\ldots \quad (162)$$

$$\frac{d}{d\tau}\left(\frac{\overline{\psi}_{\varphi\varphi}}{\overline{n}}\right) + \frac{2}{\overline{n}}\left(\frac{\overline{\psi}_{r\varphi}}{r} \cdot \frac{\partial(r\overline{v}_\varphi)}{\partial r} + \frac{\overline{\psi}_{\theta\varphi}}{r}\frac{\partial \overline{v}_\varphi}{\partial \theta} + \overline{\psi}_{\varphi\varphi}\frac{\partial \overline{v}_\varphi}{\partial \varphi}\right)$$

$$+ S_{r\varphi\varphi} + S_{\theta\varphi\varphi} + S_{\varphi\varphi\varphi} = 0\ldots\ldots \quad (163)$$

In case $j \neq k$ the x, y, and z components of Eq. 157 can be written as given in Appendix I.

The energy equations can be further simplified since the flow can be assumed to be symmetrical. The velocity distribution at every point can be assumed to be axisymmetric. This results in,

$$\overline{\psi}_{r\theta} = 0\ldots\ldots \quad (164)$$

Another important point is that the velocity in φ-direction is much smaller than the velocities in r- and θ-directions. Therefore $\overline{\psi}_{r\varphi}$ and $\overline{\psi}_{\theta\varphi}$ can be neglected when compared to the others. Considering the uniformity in φ-direction $\partial\overline{v}_\varphi/\partial\varphi$ can be assumed to be zero.

Under these conditions Eqs. 161, 162, and 163 can be written as,

$$\frac{d}{d\tau}\left(\frac{\overline{\psi}_{rr}}{\overline{n}}\right) + \frac{2}{\overline{n}}\left(\frac{\overline{\psi}_{rr}}{r}\frac{\partial(r\overline{v}_r)}{\partial r}\right) + S_{rrr} + S_{\theta rr} + S_{\varphi rr} = 0\ldots\ldots \quad (165)$$

$$\frac{d}{d\tau}\left(\frac{\bar{\psi}_{\theta\theta}}{\bar{n}}\right)+\frac{2}{\bar{n}}\left(\frac{\bar{\psi}_{\theta\theta}}{r}\cdot\frac{\partial\bar{v}_{\theta}}{\partial\theta}\right)+S_{r\theta\theta}+S_{\theta\theta\theta}+S_{\varphi\theta\theta}=0\ldots\ldots \quad (166)$$

$$\frac{d}{d\tau}\left(\frac{\bar{\psi}_{\varphi\varphi}}{\bar{n}}\right)+S_{r\varphi\varphi}+S_{\theta\varphi\varphi}+S_{\varphi\varphi\varphi}=0\ldots\ldots \quad (167)$$

$\bar{\psi}_{rr}/\bar{n}$ and $\bar{\psi}_{\theta\theta}/\bar{n}$ represent the temperature in a direction perpendicular to the magnetic field of compression i.e. to B_φ; and $\bar{\psi}_{\varphi\varphi}/\bar{n}$ gives the temperature parallel to B_φ. So from this definition, it can be written that,

$$\bar{\psi}_{rr}=\bar{\psi}_{\theta\theta}=\psi_{\perp}\ldots\ldots \quad (168)$$

$$\psi_{\varphi\varphi}=\psi_{||}\ldots\ldots \quad (169)$$

By adding Eqs. 165 and 167, and substituting the equations given above yield,

$$\frac{d}{d\tau}\left(\frac{2\psi_{\perp}}{\bar{n}}\right)+\frac{2\psi_{\perp}}{\bar{n}}\left(\frac{1}{r}\frac{\partial(r\bar{v}_r)}{\partial r}+\frac{1}{r}\frac{\partial\bar{v}_{\theta}}{\partial\theta}\right)+S_{rrr}+S_{\theta rr}+S_{\varphi rr}$$

$$+S_{r\theta\theta}+S_{\theta\theta\theta}+S_{\varphi\theta\theta}=0\ldots\ldots \quad (170)$$

$$\frac{d}{d\tau}\left(\frac{\psi_{||}}{\bar{n}}\right)+S_{r\varphi\varphi}+S_{\theta\varphi\varphi}+S_{\varphi\varphi\varphi}=0\ldots\ldots \quad (171)$$

Equation 119 now can be utilized by expressing in cylindrical coordinates. First Eq. 113 is substituted into Eq. 119 to obtain,

$$\frac{d\bar{n}}{d\tau}+\bar{n}\nabla\bar{v}+2\bar{n}\left<\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}\right>=0\ldots\ldots \quad (172)$$

in cylindrical coordinates,

$$\frac{1}{r}\frac{\partial(r\bar{v}_r)}{\partial r}+\frac{1}{r}\frac{\partial\bar{v}_{\theta}}{\partial\theta}+\frac{\partial\bar{v}_{\varphi}}{\partial\varphi}=-\frac{1}{\bar{n}}\frac{d\bar{n}}{d\tau}-2\left<\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}\right>,\ldots\ldots \quad (173)$$

$\partial v_\varphi/\partial\varphi$ is zero as mentioned earlier. The remaining terms can now be substituted into Eq. 170.

$$\frac{d}{d\tau}\left(\frac{2\bar{\psi}_\perp}{\bar{n}}\right) - \frac{2\bar{\psi}_\perp}{\bar{n}^2}\frac{d\bar{n}}{d\tau} - \frac{4\bar{\psi}_\perp}{\bar{n}}\left<\frac{\partial^2 D}{\partial v(\tau)\partial v(t)}\right>$$

$$+ S_{rrr} + S_{\theta rr} + S_{\varphi rr} + S_{r\theta\theta} + S_{\theta\theta\theta} + S_{\varphi\theta\theta} = 0 \ldots\ldots \quad (174)$$

