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ERGODIC PROPERTIES OF NONHOMOGENEOUS, CONTINUOUS-TIME
MARKOV CHAINS

Iowa State University

Ph.D. 1984

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Ergodic properties of nonhomogeneous,
continuous-time Markov chains

by

Jean Thomas Johnson

A Dissertation Submitted to the
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I. INTRODUCTION AND REVIEW

A Markov chain is a probabilistic model for representing the changes in a process over time. The Russian mathematician A. A. Markov first explicitly outlined the defining property of such chains in 1906. The theory of Markov chains has advanced continuously since that time. Applications of this type of mathematical model can be found in many diverse fields including, as a small sample, medicine, economics, chemistry and athletics.

Much of the theory which has been published on Markov chains has considered the restricted case of a homogeneous chain defined on a discrete-state space over a discrete-time domain. These three assumptions are not necessary in order to satisfy the Markovian property. One might remove the constraint of homogeneity, the assumption of a discrete-state space might be replaced by a continuous-state space or the time domain might be continuous rather than discrete. This dissertation focuses on advancing the theory of continuous-time, discrete-state, non-homogeneous Markov chains.

A Markov chain is a type of stochastic process and a stochastic process is a collection of random variables $\{X(t): t \in T\}$. T is often thought of as a collection of times. It is usually considered to be one of the two sets $[0, \infty)$ or $\{0, 1, 2, \dots\}$, giving the chain the designations continuous-time or discrete-time, respectively. Since a stochastic process includes a random variable for each t in its time domain, it can be considered a probabilistic model for something which changes over time.

The random variables in the stochastic process are functions from some probability space into the real numbers. The set of values that these functions assume is called the state space of the stochastic process. This state space can, in general, be either discrete or continuous. In this dissertation only Markov chains on a discrete-state space will be considered. The state space of such chains can always be considered as some subset, not necessarily proper, of the positive integers and will be called S .

In the type of stochastic process called a Markov chain, once the value of the chain is known at some particular time the future course of the chain, from that time on, is independent of what took place prior to that time. This is essentially the content of the following definition.

Definition I.1:

A Markov chain $\{X(t): t \in T\}$ is a stochastic process such that for any set of times $t_0 < t_1 < \dots < t_n$ and for any set of states $\{i_0, i_1, \dots, i_n\} \subset S$ the following probabilities are equal.

$$\begin{aligned} P[X(t_n) = i_n \mid X(t_0) = i_0, X(t_1) = i_1, \dots, X(t_{n-1}) = i_{n-1}] \\ = P[X(t_n) = i_n \mid X(t_{n-1}) = i_{n-1}] . \end{aligned}$$

If T is the set of nonnegative integers, $X(t)$ is a discrete-time Markov chain, and if T is the set of nonnegative reals, then $X(t)$ is a continuous-time Markov chain.

With every discrete-state chain $X(t)$, there can be associated a probability transition matrix $P(s,t)$. For $s \leq t$, let

$$p_{ij}(s,t) = P\{X(t) = j \mid X(s) = i\}.$$

The transition matrix is given by

$$P(s,t) = \begin{bmatrix} p_{11}(s,t) & p_{12}(s,t) & p_{13}(s,t) & \dots \\ p_{21}(s,t) & p_{22}(s,t) & p_{32}(s,t) & \dots \\ p_{31}(s,t) & p_{32}(s,t) & p_{33}(s,t) & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

The dimensions of this matrix will be given by the cardinality of the state space S . The elements of $P(s,t)$ will be assumed to satisfy the following properties:

- A) $p_{ij}(s,t) \geq 0$ for any $i,j \in S$ and $t \geq s \geq 0$;
- B) $\sum_{j \in S} p_{ij}(s,t) = 1$ for any $i \in S$ and $t \geq s \geq 0$;
- C) $p_{ij}(s,t) = \sum_{k \in S} p_{ik}(s,u)p_{kj}(u,t)$ for any $i,j \in S$ and $0 \leq s \leq u \leq t$;
- D) $\lim_{t \rightarrow s^+} p_{ij}(s,t) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$
for any $s \in T$ and for any $i,j \in S$.

Properties A) and B) show that each row of the transition matrix is a probability distribution on S . The i^{th} row gives the probabilities that the chain ends up in state j at time t given that it is in state i at time s . Any matrix satisfying A) and B) is called a stochastic matrix. Property C) simply states that the probability of going from state i to state j between times s and t can alternately be thought of as going from state i to some intermediate state k between times s and u and then going from the intermediate state k at time u to the final state j at time t . This is called the Chapman-Kolmogorov equation and can be written in matrix notation as

$$P(s,t) = P(s,u)P(u,t) \quad (1)$$

Property D) is not essential for a Markov process, but it is widely assumed in the literature and is mathematically convenient.

Although discrete-time chains are usually described in terms of transition matrices, an alternate approach using the derivative of the transition matrix is common for continuous-time chains. This derivative matrix is called the intensity matrix. The elements of the intensity matrix $Q(t)$ are defined as

$$\begin{aligned} q_{ii}(t) &= \lim_{h \rightarrow 0^+} \frac{p_{ii}(t,t+h) - 1}{h} \quad \text{and} \\ q_{ij}(t) &= \lim_{h \rightarrow 0^+} \frac{p_{ij}(t,t+h)}{h} \end{aligned} \quad (2)$$

for every i and j in S . It is assumed that these functions are continuous. $-q_{ii}(t)$ is called the intensity of passage out of state

i and $q_{ij}(t)$ is the intensity of passage from state i to state j . The term intensity of passage can be explained in the following informal manner. If $i = j$

$$1 - p_{ii}(t, t+h) = P(X(t+h) \neq i \mid X(t) = i).$$

Therefore, for small positive h

$$P(X(t+h) \neq i \mid X(t) = i) = -q_{ii}(t)h + o(h)$$

where $o(h)$ satisfies $o(h)/h \rightarrow 0$ as $h \rightarrow 0$. Thus $-q_{ii}(t)$ gives some measure of the "intensity" with which the chain passes out of state i at time t . Similarly, for $i \neq j$,

$$p_{ij}(t, t+h) = P(X(t+h) = j \mid X(t) = i) = q_{ij}(t)h + o(h),$$

and so $q_{ij}(t)$ indicates the "intensity" of an "instantaneous" transition out of state i into state j at time t . For each t , the elements of the intensity matrix $Q(t)$ satisfy the conditions that

$$\left. \begin{aligned} q_{ii}(t) \leq 0, \quad q_{ij}(t) \geq 0 \text{ for } i \neq j, \text{ and} \\ \sum_{j \in S} q_{ij}(t) = 0 \text{ for all } i \in S. \end{aligned} \right\} \quad (3)$$

If, in addition, the assumption is made that the limit in (2) is uniform in i then one can derive the Kolmogorov equations (for details see Feller (1950))

$$\frac{\partial}{\partial t} p_{ij}(s, t) = \sum_{k \in S} p_{ik}(s, t) q_{kj}(t) \quad \text{for any } i, j \in S \quad \text{and} \quad (4)$$

$$\frac{\partial}{\partial s} p_{ij}(s, t) = - \sum_{k \in S} q_{ik}(s) p_{kj}(s, t) \quad \text{for any } i, j \in S. \quad (5)$$

These are called the forward and backward Kolomogorov equations, respectively.

In the above, the intensities of passage are derived from the transition matrices of the Markov chain. Conversely, one can describe the Markov chain beginning with the intensity matrix and the Kolomogorov equations. Reuter and Lederman (1953) showed that for an intensity matrix with continuous elements $q_{ij}(t)$, $i, j \in S$, which satisfy (3), solutions $f_{ij}(s, t)$, $i, j \in S$, to (4) and (5) can be found such that for fixed s , $f_{ij}(s, t)$ is absolutely continuous in t , and for fixed t , $f_{ij}(s, t)$ is continuous in s and has a continuous derivative $\frac{\partial}{\partial s} f_{ij}(s, t)$. In addition

$$f_{ij}(s, t) = \delta_{ij} \quad \text{for } s = t,$$

$$f_{ij}(s, t) \geq 0,$$

$$\sum_{j \in S} f_{ij}(s, t) \leq 1 \quad \text{and} \quad (6)$$

$$f_{ij}(s, t) = \sum_{k \in S} f_{ik}(s, u) f_{kj}(u, t) \quad \text{for } s \leq u \leq t.$$

If (6) can be replaced by

$$\sum_{j \in S} f_{ij}(s, t) = 1$$

then this solution is unique and can be interpreted as the transition probabilities of a Markov chain.

The Kolmogorov differential equations can be written in matrix form as

$$\frac{\partial}{\partial t} P(s, t) = P(s, t)Q(t) \quad \text{and}$$

$$\frac{\partial}{\partial s} P(s, t) = -Q(s)P(s, t).$$

Alternately, they can be written as the integral equations

$$P(s, t) = I + \int_s^t P(s, u)Q(u)du \quad \text{and}$$

$$P(s, t) = I + \int_s^t Q(u)P(u, t)du. \quad (7)$$

$P(s, t)$ can be expressed in terms of $Q(t)$ alone. Iosifescu and Tăutu (1973) showed that if $Q(t)$ is measurable and $\sup_{i \in S} \{ |q_{ii}(t)| \}$ is integrable on every finite interval of T then

$$\begin{aligned} P(s, t) &= I + \int_s^t Q(u)du \\ &+ \sum_{n \geq 2} \int_s^t dt_n \int_s^{t_1} dt_1 \dots \int_s^{t_{n-1}} Q(t_n) \dots Q(t_1) dt_n \\ &= I + \int_s^t Q(u)du \\ &+ \sum_{n \geq 2} \int_s^t ds_1 \int_{s_1}^t ds_2 \dots \int_{s_{n-1}}^t Q(s_1) \dots Q(s_n) ds_n. \quad (8) \end{aligned}$$

One class of Markov chains which is frequently studied is that of homogeneous chains.

Definition I.2:

A Markov chain with probability transition matrix $P(s,t)$ is homogeneous in time, or stationary in time, if $P(s,s+t)$ is independent of s . If this condition does not hold the chain is nonhomogeneous.

Since the transition matrix of a homogeneous Markov chain $P(s,t)$ depends only on the difference in time $t-s$, it can be written as $P(t-s)$. This leads to a simplification of many of the previous results. For homogeneous chains, A), B), C), D) and (1) become

$$A^*) \quad p_{ij}(t) \geq 0 \quad \text{for any } i, j \in S \quad \text{and } t \geq 0 ;$$

$$B^*) \quad \sum_{j \in S} p_{ij}(t) = 1 \quad \text{for any } i \in S ;$$

$$C^*) \quad p_{ij}(s+t) = \sum_{k \in S} p_{ik}(s)p_{kj}(t) \quad \text{for any } i, j \in S \quad \text{and } s, t \geq 0 ;$$

$$D^*) \quad \lim_{t \rightarrow 0^+} p_{ij}(t) = \delta_{ij} \quad \text{for any } i, j \in S ; \quad \text{and}$$

$$P(s+t) = P(s)P(t).$$

Also, for a homogeneous chain, the intensity matrix is not dependent on t so (3), (4) and (5) become

$$\left. \begin{array}{l} q_{ii} \leq 0, \quad q_{ij} \geq 0 \quad \text{for } i \neq j, \\ \text{and } \sum_{j \in S} q_{ij} = 0 \quad \text{for all } i \in S \end{array} \right\} \quad (9)$$

$$\frac{d}{dt} p_{ij}(t) = \sum_{k \in S} p_{ik}(t) q_{kj} \quad \text{and}$$

$$\frac{d}{dt} p_{ij}(t) = - \sum_{k \in S} q_{ik} p_{kj}(t).$$

This dissertation concentrates on the ergodic properties of certain nonhomogeneous chains. These properties involve the condition of the chain after a long period of time has elapsed. It is useful in studying ergodicity to first consider whether or not all states can be reached from a given starting state. To this end the concept of irreducibility is introduced.

Definition I.3:

A Markov chain is irreducible if for any $i, j \in S$ and $s > 0$ there exist $t_1 > s$ and $t_2 > s$ such that $p_{ij}(s, t_1) > 0$ and $p_{ji}(s, t_2) > 0$. For homogeneous chains, this can be stated more simply: for any $i, j \in S$, there exist $t_1 > 0$ and $t_2 > 0$ such that $p_{ij}(t_1) > 0$ and $p_{ji}(t_2) > 0$.

The term ergodic is itself used to describe a chain which has the property that after a long period of time the probability of being in any state is independent of the starting state. This definition is given formally in the following.

Definition I.4:

A Markov chain is ergodic if there exists a collection $\{\pi_j\}_{j \in S}$ such that $\pi_j > 0$ for every $j \in S$, $\sum_{j \in S} \pi_j = 1$ and for any $i \in S$ and any $s > 0$ $\lim_{t \rightarrow \infty} p_{ij}(s, t) = \pi_j$ for every $j \in S$.

The definition of ergodicity involves only the convergence of the elements of the transition matrices $P(s,t)$. Because of the stochastic nature of $P(s,t)$ and π , this can be strengthened to the following which is also proved by Isaacson (1979).

Theorem I.5:

Suppose $\pi = \{\pi_j\}_{j \in S}$ exists and satisfies $\pi_j > 0$ for every $j \in S$ and $\sum_{j \in S} \pi_j = 1$. A Markov chain with transition matrices $P(s,t)$ is ergodic with limit π if and only if for any $i \in S$ and for any $s > 0$, $\lim_{t \rightarrow \infty} \sum_{j \in S} |p_{ij}(s,t) - \pi_j| = 0$.

Proof:

For notational convenience, this proof is written assuming the state space is infinite. If it is of cardinality $M < \infty$, then let

$$p_{ij}(s,t) = 0 \text{ for } i > M \text{ or } j > M.$$

Fix $i \in S$ and $s > 0$.

Since $|p_{ik}(s,t) - \pi_k| \leq \sum_{j \in S} |p_{ij}(s,t) - \pi_j|$ for every $k \in S$,

$$\lim_{t \rightarrow \infty} \sum_{j \in S} |p_{ij}(s,t) - \pi_j| = 0 \text{ implies } \lim_{t \rightarrow \infty} |p_{ik}(s,t) - \pi_k| = 0$$

and therefore the chain is ergodic.

Conversely, suppose the chain is ergodic. Choose $\epsilon > 0$. Since π is stochastic there exists a K such that

$$\sum_{j=K+1}^{\infty} \pi_j < \frac{\epsilon}{4}.$$

By the ergodicity of the chain, one can choose T such that for $t > T$ and $k \in \{1, 2, \dots, K\}$

$$|p_{ik}(s, t) - \pi_k| < \frac{\varepsilon}{4K}.$$

Then, for $t \geq T$

$$\begin{aligned} & \sum_{j \in S} |p_{ij}(s, t) - \pi_j| \\ &= \sum_{j=1}^K |p_{ij}(s, t) - \pi_j| + \sum_{j=K+1}^{\infty} |p_{ij}(s, t) - \pi_j| \\ &\leq \sum_{j=1}^K \frac{\varepsilon}{4K} + \sum_{j=K+1}^{\infty} p_{ij}(s, t) + \sum_{j=K+1}^{\infty} \pi_j \\ &\leq \frac{K\varepsilon}{4K} + 1 - \sum_{j=1}^K p_{ij}(s, t) + \frac{\varepsilon}{4} \\ &\leq \frac{\varepsilon}{4} + 1 + \sum_{j=1}^K \left(\frac{\varepsilon}{4K} - \pi_j \right) + \frac{\varepsilon}{4} \\ &= \frac{\varepsilon}{4} + \left(1 - \sum_{j=1}^K \pi_j \right) + \frac{K\varepsilon}{4K} + \frac{\varepsilon}{4} \\ &= \frac{\varepsilon}{4} + \sum_{j=K+1}^{\infty} \pi_j + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &< \varepsilon. \quad \blacksquare \end{aligned}$$

Thus, ergodicity can be thought of as having each row of the transition matrices converging to the same probability distribution. The rates at which different rows converge need not be the same. In order to strengthen this limiting process, a matrix norm is defined.

