

Zero mode correction for the critical coupling in (1+1)-dimensional ϕ^4 theory

by

Mengyao Huang

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Program of Study Committee:
James P. Vary, Major Professor
Chunhui Chen
Glenn R. Luecke
Kirill Tuchin

The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this thesis. The Graduate College will ensure this thesis is globally accessible and will not permit alterations after a degree is conferred.

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DEDICATION

I would like to dedicate this thesis to my parents for their allowing me to be far away from them pursuing my passion in science. And to my friends for their love and connecting with me as I am working at home during this pandemic.

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ABSTRACT

Evaluating the effect of the zero momentum mode in the discretized light cone quantization (DLCQ) approach for light front field theory is a long standing problem. Using 1+1 dimension ϕ^4 theory, we compare the critical coupling calculated in light front with zero mode excluded and included. The critical coupling without zero mode has been obtained by solving the theory in DLCQ, and the critical coupling with zero mode included was recently obtained by solving the theory in light front using a symmetric polynomial basis which was claimed to circumvent the zero mode problem ([Burkardt et al. \(2016\)](#)). The critical coupling from these two methods can be compared, and a conclusion can be drawn on whether the zero mode has a significant effect for the DLCQ critical coupling result. We then further compare the zero mode included and excluded cases after their critical couplings are converted to the corresponding values in the equal time scheme. Finally, we discuss the consistency of these converted values with the critical coupling obtained by equal time quantization approaches in literature.

CHAPTER 1. GENERAL INTRODUCTION

In modern physics, quantum field theory is the basic mathematical framework to describe elementary particles. In order to solve it computationally, one can discretize quantum field theory in equal-time frame or in light-front frame. Although equal time quantization is more commonly used, light front quantization has its unique advantages. Light front coordinates offer a clean separation between external and internal momenta, and the quantization can keep the vacuum trivial for on-mass shell massive particles. The zero mode – the lowest state in Fock space states expansion, is usually very complicated in equal time, but takes a relatively simple form in light front, which facilitates calculations in some cases.

However, there exist subtle problems in light front quantum field theory that require thorough investigation. For example, in the broken phase of the ϕ^4 theory, the physical mass can go to zero producing a degeneracy with the simple vacuum. This signals a spontaneous symmetry breaking and implies that the effects of a field zero mode should be taken into account. The critical coupling for the light-front approach with zero mode included and the light-front approach without zero mode might be different and can be calculated as shown in this work. This kind of spontaneous symmetry breaking in ϕ^4 theory also occurs in the Higgs mechanism. Since ϕ^4 theory is the simplest relativistic system in which we can see spontaneous symmetry breaking, it is an ideal test case for the various approaches to scalar zero modes on the light front.

The discretized light cone quantization (DLCQ) ([Harindranath and Vary \(1987\)](#)) is one of the approaches previously established to solve the quantum field theory in light front form. This approach generally only considers non-zero modes. With recently developed symmetric polynomial approach in [Burkardt et al. \(2016\)](#), zero modes can be included in light-front calculations as we survey in this work.

Our goal is to compare the critical coupling obtained with and without the light-front field mode and draw a conclusion of whether or not the zero mode has an effect on the critical coupling. We also calculate a correction to the light front critical coupling needed to compare with the critical coupling obtained by equal time approaches.

CHAPTER 2. OVERVIEW OF THE ZERO MODE PROBLEM

2.1 Introduction

The derivations of the key elements of 1+1 dimension ϕ^4 field theory in this chapter mainly follow [Bender et al. \(1993\)](#), [Pinsky and van de Sande \(1994\)](#), [Pinsky et al. \(1995\)](#) and additional literature reviewing the light-front field theory, such as [Harindranath \(2000\)](#) and [Burkardt \(2002\)](#).

2.2 Zero Modes in Equal Time and Light Front Time

In a Fock space expansion, one can build up the states by applying creation operators on ground state vacuum. In a free theory, the ground state is the pure vacuum. While in an interacting theory, the vacuum can involve fluctuations since the ground state of the Hamiltonian is not known *a priori*. The pure vacuum mixes with states which have zero total momentum involving various excitations. For example, in equal-time coordinates, the vacuum is a complicated state with total 3-momentum $\vec{P} = 0$ which involves creation of, in principle, arbitrary numbers of particles with compensating 3-momenta \vec{k}_i ,

$$|0\rangle_{\text{equal time}} = \sum_{\text{all possible } N, \{\vec{k}_i\}} \left(\prod_i^N a_{\vec{k}_i}^\dagger \right) \delta(\sum_i \vec{k}_i) |\text{vac}\rangle. \quad (2.1)$$

One can convert the equal time coordinates (x^0, x^3, x^\perp) to light front coordinates (x^+, x^-, x^\perp) by definition $x^\pm \equiv x^0 \pm x^3$, where $x^\perp = (x^1, x^2)$. x^+ is called light front time, and x^- light front longitudinal coordinate. Denoting the equal time coordinates as x^μ and the light front coordinates as $x^{\mu'}$, we have

$$\Lambda_{\mu}^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2.2)$$

The light-front metric is thus

$$g^{\mu'\nu'} = \Lambda_{\mu}^{\mu'} g^{\mu\nu} \Lambda_{\nu}^{\nu'} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (2.3)$$

Since $g^{\mu'\nu'} g_{\mu'\nu'} = 1$,

$$g_{\mu'\nu'} = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (2.4)$$

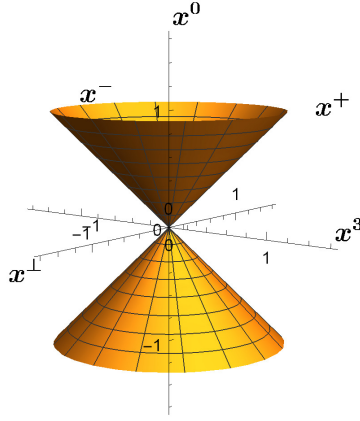


Figure 2.1 Illustration of light-front coordinates

The scalar product of the energy-momentum 4-vector and the space-time 4-vector is

$$k \cdot x = \frac{1}{2} k^+ x^- + \frac{1}{2} k^- x^+ - k^\perp \cdot x^\perp, \quad (2.5)$$

It is observed that k^- is conjugate to x^+ , the time, so we called it light front energy. k^+ which is conjugate to x^- , the longitudinal coordinate, is the light front longitudinal momentum. Using the fact that a scalar is invariant under coordinate transformations, we can expand the above expression

to get a relation between light front energy/momentum and equal time energy/momentum,

$$\begin{aligned}
k \cdot x &= \frac{1}{2}k^+(x^0 - x^3) + \frac{1}{2}k^-(x^0 + x^3) - k^\perp \cdot x^\perp \\
&= \frac{1}{2}(k^+ + k^-)x^0 - \frac{1}{2}(k^+ - k^-)x^3 - k^\perp \cdot x^\perp \\
&= k^0x^0 - k^3x^3 - k^\perp \cdot x^\perp,
\end{aligned} \tag{2.6}$$

thus $k^\pm = k^0 \pm k^3$.

Another useful scalar product gives us the invariant mass

$$k^2 = \frac{1}{2}k^+k^- + \frac{1}{2}k^-k^+ - (k^\perp)^2 = k^+k^- - (k^\perp)^2 = \mu^2 \tag{2.7}$$

which is the light front dispersion relation.

In light-front coordinates, the vacuum is defined as the state with total longitudinal light front momentum $P^+ = 0$. Since $k^\pm = k^0 \pm k^3$, for an on-mass shell particle of mass μ , $k^0 > k^3$ gives the light front energy $k^- > 0$. From the light front dispersion relation (2.7)

$$k^+ = \frac{(k^\perp)^2 + \mu^2}{k^-}, \tag{2.8}$$

the light front momentum should satisfy $k^+ \geq 0$ for $\mu^2 \geq 0$. Therefore, the only states that give $P^+ = 0$ are the states with all $k^+ = 0$ (zero modes)

$$|0\rangle_{\text{light front}} = \sum_N \left(\prod_i^N a_{0_i}^\dagger \right) |\text{vac}\rangle. \tag{2.9}$$

So in light front coordinates, the Fock space vacuum only contains the pure vacuum and the zero modes. This provides several benefits: First, if k^- is strictly > 0 and $\mu^2 > 0$, then $k^+ > 0$ and there are no zero mode quanta so the light-front vacuum is the trivial vacuum. Second, it may appear that one can sidestep the zero mode problem if one can restrict the modes to those with $k^+ > 0$, such as the case when one adopts anti-periodic boundary conditions. However, one still needs to confirm that resulting observables are independent of the chosen boundary conditions.

2.3 (1+1)-dimensional ϕ^4 Theory and the Zero Mode

Consider the 1+1 dimensional scalar field theory ϕ^4 which can be used to study the role of zero modes as will be shown here. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \partial^+ \phi \partial^- \phi - \frac{\mu^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4. \quad (2.10)$$

From the least action principle $\delta S = 0$ and $S = \int d^4 x \mathcal{L}(x)$,

$$\begin{aligned} \delta S &= \delta \int d^4 x \mathcal{L}(x) = \int d^4 x \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \delta(\partial^\mu \phi) + \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi \right] \\ &= \int d^4 x \left[\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \right) - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \right) \delta \phi + \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi \right] = 0, \end{aligned} \quad (2.11)$$

the general scalar field equation of motion is

$$\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}. \quad (2.12)$$

In 1+1 dimension ϕ^4 theory,

$$\partial^+ \left(\frac{\partial \mathcal{L}}{\partial(\partial^+ \phi)} \right) + \partial^- \left(\frac{\partial \mathcal{L}}{\partial(\partial^- \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}, \quad (2.13)$$

which is

$$\partial^+ \partial^- \phi + \mu^2 \phi + \frac{\lambda}{3!} \phi^3 = 0. \quad (2.14)$$

The field can be rewritten as

$$\phi(x^+, x^-) = \phi_0(x^+) + \Phi(x^+, x^-), \quad (2.15)$$

where ϕ_0 is the zero mode and Φ contains non-zero modes. ϕ_0 is independent of x^- since its $k^+ = 0$.

In the domain of $-L \leq x^- \leq L$, with periodic boundary condition $\phi(x^+, -L) = \phi(x^+, +L)$ and $\partial^- \phi(x^+, -L) = \partial^- \phi(x^+, +L)$, we have $\int_{-L}^L dx^- \partial^+ \partial^- \phi = 0$. We also use the fact that $\int_{-L}^L dx^- \Phi = 0$, since the mode expansion of Φ only contains $k^+ \neq 0$, resulting in a factor $\int dx^- e^{ik^+ x^-} = \delta(k^+) = 0$. Integrating the equation of motion we get

$$\mu^2 \int_{-L}^L dx^- \phi + \frac{\lambda}{3!} \int_{-L}^L dx^- \phi^3 = 0. \quad (2.16)$$

Substitute into the field with zero and non-zero modes separated,

$$\phi_0 = -\frac{1}{2L} \frac{\lambda}{\mu^2} \frac{1}{3!} \left[2L\phi_0 + \int_{-L}^L dx^- (\phi_0 \Phi^2 + \Phi^2 \phi_0 + \Phi \phi_0 \Phi + \Phi^3) \right]. \quad (2.17)$$

In the free theory, $\lambda = 0$, the zero mode vanishes. When $\lambda \neq 0$, the zero mode is a dependent field that depends on non-zero modes. In the classical theory, the zero mode can be determined in principle by solving an algebraic equation. In the quantum theory, however, one has to solve a non-linear operator equation.

2.4 The Lagrangian with Zero Mode Included

In a box of $-L \leq x^- \leq L$, we impose periodic boundary conditions. The field expansion takes the form of

$$\phi(x) = \frac{1}{\sqrt{2L}} \sum_n q_n(x^+) e^{-ik_n^+ x^-/2}, \quad (2.18)$$

where $k_n^+ = \frac{2\pi n}{L}$ and $n = 0, \pm 1, \pm 2, \dots$.

The Lagrangian L is

$$\begin{aligned} \int dx^- \mathcal{L} &= \int dx^- \left(2\partial_+ \phi \partial_- \phi - \frac{\mu^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 \right) \\ &= \int dx^- \left(\frac{2}{2L} \sum_{n,m} \left(-\frac{ik_n^+}{2} \right) q_n \dot{q}_m e^{-i(k_n^+ + k_m^+) x^-/2} \right) - \int dx^- \frac{\mu^2}{4L} \sum_{n,m} q_n q_m e^{-i(k_n^+ + k_m^+) x^-/2} \\ &\quad - \int dx^- \frac{\lambda}{4!} \frac{1}{4L^2} \sum_{nmkl} q_n q_m q_k q_l e^{-i(k_n^+ + k_m^+ + k_k^+ + k_l^+) x^-/2} \\ &= \frac{2}{2L} \sum_{n,m} \left(-\frac{ik_n^+}{2} \right) q_n \dot{q}_m 4\pi \frac{L}{2\pi} \delta_{n+m,0} - \frac{\mu^2}{4L} \sum_{n,m} q_n q_m 4\pi \frac{L}{2\pi} \delta_{n+m,0} \\ &\quad - \frac{\lambda}{4!} \frac{1}{4L^2} \sum_{nmkl} q_n q_m q_k q_l 4\pi \frac{L}{2\pi} \delta_{n+m+k+l,0} \\ &= \sum_n (-ik_{-n}^+) q_{-n} \dot{q}_n - \frac{\mu^2}{2} \sum_n q_n q_{-n} - \frac{\lambda}{4!} \frac{1}{2L} \sum_{nmkl} q_n q_m q_k q_l \delta_{n+m+k+l,0} \\ &= \sum_n ik_n^+ q_{-n} \dot{q}_n - \frac{\mu^2}{2} \sum_n q_n q_{-n} - \frac{\lambda}{4!} \frac{1}{2L} \sum_{nmkl} q_n q_m q_k q_l \delta_{n+m+k+l,0}. \end{aligned} \quad (2.19)$$

The conjugate momenta is

$$p_n = \frac{\partial L}{\partial \dot{q}_n} = ik_n^+ q_{-n}. \quad (2.20)$$

Especially,

$$p_0 = ik_0^+ q_0 = 0. \quad (2.21)$$

2.5 The Classical Hamiltonian with Zero Mode Included

The gauge invariant symmetry energy momentum tensor is

$$T^{\mu\nu} = \left(\frac{\partial}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \right) \mathcal{L}. \quad (2.22)$$

The Hamiltonian momentum P^μ is defined as

$$P^\mu = \int T^{\nu\mu} d\sigma_\nu. \quad (2.23)$$

When the integration is along $x^+ = 0$ with vanishing integral along the boundary contour,

$$P^\mu = \int T^{\nu\mu} d\sigma_\nu = \int T^{+\mu} d^2 x_\perp dx_+ = \frac{1}{2} \int [T^{+\mu}]|_{x^+=0} d^2 x^\perp dx^-. \quad (2.24)$$

Hence, the Hamiltonian is

$$\begin{aligned} P^- &= \frac{1}{2} \int dx^- d^2 x^\perp T^{+-} \\ &= \frac{1}{2} \int dx^- d^2 x^\perp \left(\frac{\partial \mathcal{L}}{\partial(\partial_+ \phi)} \partial^- \phi - g^{+-} \mathcal{L} \right) \\ &= \frac{1}{2} \int dx^- d^2 x^\perp \left(2 \frac{\partial \mathcal{L}}{\partial(\partial_+ \phi)} \partial_+ \phi - 2 \mathcal{L} \right) \\ &= \int dx^- d^2 x^\perp (\partial^+ \phi(x) \partial_+ \phi(x) - \mathcal{L}) \\ &= \int dx^- d^2 x^\perp \left[\frac{1}{2} \partial^\perp \phi^\perp \phi + V(\phi) \right]. \end{aligned} \quad (2.25)$$

In (1+1) dimension ϕ^4 theory, the Hamiltonian is

$$P^- = \int dx^- V(\phi) = \int dx^- \left(\frac{\mu^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right). \quad (2.26)$$

Following the notation of [Bender et al. \(1993\)](#), we define

$$\Sigma_n = \frac{1}{n!} \sum_{i_1, i_2, \dots, i_n \neq 0} q_{i_1} q_{i_2} \cdots q_{i_n} \delta_{i_1+i_2+\dots+i_n, 0}. \quad (2.27)$$

For the term $\int dx^- \frac{\mu^2}{2} \phi^2$,

$$\begin{aligned} \frac{\mu^2}{2} \int dx^- \phi^2 &= \frac{\mu^2}{4L} \int_{-L}^L dx^- \sum_{i,j} q_i(x^+) q_j(x^+) e^{-i(k_i^+ + k_j^+) x^- / 2} \\ &= \frac{\mu^2}{2d} \sum_{i,j} q_i(x^+) q_j(x^+) \int_{-L}^L dx^- e^{-i(k_i^+ + k_j^+) x^- / 2} \\ &= \frac{\mu^2}{4L} \sum_{i,j} q_i(x^+) q_j(x^+) \int_{-L}^L dx^- e^{-i \frac{2\pi}{L} (i+j) x^- / 2} \\ &= \frac{\mu^2}{4L} \sum_{i,j} q_i(x^+) q_j(x^+) \frac{L}{2\pi} 4\pi \delta_{i+j, 0} \\ &= \frac{\mu^2}{2} \sum_{i_1, i_2} q_{i_1}(x^+) q_{i_2}(x^+) \delta_{i_1+i_2, 0} \\ &= \frac{\mu^2 q_0^2}{2} + \frac{\mu^2}{2} 2! \sum_{i \neq 0} q_0 q_i(x^+) \delta_{i, 0} + \frac{\mu^2}{2} \sum_{i_1, i_2 \neq 0} q_{i_1} q_{i_2} \delta_{i_1+i_2, 0} \\ &= \frac{\mu^2 q_0^2}{2} + \mu^2 \Sigma_2. \end{aligned} \quad (2.28)$$

For the term $\int dx^- \frac{\lambda}{4!} \phi^4$,

$$\begin{aligned}
\frac{\lambda}{4!} \int dx^- \phi^4 &= \frac{\lambda}{4!4L^2} \int_{-L}^L dx^- \sum_{i_1, i_2, i_3, i_4} q_{i_1} q_{i_2} q_{i_3} q_{i_4} e^{-i(k_{i_1}^+ + k_{i_2}^+ + k_{i_3}^+ + k_{i_4}^+)x^-/2} \\
&= \frac{\lambda}{4!4L^2} \sum_{i_1, i_2, i_3, i_4} q_{i_1} q_{i_2} q_{i_3} q_{i_4} \int_{-L}^L dx^- e^{-i(k_{i_1}^+ + k_{i_2}^+ + k_{i_3}^+ + k_{i_4}^+)x^-/2} \\
&= \frac{\lambda}{4!4L^2} \sum_{i_1, i_2, i_3, i_4} q_{i_1} q_{i_2} q_{i_3} q_{i_4} \int_{-L}^L dx^- e^{-i\frac{2\pi}{L}(i_1 + i_2 + i_3 + i_4)x^-/2} \\
&= \frac{\lambda}{4!4L^2} \sum_{i_1, i_2, i_3, i_4} q_{i_1} q_{i_2} q_{i_3} q_{i_4} \frac{L}{2\pi} 4\pi \delta_{i_1 + i_2 + i_3 + i_4, 0} \\
&= \frac{\lambda}{4!2L} \sum_{i_1, i_2, i_3, i_4} q_{i_1} q_{i_2} q_{i_3} q_{i_4} \delta_{i_1 + i_2 + i_3 + i_4, 0} \\
&= \frac{\lambda q_0^4}{4!2L} + \frac{\lambda}{4!2L} C_4^3 \sum_{i \neq 0} q_0 q_0 q_0 q_i \delta_{i, 0} + \frac{\lambda}{4!2L} C_4^2 \sum_{i_1, i_2 \neq 0} q_0 q_0 q_{i_1} q_{i_2} \delta_{i_1 + i_2, 0} \\
&\quad + \frac{\lambda}{4!2L} C_4^1 \sum_{i_1, i_2, i_3 \neq 0} q_0 q_{i_1} q_{i_2} q_{i_3} \delta_{i_1 + i_2 + i_3, 0} + \frac{\lambda}{4!2L} \sum_{i_1, i_2, i_3, i_4 \neq 0} q_{i_1} q_{i_2} q_{i_3} q_{i_4} \delta_{i_1 + i_2 + i_3 + i_4, 0} \\
&= \frac{\lambda q_0^4}{4!2L} + \frac{\lambda}{4!2L} \frac{4!}{2!2!} \sum_{i_1, i_2 \neq 0} q_0 q_0 q_{i_1} q_{i_2} \delta_{i_1 + i_2, 0} \\
&\quad + \frac{\lambda}{4!2L} \frac{4!}{3!1!} \sum_{i_1, i_2, i_3 \neq 0} q_0 q_{i_1} q_{i_2} q_{i_3} \delta_{i_1 + i_2 + i_3, 0} + \frac{\lambda}{4!2L} \sum_{i_1, i_2, i_3, i_4 \neq 0} q_{i_1} q_{i_2} q_{i_3} q_{i_4} \delta_{i_1 + i_2 + i_3 + i_4, 0} \\
&= \frac{\lambda q_0^4}{4!2L} + \frac{\lambda q_0^2}{2!2L} \Sigma_2 + \frac{\lambda q_0}{2L} \Sigma_3 + \frac{\lambda}{2L} \Sigma_4.
\end{aligned} \tag{2.29}$$

The resulting classical Hamiltonian in light-front coordinates reads

$$P^- = \frac{\mu^2 q_0^2}{2} + \mu^2 \Sigma_2 + \frac{\lambda q_0^4}{4!2L} + \frac{\lambda q_0^2}{2!2L} \Sigma_2 + \frac{\lambda q_0}{2L} \Sigma_3 + \frac{\lambda}{2L} \Sigma_4. \tag{2.30}$$

2.6 The Commutation Relations of Zero Mode with Non-zero Modes

Besides the first constraint (2.20), a second constraint $\frac{\partial P^-}{\partial q_0} = -p_0 = 0$ is introduced, so from (2.30) we have

$$0 = \mu^2 q_0 + \frac{\lambda q_0^3}{3!2L} + \frac{\lambda q_0 \Sigma_2}{2L} + \frac{\lambda \Sigma_3}{2L}. \tag{2.31}$$

From $\phi(x) = \frac{1}{\sqrt{2L}} \sum_n q_n(x^+) e^{-ik_n^+ x^-/2}$, multiplying both sides by $e^{ik_m'^+ x^-/2}$ and integrating over x^- , we get

$$\begin{aligned}
\int \phi(x) e^{ik_m'^+ x^-/2} dx^- &= \frac{1}{\sqrt{2L}} \sum_n q_n(x^+) \int e^{i(k_m'^+ - k_n^+) x^-/2} \\
&= \frac{1}{\sqrt{2L}} \sum_n q_n(x^+) 4\pi \delta(k_m'^+ - k_n^+) \\
&= \frac{1}{\sqrt{2L}} \sum_n q_n(x^+) 4\pi \frac{L}{2\pi} \delta_{nm} \\
&= \sqrt{2L} q_m(x^+).
\end{aligned} \tag{2.32}$$

So

$$q_m(x^+) = \frac{1}{\sqrt{2L}} \int \phi(x) e^{ik_m^+ x^-} dx^-. \tag{2.33}$$

Since

$$\begin{aligned}
\Pi^+(x) &= \frac{\partial \mathcal{L}}{\partial(\partial_+ \phi)} = \partial^+ \phi = 2\partial_- \phi = \frac{2}{\sqrt{2L}} \sum_n \left(-\frac{ik_n^+}{2}\right) q_n(x^+) e^{-ik_n^+ x^-/2} \\
&= \frac{1}{\sqrt{2L}} \sum_n (-ik_n^+) q_n(x^+) e^{-ik_n^+ x^-/2},
\end{aligned} \tag{2.34}$$

multiplying both sides and integrating over x^- , we get

$$\begin{aligned}
\int \Pi^+(x) e^{ik_m'^+ x^-/2} dx^- &= \frac{1}{\sqrt{2L}} \sum_n q_n(x^+) (-ik_n^+) \int e^{i(k_m'^+ - k_n^+) x^-/2} \\
&= \frac{1}{\sqrt{2L}} \sum_n q_n(x^+) (-ik_n^+) 4\pi \delta(k_m'^+ - k_n^+) \\
&= \frac{1}{\sqrt{2L}} \sum_n q_n(x^+) (-ik_n^+) 4\pi \frac{L}{2\pi} \delta_{nm} \\
&= \sqrt{2L} (-ik_m^+) q_m(x^+).
\end{aligned} \tag{2.35}$$

So

$$q_m(x^+) = \frac{i}{\sqrt{2L} k_m^+} \int \Pi^+(x) e^{ik_m^+ x^-} dx^-. \tag{2.36}$$

From $[\phi(x), \Pi^+(y)] = i\delta(x-y) = i\delta(y-x)$,

$$\begin{aligned}
[q_m, q_n] &= \left[\frac{1}{\sqrt{2L}} \int \phi(x) e^{ik_m^+ x^- / 2} dx^-, \frac{i}{\sqrt{2L} k_n^+} \int \Pi^+(y) e^{ik_n'^+ y^- / 2} dy^- \right] \\
&= \frac{i}{2L k_n^+} \int dx^- \int dy^- e^{ik_m^+ x^- / 2} e^{ik_n'^+ y^- / 2} [\phi(x), \Pi^+(y)] \\
&= -\frac{1}{2L k_n^+} \int dx^- \int dy^- e^{ik_m^+ x^- / 2} e^{ik_n'^+ y^- / 2} \delta(y-x) \\
&= -\frac{1}{2L k_n^+} \int dx^- e^{i(k_m^+ + k_n'^+) x^- / 2} \\
&= -\frac{1}{2L k_n^+} 4\pi \delta(k_m^+ + k_n'^+) \\
&= -\frac{L}{2L 2\pi n} 4\pi \frac{L}{2\pi} \delta_{m+n,0} \\
&= -\frac{L}{2\pi n} \delta_{m+n,0} \\
&= \frac{L}{2\pi m} \delta_{m+n,0}.
\end{aligned} \tag{2.37}$$

Starting from the second constraint (2.31),

$$0 = \mu^2 q_0 + \frac{\lambda q_0^3}{3!2L} + \frac{\lambda q_0 \Sigma_2}{2L} + \frac{\lambda \Sigma_3}{2L}, \tag{2.38}$$

which is

$$0 = \mu^2 q_0 + \frac{\lambda}{3!2L} q_0^3 + \frac{\lambda}{4L} \sum_{i_1, i_2} q_0 q_{i_1} q_{i_2} \delta_{i_1+i_2,0} + \frac{\lambda}{3!2L} \sum_{j_1, j_2, j_3} q_{j_1} q_{j_2} q_{j_3} \delta_{j_1+j_2+j_3,0}. \tag{2.39}$$

We rewrite it as

$$0 = \mu^2 q_0 + \frac{\lambda}{3!2L} q_0^3 + \frac{\lambda}{4L} \sum_m q_0 q_m q_{-m} + \frac{\lambda}{3!2L} \sum_{j_1, j_2, j_3} q_{j_1} q_{j_2} q_{j_3} \delta_{j_1+j_2+j_3,0}. \tag{2.40}$$

Multiplying the above expression from the right,

$$0 = \mu^2 q_0 q_n + \frac{\lambda}{3!2L} q_0^3 q_n + \frac{\lambda}{4L} \sum_m q_0 q_m q_{-m} q_n + \frac{\lambda}{3!2L} \sum_{j_1, j_2, j_3} q_{j_1} q_{j_2} q_{j_3} \delta_{j_1+j_2+j_3,0} q_n. \tag{2.41}$$

Multiplying it from the left,

$$0 = \mu^2 q_n q_0 + \frac{\lambda}{3!2L} q_n q_0^3 + \frac{\lambda}{4L} \sum_m q_n q_0 q_m q_{-m} + \frac{\lambda}{3!2L} \sum_{j_1, j_2, j_3} q_n q_{j_1} q_{j_2} q_{j_3} \delta_{j_1+j_2+j_3,0}. \tag{2.42}$$

Let $[q_0, q_n] = K$, i.e., $q_n q_0 = q_0 q_n - K$. We substitute this expression and $q_n q_m = q_m q_n + \frac{L}{2\pi n} \delta_{n+m,0}$ into (2.42).