The second and the last terms of Eq. 156 are relatively small, therefore they can be omitted. Then the steady-state cases of Eq. 174 and Eq. 171 give,

$$\frac{d}{d\tau}\left(\frac{\bar{\psi}_\perp}{\bar{n}}\right) - \frac{\bar{\psi}_\perp}{\bar{n}^2}\frac{d\bar{n}}{d\tau} = 0 \ldots\ldots \quad (175)$$

$$\frac{d}{d\tau}\left(\frac{\bar{\psi}_{||}}{\bar{n}}\right) = 0 \ldots\ldots \quad (176)$$

The solution of Eq. 175 is given in Appendix J. From Eq. J-6 it can be written,

$$\frac{\bar{\psi}_\perp}{\psi_{\perp o}} = \frac{\bar{n}^2}{\bar{n}_o^2} \ldots\ldots \quad (177)$$

The solution of Eq. 176 gives,

$$\frac{\bar{\psi}_{||}}{\psi_{||o}} = \frac{\bar{n}}{\bar{n}_o} \ldots\ldots \quad (178)$$

According to Eq. 159, ψ and T can be related as,

$$\psi_{||} = \bar{n}T_{||} \ldots\ldots \quad (179)$$

$$\psi_\perp = \bar{n}T_\perp \ldots\ldots \quad (180)$$

Now substituting Eqs. 179 and 180 into Eqs. 177 and 178 successively, it is obtained that,

$$\frac{T_{\perp}}{T_{\perp_o}} = \frac{\bar{n}}{\bar{n}_o} \ldots\ldots \tag{181}$$

$$\frac{T_{||}}{T_{||_o}} = 1\ldots\ldots \tag{182}$$

Equation 181 is the important equation relating the temperature to the particle density in radial direction. In fact the experimental measurements of the temperature and the particle density were carried in the radial (i.e. perpendicular to the magnetic field) direction.

Equation 181 was plotted in Figs. 6, 7 and 8. First the energy curves given by Anashin, et al. [28] were plotted and then the experimental particle distributions were transposed on the same figures for three different discharge stages. The theoretical equation for particle distribution i.e. Eq. 155, was also plotted for each case for comparison.

From the inspection of Figs. 6, 7 and 8 it is seen that Eq. 181 agrees very well with the experimental results.

VIII. RESULTS

Equation 155 was solved for α = 11.5, α = 13.5 and α = 18.0. The values of $\bar{n}/\bar{n}_o$ with respect to the distance from the center of the torus are given in Table 1 for each α. These numerical values were used for plotting the theoretical results in Figs. 3-8 for the purpose of comparison between the predictions of Eqs. 155 and 181 with the experimental results.

Table 1. The calculated values of $\bar{n}/\bar{n}_o$

Distance from the center, cm	The values of $\bar{n}/\bar{n}_o$		
	α = 11.5	α = 13.5	α = 18.0
0	1.000	1.000	1.000
2	1.035	1.019	0.984
5	1.068	1.017	0.911
8	1.035	0.932	0.736
11	0.819	0.660	0.406
14	0.241	0.139	0.040
16	3.387×10^{-4}	5.165×10^{-5}	7.464×10^{-7}
17	0	0	0

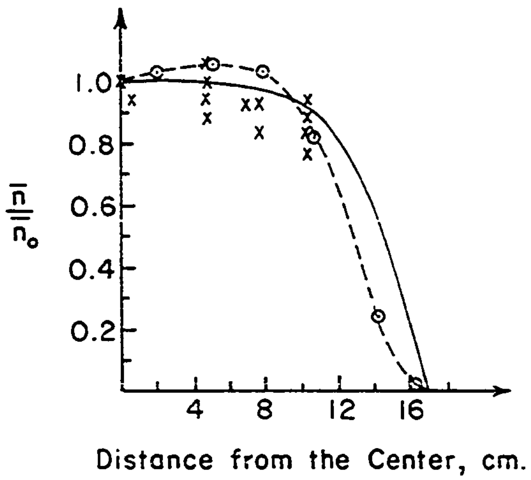

Fig. 3. Electron distribution at discharge stage of t = 4 msec. The points represent laser data and the solid curve shows the microwave measurements. The broken curve is obtained from Eq. 155 for α = 11.5

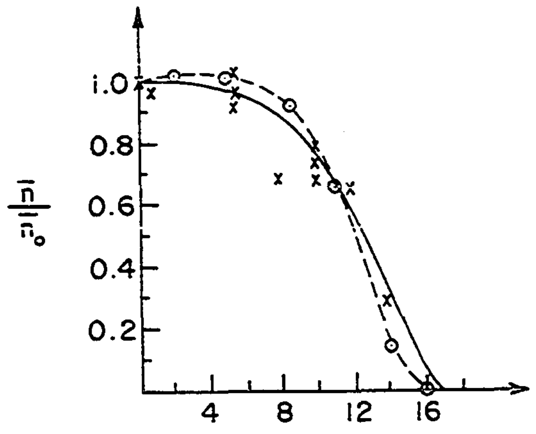

Fig. 4. Electron distribution at discharge stage of t = 16 msec. The points represent laser data and the solid curve shows the microwave measurements. The broken curve is obtained from Eq. 155 for α = 13.5

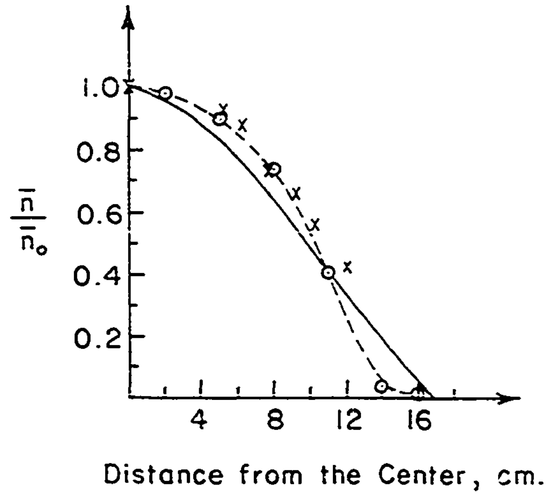

Fig. 5. Electron distribution at discharge stage of t = 30 msec. The points represent laser data and the solid curve shows the microwave measurements. The broken curve is obtained from Eq. 155 for α = 18