Definition I.6:

The norm of a vector $a = \{a_i\}_{i \in S}$ is defined as

$$\|a\| = \sum_{i \in S} |a_i|.$$

For a square matrix $A = (a_{ij})_{i,j \in S}$, the norm of A is defined as

$$\|A\| = \sup_{i \in S} \left\{ \sum_{j \in S} |a_{ij}| \right\}.$$

It is clear that for any transition matrix $P(s,t)$, $\|P(s,t)\| = 1$.

For an intensity matrix $Q(t)$ on an infinite state space, it may be that

$$\sup_{i \in S} \left\{ \sum_{j \in S} |q_{ij}(t)| \right\} = \infty, \text{ but this dissertation will be restricted to}$$

considering only intensity matrices for which $\|Q(t)\| < \infty$. Because of

the nature of intensity matrices, the following characterization of a

norm is immediate.

Theorem I.7:

For the intensity matrix $Q(t)$ of a Markov chain,

$$\|Q(t)\| = 2 \sup_{i \in S} \{ |q_{ii}(t)| \}$$

for every $t \geq 0$.

Proof:

$$\text{By (3)} \quad \sum_{j \in S - \{i\}} q_{ij}(t) = -q_{ii}(t) \text{ for every } i \in S \text{ and every}$$

$t \geq 0$. Thus,

$$\begin{aligned}
\|Q(t)\| &= \sup_{i \in S} \left\{ \sum_{j \in S} |q_{ij}(t)| \right\} \\
&= \sup_{i \in S} \left\{ \sum_{j \in S - \{i\}} |q_{ij}(t)| + |q_{ii}(t)| \right\} \\
&= 2 \sup_{i \in S} \{ |q_{ii}(t)| \}. \quad \blacksquare
\end{aligned}$$

Two immediate corollaries of this will also be useful.

Corollary 1.8:

For the intensity matrix $Q(t)$ of a Markov chain,

$$\|Q(t)\| < 2q \text{ if and only if } \sup_{i \in S} \{ |q_{ii}(t)| \} < q.$$

Corollary 1.9:

If $\|Q(t)\| < 2q$, then $|q_{ij}(t)| < q$ for every $i, j \in S$.

A stronger form of ergodicity than that defined by Definition 1.4 will be introduced below. For a strongly ergodic chain, the transition matrices converge in norm. The long-run form of the transition matrix will be called a constant stochastic matrix.

Definition 1.10:

A matrix with identical rows is called a constant matrix.

Constant matrices possess the properties given by the following lemma.

Lemma 1.11:

If L is a constant matrix and $P(s, t)$ is any stochastic matrix then $P(s, t)L = L$.

Proof:

Since L is a constant matrix, each row of L is given by the same vector $\pi = (\pi_j)_{j \in S}$. Since $P(s,t)$ is stochastic, for every $i \in S$ $\sum_{j \in S} p_{ij}(s,t) = 1$. Therefore, denoting the product $P(s,t)L$ by $C = (c_{ij})_{i,j \in S}$ one sees that

$$c_{ij} = \sum_{k \in S} p_{ik}(s,t) \pi_j = 1 \cdot \pi_j = \pi_j.$$

Thus, $C = L$.

Strong ergodicity is now defined using Definition I.10.

Definition I.12:

A Markov chain with transition matrix $P(s,t)$ is strongly ergodic if there exists a constant stochastic matrix L with rows $\pi = (\pi_j)_{j \in S}$ such that for every $s > 0$

$$\lim_{t \rightarrow \infty} \|P(s,t) - L\| = 0.$$

Clearly, strong ergodicity implies ergodicity since for any $i \in S$,

$$\sum_{j \in S} |p_{ij}(s,t) - \pi_j| \leq \sup_{i \in S} \left\{ \sum_{j \in S} |p_{ij}(s,t) - \pi_j| \right\} = \|P(s,t) - L\|.$$

For a finite state space, $\lim_{t \rightarrow \infty} \sum_{j \in S} |p_{ij}(s,t) - \pi_j| = 0$ for each $i \in S$ implies that $\lim_{t \rightarrow \infty} \sup_{i \in S} \left\{ \sum_{j \in S} |p_{ij}(s,t) - \pi_j| \right\} = 0$. Thus, ergodicity and strong ergodicity are equivalent when the state space is finite. Following a characterization of ergodicity involving invariant

distributions, Example I.21 will give a chain on an infinite state space which is ergodic but not strongly ergodic.

A third form of ergodic behavior to be considered in this dissertation is weak ergodicity. A weakly ergodic chain can be considered as one which loses memory but does not necessarily converge to a long-run distribution. A chain loses memory if, as time passes, the probability of being in the j^{th} state is independent of the initial state. Thus for any $i, k \in S$, $|p_{ij}(s, t) - p_{kj}(s, t)|$ approaches zero as t increases.

Example I.13:

$$\text{Let } P(2n, 2n+1) = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix} \text{ and } P(2n+1, 2n+2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for $n \geq 0$. Then

$$P(0, n) = \begin{cases} \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix} & \text{if } n \text{ is even} \\ \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{pmatrix} & \text{if } n \text{ is odd.} \end{cases}$$

The loss of memory in this chain is immediate since the rows are identical for every $P(0, n)$. Since for each $i, j \in S$, $p_{ij}(n)$ alternates between $1/3$ and $2/3$, there is no long-run distribution. Therefore, the chain is not ergodic and, consequently, not strongly ergodic.

For a formal definition of weak ergodicity, the δ -coefficient is first defined and several of its properties are discussed.

Definition I.14:

Let $P = (p_{ij})_{i,j \in S}$ be a stochastic matrix. The δ -coefficient is denoted and defined by

$$\delta(P) = \frac{1}{2} \sup_{i,k \in S} \left\{ \sum_{j \in S} |p_{ij} - p_{kj}| \right\}.$$

For proofs of the following lemmas see Isaacson and Madsen (1976).

Lemma I.15:

If P and R are stochastic matrices, then

$$\delta(PR) \leq \delta(P)\delta(R).$$

Lemma I.16:

If P is a stochastic matrix and $A = (a_{ij})_{i,j \in S}$ is a matrix such that $\sum_{j \in S} a_{ij} = 0$ for all $i \in S$ and $\|A\| < \infty$, then

$$\|AP\| \leq \|A\|\delta(P).$$

Lemma I.17:

If P and R are stochastic matrices, then

$$|\delta(P) - \delta(R)| \leq \|P - R\|.$$

Definition I.18:

A Markov chain is weakly ergodic if for all $s > 0$

$$\lim_{t \rightarrow \infty} \delta(P(s,t)) = 0.$$

$$\text{Since } \sum_{j \in S} |p_{ij} - p_{kj}| \leq \sum_{j \in S} |p_{ij} - \pi_j| + \sum_{j \in S} |p_{kj} - \pi_j|,$$

strong ergodicity implies weak ergodicity. Example I.13 shows that a weakly ergodic chain need not be ergodic or strongly ergodic. An ergodic chain need not be weakly ergodic. Example I.21 gives such a chain.

Homogeneous Markov chains have been widely studied. Literature on discrete-time, homogeneous chains is particularly abundant. A number of characterizations of ergodicity of such chains are available. This dissertation seeks to apply some of the results on homogeneous chains to the study of nonhomogeneous chains. Therefore, the ergodic behavior of each nonhomogeneous chain is studied in terms of the ergodic behavior of one or more associated homogeneous chains. Since the ergodicity, or strong ergodicity, of the homogeneous chain can be determined using known theorems, the ergodicity, or strong ergodicity, of the corresponding nonhomogeneous chain can also be determined. A detailed account of the characterizations which have been found for the ergodic behavior of homogeneous chains is not necessary for an understanding of this dissertation. For results in the homogeneous case, the reader is referred to works such as Isaacson and Madsen (1976), Parzen (1962) and Çinlar (1975). It will be helpful, however, to consider just a few aspects of the ergodic behavior of both discrete-time and continuous-time homogeneous Markov chains.

In the discrete-time, homogeneous case, it is possible for a chain to exhibit a periodic behavior, having positive probability of returning to a state only at some regular period of time. In this dissertation,

only chains of period one are considered. The definition of aperiodicity is given formally in the following.

Definition I.19:

State j is aperiodic if one is the greatest common divisor of all those n 's for which $p_{jj}(n) > 0$. A Markov chain is aperiodic if each of its states is aperiodic.

The major result on the ergodic behavior of discrete-time, homogeneous chains which is needed in this dissertation is given below.

Theorem I.20:

Let $X(n)$ be a discrete-time, homogeneous Markov chain with transition matrix $P = P(1) = (p_{ij})_{i,j \in S}$. If $X(n)$ is irreducible and aperiodic then a necessary and sufficient condition for it to be ergodic is the existence of a sequence $\pi = \{\pi_j\}_{j \in S}$ such that

$$\pi_j > 0, \quad \sum_{j \in S} \pi_j = 1 \quad (10)$$

and

$$\pi_j = \sum_{k \in S} \pi_k p_{kj} . \quad (11)$$

For an ergodic chain, the long-run distribution, defined by

$\pi_j = \lim_{n \rightarrow \infty} p_{ij}(n)$ for each $j \in S$, is the unique solution of (11) satisfying (10).

Any sequence $\pi = \{\pi_j\}_{j \in S}$ which satisfies (10) and (11) is called a stationary or invariant distribution of $X(n)$. (11) can be given in terms of matrix multiplication as

$$\pi P = \pi .$$

For an ergodic chain, one can form a constant stochastic matrix L , each of whose rows is π , and see that (11) gives

$$L P = L$$

Theorem 1.20 enables one to see that the following chain is ergodic.

Example 1.21:

Let

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1/2 & 0 & 1/4 & 0 & 1/4 & 0 & 0 & \dots \\ 0 & 1/2 & 0 & 1/4 & 0 & 1/4 & 0 & \dots \\ 0 & 0 & 1/2 & 0 & 1/4 & 0 & 1/4 & \dots \\ 0 & 0 & 0 & 1/2 & 0 & 1/4 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

P is irreducible and aperiodic.

$$\pi = (1/2 \quad 1/2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \dots)$$

satisfies $\pi P = \pi$. Thus, the chain is ergodic with limit π . However, $\delta(P^n) = 1$ for every $n \geq 1$ and therefore the chain is neither weakly ergodic nor strongly ergodic.

One further result on stationary distributions is provided for future reference.

Lemma I.22:

Suppose P is the transition matrix of an ergodic, discrete-time, homogeneous Markov chain with stationary distribution π . If R is a stochastic matrix, not necessarily ergodic, and R commutes with P , then $\pi R = \pi$.

Proof:

By definition π is the unique probability distribution for which $\pi P = \pi$, but $\pi R = (\pi P)R = \pi(PR) = \pi(RP) = (\pi R)P$. Since π and R are both stochastic, πR is stochastic and thus a probability distribution. Thus $\pi R = \pi$. ■

Continuous-time, homogeneous Markov chains can be handled in essentially the same way as discrete-time chains, although the theorems available to study the ergodic properties of continuous-time chains are not as numerous or varied as those discussing discrete-time chains. The theorem given here is actually a composite of statements given by Çinlar (1976) and Miller (1963). Continuous-time chains are necessarily aperiodic and so the result is slightly simpler than the corresponding discrete-time theorem.

Theorem I.23:

Let Q be the intensity matrix of a continuous-time, homogeneous Markov chain $X(t)$ and suppose $\|Q\| < q < \infty$. If $X(t)$ is an irreducible chain then it is ergodic if and only if there exists a sequence

$\pi = \{\pi_j\}_{j \in S}$ such that $\pi_j > 0$, $\sum_{j \in S} \pi_j = 1$ and $\pi Q = 0$. In the

ergodic case, π is unique, $\pi_j = \lim_{t \rightarrow \infty} p_{ij}(t)$ and $\pi P(t) = \pi$ for all t .

The equation $\pi Q = 0$ can be regarded as resulting from differentiation of $\pi P(t) = \pi$. Because $\pi P(t) = \pi$, π is called the stationary distribution of the ergodic chain. In this case, one can again define a constant stochastic matrix L with rows π such that

$$LP(t) = L. \quad (12)$$

II. NONHOMOGENEOUS MARKOV CHAINS

In this chapter, several theorems which have been proven for nonhomogeneous, discrete-time chains are extended to nonhomogeneous, continuous-time chains.

It has been stated that strong ergodicity implies weak ergodicity of a Markov chain. In the first theorem, a sufficient condition for the equality of these two types of ergodicity is given. The discrete-time version of this theorem is given by Isaacson and Madsen (1973).

Theorem II.1:

Let $X(t)$ be a Markov chain defined by the intensity matrices $Q(t)$ where $\|Q(t)\| < q < \infty$. Suppose for every $t > 0$ there exist probability distributions $\psi(t)$ such that

$$\psi(t)Q(t) = 0$$

and suppose there exists a probability distribution ψ such that

$$\int_0^{\infty} \|\psi(t) - \psi\| dt < \infty.$$

Then, if $X(t)$ is weakly ergodic, it is strongly ergodic.

Proof:

Define L as the constant stochastic matrix with $\psi = (\psi_k)_{k \in S}$ as each row. Fix $s > 0$. For any $s \leq u \leq t$

$$\|P(s,t) - L\| \leq \|P(s,t) - LP(u,t)\| + \|LP(u,t) - L\|.$$

By (1) and (7) this is

$$\begin{aligned} & \leq \|P(s,u)P(u,t) - LP(u,t)\| + \|L(I + \int_u^t Q(v)P(v,t)dv) - L\| \\ & = \| (P(s,u) - L)P(u,t) \| + \|L \int_u^t Q(v)P(v,t)dv\| . \end{aligned}$$

Consider $C(u,t) = L \int_u^t Q(v)P(v,t)dv$.

$$c_{ij}(u,t) = \sum_{k \in S} \psi_k \int_u^t (Q(v)P(v,t))_{kj} dv .$$

Since

$$\begin{aligned} \int_u^t \sum_{k \in S} |\psi_k (Q(v)P(v,t))_{kj}| dv & \leq \int_u^t \sum_{k \in S} (\psi_k \cdot q \cdot 1) dv \\ & = \int_u^t q dv = q(t-u) < \infty , \end{aligned}$$

we can use Fubini's theorem to conclude that $C(u,t) = \int_u^t LQ(v)P(v,t)dv$.

Also, since $P(s,t)$ and L are both stochastic matrices

$\sum_{j \in S} (P(s,t) - L)_{ij} = 0$ and $\|P(s,t) - L\| \leq 2$. Thus, by Lemma I.16,

$$\begin{aligned} \|P(s,t) - L\| & \leq \|P(s,u) - L\| \delta(P(u,t)) + \left\| \int_u^t LQ(v)P(v,t)dv \right\| \\ & \leq 2\delta(P(u,t)) + \int_u^t \|LQ(v)\| \|P(v,t)\| dv \\ & \leq 2\delta(P(u,t)) + \int_u^t \|LQ(v)\| dv . \end{aligned} \tag{13}$$

Choose $\varepsilon > 0$. Since L is a constant stochastic matrix,

$$\begin{aligned} \int_0^{\infty} \|LQ(t)\| dt &= \int_0^{\infty} \|\psi Q(t)\| dt \\ &= \int_0^{\infty} \|\psi Q(t) - \psi(t)Q(t)\| dt \\ &\leq \int_0^{\infty} \|\psi - \psi(t)\| \|Q(t)\| dt \\ &\leq q \int_0^{\infty} \|\psi - \psi(t)\| dt. \end{aligned}$$

By hypothesis, this is finite and therefore one can choose $T \geq s$ such that for $t \geq u \geq T$,

$$\int_u^t \|LQ(t)\| dt < \frac{1}{2} \varepsilon. \quad (14)$$

Since $X(t)$ is weakly ergodic, for any u there exists a T' such that for $t \geq T'$,

$$\delta(P(u, t)) < \frac{1}{4} \varepsilon. \quad (15)$$

Let $u = T$ and choose $t \geq \max(T, T')$. Combining (13), (14), (15) gives

$$\|P(s, t) - L\| \leq 2(1/4 \varepsilon) + 1/2 \varepsilon = \varepsilon. \quad \blacksquare$$

The above theorem gives sufficient conditions for strong ergodicity if weak ergodicity is assumed. Because continuous-time chains are defined in terms of the intensity matrices, Theorem 11.1 would be a more useful result if one could easily determine the weak ergodicity of a

continuous-time chain by using the intensity matrices $Q(t)$. Griffeath (1978) was able to classify the weak ergodicity of a continuous-time chain using an ergodic coefficient defined on the intensity matrices. For a Markov chain with intensity matrix $Q(t)$, he defined

$$\beta(t) = \inf_{\substack{i,j \in S \\ i \neq j}} \beta^{ij}(t)$$

where $\beta^{ij}(t) = q_{ij}(t) + q_{ji}(t) + \sum_{k \in S - \{i,j\}} \min(q_{ik}(t), q_{jk}(t))$. He proved that if $\lim_{t_0 \rightarrow \infty} \int_{t_0}^{\infty} \beta(t) dt = \infty$ then the chain is weakly ergodic.