For the second term on the right there results

$$\begin{aligned}
q_n q_0^3 &= (q_0 q_n - K) q_0^2 = q_0 q_n q_0^2 - K q_0^2 = q_0 (q_0 q_n - K) q_0 - K q_0^2 \\
&= q_0 q_0 q_n q_0 - 2K q_0^2 = q_0 q_0 (q_0 q_n - K) - 2K q_0^2 = q_0 q_0 q_0 q_n - 3K q_0^2.
\end{aligned} \tag{2.43}$$

For the third term on the right there results

$$\begin{aligned}
\sum_m q_n q_0 q_m q_{-m} &= \sum_m (q_0 q_n - K) q_m q_{-m} \\
&= \sum_m q_0 q_n q_m q_{-m} - K \sum_m q_m q_{-m} \\
&= \sum_m q_0 (q_m q_n + \frac{L}{2\pi n} \delta_{n+m,0}) q_{-m} - K \sum_m q_m q_{-m} \\
&= \sum_m q_0 q_m q_n q_{-m} + \sum_m q_0 \frac{L}{2\pi n} \delta_{n+m,0} q_{-m} - K \sum_m q_m q_{-m} \\
&= \sum_m q_0 q_m q_n q_{-m} + \frac{L}{2\pi n} q_0 q_n - K \sum_m q_m q_{-m} \\
&= \sum_m q_0 q_m (q_{-m} q_n + \frac{L}{2\pi n} \delta_{n-m,0}) + \frac{L}{2\pi n} q_0 q_n - K \sum_m q_m q_{-m} \\
&= \sum_m q_0 q_m q_{-m} q_n + \sum_m q_0 q_m \frac{L}{2\pi n} \delta_{n-m,0} + \frac{L}{2\pi n} q_0 q_n - K \sum_m q_m q_{-m} \\
&= \sum_m q_0 q_m q_{-m} q_n + 2 \frac{L}{2\pi n} q_0 q_n - K \sum_m q_m q_{-m}.
\end{aligned} \tag{2.44}$$

For the last term on the right there results

$$\begin{aligned}
\sum_{j_1, j_2, j_3} q_n q_{j_1} q_{j_2} q_{j_3} \delta_{j_1+j_2+j_3, 0} &= \sum_{j_1, j_2, j_3} (q_{j_1} q_n + \frac{L}{2\pi n} \delta_{n+j_1, 0}) q_{j_2} q_{j_3} \delta_{j_1+j_2+j_3, 0} \\
&= \sum_{j_1, j_2, j_3} q_{j_1} q_n q_{j_2} q_{j_3} \delta_{j_1+j_2+j_3, 0} + \sum_{j_1, j_2, j_3} \frac{L}{2\pi n} \delta_{n+j_1, 0} q_{j_2} q_{j_3} \delta_{j_1+j_2+j_3, 0} \\
&= \sum_{j_1, j_2, j_3} q_{j_1} q_n q_{j_2} q_{j_3} \delta_{j_1+j_2+j_3, 0} + \sum_{j_2, j_3} \frac{L}{2\pi n} q_{j_2} q_{j_3} \delta_{j_2+j_3, n} \\
&= \sum_{j_1, j_2, j_3} q_{j_1} (q_{j_2} q_n + \frac{L}{2\pi n} \delta_{n+j_2, 0}) q_{j_3} \delta_{j_1+j_2+j_3, 0} + \sum_{j_2, j_3} \frac{L}{2\pi n} q_{j_2} q_{j_3} \delta_{j_2+j_3, n} \\
&= \sum_{j_1, j_2, j_3} q_{j_1} q_{j_2} q_n q_{j_3} \delta_{j_1+j_2+j_3, 0} + \sum_{j_1, j_2, j_3} q_{j_1} \frac{L}{2\pi n} \delta_{n+j_2, 0} q_{j_3} \delta_{j_1+j_2+j_3, 0} \\
&\quad + \sum_{j_2, j_3} \frac{L}{2\pi n} q_{j_2} q_{j_3} \delta_{j_2+j_3, n} \\
&= \sum_{j_1, j_2, j_3} q_{j_1} q_{j_2} q_n q_{j_3} \delta_{j_1+j_2+j_3, 0} + \sum_{j_1, j_3} q_{j_1} \frac{L}{2\pi n} q_{j_3} \delta_{j_1+j_3, n} \\
&\quad + \sum_{j_2, j_3} \frac{L}{2\pi n} q_{j_2} q_{j_3} \delta_{j_2+j_3, n} \\
&= \sum_{j_1, j_2, j_3} q_{j_1} q_{j_2} q_n q_{j_3} \delta_{j_1+j_2+j_3, 0} + 2 \frac{L}{2\pi n} \sum_{j_1, j_3} q_{j_1} q_{j_3} \delta_{j_1+j_3, n} \\
&= \sum_{j_1, j_2, j_3} q_{j_1} q_{j_2} q_{j_3} q_n \delta_{j_1+j_2+j_3, 0} + 3 \frac{L}{2\pi n} \sum_{j_1, j_2} q_{j_1} q_{j_2} \delta_{j_1+j_2, n}.
\end{aligned} \tag{2.45}$$

So (2.42) becomes

$$\begin{aligned}
0 &= \mu^2 (q_0 q_n - K) + \frac{\lambda}{3!2L} (q_0 q_0 q_0 q_n - 3K q_0^2) + \frac{\lambda}{4L} \left(\sum_m q_0 q_m q_{-m} q_n + 2 \frac{L}{2\pi n} q_0 q_n - K \sum_m q_m q_{-m} \right) \\
&\quad + \frac{\lambda}{3!2L} \left(\sum_{j_1, j_2, j_3} q_{j_1} q_{j_2} q_{j_3} q_n \delta_{j_1+j_2+j_3, 0} + 3 \frac{L}{2\pi n} \sum_{j_1, j_2} q_{j_1} q_{j_2} \delta_{j_1+j_2, n} \right).
\end{aligned} \tag{2.46}$$

Subtracting (2.41) by (2.42), we get

$$0 = \mu^2 K + \frac{\lambda}{3!2L} 3K q_0^2 + \frac{\lambda}{4L} \left(-2 \frac{L}{2\pi n} q_0 q_n + K \sum_m q_m q_{-m} \right) + \frac{\lambda}{3!2L} \left(-3 \frac{L}{2\pi n} \sum_{j_1, j_2} q_{j_1} q_{j_2} \delta_{j_1+j_2, n} \right). \tag{2.47}$$

So

$$[q_0, q_n] = K = \frac{\frac{\lambda}{2L} \frac{L}{2\pi n} q_0 q_n + \frac{\lambda}{4L} \frac{L}{2\pi n} \sum_{j_1, j_2} q_{j_1} q_{j_2} \delta_{j_1+j_2, n}}{\mu^2 + \frac{\lambda}{4L} q_0^2 + \frac{\lambda}{4L} \sum_m q_m q_{-m}}, \quad (2.48)$$

where $n, m, j_1, j_2 \neq 0$.

To summarize, the commutation relations are

$$[q_m, q_n] = \frac{L}{2\pi k} \delta_{m+n, 0} \quad (2.49)$$

and

$$[q_0, q_n] = \frac{\frac{\lambda}{2L} \frac{L}{2\pi n} q_0 q_n + \frac{\lambda}{4L} \frac{L}{2\pi n} \sum_{j_1, j_2} q_{j_1} q_{j_2} \delta_{j_1+j_2, n}}{\mu^2 + \frac{\lambda}{4L} q_0^2 + \frac{\lambda}{4L} \sum_m q_m q_{-m}}, \quad (2.50)$$

where $n, m, j_1, j_2 \neq 0$.

CHAPTER 3. LIGHT FRONT CRITICAL COUPLING WITHOUT ZERO MODE

3.1 Introduction

The critical value of the coupling constant is defined as the value at which the interacting theory produces a vanishing mass gap. That is, the lowest interacting state becomes degenerate with the zero-mass free particle vacuum. We will use the Hamiltonian formalism to solve for the lowest mass eigenstate and trace its dependence on the coupling to observe the critical coupling.

We first discretize the Hamiltonian operator in light-front coordinates. Then, we use this form to solve the eigenvalue problem at a sequence of coupling strengths in order to extract the critical coupling. The following derivation mainly follows [Harindranath and Vary \(1987\)](#) and [Chakrabarti et al. \(2004\)](#).

3.2 Discretized Light Front Hamiltonian Formalism

We start from the (1+1) ϕ^4 Lagrangian density [\(2.10\)](#)

$$\mathcal{L} = \frac{1}{2} \partial^+ \phi \partial^- \phi - \frac{\mu^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4. \quad (3.1)$$

From [\(2.26\)](#), the Hamiltonian density is

$$\mathcal{H} = \frac{\mu^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4. \quad (3.2)$$

We expand the field at zero light-front time

$$\phi(x^+ = 0, x^-) = \frac{1}{2\pi} \int \frac{dk^+}{2k^+} [a(k^+) e^{-ik^+ x^-/2} + a^\dagger(k^+) e^{ik^+ x^-/2}], \quad (3.3)$$

where the commutation relation of $a(k^+)$ and $a^\dagger(k'^+)$, obtained by similar procedure as in [Section 2.6](#), is

$$[a(k^+), a^\dagger(k'^+)] = 2\pi 2k^+ \delta(k^+ - k'^+). \quad (3.4)$$

The longitudinal momentum k^+ is then discretized in the domain of $-L \leq x^- \leq +L$ by

$$k^+ \rightarrow k_n^+ = \frac{2\pi}{L}n, \quad (3.5)$$

with $n = 0, 1, 2, \dots$ for periodic boundary conditions, and $n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ for anti-periodic boundary conditions. In the following derivation, we firstly ignore the $n = 0$ point, which is the zero mode.

The field expansion becomes

$$\phi = \frac{1}{4\pi} \sum \frac{1}{n} (a(k_n^+) e^{-i\frac{n\pi}{L}x^-} + a^\dagger(k_n^+) e^{i\frac{n\pi}{L}x^-}), \quad (3.6)$$

where

$$\begin{aligned} [a(k_n^+), a^\dagger(k_m^+)] &= 2\pi 2k_n^+ \delta(k_n^+ - k_m^+) \\ &= 2\pi 2\left(\frac{2\pi}{L}n\right) \delta\left(\frac{2\pi}{L}(n - m)\right) \\ &= 2\pi 2\left(\frac{2\pi}{L}n\right) \frac{L}{2\pi} \delta(n - m) \\ &= 4\pi n \delta_{nm}. \end{aligned} \quad (3.7)$$

For convenience, we replace $\frac{n\pi}{L}x^-$ as $k_n x$, where $k_n = n$ is the longitudinal momentum quanta and $x = \frac{\pi x^-}{L}$. Then

$$\phi = \frac{1}{\sqrt{4\pi}} \sum \frac{1}{\sqrt{n}} (a_n e^{-ik_n x} + a_n^\dagger e^{ik_n x}), \quad (3.8)$$

where $a(k_n^+) = \sqrt{4\pi n} a_n$, $a^\dagger(k_n^+) = \sqrt{4\pi n} a_n^\dagger$ and

$$[a_n, a_m^\dagger] = \delta_{nm}. \quad (3.9)$$

One can also introduce the dimensionless momentum operator K

$$P^+ = \frac{2\pi}{L} K. \quad (3.10)$$

According to the dispersion relation $M^2 = P^+ P^- = K H$, Hamiltonian H has the dimension of mass squared which is defined by

$$P^- = \frac{L}{2\pi} H. \quad (3.11)$$

Therefore, the Hamiltonian H can be calculated by

$$H = \frac{2\pi}{L} P^- = \frac{2\pi}{L} \int dx^- \mathcal{H} = \frac{2\pi}{L} \int d\left(\frac{L}{\pi}x\right) \mathcal{H} = 2 \int dx \mathcal{H}. \quad (3.12)$$

For the first term $2 \int dx (\frac{\mu^2}{2} \phi^2)$,

$$\begin{aligned} 2 \int dx \phi^2 &= 2 \frac{1}{4\pi} \sum_{n,m} \frac{1}{\sqrt{nm}} \int dx (a_n e^{-ik_n x} + a_n^\dagger e^{ik_n x}) (a_m e^{-ik_m x} + a_m^\dagger e^{ik_m x}) \\ &= 2 \frac{1}{4\pi} \sum_{n,m} \frac{1}{\sqrt{nm}} \int dx (a_n a_m e^{-i(k_n+k_m)x} + a_n^\dagger a_m e^{i(k_n-k_m)x} \\ &\quad + a_n a_m^\dagger e^{i(k_m-k_n)x} + a_n^\dagger a_m^\dagger e^{i(k_n+k_m)x}) \\ &= 2 \frac{1}{4\pi} \sum_{n,m} \frac{1}{\sqrt{nm}} \int dx (a_n^\dagger a_m e^{i(k_n-k_m)x} + a_n a_m^\dagger e^{i(k_m-k_n)x}) \\ &= 2 \frac{1}{4\pi} \sum_{n,m} \frac{1}{\sqrt{nm}} (a_n^\dagger a_m + a_n a_m^\dagger) 2\pi \delta(k_n - k_m) \\ &= 2 \frac{1}{4\pi} \sum_{n,m} \frac{1}{\sqrt{nm}} (a_n^\dagger a_m + a_n a_m^\dagger) 2\pi \delta_{nm} \\ &= \sum_n \frac{1}{n} (a_n^\dagger a_n + a_n a_n^\dagger) \\ &= \sum_n \frac{1}{n} (a_n^\dagger a_n + (a_n^\dagger a_n + \delta_{n,n})) \\ &= \sum_n \frac{1}{n} (a_n^\dagger a_n + (a_n^\dagger a_n + 1)) \\ &= 2 \sum_n \frac{1}{n} a_n^\dagger a_n + \sum_n \frac{1}{n}, \end{aligned} \quad (3.13)$$

we eliminate the infinity term and get

$$\begin{aligned} H_0(1) &= \frac{\mu^2}{2} (2 \int dx \phi^2) = \frac{\mu^2}{2} (2 \sum_n \frac{1}{n} a_n^\dagger a_n + \sum_n \frac{1}{n}) = \mu^2 \sum_n \frac{1}{n} a_n^\dagger a_n - \frac{\mu^2}{2} \sum_n \frac{1}{n} \\ &= \mu^2 \sum_n \frac{1}{n} a_n^\dagger a_n + \infty = \mu^2 \sum_n \frac{1}{n} a_n^\dagger a_n. \end{aligned} \quad (3.14)$$

For the second term $2 \int dx (\frac{\lambda}{4!} \phi^4)$,

$$\begin{aligned} 2 \int dx \phi^4 &= 2 \left(\frac{1}{4\pi}\right)^2 \sum_{klmn} \frac{1}{\sqrt{klmn}} \int dx (a_k e^{-ik_k x} + a_k^\dagger e^{ik_k x}) (a_l e^{-ik_l x} + a_l^\dagger e^{ik_l x}) \\ &\quad (a_m e^{-ik_m x} + a_m^\dagger e^{ik_m x}) (a_n e^{-ik_n x} + a_n^\dagger e^{ik_n x}), \end{aligned} \quad (3.15)$$

we can eliminate all terms with just a 's and all terms with just a^\dagger 's, because these terms have the factor of $\int dx e^{\pm i(\sum k)x} = 2\pi\delta(\sum k) = 0$ due to $k > 0$. There are still four terms with three a 's with one a^\dagger and the four terms with one a with three a^\dagger 's. There are six terms with two a 's two a^\dagger 's.

- For the three a 's with one a^\dagger configuration:

i

$$\sum_{klmn} \frac{1}{\sqrt{klmn}} \int dx a_k^\dagger a_l a_m a_n e^{i(k_k - k_l - k_m - k_n)x} = \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_l a_m a_n 2\pi\delta_{k,l+m+n} \quad (3.16)$$

ii

$$\begin{aligned} & \sum_{klmn} \frac{1}{\sqrt{klmn}} \int dx a_k a_l^\dagger a_m a_n e^{i(-k_k + k_l - k_m - k_n)x} = \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k a_l^\dagger a_m a_n 2\pi\delta_{l,k+m+n} \\ &= \sum_{klmn} \frac{1}{\sqrt{klmn}} (a_l^\dagger a_k + \delta_{k,l}) a_m a_n 2\pi\delta_{l,k+m+n} \\ &= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger a_k a_m a_n 2\pi\delta_{l,k+m+n} + \sum_{klmn} \frac{1}{\sqrt{klmn}} \delta_{k,l} a_m a_n 2\pi\delta_{l,k+m+n} \\ &= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger a_k a_m a_n 2\pi\delta_{l,k+m+n} + \sum_{kmn} \frac{1}{k\sqrt{mn}} a_m a_n 2\pi\delta_{0,m+n} \\ &= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger a_k a_m a_n 2\pi\delta_{l,k+m+n} \end{aligned} \quad (3.17)$$

iii

$$\begin{aligned} & \sum_{klmn} \frac{1}{\sqrt{klmn}} \int dx a_k a_l a_m^\dagger a_n e^{i(-k_k - k_l + k_m - k_n)x} = \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k a_l a_m^\dagger a_n 2\pi\delta_{m,k+l+n} \\ &= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k (a_m^\dagger a_l + \delta_{lm}) a_n 2\pi\delta_{m,k+l+n} \\ &= \sum_{klmn} \frac{1}{\sqrt{klmn}} (a_m^\dagger a_k + \delta_{km}) a_l a_n 2\pi\delta_{m,k+l+n} \\ &= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger a_k a_l a_n 2\pi\delta_{m,k+l+n} \end{aligned} \quad (3.18)$$

iv

$$\begin{aligned}
& \sum_{klmn} \frac{1}{\sqrt{klmn}} \int dx a_k a_l a_m a_n^\dagger e^{i(-k_k - k_l - k_m + k_n)x} = \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k a_l a_m a_n^\dagger 2\pi \delta_{n,k+l+m} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k a_l (a_n^\dagger a_m + \delta_{mn}) 2\pi \delta_{n,k+l+m} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k (a_n^\dagger a_l + \delta_{ln}) a_m 2\pi \delta_{n,k+l+m} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} (a_n^\dagger a_k + \delta_{kn}) a_l a_m 2\pi \delta_{n,k+l+m} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_n^\dagger a_k a_l a_m 2\pi \delta_{n,k+l+m}.
\end{aligned} \tag{3.19}$$

So the terms with three a 's and one a^\dagger can be combined as

$$\frac{\lambda}{4!} 2 \left(\frac{1}{4\pi} \right)^2 \left[4(2\pi) \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_l a_m a_n \delta_{k,l+m+n} \right] = \frac{1}{6} \frac{\lambda}{4\pi} \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_l a_m a_n \delta_{k,l+m+n}. \tag{3.20}$$

- Similarly, the terms with three a^\dagger 's and one a provide,

$$\sum_{klmn} \frac{1}{\sqrt{klmn}} \int dx a_k^\dagger a_l^\dagger a_m^\dagger a_n e^{i(k_k + k_l + k_m - k_n)x} = \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_l^\dagger a_m^\dagger a_n 2\pi \delta_{n,k+l+m}. \tag{3.21}$$

To easily combine with the above results, we switch $n \leftrightarrow k$ and $m \leftrightarrow l$ in the following derivation:

i

$$\sum_{klmn} \frac{1}{\sqrt{klmn}} \int dx a_n^\dagger a_m^\dagger a_l^\dagger a_k e^{i(k_n + k_m + k_l - k_k)x} = \sum_{klmn} \frac{1}{\sqrt{klmn}} a_n^\dagger a_m^\dagger a_l^\dagger a_k 2\pi \delta_{k,n+m+l} \tag{3.22}$$

ii

$$\begin{aligned}
& \sum_{klmn} \frac{1}{\sqrt{klmn}} \int dx a_n^\dagger a_m^\dagger a_l a_k^\dagger e^{i(k_n+k_m-k_l+k_k)x} = \sum_{klmn} \frac{1}{\sqrt{klmn}} a_n^\dagger a_m^\dagger a_l a_k^\dagger 2\pi \delta_{l,n+m+k} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_n^\dagger a_m^\dagger (a_k^\dagger a_l + \delta_{lk}) 2\pi \delta_{l,n+m+k} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_n^\dagger a_m^\dagger a_k^\dagger a_l 2\pi \delta_{l,n+m+k}
\end{aligned} \tag{3.23}$$

iii

$$\begin{aligned}
& \sum_{klmn} \frac{1}{\sqrt{klmn}} \int dx a_n^\dagger a_m a_l^\dagger a_k^\dagger e^{i(k_n-k_m+k_l+k_k)x} = \sum_{klmn} \frac{1}{\sqrt{klmn}} a_n^\dagger a_m a_l^\dagger a_k^\dagger \delta_{m,n+l+k} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_n^\dagger (a_l^\dagger a_m + \delta_{ml}) a_k^\dagger \delta_{m,n+l+k} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_n^\dagger a_l^\dagger (a_k^\dagger a_m + \delta_{mk}) \delta_{m,n+l+k} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_n^\dagger a_l^\dagger a_k^\dagger a_m \delta_{m,n+l+k}
\end{aligned} \tag{3.24}$$

iv

$$\sum_{klmn} \frac{1}{\sqrt{klmn}} \int dx a_n a_m^\dagger a_l^\dagger a_k^\dagger e^{i(k_n-k_m+k_l+k_k)x} = \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger a_l^\dagger a_k^\dagger a_n \delta_{n,m+l+k}. \tag{3.25}$$

So the terms with three a^\dagger 's and one a can be combined as

$$\frac{\lambda}{4!} 2 \left(\frac{1}{4\pi} \right)^2 \left[4(2\pi) \sum_{klmn} \frac{1}{\sqrt{klmn}} a_n^\dagger a_m^\dagger a_l^\dagger a_k \delta_{k,n+m+l} \right] = \frac{1}{6} \frac{\lambda}{4\pi} \sum_{klmn} \frac{1}{\sqrt{klmn}} a_n^\dagger a_m^\dagger a_l^\dagger a_k \delta_{k,n+m+l}. \tag{3.26}$$

We then arrive at

$$H_2 = \frac{1}{6} \frac{\lambda}{4\pi} \sum_{klmn} \left[\frac{a_k^\dagger a_l a_m a_n + a_n^\dagger a_m^\dagger a_l^\dagger a_k}{\sqrt{klmn}} \right] \delta_{k,m+n+l}. \tag{3.27}$$

- For the terms with two a 's and two a^\dagger 's:

i

$$\sum_{klmn} \frac{1}{\sqrt{klmn}} \int dx a_k^\dagger a_l^\dagger a_m a_n e^{i(k_k+k_l-k_m-k_n)x} = \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_l^\dagger a_m a_n 2\pi \delta_{k+l,m+n} \quad (3.28)$$

ii

$$\begin{aligned} & \sum_{klmn} \frac{1}{\sqrt{klmn}} \int dx a_k^\dagger a_l a_m^\dagger a_n e^{i(k_k-k_l+k_m-k_n)x} = \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_l a_m^\dagger a_n 2\pi \delta_{k+m,l+n} \\ &= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger (a_m^\dagger a_l + \delta_{lm}) a_n 2\pi \delta_{k+m,l+n} \\ &= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_m^\dagger a_l a_n 2\pi \delta_{k+m,l+n} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger \delta_{lm} a_n 2\pi \delta_{k+m,l+n} \\ &= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_m^\dagger a_l a_n 2\pi \delta_{k+m,l+n} + \sum_{kmn} \frac{1}{m\sqrt{kn}} a_k^\dagger a_n 2\pi \delta_{kn} \\ &= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_m^\dagger a_l a_n 2\pi \delta_{k+m,l+n} + \sum_{mn} \frac{1}{mn} a_n^\dagger a_n 2\pi \\ &= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_m^\dagger a_l a_n 2\pi \delta_{k+m,l+n} + \sum_n \frac{1}{n} a_n^\dagger a_n 2\pi \sum_m \frac{1}{m} \end{aligned} \quad (3.29)$$

iii

$$\begin{aligned}
& \sum_{klmn} \frac{1}{\sqrt{klmn}} \int dx a_k^\dagger a_l a_m a_n^\dagger e^{i(k_k - k_l - k_m + k_n)x} = \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_l a_m a_n^\dagger 2\pi \delta_{k+n, l+m} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_l (a_n^\dagger a_m + \delta_{mn}) 2\pi \delta_{k+n, l+m} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_l a_n^\dagger a_m 2\pi \delta_{k+n, l+m} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_l \delta_{mn} 2\pi \delta_{k+n, l+m} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger (a_n^\dagger a_l + \delta_{ln}) a_m 2\pi \delta_{k+n, l+m} + \sum_{klmn} \frac{1}{n\sqrt{kl}} a_k^\dagger a_l 2\pi \delta_{kl} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_n^\dagger a_l a_m 2\pi \delta_{k+n, l+m} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger \delta_{ln} a_m 2\pi \delta_{k+n, l+m} + \sum_{kn} \frac{1}{nk} a_k^\dagger a_k 2\pi \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_n^\dagger a_l a_m 2\pi \delta_{k+n, l+m} + \sum_{kmn} \frac{1}{n\sqrt{km}} a_k^\dagger a_m 2\pi \delta_{km} + \sum_k \frac{1}{k} a_k^\dagger a_k 2\pi \sum_n \frac{1}{n} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_n^\dagger a_l a_m 2\pi \delta_{k+n, l+m} + \sum_{mn} \frac{1}{nm} a_m^\dagger a_m 2\pi + \sum_k \frac{1}{k} a_k^\dagger a_k 2\pi \sum_n \frac{1}{n} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_n^\dagger a_l a_m 2\pi \delta_{k+n, l+m} + \sum_m \frac{1}{m} a_m^\dagger a_m 2\pi \sum_n \frac{1}{n} + \sum_k \frac{1}{k} a_k^\dagger a_k 2\pi \sum_n \frac{1}{n}
\end{aligned} \tag{3.30}$$

iv

$$\begin{aligned}
& \sum_{klmn} \frac{1}{\sqrt{klmn}} \int dx a_k a_l^\dagger a_m^\dagger a_n e^{i(-k_k+k_l+k_m-k_n)x} = \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k a_l^\dagger a_m^\dagger a_n 2\pi \delta_{k+n,l+m} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} (a_l^\dagger a_k + \delta_{kl}) a_m^\dagger a_n 2\pi \delta_{k+n,l+m} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger a_k a_m^\dagger a_n 2\pi \delta_{k+n,l+m} + \sum_{klmn} \frac{1}{\sqrt{klmn}} \delta_{kl} a_m^\dagger a_n 2\pi \delta_{k+n,l+m} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger a_k a_m^\dagger a_n 2\pi \delta_{k+n,l+m} + \sum_{kmn} \frac{1}{k\sqrt{mn}} a_m^\dagger a_n 2\pi \delta_{nm} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger (a_m^\dagger a_k + \delta_{km}) a_n 2\pi \delta_{k+n,l+m} + \sum_{kn} \frac{1}{kn} a_n^\dagger a_n 2\pi \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger a_m^\dagger a_k a_n 2\pi \delta_{k+n,l+m} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger \delta_{km} a_n 2\pi \delta_{k+n,l+m} + \sum_{kn} \frac{1}{kn} a_n^\dagger a_n 2\pi \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger a_m^\dagger a_k a_n 2\pi \delta_{k+n,l+m} + \sum_{lmn} \frac{1}{m\sqrt{ln}} a_l^\dagger a_n 2\pi \delta_{nl} + \sum_{kn} \frac{1}{kn} a_n^\dagger a_n 2\pi \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger a_m^\dagger a_k a_n 2\pi \delta_{k+n,l+m} + \sum_{mn} \frac{1}{mn} a_n^\dagger a_n 2\pi + \sum_{kn} \frac{1}{kn} a_n^\dagger a_n 2\pi \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger a_m^\dagger a_k a_n 2\pi \delta_{k+n,l+m} + \sum_n \frac{1}{n} a_n^\dagger a_n 2\pi \sum_m \frac{1}{m} + \sum_n \frac{1}{n} a_n^\dagger a_n 2\pi \sum_k \frac{1}{k}
\end{aligned} \tag{3.31}$$

v

$$\begin{aligned}
& \sum_{klmn} \frac{1}{\sqrt{klmn}} \int dx a_k a_l^\dagger a_m a_n^\dagger e^{i(-k_k+k_l-k_m+k_n)x} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k a_l^\dagger a_m a_n^\dagger 2\pi \delta_{k+m,l+n} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} (a_l^\dagger a_k + \delta_{kl}) (a_n^\dagger a_m + \delta_{mn}) 2\pi \delta_{k+m,l+n} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger a_k a_n^\dagger a_m 2\pi \delta_{k+m,l+n} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger a_k \delta_{mn} 2\pi \delta_{k+m,l+n} \\
&\quad + \sum_{klmn} \frac{1}{\sqrt{klmn}} \delta_{kl} a_n^\dagger a_m 2\pi \delta_{k+m,l+n} + \sum_{klmn} \frac{1}{\sqrt{klmn}} \delta_{kl} \delta_{mn} 2\pi \delta_{k+m,l+n} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger (a_n^\dagger a_k + \delta_{kn}) a_m 2\pi \delta_{k+m,l+n} + 2 \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger a_k \delta_{mn} 2\pi \delta_{kl} + 2\pi \sum_{kn} \frac{1}{kn} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger a_n^\dagger a_k a_m 2\pi \delta_{k+m,l+n} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger \delta_{kn} a_m 2\pi \delta_{k+m,l+n} \\
&\quad + 2 \sum_{kmn} \frac{1}{k\sqrt{mn}} a_k^\dagger a_k \delta_{mn} 2\pi + 2\pi \sum_{kn} \frac{1}{kn} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_l^\dagger a_n^\dagger a_k a_m 2\pi \delta_{k+m,l+n} + 3 \sum_{kn} \frac{1}{kn} a_k^\dagger a_k 2\pi + 2\pi \sum_{kn} \frac{1}{kn}
\end{aligned} \tag{3.32}$$

vi

$$\begin{aligned}
& \sum_{klmn} \frac{1}{\sqrt{klmn}} \int dx a_k a_l a_m^\dagger a_n^\dagger e^{i(-k_k - k_l + k_m + k_n)x} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k a_l a_m^\dagger a_n^\dagger \delta_{k+l, m+n} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k (a_m^\dagger a_l + \delta_{lm}) a_n^\dagger \delta_{k+l, m+n} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k a_m^\dagger a_l a_n^\dagger \delta_{k+l, m+n} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k \delta_{lm} a_n^\dagger \delta_{k+l, m+n} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} (a_m^\dagger a_k + \delta_{km}) (a_n^\dagger a_l + \delta_{ln}) \delta_{k+l, m+n} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k \delta_{lm} a_n^\dagger \delta_{k+l, m+n} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger a_k a_n^\dagger a_l \delta_{k+l, m+n} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger a_k \delta_{ln} \delta_{k+l, m+n} \\
&\quad + \sum_{klmn} \frac{1}{\sqrt{klmn}} \delta_{km} a_n^\dagger a_l \delta_{k+l, m+n} + \sum_{klmn} \frac{1}{\sqrt{klmn}} \delta_{km} \delta_{ln} \delta_{k+l, m+n} \\
&\quad + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k \delta_{lm} a_n^\dagger \delta_{k+l, m+n} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger (a_n^\dagger a_k + \delta_{kn}) a_l \delta_{k+l, m+n} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger a_k \delta_{ln} \delta_{k+l, m+n} \\
&\quad + \sum_{klmn} \frac{1}{\sqrt{klmn}} \delta_{km} a_n^\dagger a_l \delta_{k+l, m+n} + \sum_{klmn} \frac{1}{\sqrt{klmn}} \delta_{km} \delta_{ln} \delta_{k+l, m+n} \\
&\quad + \sum_{klmn} \frac{1}{\sqrt{klmn}} (a_n^\dagger a_k + \delta_{kn}) \delta_{lm} \delta_{k+l, m+n} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger a_n^\dagger a_k a_l \delta_{k+l, m+n} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger \delta_{kn} a_l \delta_{k+l, m+n} \\
&\quad + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger a_k \delta_{ln} \delta_{k+l, m+n} + \sum_{klmn} \frac{1}{\sqrt{klmn}} \delta_{km} a_n^\dagger a_l \delta_{k+l, m+n} \\
&\quad + \sum_{klmn} \frac{1}{\sqrt{klmn}} \delta_{km} \delta_{ln} \delta_{k+l, m+n} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_n^\dagger a_k \delta_{lm} \delta_{k+l, m+n} \\
&\quad + \sum_{klmn} \frac{1}{\sqrt{klmn}} \delta_{kn} \delta_{lm} \delta_{k+l, m+n} \\
&= \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger a_n^\dagger a_k a_l \delta_{k+l, m+n} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger a_l \delta_{kn} \delta_{lm} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger a_k \delta_{ln} \delta_{km}
\end{aligned}$$

vii

$$\begin{aligned}
& + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_n^\dagger a_l \delta_{km} \delta_{ln} + \sum_{klmn} \frac{1}{\sqrt{klmn}} \delta_{km} \delta_{ln} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_n^\dagger a_k \delta_{lm} \delta_{kn} \\
& + \sum_{klmn} \frac{1}{\sqrt{klmn}} \delta_{kn} \delta_{lm} \\
& = \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger a_n^\dagger a_k a_l \delta_{k+l, m+n} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger a_l \delta_{kn} \delta_{lm} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger a_k \delta_{ln} \delta_{km} \\
& + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_n^\dagger a_l \delta_{km} \delta_{ln} + \sum_{klmn} \frac{1}{\sqrt{klmn}} \delta_{km} \delta_{ln} + \sum_{klmn} \frac{1}{\sqrt{klmn}} a_n^\dagger a_k \delta_{lm} \delta_{kn} \\
& + \sum_{klmn} \frac{1}{\sqrt{klmn}} \delta_{kn} \delta_{lm} \\
& = \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger a_n^\dagger a_k a_l \delta_{k+l, m+n} + 4 \sum_{klmn} \frac{1}{\sqrt{klmn}} a_m^\dagger a_l \delta_{kn} \delta_{lm} + 2 \sum_{klmn} \frac{1}{\sqrt{klmn}} \delta_{km} \delta_{ln}.
\end{aligned} \tag{3.33}$$

Collecting the terms, we get

$$H_1 = \frac{\lambda}{4!} 2 \left(\frac{1}{4\pi} \right)^2 \left[6(2\pi) \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_l^\dagger a_m a_n \delta_{k+l, m+n} \right] = \frac{1}{4} \frac{\lambda}{4\pi} \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_l^\dagger a_m a_n \delta_{k+l, m+n}, \tag{3.34}$$

$$\begin{aligned}
H_0(2) & = \frac{\lambda}{4!} 2 \left(\frac{1}{4\pi} \right)^2 \left[12(2\pi) \sum_n \frac{1}{n} a_n^\dagger a_n \sum_m \frac{1}{m} \right] = \frac{\lambda}{4\pi} \frac{1}{2} \sum_n \frac{1}{n} a_n^\dagger a_n \sum_m \frac{1}{m} \\
& = \sum_n \frac{1}{n} a_n^\dagger a_n \left(\frac{\lambda}{4\pi} \frac{1}{2} \sum_k \frac{1}{k} \right).
\end{aligned} \tag{3.35}$$

Combing $H_0(1)$ and $H_0(2)$,

$$H_0 = \sum_n \frac{1}{n} a_n^\dagger a_n \left(\mu^2 + \frac{\lambda}{4\pi} \frac{1}{2} \sum_k \frac{1}{k} \right). \tag{3.36}$$

In summary,

$$H_0 = \sum_n \frac{1}{n} a_n^\dagger a_n \left(\mu^2 + \frac{\lambda}{4\pi} \frac{1}{2} \sum_k \frac{1}{k} \right), \tag{3.37}$$

$$H_1 = \frac{1}{4} \frac{\lambda}{4\pi} \sum_{klmn} \frac{1}{\sqrt{klmn}} a_k^\dagger a_l^\dagger a_m a_n \delta_{k+l, m+n}, \tag{3.38}$$

$$H_2 = \frac{1}{6} \frac{\lambda}{4\pi} \sum_{klmn} \left[\frac{a_k^\dagger a_l a_m a_n + a_n^\dagger a_m^\dagger a_l^\dagger a_k}{\sqrt{klmn}} \right] \delta_{k,m+n+l}. \quad (3.39)$$

Considering that, due to boson symmetry, there is double counting in the summation, we can rewrite the above equations in a more convenient way while adopting the normal-ordered Hamiltonian equivalent to adopting a mass counterterm to remove the logarithmic term,

$$H_0 = \mu^2 \sum_n^K \frac{1}{n} a_n^\dagger a_n, \quad (3.40)$$

$$H_1 = \frac{\lambda}{4\pi} \sum_{k \leq l, m \leq n}^K \frac{1}{N_{kl}^2} \frac{1}{N_{mn}^2} \frac{a_k^\dagger a_l^\dagger a_m a_n}{\sqrt{klmn}} \delta_{m+n, k+l}, \quad (3.41)$$

$$H_2 = \frac{\lambda}{4\pi} \sum_{k, l \leq m \leq n}^K \frac{1}{N_{lmn}^2} \left[\frac{a_k^\dagger a_l a_m a_n + a_n^\dagger a_m^\dagger a_l^\dagger a_k}{\sqrt{klmn}} \right] \delta_{k, m+n+l}, \quad (3.42)$$

where $N_{kl}^2 = 1, k \neq l$; $N_{kl}^2 = 2!, k = l$. And $N_{lmn}^2 = 1, l \neq m \neq n$; $N_{lmn}^2 = 2!, l = m \neq n, l \neq m = n$; $N_{lmn}^2 = 3!, l = m = n$. K is the dimensionless total light-front momentum defined as before.