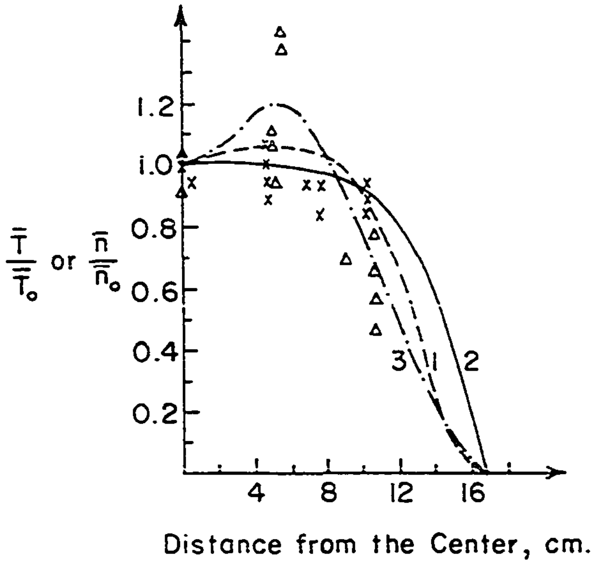

Fig. 6. Electron temperature and density distributions at discharge stage of t = 4 msec. The triangular points show laser data and curve #3 passes through these points. The rest are the same as in Fig. 3

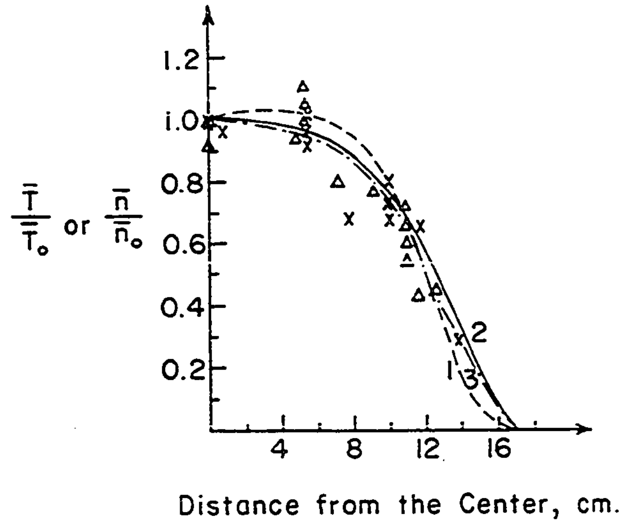

Fig. 7. Electron temperature and density distributions at discharge stage of t = 16 msec. The triangular points show laser data and curve #3 passes through these points. The rest are the same as in Fig. 4

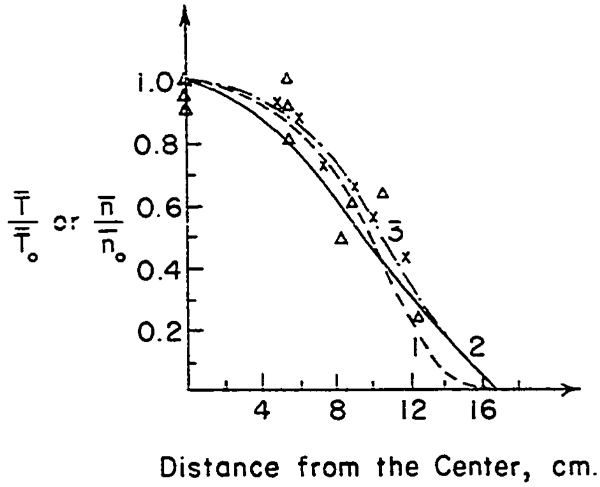

Fig. 8. Electron temperature and density distributions at discharge stage of t = 30 msec. The triangular points show laser data and curve #3 passes through these points. The rest are the same as in Fig. 5

IX. DISCUSSIONS

A. Discussion of the General Formulation

In the development of our strong turbulence equations we started with Boltzmann equation and separated it into average and fluctuating parts by using an averaging operator "A" which takes the statistical ensemble average of each term it operates on.

The solutions for the average and fluctuating distribution functions were obtained in terms of the time propagator $U(\tau, t)$. An important point to mention here is that $U(\tau, t)$ is a microscopic quantity while $\langle U(\tau, t) \rangle$ is a macroscopic one since it gives the statistically ensemble average of $U(\tau, t)$. Therefore in the kinetic equation for average particles (i.e. Eq. 57) where the time propagator is involved in the diffusional term "D", the contributions of the fluctuating fields are obtained in the form of statistical averages as it is seen from Eq. 64.

Since the linearization of the distribution function is avoided, the nonlinear effects gave rise to a diffusional term in strong turbulence equations. In fact it is this term which makes the real difference between the weak and the strong turbulence. The nonlinearities which can be caused by the fluctuation of the fields are expressed by this term as diffusional processes.

The solution of the very general equations for $\overline{f}$ and f' (i.e. Eq. 32 and Eqs. 31 or 37) is not easily obtained. For instance in order to solve Eq. 37 for f' an iteration process is needed. In order to have a convergent perturbation series of the iteration process, the propagator must be defined in such a way that it includes part of the nonlinearity of the system. The definition of the propagator in the literature was arbitrary as in the cases of Dupree [14], Weinstock [21] and Benford and Thomson [23]. This arbitrariness was removed by requiring the convergence condition to depend on the parameters of the plasma. According to Eq. 42 this dependence was related to the collision frequencies. This natural convergence condition actually comes from the starting equation i.e. the Boltzmann equation rather than the Vlasov equation which is the one used by all others.