Theorem II.1 uses only properties of $Q(t)$ to determine the strong ergodicity of the Markov chain it defines. Since methods for determining the ergodicity of homogeneous chains are more widely known than those for nonhomogeneous chains, another method for demonstrating strong ergodicity of nonhomogeneous chains is to relate it to the ergodicity of a corresponding homogeneous chain. Isaacson and Madsen (1976) gave a result of this type by comparing the transition matrices of a nonhomogeneous, discrete-time chain to that of a homogeneous, discrete-time chain. The theorem which follows is a similar result in the continuous-time case, but it is the intensity matrices of the homogeneous and nonhomogeneous chains which are used for the comparison.

Two lemmas will be needed.

Lemma II.2:

Let $Q(s_1), Q(s_2), \dots, Q(s_n)$ and Q be intensity matrices with

$$\|Q(s_i)\| < M \text{ for every } 1 \leq i \leq n \text{ and}$$

$$\|Q\| < M.$$

$$\text{Then } \|Q(s_1)Q(s_2)\dots Q(s_n) - Q^n\| \leq M^{n-1} \sum_{i=1}^n \|Q(s_i) - Q\|.$$

Proof:

$$\begin{aligned} & \|Q(s_1)Q(s_2)\dots Q(s_n) - Q^n\| \\ & \leq \|Q(s_1)Q(s_2)\dots Q(s_n) - Q(s_1)Q(s_2)\dots Q(s_{n-1})\| \\ & \quad + \|Q(s_1)Q(s_2)\dots Q(s_{n-1})Q - Q(s_1)Q(s_2)\dots Q(s_{n-2})Q^2\| \\ & \quad + \dots \\ & \quad + \|Q(s_1)Q^{n-1} - Q^n\| \\ & \leq \|Q(s_1)\| \|Q(s_2)\| \dots \|Q(s_{n-1})\| \|Q(s_n) - Q\| \\ & \quad + \|Q(s_1)\| \|Q(s_2)\| \dots \|Q(s_{n-2})\| \|Q(s_{n-1}) - Q\| \|Q\| \\ & \quad + \dots + \|Q(s_1) - Q\| \|Q\|^{n-1} \\ & \leq M^{n-1} \sum_{i=1}^n \|Q(s_i) - Q\|. \quad \blacksquare \end{aligned}$$

Lemma II.3:

Suppose $\lim_{t \rightarrow \infty} \|Q(t) - Q\| = 0$ where $\sup_{t \geq 0} \|Q(t)\| < \infty$ and $\sup_{i \in S} \{ |q_{ii}| \} < q < \infty$. There exist M such that

$$\|Q\| \leq M \text{ and } \|Q(t)\| \leq M \text{ for every } t \geq 0.$$

Also, for every $U > 0$ and $\delta > 0$ there exists a $T = T(\delta)$ such that for $t \geq T$

$$\|P(t, t+U) - P(U)\| \leq \delta U e^{UM}.$$

Proof:

by Corollary I.8, $\|Q\| < 2q$. Therefore

$$\|Q(t)\| \leq \|Q(t) - Q\| + \|Q\| \leq \|Q(t) - Q\| + 2q.$$

By hypothesis, one can choose S^* such that for $t \geq S^*$, $\|Q(t) - Q\| \leq 1$.

Let $M^* = \sup_{0 \leq t \leq S^*} \{\|Q(t) - Q\|\}$ and let $M = M^* + 1 + 2q$. Then, $\|Q(t)\| \leq M$

for any t and $\|Q\| \leq 2q \leq M$.

According to (8), the transition matrix of the homogeneous chain can be written as

$$\begin{aligned} P(U) &= I + \int_0^U Q ds \\ &\quad + \sum_{n \geq 2} \int_0^U ds_1 \int_{s_1}^U ds_2 \dots \int_{s_{n-1}}^U Q^n ds_n \\ &= I + \int_0^U Q ds + \sum_{n \geq 2} \frac{Q^n}{n!} \left(\int_0^U 1 ds \right)^n \\ &= I + \int_t^{t+U} Q ds + \sum_{n \geq 2} \frac{Q^n}{n!} \left(\int_t^{t+U} 1 ds \right)^n \end{aligned}$$

$$\begin{aligned}
&= I + \int_t^{t+U} Q \, ds \\
&\quad + \sum_{n \geq 2} \int_t^{t+U} ds_1 \int_{s_1}^{t+U} ds_2 \dots \int_{s_{n-1}}^{t+U} Q^n \, ds_n .
\end{aligned}$$

Thus, again by (8)

$$\|P(t, t+U) - P(U)\|$$

$$\begin{aligned}
&= \|I + \int_t^{t+U} Q(s_1) \, ds_1 \\
&\quad + \sum_{n \geq 2} \int_t^{t+U} ds_1 \int_{s_1}^{t+U} ds_2 \dots \int_{s_{n-1}}^{t+U} Q(s_1) \dots Q(s_n) \, ds_n \\
&\quad - I - \int_t^{t+U} Q \, ds_1 \\
&\quad - \sum_{n \geq 2} \int_t^{t+U} ds_1 \int_{s_1}^{t+U} ds_2 \dots \int_{s_{n-1}}^{t+U} Q^n \, ds_n \| \\
&= \| \int_t^{t+U} [Q(s_1) - Q] \, ds_1 \\
&\quad + \sum_{n \geq 2} \int_t^{t+U} ds_1 \int_{s_1}^{t+U} ds_2 \dots \int_{s_{n-1}}^{t+U} [Q(s_1) \dots Q(s_n) - Q^n] \, ds_n \| \\
&\leq \int_t^{t+U} \|Q(s_1) - Q\| \, ds_1 \\
&\quad + \sum_{n \geq 2} \int_t^{t+U} ds_1 \int_{s_1}^{t+U} ds_2 \dots \int_{s_{n-1}}^{t+U} \|Q(s_1) \dots Q(s_n) - Q^n\| \, ds_n .
\end{aligned}$$

By Lemma II.2, this is

$$\begin{aligned} & \leq \int_t^{t+U} \|Q(s_1) - Q\| ds_1 \\ & \quad + \sum_{n \geq 2} \int_t^{t+U} ds_1 \int_{s_1}^{t+U} ds_2 \dots \int_{s_{n-1}}^{t+U} M^{n-1} \sum_{i=1}^n \|Q(s_i) - Q\| ds_n . \end{aligned}$$

Since $\lim_{t \rightarrow \infty} \|Q(t) - Q\| = 0$, one can find a T such that for $t \geq T$,

$$\|Q(t) - Q\| < \delta .$$

Therefore, for $t \geq T$ the above becomes

$$\|P(t, t+U) - P(U)\|$$

$$\begin{aligned} & \leq \int_t^{t+U} \delta ds_1 \\ & \quad + \sum_{n \geq 2} \int_t^{t+U} ds_1 \int_{s_1}^{t+U} ds_2 \dots \int_{s_{n-1}}^{t+U} M^{n-1} \sum_{i=1}^n \delta ds_n . \\ & = \delta U + \delta \sum_{n \geq 2} \frac{M^{n-1}}{n!} \left(\int_t^{t+U} 1 ds \right)^n \\ & = \delta U + \delta \sum_{n \geq 2} \frac{M^{n-1}}{(n-1)!} U^n \\ & = \delta U \left(1 + \sum_{n \geq 2} \frac{M^{n-1} U^{n-1}}{(n-1)!} \right) = \delta U e^{MU} . \quad \blacksquare \end{aligned}$$

Theorem II.4:

Suppose $\lim_{t \rightarrow \infty} \|Q(t) - Q\| = 0$ where $\sup_{t > 0} \|Q(t)\| < \infty$ and $\sup_{i \in S} \{ |q_{ii}| \} < q < \infty$. If the homogeneous Markov chain with intensity matrix Q and transition matrix $P(t)$ is strongly ergodic with limit L , then the nonhomogeneous Markov chain with intensity matrices $Q(t)$ and transition matrices $P(s,t)$ is strongly ergodic with limit matrix L .

Proof:

Choose $\varepsilon > 0$. It follows from (1) and Lemma I.11 that

$$\begin{aligned} \|P(s,t) - L\| &\leq \|P(s,t-u)P(t-u,t) - P(s,t-u)P(u)\| \\ &\quad + \|P(s,t-u)P(u) - P(s,t-u)L\| \\ &\leq \|P(s,t-u)\| \|P(t-u,t) - P(u)\| + \|P(s,t-u)\| \|P(u) - L\| \\ &\leq 1 \cdot \|P(t-u,t) - P(u)\| + 1 \cdot \|P(u) - L\|. \end{aligned}$$

By hypothesis, one can find U such that

$$\|P(U) - L\| \leq \varepsilon/2.$$

Thus, for $t > U$,

$$\|P(s,t) - L\| \leq \|P(t-U,t) - P(U)\| + \varepsilon/2. \quad (16)$$

By Lemma II.3, one can find M and, once U and M are fixed, choose

$$\delta = \frac{\varepsilon}{2U} e^{-UM} \text{ such that for } t \text{ large}$$

$$\|P(t-U,t) - P(U)\| = \|P(t-U,(t-U)+U) - P(U)\| \leq \varepsilon/2.$$

Combining this with (16) shows that there exists a T such that $t > T$ implies

$$\|P(s, t) - L\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad \blacksquare$$

When the conditions of this theorem are met, one can determine the rate of convergence of the nonhomogeneous chain in terms of the rate at which the intensity matrices of the nonhomogeneous chain converge to the intensity matrix of the homogeneous chain. The theorem and the proof are similar to results on discrete-time chains given by Huang, Isaacson and Vinograd (1976). Before the proof two more lemmas are needed.

Lemma II.5:

If $P(t)$ is the transition matrix of a homogeneous, continuous-time Markov chain which is strongly ergodic with limit L , then there exist constants $c > 0$ and $\beta > 0$ such that

$$\|P(t) - L\| \leq c e^{-\beta t} \quad \text{for any } t > 0.$$

Proof:

Since $P(t)$ is strongly ergodic it is also weakly ergodic and so by the definition of weak ergodicity of a homogeneous chain there exist $T > 0$ and d such that $0 < d < 1$ and $\delta(P(T)) \leq d$.

For $t > T$, write t as $t = mT + r$ where $m = [t/T]$, the greatest integer in t/T . By (1) and (12),

$$\begin{aligned} \|P(t) - L\| &= \|P(mT + r) - L\| = \|P(r)P(mT) - LP(mT)\| \\ &= \|(P(r) - L)P(mT)\|. \end{aligned}$$

By Lemma I.14 and Lemma I.16 this is

$$\begin{aligned} &\leq \|P(r) - L\| \delta(P(mT)) \leq 2\delta((P(T))^m) \\ &\leq 2(\delta(P(T)))^m \leq 2d^m \leq (2/d)(d^{1/T})^t. \end{aligned}$$

Since $0 < d < 1$, one can set $c = 2/d$ and $e^{-\beta} = d^{1/T}$. Thus for $t > T$, $\|P(t) - L\| \leq c e^{-\beta t}$.

For $t < T$,

$$\begin{aligned} \|P(t) - L\| &\leq 2 = \frac{2}{\exp\{-\beta t\}} \exp\{-\beta t\} \leq \frac{2}{\exp\{-\beta T\}} \exp\{-\beta t\} \\ &= (2/d) \exp\{-\beta t\} = c e^{-\beta t}. \quad \blacksquare \end{aligned}$$

Lemma II.6:

Suppose $\lim_{t \rightarrow \infty} \|Q(t) - Q\| = 0$ where $\sup_{i \in S} \{ |q_{ii}| \} < q < \infty$ and $\sup_{t \geq 0} \|Q(t)\| < \infty$. If $\delta(P(U)) = d < 1$ for some U , then there exist $T > 0$ and $\gamma < 1$ such that $\delta(P(t, t+U)) < \gamma$ for all $t \geq T$.

Proof:

By choosing $\delta = \frac{1-d}{2U \exp\{UM\}}$ in Lemma II.3, one can see that there exists a T such that for $t \geq T$

$$\|P(t, t+U) - P(U)\| \leq 1/2 (1 - d).$$

Also, by Lemma I.17

$$|\delta(P(s, t)) - \delta(P(U))| \leq \|P(s, t) - P(U)\|.$$

Thus, letting $\gamma = (1 - d)/2 + d = 1/2 + 1/2 d < 1$, these two inequalities give

$$\begin{aligned} \delta(P(t, t+U)) &\leq |\delta(P(t, t+U)) - \delta(P(U))| + |\delta(P(U))| \\ &\leq \|P(t, t+U) - P(U)\| + \delta(P(U)) \\ &\leq 1/2(1 - d) + d = \gamma \quad \text{for } t > T. \quad \blacksquare \end{aligned}$$

Theorem II.7:

Let $\lim_{t \rightarrow \infty} \|Q(t) - Q\| = 0$ where Q is strongly ergodic,
 $\sup_{i \in S} \{ |q_{ii}| \} < q < \infty$ and $\sup_{t > 0} \|Q(t)\| < \infty$. Let $g(t)$ be a monotonically increasing function from \mathbb{R}^+ to \mathbb{R}^+ . If $\lim_{t \rightarrow \infty} g(2t) \|Q(t) - Q\| = 0$,
then $\lim_{t \rightarrow \infty} \sup_{s > 0} \{ \min(e^{\lambda t}, g(t)) \|P(s, s+t) - L\| \} = 0$ where L is the limit matrix for Q and $0 < \lambda < 1/2\beta$ where β is chosen as in Lemma II.5.

Proof:

By Definition I.i8, since Q is strongly ergodic, there exists a U such that $\delta(P(U)) < 1$. By Lemma II.6, there exist S^* and $\gamma < 1$ such that

$$\delta(P(s, s+U)) < \gamma \quad \text{for any } s > S^*. \quad (17)$$

By hypothesis, given any $\varepsilon > 0$ there exists a T such that $t > T$ implies

$$g(2t) \|Q(t) - Q\| < \varepsilon. \quad (18)$$

Choose t such that $1/2 t > \max(U, S^*, T)$.