3.3 Numerical Procedure without Zero Mode

The light-front longitudinal momentum can be expressed more explicitly. Since

$$\begin{aligned} P^+ &= \frac{1}{2} \int dx^- T^{++} = \frac{1}{2} \int dx^- \partial^+ \phi \partial^+ \phi \\ &= \frac{1}{2} \frac{1}{4\pi} \int dx^- \sum_n \frac{1}{\sqrt{n}} [(-ik_n^+) a_n e^{-ik_n^+ x^+} + (ik_n^+) a_n^\dagger e^{ik_n^+ x^+}] \\ &\quad \times \sum_m \frac{1}{\sqrt{m}} [(-ik_m^+) a_m e^{-ik_m^+ x^+} + (ik_m^+) a_m^\dagger e^{ik_m^+ x^+}] \\ &= \frac{1}{2} \frac{1}{4\pi} \sum_n \sum_m \frac{1}{\sqrt{nm}} \int dx^- [(-ik_n^+) a_n e^{-\frac{i}{2} k_n^+ x^-} + (ik_n^+) a_n^\dagger e^{\frac{i}{2} k_n^+ x^-}] \\ &\quad \times [(-ik_m^+) a_m e^{-\frac{i}{2} k_m^+ x^-} + (ik_m^+) a_m^\dagger e^{\frac{i}{2} k_m^+ x^-}] \\ &= \frac{1}{2} \frac{1}{4\pi} \sum_n \sum_m \frac{1}{\sqrt{nm}} \int dx^- [k_n^+ k_m^+ a_n^\dagger a_m e^{\frac{i}{2} (k_n^+ - k_m^+) x^-} + k_n^+ k_m^+ a_n a_m^\dagger e^{-\frac{i}{2} (k_n^+ - k_m^+) x^-}], \end{aligned} \quad (3.43)$$

where we used the fact that $k_n^+ + k_m^+ > 0$ to eliminate two out of four terms in the expansion. Using

$$k_n^+ = \frac{2\pi n}{L},$$

$$\begin{aligned}
P^+ &= \frac{1}{2} \frac{1}{4\pi} \sum_n \sum_m \frac{1}{\sqrt{nm}} [k_n^+ k_m^+ a_n^\dagger a_m 4\pi \delta(k_n^+ - k_m^+) + k_n^+ k_m^+ a_n a_m^\dagger 4\pi \delta(k_n^+ - k_m^+)] \\
&= \frac{1}{2} \frac{1}{4\pi} \sum_n \sum_m \frac{1}{\sqrt{nm}} [k_n^+ k_m^+ a_n^\dagger a_m 4\pi \frac{L}{2\pi} \delta_{nm} + k_n^+ k_m^+ a_n a_m^\dagger 4\pi \frac{L}{2\pi} \delta_{nm}] \\
&= \frac{1}{2} \frac{1}{4\pi} \sum_n \frac{2L}{n} k_n^+ k_n^+ [a_n^\dagger a_n + a_n a_n^\dagger] \\
&= \frac{1}{2} \frac{1}{4\pi} \sum_n \frac{2L}{n} \frac{2\pi n}{L} \frac{2\pi n}{L} [a_n^\dagger a_n + (a_n^\dagger a_n + \delta_{n,n})] \\
&= \frac{\pi}{L} \sum_n n [2a_n^\dagger a_n + 1] \\
&= \frac{2\pi}{L} \sum_n n a_n^\dagger a_n + \frac{\pi}{L} \sum_{n=1} n \\
&= \frac{2\pi}{L} \sum_n n a_n^\dagger a_n + \infty \\
&= \frac{2\pi}{L} \sum_n n a_n^\dagger a_n.
\end{aligned} \tag{3.44}$$

Therefore,

$$K = \frac{L}{2\pi} P^+ = \sum_n n a_n^\dagger a_n, \tag{3.45}$$

which is just the summation of the product of momentum quanta n and the number operator $\hat{m} = a_n^\dagger a_n$.

The Hamiltonian is

$$H = H_0 + H_1 + H_2. \tag{3.46}$$

The mass square is related to the longitudinal momentum and Hamiltonian by

$$M^2 = P^+ P^- = K H. \tag{3.47}$$

Many body states are represented by Fock-space basis $|n_1^{m_1}, n_2^{m_2}, n_3^{m_3}, \dots, n_i^{m_i}, \dots\rangle$ for m_1 quanta with n_1 units of momentum and so on.

The annihilation operator and the creation operator acting on the basis results in

$$a_{n_i}|n_1^{m_1}, n_2^{m_2}, n_3^{m_3}, \dots, n_i^{m_i}, \dots\rangle = \sqrt{m_i}|n_1^{m_1}, n_2^{m_2}, n_3^{m_3}, \dots, n_i^{m_i-1}, \dots\rangle \quad (3.48)$$

and

$$a_{n_i}^\dagger|n_1^{m_1}, n_2^{m_2}, n_3^{m_3}, \dots, n_i^{m_i}, \dots\rangle = \sqrt{m_i+1}|n_1^{m_1}, n_2^{m_2}, n_3^{m_3}, \dots, n_i^{m_i+1}, \dots\rangle. \quad (3.49)$$

Let us consider cases with specific values of total light-front momentum, K , defined as $K = \sum_i n_i \cdot m_i$.

- $K=0$. $M^2 = KH = 0$. Since we have neglected zero modes ($k^+=0$ states), the only basis state is the vacuum state $|\text{vac}\rangle$,

$$M^2|\text{vac}\rangle = 0|\text{vac}\rangle. \quad (3.50)$$

- $K=1$. $M^2 = KH = H$. We have a single state $|1^1\rangle$.

$$\langle 1^1|M^2|1^1\rangle = \langle 1^1|H|1^1\rangle = \langle 1^1|H_0|1^1\rangle = \langle 1^1|\sum_n \frac{1}{n}a_n^\dagger a_n \mu^2|1^1\rangle = \langle 1^1|\frac{1}{1}a_1^\dagger a_1 \mu^2|1^1\rangle = \mu^2. \quad (3.51)$$

- $K=2$. $M^2 = KH = 2H$. We have two states $|2^1\rangle$ and $|1^2\rangle$.

$$\langle 2^1|M^2|2^1\rangle = 2\langle 2^1|H|2^1\rangle = 2\langle 2^1|H_0|2^1\rangle = 2\langle 2^1|\sum_n \frac{1}{n}a_n^\dagger a_n \mu^2|2^1\rangle = 2\langle 2^1|\frac{1}{2}a_2^\dagger a_2 \mu^2|2^1\rangle = \mu^2; \quad (3.52)$$

$$\langle 1^2|M^2|1^2\rangle = 2\langle 1^2|H|1^2\rangle = 2\langle 1^2|H_0 + H_1|1^2\rangle \quad (3.53)$$

has two terms,

$$\langle 1^2|H_0|1^2\rangle = \langle 1^2|\sum_n \frac{1}{n}a_n^\dagger a_n \mu^2|1^2\rangle = \langle 1^2|\frac{1}{1}a_1^\dagger a_1 \mu^2|1^2\rangle = \langle 1^2|\frac{1}{1}2\mu^2|1^2\rangle = 2\mu^2 \quad (3.54)$$

and

$$\begin{aligned} \langle 1^2|H_1|1^2\rangle &= \langle 1^2|\frac{1}{4}\frac{\lambda}{4\pi}\sum_{klmn}\frac{a_k^\dagger a_l^\dagger a_m a_n}{\sqrt{klmn}}\delta_{m+n,k+l}|1^2\rangle = \langle 1^2|\frac{1}{4}\frac{\lambda}{4\pi}\frac{a_1^\dagger a_1^\dagger a_1 a_1}{\sqrt{1\times 1\times 1\times 1}}\delta_{1+1,1+1}|1^2\rangle \\ &= \langle 1^2|\frac{1}{4}\frac{\lambda}{4\pi}\sqrt{2}\sqrt{1}\sqrt{1}\sqrt{2}|1^2\rangle = \frac{1}{2}\frac{\lambda}{4\pi}, \end{aligned} \quad (3.55)$$

so

$$\langle 1^2 | M^2 | 1^2 \rangle = 4\mu^2 + \frac{\lambda}{4\pi}; \quad (3.56)$$

for the off-diagonal terms,

$$\langle 2^1 | M^2 | 1^2 \rangle = 2\langle 2^1 | H | 1^2 \rangle = 2\langle 2^1 | H_1 + H_2 | 1^2 \rangle, \quad (3.57)$$

for H_1 , $m = 1$, $n = 1$, and $k + l = m + n = 2$. In order to matching the right bracket, we should let $l = 2$. This will impose $k = 0$ which is impossible because we omit zero modes in the dynamics (to be reconsidered as later with the constraint equation). For H_2 , consider $a_n^\dagger a_m^\dagger a_l^\dagger a_k$. $k = 1$, $l = 2$, and $k = m + n + l$ is impossible. Therefore, in general, the even boson and odd boson sectors are decoupled and, in this particular case, we see this decoupling with

$$\langle 2^1 | M^2 | 1^2 \rangle = \langle 1^2 | M^2 | 2^1 \rangle = 0. \quad (3.58)$$

- K=3. $K = n_1 m_1 + n_2 m_2 + \dots$, so $3 = 1 \times 3 = 1 \times 1 + 2 \times 1 = 3 \times 1$. This case has three states: $|1^3\rangle$, $|1^1, 2^1\rangle$ and $|3^1\rangle$.

$$\begin{aligned} \langle 1^3 | M^2 | 1^3 \rangle &= 3\langle 1^3 | H | 1^3 \rangle = 3\langle 1^3 | H_0 + H_1 | 1^3 \rangle \\ &= 3\langle 1^3 | H_0 | 1^3 \rangle + 3\langle 1^3 | H_1 | 1^3 \rangle \\ &= 3\langle 1^3 | \mu^2 \sum_n \frac{1}{n} a_n^\dagger a_n | 1^3 \rangle + 3\langle 1^3 | \frac{1}{4} \frac{\lambda}{4\pi} \sum_{klmn} \frac{a_k^\dagger a_l^\dagger a_m a_n}{\sqrt{klmn}} \delta_{m+n, k+l} | 1^3 \rangle \\ &= 3\langle 1^3 | \mu^2 \frac{1}{1} a_1^\dagger a_1 | 1^3 \rangle + 3\langle 1^3 | \frac{1}{4} \frac{\lambda}{4\pi} \frac{a_1^\dagger a_1^\dagger a_1 a_1}{1} | 1^3 \rangle \\ &= 3\langle 1^3 | \mu^2 \frac{1}{1} 3 | 1^3 \rangle + 3\langle 1^3 | \frac{1}{4} \frac{\lambda}{4\pi} \frac{2 \times 3}{1} | 1^3 \rangle \\ &= 9\mu^2 + \frac{\lambda}{4\pi} \frac{9}{2} \end{aligned} \quad (3.59)$$

$$\begin{aligned}
\langle 1^1, 2^1 | M^2 | 1^1, 2^1 \rangle &= 3 \langle 1^1, 2^1 | H | 1^1, 2^1 \rangle = 3 \langle 1^1, 2^1 | H_0 + H_1 | 1^1, 2^1 \rangle \\
&= 3 \langle 1^1, 2^1 | H_0 | 1^1, 2^1 \rangle + 3 \langle 1^1, 2^1 | H_1 | 1^1, 2^1 \rangle \\
&= 3 \langle 1^1, 2^1 | \mu^2 \sum_n \frac{1}{n} a_n^\dagger a_n | 1^1, 2^1 \rangle \\
&\quad + 3 \langle 1^1, 2^1 | \frac{1}{4} \frac{\lambda}{4\pi} \sum_{klmn} \frac{a_k^\dagger a_l^\dagger a_m a_n}{\sqrt{klmn}} \delta_{m+n, k+l} | 1^1, 2^1 \rangle \\
&= 3 \langle 1^1, 2^1 | \mu^2 (\frac{1}{1} a_1^\dagger a_1 + \frac{1}{2} a_2^\dagger a_2) | 1^1, 2^1 \rangle + 3 \langle 1^1, 2^1 | \frac{1}{4} \frac{\lambda}{4\pi} (\frac{a_1^\dagger a_2^\dagger a_2 a_1}{2} \\
&\quad + \frac{a_1^\dagger a_2^\dagger a_1 a_2}{2} + \frac{a_2^\dagger a_1^\dagger a_2 a_1}{2} + \frac{a_2^\dagger a_1^\dagger a_1 a_2}{2}) | 1^1, 2^1 \rangle \\
&= 3 \langle 1^1, 2^1 | \mu^2 (\frac{1}{1} 1 + \frac{1}{2} 1) | 1^1, 2^1 \rangle + 3 \langle 1^1, 2^1 | \frac{1}{4} \frac{\lambda}{4\pi} (\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}) | 1^1, 2^1 \rangle \\
&= \frac{9}{2} \mu^2 + \frac{\lambda}{4\pi} \frac{3}{2}
\end{aligned} \tag{3.60}$$

$$\begin{aligned}
\langle 3^1 | M^2 | 1^3 \rangle &= 3 \langle 3^1 | H | 1^3 \rangle = 3 \langle 3^1 | H_2 | 1^3 \rangle \\
&= 3 \langle 3^1 | \frac{1}{6} \frac{\lambda}{4\pi} \sum_{klmn} [\frac{a_k^\dagger a_l^\dagger a_m a_n + a_n^\dagger a_m^\dagger a_l^\dagger a_k}{\sqrt{klmn}}] \delta_{k, m+n+l} | 1^3 \rangle \\
&= 3 \langle 3^1 | \frac{1}{6} \frac{\lambda}{4\pi} \sum_{klmn} [\frac{a_k^\dagger a_l^\dagger a_m a_n}{\sqrt{klmn}}] \delta_{k, m+n+l} | 1^3 \rangle \\
&= 3 \langle 3^1 | \frac{1}{6} \frac{\lambda}{4\pi} [\frac{a_3^\dagger a_1 a_1 a_1}{\sqrt{3}}] | 1^3 \rangle \\
&= 3 \langle 3^1 | \frac{1}{6} \frac{\lambda}{4\pi} [\frac{\sqrt{1}\sqrt{1}\sqrt{2}\sqrt{3}}{\sqrt{3}}] | 3^1 \rangle \\
&= \frac{\lambda}{4\pi} \frac{\sqrt{2}}{2}
\end{aligned} \tag{3.61}$$

We now arrange the results in a matrix with the order of the basis states $|3^1\rangle, |1^1, 2^1\rangle, |1^3\rangle$,

$$\langle M^2 \rangle = \begin{bmatrix} \mu^2 & 0 & \frac{\lambda}{4\pi} \frac{\sqrt{2}}{2} \\ 0 & \frac{9}{2} \mu^2 + \frac{\lambda}{4\pi} \frac{3}{2} & 0 \\ \frac{\lambda}{4\pi} \frac{\sqrt{2}}{2} & 0 & 9\mu^2 + \frac{\lambda}{4\pi} \frac{9}{2} \end{bmatrix}. \tag{3.62}$$

Direct diagonalization of this Hamiltonian produces a ground state mass square eigenvalue (after multiplication by K) as shown in Figure 3.1 and Figure 3.2. This shows the vanishing

mass gap that signals a critical coupling at this value of K . Note that, as expected, the vanishing mass gap depends only on the ratio of $\lambda/\mu^2 = 134.2673778251021$ to within numerical precision.

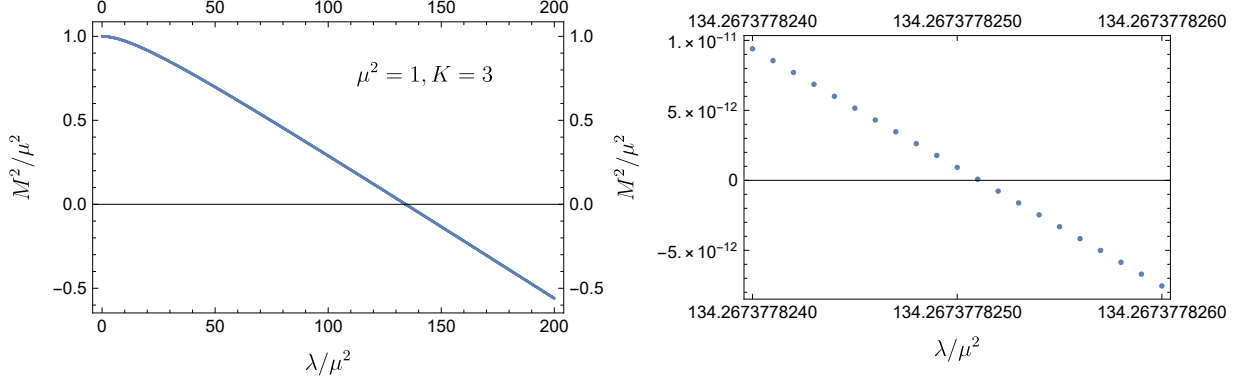


Figure 3.1 The physical mass square M^2 as a function of λ (both scaled by μ^2) with $\mu^2=1.0$ for $K = 3$. The right panel is an enlarged graph near the x-intercept (134.2673778251).

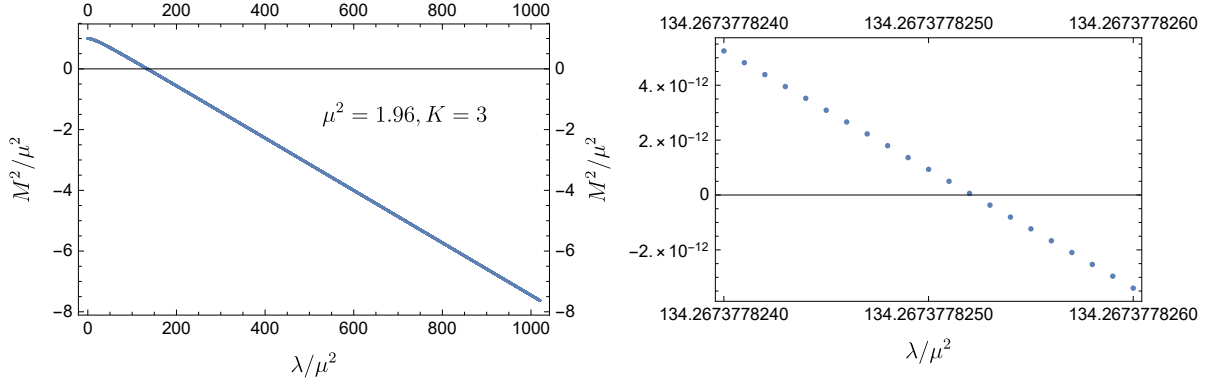


Figure 3.2 The physical mass square M^2 as a function of λ (both scaled by μ^2) with $\mu^2=1.96$ for $K = 3$. The right panel is an enlarged graph near the x-intercept (134.2673778251021).

- $K=4$. $4 = 1 \times 4 = 1 \times 2 + 2 \times 1 = 1 \times 1 + 3 \times 1 = 4 \times 1 = 2 \times 2$, corresponding to $|1^4\rangle$, $|1^2, 2^1\rangle$, $|1^1, 3^1\rangle$, $|4^1\rangle$, $|2^2\rangle$. There are 2 two-particle states $|1^1, 3^1\rangle$ and $|2^2\rangle$, 1 three particle state $|1^2, 2^1\rangle$ and 1 four-particle state $|1^4\rangle$.

$$\begin{aligned}
\langle 2^2 | M^2 | 2^2 \rangle &= 4 \langle 2^2 | H | 2^2 \rangle = 4 \langle 2^2 | H_0 + H_1 | 2^2 \rangle \\
&= 4 \langle 2^2 | H_0 | 2^2 \rangle + 4 \langle 2^2 | H_1 | 2^2 \rangle \\
&= 4 \langle 2^2 | \mu^2 \sum_n \frac{1}{n} a_n^\dagger a_n | 2^2 \rangle + 4 \langle 2^2 | \frac{1}{4} \frac{\lambda}{4\pi} \sum_{klmn} \frac{a_k^\dagger a_l^\dagger a_m a_n}{\sqrt{klmn}} \delta_{m+n, k+l} | 2^2 \rangle \\
&= 4 \langle 2^2 | \mu^2 \frac{1}{2} a_2^\dagger a_2 | 2^2 \rangle + 4 \langle 2^2 | \frac{1}{4} \frac{\lambda}{4\pi} \frac{a_2^\dagger a_2^\dagger a_2 a_2}{\sqrt{16}} | 2^2 \rangle \\
&= 4 \langle 2^2 | \mu^2 \frac{1}{2} 2 | 2^2 \rangle + 4 \langle 2^2 | \frac{1}{4} \frac{\lambda}{4\pi} \frac{2}{4} | 2^2 \rangle \\
&= 4\mu^2 + \frac{\lambda}{4\pi} \frac{1}{2},
\end{aligned} \tag{3.63}$$

$$\begin{aligned}
\langle 1^1, 3^1 | M^2 | 1^1, 3^1 \rangle &= 4 \langle 1^1, 3^1 | H | 1^1, 3^1 \rangle = 4 \langle 1^1, 3^1 | H_0 | 1^1, 3^1 \rangle + 4 \langle 1^1, 3^1 | H_1 | 1^1, 3^1 \rangle \\
&= 4 \langle 1^1, 3^1 | \mu^2 \sum_n \frac{1}{n} a_n^\dagger a_n | 1^1, 3^1 \rangle \\
&\quad + 4 \langle 1^1, 3^1 | \frac{1}{4} \frac{\lambda}{4\pi} \sum_{klmn} \frac{a_k^\dagger a_l^\dagger a_m a_n}{\sqrt{klmn}} \delta_{m+n, k+l} | 1^1, 3^1 \rangle \\
&= 4 \langle 1^1, 3^1 | \mu^2 \left(\frac{1}{1} a_1^\dagger a_1 + \frac{1}{3} a_3^\dagger a_3 \right) | 1^1, 3^1 \rangle + 4 \langle 1^1, 3^1 | \frac{1}{4} \frac{\lambda}{4\pi} \\
&\quad \left(\frac{a_1^\dagger a_3^\dagger a_3 a_1}{3} + \frac{a_3^\dagger a_1^\dagger a_3 a_1}{3} + \frac{a_3^\dagger a_1^\dagger a_1 a_3}{3} + \frac{a_1^\dagger a_3^\dagger a_1 a_3}{3} \right) | 1^1, 3^1 \rangle \\
&= 4 \langle 1^1, 3^1 | \mu^2 \left(\frac{1}{1} 1 + \frac{1}{3} 1 \right) | 1^1, 3^1 \rangle + 4 \langle 1^1, 3^1 | \frac{1}{4} \frac{\lambda}{4\pi} \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \right) | 1^1, 3^1 \rangle \\
&= \frac{16}{3} \mu^2 + \frac{\lambda}{4\pi} \frac{4}{3},
\end{aligned} \tag{3.64}$$

$$\begin{aligned}
\langle 1^2, 2^1 | M^2 | 1^2, 2^1 \rangle &= 4 \langle 1^2, 2^1 | H | 1^2, 2^1 \rangle = 4 \langle 1^2, 2^1 | H_0 + H_1 | 1^2, 2^1 \rangle \\
&= 4 \langle 1^2, 2^1 | \mu^2 \sum_n \frac{1}{n} a_n^\dagger a_n | 1^2, 2^1 \rangle \\
&\quad + 4 \langle 1^2, 2^1 | \frac{1}{4} \frac{\lambda}{4\pi} \sum_{klmn} \frac{a_k^\dagger a_l^\dagger a_m a_n}{\sqrt{klmn}} \delta_{m+n, k+l} | 1^2, 2^1 \rangle \\
&= 4 \langle 1^2, 2^1 | \mu^2 (\frac{1}{1} a_1^\dagger a_1 + \frac{1}{2} a_2^\dagger a_2) | 1^2, 2^1 \rangle + 4 \langle 1^2, 2^1 | \frac{1}{4} \frac{\lambda}{4\pi} \\
&\quad (\frac{a_1^\dagger a_1^\dagger a_1 a_1}{1} + \frac{a_1^\dagger a_2^\dagger a_2 a_1}{2} + \frac{a_2^\dagger a_1^\dagger a_2 a_1}{2} + \frac{a_2^\dagger a_1^\dagger a_1 a_2}{2} + \frac{a_1^\dagger a_2^\dagger a_1 a_2}{2}) | 1^2, 2^1 \rangle \\
&= 4 \langle 1^2, 2^1 | \mu^2 (\frac{1}{1} 2 + \frac{1}{2} 1) | 1^2, 2^1 \rangle + 4 \langle 1^2, 2^1 | \frac{1}{4} \frac{\lambda}{4\pi} (\frac{2}{1} + \frac{2}{2} + \frac{2}{2} + \frac{2}{2} + \frac{2}{2}) | 1^2, 2^1 \rangle \\
&= 10\mu^2 + \frac{\lambda}{4\pi} 6,
\end{aligned} \tag{3.65}$$

$$\begin{aligned}
\langle 1^4 | M^2 | 1^4 \rangle &= 4 \langle 1^4 | H | 1^4 \rangle = 4 \langle 1^4 | H_0 + H_1 | 1^4 \rangle \\
&= 4 \langle 1^4 | \mu^2 \sum_n \frac{1}{n} a_n^\dagger a_n | 1^4 \rangle + 4 \langle 1^4 | \frac{1}{4} \frac{\lambda}{4\pi} \sum_{klmn} \frac{a_k^\dagger a_l^\dagger a_m a_n}{\sqrt{klmn}} \delta_{m+n, k+l} | 1^4 \rangle \\
&= 4 \langle 1^4 | \mu^2 \frac{1}{1} a_1^\dagger a_1 | 1^4 \rangle + 4 \langle 1^4 | \frac{1}{4} \frac{\lambda}{4\pi} \frac{a_1^\dagger a_1^\dagger a_1 a_1}{1} | 1^4 \rangle \\
&= 4 \langle 1^4 | \mu^2 \frac{1}{1} 4 | 1^4 \rangle + 4 \langle 1^4 | \frac{1}{4} \frac{\lambda}{4\pi} \frac{3 \times 4}{1} | 1^4 \rangle \\
&= 16\mu^2 + \frac{\lambda}{4\pi} 12,
\end{aligned} \tag{3.66}$$

$$\begin{aligned}
\langle 4^1 | M^2 | 1^2, 2^1 \rangle &= 4 \langle 4^1 | H | 1^2, 2^1 \rangle = 4 \langle 4^1 | H_2 | 1^2, 2^1 \rangle \\
&= 4 \langle 4^1 | \frac{1}{6} \frac{\lambda}{4\pi} \sum_{klmn} [\frac{a_k^\dagger a_l^\dagger a_m a_n + a_n^\dagger a_m^\dagger a_l^\dagger a_k}{\sqrt{klmn}}] \delta_{k, m+n+l} | 1^2, 2^1 \rangle \\
&= 4 \langle 4^1 | \frac{1}{6} \frac{\lambda}{4\pi} \sum_{klmn} [\frac{a_k^\dagger a_l^\dagger a_m a_n}{\sqrt{klmn}}] \delta_{k, m+n+l} | 1^2, 2^1 \rangle \\
&= 4 \langle 4^1 | \frac{1}{6} \frac{\lambda}{4\pi} [\frac{a_4^\dagger a_2 a_1 a_1}{\sqrt{8}} + \frac{a_4^\dagger a_1 a_2 a_1}{\sqrt{8}} + \frac{a_4^\dagger a_1 a_1 a_2}{\sqrt{8}}] | 1^2, 2^1 \rangle \\
&= 4 \langle 4^1 | \frac{1}{6} \frac{\lambda}{4\pi} [\frac{\sqrt{2}}{\sqrt{8}} + \frac{\sqrt{2}}{\sqrt{8}} + \frac{\sqrt{2}}{\sqrt{8}}] | 1^2, 2^1 \rangle \\
&= 4 \frac{1}{6} \frac{\lambda}{4\pi} \frac{3}{2} = \frac{\lambda}{4\pi},
\end{aligned} \tag{3.67}$$

$$\begin{aligned}
\langle 1^1, 3^1 | M^2 | 1^4 \rangle &= 4 \langle 1^1, 3^1 | H | 1^4 \rangle = 4 \langle 1^1, 3^1 | H_2 | 1^4 \rangle \\
&= 4 \langle 1^1, 3^1 | \frac{1}{6} \frac{\lambda}{4\pi} \sum_{klmn} [\frac{a_k^\dagger a_l a_m a_n + a_n^\dagger a_m^\dagger a_l^\dagger a_k}{\sqrt{klmn}}] \delta_{k,m+n+l} | 1^4 \rangle \\
&= 4 \langle 1^1, 3^1 | \frac{1}{6} \frac{\lambda}{4\pi} \sum_{klmn} [\frac{a_k^\dagger a_l a_m a_n}{\sqrt{klmn}}] \delta_{k,m+n+l} | 1^4 \rangle \\
&= 4 \langle 1^1, 3^1 | \frac{1}{6} \frac{\lambda}{4\pi} \sum_{klmn} [\frac{a_3^\dagger a_1 a_1 a_1}{\sqrt{3}}] | 1^4 \rangle \\
&= 4 \langle 1^1, 3^1 | \frac{1}{6} \frac{\lambda}{4\pi} \sum_{klmn} [\frac{\sqrt{2 \times 3 \times 4}}{\sqrt{3}}] | 1^4 \rangle \\
&= 4 \frac{1}{6} \frac{\lambda}{4\pi} 2\sqrt{2} = \frac{\lambda}{4\pi} \frac{4\sqrt{2}}{3},
\end{aligned} \tag{3.68}$$

$$\begin{aligned}
\langle 1^1, 3^1 | M^2 | 2^2 \rangle &= 4 \langle 1^1, 3^1 | H | 2^2 \rangle = 4 \langle 1^1, 3^1 | H_1 | 2^2 \rangle \\
&= 4 \langle 1^1, 3^1 | \frac{1}{4} \frac{\lambda}{4\pi} \sum_{klmn} \frac{a_k^\dagger a_l^\dagger a_m a_n}{\sqrt{klmn}} \delta_{m+n,k+l} | 2^2 \rangle \\
&= 4 \langle 1^1, 3^1 | \frac{1}{4} \frac{\lambda}{4\pi} (\frac{a_1^\dagger a_3^\dagger a_2 a_2}{\sqrt{1 \times 3 \times 2 \times 2}} + \frac{a_3^\dagger a_1^\dagger a_2 a_2}{\sqrt{3 \times 1 \times 2 \times 2}}) | 2^2 \rangle \\
&= 4 \frac{1}{4} \frac{\lambda}{4\pi} \langle 1^1, 3^1 | (\frac{\sqrt{1}\sqrt{1}\sqrt{1}\sqrt{2}}{2\sqrt{3}} + \frac{\sqrt{1}\sqrt{1}\sqrt{1}\sqrt{2}}{2\sqrt{3}}) | 2^2 \rangle \\
&= 4 \frac{1}{4} \frac{\lambda}{4\pi} \sqrt{\frac{2}{3}} = \frac{\lambda}{4\pi} \sqrt{\frac{2}{3}}.
\end{aligned} \tag{3.69}$$

We now arrange the results in a matrix with the order of the basis states $|4^1\rangle$, $|2^2\rangle$, $|1^1, 3^1\rangle$, $|1^2, 2^1\rangle$, $|1^4\rangle$,

$$\langle M^2 \rangle = \begin{bmatrix} \mu^2 & 0 & 0 & \frac{\lambda}{4\pi} & 0 \\ 0 & 4\mu^2 + \frac{\lambda}{4\pi} \frac{1}{2} & \frac{\lambda}{4\pi} \sqrt{\frac{2}{3}} & 0 & 0 \\ 0 & \frac{\lambda}{4\pi} \sqrt{\frac{2}{3}} & \frac{16}{3} \mu^2 + \frac{\lambda}{4\pi} \frac{4}{3} & 0 & \frac{\lambda}{4\pi} \frac{4\sqrt{2}}{3} \\ \frac{\lambda}{4\pi} & 0 & 0 & 10\mu^2 + \frac{\lambda}{4\pi} 6 & 0 \\ 0 & 0 & \frac{\lambda}{4\pi} \frac{4\sqrt{2}}{3} & 0 & 16\mu^2 + \frac{\lambda}{4\pi} 12 \end{bmatrix}. \tag{3.70}$$

Direct diagonalization of this Hamiltonian produces a ground state mass square eigenvalue (after multiplication by K) as shown in Figure 3.3 and Figure 3.4. This shows the vanishing mass gap that signals a critical coupling at this value of K. Note that, as expected, the

vanishing mass gap depends only on the ratio of $\lambda/\mu^2 = 92.4746514381633$ to within numerical precision.