B. Discussion of the Results

Equation 155 gives the distribution of particles along the minor radius of the torus. As it is seen from Eq. 153, the parameter α which appears in Eq. 155 is a function of time. So for different times, it will have different values.

When the Figs. 3, 4 and 5 are examined it is seen that the theoretical results obtained from Eq. 155 are in good agreement with the experimental results.

The experimental data were obtained by laser and microwave probing. But it is difficult to interpret which method gives the more reliable physical conditions. In Fig. 3 almost all points obtained from laser method lie below the curve obtained from microwave data. It is just the opposite in Fig. 5, while some of the points lie above and some below in Fig. 4. So it seems that depending on the time of the discharge stage laser and microwave methods give different values.

In the early times of the discharge stages the $\overline{n}/\overline{n}_o$ ratio is more than one as it is seen in Fig. 3. This situation is much better seen in Fig. 6. Since $\overline{T} \sim \overline{n}$ according to Eq. 181, so $\overline{T}/\overline{T}_o \sim \overline{n}/\overline{n}_o$ is more than one. Equation 155 describes this behavior very well as it can be seen from Figs. 3 and 6.

The phenomenological reason of this situation can be explained by considering the time of the discharge stage. At early times the electrons pushed by B_φ from the wall to the center may undergo collisions with other particles. As a result of collisions the electrons which go to the central region are retarded. This gives rise to a temporary increase of particle density in the region where collisions take place. Figures 3 and 6 indicate that 4 to 8 cm from the center corresponds to the maximum collision region for the discharge stage time of 4 msec.

Once the effect of collisions is considered, then Eq. 155 will not be very valid, because it was derived starting with Eq. 79 which does not include collisional term. The time dependent term of number density was also omitted in the derivation of Eq. 155. However the inspection of Figs. 4 and 5 tells that the collisional term can be omitted and a steady-state condition can be assumed when the discharge stage times increased to t = 16 msec (Fig. 4) and t = 30 msec (Fig. 5).

The experiments show that the Tokamak plasma is usually near Maxwellian. Therefore this is the basis of neglecting the collisional term while solving the continuity equation. The collisional term becomes less important as the discharge stage time increased. Because the collisions become less effective in altering the particle density as the time passes. So the distribution gets more Maxwellian with time. This is better seen when Fig. 5 is compared with Fig. 3 and even with Fig. 4. A similar comparison can be made between Fig. 8 and Fig. 6 or Fig. 7. At the longest time (i.e. Fig. 5 and Fig. 8) the theoretical curve gets closer to the experimental curve.

The data for the energy curves were obtained only from laser measurements. So a less confusing comparison can be made among Figs. 6, 7 and 8. As it is seen the experimental and the theoretical differences is the least in Fig. 8 and the largest in Fig. 6.

Another point observed in Figs. 3, 4 and 5 is that the theoretical distributions slightly fall below the experimental ones in about the one third portion of the torus close to the wall. This anomaly can be explained with the well-known neo-classical theory. It says that the presence of local magnetic mirrors coupled to the inevitable inhomogeneity of the toroidal magnetic field gives rise to the appearance of trapped particles. It was found that the difference in the trajectories of trapped and transit particles results in enhanced diffusion and thermal conductivity. Therefore the strong turbulence theory which does not include the neo-classical effect give smaller diffusion and thermal conductivity which in turn gives less number of particles and particle energies in the region where neo-classical effect makes contribution. Since the trapped particles are concentrated around the wall, the neo-classical effect becomes more important when one gets closer to the wall. That is why the slightly lower values of the theoretical results are obtained in this region.

In order to arrive at the result of $\overline{T} \sim \overline{n}$ as given by Eq. 181 some assumptions were made. The contribution of collisions is considerable for the case of t = 4 msec, as shown in Fig. 6. At early times of the discharge the collisions change the total energy of the particles. The last term of Eq. 156 gives this change. But all S terms of Eq. 174 were dropped while solving it for steady-state case. In fact the requirement for steady-state solution does not necessitate dropping the collisional term, but it was assumed to be small. Actually it has some considerable contribution at the early stages of the discharge.

Another term that was omitted is the first term i.e. heat flux term of Eq. 156. This is actually the well-known adiabatic law assumption. In other words, to assume $\partial \overline{Q}_i / \partial x_i$ to be zero means to assume adiabatic compression inside the plasma. Since in the torus the electrons are compressed to the center by the magnetic field B_φ, this compression was assumed to be adiabatic. So the result given in Eq. 181 as $\overline{T} \sim \overline{n}$ is the result of adiabatic compression perpendicular to magnetic field.

In fact, there is another criterion to assume the heat flux term to be zero. This comes from the isotropic property of the plasma. When the distribution function is isotropic the average quantity $\langle v_i' v_j' v_j' \rangle$ is zero. Hence Q_i becomes zero according to Eq. 108. Since the Tokamak plasma shows a near-Maxwellian distribution it can be said that the distribution function is close to being isotropic. This also justifies why the heat flux term can be omitted.

Under isotropic conditions the second, third, and fourth terms can also be dropped; while the sixth and seventh terms cancel each other.

From the inspection of Figs. 6, 7 and 8 it is seen that $\overline{T} \sim \overline{n}$ relation holds best in Fig. 8, which corresponds to the longest time of discharge. The results shown in Fig. 7 stands between those in Figs. 6 and 8 as expected.

X. CONCLUSIONS

1. A strong turbulence theory was developed starting with Boltzmann transport equation.

2. Time propagator technique is utilized in the derivation of the equations.

3. The convergence condition for the iteration series is imposed by employing the collision frequencies of the average and the fluctuating particles.