Now, for any $s \geq 0$, one has

$$\begin{aligned}
& \|F(s, s+t) - L\| \\
& \leq \|P(s, s+t) - P(s, s+\frac{1}{2}t)P(\frac{1}{2}t)\| + \|P(s, s+\frac{1}{2}t)P(\frac{1}{2}t) - L\| \\
& \quad \|P(s, s+\frac{1}{2}t)P(s+\frac{1}{2}t, t) - P(s, s+\frac{1}{2}t)P(\frac{1}{2}t)\| \\
& \quad + \|P(s, s+\frac{1}{2}t)P(\frac{1}{2}t) - P(s, s+\frac{1}{2}t)L\| \\
& \leq \|P(s, s+\frac{1}{2}t)\| \|P(s+\frac{1}{2}t, s+t) - P(\frac{1}{2}t)\| \\
& \quad + \|P(s, s+\frac{1}{2}t)\| \|P(\frac{1}{2}t) - L\| \\
& \leq \|P(s+\frac{1}{2}t, s+t) - P(\frac{1}{2}t)\| + \|P(\frac{1}{2}t) - L\|. \tag{19}
\end{aligned}$$

Next, consider

$$\begin{aligned}
& \|P(s+\frac{1}{2}t, s+t) - P(\frac{1}{2}t)\| \\
& \leq \|P(s+\frac{1}{2}t, s+t) - P(\frac{1}{2}t-U)P(s+t-U, s+t)\| \\
& \quad + \|P(\frac{1}{2}t-U)P(s+t-U, s+t) - P(\frac{1}{2}t)\| \\
& = \|P(s+\frac{1}{2}t, s+t-U)P(s+t-U, s+t) - P(\frac{1}{2}t-U)P(s+t-U, s+t)\| \\
& \quad + \|P(\frac{1}{2}t-U)P(s+t-U, s+t) - P(\frac{1}{2}t-U)P(U)\| \\
& \leq \|P(s+\frac{1}{2}t, s+t-U) - P(\frac{1}{2}t-U)\| \delta(P(s+t-U, s+t)) \\
& \quad + \|P(s+t-U, s+t) - P(U)\|.
\end{aligned}$$

Repeating this procedure on $\|P(s+\frac{1}{2}t, s+t-U) - P(\frac{1}{2}t-U)\|$ several times and using (17) gives

$$\begin{aligned}
& \|P(s + \frac{1}{2}t, s+t) - P(\frac{1}{2}t)\| \\
& \leq \|P(s + \frac{1}{2}t, s+t-(m-1)U) - P(\frac{1}{2}t - (m-1)U)\| \gamma^{m-1} \\
& \quad + \sum_{i=1}^{m-1} \|P(s+t-iU, s+t-(i-1)U) - P(U)\| \gamma^{i-1} \quad (20)
\end{aligned}$$

where $m = [t/2U]$, the greatest integer in $t/2U$.

By Lemma II.3, there exists an M such that $\|Q\| < M$ and $\|Q(t)\| < M$ for any $t \geq 0$. Consider

$$\begin{aligned}
& g(t) \|P(s + \frac{1}{2}t, s+t-(m-1)U) - P(\frac{1}{2}t - (m-1)U)\| \\
& = g(t) \left\| \int_{s + \frac{1}{2}t}^{s+t-(m-1)U} [Q(s_1) - Q] ds_1 \right. \\
& \quad + \sum_{n \geq 2} \int_{s + \frac{1}{2}t}^{s+t-(m-1)U} ds_1 \dots \int_{s_{n-1}}^{s+t-(m-1)U} [Q(s_1) \dots Q(s_n) - Q^n] ds_n \Big\| \\
& \leq \int_{s + \frac{1}{2}t}^{s+t-(m-1)U} g(t) \|Q(s_1) - Q\| ds_1 \\
& \quad + \sum_{n \geq 2} \int_{s + \frac{1}{2}t}^{s+t-(m-1)U} ds_1 \dots \\
& \quad \int_{s_{n-1}}^{s+t-(m-1)U} g(t) \|Q(s_1) \dots Q(s_n) - Q^n\| ds_n
\end{aligned}$$

which by Lemma II.2 is

$$\begin{aligned}
& \leq \int_{s+1/2 t}^{s+t-(m-1)U} g(t) \|Q(s_1) - Q\| ds_1 \\
& + \sum_{n \geq 2} \int_{s+1/2 t}^{s+t-(m-1)U} ds_1 \dots \\
& \int_{s_{n-1}}^{s+t-(m-1)U} M^{n-1} g(t) \sum_{k=1}^n \|Q(s_k) - Q\| ds_n .
\end{aligned}$$

But $t \leq s+t \leq 2s+t = 2(s+1/2 t) \leq 2s_i$ for every i , and since $g(t)$ is monotonically increasing $g(t) \leq g(2s_i)$ for every i . The above is therefore

$$\begin{aligned}
& \leq \int_{s+1/2 t}^{s+t-(m-1)U} g(2s_1) \|Q(s_1) - Q\| ds_1 \\
& + \sum_{n \geq 2} \int_{s+1/2 t}^{s+t-(m-1)U} ds_1 \dots \\
& \int_{s_{n-1}}^{s+t-(m-1)U} M^{n-1} \sum_{k=1}^n g(2s_k) \|Q(s_k) - Q\| ds_n .
\end{aligned}$$

Since $1/2 t > T$, by (18) this is

$$\begin{aligned}
& \leq \int_{s+1/2 t}^{s+t-(m-1)U} \varepsilon ds_1 \\
& + \sum_{n \geq 2} \int_{s+1/2 t}^{s+t-(m-1)U} ds_1 \dots \int_{s_{n-1}}^{s+t-(m-1)U} M^{n-1} \varepsilon ds_n \\
& = \varepsilon (1/2 t - (m-1)U) + \sum_{n \geq 2} \frac{M^{n-1} \varepsilon}{n!} \left(\int_{s+1/2 t}^{s+t-(m-1)U} ds_1 \right)^n \\
& = \varepsilon (1/2 t - (m-1)U) + \sum_{n \geq 2} \frac{M^{n-1} \varepsilon}{(n-1)!} (1/2 t - (m-1)U)^n .
\end{aligned}$$

But $m = [t/2U]$ implies that $1/2 t - (m-1)U < 2U$, so the above becomes

$$\begin{aligned} &\leq \varepsilon (2U) + \varepsilon \sum_{n \geq 2} \frac{M^{n-1}}{(n-1)!} (2U)^n \\ &= \varepsilon 2U e^{2UM}. \end{aligned}$$

Similarly, for $i = 1, 2, \dots, m-1$

$$\begin{aligned} &g(t) \| P(s+t-iU, s+t-(i-1)U) - P(U) \| \\ &\leq \int_{s+t-iU}^{s+t-(i-1)U} g(t) \| Q(s_1) - Q \| ds_1 \\ &\quad + \sum_{n \geq 2} \int_{s+t-iU}^{s+t-(i-1)U} ds_1 \dots \\ &\quad \int_{s_{n-1}}^{s+t-(i-1)U} M^{n-1} g(t) \sum_{k=1}^n \| Q(s_k) - Q \| ds_n. \end{aligned}$$

But $1 \leq i \leq m-1 = [t/2U] - 1$, implies that $U - 1/2 t \leq -iU$ or, equivalently, $2U - t \leq -2iU$. Thus, $t \leq t+2s+2U = 2t+2s+2U-t \leq 2t+2s-2iU \leq 2(t+s-iU) \leq 2s_i$. Therefore, the above is

$$\begin{aligned} &\leq \int_{s+t-iU}^{s+t-(i-1)U} g(2s_1) \| Q(s_1) - Q \| ds_1 \\ &\quad + \sum_{n \geq 2} \int_{s+t-iU}^{s+t-(i-1)U} ds_1 \dots \\ &\quad \int_{s_{n-1}}^{s+t-(i-1)U} M^{n-1} \sum_{k=1}^n g(2s_k) \| Q(s_k) - Q \| ds_n. \end{aligned}$$

Again since $U - \frac{1}{2}t \leq -iU$, $s+t-iU \geq s+t-\frac{1}{2}t+U = s+\frac{1}{2}t+U \geq \frac{1}{2}t > T$.

Therefore, by (18), the above is

$$\begin{aligned} & \leq \varepsilon \int_{s+t-iU}^{s+t-(i-1)U} 1 \, ds_1 + \sum_{n \geq 2} \frac{M^{n-1} n \varepsilon}{n!} \left(\int_{s+t-iU}^{s+t-(i-1)U} 1 \, ds_1 \right)^n \\ & = \varepsilon U e^{MU} \\ & \leq \varepsilon 2U e^{2MU}. \end{aligned}$$

Therefore, from (20)

$$\begin{aligned} & g(t) \|P(s+\frac{1}{2}t, s+t) - P(\frac{1}{2}t)\| \\ & \leq \varepsilon 2U e^{2UM} \gamma^{m-1} + \sum_{i=1}^{m-1} \varepsilon 2U e^{2UM} \gamma^{i-1} \\ & = \varepsilon 2U e^{2UM} \sum_{i=0}^{m-1} \gamma^i \\ & < \varepsilon \frac{2U e^{2UM}}{1-\gamma}. \end{aligned}$$

Next consider the second term on the right-hand side of (19). By Lemma II.5, there exist constants $c > 0$ and $\beta > 0$ such that

$\|P(\frac{1}{2}t) - L\| \leq c e^{-\frac{1}{2}t\beta}$. Hence, if $0 < \lambda < \frac{1}{2}\beta$,
 $\min(g(t), e^{\lambda t}) \|P(\frac{1}{2}t) - L\| \leq e^{\lambda t} c e^{-\frac{1}{2}t\beta} = c e^{(\lambda - \frac{1}{2}\beta)t}$ which goes
to 0 as t approaches ∞ . It follows, therefore, from (19) that

$$\lim_{t \rightarrow \infty} \min(g(t), e^{\lambda t}) \|P(s, s+t) - L\| = 0 \text{ uniformly in } s. \quad \blacksquare$$

Slight alterations to this theorem can be made. If it is assumed that $\lim_{t \rightarrow \infty} g(kt) \|Q(t) - Q\| = 0$ where $k > 1$, then the result holds with $0 < \lambda < \beta(k-1)/k$. The proof would have in place of (19)

$$\begin{aligned} & \|P(s, s+t) - L\| \\ & \leq \|P(s, s+t) - P(s, s + t/k)P(t - t/k)\| \\ & \quad + \|P(s, s + t/k)P(t - t/k) - L\| \\ & \leq \|P(s + t/k, s + t) - P(t - t/k)\| + \|P(t - t/k) - L\|. \end{aligned}$$

The first term on the right-hand side is handled the same way as before except that $m = [(k-1)t/k]$. For the second term, by Lemma II.5,

$$\|P((k-1)/k)t - L\| \leq c e^{-\beta t(k-1)/k}.$$

Thus

$$\begin{aligned} & \min(g(t), e^{\lambda t}) \|P(t - t/k) - L\| \\ & \leq c e^{(\lambda - \beta(k-1)/k)t}. \end{aligned}$$

This goes to zero for $\lambda < \beta(k-1)/k$. If $\lim_{t \rightarrow \infty} g(kt) \|Q(t) - Q\| = 0$ for all $k > 1$, then the result is true for $0 < \lambda < \beta$.

III. CONSTANT CAUSATIVE MARKOV CHAINS

A. The General Case

The remainder of this dissertation will concentrate on various classes of nonhomogeneous, continuous-time Markov chains rather than the general chains studied in Chapter II. The first class to be considered is that of constant causative chains. Lipstein (1965) was apparently the first to introduce the notion of a constant causative chain. He used this type of nonhomogeneous, discrete-time chain to study consumer behavior after a new brand is introduced in the marketplace (see also Lipstein (1968)).

Definition III.1:

A discrete-time, nonhomogeneous Markov chain with transition matrix $P(m,n)$ is a constant causative chain if there exists a matrix C for which

$$P(n,n+1) = P(n-1,n)C = P(0,1)C^n .$$

C is called a causative matrix.

Although constant causative chains have appeared occasionally in the literature, applications of these chains are still infrequent. Besides Lipsteins work, Franklin (1976) used a constant causative model to study the work patterns of the disabled. Sociologists commonly use discrete-time, homogeneous Markov chains for such studies of shifts in populations. These have often been found to fit the data inadequately. Two possible remedies that have been suggested are to look instead at either

nonhomogeneous chains, as Franklin did, or at continuous-time chains (e.g. see Singer and Spilerman (1976)). Some authors have even tried both. Sørensen (1975) argues that his continuous-time model is not sufficient and that either the homogeneity or the Markovian assumption needs to be dropped in order to fit his data. He chose the former and looks at a nonhomogeneous Poisson process, which is a well-known nonhomogeneous, continuous-time chain. All this indicates that there are many possible applications of nonhomogeneous, continuous-time chains. The work of Lipstein and Franklin indicates that there may even be applications of some form of continuous-time, constant causative chain.

The following definition extends the constant causative property to the continuous-time case and quite a number of results follow.

Definition III.2:

A continuous-time Markov chain is constant causative if there exists a matrix $C(s,t)$ which satisfies the following properties:

$$i) \quad C(s, h+k) = C(s, h)C(s, k) \quad \text{whenever} \quad \{s, h, k\} \subset [0, \infty) \quad (21)$$

ii) For any s and t such that $0 \leq s \leq t$, the transition matrix $P(s, t)$ can be written as

$$P(s, t) = P(s-h, t-h)C(t-s, h) \quad \text{whenever} \quad 0 \leq h \leq s. \quad (22)$$

This definition coincides with the discrete-time definition of a constant causative chain if only integer-valued times are considered. In this case, $C = C(1,1)$ and (21) and (22) give

$$P(n, n+1) = P(n-1, n)C(1,1) = P(0,1)C(1,n) = P(0,1)(C(1,1))^n.$$

It is interesting to note that homogeneous Markov chains are the special case of constant causative chains where $C(s,t) = I$, the identity matrix.

Lipstein (1965) conjectured that if $\lim_{n \rightarrow \infty} C^n = L$ where L is a constant stochastic matrix, then $\lim_{n \rightarrow \infty} P(0,n) = L$. Proofs of this in somewhat restrictive cases are given by Harary et al. (1970) and by Le Maire and Mauffrey (1977). Pullman and Styan (1973) and Huang et al. (1976) both present stronger results which have Lipstein's conjecture as a consequence. The following theorem implies a continuous-time version of Lipstein's conjecture.

Theorem III.3:

If there exists an x and an M such that $\|C(x,t)\| \leq M$ for any $t \geq 0$ and $\lim_{t \rightarrow \infty} \|C(x,t) - L\| = 0$ where L is a constant stochastic matrix, $\int_0^{\infty} \|P(s,s+t) - L\| dt$ converges for any $s \geq 0$.

Proof:

By (21) for any $s \geq 0$, $C(x,s+t) = C(x,s)C(x,t) = C(x,t)C(x,s)$. Since C is bounded, $\lim_{t \rightarrow \infty} C(x,s+t) = C(x,s) \lim_{t \rightarrow \infty} C(x,t) = (\lim_{t \rightarrow \infty} C(x,t))C(x,s)$. Thus, since L is a constant stochastic matrix

$$L = C(x,s)L = LC(x,s) = L^2 \quad \text{for any } s \geq 0. \quad (23)$$

For any $s \geq 0$, consider

$$\begin{aligned}
& \int_0^{\infty} \|P(s, s+t) - L\| dt \\
&= \int_0^x \|P(s, s+t) - L\| dt + \int_x^{\infty} \|P(s, s+t) - L\| dt \\
&\leq \int_0^x 2 dt + \int_x^{\infty} \|P(s, s+t-x)P(s+t-x, s+t) - L\| dt \\
&= 2x + \int_x^{\infty} \|P(s, s+t-x)P(0, x)C(x, s+t-x) - L\| dt
\end{aligned}$$

which by Lemma I.11 is

$$\begin{aligned}
&= 2x + \int_x^{\infty} \|P(s, s+t-x)P(0, x)(C(x, s+t-x) - L)\| dt \\
&\leq 2x + \int_x^{\infty} 1 \cdot \|C(x, s+t-x) - L\| dt \\
&= 2x + \int_0^{\infty} \|C(x, s+t) - L\| dt .
\end{aligned}$$

Since $\lim_{t \rightarrow \infty} \|C(x, t) - L\| = 0$, there exists a T such that

$\|C(x, T) - L\| = \alpha < 1$. Therefore, the above is

$$\begin{aligned}
&= 2x + \sum_{k=0}^{\infty} \int_0^T \|C(x, kT+u+s) - L\| du \\
&= 2x + \sum_{k=0}^{\infty} \int_0^T \|C(x, kT)C(x, u+s) - L\| du
\end{aligned}$$

which by (23) is

$$\begin{aligned}
&= 2x + \sum_{k=0}^{\infty} \int_0^T \| (C(x, kT) - L)(C(x, u+s) - L) \| du \\
&\leq 2x + \sum_{k=0}^{\infty} \int_0^T \| C(x, kT) - L \| \| C(x, u+s) - L \| du .
\end{aligned}$$

Using (21) and the triangle inequality this is

$$\leq 2x + \sum_{k=0}^{\infty} \| (C(x, T))^k - L \| \int_0^T \| C(x, u+s) \| + \| L \| du$$

which by (23) and hypothesis is

$$\begin{aligned}
&\leq 2x + \sum_{k=0}^{\infty} \| (C(x, T) - L)^k \| \int_0^T (M+1) du \\
&\leq 2x + \sum_{k=0}^{\infty} \| C(x, T) - L \|^k (M+1) T \\
&\leq 2x + \sum_{k=0}^{\infty} \alpha^k (M+1) T .
\end{aligned}$$

Since $\alpha < 1$, $\int_0^{\infty} \| P(s, s+t) - L \| dt$ converges. ■

In the discrete-time case, it is clear that for a constant causative chain the changes in the transition matrix are the same at each step. It seems, therefore, not unreasonable to guess that constant causative chains might be described in terms of linear equations. The next theorem shows that this is in fact the case. A simple lemma about continuous functions is given first.