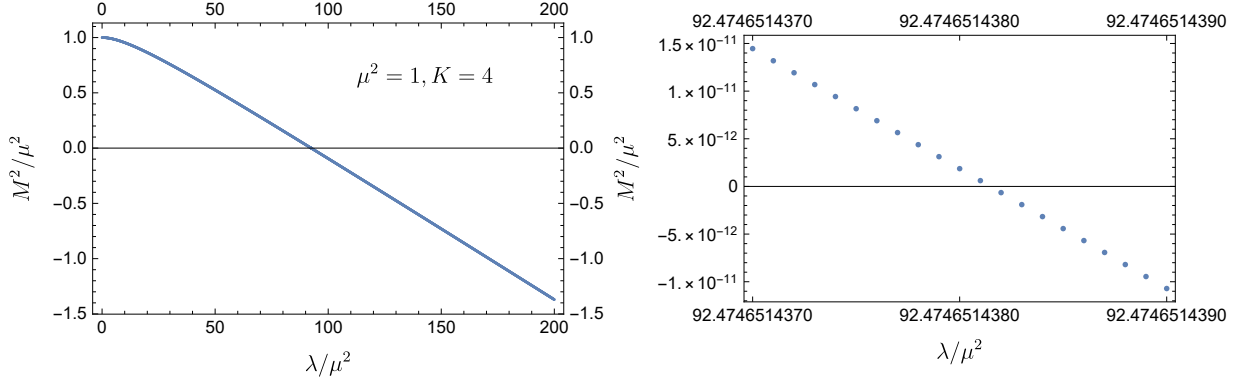


Figure 3.3 The physical mass square M^2 as a function of λ (both scaled by μ^2) with $\mu^2=1.0$ for $K = 4$. The right panel is an enlarged graph near the x-intercept (92.4746514381).

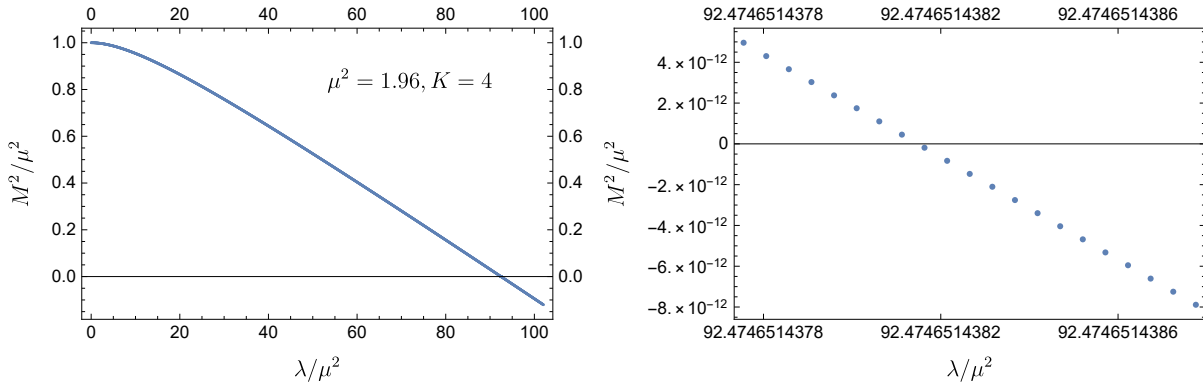


Figure 3.4 The physical mass square M^2 as a function of λ (both scaled by μ^2) with $\mu^2=1.96$ for $K = 4$. The right panel is an enlarged graph near the x-intercept (92.4746514381633).

If we fixed the physical mass (such as $\mu^2 = 1$), and iterate to get the μ^2 that produces this fixed μ^2 , we would be engaged in the process of mass renormalization. With such a mass renormalization method, the mass gap does not vanish and λ/μ^2 saturates at sufficiently large λ , as shown in Figure 3.5, restraining the theory to the weak coupling regime. Since our goal is to determine the critical coupling, we will perform calculations without mass renormalization.

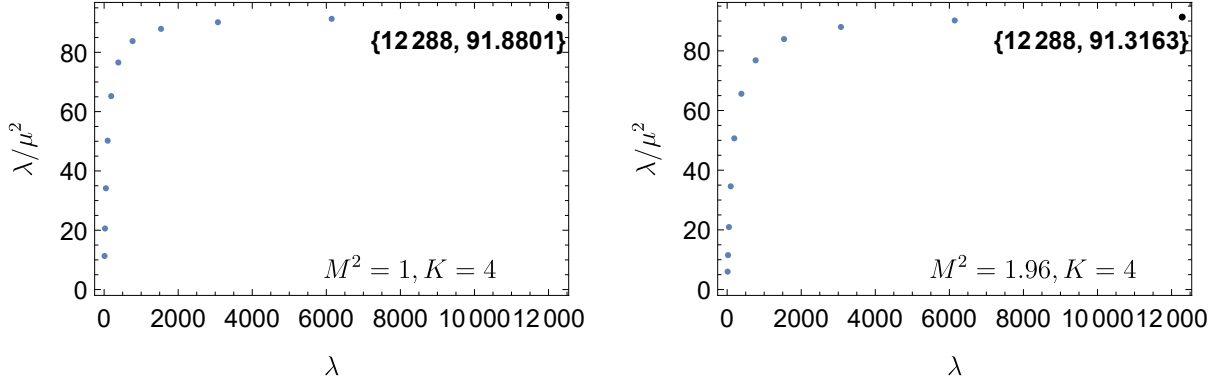


Figure 3.5 The intrinsic dimensionless coupling λ/μ^2 as a function of λ with $\mu^2 = 1.0$ (left panel) and $\mu^2 = 1.96$ (right panel) for $K=4$. The coordinates of the last point are shown in the graph.

3.4 Using MPI

One way to go to higher K , which consumes more computational memory space, is to parallelize the code. I wrote a C++ code to obtain the Hamiltonian matrix elements. Then the matrix diagonalization is done with Intel[®] MKL library, and is parallelized using MPI routines in Fortran. The computational resource I used here is the high performance computing facility in Durham Center in Iowa State University. The critical coupling can be found by iteration around $M^2 = 0$.

For example, part of the $K = 16$ result is shown in Figure 3.6. The discontinuous trend seen at the last few points might indicate that the previous ground state switches to a different state, which we will explore in future.

Changing the number of MPI processors, we can see a clear speed up in calculation in Figure 3.7.

In Figure 3.8, the critical coupling for $K=14, 16, 18, 20, 24, 28, 32$ are plotted. A quadratic extrapolation is performed to get the critical coupling at $K \rightarrow \infty$, which is $\lambda_c = 30.2547$.

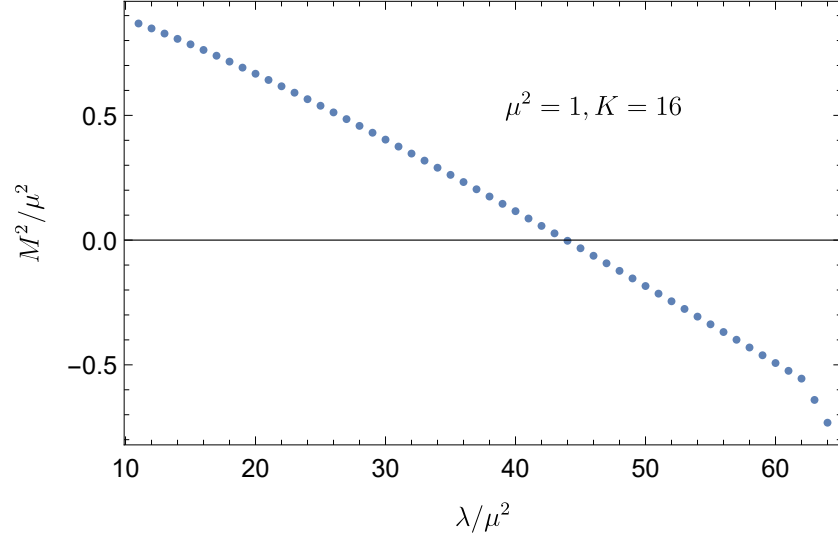


Figure 3.6 The physical mass square M^2 as a function of λ with $\mu^2=1$ for $K = 16$ in the first iteration. After ten iterations the x-intercept is found to be 43.908354957.

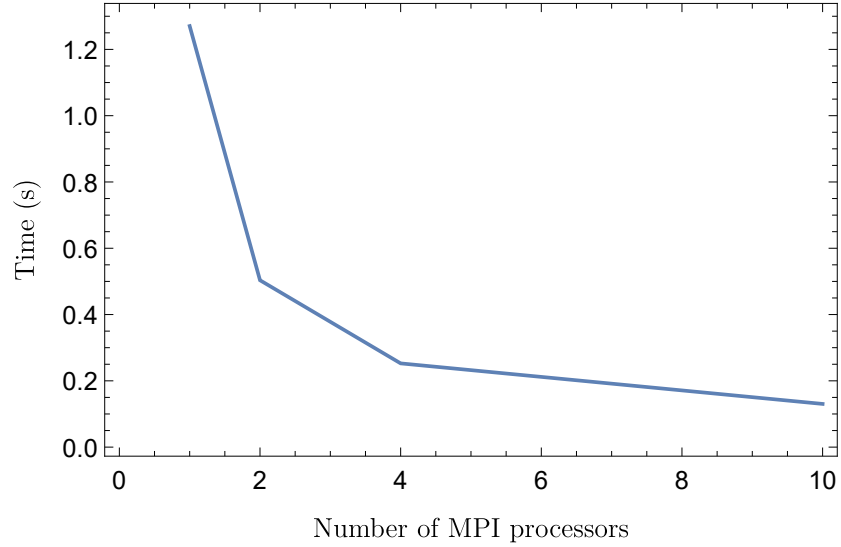


Figure 3.7 The calculation time with number of MPI processors for $K = 16$ matrix diagonalization

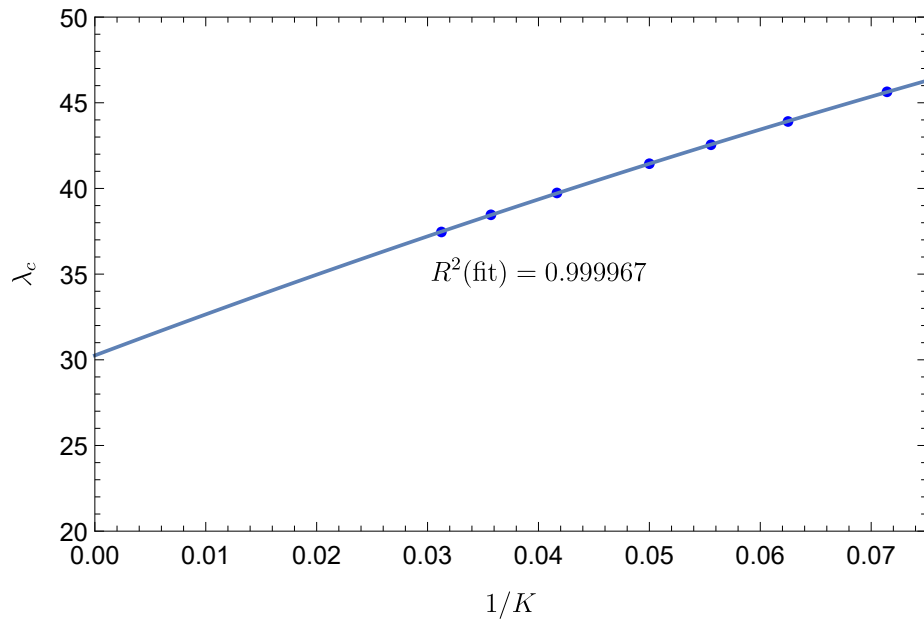


Figure 3.8 The critical coupling extrapolated at $K \rightarrow \infty$ by quadratic fit. The y-intercept is 30.2547.

CHAPTER 4. LIGHT FRONT CRITICAL COUPLING WITH ZERO MODE

4.1 Introduction

The following derivations mainly follow [Burkardt et al. \(2016\)](#), [Chabysheva and Hiller \(2014\)](#) and [Chabysheva et al. \(2013\)](#). These papers claimed that the a symmetric polynomial basis is able to extend the basis function approach to include the effects of the zero mode. Such an extension is believed to facilitate convergence in numerical solutions for the low-lying mass square eigenstates. Eventually, this claim should be checked by an application of the methods of the previous chapter. The key issue, for example, is the quantitative value for the critical coupling in the continuum limit (K to be taken to infinity in the future application of the method of the previous chapter).

4.2 Continuous Light Front Hamiltonian Formalism

We start from the commutation relation [\(3.4\)](#)

$$\left[a(k), a^\dagger(k') \right] = 2\pi 2k^+ \delta(k^+ - k'^+), \quad (4.1)$$

where we change the notation of $a(k^+)$ and $a^\dagger(k'^+)$ to $a(k)$ and $a^\dagger(k')$ for convenience. Since $a^\dagger(k)$ is the conjugate of $a(k)$, both determined by the Fourier transform of the field in 1+1 dimensions (there is no role of k^- or k^\perp), $a(k) = a(k^+)$ and $a^\dagger(k') = a^\dagger(k'^+)$.

We redefine $\tilde{a}(k) = \frac{a(k)}{\sqrt{4\pi k^+}}$ and then change the notation of \tilde{a} to a , k to p , and do the similar replacement for $a^\dagger(k')$. Then

$$\left[a(p), a^\dagger(p') \right] = \delta(p - p'), \quad (4.2)$$

and the mode expansion for the field at zero light-front time [\(3.3\)](#) becomes

$$\phi(0, x^-) = \int \frac{dp}{\sqrt{4\pi p}} [a(p)e^{-\frac{i}{2}px^-} + a^\dagger(p)e^{\frac{i}{2}px^-}]. \quad (4.3)$$

m-particles (constituents) at momentum p_i Fock space is represented

$$|p_i; m\rangle = \frac{1}{\sqrt{m!}} \prod_{i=1}^m a^\dagger(p_i) |0\rangle. \quad (4.4)$$

Defining $y_i \equiv \frac{p_i}{P}$ to be the longitudinal momentum fraction for the i -th constituent,

$$|y_i P; P, m\rangle = \frac{1}{\sqrt{m!}} \prod_{i=1}^m a^\dagger(y_i P) |0\rangle. \quad (4.5)$$

The light-front Hamiltonian density is

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \\ &= \frac{1}{2} \mu^2 \left(\int \frac{dp}{\sqrt{4\pi p}} [a(p) e^{-\frac{i}{2} p x^-} + a^\dagger(p) e^{\frac{i}{2} p x^-}] \right)^2 + \frac{\lambda}{4!} \left(\int \frac{dp}{\sqrt{4\pi p}} [a(p) e^{-\frac{i}{2} p x^-} + a^\dagger(p) e^{\frac{i}{2} p x^-}] \right)^4. \end{aligned} \quad (4.6)$$

The light-front Hamiltonian is

$$\mathcal{P}^- = \int dx^- \mathcal{H}. \quad (4.7)$$

For the first term,

$$\begin{aligned} & \int dx^- \int \frac{dp_1}{\sqrt{4\pi p_1}} [a(p_1) e^{-\frac{i}{2} p_1 x^-} + a^\dagger(p_1) e^{\frac{i}{2} p_1 x^-}] \int \frac{dp_2}{\sqrt{4\pi p_2}} [a(p_2) e^{-\frac{i}{2} p_2 x^-} + a^\dagger(p_2) e^{\frac{i}{2} p_2 x^-}] \\ &= \int dx^- \int \frac{dp_1 dp_2}{4\pi \sqrt{p_1 p_2}} [a(p_1) e^{-\frac{i}{2} p_1 x^-} + a^\dagger(p_1) e^{\frac{i}{2} p_1 x^-}] [a(p_2) e^{-\frac{i}{2} p_2 x^-} + a^\dagger(p_2) e^{\frac{i}{2} p_2 x^-}] \\ &= \int dx^- \int \frac{dp_1 dp_2}{4\pi \sqrt{p_1 p_2}} [a(p_1) e^{-\frac{i}{2} p_1 x^-} a(p_2) e^{-\frac{i}{2} p_2 x^-} + a(p_1) e^{-\frac{i}{2} p_1 x^-} a^\dagger(p_2) e^{\frac{i}{2} p_2 x^-} \\ &\quad + a^\dagger(p_1) e^{\frac{i}{2} p_1 x^-} a(p_2) e^{-\frac{i}{2} p_2 x^-} + a^\dagger(p_1) e^{\frac{i}{2} p_1 x^-} a^\dagger(p_2) e^{\frac{i}{2} p_2 x^-}] \\ &= \int dx^- \int \frac{dp_1 dp_2}{4\pi \sqrt{p_1 p_2}} [a(p_1) a(p_2) e^{-\frac{i}{2} (p_1 + p_2) x^-} + a(p_1) a^\dagger(p_2) e^{-\frac{i}{2} (p_1 - p_2) x^-} \\ &\quad + a^\dagger(p_1) a(p_2) e^{\frac{i}{2} (p_1 - p_2) x^-} + a^\dagger(p_1) a^\dagger(p_2) e^{\frac{i}{2} (p_1 + p_2) x^-}] \\ &= \int \frac{dp_1 dp_2}{\sqrt{p_1 p_2}} [a(p_1) a(p_2) \delta(p_1 + p_2) + a(p_1) a^\dagger(p_2) \delta(p_1 - p_2) \\ &\quad + a^\dagger(p_1) a(p_2) \delta(p_1 - p_2) + a^\dagger(p_1) a^\dagger(p_2) \delta(p_1 + p_2)], \end{aligned} \quad (4.8)$$

where we use $\int dx^- e^{\pm \frac{i}{2} (k^+ - k'^+) x^-} = (2)(2\pi) \delta(k^+ - k'^+)$.

If $p_1 + p_2 > 0$, then the above equation becomes

$$\int \frac{dp_1 dp_2}{\sqrt{p_1 p_2}} [a(p_1) a^\dagger(p_2) \delta(p_1 - p_2) + a^\dagger(p_1) a(p_2) \delta(p_1 - p_2)] = \int \frac{dp}{p} [a(p) a^\dagger(p) + a^\dagger(p) a(p)]. \quad (4.9)$$

After normal ordering, the above result

$$= \int \frac{dp}{p} [2a^\dagger(p)a(p)], \quad (4.10)$$

so

$$\begin{aligned} \mathcal{P}_{11}^- &= \frac{1}{2} \mu^2 \int \frac{dp}{p} 2a^\dagger(p)a(p) \\ &= \int dp \frac{\mu^2}{p} a^\dagger(p)a(p). \end{aligned} \quad (4.11)$$

For the second term,

$$\begin{aligned} & \int dx^- \int \frac{dp_1 dp_2 dp_3 dp_4}{(4\pi)^2 \sqrt{p_1 p_2 p_3 p_4}} [a(p_1) e^{-\frac{i}{2} p_1 x^-} + a^\dagger(p_1) e^{\frac{i}{2} p_1 x^-}] [a(p_2) e^{-\frac{i}{2} p_2 x^-} + a^\dagger(p_2) e^{\frac{i}{2} p_2 x^-}] \\ & [a(p_3) e^{-\frac{i}{2} p_3 x^-} + a^\dagger(p_3) e^{\frac{i}{2} p_3 x^-}] [a(p_4) e^{-\frac{i}{2} p_4 x^-} + a^\dagger(p_4) e^{\frac{i}{2} p_4 x^-}] \\ &= \int dx^- \int \frac{dp_1 dp_2 dp_3 dp_4}{(4\pi)^2 \sqrt{p_1 p_2 p_3 p_4}} [a(p_1) a(p_2) a(p_3) a(p_4) e^{-\frac{i}{2} (p_1 + p_2 + p_3 + p_4) x^-} \\ & + a^\dagger(p_1) a(p_2) a(p_3) a(p_4) e^{-\frac{i}{2} (-p_1 + p_2 + p_3 + p_4) x^-} + a(p_1) a^\dagger(p_2) a(p_3) a(p_4) e^{-\frac{i}{2} (p_1 - p_2 + p_3 + p_4) x^-} \\ & + a(p_1) a(p_2) a^\dagger(p_3) a(p_4) e^{-\frac{i}{2} (p_1 + p_2 - p_3 + p_4) x^-} + a(p_1) a(p_2) a(p_3) a^\dagger(p_4) e^{-\frac{i}{2} (p_1 + p_2 + p_3 - p_4) x^-} + \dots], \end{aligned} \quad (4.12)$$

After normal ordering and changing internal variables, and assuming $p_1 + p_2 + p_3 + p_4 > 0$,

$$\begin{aligned} &= \int dx^- \int \frac{dp_1 dp_2 dp_3 dp_4}{(4\pi)^2 \sqrt{p_1 p_2 p_3 p_4}} [a(p_1) a(p_2) a(p_3) a(p_4) e^{-\frac{i}{2} (p_1 + p_2 + p_3 + p_4) x^-} \\ & + C_4^1 a^\dagger(p_1) a(p_2) a(p_3) a(p_4) e^{-\frac{i}{2} (-p_1 + p_2 + p_3 + p_4) x^-} + C_4^2 a^\dagger(p_1) a^\dagger(p_2) a(p_3) a(p_4) e^{-\frac{i}{2} (-p_1 - p_2 + p_3 + p_4) x^-} \\ & + C_4^3 a^\dagger(p_1) a^\dagger(p_2) a^\dagger(p_3) a(p_4) e^{-\frac{i}{2} (-p_1 - p_2 - p_3 + p_4) x^-} + a^\dagger(p_1) a^\dagger(p_2) a^\dagger(p_3) a^\dagger(p_4) e^{\frac{i}{2} (p_1 + p_2 + p_3 + p_4) x^-}] \\ &= \int \frac{dp_1 dp_2 dp_3 dp_4}{4\pi \sqrt{p_1 p_2 p_3 p_4}} [4a^\dagger(p_1) a(p_2) a(p_3) a(p_4) \delta(-p_1 + p_2 + p_3 + p_4) \\ & + 6a^\dagger(p_1) a^\dagger(p_2) a(p_3) a(p_4) \delta(-p_1 - p_2 + p_3 + p_4) \\ & + 4a^\dagger(p_1) a^\dagger(p_2) a^\dagger(p_3) a(p_4) \delta(-p_1 - p_2 - p_3 + p_4)], \end{aligned} \quad (4.13)$$

so

$$\begin{aligned}
\mathcal{P}_{13}^- &= \frac{\lambda}{4!} \int \frac{dp_1 dp_2 dp_3 dp_4}{4\pi \sqrt{p_1 p_2 p_3 p_4}} 4a^\dagger(p_1)a(p_2)a(p_3)a(p_4)\delta(-p_1 + p_2 + p_3 + p_4) \\
&= \frac{\lambda}{6} \int \frac{dp_2 dp_3 dp_4}{4\pi \sqrt{(p_2 + p_3 + p_4)p_2 p_3 p_4}} a^\dagger(p_2 + p_3 + p_4)a(p_2)a(p_3)a(p_4) \\
&= \frac{\lambda}{6} \int \frac{dp_1 dp_2 dp_3}{4\pi \sqrt{(p_1 + p_2 + p_3)p_1 p_2 p_3}} a^\dagger(p_1 + p_2 + p_3)a(p_1)a(p_2)a(p_3),
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
\mathcal{P}_{22}^- &= \frac{\lambda}{4!} \int \frac{dp_1 dp_2 dp_3 dp_4}{4\pi \sqrt{p_1 p_2 p_3 p_4}} 6a^\dagger(p_1)a^\dagger(p_2)a(p_3)a(p_4)\delta(-p_1 - p_2 + p_3 + p_4) \\
&= \frac{\lambda}{4} \int \frac{dp_1 dp_2 dp_3 dp_4}{4\pi \sqrt{p_1 p_2 p_3 p_4}} a^\dagger(p_1)a^\dagger(p_2)a(p_3)a(p_4)\delta(-p_1 - p_2 + p_3 + p_4) \\
&= \frac{\lambda}{4} \int \frac{dp_1 dp_2 dp'_1 dp'_2}{4\pi \sqrt{p_1 p_2 p'_1 p'_2}} a^\dagger(p_1)a^\dagger(p_2)a(p'_1)a(p'_2)\delta(p_1 + p_2 - p'_1 - p'_2),
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
\mathcal{P}_{31}^- &= \frac{\lambda}{4!} \int \frac{dp_1 dp_2 dp_3 dp_4}{4\pi \sqrt{p_1 p_2 p_3 p_4}} 4a^\dagger(p_1)a^\dagger(p_2)a^\dagger(p_3)a(p_4)\delta(-p_1 - p_2 - p_3 + p_4) \\
&= \frac{\lambda}{6} \int \frac{dp_1 dp_2 dp_3}{4\pi \sqrt{p_1 p_2 p_3 (p_1 + p_2 + p_3)}} a^\dagger(p_1)a^\dagger(p_2)a^\dagger(p_3)a(p_1 + p_2 + p_3),
\end{aligned} \tag{4.16}$$

and the light-front Hamiltonian is $\mathcal{P}^- = \mathcal{P}_{11}^- + \mathcal{P}_{22}^- + \mathcal{P}_{13}^- + \mathcal{P}_{31}^-$.

4.3 Eigenvalue Problem

The Fock-state expansion of an eigenstate can be written,

$$|\Psi(P)\rangle = \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta\left(1 - \sum_i^m y_i\right) \Psi_m(y_i) |y_i P; P, m\rangle, \tag{4.17}$$

where Ψ_m is the wave function for m constituents.