4. The general moment equations were derived.

5. The continuity and energy equations were obtained for Tokamak thermonuclear devices.

6. The particle and energy distributions in these devices were predicted by solving the continuity and the energy equations under steady-state and collisionless conditions.

7. At early stages of discharge times, the effect of collisions becomes relatively important.

8. The adiabatic compression assumption is a very good approximation to relate the density to the energy of the particle.

9. The comparison of the theoretical predictions with experimental results shows a very good agreement.

10. The assumptions made become more sound as the discharge stage time increases.

XI. FUTURE STUDIES

The toroidal thermonuclear reactors probably will be the energy generators in the near future. So there will be more studies on the particle and energy transport mechanisms in these devices. The equations derived in this research work can be solved also for time dependent and collisional cases. This can be achieved through numerical solution by using computers.

The transport coefficients also can be obtained by making use of the equations derived.

It is worth mentioning that Eq. 77 is the most general form of moment equations for strong turbulence. So depending on the physical conditions of the problem some assumptions can be imposed on Eq. 77 and be used anywhere strong turbulence involved to predict particle, momentum, and energy transport mechanisms.

Since the strong turbulence equations contain the nonlinearities in the time propagator many instability problems can be handled by a better technique.

XII. LITERATURE CITED

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XIV. APPENDIX A. THE SIMPLIFICATIONS OF EQ. 27 AND EQ. 29

By substituting Eq. 30 into Eq. 27 we obtain,

$$\frac{\partial f'}{\partial t} = -(1-A)(\bar{L}+L')\bar{f} - (1-A)(\bar{L}+L')f' + \nu' f' \ldots\ldots \quad \text{(A-1)}$$

$$= -\overline{\bar{L}\bar{f}} - L'\bar{f} + A\overline{\bar{L}\bar{f}} + AL'\bar{f} - \bar{L}f' - L'f' + A\bar{L}f' + AL'f' + \nu' f'$$

$$= -\overline{\bar{L}\bar{f}} - L'\bar{f} + \langle\overline{\bar{L}\bar{f}}\rangle + \langle L'\bar{f}\rangle + \bar{L}f' - L'f'$$

$$+ \langle\bar{L}f'\rangle + \langle L'f'\rangle + \nu' f'$$

$$= -\overline{\bar{L}\bar{f}} - L'\bar{f} + \overline{\bar{L}\bar{f}} + \langle L'\rangle\bar{f} + \bar{L}f' - L'f'$$

$$+ \bar{L}\langle f'\rangle + \langle L'f'\rangle + \nu' f'$$

Since $\langle f'\rangle = 0$ and $\langle L'\rangle = 0$,

$$\frac{\partial f'}{\partial t} = -L'\bar{f} - \bar{L}f' - L'f' + \langle L'f'\rangle + \nu' f' \ldots\ldots \quad \text{(A-2)}$$

$$\frac{\partial f'}{\partial t} = -L'\bar{f} - \bar{L}f' - (1-A)L'f' + \nu' f' \ldots\ldots \quad \text{(A-3)}$$

Now by substituting Eq. 30 into Eq. 29 it is obtained that,

$$\frac{\partial \bar{f}}{\partial t} = -A(\bar{L}+L')\bar{f} - A(\bar{L}+L')f' + \overline{\nu f} \ldots\ldots \quad \text{(A-4)}$$

$$= -A\overline{\bar{L}\bar{f}} - AL'\bar{f} - A\bar{L}f' - AL'f' + \overline{\nu f}$$

$$= -\langle\overline{\bar{L}\bar{f}}\rangle - \langle L'\bar{f}\rangle - \langle\bar{L}f'\rangle - \langle L'f'\rangle + \overline{\nu f}$$

$$= -\overline{\bar{L}\bar{f}} - \langle L'\rangle\bar{f} - \bar{L}\langle f'\rangle - \langle L'f'\rangle + \overline{\nu f}$$

$$\frac{\partial \bar{f}}{\partial t} = -\overline{\bar{L}\bar{f}} - AL'f' + \overline{\nu f} \ldots\ldots \quad \text{(A-5)}$$

XV. APPENDIX B. THE INTEGRATION OF EQ. 36

The term on the left side of Eq. 36 can be written as,

$$U(\tau,\ t)\ \frac{\partial f'(t)}{\partial t} = \frac{\partial}{\partial t}\ (U(\tau,\ t)f'(t)) - (\frac{\partial U(\tau,\ t)}{\partial t}) \cdot f'(t)\ldots\ldots \tag{B-1}$$

Equation 36 now becomes,

$$\frac{\partial}{\partial t}\ (U(\tau,\ t)f'(t)) = (\frac{\partial U(\tau,\ t)}{\partial t})\ f'(t) - U(\tau,\ t)L'(t)\overline{f}(t)$$

$$- U(\tau,\ t)\overline{L}(t)f'(t) - (1 - A)U(\tau,\ t)L'(t)f'(t)$$

$$+ U(\tau,\ t)\nu'f'(t)\ldots\ldots \tag{B-2}$$

The third term on the right side can be written as,

$$U(\tau,\ t)\overline{L}(t)f'(t) = \overline{L}(t)U(\tau,\ t)f'(t) - (\overline{L}(t)U(\tau,\ t))f'(t)\ldots\ldots \tag{B-3}$$

According to Eq. 35,

$$U(\tau,\ t)f'(t) = f'(\tau)\ldots\ldots \tag{B-4}$$

then it is seen that,

$$L(t)U(\tau,\ t)f'(t) = L(t)f'(\tau) = 0\ldots\ldots \tag{B-5}$$

So, Eq. B-3 becomes,

$$U(\tau,\ t)\overline{L}(t)f'(t) = -(\overline{L}(t)U(\tau,\ t))f'(t)\ldots\ldots \tag{B-6}$$