Lemma III.4:

Let f be a continuous function from the nonnegative reals into the reals. If for every $x \geq 0$ and $y \geq 0$,

$$f(x + y) = f(x) + f(y)$$

then there exists an a such that

$$f(x) = ax.$$

Proof:

Let p be a positive integer. $f(p) = f(p \cdot 1) = f(1+1+\dots+1)$
 $= f(1) + f(1) + \dots + f(1) = pf(1)$. Also $pf(1/p) = f(1/p) + f(1/p) + \dots + f(1/p) = f(p(1/p)) = f(1)$ implies that $f(1/p) = (1/p)f(1)$.
 Therefore, for any positive rational p/q , $f(p/q) = (p/q)f(1)$. For any $x \geq 0$, there exists a sequence of positive rationals p_n/q_n such that $\lim_{n \rightarrow \infty} p_n/q_n = x$. Thus, since f is continuous, $f(x) = f(\lim_{n \rightarrow \infty} p_n/q_n) = \lim_{n \rightarrow \infty} f(p_n/q_n) = \lim_{n \rightarrow \infty} (p_n/q_n)f(1) = xf(1)$. Letting $a = f(1)$, the proof is complete. ■

Theorem III.5:

Suppose $X(t)$ is a constant causative Markov chain whose causative matrix $C(x,t)$ has elements with continuous first order partials. Then, the intensity matrices of $X(t)$ are given by $Q(t) = tC + Q$ where C and Q are the intensity matrices of two continuous-time, homogeneous chains.

Proof:

First, since for any $t \geq 0$ $P(t,t) = I$, (22) implies

$$I = P(t,t) = P(0,0)C(0,t) = C(0,t) \quad \text{for any } t \geq 0. \quad (24)$$

By definition, the intensity matrix of $X(t)$ is given by

$$\begin{aligned} Q(t) &= \lim_{h \rightarrow 0} \frac{P(t,t+h) - I}{h} = \lim_{h \rightarrow 0} \frac{P(0,h)C(h,t) - P(0,0)C(0,t)}{h} \\ &= \lim_{h \rightarrow 0} \left[P(0,h) \frac{(C(h,t) - C(0,t))}{h} + \frac{(P(0,h) - P(0,0))}{h} C(0,t) \right] \\ &= IC_1(0,t) + Q(0)I \\ &= C_1(0,t) + Q(0) \end{aligned}$$

where $C_1(0,t) = \frac{\partial}{\partial s} C(s,t) \big|_{(0,t)}$.

Using (21) and (24), one sees that

$$\begin{aligned} C_1(0,s+t) &= \lim_{h \rightarrow 0} \frac{C(h,s+t) - C(0,s+t)}{h} \\ &= \lim_{h \rightarrow 0} \frac{C(h,s)C(h,t) - C(0,s)C(0,t)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{(C(h,s) - C(0,s))}{h} C(h,t) \right. \\ &\quad \left. + C(0,s) \frac{(C(h,t) - C(0,t))}{h} \right] \\ &= C_1(0,s) + C_1(0,t). \end{aligned}$$

Therefore, applying Lemma III.4 to the elements of $C_1(0,t)$ it follows that $C_1(0,t) = tC$ for some matrix C . Thus, $Q(t) = tC + Q$.

Since the intensity matrices $Q(t)$ must satisfy (3), it is clear that C and Q both satisfy (9) are therefore intensity matrices for two homogeneous chains. ■

In the case where the C and Q found in the above theorem commute with each other, a great deal can be determined about the chain. The rest of this chapter will consider this case.

B. The Commutative Case

In Theorem III.5 two matrices, C and Q , associated with the constant causative chain were introduced. When C and Q commute and are bounded, one can solve the Kolmogorov differential equations to find the transition matrix $P(s,t)$. Recall that the intensity matrix Q of a continuous-time homogeneous Markov chain satisfies (9),

$$q_{ii} \leq 0, \quad q_{ij} \geq 0 \quad \text{for } i \neq j$$

$$\text{and } \sum_{j \in S} q_{ij} = 0 \quad \text{for every } i \in S.$$

For any matrix A it is assumed that $A^0 = I$, the identity matrix. The elements of I are given by the Kronecker delta

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Theorem III.6:

Let $Q(t) = tC + Q$ where C and Q are the intensity matrices of two homogeneous chains such that $\sup_{i \in S} \{ |q_{ii}| \} < q < \infty$, $\sup_{i \in S} \{ |c_{ii}| \} < c < \infty$ and $CQ = QC$. Then, $\{Q(t)\}$ are the intensity matrices of a Markov chain and the transition matrices are determined uniquely by

$$\begin{aligned}
 P(s,t) &= I + \sum_{n=1}^{\infty} \frac{1}{n!} \{ (t-s)Q + 1/2 (t^2-s^2)C \}^n \\
 &= \exp[(t-s)Q + 1/2 (t^2-s^2)C] \\
 &= \exp[(t-s)Q] \exp[1/2 (t^2-s^2)C] .
 \end{aligned} \tag{25}$$

Proof:

Notation: $(A)_{ij}^n = a_{ij}(n)$ is the ij^{th} element of the matrix A^n .

$$\text{Let } F(s,t) = I + \sum_{n=1}^{\infty} \frac{1}{n!} \{ (t-s)Q + 1/2 (t^2-s^2)C \}^n.$$

To show that $F(s,t)$ is actually the transition matrix $P(s,t)$ it is sufficient to show that $F(s,t)$ satisfies the Kolmogorov differential equations (4) and (5) and that $\sum_{j \in S} f_{ij}(s,t) = 1$ for every $i \in S$ and all $0 \leq s \leq t$.

The backward Kolmogorov equation (5) can be given in matrix form as

$$\frac{\partial}{\partial s} F(s,t) = -Q(s)F(s,t) .$$

Since $F(s,t) = \exp\left\{\int_s^t Q(u)du\right\}$, the fact that

$$\begin{aligned}\frac{\partial}{\partial s} F(s,t) &= -Q(s) \exp\left\{\int_s^t Q(u)du\right\} \\ &= -Q(s)F(s,t)\end{aligned}$$

follows from advanced calculus techniques. The details are given for completeness.

In order to show that $F(s,t)$ is a solution to the backward Kolomogorov differential equation, first consider for $t > s > 0$

$$\begin{aligned}\frac{\partial}{\partial s} f_{ij}(s,t) &= \frac{\partial}{\partial s} \left(\delta_{ij} + \sum_{n=1}^{\infty} \frac{1}{n!} ((t-s)Q + 1/2 (t^2-s^2)C)_{ij}^n \right) \\ &= \frac{\partial}{\partial s} \left(\sum_{n=1}^{\infty} \frac{1}{n!} ((t-s)Q + 1/2 (t^2-s^2)C)_{ij}^n \right)\end{aligned}\quad (26)$$

In order to differentiate the series termwise it is sufficient to show that the series of partial derivatives converges absolutely. From Corollaries I.8 and I.9 applied to Q and C

$$\|Q\|^n \leq (2q)^n \quad \text{and}$$

$$\|C\|^n \leq (2c)^n.$$

Therefore,

$$\begin{aligned}|(C^n Q^m)_{ij}| &\leq \sum_{k \in S} |(C^n Q^m)_{ik}| \leq \|C^n Q^m\| \leq \|C\|^n \|Q\|^m \\ &\leq (2c)^n (2q)^m.\end{aligned}\quad (27)$$

Thus, consider

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \left| \frac{\partial}{\partial s} \left((t-s)q + \frac{1}{2} (t^2 - s^2)c \right)_{ij}^n \frac{1}{n!} \right| \\
 &= \sum_{n=1}^{\infty} \frac{1}{n!} \left| \frac{\partial}{\partial s} \sum_{k=0}^n \binom{n}{k} (t-s)^k \left(\frac{1}{2} (t^2 - s^2) \right)^{n-k} (q^k c^{n-k})_{ij} \right| \\
 &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \left| \frac{\partial}{\partial s} \left((t-s)^k \left(\frac{t^2 - s^2}{2} \right)^{n-k} \right) \right| |(q^k c^{n-k})_{ij}|
 \end{aligned}$$

which according to (27) is

$$\begin{aligned}
 &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \left| -k(t-s)^{k-1} \left(\frac{t^2 - s^2}{2} \right)^{n-k} \right. \\
 &\quad \left. - s(n-k)(t-s)^k \left(\frac{t^2 - s^2}{2} \right)^{n-k-1} \right| (2q)^k (2c)^{n-k} \\
 &= \sum_{n=1}^{\infty} \frac{2^n}{n!} \left\{ \sum_{k=0}^n \frac{n!}{k!(n-k)!} k(t-s)^{k-1} \left(\frac{t^2 - s^2}{2} \right)^{n-k} q^k c^{n-k} \right. \\
 &\quad \left. + \sum_{k=0}^n \frac{n!}{k!(n-k)!} s(n-k)(t-s)^k \left(\frac{t^2 - s^2}{2} \right)^{n-k-1} q^k c^{n-k} \right\} \\
 &= \sum_{n=1}^{\infty} \frac{2^n}{n!} n(q+sc) \left\{ \sum_{k=0}^{n-1} \binom{n-1}{k} (t-s)^k q^k \left(\frac{t^2 - s^2}{2} \right)^{n-1-k} c^{n-1-k} \right\} \\
 &= 2(q+sc) \sum_{n=1}^{\infty} \frac{2^{n-1}}{(n-1)!} \left((t-s)q + \frac{1}{2} (t^2 - s^2)c \right)^{n-1} \\
 &= 2(q+sc) \exp \left\{ 2 \left((t-s)q + \frac{1}{2} (t^2 - s^2)c \right) \right\} < \infty.
 \end{aligned}$$

Therefore, (26) can be differentiated termwise.

$$\begin{aligned}
\frac{\partial}{\partial s} f_{ij}(s, t) &= \frac{\partial}{\partial s} \left(\sum_{n=1}^{\infty} ((t-s)Q + 1/2 (t^2 - s^2)C)_{ij}^n \frac{1}{n!} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial}{\partial s} \left(\sum_{k=0}^n \binom{n}{k} (t-s)^k \left(\frac{t^2 - s^2}{2} \right)^{n-k} (Q^k C^{n-k})_{ij} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \left\{ -k(t-s)^{k-1} \left(\frac{t^2 - s^2}{2} \right)^{n-k} \right. \\
&\quad \left. - s(n-k)(t-s)^k \left(\frac{t^2 - s^2}{2} \right)^{n-k-1} \right\} (Q^k C^{n-k})_{ij} \\
&= - \sum_{n=1}^{\infty} \frac{1}{n!} \left\{ \sum_{k=0}^n \frac{n!k(t-s)^{k-1}}{k!(n-k)!} \left(\frac{t^2 - s^2}{2} \right)^{n-k} \sum_{m \in S} q_{im} (Q^{k-1} C^{n-k})_{mj} \right. \\
&\quad \left. + \sum_{k=0}^n \frac{n!s(n-k)(t-s)^k}{k!(n-k)!} \left(\frac{t^2 - s^2}{2} \right)^{n-k-1} \sum_{m \in S} c_{im} (Q^k C^{n-1-k})_{mj} \right\}
\end{aligned}$$

which by interchanging the order of summation is

$$\begin{aligned}
&= - \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left\{ \sum_{m \in S} (q_{im} + c_{im}s) \right. \\
&\quad \left. \cdot \left(\sum_{k=0}^{n-1} \binom{n-1}{k} (t-s)^k \left(\frac{t^2 - s^2}{2} \right)^{n-1-k} (Q^k C^{n-1-k})_{mj} \right) \right\} \\
&= - \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \sum_{m \in S} (q_{im} + c_{im}s) \left((t-s)Q + \left(\frac{t^2 - s^2}{2} \right) C \right)_{mj}^{n-1}.
\end{aligned}$$

However,

$$\begin{aligned}
& \sum_{m \in S} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} |q_{im} + c_{im}s| \left| \left((t-s)Q + \left(\frac{t^2-s^2}{2} \right) C \right)_{mj}^{n-1} \right| \\
&= \sum_{m \in S} |q_{im} + c_{im}s| \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left| \left((t-s)Q + \frac{1}{2} (t^2-s^2)C \right)_{mj}^{n-1} \right| \\
&\leq \|Q + Cs\| \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \| (t-s)Q + \frac{1}{2} (t^2-s^2)C \|^{n-1} \\
&\|Q + Cs\| \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (|t-s| \|Q\| + \frac{1}{2} |t^2-s^2| \|C\|)^{n-1} \\
&\leq \|Q + Cs\| \sum_{n=1}^{\infty} \frac{1}{(n-1)!} ((t-s)2q + \frac{1}{2} (t^2-s^2)2c)^{n-1} \\
&\leq \|Q + Cs\| \exp\{ (t-s)2q + (t^2-s^2)c \} < \infty .
\end{aligned}$$

Therefore, one can interchange the order of summation to get

$$\begin{aligned}
\frac{\partial}{\partial s} f_{ij}(s,t) &= - \sum_{m \in S} (q_{im} + c_{im}s) \sum_{n=1}^{\infty} \frac{1}{(n-1)!} ((t-s)Q + \frac{1}{2} (t^2-s^2)C)_{mj}^{n-1} \\
&= - \sum_{m \in S} (q_{im} + c_{im}s) \sum_{n=0}^{\infty} \frac{1}{n!} ((t-s)Q + \frac{1}{2} (t^2-s^2)C)_{mj}^n \\
&= \sum_{m \in S} -q_{im}(s) f_{mj}(s,t) .
\end{aligned}$$

The last equality holds since $Q(t) = tC + Q$ and

$F(s,t) = \exp\{ (t-s)Q + \frac{1}{2} (t^2-s^2)C \}$. Thus, the backward Kolmogorov

differential equation holds. The proof that $F(s,t)$ satisfies the

forward equation is similar.

To see that this is the unique solution for the transition matrix, one must show that $\sum_{j \in S} f_{ij}(s, t) = 1$. Therefore, consider

$$\begin{aligned} \sum_{j \in S} f_{ij}(s, t) &= \sum_{j \in S} \left\{ \delta_{ij} + \sum_{n=1}^{\infty} \frac{1}{n!} ((t-s)Q + 1/2 (t^2-s^2)C)_{ij}^n \right\} \\ &= 1 + \sum_{j \in S} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} (t-s)^k \left(\frac{t^2-s^2}{2} \right)^{n-k} (Q^k C^{n-k})_{ij}. \end{aligned}$$

Since

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=0}^n \sum_{j \in S} \left| \binom{n}{k} (t-s)^k \left(\frac{t^2-s^2}{2} \right)^{n-k} (Q^k C^{n-k})_{ij} \right| \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} (t-s)^k \left(\frac{t^2-s^2}{2} \right)^{n-k} \sum_{j \in S} |(Q^k C^{n-k})_{ij}| \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} (t-s)^k \left(\frac{t^2-s^2}{2} \right)^{n-k} (2q)^k (2c)^{n-k} \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} ((t-s)2q + 1/2 (t^2-s^2)2c)^n \\ &= \exp\{(t-s)2q + (t^2-s^2)c\} - 1 < \infty, \end{aligned}$$

the above becomes

$$\sum_{j \in S} f_{ij}(s, t) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} (t-s)^k \left(\frac{t^2-s^2}{2} \right)^{n-k} \sum_{j \in S} (Q^k C^{n-k})_{ij}. \quad (28)$$

Consider $\sum_{j \in S} (Q^k C^{n-k})_{ij}$. Since $n \geq 1$, either $k > 0$ or $n-k > 0$.