Since (4.2)

$$[a(p), a^\dagger(p')] = \delta(p - p') \tag{4.18}$$

and (4.5)

$$|y_i P; P, m\rangle = \frac{1}{\sqrt{m!}} \prod_{i=1}^m a^\dagger(y_i P) |0\rangle, \tag{4.19}$$

the normalization is not unity, but instead,

$$\langle y'_j P; P, m' | y_i P; P, m \rangle = P^{-m} \delta_{m, m'} \delta(y'_j - y_i). \quad (4.20)$$

This is because $p = k^+ = yP$, (4.2) gives

$$[a(y), a^\dagger(y')] = \delta(y - y'). \quad (4.21)$$

Thus we have

$$a^\dagger(y)|0\rangle = \sqrt{1}|1\rangle, \quad (4.22)$$

$$a^\dagger(y)|m\rangle = \sqrt{m+1}|m+1\rangle. \quad (4.23)$$

And since $a^\dagger(p) = \tilde{a}^\dagger(k^+)$ is a normalized $a^\dagger(k^+)$, $a^\dagger(k^+) = \sqrt{4\pi k^+} \tilde{a}^\dagger(k^+) = \sqrt{4\pi y} a^\dagger(y)$,

$$\tilde{a}^\dagger(k^+) = \sqrt{\frac{y}{k^+}} a^\dagger(y) = \sqrt{\frac{y}{yP}} a^\dagger(y) = \frac{1}{\sqrt{P}} a^\dagger(y). \quad (4.24)$$

So

$$\tilde{a}^\dagger(k^+)|0\rangle = \frac{1}{\sqrt{P}} \sqrt{1}|1\rangle, \quad (4.25)$$

and

$$\tilde{a}^\dagger(k^+)|m\rangle = \frac{1}{\sqrt{P}} \sqrt{m+1}|m+1\rangle. \quad (4.26)$$

The definition of the m-constituent Fock state is thus

$$\begin{aligned} |y_i P; P, m\rangle &= \frac{1}{\sqrt{m!}} \prod_{i=1}^m a^\dagger(y_i P) |0\rangle \\ &= \frac{1}{\sqrt{m!}} \frac{1}{\sqrt{P}} \sqrt{m} \frac{1}{\sqrt{P}} \sqrt{m-1} \cdots \frac{1}{\sqrt{P}} \sqrt{2} \frac{1}{\sqrt{P}} \sqrt{1} |0\rangle \\ &= \frac{\sqrt{m!}}{\sqrt{m!}} \frac{1}{\sqrt{P^m}} |m\rangle = P^{-\frac{m}{2}} |m\rangle. \end{aligned} \quad (4.27)$$

Similarly, the annihilation operator acting to the left should be equivalent to creation operator acting to the right,

$$\langle y_i P; P, m| = \langle 0| \frac{1}{\sqrt{m!}} \prod_{i=1}^m a(y_i P) = \langle m| P^{-\frac{m}{2}}. \quad (4.28)$$

Therefore, the normalization is

$$\langle y'_j P; P, m' | y_i P; P, m \rangle = \delta_{m, m'} \langle y'_j; m' | P^{-\frac{m'}{2}} P^{-\frac{m}{2}} | y_i; m \rangle = P^{-m} \delta_{m, m'} \delta(y'_j - y_i). \quad (4.29)$$

Otherwise, if $P' \neq P$, and $\sum_j^{m'} y'_j = \sum_{j=1}^{m'} y'_j = 1$ but $\sum_i^m y_i \neq 1$ ($\sum_{i=1}^m y_i = 1$), let $\Delta = 1 - \sum_i^m y_i$,

$$\begin{aligned} \langle y'_j P'; P', m' | y_i P; P, m \rangle &= \delta_{m, m'} \langle y'_j P'; m' | P'^{-\frac{m'+1}{2}} P'^{-\frac{m+1}{2}} | y_i P; m \rangle \\ &= P'^{-m'+1} \delta_{m, m'} \delta(y'_j - y_i) \delta\left(\sum_j^{m'} y'_j P' - \sum_i^m y_i P\right) \\ &= P'^{-m'+1} \delta_{m, m'} \delta(y'_j - y_i) \delta(P' - (1 - \Delta)P). \end{aligned} \quad (4.30)$$

Then, if $P' \neq P$, and $\sum_j^{m'} y'_j \neq 1$ ($\sum_{j=1}^{m'} y'_j = 1$) but $\sum_i^m y_i = \sum_{i=1}^m y_i = 1$, let $\Delta = 1 - \sum_j^{m'} y'_j$,

$$\begin{aligned} \langle y'_j P'; P', m' | y_i P; P, m \rangle &= \delta_{m, m'} \langle y'_j P'; m' | P^{-\frac{m'+1}{2}} P^{-\frac{m+1}{2}} | y_i P; m \rangle \\ &= P^{-m'+1} \delta_{m, m'} \delta(y'_j - y_i) \delta\left(\sum_j^{m'} y'_j P' - \sum_i^m y_i P\right) \\ &= P^{-m'+1} \delta_{m, m'} \delta(y'_j - y_i) \delta((1 - \Delta)P' - P). \end{aligned} \quad (4.31)$$

The creation and annihilation operators acting on the m-constituent Fock state becomes

$$\begin{aligned} \tilde{a}^\dagger(k^+) |y_i P; P, m\rangle &= \tilde{a}^\dagger(k^+) P^{-\frac{m}{2}} |m\rangle = \frac{1}{\sqrt{P}} a^\dagger(y) P^{-\frac{m}{2}} |m\rangle = \frac{1}{\sqrt{P}} \sqrt{m+1} P^{-\frac{m}{2}} |m+1\rangle \\ &= \sqrt{m+1} P^{-\frac{(m+1)}{2}} |m+1\rangle \\ &= \sqrt{m+1} |y_i P; P, m+1\rangle, \end{aligned} \quad (4.32)$$

and

$$\begin{aligned} \tilde{a}(k^+) |y_i P; P, m\rangle &= \tilde{a}(k^+) P^{-\frac{m}{2}} |m\rangle = \frac{1}{\sqrt{P}} a(y) P^{-\frac{m}{2}} |m\rangle = \frac{1}{\sqrt{P}} \sqrt{m} P^{-\frac{m}{2}} |m-1\rangle \\ &= \sqrt{m} P^{-\frac{m-1}{2}} |m-1\rangle = \frac{1}{P} \sqrt{m} P^{-\frac{(m-1)}{2}} |m-1\rangle = \frac{1}{P} \sqrt{m} |y_i P; P, m-1\rangle \\ &= \sqrt{m} \delta(k^+ - y_1 P) |y_1 P \cdots y_m P; P, m-1\rangle. \end{aligned} \quad (4.33)$$

Thus, the annihilation operator's rule of acting on the states will change compared with the standard definition, with a $\frac{1}{P}$ factor absorbed in δ .

To summarize, the light-front Hamiltonian eigenvalue problem is

$$\mathcal{P}^- |\Psi(P)\rangle = \frac{M^2}{P} |\Psi(P)\rangle, \quad (4.34)$$

with light-front Hamiltonian $\mathcal{P}^- = \mathcal{P}_{11}^- + \mathcal{P}_{22}^- + \mathcal{P}_{13}^- + \mathcal{P}_{31}^-$, where

$$\mathcal{P}_{11}^- = \int dp \frac{\mu^2}{p} a^\dagger(p) a(p), \quad (4.35)$$

$$\mathcal{P}_{22}^- = \frac{\lambda}{4} \int \frac{dp_1 dp_2 dp'_1 dp'_2}{4\pi \sqrt{p_1 p_2 p'_1 p'_2}} a^\dagger(p_1) a^\dagger(p_2) a(p'_1) a(p'_2) \delta(p_1 + p_2 - p'_1 - p'_2), \quad (4.36)$$

$$\mathcal{P}_{13}^- = \frac{\lambda}{6} \int \frac{dp_1 dp_2 dp_3}{4\pi \sqrt{(p_1 + p_2 + p_3) p_1 p_2 p_3}} a^\dagger(p_1 + p_2 + p_3) a(p_1) a(p_2) a(p_3), \quad (4.37)$$

$$\mathcal{P}_{31}^- = \frac{\lambda}{6} \int \frac{dp_1 dp_2 dp_3}{4\pi \sqrt{p_1 p_2 p_3 (p_1 + p_2 + p_3)}} a^\dagger(p_1) a^\dagger(p_2) a^\dagger(p_3) a(p_1 + p_2 + p_3). \quad (4.38)$$

Projecting the eigenvalue equation to m -boson sector reduces the equation

$$\langle y'_i P; P, m' | \mathcal{P}^- | \Psi(P) \rangle = \frac{M^2}{P} \langle y'_i P; P, m' | \Psi(P) \rangle. \quad (4.39)$$

The right-hand-side becomes

$$\begin{aligned} \langle y'_j P; P, m' | \Psi(P) \rangle &= \langle y'_j P; P, m' | \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta\left(1 - \sum_i^m y_i\right) \Psi_m(y_i) | y_i P; P, m \rangle \\ &= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta\left(1 - \sum_i^m y_i\right) \Psi_m(y_i) \langle y'_j P; P, m' | y_i P; P, m \rangle \\ &= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta\left(1 - \sum_i^m y_i\right) \Psi_m(y_i) P^{-m} \delta_{m,m'} \delta(y'_j - y_i) \\ &= P^{\frac{m'-1}{2}} P^{-m'} \int \prod_i^{m'} dy_i \delta(y'_j - y_i) \Psi_{m'}(y_i) \delta\left(1 - \sum_i^{m'} y_i\right) \\ &= P^{\frac{m'-1}{2}} P^{-m'} \Psi_{m'}(y'_j) \delta\left(1 - \sum_j^{m'} y'_j\right) \\ &= P^{\frac{m'-1}{2}} P^{-m'} \Psi_{m'}(y'_j), \end{aligned} \quad (4.40)$$

where y'_1, y'_2, \dots, y'_m should add up to 1.

The left-hand-side:

$$\begin{aligned}
& \langle y'_j P; P, m' | \mathcal{P}_{11}^- | \Psi(P) \rangle \\
&= \langle y'_j P; P, m' | \int dp \frac{\mu^2}{p} a^\dagger(p) a(p) \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta \left(1 - \sum_i^m y_i \right) \Psi_m(y_i) | y_i P; P, m \rangle \\
&= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta \left(1 - \sum_i^m y_i \right) \int dp \frac{\mu^2}{p} \Psi_m(y_i) \langle y'_j P; P, m' | a^\dagger(p) a(p) | y_i P; P, m \rangle \\
&= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta \left(1 - \sum_i^m y_i \right) \int dp \frac{\mu^2}{p} \Psi_m(y_i) \\
&\quad \langle y'_j P; P, m' - 1 | \frac{1}{P} \sqrt{m'} \delta(y'_1 - x) a(p) | y_i P; P, m \rangle \\
&= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta \left(1 - \sum_i^m y_i \right) \int dp \frac{\mu^2}{p} \Psi_m(y_i) \\
&\quad \langle y'_j P; P, m' | \frac{1}{P} \sqrt{m'} \delta(y'_1 - x) \sqrt{m'} | y_i P; P, m \rangle \\
&= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta \left(1 - \sum_i^m y_i \right) \int dx \frac{\mu^2}{x} \Psi_m(y_i) \frac{1}{P} m' \delta(y'_1 - x) \\
&\quad \langle y'_j P; P, m' | y_i P; P, m \rangle \tag{4.41} \\
&= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta \left(1 - \sum_i^m y_i \right) \int dx \frac{\mu^2}{x} \Psi_m(y_i) \frac{1}{P} m' \delta(y'_1 - x) \\
&\quad P^{-m} \delta_{m,m'} \delta(y'_j - y_i) \\
&= P^{\frac{m'-1}{2}} P^{-m'} \frac{1}{P} m' \int dx \frac{\mu^2}{x} \delta(y'_1 - x) \int \prod_i^{m'} dy_i \delta(y'_j - y_i) \Psi_{m'}(y_i) \delta \left(1 - \sum_i^{m'} y_i \right) \\
&= P^{\frac{m'-1}{2}} P^{-m'} \frac{1}{P} m' \int dx \frac{\mu^2}{x} \delta(y'_1 - x) \Psi_{m'}(y'_j) \delta \left(1 - \sum_j^{m'} y'_j \right) \\
&= P^{\frac{m'-1}{2}} P^{-m'} \frac{1}{P} m' \frac{\mu^2}{y'_1} \Psi_{m'}(y'_j) \delta \left(1 - \sum_j^{m'} y'_j \right) \\
&= P^{\frac{m'-1}{2}} P^{-m'} m' \frac{\mu^2}{y'_1 P} \Psi_{m'}(y'_j) \delta \left(1 - \sum_j^{m'} y'_j \right) \\
&= P^{\frac{m'-1}{2}} P^{-m'} m' \frac{\mu^2}{y'_1 P} \Psi_{m'}(y'_j),
\end{aligned}$$

where y'_1, y'_2, \dots, y'_m should add up to 1.

$$\begin{aligned}
& \langle y'_j P; P, m' | \mathcal{P}_{22}^- | \Psi(P) \rangle \\
&= \langle y'_j P; P, m' | \frac{\lambda}{4} \int \frac{dp_1 dp_2 dp'_1 dp'_2}{4\pi \sqrt{p_1 p_2 p'_1 p'_2}} a^\dagger(p_1) a^\dagger(p_2) a(p'_1) a(p'_2) \delta(p_1 + p_2 - p'_1 - p'_2) \\
&\quad \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta\left(1 - \sum_i^m y_i\right) \Psi_m(y_i) |y_i P; P, m\rangle \\
&= \frac{\lambda}{4} \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta\left(1 - \sum_i^m y_i\right) \Psi_m(y_i) \int \frac{P^2 dx_1 dx_2 dx'_1 dx'_2}{4\pi \sqrt{x_1 x_2 x'_1 x'_2}} \\
&\quad \frac{1}{P} \delta(x_1 + x_2 - x'_1 - x'_2) \langle y'_j P; P, m' | a^\dagger(x_1 P) a^\dagger(x_2 P) a(x'_1 P) a(x'_2 P) |y_i P; P, m\rangle \\
&= \frac{\lambda}{4} \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta\left(1 - \sum_i^m y_i\right) \Psi_m(y_i) \int \frac{P dx_1 dx_2 dx'_1 dx'_2}{4\pi \sqrt{x_1 x_2 x'_1 x'_2}} \delta(x_1 + x_2 - x'_1 - x'_2) \\
&\quad \langle y'_j P; P, m' - 1 | \frac{1}{P} \sqrt{m'} \delta(x_1 - y'_1) a^\dagger(x_2 P) a(x'_1 P) a(x'_2 P) |y_i P; P, m\rangle \\
&= \frac{\lambda}{4} \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta\left(1 - \sum_i^m y_i\right) \Psi_m(y_i) \int \frac{P dx_1 dx_2 dx'_1 dx'_2}{4\pi \sqrt{x_1 x_2 x'_1 x'_2}} \delta(x_1 + x_2 - x'_1 - x'_2) \\
&\quad \langle y'_j P; P, m' - 2 | \frac{1}{P} \sqrt{m'} \delta(x_1 - y'_1) \frac{1}{P} \sqrt{m' - 1} \delta(x_2 - y'_2) a(x'_1 P) a(x'_2 P) |y_i P; P, m\rangle \\
&= \frac{\lambda}{4P} \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta\left(1 - \sum_i^m y_i\right) \Psi_m(y_i) \int \frac{dx_1 dx_2 dx'_1 dx'_2}{4\pi \sqrt{x_1 x_2 x'_1 x'_2}} \delta(x_1 + x_2 - x'_1 - x'_2) \\
&\quad \langle y'_j P; P, m' - 2 | \sqrt{m'(m' - 1)} \delta(x_1 - y'_1) \delta(x_2 - y'_2) a(x'_1 P) a(x'_2 P) |y_i P; P, m\rangle \\
&= \frac{\lambda}{4P} \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta\left(1 - \sum_i^m y_i\right) \Psi_m(y_i) \int \frac{dx_1 dx_2 dx'_1 dx'_2}{4\pi \sqrt{x_1 x_2 x'_1 x'_2}} \delta(x_1 + x_2 - x'_1 - x'_2) \\
&\quad \langle y'_j P \equiv x'_1 P, x'_2 P, y'_3 P, \dots, y'_m P; P, m' | m'(m' - 1) \delta(x_1 - y'_1) \delta(x_2 - y'_2) |y_i P; P, m\rangle \\
&= \frac{\lambda}{4P} \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta\left(1 - \sum_i^m y_i\right) \Psi_m(y_i) \int \frac{dx_1 dx_2 dx'_1 dx'_2}{4\pi \sqrt{x_1 x_2 x'_1 x'_2}} \\
&\quad \delta(x_1 + x_2 - x'_1 - x'_2) m'(m' - 1) \delta(x_1 - y'_1) \delta(x_2 - y'_2) \langle y'_j P; P, m' | y_i P; P, m\rangle \\
&= \frac{\lambda}{4P} \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta\left(1 - \sum_i^m y_i\right) \Psi_m(y_i) \int \frac{dx_1 dx_2 dx'_1 dx'_2}{4\pi \sqrt{x_1 x_2 x'_1 x'_2}} \\
&\quad \delta(x_1 + x_2 - x'_1 - x'_2) m'(m' - 1) \delta(x_1 - y'_1) \delta(x_2 - y'_2) P^{-m} \delta_{m, m'} \delta(y'_j - y_i)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\lambda}{4P} P^{\frac{m'-1}{2}} \int \prod_i^{m'} dy_i \delta \left(1 - \sum_i^{m'} y_i \right) \Psi_{m'}(y_i) \int \frac{dx_1 dx_2 dx'_1 dx'_2}{4\pi \sqrt{x_1 x_2 x'_1 x'_2}} \\
&\quad \delta(x_1 + x_2 - x'_1 - x'_2) m'(m' - 1) \delta(x_1 - y'_1) \delta(x_2 - y'_2) P^{-m'} \delta(y'_j - y_i) \\
&= \frac{\lambda}{4P} P^{\frac{m'-1}{2}} P^{-m'} m'(m' - 1) \int \frac{dx_1 dx_2 dx'_1 dx'_2}{4\pi \sqrt{x_1 x_2 x'_1 x'_2}} \delta(x_1 + x_2 - x'_1 - x'_2) \delta(x_1 - y'_1) \delta(x_2 - y'_2) \\
&\quad \int \prod_i^{m'} dy_i \delta(y'_j - y_i) \Psi_{m'}(y_i) \delta \left(1 - \sum_i^{m'} y_i \right) \\
&= \frac{\lambda}{4P} P^{\frac{m'-1}{2}} P^{-m'} m'(m' - 1) \int \frac{dx_1 dx_2 dx'_1 dx'_2}{4\pi \sqrt{x_1 x_2 x'_1 x'_2}} \delta(x_1 + x_2 - x'_1 - x'_2) \delta(x_1 - y'_1) \delta(x_2 - y'_2) \\
&\quad \Psi_{m'}(y'_j) \delta \left(1 - \sum_j^{m'} y'_j \right) \\
&= \frac{\lambda}{4P} P^{\frac{m'-1}{2}} P^{-m'} m'(m' - 1) \int \frac{dx'_1 dx'_2}{4\pi \sqrt{y'_1 y'_2 x'_1 x'_2}} \delta(y'_1 + y'_2 - x'_1 - x'_2) \Psi_{m'}(y'_j) \delta \left(1 - \sum_i^{m'} y'_j \right) \\
&= P^{\frac{m'-1}{2}} P^{-m'} \frac{\lambda}{4\pi P} \frac{m'(m' - 1)}{4\sqrt{y'_1 y'_2}} \int \frac{dx'_1 dx'_2}{\sqrt{x'_1 x'_2}} \delta(y'_1 + y'_2 - x'_1 - x'_2) \Psi_{m'}(y'_j) \delta \left(1 - \sum_i^{m'} y'_j \right) \\
&= P^{\frac{m'-1}{2}} P^{-m'} \frac{\lambda}{4\pi P} \frac{m'(m' - 1)}{4\sqrt{y'_1 y'_2}} \int \frac{dx'_1 dx'_2}{\sqrt{x'_1 x'_2}} \delta(y'_1 + y'_2 - x'_1 - x'_2) \Psi_{m'}(x'_1, x'_2, y'_3, \dots, y'_m),
\end{aligned} \tag{4.42}$$

where $x'_1, x'_2, y'_3, \dots, y'_m$ should add up to 1.

Notice that $y'_j P$ is redefined as $y'_j P \equiv x'_1 P, x'_2 P, y'_3 P, \dots, y'_m P$ after applying the annihilation operators to the left. The two creation operators acting on the left annihilate $y'_1 P$ and $y'_2 P$, and the two creation operators acting on the left create $x'_1 P$ and $x'_2 P$. Similar redefinition also occur in later derivations.

Here y'_1 is again picked by a^\dagger as in \mathcal{P}_{11}^- case, because we assume that the values of y'_i are the same, as for the normalization condition.

$$\begin{aligned}
& \langle y'_j P; P, m' | \mathcal{P}_{13}^- | \Psi(P) \rangle \\
&= \langle y'_j P; P, m' | \frac{\lambda}{6} \int \frac{dp_1 dp_2 dp_3}{4\pi \sqrt{(p_1 + p_2 + p_3)p_1 p_2 p_3}} a^\dagger(p_1 + p_2 + p_3) a(p_1) a(p_2) a(p_3) \\
&\quad \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta \left(1 - \sum_i^m y_i \right) \Psi_m(y_i) | y_i P; P, m \rangle \\
&= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta \left(1 - \sum_i^m y_i \right) \frac{\lambda}{6} \int \frac{dp_1 dp_2 dp_3}{4\pi \sqrt{(p_1 + p_2 + p_3)p_1 p_2 p_3}} \\
&\quad \Psi_m(y_i) \langle y'_j P; P, m' | a^\dagger(p_1 + p_2 + p_3) a(p_1) a(p_2) a(p_3) | y_i P; P, m \rangle \\
&= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta \left(1 - \sum_i^m y_i \right) \frac{\lambda}{6} \int \frac{P dx_1 dx_2 dx_3}{4\pi \sqrt{(x_1 + x_2 + x_3)x_1 x_2 x_3}} \\
&\quad \Psi_m(y_i) \langle y'_j P; P, m' - 1 | \frac{1}{P} \sqrt{m'} \delta(x_1 + x_2 + x_3 - y'_1) a(p_1) a(p_2) a(p_3) | y_i P; P, m \rangle \\
&= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta \left(1 - \sum_i^m y_i \right) \frac{\lambda}{6} \int \frac{P dx_1 dx_2 dx_3}{4\pi \sqrt{(x_1 + x_2 + x_3)x_1 x_2 x_3}} \\
&\quad \Psi_m(y_i) \langle y'_j P \equiv x_1 P, x_2 P, x_3 P, y_2 P, \dots, y'_m P; P, m' + 2 | \frac{1}{P} \sqrt{m'} \delta(x_1 + x_2 + x_3 - y'_1) \\
&\quad \sqrt{m'} \sqrt{m' + 1} \sqrt{m' + 2} | y_i P; P, m \rangle \\
&= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \Psi_m(y_i) \delta \left(1 - \sum_i^m y_i \right) \frac{\lambda}{6} m' \sqrt{(m' + 1)(m' + 2)} \\
&\quad \int \frac{dx_1 dx_2 dx_3}{4\pi \sqrt{(x_1 + x_2 + x_3)x_1 x_2 x_3}} \delta(x_1 + x_2 + x_3 - y'_1) \langle y'_j P; P, m' + 2 | y_i P; P, m \rangle \\
&= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \Psi_m(y_i) \delta \left(1 - \sum_i^m y_i \right) \frac{\lambda}{6} m' \sqrt{(m' + 1)(m' + 2)} \\
&\quad \int \frac{dx_1 dx_2 dx_3}{4\pi \sqrt{(x_1 + x_2 + x_3)x_1 x_2 x_3}} \delta(x_1 + x_2 + x_3 - y'_1) P^{-m} \delta_{m'+2, m} \delta(y'_j - y_i) \\
&= P^{\frac{m'+1}{2}} \int \prod_i^{m'+2} dy_i \delta(y'_j - y_i) \Psi_{m'+2}(y_i) \delta \left(1 - \sum_i^{m'+2} y_i \right) \frac{\lambda}{6} m' \sqrt{(m' + 1)(m' + 2)} \\
&\quad \int \frac{dx_1 dx_2 dx_3}{4\pi \sqrt{(x_1 + x_2 + x_3)x_1 x_2 x_3}} \delta(x_1 + x_2 + x_3 - y'_1) P^{-(m'+2)}
\end{aligned}$$

$$\begin{aligned}
&= P^{\frac{m'+1}{2}} P^{-(m'+2)} \frac{\lambda}{P} m' \sqrt{(m'+1)(m'+2)} \int \frac{dx_1 dx_2 dx_3}{4\pi \sqrt{(x_1+x_2+x_3)x_1 x_2 x_3}} \\
&\quad \delta(x_1+x_2+x_3-y'_1) \int \prod_i^{m'+2} dy_i \delta(y'_j - y_i) \Psi_{m'+2}(y_i) \delta \left(1 - \sum_i^{m'+2} y_i \right) \\
&= P^{\frac{m'-1}{2}} P P^{-m'} P^{-2} \frac{\lambda}{P} m' \sqrt{(m'+1)(m'+2)} \\
&\quad \int \frac{dx_1 dx_2 dx_3}{4\pi \sqrt{(x_1+x_2+x_3)x_1 x_2 x_3}} \delta(x_1+x_2+x_3-y'_1) \Psi_{m'+2}(y'_j) \delta \left(1 - \sum_j^{m'+2} y'_j \right) \\
&= P^{\frac{m'-1}{2}} P^{-m'} \frac{1}{P} \frac{\lambda}{6} m' \sqrt{(m'+1)(m'+2)} \\
&\quad \int \frac{dx_1 dx_2 dx_3}{4\pi \sqrt{(x_1+x_2+x_3)x_1 x_2 x_3}} \delta(x_1+x_2+x_3-y'_1) \Psi_{m'+2}(y'_j) \delta \left(1 - \sum_j^{m'+2} y'_j \right) \\
&= P^{\frac{m'-1}{2}} P^{-m'} \frac{\lambda}{4\pi P} \frac{m'}{6} \sqrt{(m'+1)(m'+2)} \\
&\quad \int \frac{dx_1 dx_2 dx_3}{\sqrt{y'_1 x_1 x_2 x_3}} \delta(x_1+x_2+x_3-y'_1) \Psi_{m'+2}(y'_j) \delta \left(1 - \sum_j^{m'+2} y'_j \right) \\
&= P^{\frac{m'-1}{2}} P^{-m'} \frac{\lambda}{4\pi P} \frac{m'}{6} \sqrt{(m'+1)(m'+2)} \\
&\quad \int \frac{dx_1 dx_2 dx_3}{\sqrt{y'_1 x_1 x_2 x_3}} \delta(x_1+x_2+x_3-y'_1) \Psi_{m'+2}(x_1, x_2, x_3, y_2, \dots, y'_m),
\end{aligned} \tag{4.43}$$

where $x_1, x_2, x_3, y_2, \dots, y'_m$ should add up to 1.

$$\begin{aligned}
& \langle y'_j P; P, m' | \mathcal{P}_{31}^- | \Psi(P) \rangle \\
&= \langle y'_j P; P, m' | \frac{\lambda}{6} \int \frac{dp_1 dp_2 dp_3}{4\pi \sqrt{p_1 p_2 p_3 (p_1 + p_2 + p_3)}} a^\dagger(p_1) a^\dagger(p_2) a^\dagger(p_3) a(p_1 + p_2 + p_3) \\
&\quad \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta \left(1 - \sum_i^m y_i \right) \Psi_m(y_i) |y_i P; P, m \rangle \\
&= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta \left(1 - \sum_i^m y_i \right) \frac{\lambda}{6} \int \frac{dp_1 dp_2 dp_3}{4\pi \sqrt{p_1 p_2 p_3 (p_1 + p_2 + p_3)}} \Psi_m(y_i) \\
&\quad \langle y'_j P; P, m' | a^\dagger(p_1) a^\dagger(p_2) a^\dagger(p_3) a(p_1 + p_2 + p_3) |y_i P; P, m \rangle \\
&= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta \left(1 - \sum_i^m y_i \right) \frac{\lambda}{6} \int \frac{P dx_1 dx_2 dx_3}{4\pi \sqrt{x_1 x_2 x_3 (x_1 + x_2 + x_3)}} \Psi_m(y_i) \\
&\quad \langle y'_j P; P, m' - 3 | \frac{1}{P} \sqrt{m'} \delta(x_1 - y'_1) \frac{1}{P} \sqrt{m' - 1} \delta(x_2 - y'_2) \frac{1}{P} \sqrt{m' - 2} \delta(x_3 - y'_3) \\
&\quad a(p_1 + p_2 + p_3) |y_i P; P, m \rangle \\
&= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta \left(1 - \sum_i^m y_i \right) \frac{\lambda}{6} \int \frac{P dx_1 dx_2 dx_3}{4\pi \sqrt{x_1 x_2 x_3 (x_1 + x_2 + x_3)}} \Psi_m(y_i) \\
&\quad \langle y'_j P \equiv x_1 P + x_2 P + x_3 P, y'_4 P, \dots, y'_m P; P, m' - 2 | \frac{1}{P^3} \\
&\quad \sqrt{m'} \delta(x_1 - y'_1) \sqrt{m' - 1} \delta(x_2 - y'_2) \sqrt{m' - 2} \delta(x_3 - y'_3) \sqrt{m' - 2} |y_i P; P, m \rangle \\
&= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta \left(1 - \sum_i^m y_i \right) \frac{\lambda}{6 P^2} (m' - 2) \sqrt{m' (m' - 1)} \int \frac{dx_1 dx_2 dx_3}{4\pi \sqrt{x_1 x_2 x_3 (x_1 + x_2 + x_3)}} \\
&\quad \Psi_m(y_i) \delta(x_1 - y'_1) \delta(x_2 - y'_2) \delta(x_3 - y'_3) \langle y'_j P; P, m' - 2 | y_i P; P, m \rangle \\
&= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta \left(1 - \sum_i^m y_i \right) \frac{\lambda}{6 P^2} (m' - 2) \sqrt{m' (m' - 1)} \\
&\quad \int \frac{dx_1 dx_2 dx_3}{4\pi \sqrt{x_1 x_2 x_3 (x_1 + x_2 + x_3)}} \Psi_m(y_i) \delta(x_1 - y'_1) \delta(x_2 - y'_2) \delta(x_3 - y'_3) P^{-m} \delta_{m'-2, m} \delta(y'_j - y_i) \\
&= P^{\frac{m'-3}{2}} \int \prod_i^{m'-2} dy_i \delta \left(1 - \sum_i^{m'-2} y_i \right) \frac{\lambda}{6 P^2} (m' - 2) \sqrt{m' (m' - 1)} \\
&\quad \int \frac{dx_1 dx_2 dx_3}{4\pi \sqrt{x_1 x_2 x_3 (x_1 + x_2 + x_3)}} \Psi_{m'-2}(y_i) \delta(x_1 - y'_1) \delta(x_2 - y'_2) \delta(x_3 - y'_3) P^{-(m'-2)} \delta(y'_j - y_i)
\end{aligned}$$

$$\begin{aligned}
&= P^{\frac{m'-3}{2}} P^{-(m'-2)} \frac{\lambda}{6P^2} (m'-2) \sqrt{m'(m'-1)} \int \frac{dx_1 dx_2 dx_3}{4\pi \sqrt{x_1 x_2 x_3 (x_1 + x_2 + x_3)}} \\
&\quad \delta(x_1 - y'_1) \delta(x_2 - y'_2) \delta(x_3 - y'_3) \int \prod_i^{m'-2} dy_i \delta(y'_j - y_i) \Psi_{m'-2}(y_i) \delta \left(1 - \sum_i^{m'-2} y_i \right) \\
&= P^{\frac{m'-1}{2}} \frac{1}{P} P^{-m'} P^2 \frac{\lambda}{6P^2} (m'-2) \sqrt{m'(m'-1)} \\
&\quad \int \frac{dx_1 dx_2 dx_3}{4\pi \sqrt{x_1 x_2 x_3 (x_1 + x_2 + x_3)}} \delta(x_1 - y'_1) \delta(x_2 - y'_2) \delta(x_3 - y'_3) \Psi_{m'-2}(y'_j) \delta \left(1 - \sum_j^{m'-2} y'_j \right) \\
&= P^{\frac{m'-1}{2}} P^{-m'} \frac{1}{P} \frac{\lambda}{6} (m'-2) \sqrt{m'(m'-1)} \frac{1}{4\pi \sqrt{y'_1 y'_2 y'_3 (y'_1 + y'_2 + y'_3)}} \Psi_{m'-2}(y'_j) \delta \left(1 - \sum_j^{m'-2} y'_j \right) \\
&= P^{\frac{m'-1}{2}} P^{-m'} \frac{\lambda}{4\pi P} \frac{m'-2}{6} \frac{\sqrt{m'(m'-1)}}{\sqrt{y'_1 y'_2 y'_3 (y'_1 + y'_2 + y'_3)}} \Psi_{m'-2}(y'_j) \delta \left(1 - \sum_j^{m'-2} y'_j \right) \\
&= P^{\frac{m'-1}{2}} P^{-m'} \frac{\lambda}{4\pi P} \frac{m'-2}{6} \frac{\sqrt{m'(m'-1)}}{\sqrt{y'_1 y'_2 y'_3 (y'_1 + y'_2 + y'_3)}} \Psi_{m'-2}(y_1 + y_2 + y_3, y'_4, \dots, y'_m),
\end{aligned} \tag{4.44}$$

where $y_1 + y_2 + y_3, y'_4, \dots, y'_m$ should add up to 1.