Substituting Eq. B-6 into B-2, it is obtained that,

$$\frac{\partial}{\partial t}\ U(\tau,\ t)f'(t) = [\frac{\partial U(\tau,\ t)}{\partial t} + \overline{L}(t)U(\tau,\ t)]f'(t) - U(\tau,\ t)L'(t)\overline{f}(t)$$

$$- (1 - A)U(\tau,\ t)L'(t)f'(t) + U(\tau,\ t)\nu'f'(t)\ldots\ldots \tag{B-7}$$

The first two terms on the right side can be substituted by $\bar{\nu}U(\tau, t)$ as follows from Eq. 33 which is,

$$\frac{\partial U(\tau, t)}{\partial t} + \bar{L}U(\tau, t) = \bar{\nu}U(\tau, t) \ldots\ldots \qquad \text{(B-8)}$$

Hence Eq. B-7 now becomes,

$$\frac{\partial}{\partial t}(U(\tau, t)f'(t)) = - U(\tau, t)L'(t)\bar{f}(t) - (1 - A)U(\tau, t)L'(t)f'(t) + U(\tau, t)(\bar{\nu} + \nu')f'(t) \ldots\ldots \qquad \text{(B-9)}$$

The integration can be carried out with the boundary conditions of,

As $t \to t_o$ $\quad f'(t) \to f'(t_o)$

$t \to t$ $\quad f'(t) \to f'(t)$

The integrated form of Eq. B-9 can now be written as,

$$f'(\tau) = U(\tau, t_o)f'(t_o) - \int dt\, U(\tau, t)L'(t)\bar{f}(t) - \int dt(1 - A)U(\tau, t)L'(t)f'(t) + \int dt\, U(\tau, t)(\bar{\nu} + \nu')f'(t) \ldots\ldots \qquad \text{(B-10)}$$

XVI. APPENDIX C. THE EVALUATION OF THE FIRST TERM OF EQ. 59

The first term of Eq. 59 is evaluated as follows,

$$\int \phi \frac{\partial \bar{f}}{\partial \tau} dv = \int \frac{\partial}{\partial \tau} (\phi \bar{f}) dv - \int \bar{f} \frac{\partial \phi}{\partial \tau} dv \ldots\ldots \quad \text{(C-1)}$$

From the definition, the mean value of any function F is found as,

$$\langle F \rangle = \frac{\int F f dv}{\int f dv} = \frac{\int F f dv}{n} \ldots\ldots \quad \text{(C-2)}$$

where n is the particle density. So Eq. C-1 can now be written as,

$$\int \phi \frac{\partial \bar{f}}{\partial \tau} dv = \frac{\partial}{\partial \tau} \langle \phi \bar{n} \rangle - \bar{n} \langle \frac{\partial \phi}{\partial \tau} \rangle \ldots\ldots \quad \text{(C-3)}$$

where $\bar{n}$ is the density of the nonfluctuating particles.

XVII. APPENDIX D. THE EVALUATION OF THE SECOND TERM OF EQ. 59

The second term of Eq. 59 is evaluated as follows,

$$\int \phi \, v_i \frac{\partial \bar{f}}{\partial x_i} \, dv = \int v_i \frac{\partial}{\partial x_i} (\phi \bar{f}) dv - \int v_i \bar{f} \frac{\partial \phi}{\partial x_i} dv \ldots\ldots \quad \text{(D-1)}$$

$$= \int \frac{\partial}{\partial x_i} (v_i \phi \bar{f}) dv - \int \phi \bar{f} \frac{\partial v_i}{\partial x_i} \, dv - \int v_i \bar{f} \frac{\partial \phi}{\partial x_i} \, dv \ldots\ldots \quad \text{(D-2)}$$

Since x and v are independent variables the second term drops,

$$\int \phi v_i \frac{\partial \bar{f}}{\partial x_i} \, dv = \int \frac{\partial}{\partial x_i} (v_i \phi \bar{f}) dv - \int v_i \bar{f} \frac{\partial \phi}{\partial x_i} \, dv \ldots\ldots \quad \text{(D-3)}$$

$$\int \phi v_i \frac{\partial \bar{f}}{\partial x_i} \, dv = \frac{\partial}{\partial x_i} (\bar{n} \langle v_i \phi \rangle) - \bar{n} \langle v_i \frac{\partial \phi}{\partial x_i} \rangle \ldots\ldots \quad \text{(D-4)}$$

XVIII. APPENDIX E. THE EVALUATION OF THE THIRD TERM OF EQ. 59

The third term of Eq. 59 is evaluated as follows,

$$\int \phi \bar{R}_i \frac{\partial \bar{f}}{\partial v_i} dv = \int \bar{R}_i \frac{\partial}{\partial v_i} (\phi \bar{f}) dv - \int \bar{R}_i \bar{f} \frac{\partial \phi}{\partial v_i} dv \ldots\ldots \qquad \text{(E-1)}$$

$$= \int \frac{\partial}{\partial v_i} (\phi \bar{f} \bar{R}_i) dv - \int \phi \bar{f} \frac{\partial \bar{R}_i}{\partial v_i} dv - \int \bar{R}_i \bar{f} \frac{\partial \phi}{\partial v_i} dv \ldots\ldots \qquad \text{(E-2)}$$

The first term on the right side is integrated as,

$$\int \frac{\partial}{\partial v_i} (\phi \overline{fR}_i) dv = \phi \overline{fR}_i \Big]_{-\infty}^{+\infty} = 0 \ldots\ldots \qquad \text{(E-3)}$$

This result is achieved by considering the fact that $\bar{f}$ goes to zero at $\pm\infty$. In other words, it is assumed that no molecules have infinite velocities.