Therefore, the following argument shows that $\sum_{j \in S} (Q^k C^{n-k})_{ij} = 0$. By hypothesis $\sum_{j \in S} q_{ij} = 0$ and $\sum_{j \in S} c_{ij} = 0$. For $n > 0$ or $m > 0$ consider $\sum_{j \in S} (Q^n C^m)_{ij}$. Since C and Q commute one can assume without loss of generality that $m > 0$. Since

$$\begin{aligned} \sum_{k \in S} \sum_{j \in S} |(Q^n C^{m-1})_{ik}| |c_{kj}| &\leq \sum_{k \in S} |(Q^n C^{m-1})_{ik}| \sum_{j \in S} |c_{kj}| \\ &\leq \|C\| \sum_{k \in S} |(Q^n C^{m-1})_{ik}| \leq \|C\| \|Q^n C^{m-1}\| \leq \|C\|^m \|Q\|^n < \infty, \end{aligned}$$

one can interchange sums to get

$$\begin{aligned} \sum_{j \in S} (Q^n C^m)_{ij} &= \sum_{j \in S} \sum_{k \in S} (Q^n C^{m-1})_{ik} c_{kj} \\ &= \sum_{k \in S} (Q^n C^{m-1})_{ik} \sum_{j \in S} c_{kj} = 0. \end{aligned}$$

Therefore, (28) gives

$$\sum_{j \in S} f_{ij}(s, t) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} (t-s)^k \left(\frac{t^2 - s^2}{2} \right)^{n-k} (0) = 1.$$

Thus, the solution to the Kolmogorov differential equations is unique. In other words, the Markov chain given by the intensity matrices $Q(t) = tC + Q$ has the transition matrices

$$P(s, t) = I + \sum_{n=1}^{\infty} \frac{1}{n!} ((t-s)Q + 1/2 (t^2 - s^2)C)^n$$

$$= \exp\{(t-s)Q + 1/2 (t^2-s^2)C\} . \quad \blacksquare$$

In the commutative case, it is also possible to recover the causative matrix when $Q(t) = tC + Q$ is given.

Theorem III.7:

If $Q(t) = tC + Q$ are the intensity matrices of a Markov chain $X(t)$ where $\|C\| < 2c < \infty$, $\|Q\| < 2q < \infty$ and $QC = CQ$, then $X(t)$ is constant causative and

$$C(s,t) = \exp\{stC\} .$$

Proof:

$$\begin{aligned} C(s, h+k) &= \exp\{s(h+k)C\} = \exp\{shC\} \exp\{skC\} \\ &= C(s, h)C(s, k) \end{aligned}$$

so (21) is satisfied. To see that (22) is also satisfied, use (25).

$$\begin{aligned} &F(s-h, t-h)C(t-s, h) \\ &= \exp\{((t-h)-(s-h))Q + 1/2 ((t-h)^2 - (s-h)^2)C\} \\ &\quad \cdot \exp\{(t-s)hC\} \\ &= \exp\{(t-s)Q + 1/2 (t^2 - 2th + h^2 - s^2 + 2sh - h^2 + 2th - 2sh)C\} \\ &= \exp\{(t-s)Q + 1/2 (t^2 - s^2)C\} \\ &= P(s, t) . \quad \blacksquare \end{aligned}$$

Example III.8:

$$\text{Let } C = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} -3 & 1 & 2 \\ 2 & -4 & 2 \\ 2 & 1 & -3 \end{bmatrix} \quad \text{then } C^2 = -2C,$$

$$Q^2 = -5Q \quad \text{and} \quad CQ = QC = -5C.$$

$$\begin{aligned} \exp(tQ) &= I + \sum_{n=1}^{\infty} \frac{(tQ)^n}{n!} = I + \sum_{n=1}^{\infty} \frac{t^n (-5)^{n-1}}{n!} Q \\ &= I - \frac{Q}{5} \sum_{n=1}^{\infty} \frac{(-5t)^n}{n!} = I - \frac{Q}{5} (\exp(-5t) - 1) \\ &= (1/5) \begin{bmatrix} 2+3e^{-5t} & 1-1e^{-5t} & 2-2e^{-5t} \\ 2-2e^{-5t} & 1+4e^{-5t} & 2-2e^{-5t} \\ 2-2e^{-5t} & 1-1e^{-5t} & 2+3e^{-5t} \end{bmatrix}. \end{aligned}$$

Similarly,

$$\exp(tC) = 1/2 \begin{bmatrix} 1+1e^{-2t} & 0 & 1-1e^{-2t} \\ 0 & 2 & 0 \\ 1-1e^{-2t} & 0 & 1+1e^{-2t} \end{bmatrix}.$$

Thus ,

$$C(s,t) = \exp\{stC\} = \begin{bmatrix} 1/2 + 1/2 e^{-2st} & 0 & 1/2 - 1/2 e^{-2st} \\ 0 & 1 & 0 \\ 1/2 - 1/2 e^{-2st} & 0 & 1/2 + 1/2 e^{-2st} \end{bmatrix}.$$

And $P(s,t) = \exp\{(t-s)Q\} \exp\{1/2 (t^2 - s^2)C\}$ is given in Table 1.

Table 1. $P(s,t)$ for Example III.8

$$(1/10) \begin{bmatrix} 4+e^{-5(t-s)}+5e^{-5(t-s)-(t^2-s^2)} & 2-2e^{-5(t-s)} & 4+e^{-5(t-s)}-5e^{-5(t-s)-(t^2-s^2)} \\ 4-4e^{-5(t-s)} & 2+8e^{-5(t-s)} & 4-4e^{-4(t-s)} \\ 4+e^{-5(t-s)}-5e^{-5(t-s)-(t^2-s^2)} & 2-2e^{-5(t-s)} & 4+e^{-5(t-s)}+5e^{-5(t-s)-(t^2-s^2)} \end{bmatrix}$$

It is interesting to notice that this example shows that a converse to Theorem III.3 does not hold.

$$L = \begin{bmatrix} 2/5 & 1/5 & 2/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & 1/5 & 2/5 \end{bmatrix},$$

$$\text{then } \int_0^{\infty} \|P(s, s+t) - U\| dt < \int_0^{\infty} \frac{8}{5} e^{-5t} + e^{-5t-2st-t^2} dt < \infty \text{ for any}$$

$s > 0$. On the other hand,

$$\lim_{t \rightarrow \infty} C(s, t) = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}.$$

The remainder of the work in this chapter will relate the commutative case of constant causative chains to two discrete-time, homogeneous Markov chains. This is done in the following way. Define

$$\bar{P} = I + \frac{Q}{q} \quad \text{where} \quad \sup_{i \in S} \{ |q_{ii}| \} < q < \infty \quad \text{and} \quad (29)$$

$$\bar{R} = I + \frac{C}{c} \quad \text{where} \quad \sup_{i \in S} \{ |c_{ii}| \} < c < \infty. \quad (30)$$

Since $\sup_{i \in S} \{ |q_{ii}| \} < q$, $0 \leq q_{ij}/q < 1$ for $i \neq j$ and

$-1 < q_{ii}/q \leq 0$. Thus by (29), $0 \leq \delta_{ij} + (q_{ij}/q) = \bar{p}_{ij} \leq 1$ for all $i, j \in S$. Also,

$$\sum_{j \in S} \bar{p}_{ij} = \sum_{j \in S} (\delta_{ij} + (q_{ij}/q)) = 1 + \frac{1}{q} \sum_{j \in S} q_{ij} = 1$$

for all $i \in S$. In the same way, $0 \leq \bar{r}_{ij} \leq 1$ for all $i, j \in S$ and

$$\sum_{j \in S} \bar{r}_{ij} = 1 \text{ for all } i \in S. \text{ Thus, } \bar{P} \text{ and } \bar{R} \text{ are both stochastic}$$

and can be viewed as the transition matrices of two discrete-time, homogeneous Markov chains which are referred to as $\bar{X}(k)$ and $\bar{Y}(k)$, respectively.

Since $CQ = QC$, it is easy to see that \bar{P} and \bar{R} also commute as follows

$$\begin{aligned} \overline{PK} &= \left(I + \frac{Q}{q} \right) \left(I + \frac{C}{c} \right) = I + \frac{Q}{q} + \frac{C}{c} + \frac{QC}{qc} \\ &= I + \frac{Q}{q} + \frac{C}{c} + \frac{CQ}{cq} = \left(I + \frac{C}{c} \right) \left(I + \frac{Q}{q} \right) \\ &= \overline{RP}. \end{aligned}$$

Note also that $\sup_{i \in S} \{ |q_{ii}| \} < q$ implies that

$$\bar{p}_{ii} = 1 + (q_{ii}/q) > 0 \text{ for each } i \in S. \text{ Therefore, } \bar{P} \text{ is aperiodic.}$$

Similarly, \bar{R} is aperiodic. Since periodicity is never present in continuous-time chains, it is logical to compare the continuous-time chain to aperiodic, discrete-time chains in order to study ergodic behavior.

The transition matrix for the constant causative chain, which will now be referred to as $X(t)$, can be written in terms of \bar{P} and \bar{R} .

Since $Q = q(\bar{P} - I)$ and $C = c(\bar{R} - I)$, (25) gives

$$\begin{aligned}
 P(s,t) &= \exp\{(t-s)q(\bar{P}-I)\} \exp\{1/2 (t^2-s^2)c(\bar{R}-I)\} \\
 &= \exp\{(t-s)q\bar{P}\} \exp\{-q(t-s)I\} \\
 &\quad \cdot \exp\{1/2 c(t^2-s^2)\bar{R}\} \exp\{-1/2 (t^2-s^2)I\} \\
 &= \exp\{q(t-s)\bar{P}\} \exp\{-q(t-s)\} I \\
 &\quad \cdot \exp\{1/2 c(t^2-s^2)\bar{R}\} \exp\{-1/2 c(t^2-s^2)\} I \\
 &= \exp\{-q(t-s) - 1/2 c(t^2-s^2)\} \exp\{q(t-s)\bar{P}\} \exp\{1/2 c(t^2-s^2)\bar{R}\} .
 \end{aligned}$$

Since C and Q commute, this can also be written as

$$\begin{aligned}
 P(s,t) &= \exp\{-q(t-s) - 1/2 c(t^2-s^2)\} \\
 &\quad \cdot \exp\{1/2 c(t^2-s^2)\bar{R}\} \exp\{q(t-s)\bar{P}\}
 \end{aligned}$$

or as

$$\begin{aligned}
 P(s,t) &= \exp\{-q(t-s) - 1/2 c(t^2-s^2)\} \\
 &\quad \cdot \exp\{q(t-s)\bar{P} + 1/2 c(t^2-s^2)\bar{R}\} . \tag{31}
 \end{aligned}$$

Using the notation $\bar{p}_{ij}(n)$ to indicate the ij^{th} element of \bar{P}^n and $\bar{r}_{ij}(n)$ to indicate the ij^{th} elements of \bar{R}^n , (31) can be written in terms of elements as

$$p_{ij}(s,t) = \exp\{-q(t-s) - 1/2 c(t^2-s^2)\} \\ \cdot \sum_{k \in S} \left(\sum_{n=0}^{\infty} \frac{q^n(t-s)^n}{n!} \bar{p}_{ik}(n) \right) \left(\sum_{n=0}^{\infty} \frac{c^n}{n!} \left(\frac{t^2-s^2}{2} \right)^n \bar{r}_{kj}(n) \right) \quad (32)$$

$$\text{or } p_{ij}(s,t) = \exp\{-q(t-s) - 1/2 c(t^2-s^2)\} \\ \cdot \sum_{k \in S} \left(\sum_{n=0}^{\infty} \frac{c^n}{n!} \left(\frac{t^2-s^2}{2} \right)^n \bar{r}_{ik}(n) \right) \left(\sum_{n=0}^{\infty} \frac{q^n(t-s)^n}{n!} \bar{p}_{kj}(n) \right)$$

$$\text{or } p_{ij}(s,t) = \exp\{-q(t-s) - 1/2 c(t^2-s^2)\} \\ \cdot \sum_{n=0}^{\infty} \frac{1}{n!} (q(t-s)\bar{P} + 1/2 c(t^2-s^2)\bar{R})^n_{ij} \\ = \exp\{-q(t-s) - 1/2 c(t^2-s^2)\} \\ \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} (q(t-s))^k (1/2 c(t^2-s^2))^{n-k} (\bar{P}^k \bar{R}^{n-k})_{ij} . \quad (33)$$

This way of expressing $P(s,t)$ becomes very useful in the following theorems, which relate properties of $X(t)$ to the corresponding properties of $\bar{X}(k)$ and $\bar{Y}(k)$ and other related discrete-time, homogeneous chains.

Lemma III.9:

Let \bar{P} and \bar{R} be given as in (29) and (30). If for some $i, j \in S$ there are nonnegative integers m and n for which $(\bar{P}^m \bar{R}^n)_{ij} > 0$, then for any integers $u \geq m$ and $v \geq n$ $(\bar{P}^u \bar{R}^v)_{ij} > 0$.

Proof:

By definition $\bar{p}_{ii} > 0$ and $\bar{r}_{jj} > 0$. Thus,

$$(\bar{P}^{m+1} \bar{R}^n)_{ij} = \sum_{k \in S} \bar{p}_{ik} (\bar{P}^m \bar{R}^n)_{kj} > \bar{p}_{ii} (\bar{P}^m \bar{R}^n)_{ij} > 0$$

and

$$(\bar{P}^m \bar{R}^{n+1})_{ij} = \sum_{k \in S} (\bar{P}^m \bar{R}^n)_{ik} \bar{r}_{kj} > (\bar{P}^m \bar{R}^n)_{ij} \bar{r}_{jj} > 0.$$

The lemma follows by induction. ■

Theorem III.10:

Let $X(t)$ be a Markov chain defined by the intensity matrix

$$Q(t) = tQ + C \quad \text{where} \quad \sup_{i \in S} \{ |q_{ii}| \} < q < \infty \quad \text{and} \quad \sup_{i \in S} \{ |c_{ii}| \} < c < \infty$$

and $CQ = QC$. Let $\bar{P} = I + \frac{Q}{q}$ and $\bar{R} = I + \frac{C}{c}$. $X(t)$ is irreducible if and only if $\bar{P}\bar{R}$ is the transition matrix of an irreducible discrete-time, homogeneous Markov chain.

Proof:

Suppose $X(t)$ is irreducible. Consider any $i, j \in S$. By the irreducibility of $X(t)$, there exists a $t > 0$ such that

$0 < p_{ij}(0, t)$. By (33), this implies that there exists some $k > 0$ and some $n > 0$, which depend on i and j , such that $(\bar{P}^k \bar{R}^{n-k})_{ij} > 0$.

Let $m = \max\{k, n-k\}$. By Lemma III.9 $0 < (\bar{P}^m \bar{R}^m)_{ij} = (\bar{P}\bar{R})_{ij}^m$. In other words, $\bar{P}\bar{R}$ is the transition matrix of an irreducible chain.

Suppose \overline{PR} is the transition matrix of an irreducible chain.

Choose any $i, j \in S$ and any $s > 0$. Since \overline{PR} is irreducible, one can find an m such that $(\overline{PR})_{ij}^m > 0$. Let $t = s+1$, then $t-s = 1$ and $t^2 - s^2 = 2s+1$. By (33),

$$p_{ij}(s, t) = \exp\{-q - \frac{1}{2} c(2s+1)\}$$

$$\cdot \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} q^k \left(\frac{1}{2} c(2s+1) \right)^{n-k} (\overline{P}^k \overline{R}^{n-k})_{ij} \right\}$$

$$> \exp\{-q - \frac{1}{2} c(2s+1)\}$$

$$\cdot \frac{1}{(2m)!} \binom{2m}{m} q^m \left(\frac{1}{2} c(2s+1) \right)^m (\overline{P}^m \overline{R}^m)_{ij} > 0.$$

Thus, $X(t)$ is irreducible. ■

It is often not practical to find \overline{Pk} . However, the following corollary, when applicable, will affirm the irreducibility of $X(t)$ more readily than by using Theorem III.10.

Corollary III.11:

$X(t)$ is irreducible if either $\overline{X}(k)$ or $\overline{Y}(k)$ is irreducible.