Therefore, the eigenvalue equation becomes

$$\begin{aligned}
&P^{\frac{m'-1}{2}} P^{-m'} m' \frac{\mu^2}{y'_1 P} \Psi_{m'}(y'_j) \\
&+ P^{\frac{m'-1}{2}} P^{-m'} \frac{\lambda}{4\pi P} \frac{m'(m'-1)}{4\sqrt{y'_1 y'_2}} \int \frac{dx'_1 dx'_2}{\sqrt{x'_1 x'_2}} \delta(y'_1 + y'_2 - x'_1 - x'_2) \Psi_{m'}(x'_1, x'_2, y'_3, \dots, y'_m) \\
&+ P^{\frac{m'-1}{2}} P^{-m'} \frac{\lambda}{4\pi P} \frac{m'}{6} \sqrt{(m'+1)(m'+2)} \int \frac{dx_1 dx_2 dx_3}{\sqrt{y'_1 x_1 x_2 x_3}} \delta(x_1 + x_2 + x_3 - y'_1) \\
&\quad \Psi_{m'+2}(x_1, x_2, x_3, y_2, \dots, y'_m) \\
&+ P^{\frac{m'-1}{2}} P^{-m'} \frac{\lambda}{4\pi P} \frac{m'-2}{6} \frac{\sqrt{m'(m'-1)}}{\sqrt{y'_1 y'_2 y'_3 (y'_1 + y'_2 + y'_3)}} \Psi_{m'-2}(y_1 + y_2 + y_3, y'_4, \dots, y'_m) \\
&= P^{\frac{m'-1}{2}} P^{-m'} \frac{M^2}{P} \Psi_{m'}(y'_j).
\end{aligned} \tag{4.45}$$

Eliminating the common factor, we get

$$\begin{aligned}
& m' \frac{\mu^2}{y_1' P} \Psi_{m'}(y_j') \\
& + \frac{\lambda}{4\pi P} \frac{m'(m'-1)}{4\sqrt{y_1' y_2'}} \int \frac{dx_1' dx_2'}{\sqrt{x_1' x_2'}} \delta(y_1' + y_2' - x_1' - x_2') \Psi_{m'}(x_1', x_2', y_3', \dots, y_m') \\
& + \frac{\lambda}{4\pi P} \frac{m'}{6} \sqrt{(m'+1)(m'+2)} \int \frac{dx_1 dx_2 dx_3}{\sqrt{y_1' x_1 x_2 x_3}} \delta(x_1 + x_2 + x_3 - y_1') \Psi_{m'+2}(x_1, x_2, x_3, y_2, \dots, y_m') \\
& + \frac{\lambda}{4\pi P} \frac{m'-2}{6} \frac{\sqrt{m'(m'-1)}}{\sqrt{y_1' y_2' y_3' (y_1' + y_2' + y_3')}} \Psi_{m'-2}(y_1 + y_2 + y_3, y_4', \dots, y_m') \\
& = \frac{M^2}{P} \Psi_{m'}(y_j').
\end{aligned} \tag{4.46}$$

We can eliminate the primes for convenience, so the equation can be written

$$\begin{aligned}
& m \frac{\mu^2}{y_1 P} \Psi_m(y_j) \\
& + \frac{\lambda}{4\pi P} \frac{m(m-1)}{4\sqrt{y_1 y_2}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2}} \delta(y_1 + y_2 - x_1 - x_2) \Psi_m(x_1, x_2, y_3, \dots, y_m) \\
& + \frac{\lambda}{4\pi P} \frac{m}{6} \sqrt{(m+1)(m+2)} \int \frac{dx_1 dx_2 dx_3}{\sqrt{y_1 x_1 x_2 x_3}} \delta(x_1 + x_2 + x_3 - y_1) \Psi_{m+2}(x_1, x_2, x_3, y_2, \dots, y_m) \\
& + \frac{\lambda}{4\pi P} \frac{m-2}{6} \frac{\sqrt{m(m-1)}}{\sqrt{y_1 y_2 y_3 (y_1 + y_2 + y_3)}} \Psi_{m-2}(y_1 + y_2 + y_3, y_4, \dots, y_m) \\
& = \frac{M^2}{P} \Psi_m(y_j),
\end{aligned} \tag{4.47}$$

where the arguments in Ψ should add up to 1.

Multiplying the above equation by a factor P/μ^2 , we get

$$\begin{aligned}
& m \frac{1}{y_1} \Psi_m(y_j) \\
& + \frac{\lambda}{4\pi \mu^2} \frac{m(m-1)}{4\sqrt{y_1 y_2}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2}} \delta(y_1 + y_2 - x_1 - x_2) \Psi_m(x_1, x_2, y_3, \dots, y_m) \\
& + \frac{\lambda}{4\pi \mu^2} \frac{m}{6} \sqrt{(m+1)(m+2)} \int \frac{dx_1 dx_2 dx_3}{\sqrt{y_1 x_1 x_2 x_3}} \delta(x_1 + x_2 + x_3 - y_1) \Psi_{m+2}(x_1, x_2, x_3, y_2, \dots, y_m) \\
& + \frac{\lambda}{4\pi \mu^2} \frac{m-2}{6} \frac{\sqrt{m(m-1)}}{\sqrt{y_1 y_2 y_3 (y_1 + y_2 + y_3)}} \Psi_{m-2}(y_1 + y_2 + y_3, y_4, \dots, y_m) \\
& = \frac{M^2}{\mu^2} \Psi_m(y_j).
\end{aligned} \tag{4.48}$$

The equation can be simplified further by the introduction of a dimensionless coupling

$$g \equiv \frac{\lambda}{4\pi\mu^2}. \quad (4.49)$$

Then

$$\begin{aligned} & \frac{m}{y_1} \Psi_m(y_j) \\ & + g \frac{m(m-1)}{4\sqrt{y_1 y_2}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2}} \delta(y_1 + y_2 - x_1 - x_2) \Psi_m(x_1, x_2, y_3, \dots, y_m) \\ & + g \frac{m}{6} \sqrt{(m+1)(m+2)} \int \frac{dx_1 dx_2 dx_3}{\sqrt{y_1 x_1 x_2 x_3}} \delta(x_1 + x_2 + x_3 - y_1) \Psi_{m+2}(x_1, x_2, x_3, y_2, \dots, y_m) \quad (4.50) \\ & + g \frac{m-2}{6} \frac{\sqrt{m(m-1)}}{\sqrt{y_1 y_2 y_3 (y_1 + y_2 + y_3)}} \Psi_{m-2}(y_1 + y_2 + y_3, y_4, \dots, y_m) \\ & = \frac{M^2}{\mu^2} \Psi_m(y_j), \end{aligned}$$

which is the same result as [Burkardt et al. \(2016\)](#).

To summarize, we get a coupled set of integral equations for the Fock state wave functions which can be solved numerically,

$$\begin{aligned} & \frac{m}{y_1} \Psi_m(y_j) \\ & + \frac{g}{4} \frac{m(m-1)}{\sqrt{y_1 y_2}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2}} \delta(y_1 + y_2 - x_1 - x_2) \Psi_m(x_1, x_2, y_3, \dots, y_m) \\ & + \frac{g}{6} m \sqrt{(m+1)(m+2)} \int \frac{dx_1 dx_2 dx_3}{\sqrt{y_1 x_1 x_2 x_3}} \delta(x_1 + x_2 + x_3 - y_1) \Psi_{m+2}(x_1, x_2, x_3, y_2, \dots, y_m) \quad (4.51) \\ & + \frac{g}{6} \frac{(m-2)\sqrt{m(m-1)}}{\sqrt{y_1 y_2 y_3 (y_1 + y_2 + y_3)}} \Psi_{m-2}(y_1 + y_2 + y_3, y_4, \dots, y_m) \\ & = \frac{M^2}{\mu^2} \Psi_m(y_j), \end{aligned}$$

where the arguments in Ψ should add up to 1, and $\Psi_m(y_j) \equiv \Psi_m(\{y_j\}) \equiv \Psi_m(y_1, \dots, y_m)$. Up to now the wave function Ψ is quite general.

4.4 Symmetric Polynomial Basis

While (4.51) seems hopeless for including the zero mode because the denominators can not be zero, the story changes after expanding $\Psi_m(y_j)$ in terms of polynomials in a specific form

$$\Psi_m(y_j) = \sqrt{\prod_j y_j} \sum_{ni} c_{ni}^{(m)} P_{ni}^{(m)}(y_1, \dots, y_m), \quad (4.52)$$

for the square of the pre-factor $\sqrt{\prod_j y_j}$ cancels all the momentum fractions in the denominators in (4.51). $P_{ni}^{(m)}$ are polynomials in the m momentum fractions $\{y_j\}$ of order n and the $c_{ni}^{(m)}$ are the expansion coefficients. The index i indicates there are multiple fully symmetric polynomials at the same order n and the same number of constituent m . The polynomials are fully symmetric with respect to interchange of the momenta and restricted by $\sum_j y_j = 1$ (momentum conservation).

After some math done in Chabysheva et al. (2013), one can figure out that such polynomials can be written as a product of powers of simpler polynomials.

$$P_{ni}^{(m)} = (C_2)^{n_2} (C_3)^{n_3} \dots (C_m)^{n_m}, \quad (4.53)$$

where $n = \sum_j j n_j$. C_j is a symmetric polynomial of order j made by sum of simple monomials, and C_j should also satisfy the constraint $\sum_j y_j = 1$. For example, $C_1 = y_1 + y_2 + \dots + y_m = 1$ which happens to be a constant so it is not included as one of the factor in $P_{ni}^{(m)}$. C_2 for three particles can be $y_1^2 + y_2^2 + (1 - y_1 - y_2)^2$ or $y_1 y_2 + y_1(1 - y_1 - y_2) + y_2(1 - y_1 - y_2)$. C_3 for three particles can be $y_1 y_2 (1 - y_1 - y_2)$ and so on.

We manipulate (4.51) by firstly substituting (4.52), then multiplying both sides by $\sqrt{\prod_j y_j} P_{ni}^{(m)}(y_j)$ and doing the integration $\int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right)$.

The kinetic energy term of (4.51) becomes

$$\begin{aligned}
& \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \sqrt{\prod_j y_j} P_{ni}^{(m)}(y_j) \frac{m}{y_1} \Psi_m(y_j) \\
&= \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \sqrt{\prod_j y_j} P_{ni}^{(m)}(y_j) \frac{m}{y_1} \sqrt{\prod_j y_j} \sum_{n'i'} c_{n'i'}^{(m)} P_{n'i'}^{(m)}(y_1, \dots, y_m) \\
&= m \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \left(\prod_{j=2} y_j \right) P_{ni}^{(m)}(y_j) \sum_{n'i'} c_{n'i'}^{(m)} P_{n'i'}^{(m)}(y_1, \dots, y_m) \quad (4.54) \\
&= \sum_{n'i'} c_{n'i'}^{(m)} m \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \left(\prod_{j=2} y_j \right) P_{ni}^{(m)}(y_j) P_{n'i'}^{(m)}(y_1, \dots, y_m) \\
&= \sum_{n'i'} c_{n'i'}^{(m)} T_{ni, n'i'}^{(m)}.
\end{aligned}$$

The potential energy terms are

$$\begin{aligned}
& \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \sqrt{\prod_j y_j} P_{ni}^{(m)}(y_j) \frac{g}{4} \frac{m(m-1)}{\sqrt{y_1 y_2}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2}} \delta(y_1 + y_2 - x_1 - x_2) \\
& \Psi_m(x_1, x_2, y_3, \dots, y_m) \\
& = \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \sqrt{\prod_j y_j} P_{ni}^{(m)}(y_j) \frac{g}{4} \frac{m(m-1)}{\sqrt{y_1 y_2}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2}} \delta(y_1 + y_2 - x_1 - x_2) \\
& \quad \sqrt{x_1 x_2} \prod_{j=3} y_j \sum_{n'i'} c_{n'i'}^{(m)} P_{n'i'}^{(m)}(x_1, x_2, y_3, \dots, y_m) \\
& = \frac{g}{4} m(m-1) \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \sqrt{\prod_{j=3} y_j} P_{ni}^{(m)}(y_j) \int dx_1 dx_2 \delta(y_1 + y_2 - x_1 - x_2) \\
& \quad \sqrt{\prod_{j=3} y_j} \sum_{n'i'} c_{n'i'}^{(m)} P_{n'i'}^{(m)}(x_1, x_2, y_3, \dots, y_m) \\
& = \frac{g}{4} m(m-1) \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) P_{ni}^{(m)}(y_j) \int dx_1 dx_2 \delta(y_1 + y_2 - x_1 - x_2) \left(\prod_{j=3} y_j \right) \\
& \quad \sum_{n'i'} c_{n'i'}^{(m)} P_{n'i'}^{(m)}(x_1, x_2, y_3, \dots, y_m) \\
& = \sum_{n'i'} c_{n'i'}^{(m)} \frac{g}{4} m(m-1) \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \int dx_1 dx_2 \delta(y_1 + y_2 - x_1 - x_2) \left(\prod_{j=3} y_j \right) \\
& \quad P_{ni}^{(m)}(y_j) P_{n'i'}^{(m)}(x_1, x_2, y_3, \dots, y_m) \\
& = \sum_{n'i'} c_{n'i'}^{(m)} V_{ni, n'i'}^{(m, m)},
\end{aligned} \tag{4.55}$$

$$\begin{aligned}
& \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \sqrt{\prod_j y_j} P_{ni}^{(m)}(y_j) \frac{g}{6} m \sqrt{(m+1)(m+2)} \\
& \int \frac{dx_1 dx_2 dx_3}{\sqrt{y_1 x_1 x_2 x_3}} \delta(x_1 + x_2 + x_3 - y_1) \Psi_{m+2}(x_1, x_2, x_3, y_2, \dots, y_m) \\
& = \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \sqrt{\prod_j y_j} P_{ni}^{(m)}(y_j) \frac{g}{6} m \sqrt{(m+1)(m+2)} \\
& \int \frac{dx_1 dx_2 dx_3}{\sqrt{y_1 x_1 x_2 x_3}} \delta(x_1 + x_2 + x_3 - y_1) \sqrt{x_1 x_2 x_3 \prod_{j=2} y_j} \sum_{n'i'} c_{n'i'}^{(m+2)} P_{n'i'}^{(m+2)}(x_1, x_2, x_3, y_2, \dots, y_m) \\
& = \frac{g}{6} m \sqrt{(m+1)(m+2)} \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \sqrt{\prod_{j=2} y_j} P_{ni}^{(m)}(y_j) \\
& \int dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - y_1) \sqrt{\prod_{j=2} y_j} \sum_{n'i'} c_{n'i'}^{(m+2)} P_{n'i'}^{(m+2)}(x_1, x_2, x_3, y_2, \dots, y_m) \\
& = \frac{g}{6} m \sqrt{(m+1)(m+2)} \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) P_{ni}^{(m)}(y_j) \\
& \int dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - y_1) \left(\prod_{j=2} y_j \right) \sum_{n'i'} c_{n'i'}^{(m+2)} P_{n'i'}^{(m+2)}(x_1, x_2, x_3, y_2, \dots, y_m) \\
& = \sum_{n'i'} c_{n'i'}^{(m+2)} \frac{g}{6} m \sqrt{(m+1)(m+2)} \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \int dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - y_1) \\
& \left(\prod_{j=2} y_j \right) P_{ni}^{(m)}(y_j) P_{n'i'}^{(m+2)}(x_1, x_2, x_3, y_2, \dots, y_m) \\
& = \sum_{n'i'} c_{n'i'}^{(m+2)} V_{ni, n'i'}^{(m, m+2)},
\end{aligned}$$

(4.56)

$$\begin{aligned}
& \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \sqrt{\prod_j y_j} P_{ni}^{(m)}(y_j) \frac{g}{6} \frac{(m-2)\sqrt{m(m-1)}}{\sqrt{y_1 y_2 y_3 (y_1 + y_2 + y_3)}} \\
& \Psi_{m-2}(y_1 + y_2 + y_3, y_4, \dots, y_m) \\
& = \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \sqrt{\prod_j y_j} P_{ni}^{(m)}(y_j) \frac{g}{6} \frac{(m-2)\sqrt{m(m-1)}}{\sqrt{y_1 y_2 y_3 (y_1 + y_2 + y_3)}} \\
& \quad \sqrt{(y_1 + y_2 + y_3) \prod_{j=4} y_j} \sum_{n'i'} c_{n'i'}^{(m-2)} P_{n'i'}^{(m-2)}(y_1 + y_2 + y_3, y_4, \dots, y_m) \\
& = \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \sqrt{\prod_{j=4} y_j} P_{ni}^{(m)}(y_j) \frac{g}{6} (m-2) \sqrt{m(m-1)} \\
& \quad \sqrt{\prod_{j=4} y_j} \sum_{n'i'} c_{n'i'}^{(m-2)} P_{n'i'}^{(m-2)}(y_1 + y_2 + y_3, y_4, \dots, y_m) \tag{4.57} \\
& = \frac{g}{6} (m-2) \sqrt{m(m-1)} \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \left(\prod_{j=4} y_j \right) P_{ni}^{(m)}(y_j) \\
& \quad \sum_{n'i'} c_{n'i'}^{(m-2)} P_{n'i'}^{(m-2)}(y_1 + y_2 + y_3, y_4, \dots, y_m) \\
& = \sum_{n'i'} c_{n'i'}^{(m-2)} \frac{g}{6} (m-2) \sqrt{m(m-1)} \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \left(\prod_{j=4} y_j \right) P_{ni}^{(m)}(y_j) \\
& \quad P_{n'i'}^{(m-2)}(y_1 + y_2 + y_3, y_4, \dots, y_m) \\
& = \sum_{n'i'} c_{n'i'}^{(m-2)} V_{ni, n'i'}^{(m, m-2)}.
\end{aligned}$$

And the right-hand-side

$$\begin{aligned}
& \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \sqrt{\prod_j y_j} P_{ni}^{(m)}(y_j) \frac{M^2}{\mu^2} \Psi_m(y_j) \\
&= \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \sqrt{\prod_j y_j} P_{ni}^{(m)}(y_j) \frac{M^2}{\mu^2} \sqrt{\prod_j y_j} \sum_{n'i'} c_{n'i'}^{(m)} P_{n'i'}^{(m)}(y_1, \dots, y_m) \\
&= \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \left(\prod_j y_j \right) P_{ni}^{(m)}(y_j) \frac{M^2}{\mu^2} \sum_{n'i'} c_{n'i'}^{(m)} P_{n'i'}^{(m)}(y_1, \dots, y_m) \quad (4.58) \\
&= \frac{M^2}{\mu^2} \sum_{n'i'} c_{n'i'}^{(m)} \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \left(\prod_j y_j \right) P_{ni}^{(m)}(y_j) P_{n'i'}^{(m)}(y_1, \dots, y_m) \\
&= \frac{M^2}{\mu^2} \sum_{n'i'} c_{n'i'}^{(m)} B_{ni, n'i'}^{(m)}.
\end{aligned}$$

So the coupled equation becomes

$$\begin{aligned}
& \sum_{n'i'} c_{n'i'}^{(m)} T_{ni, n'i'}^{(m)} + \sum_{n'i'} c_{n'i'}^{(m)} V_{ni, n'i'}^{(m, m)} + \sum_{n'i'} c_{n'i'}^{(m+2)} V_{ni, n'i'}^{(m, m+2)} + \sum_{n'i'} c_{n'i'}^{(m-2)} V_{ni, n'i'}^{(m, m-2)} \\
&= \frac{M^2}{\mu^2} \sum_{n'i'} c_{n'i'}^{(m)} B_{ni, n'i'}^{(m)}. \quad (4.59)
\end{aligned}$$

All of the integrals can be done analytically in terms of the generalized beta function

$$\begin{aligned}
B_m(m_1 + 1, m_2 + 1, \dots, m_m + 1) &= \int \left(\prod_j y_j \right) \delta \left(1 - \sum_j y_j \right) \left(\prod_j y_j^{m_j} \right) \\
&= \frac{m_1! m_2! \dots m_m!}{(m_1 + m_2 + \dots + m_m + m - 1)!}. \quad (4.60)
\end{aligned}$$

4.4.1 Fully Symmetric Polynomials for Three Bosons

We can illustrate how to build up such polynomials by the example of three bosons. At order $N = i + j + k$, the fully symmetric polynomials are linear combinations of the form

$$x^i y^j z^k + x^j y^k z^i + x^k y^i z^j + x^j y^i z^k + x^i y^k z^j + x^k y^j z^i. \quad (4.61)$$

For $N=1$, the above expression reduces to $x + y + z$, which is a constant due to the constraint $x + y + z = 1$. For $N=2$, we have two candidates

$$x^2 + y^2 + z^2, \quad xy + xz + yz. \quad (4.62)$$

We substitute $z = 1 - x - y$ into the above expressions,

$$\begin{aligned} x^2 + y^2 + z^2 &= x^2 + y^2 + (1 - x - y)^2 = 2x^2 + 2y^2 + 1 - 2x - 2y + 2xy \\ &= 1 - 2x - 2y + 2x^2 + 2xy + 2y^2, \end{aligned} \quad (4.63)$$

$$\begin{aligned} xy + xz + yz &= xy + x(1 - x - y) + y(1 - x - y) = xy + x - x^2 - xy + y - yx - y^2 \\ &= x - x^2 + y - xy - y^2 \\ &= x + y - x^2 - xy - y^2. \end{aligned} \quad (4.64)$$

For $N=3$, the possible combinations are $\{0, 0, 3\}$, $\{0, 1, 2\}$ and $\{1, 1, 1\}$

$$x^3 + y^3 + z^3, \quad x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2, \quad xyz. \quad (4.65)$$

Substituting $z = 1 - x - y$,

$$\begin{aligned} x^3 + y^3 + z^3 &= x^3 + y^3 + (1 - x - y)^3 \\ &= x^3 + y^3 + 1 - x^3 - y^3 + 3x^2 + 3y^2 - 3x - 3y + 3xy - 3x^2y - 3xy^2 \\ &= 1 + 3x^2 + 3y^2 - 3x - 3y + 3xy - 3x^2y - 3xy^2 \\ &= 1 - 3x - 3y + 3x^2 + 3y^2 + 3xy - 3x^2y - 3xy^2, \end{aligned} \quad (4.66)$$

$$\begin{aligned} &x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 \\ &= x^2y + x^2(1 - x - y) + xy^2 + x(1 - x - y)^2 + y^2(1 - x - y) + y(1 - x - y)^2 \\ &= x^2y + x^2 - x^3 - x^2y + xy^2 + x(1 + x^2 + y^2 - 2x - 2y + 2xy) \\ &\quad + y^2 - xy^2 - y^3 + y(1 + x^2 + y^2 - 2x - 2y + 2xy) \\ &= x + y - x^2 - y^2 - 4xy + xy^2 + yx^2 + 2x^2y + 2xy^2, \end{aligned} \quad (4.67)$$

$$xyz = xy(1 - x - y) = xy - x^2y - xy^2. \quad (4.68)$$

For full symmetry in three coordinates with the constraint of triangulation, the coefficient of the polynomial

$$P(u, v) = \sum_{n=0}^N c_n u^n v^{N-n} \quad (4.69)$$

must satisfy the linear equations

$$c_n = c_{N-n}, \quad \sum_{m=n}^N (-1)^m \frac{m!}{(m-n)!n!} c_m = c_n. \quad (4.70)$$

For $N=6$,

$$c_n = c_{6-n}, \quad \sum_{m=n}^6 (-1)^m \frac{m!}{(m-n)!n!} c_m = c_n. \quad (4.71)$$

- $n = 0$:

$$\begin{aligned} c_0 &= c_6, \\ \sum_{m=0}^6 (-1)^m c_m &= c_0 \Rightarrow c_0 - c_1 + c_2 - c_3 + c_4 - c_5 + c_6 = c_0 \\ &\Rightarrow -c_1 + c_2 - c_3 + c_4 - c_5 + c_6 = 0. \end{aligned} \quad (4.72)$$

- $n = 1$:

$$\begin{aligned} c_1 &= c_5, \\ \sum_{m=1}^6 (-1)^m m c_m &= c_1 \Rightarrow -c_1 + 2c_2 - 3c_3 + 4c_4 - 5c_5 + 6c_6 = c_1 \\ &\Rightarrow -2c_1 + 2c_2 - 3c_3 + 4c_4 - 5c_5 + 6c_6 = 0. \end{aligned} \quad (4.73)$$

- $n = 2$:

$$\begin{aligned} c_2 &= c_4, \\ \sum_{m=2}^6 (-1)^m \frac{m(m-1)}{2} c_m &= c_2 \Rightarrow c_2 - 3c_3 + 6c_4 - 10c_5 + 15c_6 = c_2 \\ &\Rightarrow -3c_3 + 6c_4 - 10c_5 + 15c_6 = 0. \end{aligned} \quad (4.74)$$

- $n = 3$:

$$\begin{aligned} c_3 &= c_3, \\ \sum_{m=3}^6 (-1)^m \frac{m(m-1)(m-2)}{6} c_m &= c_3 \Rightarrow -c_3 + 4c_4 - 10c_5 + 20c_6 = c_3 \\ &\Rightarrow -2c_3 + 4c_4 - 10c_5 + 20c_6 = 0. \end{aligned} \quad (4.75)$$

- $n = 4$:

$$c_4 = c_2,$$

$$\sum_{m=4}^6 (-1)^m \frac{m(m-1)(m-2)(m-3)}{4!} c_m = c_4 \Rightarrow c_4 - 5c_5 + 15c_6 = c_4 \quad (4.76)$$

$$\Rightarrow -5c_5 + 15c_6 = 0.$$

- $n = 5$:

$$c_5 = c_1,$$

$$\sum_{m=5}^6 (-1)^m \frac{m!}{(m-5)!5!} c_m = c_5 \Rightarrow -c_5 + 6c_6 = c_5 \quad (4.77)$$

$$\Rightarrow -2c_5 + 6c_6 = 0.$$

- $n = 6$:

$$c_6 = c_0,$$

$$\sum_{m=6}^6 (-1)^m \frac{m!}{(m-6)!6!} c_m = c_6 \Rightarrow c_6 = c_6 \quad (4.78)$$

$$\Rightarrow 0 = 0.$$

This is equivalent to the matrix problem

$$\begin{pmatrix} 0 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & -2 & 2 & -3 & 4 & -5 & 6 \\ 0 & 0 & 0 & -3 & 6 & -10 & 15 \\ 0 & 0 & 0 & -2 & 4 & -10 & 20 \\ 0 & 0 & 0 & 0 & 0 & -5 & 15 \\ 0 & 0 & 0 & 0 & 0 & -2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (4.79)$$

which can be simplified as

$$\begin{pmatrix} 0 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & -2 & 2 & -3 & 4 & -5 & 6 \\ 0 & 0 & 0 & -3 & 6 & -10 & 15 \\ 0 & 0 & 0 & -1 & 2 & -5 & 10 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (4.80)$$

$$\Rightarrow \begin{pmatrix} 0 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 2 & -3 & 4 \\ 0 & 0 & 0 & -1 & 2 & -5 & 10 \\ 0 & 0 & 0 & 0 & 0 & 5 & -15 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (4.81)$$

$$\Rightarrow \begin{pmatrix} 0 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 & 0 & -2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (4.82)$$

$$\Rightarrow \begin{pmatrix} 0 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (4.83)$$

The columns are for $a_0, a_1, a_2, a_3, a_2, a_1, a_0$, which means we can collapse the first three columns to the last three columns.

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 & -1 & 2 & -2 & 1 \\ 0 & 0 & 0 & -1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (4.84)$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 & -1 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (4.85)$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 & -1 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (4.86)$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 & -1 & 2 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (4.87)$$

Therefore, two equations emerge for four unknowns, leading to two linearly independent solutions.

4.4.2 Eigenvalue Problem for Three Bosons

From the coupled equation (4.51),

$$\begin{aligned} & \frac{m}{y_1} \Psi_m(y_j) \\ & + \frac{g}{4} \frac{m(m-1)}{\sqrt{y_1 y_2}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2}} \delta(y_1 + y_2 - x_1 - x_2) \Psi_m(x_1, x_2, y_3, \dots, y_m) \\ & + \frac{g}{6} m \sqrt{(m+1)(m+2)} \int \frac{dx_1 dx_2 dx_3}{\sqrt{y_1 x_1 x_2 x_3}} \delta(x_1 + x_2 + x_3 - y_1) \Psi_{m+2}(x_1, x_2, x_3, y_2, \dots, y_m) \\ & + \frac{g}{6} \frac{(m-2)\sqrt{m(m-1)}}{\sqrt{y_1 y_2 y_3 (y_1 + y_2 + y_3)}} \Psi_{m-2}(y_1 + y_2 + y_3, y_4, \dots, y_m) \\ & = \frac{M^2}{\mu^2} \Psi_m(y_j). \end{aligned} \quad (4.88)$$

When $m = 1$, then $p_1 = y_1 P^+ = P^+$ and $y_1 = 1$, the above equation becomes

$$\frac{1}{y_1} \Psi_1(y_j) + \frac{g}{6} \sqrt{(2)(3)} \int \frac{dx_1 dx_2 dx_3}{\sqrt{y_1 x_1 x_2 x_3}} \delta(x_1 + x_2 + x_3 - y_1) \Psi_3(x_1, x_2, x_3, y_2, \dots, y_m) = \frac{M^2}{\mu^2} \Psi_1(y_j), \quad (4.89)$$

which is

$$\Psi_1(y_1) + \frac{g}{\sqrt{6}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \Psi_3(x_1, x_2, x_3) = \frac{M^2}{\mu^2} \Psi_1(y_1), \quad (4.90)$$

where $x_3 = 1 - x_1 - x_2$.