Using Eq. 52 the second term on the right side of Eq. E-2 can be written as follows,

$$\int \phi \bar{f} \frac{\partial \bar{R}_i}{\partial v_i} dv = \int \phi \bar{f} \frac{\partial}{\partial v_i} \frac{q}{m} (\bar{E}_i + v \times \bar{B}_i) dv \ldots\ldots \qquad \text{(E-4)}$$

$$= \frac{q}{m} \int \phi \bar{f} \left[\frac{\partial \bar{E}_i}{\partial v_i} + \frac{\partial}{\partial v_i} (v \times \bar{B}_i) \right] dv \ldots\ldots \qquad \text{(E-5)}$$

The fields $\bar{E}$ and $\bar{B}$ are independent of velocity by definition, so that,

$$\frac{\partial \bar{E}}{\partial v} = 0 \ldots\ldots \qquad \text{(E-6)}$$

The magnetic field term in Eq. E-5 can be written as,

$$\frac{\partial}{\partial v_i} (v \times B) = \frac{\partial}{\partial v_i} (\varepsilon_{ijk} v_j \bar{B}_k) \ldots\ldots \qquad \text{(E-7)}$$

$$\frac{\partial v_j}{\partial v_i} = \delta_{ij}$$

so,

$$\frac{\partial}{\partial v_i}(v \times \overline{B}) = \varepsilon_{ijk}\delta_{ij}\overline{B}_k \ldots\ldots \quad \text{(E-8)}$$

$$= \varepsilon_{iik}B_k$$

$$= 0 \ldots\ldots \quad \text{(E-9)}$$

With the results of Eqs. E-3, E-6 and E-9 only the third term remains on the right of Eq. E-1, so that,

$$\int \phi \overline{R}_i \frac{\partial \overline{f}}{\partial v_i}\, dv = -\int \overline{R}_i \overline{f} \frac{\partial \phi}{\partial v_i}\, dv \ldots\ldots \quad \text{(E-10)}$$

$$\int \phi \overline{R}_i \frac{\partial \overline{f}}{\partial v_i}\, dv = -\,\overline{n} \langle \overline{R}_i \frac{\partial \phi}{\partial v_i} \rangle \ldots\ldots \quad \text{(E-11)}$$

XIX. APPENDIX F. THE EVALUATION OF $D_{B'B'}$

$D_{B'B'}$ can be written as follows,

$$D_{B'B'} = \frac{q^2}{m^2}\int dt \frac{\partial^2}{\partial v_i^2} < (v \times B')(v \times B') > \ldots\ldots \qquad \text{(F-1)}$$

$$\frac{\partial^2}{\partial v_i^2} (v \times B')(v \times B') = \frac{\partial^2}{\partial v_i^2} \begin{vmatrix} i & j & k \\ v_i & v_j & v_k \\ B'_i & B'_j & B'_k \end{vmatrix} \begin{vmatrix} i & j & k \\ v_i & v_j & v_k \\ B'_i & B'_j & B'_k \end{vmatrix} \ldots\ldots \qquad \text{(F-2)}$$

$$= \frac{\partial^2}{\partial v_i^2} [(v_j B'_k - v_k B'_j)^2 + (v_i B'_k - v_k B'_i)^2 + (v_i B'_j - v_j B'_i)^2]$$

$$= 2(B_j'^2 + B_k'^2) \ldots\ldots \qquad \text{(F-3)}$$

Therefore Eq. F-1 now becomes,

$$D_{B'B'} = 2\frac{q^2}{m^2}\int dt < B_j'^2 + B_k'^2 > \ldots\ldots \qquad \text{(F-4)}$$

XX. APPENDIX G. DEPENDENCE OF B' ON RADIUS

The plasma current in the torus runs only when the poloidal magnetic flux is changing. But the fluctuation in B_θ is much smaller than the fluctuation in B_φ. Therefore B'_θ can be neglected in Eq. 147.

B'_φ can be expressed in terms of radius using the Maxwell equations

$$\nabla \times B' = \frac{4\pi}{c} j' + \frac{1}{c}\frac{\partial E'}{\partial t} \quad \ldots\ldots \tag{G-1}$$

Assuming E' to be independent of time and integrating over an area element dA, it is obtained that,

$$\int \nabla \times B' dA = \frac{4\pi}{c} \int j' dA \ldots\ldots \tag{G-2}$$

The first term reduces to a contour integral,

$$\oint B' d\ell = \frac{4\pi}{c} \int j' dA \ldots\ldots \tag{G-3}$$

The integral over j'dA is constant. Then Eq. G-3 can be written as,

$$\oint B' r d\theta = Cst \ldots\ldots \tag{G-4}$$

then,

$$B' = \frac{Cst}{r} \quad \ldots\ldots \tag{G-5}$$

XXI. APPENDIX H. DECOMPOSITION OF EQ. 160 INTO ITS COMPONENTS

Equation 160 can be written in x, y, and z components as follows,

$$\frac{d}{d\tau}\left(\frac{\bar{\psi}_{xx}}{\bar{\bar{n}}}\right) + \frac{2}{\bar{\bar{n}}}\left(\bar{\psi}_{xx}\frac{\partial \bar{v}_x}{\partial x} + \bar{\psi}_{yx}\frac{\partial \bar{v}_x}{\partial y} + \bar{\psi}_{zx}\frac{\partial \bar{v}_x}{\partial z}\right)$$

$$+ S_{xxx} + S_{yxx} + S_{zxx} = 0\ldots\ldots \qquad \text{(H-1)}$$

$$\frac{d}{d\tau}\left(\frac{\bar{\psi}_{yy}}{\bar{\bar{n}}}\right) + \frac{2}{\bar{\bar{n}}}\left(\bar{\psi}_{xy}\frac{\partial \bar{v}_y}{\partial x} + \bar{\psi}_{yy}\frac{\partial \bar{v}_y}{\partial y} + \bar{\psi}_{zy}\frac{\partial \bar{v}_y}{\partial z}\right)$$