Proof:

Suppose $\overline{X}(k)$ is irreducible. For any $i, j \in S$, there exists an m such that $0 < \overline{p}_{ij}(m) = (\overline{P}^m \overline{R}^0)_{ij}$. By Lemma III.9, $0 < (\overline{P}^m \overline{R}^m)_{ij} = (\overline{PK})_{ij}^m$. Therefore, \overline{PR} is the transition matrix of an irreducible chain. By Theorem III.10, $X(t)$ is irreducible. ■

The proof is similar if $\overline{Y}(k)$ is irreducible.

The converse of the corollary is not true as demonstrated by the following example.

Example III.12:

$$\text{Let } Q = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \text{ then}$$

$$CQ = QC = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}, \quad C^2 = -2C \quad \text{and} \quad Q^2 = -2Q.$$

$$e^{tC} = \begin{pmatrix} 1/2 + 1/2 e^{-2t} & 1/2 - 1/2 e^{-2t} & 0 & 0 \\ 1/2 - 1/2 e^{-2t} & 1/2 + 1/2 e^{-2t} & 0 & 0 \\ 0 & 0 & 1/2 + 1/2 e^{-2t} & 1/2 - 1/2 e^{-2t} \\ 0 & 0 & 1/2 - 1/2 e^{-2t} & 1/2 + 1/2 e^{-2t} \end{pmatrix}.$$

$$e^{tQ} = \begin{pmatrix} 1/2 + 1/2 e^{-2t} & 0 & 0 & 1/2 - 1/2 e^{-2t} \\ 0 & 1/2 + 1/2 e^{-2t} & 1/2 - 1/2 e^{-2t} & 0 \\ 0 & 1/2 - 1/2 e^{-2t} & 1/2 + 1/2 e^{-2t} & 0 \\ 1/2 - 1/2 e^{-2t} & 0 & 0 & 1/2 + 1/2 e^{-2t} \end{pmatrix}.$$

Two forms of $P(s,t) = \exp[(t-s)Q] \exp[1/2(t^2-s^2)C]$ are given in Tables 2 and 3. From the form of $P(s,t)$ given in Table 3, it is clear that for $s < t < \infty$ each element is greater than zero and thus $X(t)$ is irreducible. On the other hand, letting $q = c = 2$ gives

Table 2. $P(s, t)$ for Example III.12

$\begin{bmatrix} 1/4 + 1/4 e^{-2(t-s)} \\ + 1/4 e^{-(t-s)^2} \\ + 1/4 e^{-2(t-s)-(t-s)^2} \end{bmatrix}$	$\begin{bmatrix} 1/4 + 1/4 e^{-2(t-s)} \\ - 1/4 e^{-(t-s)^2} \\ - 1/4 e^{-2(t-s)-(t-s)^2} \end{bmatrix}$	$\begin{bmatrix} 1/4 - 1/4 e^{-2(t-s)} \\ - 1/4 e^{-(t-s)^2} \\ + 1/4 e^{-2(t-s)-(t-s)^2} \end{bmatrix}$	$\begin{bmatrix} 1/4 - 1/4 e^{-2(t-s)} \\ + 1/4 e^{-(t-s)^2} \\ - 1/4 e^{-2(t-s)-(t-s)^2} \end{bmatrix}$
$\begin{bmatrix} 1/4 + 1/4 e^{-2(t-s)} \\ - 1/4 e^{-(t-s)^2} \\ - 1/4 e^{-2(t-s)-(t-s)^2} \end{bmatrix}$	$\begin{bmatrix} 1/4 + 1/4 e^{-2(t-s)} \\ + 1/4 e^{-(t-s)^2} \\ + 1/4 e^{-2(t-s)-(t-s)^2} \end{bmatrix}$	$\begin{bmatrix} 1/4 - 1/4 e^{-2(t-s)} \\ + 1/4 e^{-(t-s)^2} \\ - 1/4 e^{-2(t-s)-(t-s)^2} \end{bmatrix}$	$\begin{bmatrix} 1/4 - 1/4 e^{-2(t-s)} \\ - 1/4 e^{-(t-s)^2} \\ + 1/4 e^{-2(t-s)-(t-s)^2} \end{bmatrix}$
$\begin{bmatrix} 1/4 - 1/4 e^{-2(t-s)} \\ - 1/4 e^{-(t-s)^2} \\ + 1/4 e^{-2(t-s)-(t-s)^2} \end{bmatrix}$	$\begin{bmatrix} 1/4 - 1/4 e^{-2(t-s)} \\ + 1/4 e^{-(t-s)^2} \\ - 1/4 e^{-2(t-s)-(t-s)^2} \end{bmatrix}$	$\begin{bmatrix} 1/4 + 1/4 e^{-2(t-s)} \\ - 1/4 e^{-(t-s)^2} \\ + 1/4 e^{-2(t-s)-(t-s)^2} \end{bmatrix}$	$\begin{bmatrix} 1/4 + 1/4 e^{-2(t-s)} \\ + 1/4 e^{-(t-s)^2} \\ - 1/4 e^{-2(t-s)-(t-s)^2} \end{bmatrix}$
$\begin{bmatrix} 1/4 - 1/4 e^{-2(t-s)} \\ + 1/4 e^{-(t-s)^2} \\ - 1/4 e^{-2(t-s)-(t-s)^2} \end{bmatrix}$	$\begin{bmatrix} 1/4 - 1/4 e^{-2(t-s)} \\ - 1/4 e^{-(t-s)^2} \\ + 1/4 e^{-2(t-s)-(t-s)^2} \end{bmatrix}$	$\begin{bmatrix} 1/4 + 1/4 e^{-2(t-s)} \\ + 1/4 e^{-(t-s)^2} \\ - 1/4 e^{-2(t-s)-(t-s)^2} \end{bmatrix}$	$\begin{bmatrix} 1/4 + 1/4 e^{-2(t-s)} \\ - 1/4 e^{-(t-s)^2} \\ + 1/4 e^{-2(t-s)-(t-s)^2} \end{bmatrix}$

Table 3. Simplified form of $P(s,t)$ for Example III.12

$\frac{1}{4} (1+e^{-2(t-s)})$ • $(1+e^{-(t^2-s^2)})$	$\frac{1}{4} (1+e^{-2(t-s)})$ • $(1-e^{-(t^2-s^2)})$	$\frac{1}{4} (1-e^{-2(t-s)})$ • $(1-e^{-(t^2-s^2)})$	$\frac{1}{4} (1-e^{-2(t-s)})$ • $(1+e^{-(t^2-s^2)})$
$\frac{1}{4} (1+e^{-2(t-s)})$ • $(1-e^{-(t^2-s^2)})$	$\frac{1}{4} (1+e^{-2(t-s)})$ • $(1+e^{-(t^2-s^2)})$	$\frac{1}{4} (1-e^{-2(t-s)})$ • $(1+e^{-(t^2-s^2)})$	$\frac{1}{4} (1-e^{-2(t-s)})$ • $(1-e^{-(t^2-s^2)})$
$\frac{1}{4} (1-e^{-2(t-s)})$ • $(1-e^{-(t^2-s^2)})$	$\frac{1}{4} (1-e^{-2(t-s)})$ • $(1+e^{-(t^2-s^2)})$	$\frac{1}{4} (1+e^{-2(t-s)})$ • $(1+e^{-(t^2-s^2)})$	$\frac{1}{4} (1+e^{-2(t-s)})$ • $(1-e^{-(t^2-s^2)})$
$\frac{1}{4} (1-e^{-2(t-s)})$ • $(1+e^{-(t^2-s^2)})$	$\frac{1}{4} (1-e^{-2(t-s)})$ • $(1-e^{-(t^2-s^2)})$	$\frac{1}{4} (1+e^{-2(t-s)})$ • $(1-e^{-(t^2-s^2)})$	$\frac{1}{4} (1+e^{-2(t-s)})$ • $(1+e^{-(t^2-s^2)})$

$$\bar{P} = I + \frac{Q}{q} = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix} \quad \text{and} \quad \bar{R} = I + \frac{C}{c} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}.$$

Neither $\bar{X}(k)$ nor $\bar{Y}(k)$ is irreducible.

Therefore, since neither $\bar{X}(k)$ nor $\bar{Y}(k)$ is irreducible, Corollary III.11 is not applicable. There is, however, another corollary of Theorem III.10 which characterizes the irreducibility of $X(t)$ by using a discrete-time chain whose transition matrix is easier to find than \overline{PR} .

Corollary III.13:

$X(t)$ is irreducible if and only if $1/2(\bar{P} + \bar{R})$ is the transition matrix of an irreducible, homogeneous, discrete-time Markov chain.

Proof:

If $X(t)$ is irreducible then, by Theorem III.10, for any $i, j \in S$, there exists an m such that $(\overline{PR})_{ij}^m > 0$. Therefore,

$$0 < (\overline{PR})_{ij}^m < \sum_{k=0}^{2m} \binom{2m}{k} (\bar{P}^k \bar{R}^{2m-k})_{ij} = (\bar{P} + \bar{R})_{ij}^{2m}.$$

Thus, $1/2(\bar{P} + \bar{R})$ is the transition matrix of an irreducible Markov chain.

Conversely, suppose $1/2(\bar{P} + \bar{R})$ is irreducible and choose $i, j \in S$. There exists an m such that $(1/2(\bar{P} + \bar{R}))_{ij}^m > 0$. Therefore,

$$0 < \left(\frac{1}{2}\right)^m \sum_{k=0}^m \binom{m}{k} (\bar{P}^k \bar{R}^{m-k})_{ij}.$$

This implies that there exists some $k_0 \leq m$ such that $(\bar{P}^{k_0} \bar{R}^{m-k_0})_{ij} > 0$.

Let $n = \max\{k_0, m-k_0\}$. By Lemma III.9

$$0 < (\bar{P}^n \bar{R}^n)_{ij} = (\overline{PR})_{ij}^n.$$

Thus \overline{PR} is irreducible and by Theorem III.10 so is $X(t)$. ■

The major reason for introducing the discrete-time chains $\bar{X}(k)$ and $\bar{Y}(k)$ is that in many cases the ergodicity of the constant causative chain can be directly related to the ergodicity of these discrete-time chains. In the cases where either discrete-time chain is ergodic, the constant causative chain will also be ergodic and its long run distribution will be the same as that of the ergodic discrete-time chain. An obvious question arises if both discrete-time chains are ergodic since it is not clear which of the two corresponding long-run distributions would be assumed by the continuous-time chain. Fortunately, since \bar{P} and \bar{R} commute, Lemma I.22 shows that $\bar{X}(k)$ and $\bar{Y}(k)$ must have the same long run distribution whenever they are both ergodic.

Because the stationary distributions of $\bar{X}(k)$ and $\bar{Y}(k)$ coincide when they are both ergodic, it is reasonable to relate the ergodicity of these discrete-time chains to that of the constant causative chain. This is done in the following theorem.

Theorem III.14:

Suppose either $\bar{X}(k)$ or $\bar{Y}(k)$ is ergodic with limit distribution π , then $X(t)$ is ergodic with the same limit distribution.

Proof:

Suppose $\bar{X}(k)$ is ergodic and $\lim_{n \rightarrow \infty} \bar{p}_{ij}(n) = \pi_j$ for $i \in S$ where $\pi_j > 0$ and $\sum_{j \in S} \pi_j = 1$. First consider

$$\begin{aligned} & \exp\{-q(t-s) - 1/2 c(t^2 - s^2)\} \\ & \cdot \sum_{k \in S} \left(\sum_{n=0}^{\infty} \frac{(t-s)^n q^n}{n!} (\bar{p}_{ik}(n) - \pi_k) \right) \left(\sum_{n=0}^{\infty} \left(\frac{t^2 - s^2}{2} \right)^n \frac{c^n}{n!} \bar{r}_{kj} \right) \\ & = \exp\{-q(t-s) - 1/2 c(t^2 - s^2)\} \\ & \cdot \left\{ \sum_{k \in S} \left(\sum_{n=0}^{\infty} \frac{(t-s)^n q^n}{n!} \bar{p}_{ik}(n) \right) \left(\sum_{n=0}^{\infty} \left(\frac{t^2 - s^2}{2} \right)^n \frac{c^n}{n!} \bar{r}_{kj} \right) \right. \\ & \quad \left. - \sum_{k \in S} \left(\sum_{n=0}^{\infty} \frac{(t-s)^n q^n}{n!} \pi_k \right) \left(\sum_{n=0}^{\infty} \left(\frac{t^2 - s^2}{2} \right)^n \frac{c^n}{n!} \bar{r}_{kj} \right) \right\}. \end{aligned}$$

By Lemma I.22, $\sum_{k \in S} \pi_k \bar{r}_{kj} = \pi_j$. Therefore, using (32), the above is

$$\begin{aligned} & = p_{ij}(s, t) - \exp\{-q(t-s) - 1/2 c(t^2 - s^2)\} \pi_j \exp\{q(t-s) + 1/2 c(t^2 - s^2)\} \\ & = p_{ij}(s, t) - \pi_j. \end{aligned} \tag{34}$$

Choose $\varepsilon > 0$. Since $\bar{X}(k)$ is ergodic, Theorem 1.5 implies that there exists an N such that for $n > N$

$$\sum_{j \in S} |\bar{p}_{ij}(n) - \pi_j| < \frac{\varepsilon}{2}.$$

Therefore, from (34)

$$\begin{aligned} |p_{ij}(s, t) - \pi_j| &\leq \exp\{-q(t-s) - 1/2 c(t^2 - s^2)\} \\ &\cdot \sum_{k \in S} \left(\sum_{n=0}^{\infty} \frac{(t-s)^n q^n}{n!} |\bar{p}_{ik}(n) - \pi_k| \right) \left(\sum_{n=0}^{\infty} \left(\frac{t^2 - s^2}{2} \right)^n \frac{c^n}{n!} \bar{r}_{kj}(n) \right) \\ &\leq \exp\{-q(t-s) - 1/2 c(t^2 - s^2)\} \\ &\cdot \sum_{k \in S} \left(\sum_{n=0}^{\infty} \frac{(t-s)^n q^n}{n!} |\bar{p}_{ik}(n) - \pi_k| \right) \left(\sum_{n=0}^{\infty} \left(\frac{t^2 - s^2}{2} \right)^n \frac{c^n}{n!} \cdot 1 \right) \\ &\leq \exp\{-q(t-s) - 1/2 c(t^2 - s^2)\} \exp\{1/2 c(t^2 - s^2)\} \\ &\cdot \sum_{n=0}^{\infty} \frac{(t-s)^n q^n}{n!} \sum_{k \in S} |\bar{p}_{ik}(n) - \pi_k| \\ &\leq \exp\{-q(t-s)\} \left\{ \sum_{n=0}^{N-1} \frac{(t-s)^n q^n}{n!} \cdot 2 + \sum_{n=N}^{\infty} \frac{(t-s)^n q^n}{n!} \cdot \frac{\varepsilon}{2} \right\} \\ &\leq 2 \sum_{n=0}^{N-1} \exp\{-q(t-s)\} \frac{(t-s)^n q^n}{n!} \\ &\quad + \exp\{-q(t-s)\} \sum_{n=0}^{\infty} \frac{(t-s)^n q^n}{n!} \cdot \frac{\varepsilon}{2}. \end{aligned}$$

Since for fixed n , $\lim_{t \rightarrow \infty} \exp\{-q(t-s)\} \frac{(t-s)^n q^n}{n!} = 0$, one can find a T such that for $t > T$ and $n = 1, 2, \dots, N-1$

$$\exp\{-q(t-s)\} \frac{(t-s)^n q^n}{n!} < \frac{\varepsilon}{4N}.$$

Thus, for $t > T$

$$\begin{aligned} |p_{ij}(s, t) - \pi_j| \\ \leq 2 \sum_{n=0}^{N-1} \frac{\varepsilon}{4N} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

If $\bar{Y}(k)$ is ergodic the proof is similar. ■

Example III.12 shows that the converse of this theorem is not true.

It is also possible to relate the strong ergodicity of the constant causative chain to that of the corresponding discrete-time chains as demonstrated in the following theorem.

Theorem III.15:

If either $\bar{X}(k)$ or $\bar{Y}(k)$ is strongly ergodic with limit distribution π , then $X(t)$ is strongly ergodic with the same limit distribution.