When $m = 3$, it becomes

$$\begin{aligned}
& \frac{3}{y_1} \Psi_3(y_j) \\
& + \frac{g}{4} \frac{3(2)}{\sqrt{y_1 y_2}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2}} \delta(y_1 + y_2 - x_1 - x_2) \Psi_3(x_1, x_2, y_3, \dots, y_m) \\
& + \frac{g}{6} 3\sqrt{(4)(5)} \int \frac{dx_1 dx_2 dx_3}{\sqrt{y_1 x_1 x_2 x_3}} \delta(x_1 + x_2 + x_3 - y_1) \Psi_5(x_1, x_2, x_3, y_2, \dots, y_m) \\
& + \frac{g}{6} \frac{\sqrt{3(2)}}{\sqrt{y_1 y_2 y_3 (y_1 + y_2 + y_3)}} \Psi_1(y_1 + y_2 + y_3, y_4, \dots, y_m) \\
& = \frac{M^2}{\mu^2} \Psi_3(y_j).
\end{aligned} \tag{4.91}$$

Now, consider what emerges when we drop Ψ_5 ,

$$\begin{aligned}
& \frac{3}{y_1} \Psi_3(y_j) + \frac{3g}{2} \frac{1}{\sqrt{y_1 y_2}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2}} \delta(y_1 + y_2 - x_1 - x_2) \Psi_3(x_1, x_2, y_3) \\
& + \frac{g}{\sqrt{6}} \frac{1}{\sqrt{y_1 y_2 y_3 (y_1 + y_2 + y_3)}} \Psi_1(y_1 + y_2 + y_3) = \frac{M^2}{\mu^2} \Psi_3(y_j).
\end{aligned} \tag{4.92}$$

Further dropping the two-body scattering term \mathcal{P}_{22}^- , and applying $y_1 + y_2 + y_3 = 1$.

$$\frac{3}{y_1} \Psi_3(y_j) + \frac{g}{\sqrt{6}} \frac{1}{\sqrt{y_1 y_2 y_3}} \Psi_1(y_1 + y_2 + y_3) = \frac{M^2}{\mu^2} \Psi_3(y_j). \tag{4.93}$$

Thus, combining (4.90) and (4.93), we get

$$\Psi_1(y_1) + \frac{g}{\sqrt{6}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \Psi_3(x_1, x_2, x_3) = \frac{M^2}{\mu^2} \Psi_1(y_1) \tag{4.94}$$

where $x_3 = 1 - x_1 - x_2$, and

$$\frac{3}{y_1} \Psi_3(y_1, y_2, y_3) + \frac{g}{\sqrt{6}} \frac{1}{\sqrt{y_1 y_2 y_3}} \Psi_1(y_1 + y_2 + y_3) = \frac{M^2}{\mu^2} \Psi_3(y_1, y_2, y_3). \tag{4.95}$$

We substitute $g = \frac{\lambda}{4\pi\mu^2}$ and symmetrize the equations for exchanging y_1, y_2 and y_3 ,

$$\mu^2 \Psi_1 + \frac{\lambda}{\sqrt{6}} \int \frac{dx_1 dx_2}{4\pi \sqrt{x_1 x_2 x_3}} \Psi_3(x_1, x_2, x_3) = M^2 \Psi_1, \tag{4.96}$$

where $x_3 = 1 - x_1 - x_2$, and

$$\mu^2 \left(\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} \right) \Psi_3(y_1, y_2, y_3) + \frac{\lambda}{\sqrt{6}} \frac{1}{4\pi \sqrt{y_1 y_2 y_3}} \Psi_1 = M^2 \Psi_3(y_1, y_2, y_3). \tag{4.97}$$

From (4.96) we get

$$\Psi_1 = -\frac{\lambda}{\sqrt{6}} \frac{1}{\mu^2 - M^2} \int \frac{dx_1 dx_2}{4\pi \sqrt{x_1 x_2 x_3}} \Psi_3(x_1, x_2, x_3) \quad (4.98)$$

Substitute it into (4.97) to obtain

$$\begin{aligned} & \mu^2 \left(\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} \right) \Psi_3(y_1, y_2, y_3) - \frac{\lambda^2}{6(4\pi)^2} \frac{1}{\mu^2 - M^2} \frac{1}{\sqrt{y_1 y_2 y_3}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \Psi_3(x_1, x_2, x_3) \\ &= M^2 \Psi_3(y_1, y_2, y_3). \end{aligned} \quad (4.99)$$

Then

$$\left[\mu^2 \left(\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} \right) - M^2 \right] \Psi_3(y_1, y_2, y_3) = \frac{\lambda^2}{6(4\pi)^2} \frac{1}{\mu^2 - M^2} \frac{1}{\sqrt{y_1 y_2 y_3}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \Psi_3(x_1, x_2, x_3), \quad (4.100)$$

and

$$\frac{6(4\pi)^2}{\lambda^2} (\mu^2 - M^2) \left[\mu^2 \left(\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} \right) - M^2 \right] \Psi_3(y_1, y_2, y_3) = \frac{1}{\sqrt{y_1 y_2 y_3}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \Psi_3(x_1, x_2, x_3), \quad (4.101)$$

$$\frac{6(4\pi)^2}{\lambda^2} (\mu^2 - M^2) \mu^2 \left[\left(\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} \right) - \frac{M^2}{\mu^2} \right] \Psi_3(y_1, y_2, y_3) = \frac{1}{\sqrt{y_1 y_2 y_3}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \Psi_3(x_1, x_2, x_3), \quad (4.102)$$

$$\xi \Psi_3(y_1, y_2, y_3) = \left[\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} - \frac{M^2}{\mu^2} \right]^{-1} \frac{1}{\sqrt{y_1 y_2 y_3}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \Psi_3(x_1, x_2, x_3), \quad (4.103)$$

where we define

$$\xi = \frac{6(4\pi)^2}{\lambda^2} (\mu^2 - M^2) \mu^2 = \frac{6(4\pi)^2}{\lambda^2} \left(1 - \frac{M^2}{\mu^2} \right) \mu^4, \quad (4.104)$$

which is a dimensionless coupling.

To symmetrize the kernel of this equation, we separate the power of -1 to two $-1/2$ by replacing Ψ_3 as

$$\Psi_3(x_1, x_2, x_3) = \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1/2} f_3(x_1, x_2, x_3) \quad (4.105)$$

Then the equation (4.103) becomes

$$\begin{aligned} & \xi \left[\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} - \frac{M^2}{\mu^2} \right]^{-1/2} f_3(y_1, y_2, y_3) \\ &= \left[\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} - \frac{M^2}{\mu^2} \right]^{-1} \frac{1}{\sqrt{y_1 y_2 y_3}} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1/2} f_3(x_1, x_2, x_3) \end{aligned} \quad (4.106)$$

Thus,

$$\begin{aligned} & \xi f_3(y_1, y_2, y_3) \\ &= \frac{1}{\sqrt{y_1 y_2 y_3}} \left[\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} - \frac{M^2}{\mu^2} \right]^{-1/2} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1/2} f_3(x_1, x_2, x_3). \end{aligned} \quad (4.107)$$

From the above expression, we can see that f_3 must be of the form

$$f_3(x_1, x_2, x_3) = \frac{A}{\sqrt{x_1 x_2 x_3}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1/2}, \quad (4.108)$$

with a normalization A . Substitution of this form into the equation produces

$$\begin{aligned} & \xi \frac{A}{\sqrt{y_1 y_2 y_3}} \left[\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} - \frac{M^2}{\mu^2} \right]^{-1/2} = \frac{1}{\sqrt{y_1 y_2 y_3}} \left[\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} - \frac{M^2}{\mu^2} \right]^{-1/2} \\ & \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1/2} \frac{A}{\sqrt{x_1 x_2 x_3}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1/2}. \end{aligned} \quad (4.109)$$

Then

$$\begin{aligned} \xi &= \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1/2} \frac{1}{\sqrt{x_1 x_2 x_3}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1/2} \\ &= \int \frac{dx_1 dx_2}{x_1 x_2 x_3} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1}. \end{aligned} \quad (4.110)$$

A value can be computed when the ratio M/μ is specified.

To solve the equation for f_3 with the symmetric polynomial basis, we substitute the truncated expansion

$$f_3 = \sum_{n,i}^N a_{ni} P_n^{(i)}. \quad (4.111)$$

Then

$$\begin{aligned}
& \xi \sum_{n,i}^N a_{ni} P_n^{(i)} \\
&= \frac{1}{\sqrt{y_1 y_2 y_3}} \left[\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} - \frac{M^2}{\mu^2} \right]^{-1/2} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1/2} \sum_{m,j}^N a_{mj} P_m^{(j)}
\end{aligned} \tag{4.112}$$

which is

$$\begin{aligned}
& \sum_{n,i}^N \xi a_{ni} P_n^{(i)} \\
&= \frac{1}{\sqrt{y_1 y_2 y_3}} \left[\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} - \frac{M^2}{\mu^2} \right]^{-1/2} \sum_{m,j}^N \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1/2} a_{mj} P_m^{(j)}
\end{aligned} \tag{4.113}$$

Using the orthonormal condition

$$\int_0^1 dx \int_0^{1-x} dy P_n^{(i)}(x, y) P_m^{(j)}(x, y) = \delta_{nm} \delta_{ij}, \tag{4.114}$$

then

$$\begin{aligned}
& \sum_{n,i}^N \xi a_{ni} P_n^{(i)} P_{n'}^{(i')} \\
&= \frac{1}{\sqrt{y_1 y_2 y_3}} \left[\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} - \frac{M^2}{\mu^2} \right]^{-1/2} P_{n'}^{(i')} \sum_{m,j}^N \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1/2} P_m^{(j)} a_{mj}
\end{aligned} \tag{4.115}$$

$$\begin{aligned}
& \sum_{n,i}^N \xi a_{ni} \int dy_1 dy_2 P_n^{(i)} P_{n'}^{(i')} = \int dy_1 dy_2 \frac{1}{\sqrt{y_1 y_2 y_3}} \left[\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} - \frac{M^2}{\mu^2} \right]^{-1/2} \\
& P_{n'}^{(i')} \sum_{m,j}^N \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1/2} P_m^{(j)} a_{mj}
\end{aligned} \tag{4.116}$$

$$\begin{aligned}
& \sum_{n,i}^N \xi a_{ni} \delta_{nn'} \delta_{ii'} = \int dy_1 dy_2 \frac{1}{\sqrt{y_1 y_2 y_3}} \left[\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} - \frac{M^2}{\mu^2} \right]^{-1/2} \\
& P_{n'}^{(i')} \sum_{m,j}^N \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1/2} P_m^{(j)} a_{mj}
\end{aligned} \tag{4.117}$$

$$\begin{aligned} \xi a_{n'i'} &= \int dy_1 dy_2 \frac{1}{\sqrt{y_1 y_2 y_3}} \left[\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} - \frac{M^2}{\mu^2} \right]^{-1/2} \\ &P_{n'}^{(i')} \sum_{m,j}^N \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1/2} P_m^{(j)} a_{mj} \end{aligned} \quad (4.118)$$

which is

$$\begin{aligned} \xi a_{ni} &= \sum_{m,j}^N \int dy_1 dy_2 \frac{1}{\sqrt{y_1 y_2 y_3}} \left[\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} - \frac{M^2}{\mu^2} \right]^{-1/2} \\ &P_n^{(i)} \int \frac{dx_1 dx_2}{\sqrt{x_1 x_2 x_3}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1/2} P_m^{(j)} a_{mj} \\ &= \sum_{m,j}^N b_{ni} b_{mj} a_{mj} \end{aligned} \quad (4.119)$$

with

$$b_{ni} \equiv \int dy_1 dy_2 \frac{1}{\sqrt{y_1 y_2 y_3}} \left[\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} - \frac{M^2}{\mu^2} \right]^{-1/2} P_n^{(i)}. \quad (4.120)$$

4.4.3 Amplitude for Three Bosons

Instead of solving the polynomial equation, we can have a glance of the wavefunction amplitudes of the continuous Hamiltonian equation by numerical integration in three bosons' case. From (4.105) and (4.107),

$$\Psi_3(x_1, x_2, x_3) = \frac{A}{\sqrt{x_1 x_2 x_3}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1}, \quad (4.121)$$

and

$$\begin{aligned}
\Psi_1 &= -\frac{\lambda}{\sqrt{6}} \frac{1}{\mu^2 - M^2} \int \frac{dx_1 dx_2}{4\pi \sqrt{x_1 x_2 x_3}} \Psi_3(x_1, x_2, x_3) \\
&= -\frac{\lambda}{\sqrt{6}} \frac{1}{\mu^2 - M^2} \int \frac{dx_1 dx_2}{4\pi \sqrt{x_1 x_2 x_3}} \frac{A}{\sqrt{x_1 x_2 x_3}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1} \\
&= -\frac{\lambda}{\sqrt{6}} \frac{1}{\mu^2 - M^2} \int dx_1 dx_2 \frac{A}{4\pi x_1 x_2 x_3} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{M^2}{\mu^2} \right]^{-1} \\
&= -\frac{\lambda}{4\pi \sqrt{6}} \frac{A}{\mu^2 (1 - \frac{M^2}{\mu^2})} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{x_1 x_2 (1 - x_1 - x_2)} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{1 - x_1 - x_2} - \frac{M^2}{\mu^2} \right]^{-1} \\
&= -A \frac{g}{\sqrt{6}} \left(1 - \frac{M^2}{\mu^2} \right)^{-1} \\
&\quad \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{x_1 x_2 (1 - x_1 - x_2)} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{1 - x_1 - x_2} - \frac{M^2}{\mu^2} \right]^{-1},
\end{aligned} \tag{4.122}$$

where $g \equiv \frac{\lambda}{4\pi \mu^2}$.

The normalization constant A can be obtained by

$$\begin{aligned}
1 &= \int_0^1 dx_1 \int_0^{1-x_1} dx_2 (|\Psi_3|^2 + |\Psi_1|^2) \\
&= A^2 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left\{ \frac{1}{x_1 x_2 (1 - x_1 - x_2)} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{1 - x_1 - x_2} - \frac{M^2}{\mu^2} \right]^{-2} \right. \\
&\quad \left. + \frac{g^2}{6} \left(1 - \frac{M^2}{\mu^2} \right)^{-2} \left[\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{x_1 x_2 (1 - x_1 - x_2)} \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{1 - x_1 - x_2} - \frac{M^2}{\mu^2} \right)^{-1} \right]^2 \right\}.
\end{aligned} \tag{4.123}$$

We get

$$\begin{aligned}
\Psi_1 &= -A \frac{g}{\sqrt{6}} \left(1 - \frac{M^2}{\mu^2} \right)^{-1} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{x_1 x_2 (1 - x_1 - x_2)} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{1 - x_1 - x_2} - \frac{M^2}{\mu^2} \right]^{-1}
\end{aligned} \tag{4.124}$$

and

$$\Psi_3(x_1, x_2) = \frac{A}{\sqrt{x_1 x_2 (1 - x_1 - x_2)}} \left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{1 - x_1 - x_2} - \frac{M^2}{\mu^2} \right]^{-1}. \tag{4.125}$$

Ψ_1 changing with λ for different value of M^2/μ^2 is shown in Figure 4.1 for $M^2/\mu^2 > 0$ and Figure 4.2 for $M^2/\mu^2 < 0$. Ψ_1 for different value of M^2/μ^2 when $M^2/\mu^2 < 1$ eventually reach $\Psi_1 = -1.41421 = -\sqrt{2}$ at sufficiently large λ . While Ψ_1 for different value of M^2/μ^2 when $M^2/\mu^2 > 1$ eventually reach $\Psi_1 = 1.41421 = \sqrt{2}$ at sufficiently large λ .

When $\Psi_1 = \pm\sqrt{2}$,

$$\int_0^1 dx_1 \int_0^{1-x_1} dx_2 |\Psi_1|^2 = 1. \quad (4.126)$$

The probability to find Ψ_1 at sufficiently large λ is $\sim 100\%$. Larger λ gives deeper potential well, and ground state is more probable.

We also plot Ψ_1 changing with M^2/μ^2 for different value of λ in Figure 4.3. $M^2/\mu^2 \gtrsim 9$ is when the integral cannot converge in this numerical integration. The reason for this is that for the three bosons' case, $|M/\mu|$ should be smaller than three. As we have already seen in Fig. 4.1 and Fig. 4.2, Fig. 4.3 shows the same break close to $M^2/\mu^2 = 1$, with $\Psi_1 = \pm\sqrt{2}$ on either side of the break. This is due to that Ψ_1 has a pole at $M^2/\mu^2 = 1$ as can be seen in equation (4.124).

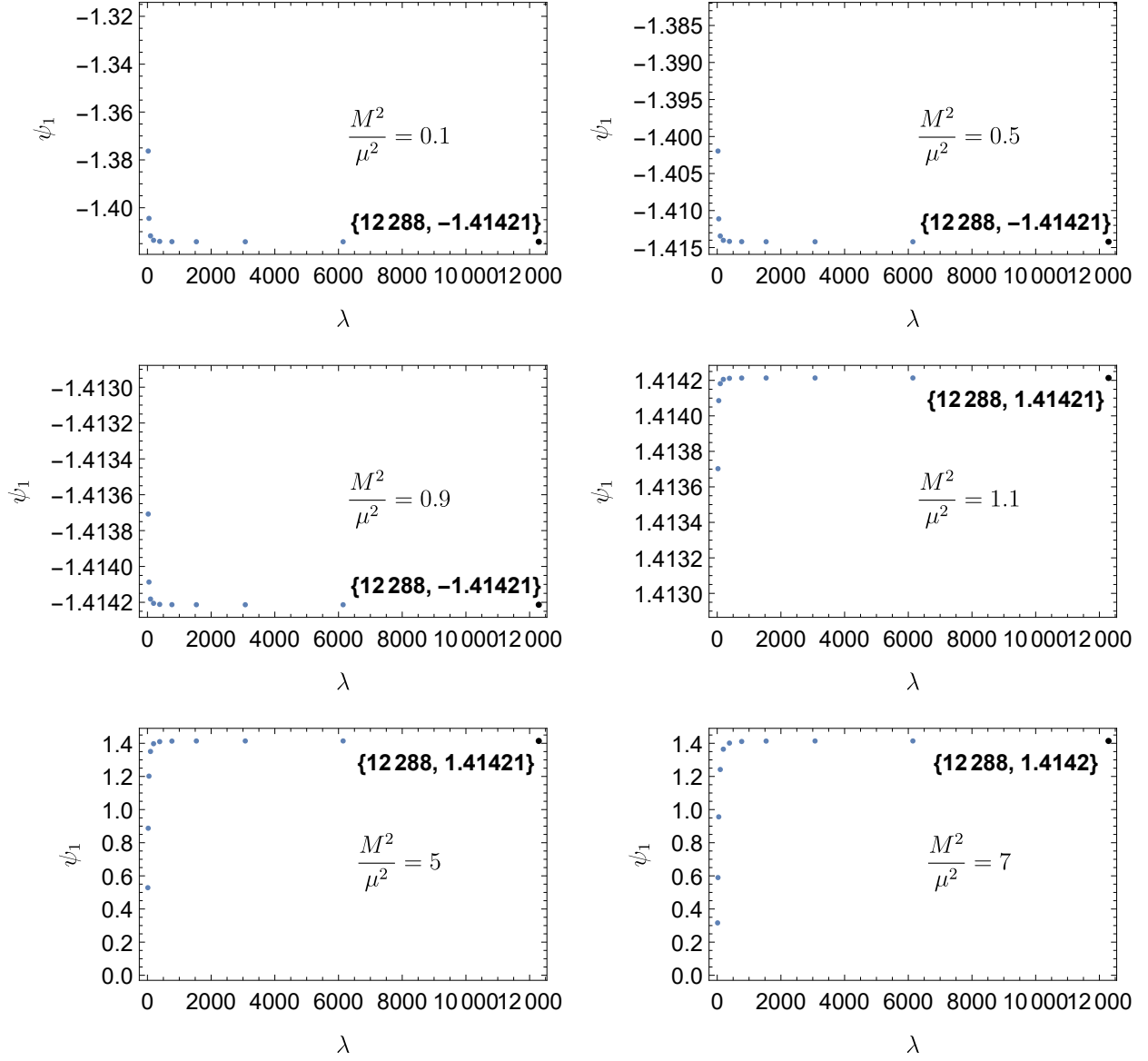


Figure 4.1 Ψ_1 changing with λ for different value of $M^2/\mu^2 > 0$. The coordinates of the last point are shown in the graph.

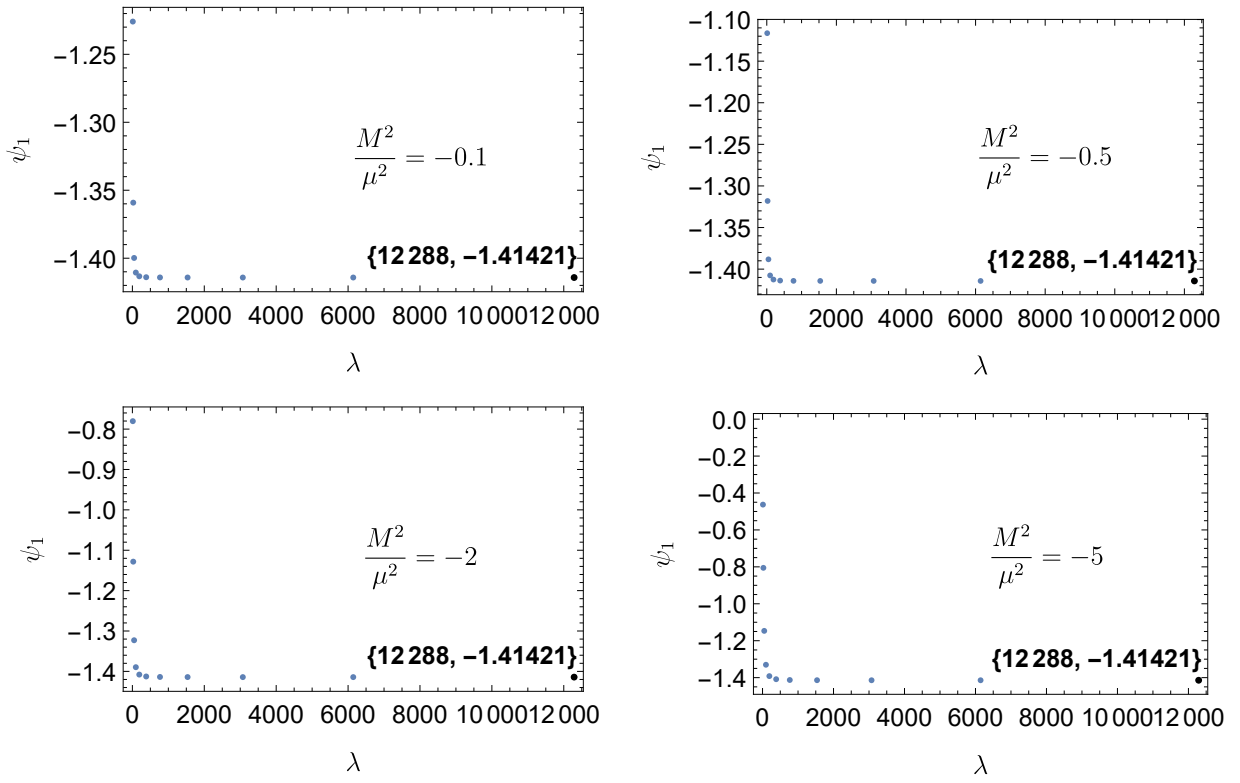


Figure 4.2 Ψ_1 changing with λ for different value of $M^2/\mu^2 < 0$. The coordinates of the last point are shown in the graph.

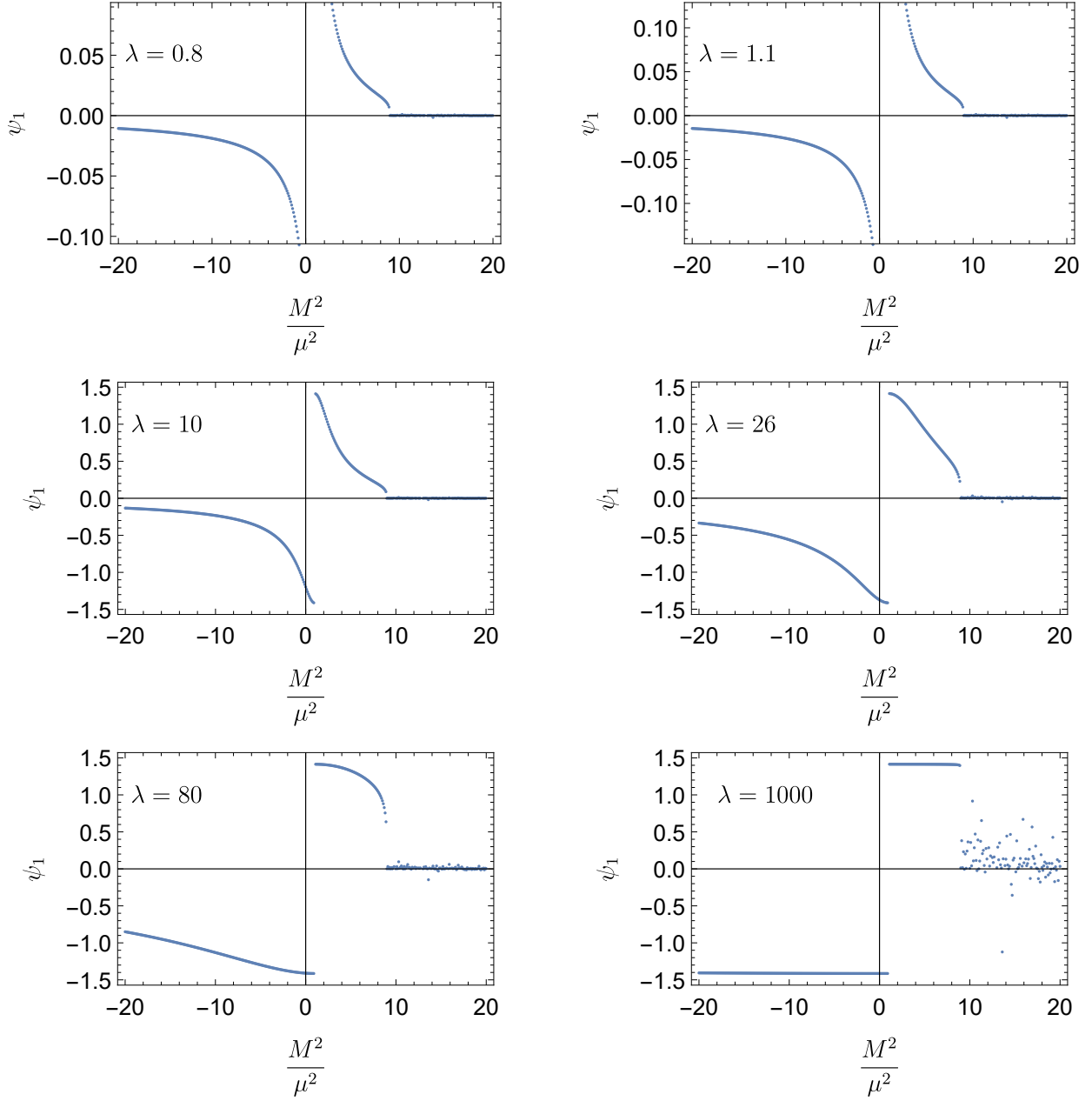


Figure 4.3 Ψ_1 changing with M^2/μ^2 for different value of λ

CHAPTER 5. SUMMARY AND DISCUSSION

We have reviewed the critical coupling calculational frameworks without zero mode (DLCQ) and with zero mode (polynomial basis). We can also compare the critical coupling values and see if they are consistent.

5.1 Introduction

For comparison, we can also convert the critical coupling to equal time using the mass renormalization described in [Burkardt et al. \(2016\)](#).

5.2 Mass Renormalization

The bare mass in ϕ_{1+1}^4 theory is renormalized by tadpole contributions in equal-time quantization, but not in light-front quantization. This amounts to a shift in the mass entering the dynamical calculation. This same shift needs to be inserted into the mass of the light-front approach in order to compare extracted "observables" such as the critical coupling.

The light front mass and the equal time mass (bare mass) are related by

$$\mu_{LF}^2 = \mu_{ET}^2 + \lambda \left[\langle 0 | \frac{\phi^2}{2} | 0 \rangle - \langle 0 | \frac{\phi^2}{2} | 0 \rangle_{\text{free}} \right]. \quad (5.1)$$

This is from adding vacuum loop diagrams to the original lagrangian, i.e.,

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 \quad (5.2)$$

becomes

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{\lambda}{2!}\phi^2 \frac{\langle 0|\phi^2|0\rangle}{2!} \\ &= \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\left(\mu^2 + \lambda\langle 0|\frac{\phi^2}{2}|0\rangle\right)\phi^2 - \frac{\lambda}{4!}\phi^4.\end{aligned}\tag{5.3}$$

So

$$\mu_{LF}^2 = \mu^2 + \lambda\langle 0|\frac{\phi^2}{2}|0\rangle.\tag{5.4}$$

where $\mu = \mu_{ET}$ is the bare mass and $\langle 0|\phi^2|0\rangle$ is calculated in equal time quantization and signals the shift in the mass. If this vacuum expectation value were calculated with the perturbative light front vacuum, it would be zero. Hence, to make a comparison between equal-time and light-front time results, we need to employ the corresponding mass shift in both cases. In the light-front quantized theory we identified the zero mode as a constraint equation and can use it to identify the appropriate mass shift for the light-front calculations.

We subtracted the divergent free field contribution in order to obtain a finite result,

$$\mu_{LF}^2 = \mu_{ET}^2 + \lambda \left[\langle 0|\frac{\phi^2}{2}|0\rangle - \langle 0|\frac{\phi^2}{2}|0\rangle_{\text{free}} \right].\tag{5.5}$$

The vacuum contribution should not depend on the starting point of the field line (similar to mass line, or a propagator). So we assume it starts at $(x^+, x^-) = (0, 0)$. At this point, the field is $\phi(0, 0)$. The end point is $(x^+, x^-) = (\epsilon^+, \epsilon^-)$. At this point, the field is $\phi(\epsilon^+, \epsilon^-)$. We inserted the identity operator for expansion of mass eigenstates into the expression, then

$$\langle 0|\frac{\phi^2}{2}|0\rangle = \frac{1}{2}\langle 0|\phi(\epsilon^+, \epsilon^-) \int_0^\infty dP \sum_n |\Psi_n(P)\rangle \langle \Psi_n(P)|\phi(0, 0)|0\rangle,\tag{5.6}$$

where

$$|\Psi(P)\rangle = \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta\left(1 - \sum_i^m y_i\right) \Psi_m(y_i) |y_i P; P, m\rangle\tag{5.7}$$

with (4.27)

$$|y_i P; P, m\rangle = \frac{1}{\sqrt{m!}} \prod_{i=1}^m a^\dagger(y_i P) |0\rangle = P^{\frac{-m}{2}} |m\rangle,\tag{5.8}$$

i.e.,

$$\begin{aligned} |\Psi_n(P)\rangle &= \sum_m P^{\frac{m-1}{2}} \int \prod_i^m dy_i \delta\left(1 - \sum_i^m y_i\right) \Psi_m(y_i) P^{-\frac{m}{2}} |m\rangle \\ &= \sum_m P^{\frac{-1}{2}} \int \prod_i^m dy_i \delta\left(1 - \sum_i^m y_i\right) \Psi_{nm}(y_i) |m\rangle. \end{aligned} \quad (5.9)$$

Similarly,

$$\langle \Psi_n(P) | = \langle m | \sum_m P^{\frac{-1}{2}} \int \prod_i^m dy_i \delta\left(1 - \sum_i^m y_i\right) \Psi_{nm}^*(y_i). \quad (5.10)$$

At $x^+ = 0$

$$\phi(x^+ = 0, x^-) = \int \frac{dp}{\sqrt{4\pi p}} \{a(p)e^{-ipx^-/2} + a^\dagger(p)e^{ipx^-/2}\}. \quad (5.11)$$

Now, with

$$\phi(0, 0) = \int \frac{dp}{\sqrt{4\pi p}} \{a(p) + a^\dagger(p)\}, \quad (5.12)$$

$$\begin{aligned} \langle \Psi_n(P) | \phi(0, 0) | 0 \rangle &= \langle \Psi_n(P) | \int \frac{dp}{\sqrt{4\pi p}} \{a(p) + a^\dagger(p)\} | 0 \rangle \\ &= \langle \Psi_n(P) | \int \frac{dp}{\sqrt{4\pi p}} a^\dagger(p) | 0 \rangle. \end{aligned} \quad (5.13)$$

Only $\langle m | = \langle 1 |$ piece in $\langle \Psi_n(P) |$ will remain

$$\langle 1 | P^{\frac{-1}{2}} \int dy_1 \delta(1 - y_1) \Psi_{n1}^*(y_1) = \langle 1 | P^{\frac{-1}{2}} \Psi_{n1}^*(1) = \langle 0 | a(P) \Psi_{n1}^*, \quad (5.14)$$

where we used (4.25)

$$a^\dagger(y_i P) | 0 \rangle = \frac{1}{\sqrt{P}} \sqrt{1} | 1 \rangle \quad \langle 0 | a(y_i P) = \langle 1 | \frac{1}{\sqrt{P}} \sqrt{1}. \quad (5.15)$$

Thus,

$$\begin{aligned} \langle \Psi_n(P) | \phi(0, 0) | 0 \rangle &= \langle \Psi_n(P) | \int \frac{dp}{\sqrt{4\pi p}} a^\dagger(p) | 0 \rangle \\ &= \langle 0 | a(P) \Psi_{n1}^* \int \frac{dp}{\sqrt{4\pi p}} a^\dagger(p) | 0 \rangle \\ &= \langle 0 | \Psi_{n1}^* \int \frac{dp}{\sqrt{4\pi p}} \delta(P - p) | 0 \rangle = \frac{\Psi_{n1}^*}{\sqrt{4\pi P}}. \end{aligned} \quad (5.16)$$

From [Peskin and Schroeder \(1995\)](#), P25, in the Heisenberg picture, we make the operators ϕ time-dependent in the usual way:

$$\phi(x, t) = e^{iHt} \phi(x) e^{-iHt}, \quad (5.17)$$

with Heisenberg equation of motion,

$$i \frac{\partial}{\partial t} \mathcal{O} = [\mathcal{O}, H], \quad \mathcal{O} = \phi(x, t). \quad (5.18)$$

Therefore, in light-front coordinate, we similarly have

$$\phi(\epsilon^+, \epsilon^-) = e^{i\mathcal{P}^- \epsilon^+ / 2} \phi(0, \epsilon^-) e^{-i\mathcal{P}^- \epsilon^+ / 2}, \quad (5.19)$$

where

$$\phi(0, \epsilon^-) = \int \frac{dp}{\sqrt{4\pi p}} \{a(p) e^{-ip\epsilon^- / 2} + a^\dagger(p) e^{ip\epsilon^- / 2}\}, \quad (5.20)$$

i.e.,

$$\phi(\epsilon^+, \epsilon^-) = e^{i\mathcal{P}^- \epsilon^+ / 2} \int \frac{dp}{\sqrt{4\pi p}} \{a(p) e^{-ip\epsilon^- / 2} + a^\dagger(p) e^{ip\epsilon^- / 2}\} e^{-i\mathcal{P}^- \epsilon^+ / 2}. \quad (5.21)$$

This provides

$$\begin{aligned} \langle 0 | \phi(\epsilon^+, \epsilon^-) | \Psi_n(P) \rangle &= \langle 0 | e^{i\mathcal{P}^- \epsilon^+ / 2} \int \frac{dp}{\sqrt{4\pi p}} \{a(p) e^{-ip\epsilon^- / 2} + a^\dagger(p) e^{ip\epsilon^- / 2}\} e^{-i\mathcal{P}^- \epsilon^+ / 2} | \Psi_n(P) \rangle \\ &= \langle 0 | e^{i0\epsilon^+ / 2} \int \frac{dp}{\sqrt{4\pi p}} a(p) e^{-ip\epsilon^- / 2} e^{-iM_n^2 \epsilon^+ / 2P} | \Psi_n(P) \rangle, \end{aligned} \quad (5.22)$$

where we make use of the eigenvalue problem

$$\mathcal{P}^- | \Psi_n(P) \rangle = \frac{M_n^2}{P} | \Psi_n(P) \rangle \quad (5.23)$$

to replace the right \mathcal{P}^- as M_n^2/P . And since \mathcal{P}^- is normal-ordering, there will always be an annihilator on the right in the expression of \mathcal{P}^- . Hence,

$$\mathcal{P}^- | 0 \rangle = 0 = 0 | 0 \rangle. \quad (5.24)$$

and we replaced the left \mathcal{P}^- as 0.