$$+ S_{xyy} + S_{yyy} + S_{zyy} = 0\ldots\ldots \qquad \text{(H-2)}$$

$$\frac{d}{d\tau}\left(\frac{\bar{\psi}_{zz}}{\bar{\bar{n}}}\right) + \frac{2}{\bar{\bar{n}}}\left(\bar{\psi}_{xz}\frac{\partial \bar{v}_z}{\partial x} + \bar{\psi}_{yz}\frac{\partial \bar{v}_z}{\partial y} + \bar{\psi}_{zz}\frac{\partial \bar{v}_z}{\partial z}\right)$$

$$+ S_{xzz} + S_{yzz} + S_{zzz} = 0\ldots\ldots \qquad \text{(H-3)}$$

XXII. APPENDIX I. DECOMPOSITION OF EQ. 157 INTO ITS COMPONENTS WHEN $j \neq k$

When $j \neq k$ Eq. 157 can be written in its components as follows,

$$\frac{d}{d\tau}\left(\frac{\bar{\psi}_{xy}}{\bar{n}}\right) + \frac{\bar{\psi}_{xy}}{\bar{n}}\left(\frac{\partial \bar{v}_x}{\partial x} + \frac{\partial \bar{v}_y}{\partial y}\right) + \frac{\bar{\psi}_{xx}}{\bar{n}}\frac{\partial \bar{v}_y}{\partial x} + \frac{\bar{\psi}_{yy}}{\bar{n}} \cdot \frac{\partial \bar{v}_x}{\partial y} + \frac{\bar{\psi}_{xz}}{\bar{n}}\frac{\partial \bar{v}_y}{\partial z} + \frac{\bar{\psi}_{yz}}{\bar{n}}\frac{\partial \bar{v}_x}{\partial z} + S_{xxy} + S_{yxy} + S_{zxy} \ldots\ldots \qquad \text{(I-1)}$$

$$\frac{d}{d\tau}\left(\frac{\bar{\psi}_{yz}}{\bar{n}}\right) + \frac{\bar{\psi}_{yz}}{\bar{n}}\left(\frac{\partial \bar{v}_y}{\partial y} + \frac{\partial \bar{v}_z}{\partial z}\right) + \frac{\bar{\psi}_{yy}}{\bar{n}}\frac{\partial \bar{v}_z}{\partial y} + \frac{\bar{\psi}_{zz}}{\bar{n}}\frac{\partial \bar{v}_y}{\partial z} + \frac{\bar{\psi}_{yx}}{\bar{n}}\frac{\partial \bar{v}_z}{\partial x} + \frac{\bar{\psi}_{zx}}{\bar{n}}\frac{\partial \bar{v}_y}{\partial x} + S_{xyx} + S_{yyz} + S_{zyz} \ldots\ldots \qquad \text{(I-2)}$$

$$\frac{d}{d\tau}\left(\frac{\bar{\psi}_{zx}}{\bar{n}}\right) + \frac{\bar{\psi}_{zx}}{\bar{n}}\left(\frac{\partial \bar{v}_z}{\partial z} + \frac{\partial \bar{v}_x}{\partial x}\right) + \frac{\bar{\psi}_{zz}}{\bar{n}}\frac{\partial \bar{v}_x}{\partial z} + \frac{\bar{\psi}_{xx}}{\bar{n}} \cdot \frac{\partial \bar{v}_z}{\partial x} + \frac{\bar{\psi}_{zy}}{\bar{n}}\frac{\partial \bar{v}_x}{\partial y} + \frac{\bar{\psi}_{xy}}{\bar{n}}\frac{\partial \bar{v}_z}{\partial y} + S_{xzx} + S_{yzx} + S_{zzx} \ldots\ldots \qquad \text{(I-3)}$$

XXIII. APPENDIX J. THE EVALUATION OF EQ. 175

Equation 175 can be written as,

$$\frac{d}{d\tau}\left(\frac{\bar{\psi}_\perp}{\bar{\bar{n}}}\right) - \frac{\bar{\psi}_\perp}{\bar{n}^2}\frac{d\bar{n}}{d\tau} = \frac{1}{\bar{\bar{n}}}\frac{d\bar{\psi}_\perp}{d\tau} - \frac{\bar{\psi}_\perp}{\bar{n}^2}\frac{d\bar{n}}{d\tau} - \frac{\bar{\psi}_\perp}{\bar{n}^2}\frac{d\bar{n}}{d\tau}$$

$$= \frac{1}{\bar{\bar{n}}}\frac{d\bar{\psi}_\perp}{d\tau} - \frac{2\bar{\psi}_\perp}{\bar{n}^2}\frac{d\bar{n}}{d\tau} \; ..... \qquad \text{(J-1)}$$

Since it is equal to zero,

$$\frac{1}{\bar{\bar{n}}}\frac{d\bar{\psi}_\perp}{d\tau} = \frac{2\bar{\psi}_\perp}{\bar{n}^2}\frac{d\bar{n}}{d\tau} \; ..... \qquad \text{(J-2)}$$

$$\frac{d\bar{\psi}_\perp}{\bar{\psi}_\perp} = 2\,\frac{d\bar{n}}{\bar{n}} \; ..... \qquad \text{(J-3)}$$

$$\bar{\psi}_\perp = \text{Const.}\ \bar{n}^2 \; ..... \qquad \text{(J-4)}$$

When $\bar{n} = \bar{n}_o$

$$\bar{\psi}_\perp = \bar{\psi}_{\perp o}$$

So,

$$\text{Const.} = \frac{\bar{\psi}_{\perp o}}{\bar{\bar{n}}_o} \; ..... \qquad \text{(J-5)}$$

Now Eq. J-4 becomes,

$$\frac{\bar{\psi}_\perp}{\bar{\psi}_{\perp o}} = \frac{\bar{n}^2}{\bar{\bar{n}}_o^2} \; ..... \qquad \text{(J-6)}$$