Only the $|m\rangle = |1\rangle$ piece in $|\Psi_n(P)\rangle$ will remain, similar to (5.14),

$$P^{-\frac{1}{2}} \int dy_1 \delta(1 - y_1) \Psi_{n1}(y_1) |1\rangle = \Psi_{n1}(1) P^{-\frac{1}{2}} |1\rangle = \Psi_{n1} a^\dagger(P) |0\rangle. \quad (5.25)$$

Therefore,

$$\begin{aligned} \langle 0 | \phi(\epsilon^+, \epsilon^-) | \Psi_n(P) \rangle &= \langle 0 | \int \frac{dp}{\sqrt{4\pi p}} a(p) e^{-ip\epsilon^-/2} e^{-iM_n^2 \epsilon^+/2P} \Psi_{n1} a^\dagger(P) | 0 \rangle \\ &= \langle 0 | \int \frac{dp}{\sqrt{4\pi p}} \delta(p - P) e^{-ip\epsilon^-/2} e^{-iM_n^2 \epsilon^+/2P} \Psi_{n1} | 0 \rangle \\ &= \langle 0 | \frac{1}{\sqrt{4\pi P}} e^{-iP\epsilon^-/2} e^{-iM_n^2 \epsilon^+/2P} \Psi_{n1} | 0 \rangle \\ &= \frac{\Psi_{n1}}{\sqrt{4\pi P}} e^{-i(P\epsilon^- + M_n^2 \epsilon^+/P)/2}. \end{aligned} \quad (5.26)$$

with M_n the mass of the n th state.

For the free case, the coupled equation (4.51) only has the first term on the left hand side,

$$\frac{m}{y_1} \Psi_{nm}(y_j) = \frac{M_n^2}{\mu^2} \Psi_{nm}(y_j) \quad (5.27)$$

Therefore,

$$\frac{1}{1} \Psi_{n1}(1) = \frac{M_n^2}{\mu^2} \Psi_{n1}(1). \quad (5.28)$$

So $M_n^2 = \mu^2$. And since there will be only one wave function in the free field case, $n = 1$, the normalization yields $\Psi_{n1} = 1$.

From (5.16) and (5.26)

$$\langle \Psi_n(P) | \phi(0, 0) | 0 \rangle_{\text{free}} = \langle 0 | \Psi_{n1}^* \int \frac{dp}{\sqrt{4\pi p}} \delta(P - p) | 0 \rangle = \frac{1}{\sqrt{4\pi P}} \quad (5.29)$$

and

$$\langle 0 | \phi(\epsilon^+, \epsilon^-) | \Psi_n(P) \rangle_{\text{free}} = \frac{1}{\sqrt{4\pi P}} e^{-i(P\epsilon^- + \mu^2 \epsilon^+/P)/2}, \quad (5.30)$$

we get,

$$\begin{aligned} \langle 0 | \frac{\phi^2}{2} | 0 \rangle &= \frac{1}{2} \sum_n \int_0^\infty dP \langle 0 | \phi(\epsilon^+, \epsilon^-) | \Psi_n(P) \rangle \langle \Psi_n(P) | \phi(0, 0) | 0 \rangle \\ &= \frac{1}{2} \sum_n \int_0^\infty dP \frac{\Psi_{n1}}{\sqrt{4\pi P}} e^{-i(P\epsilon^- + M_n^2 \epsilon^+/P)/2} \frac{\Psi_{n1}^*}{\sqrt{4\pi P}} \\ &= \frac{1}{2} \sum_n \int_0^\infty dP \frac{|\Psi_{n1}|^2}{4\pi P} e^{-i(P\epsilon^- + M_n^2 \epsilon^+/P)/2} \end{aligned} \quad (5.31)$$

and

$$\begin{aligned}
\langle 0 | \frac{\phi^2}{2} | 0 \rangle_{\text{free}} &= \frac{1}{2} \sum_n \int_0^\infty dP \langle 0 | \phi(\epsilon^+, \epsilon^-) | \Psi_n(P) \rangle_{\text{free}} \langle \Psi_n(P) | \phi(0, 0) | 0 \rangle_{\text{free}} \\
&= \frac{1}{2} \sum_n \int_0^\infty dP \frac{1}{\sqrt{4\pi P}} e^{-i(P\epsilon^- + \mu^2\epsilon^+)/2} \frac{1}{\sqrt{4\pi P}} \\
&= \frac{1}{2} \sum_{n=1}^\infty \int_0^\infty dP \frac{1}{4\pi P} e^{-i(P\epsilon^- + \mu^2\epsilon^+)/2} = \frac{1}{2} \int_0^\infty dP \frac{1}{4\pi P} e^{-i(P\epsilon^- + \mu^2\epsilon^+)/2}.
\end{aligned} \tag{5.32}$$

The completeness of the eigenstates allows us to replace 1 in the free case to the sum $1 = \sum_n |\Psi_n|^2$.

The difference can be written

$$\begin{aligned}
&\langle 0 | \frac{\phi^2}{2} | 0 \rangle - \langle 0 | \frac{\phi^2}{2} | 0 \rangle_{\text{free}} \\
&= \frac{1}{2} \sum_n \int_0^\infty dP \frac{|\Psi_{n1}|^2}{4\pi P} e^{-i(P\epsilon^- + M_n^2\epsilon^+)/2} - \frac{1}{2} \int_0^\infty dP \frac{\sum_n |\Psi_{n1}|^2}{4\pi P} e^{-i(P\epsilon^- + \mu^2\epsilon^+)/2} \\
&= \frac{1}{2} \sum_n \int_0^\infty dP \frac{|\Psi_{n1}|^2}{4\pi P} e^{-i(P\epsilon^- + M_n^2\epsilon^+)/2} - \frac{1}{2} \sum_n \int_0^\infty dP \frac{|\Psi_{n1}|^2}{4\pi P} e^{-i(P\epsilon^- + \mu^2\epsilon^+)/2} \\
&= \frac{1}{2} \sum_n \int_0^\infty dP \frac{|\Psi_{n1}|^2}{4\pi P} (e^{-i(P\epsilon^- + M_n^2\epsilon^+)/2} - e^{-i(P\epsilon^- + \mu^2\epsilon^+)/2}) \\
&= \frac{1}{2} \sum_n \int_0^\infty dP \frac{|\Psi_{n1}|^2}{4\pi P} (e^{-iP\epsilon^-/2} e^{-iM_n^2\epsilon^+/2P} - e^{-iP\epsilon^-/2} e^{-i\mu^2\epsilon^+/2P}) \\
&= \frac{1}{2} \sum_n \int_0^\infty dP \frac{|\Psi_{n1}|^2}{4\pi P} e^{-iP\epsilon^-/2} (e^{-iM_n^2\epsilon^+/2P} - e^{-i\mu^2\epsilon^+/2P}) \\
&= \sum_n \frac{|\Psi_{n1}|^2}{8\pi} \int_0^\infty \frac{dP}{P} e^{-iP\epsilon^-/2} (e^{-iM_n^2\epsilon^+/2P} - e^{-i\mu^2\epsilon^+/2P}).
\end{aligned} \tag{5.33}$$

With the change of variable $P = 2z\epsilon^+$ ($2z = P/\epsilon^+$) and the introduction of a convergence factor $e^{-\eta z}$, the expression becomes

$$\begin{aligned}
\langle 0 | \frac{\phi^2}{2} | 0 \rangle - \langle 0 | \frac{\phi^2}{2} | 0 \rangle_{\text{free}} &= \sum_n \frac{|\Psi_{n1}|^2}{8\pi} \int_0^\infty \frac{dz}{z} e^{-iz\epsilon^+\epsilon^-} e^{-\eta z} (e^{-iM_n^2/4z} - e^{-i\mu^2/4z}) \\
&= \sum_n \frac{|\Psi_{n1}|^2}{8\pi} \int_0^\infty \frac{dz}{z} e^{iz(-\epsilon^2 + i\eta z)} (e^{-iM_n^2/4z} - e^{-i\mu^2/4z}),
\end{aligned} \tag{5.34}$$

where we used $\epsilon^+\epsilon^- = \frac{1}{2}\epsilon^+\epsilon^- + \frac{1}{2}\epsilon^+\epsilon^- = \epsilon^+\epsilon_+ + \epsilon_-\epsilon^- = \epsilon^2$.

Each term is an integral representation of the modified Bessel function K_0 ,

$$K_0(\alpha\beta) = \frac{1}{2} \int \frac{dx}{x} \exp \left(i \frac{\alpha}{2} \left[x - \frac{\beta^2}{x} \right] \right). \tag{5.35}$$

With $-\epsilon^2 + i\eta = \alpha/2$ and $M_n^2 = 2\alpha\beta^2$, we have

$$\begin{aligned} \int_0^\infty \frac{dz}{z} e^{iz(-\epsilon^2 + i\eta z)} e^{-iM_n^2/4z} &= \int_0^\infty \frac{dz}{z} e^{iz\alpha/2} e^{-iM_n^2/4z} = \int_0^\infty \frac{dz}{z} \exp\left(iz\frac{\alpha}{2} - i\frac{\alpha\beta^2}{2z}\right) \\ &= \int_0^\infty \frac{dz}{z} \exp\left(i\frac{\alpha}{2}\left[z - \frac{\beta^2}{z}\right]\right) = 2K_0(\alpha\beta) = 2K_0(M_n\sqrt{-\epsilon^2 + i\eta}), \end{aligned} \quad (5.36)$$

where we used $\alpha = 2(-\epsilon^2 + i\eta)$, $\beta = M_n/\sqrt{2\alpha} = M_n/(2\sqrt{-\epsilon^2 + i\eta})$, and $\alpha\beta = M_n\sqrt{-\epsilon^2 + i\eta}$. So

$$\begin{aligned} \langle 0|\frac{\phi^2}{2}|0\rangle - \langle 0|\frac{\phi^2}{2}|0\rangle_{\text{free}} &= \sum_n \frac{|\Psi_{n1}|^2}{8\pi} \int_0^\infty \frac{dz}{z} e^{iz(-\epsilon^2 + i\eta z)} (e^{-iM_n^2/4z} - e^{-i\mu^2/4z}) \\ &= \sum_n \frac{|\Psi_{n1}|^2}{4\pi} \left[K_0(M_n\sqrt{-\epsilon^2 + i\eta}) - K_0(\mu\sqrt{-\epsilon^2 + i\eta}) \right]. \end{aligned} \quad (5.37)$$

As $-\epsilon^2 + i\eta$ goes to zero (i.e., $\epsilon \rightarrow 0$ and $\eta \rightarrow 0$), the only contribution from K_0 is a simple logarithm,

$$K_0(z \rightarrow 0) = -\ln\left(\frac{z}{2}\right) - \gamma. \quad (5.38)$$

$$\begin{aligned} \langle 0|\frac{\phi^2}{2}|0\rangle - \langle 0|\frac{\phi^2}{2}|0\rangle_{\text{free}} &= \sum_n \frac{|\Psi_{n1}|^2}{4\pi} \left[\ln\left(\frac{\mu\sqrt{-\epsilon^2 + i\eta}}{2}\right) - \ln\left(\frac{M_n\sqrt{-\epsilon^2 + i\eta}}{2}\right) \right] \\ &= \sum_n \frac{|\Psi_{n1}|^2}{4\pi} \left[\left(-\ln\frac{M_n\sqrt{-\epsilon^2 + i\eta}}{2}\right) - \left(-\ln\frac{\mu\sqrt{-\epsilon^2 + i\eta}}{2}\right) \right] \\ &= -\sum_n \frac{|\Psi_{n1}|^2}{4\pi} \left[\left(\ln\frac{M_n\sqrt{-\epsilon^2 + i\eta}}{2}\right) - \left(\ln\frac{\mu\sqrt{-\epsilon^2 + i\eta}}{2}\right) \right] \\ &= -\frac{1}{4\pi} \sum_n |\Psi_{n1}|^2 \ln \frac{M_n}{\mu} = -\frac{1}{4\pi} \Delta, \end{aligned} \quad (5.39)$$

where

$$\Delta = \sum_n |\Psi_{n1}|^2 \ln \frac{M_n}{\mu} = \sum_n |\Psi_{n1}|^2 \ln \frac{M_n}{\mu_{LF}}. \quad (5.40)$$

This expression is consistent with Eq. (3.14) in [Burkardt et al. \(2016\)](#), and defines the quantity that can be evaluated after solving for the DLCQ spectra omitting zero modes. Then this quantity can be used to relate the bare masses on ET and LF quantized approaches. This quantity is

currently under investigation using results from DLCQ calculations and the results will be reported separately.

The bare masses in the two quantizations are then related by (5.5)

$$\begin{aligned}\mu_{LF}^2 &= \mu_{ET}^2 + \lambda \left[\langle 0 | \frac{\phi^2}{2} | 0 \rangle - \langle 0 | \frac{\phi^2}{2} | 0 \rangle_{\text{free}} \right] \\ &= \mu_{ET}^2 + \lambda \left(-\frac{1}{4\pi} \Delta \right) = \mu_{ET}^2 - \frac{\lambda}{4\pi} \Delta.\end{aligned}\tag{5.41}$$

Divide both sides by μ_{LF}^2 ,

$$\frac{\mu_{ET}^2}{\mu_{LF}^2} = 1 + \frac{\lambda}{4\pi\mu_{LF}^2} \Delta = 1 + g_{LF} \Delta,\tag{5.42}$$

where we defined $g_{LF} = \frac{\lambda}{4\pi\mu_{LF}^2}$.

Thus we get the transformation between light front and equal time critical coupling,

$$g_{ET} = \frac{g_{LF}}{\mu_{ET}^2/\mu_{LF}^2} = \frac{g_{LF}}{1 + g_{LF} \Delta}.\tag{5.43}$$

Other light front quantities which scaled by μ_{LF}^2 can also be transformed into equal time quantity.

For example,

$$\frac{M^2}{\mu_{ET}^2} = \frac{M^2/\mu_{LF}^2}{\mu_{ET}^2/\mu_{LF}^2} = \frac{1}{1 + g_{LF} \Delta} \frac{M^2}{\mu_{LF}^2}.\tag{5.44}$$

5.3 Numerical Results

Table 5.1 Δ Obtained by DLCQ. (collaborated with Shreeram Jawadekar, Mamoon Sharaf, and James P. Vary)

K	$\lambda = 25$	$\lambda = 27.8$
16	-0.282872	-0.354201
20	-0.313787	-0.397386
24	-0.339170	-0.433876
28	-0.360677	-0.465788

We draw the result of Δ as a function of K for two values of coupling $\lambda = 25$ and $\lambda = 27.8$, which is 26.4 ± 1.4 , where $\lambda = 26.4$ is the critical coupling obtained in [Burkardt et al. \(2016\)](#). Since the critical coupling value is not determined to high accuracy in the literature, we choose this range

for calculating Δ and estimate that the final value of Δ should be in this range. Figure 5.1 shows a linear fit for Δ changing with $1/K$, the y-intercept indicates the value of Δ at $K \rightarrow \infty$. The red bar at $1/K = 0$ illustrates the range of Δ obtained by Burkardt et al. (2016). Figure 5.2 shows the same graph with quadratic fit. Figure 5.3 plots Δ changing with $1/K^2$, and fits the curve by quadratic fit. As we can see in all these extrapolation methods we use, the range of the y-intercept overlaps with the Burkardt et al. (2016) range $\Delta = (-0.59, -0.35)$.

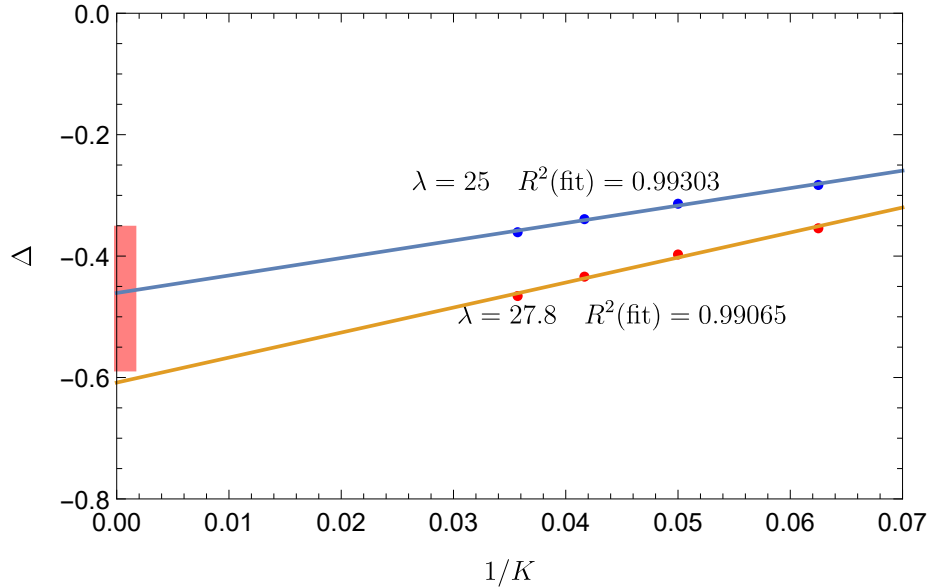


Figure 5.1 Δ changing with $1/K$ by linear fit. The y-intercept is in the range of $(-0.6084, -0.4607)$. (collaborated with Shreeram Jawadekar, Mamoon Sharaf, and James P. Vary)

5.4 Comparison

Table 5.2 shows the critical coupling obtained by different light front or equal time methods. The dimensionless coupling constant is defined as $\bar{g} = \frac{\pi}{6}g$, where $g = \frac{\lambda}{4\pi\mu^2}$. The critical coupling obtained in light front $\bar{g}_c(LF)$ can be converted to $\bar{g}_c(ET)$ using equation (5.43)

$$\bar{g}_c(ET) = \frac{\bar{g}_c(LF)}{1 + \bar{g}_c(LF)(\frac{6}{\pi}\Delta)}, \quad (5.45)$$

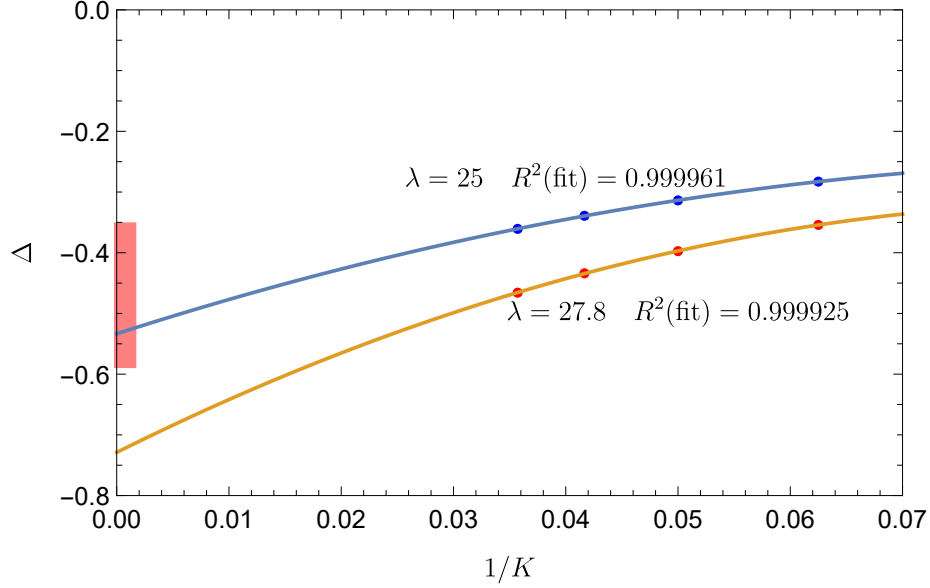


Figure 5.2 Δ changing with $1/K$ by quadratic fit. The y-intercept is in the range of $(-0.7289, -0.5334)$. (collaborated with Shreeram Jawadekar, Mamoon Sharaf, and James P. Vary)

with $\Delta = -0.3$ which is the value of the Δ gotten by substituting $\bar{g}_c(ET)$ (Rychkov and Vitale (2015)) into (5.43) in Burkardt et al. (2016). This value seems out of the range Burkardt et al. (2016) obtained by itself, which is $\Delta = -0.47 \pm 0.12$, so we assign a rather larger uncertainty $\delta\Delta = 0.29$ to include this range. Since we have good agreement with their range as shown in Fig. 5.1, Fig. 5.2 and Fig. 5.3, we will take this value and uncertainty of Δ for current analysis. Of course this is just a rough estimation, and we need to re-evaluate Δ and shrink the uncertainty range with extended calculations in the future.

The critical coupling obtained by this work using quadratic extrapolation as shown in Fig. 3.8 is $\lambda_c = 30.2547$, which corresponds to $\bar{g}_c(LF) = 1.26$.

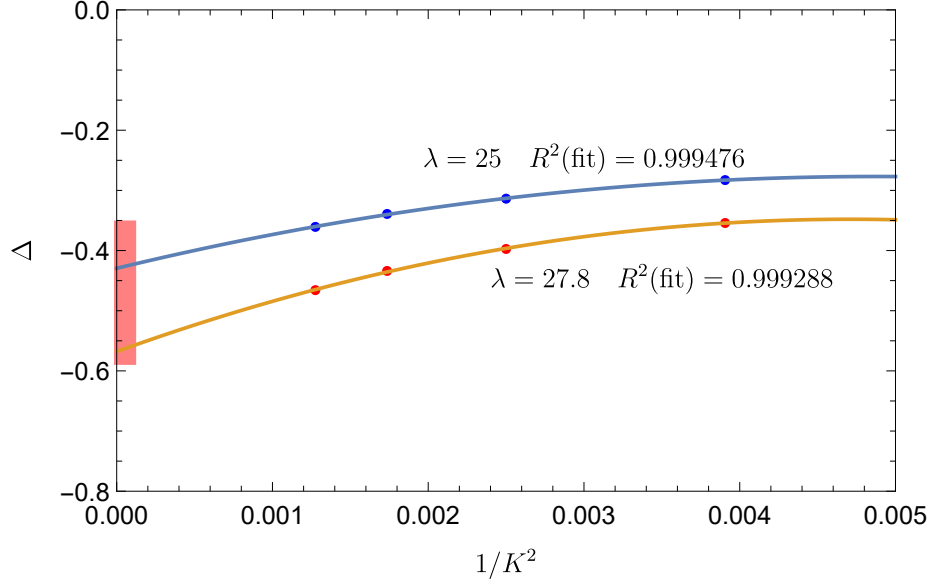


Figure 5.3 Δ changing with $1/K^2$ by quadratic fit. The y-intercept is in the range of $(-0.5679, -0.4295)$. (collaborated with Shreeram Jawadekar, Mamoon Sharaf, and James P. Vary)

According to propagation of uncertainty,

$$\begin{aligned}
 \delta g_{ET} &= \sqrt{\left(\frac{\partial g_{ET}}{\partial g_{LF}} \delta g_{LF}\right)^2 + \left(\frac{\partial g_{ET}}{\partial \Delta} \delta \Delta\right)^2} \\
 &= \sqrt{\left(\frac{1}{(1 + \partial g_{LF} \Delta)^2} \delta g_{LF}\right)^2 + \left(\frac{-g_{LF}^2}{(1 + \partial g_{LF} \Delta)^2} \delta \Delta\right)^2} \\
 &= \frac{1}{1 + \partial g_{LF} \Delta} \sqrt{(\delta g_{LF})^2 + (g_{LF} \delta \Delta)^2},
 \end{aligned} \tag{5.46}$$

and

$$\delta \bar{g}_c(ET) = \frac{1}{1 + \partial \bar{g}_c(LF) \left(\frac{6}{\pi} \Delta\right)} \sqrt{(\delta \bar{g}_c(LF))^2 + [\bar{g}_c(LF) \delta \left(\frac{6}{\pi} \Delta\right)]^2}. \tag{5.47}$$

We can see that the critical coupling with DLCQ by this work overlaps with [Burkardt et al. \(2016\)](#) result after converted to equal time, and they both consistent with the newest equal time critical coupling by [Rychkov and Vitale \(2015\)](#). [Harindranath and Vary \(1987\)](#) extrapolated the critical coupling using the DLCQ method after mass renormalization, as illustrated in Fig. 3.5. In that method, the points can only go through one phase, and the horizontal asymptote corresponding to the critical coupling is difficult to determine.

Table 5.2 Critical Coupling by Different Methods

Method	$\bar{g}_c(LF)$	$\bar{g}_c(ET)$
DLCQ (This work)	1.26	4.53 ± 2.51
DLCQ (Harindranath and Vary (1987))	1.38	6.59 ± 3.65
Light-front symmetric polynomials (Burkardt et al. (2016))	1.1 ± 0.03	2.98 ± 1.65
Quasiparse eigenvector (Lee et al. (2001))	–	2.5
Density matrix renormalization group (Sugihara (2004))	–	$2.4954(4)$
Lattice Monte Carlo (Schaich and Loinaz (2009))	–	$2.70^{+0.025}_{-0.013}$
Lattice Monte Carlo (Bosetti et al. (2015))	–	2.79 ± 0.02
Uniform matrix product (Milsted et al. (2013))	–	$2.766(5)$
Renormalized Hamiltonian truncation (Rychkov and Vitale (2015))	–	$2.97(14)$

However, this is not sufficient to conclude that the zero mode has no effect on the critical coupling, because the uncertainty associated with both DLCQ and Light-front symmetric polynomials are still rather large. With better computational power by using parallel programming high-performance computers in the future, we hope to obtain both the critical coupling and Δ with significantly improved precision leading to a conclusion of whether the zero mode plays an essential role in the critical coupling of 1+1 dimensional ϕ^4 theory.

BIBLIOGRAPHY

- Bender, C. M., Pinsky, S., and van de Sande, B. (1993). Spontaneous symmetry breaking of (1+1)-dimensional φ^4 theory in light-front field theory. *Phys. Rev. D*, 48:816–821.
- Bosetti, P., De Palma, B., and Guagnelli, M. (2015). Monte carlo determination of the critical coupling in ϕ_2^4 theory. *Phys. Rev. D*, 92:034509.
- Burkardt, M. (2002). *Light Front Quantization*, pages 1–74. Springer US, Boston, MA.
- Burkardt, M., Chabysheva, S. S., and Hiller, J. R. (2016). Two-dimensional light-front ϕ^4 theory in a symmetric polynomial basis. *Phys. Rev. D*, 94:065006.
- Chabysheva, S. S., Elliott, B., and Hiller, J. R. (2013). Symmetric multivariate polynomials as a basis for three-boson light-front wave functions. *Phys. Rev. E*, 88:063307.
- Chabysheva, S. S. and Hiller, J. R. (2014). Basis of symmetric polynomials for many-boson light-front wave functions. *Phys. Rev. E*, 90:063310.
- Chakrabarti, D., Harindranath, A., Martinovič, L., and Vary, J. (2004). Kinks in discrete light cone quantization. *Physics Letters B*, 582(3):196 – 202.
- Harindranath, A. (2000). Light front field theory and light front qcd. *Lecture Notes*.
- Harindranath, A. and Vary, J. P. (1987). Solving two-dimensional φ^4 theory by discretized light-front quantization. *Phys. Rev. D*, 36:1141–1147.
- Lee, D., Salwen, N., and Lee, D. (2001). The diagonalization of quantum field hamiltonians. *Physics Letters B*, 503(1):223 – 235.
- Milsted, A., Haegeman, J., and Osborne, T. J. (2013). Matrix product states and variational methods applied to critical quantum field theory. *Phys. Rev. D*, 88:085030.
- Peskin, M. E. and Schroeder, D. V. (1995). Quantum field theory. the advanced book program.
- Pinsky, S. S. and van de Sande, B. (1994). Spontaneous symmetry breaking of (1+1)-dimensional φ^4 theory in light-front field theory. ii. *Phys. Rev. D*, 49:2001–2013.
- Pinsky, S. S., van de Sande, B., and Hiller, J. R. (1995). Spontaneous symmetry breaking of (1+1)-dimensional φ^4 theory in light-front field theory. iii. *Phys. Rev. D*, 51:726–733.

- Rychkov, S. and Vitale, L. G. (2015). Hamiltonian truncation study of the φ^4 theory in two dimensions. *Phys. Rev. D*, 91:085011.
- Schaich, D. and Loinaz, W. (2009). Improved lattice measurement of the critical coupling in ϕ_2^4 theory. *Phys. Rev. D*, 79:056008.
- Sugihara, T. (2004). Density matrix renormalization group in a two-dimensional ϕ_4 hamiltonian lattice model. *Journal of High Energy Physics*, 2004(05):007–007.