Proof:

Suppose $\bar{X}(k)$ is strongly ergodic with stationary distribution π .

Let L be the constant matrix which has π for each of its rows.

Then, L satisfies $\lim_{n \rightarrow \infty} \|\bar{P}^n - L\| = 0$. A derivation similar to that of (34) implies

$$|p_{ij}(s, t) - \pi_j| \leq \exp\{-q(t-s) - 1/2 c(t^2 - s^2)\} \\ \cdot \sum_{k \in S} \left(\sum_{n=0}^{\infty} \left(\frac{t^2 - s^2}{2} \right)^n \frac{c^n}{n!} \bar{r}_{ik}(n) \right) \left(\sum_{n=0}^{\infty} \frac{(t-s)^n q^n}{n!} |\bar{p}_{kj}(n) - \pi_j| \right).$$

Therefore,

$$\|P(s, t) - L\| = \sup_{i \in S} \left\{ \sum_{j \in S} |p_{ij}(s, t) - \pi_j| \right\} \\ \leq \sup_{i \in S} \left\{ \exp\{-q(t-s) - 1/2 c(t^2 - s^2)\} \right. \\ \cdot \sum_{j \in S} \sum_{k \in S} \left(\sum_{n=0}^{\infty} \left(\frac{t^2 - s^2}{2} \right)^n \frac{c^n}{n!} \bar{r}_{ik}(n) \right) \\ \cdot \left. \left(\sum_{n=0}^{\infty} \frac{(t-s)^n q^n}{n!} |\bar{p}_{kj}(n) - \pi_j| \right) \right\}.$$

Since one can interchange the order of summation this is

$$= \sup_{i \in S} \left\{ \exp\{-q(t-s) - 1/2 c(t^2 - s^2)\} \right. \\ \cdot \sum_{k \in S} \left(\sum_{n=0}^{\infty} \left(\frac{t^2 - s^2}{2} \right)^n \frac{c^n}{n!} \bar{r}_{ik}(n) \right) \\ \cdot \left. \left(\sum_{n=0}^{\infty} \frac{(t-s)^n q^n}{n!} \sum_{j \in S} |\bar{p}_{kj}(n) - \pi_j| \right) \right\}$$

$$\begin{aligned}
&< \sup_{i \in S} \left\{ \exp\{-q(t-s) - 1/2 c(t^2-s^2)\} \right. \\
&\quad \cdot \sum_{k \in S} \left(\sum_{n=0}^{\infty} \left(\frac{t^2-s^2}{2} \right)^n \frac{c^n}{n!} \bar{r}_{ik}(n) \right) \\
&\quad \cdot \left(\sum_{n=0}^{\infty} \frac{(t-s)^n q^n}{n!} \| \bar{P}^n - U \| \right) \left. \right\} .
\end{aligned}$$

Since $\lim_{n \rightarrow \infty} \| \bar{P}^n - U \| = 0$, there exists an N such that for $n > N$

$\| \bar{P}^n - U \| < 1/2 \varepsilon$. Also, since for fixed n $\lim_{t \rightarrow \infty} e^{-q(t-s)} \frac{(t-s)^n q^n}{n!} = 0$

there exists a T such that $t > T$ implies

$e^{-q(t-s)} \frac{(t-s)^n q^n}{n!} < \frac{\varepsilon}{(N+1)4}$ for $n = 0, 1, 2, \dots, N$. Thus, for $t > T$

$$\begin{aligned}
\| P(s, t) - U \| &< \sup_{i \in S} \left\{ \exp\{-1/2 c(t^2-s^2)\} \right. \\
&\quad \cdot \sum_{k \in S} \left(\sum_{n=0}^{\infty} \left(\frac{t^2-s^2}{2} \right)^n \frac{c^n}{n!} \bar{r}_{ik}(n) \right) \left(\sum_{n=0}^N \frac{2\varepsilon}{(N+1)4} + 1/2 \varepsilon \right) \left. \right\} \\
&= \sup_{i \in S} \{ \exp\{-1/2 c(t^2-s^2)\} \exp\{1/2 c(t^2-s^2)\} \varepsilon \} \\
&= \varepsilon .
\end{aligned}$$

The proof is similar if $\bar{Y}(k)$ is strongly ergodic. ■

IV. SERIES OF PROPORTIONAL INTENSITIES

Markov chains with proportional intensities, $Q(t) = h(t)Q$, have been studied by Scott and Isaacson (1983). In this chapter, chains with intensity matrices defined by a series of proportional intensities,

$Q(t) = \sum_{n=0}^{\infty} h_n(t)A_n$, are investigated. Recall that the defining

properties of an intensity matrix given in (3) are that $q_{ij}(t) \geq 0$ if $i \neq j$, $q_{ii}(t) \leq 0$ and $\sum_{j \in S} q_{ij}(t) = 0$ for every $i \in S$ and every $t \geq 0$. The zero matrix clearly satisfies these conditions and is an

intensity matrix. Thus, A_n can be chosen to be the zero matrix for n

larger than some N , and $Q(t) = \sum_{n=1}^{\infty} h_n(t)A_n$ can be used to represent

finite sums, $Q(t) = \sum_{n=1}^N h_n(t)A_n$, as well as an infinite series.

Choosing $h_n(t)$ appropriately, one can see that the class of chains with

$Q(t) = \sum_{n=1}^{\infty} h_n(t)A_n$ has as special cases the following:

- 1) homogeneous chains, where $Q(t) = Q$;
- 2) chains with proportional intensities, where $Q(t) = h(t)Q$;
- 3) constant causative chains, where $Q(t) = tC + Q$ (discussed in Chapter III);
- 4) chains with intensity matrices defined by a finite sum of pro-

portional intensities, $Q(t) = \sum_{n=1}^N h_n(t)A_n$.

Some restrictions are placed on the terms $h_n(t)A_n$. Each A_n is taken to be an intensity matrix with $\|A_n\| < \infty$ and each $h_n(t)$ is continuous and nonnegative so that $h_n(t)A_n$ is an intensity matrix. In order to ensure the convergence of the series $Q(t) = \sum_{n=1}^{\infty} h_n(t)A_n$, it is assumed that there exist $b_n > 0$ such that $\sum_{n=1}^{\infty} b_n < \infty$ and $\sup_{t \geq 0} \|h_n(t)A_n\| \leq b_n$. Finally, as for the constant causative chains of Chapter III, the A_n 's are assumed to commute. This condition ensures that the long-run distribution will be the same for any of the homogeneous chains which happen to be ergodic. In this chapter, $X(t)$ will be used to designate chains with

$$Q(t) = \sum_{n=1}^{\infty} h_n(t)A_n \text{ which satisfy the above restrictions.}$$

Most of the results from Chapter III can be extended to $X(t)$ and several new results are possible. Theorem IV.1 shows how to find the transition matrix for the chains in question.

Theorem IV.1:

With the assumptions listed above $Q(t) = \sum_{n=1}^{\infty} h_n(t)A_n$ is the intensity matrix for a Markov chain $X(t)$ which has as its transition matrix

$$P(s,t) = \exp \left\{ \sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right\}.$$

Proof:

It will be shown that Kolmogorov's backward equation holds and that

$\sum_{j \in S} p_{ij}(s, t) = 1$ for each $i \in S$. The proof for Kolmogorov's forward equation is similar to the backward equation and will not be given.

Let $M = \sum_{n=1}^{\infty} b_n$. By hypothesis, $M < \infty$. Note that

$$\|Q(t)\| \leq \sum_{n=1}^{\infty} h_n(t) \|A_n\| \leq \sum_{n=1}^{\infty} b_n = M \quad \text{and}$$

$$\begin{aligned} \left\| \exp \left\{ \sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right\} \right\| &\leq \sum_{m=0}^{\infty} \frac{1}{m!} \left\| \sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right\|^m \\ &\leq \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{n=1}^{\infty} (t-s) b_n \right)^m \\ &\leq \sum_{m=0}^{\infty} \frac{1}{m!} (t-s)^m M^m = \exp\{M(t-s)\}. \end{aligned} \quad (35)$$

For Kolmogorov's backward equation, one must show

$$\begin{aligned} \frac{\partial}{\partial s} p_{ij}(s, t) &= \frac{\partial}{\partial s} \left(\delta_{ij} + \left(\sum_{m=1}^{\infty} \frac{1}{m!} \sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ij} \right) \\ &= \sum_{k \in S} \left(- \sum_{n=1}^{\infty} h_n(s) A_n \right)_{ik} \left(\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=1}^{\infty} \left(\int_s^t h_n(u) du A_n \right)_{kj}^m \right). \end{aligned} \quad (36)$$

The proof uses advanced calculus techniques. The details are included for completeness. First a bound is established.

$$\begin{aligned}
& \sum_{k \in S} \left| \left(- \sum_{n=1}^{\infty} h_n(s) A_n \right)_{ik} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ik}^m \right| \\
& \leq \sum_{k \in S} \left| \left(\sum_{n=1}^{\infty} h_n(s) A_n \right)_{ik} \right| \sum_{j \in S} \left| \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj}^m \right| \\
& \leq \sum_{k \in S} \left(\sum_{n=1}^{\infty} h_n(s) |A_n|_{ik} \right) \left\| \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)^m \right\| \\
& \leq \sum_{n=1}^{\infty} h_n(s) \|A_n\| \left(\sum_{n=1}^{\infty} \left| \int_s^t h_n(u) du \right| \|A_n\| \right)^m \\
& \leq \left(\sum_{n=1}^{\infty} b_n \right) \left(\sum_{n=1}^{\infty} (t-s) b_n \right)^m \\
& \leq M(t-s)^{m_M m} \\
& = (t-s)^{m_M m+1} .
\end{aligned} \tag{37}$$

To prove (36), first it is shown by induction that for $m \geq 1$

$$\begin{aligned}
& \frac{\partial}{\partial s} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ij}^m \\
& = m \sum_{k \in S} \left(\sum_{n=1}^{\infty} (-h_n(s)) A_n \right)_{ik} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj}^{m-1} .
\end{aligned} \tag{38}$$

For $m = 1$

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left| \frac{\partial}{\partial s} \int_s^t h_n(u) du (A_n)_{ij} \right| \\
& \leq \sum_{n=1}^{\infty} |h_n(s)| \|A_n\| \\
& \leq \sum_{n=1}^{\infty} b_n = M < \infty .
\end{aligned}$$

Thus, one can differentiate the following termwise.

$$\begin{aligned}
& \frac{\partial}{\partial s} \sum_{n=1}^{\infty} \int_s^t h_n(u) du (A_n)_{ij} \\
& = \sum_{n=1}^{\infty} \frac{\partial}{\partial s} \int_s^t h_n(u) du (A_n)_{ij} \\
& = 1 \cdot \sum_{k \in S} \left(\sum_{n=1}^{\infty} (-h_n(s)) A_n \right)_{ik} \delta_{kj} .
\end{aligned}$$

Therefore, (38) holds for $m = 1$. Suppose this induction hypothesis is true for $m \leq N$. For $N + 1$,

$$\begin{aligned}
& \frac{\partial}{\partial s} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ij}^{N+1} \\
& = \frac{\partial}{\partial s} \sum_{k \in S} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ik} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj}^N . \quad (39)
\end{aligned}$$

To see that one can differentiate termwise, first consider

$$\begin{aligned} & \sum_{k \in S} \left| \frac{\partial}{\partial s} \left[\left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ik} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj}^N \right] \right| \\ &= \sum_{k \in S} \left| \left(\frac{\partial}{\partial s} \sum_{n=1}^{\infty} \int_s^t h_n(u) du (A_n)_{ik} \right) \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj}^N \right. \\ & \quad \left. + \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ik} \left(\frac{\partial}{\partial s} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj}^N \right) \right|. \end{aligned}$$

By the induction hypothesis, this is

$$\begin{aligned} &= \sum_{k \in S} \left| \left(\sum_{n=1}^{\infty} (-h_n(s)) A_n \right)_{ik} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj}^N \right. \\ & \quad \left. + \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ik} \right. \\ & \quad \left. \cdot N \sum_{\ell \in S} \left(\sum_{n=1}^{\infty} (-h_n(s)) A_n \right)_{k\ell} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{\ell j}^{N-1} \right|. \end{aligned}$$

Since the A_n 's commute, this is

$$\begin{aligned} &= \sum_{k \in S} \left| \left(\sum_{n=1}^{\infty} (-h_n(s)) A_n \right)_{ik} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj}^N \right. \\ & \quad \left. + N \left(\sum_{n=1}^{\infty} (-h_n(s)) A_n \right)_{ik} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj}^N \right| \\ &= \sum_{k \in S} \left| (N+1) \left(\sum_{n=1}^{\infty} (-h_n(s)) A_n \right)_{ik} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj}^N \right|. \end{aligned}$$

By (37), this is

$$\begin{aligned} &< (N+1)(t-s)^N M^{N+1} \\ &< \infty. \end{aligned} \quad (40)$$

Therefore, (39) can be differentiated termwise.

$$\begin{aligned} &\frac{\partial}{\partial s} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ij}^{N+1} \\ &= \sum_{k \in S} \left\{ \left[\frac{\partial}{\partial s} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ik} \right] \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj}^N \right. \\ &\quad \left. + \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ik} \left[\frac{\partial}{\partial s} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj}^N \right] \right\}. \end{aligned}$$

By the induction hypothesis, this is

$$\begin{aligned} &= \sum_{k \in S} \left\{ \left(\sum_{n=1}^{\infty} (-h_n(s)) A_n \right)_{ik} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj}^N \right. \\ &\quad \left. + \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ik} \right. \\ &\quad \left. \cdot N \sum_{\ell \in S} \left(\sum_{n=1}^{\infty} (-h_n(s)) A_n \right)_{k\ell} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{\ell j}^{N-1} \right\}. \end{aligned}$$

By the commutativity of the A_n 's, this is

$$= (N+1) \sum_{k \in S} \left(\sum_{n=1}^{\infty} (-h_n(s)) A_n \right)_{ik} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj}^N.$$

This proves (38).

Returning to the proof of (36),

$$\frac{\partial}{\partial s} p_{ij}(s, t) = \frac{\partial}{\partial s} \sum_{m=1}^{\infty} \frac{1}{m!} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ij}^m.$$

This can be differentiated termwise since (40) gives

$$\begin{aligned} & \sum_{m=1}^{\infty} \frac{1}{m!} \left| \frac{\partial}{\partial s} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ij}^m \right| \\ & \leq \sum_{m=1}^{\infty} \frac{1}{m!} m(t-s)^{m-1} M^m \\ & = M \sum_{m=1}^{\infty} \frac{(M(t-s))^{m-1}}{(m-1)!} = M \exp\{M(t-s)\} < \infty. \end{aligned}$$

Thus, using (38),

$$\begin{aligned} \frac{\partial}{\partial s} p_{ij}(s, t) &= \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial}{\partial s} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ij}^m \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} m \sum_{k \in S} \left(\sum_{n=1}^{\infty} (-h_n(s)) A_n \right)_{ik} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj}^{m-1}. \end{aligned}$$

Once again, one can interchange summations since by (37)

$$\begin{aligned} & \sum_{m=1}^{\infty} \frac{1}{m!} m \sum_{k \in S} \left| \sum_{n=1}^{\infty} ((-h_n(s)) A_n)_{ik} \sum_{n=1}^{\infty} \left(\int_s^t h_n(u) du A_n \right)_{kj}^{m-1} \right| \\ & \leq \sum_{m=1}^{\infty} \frac{1}{(m-1)!} (t-s)^{m-1} M^m = M \exp\{M(t-s)\} < \infty. \end{aligned}$$

Therefore,

$$\frac{\partial}{\partial s} p_{ij}(s, t) = \sum_{k \in S} \left(-\sum_{n=1}^{\infty} h_n(s) A_n \right)_{ik} \left(\sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj}^m \right)$$

which is (36), the backward Kolomogorov equation.

Using (35) in order to show that the limits can be interchanged gives

$$\begin{aligned} \sum_{j \in S} p_{ij}(s, t) &= \sum_{j \in S} \left(\delta_{ij} + \sum_{m=1}^{\infty} \frac{1}{m!} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ij}^m \right) \\ &= 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{j \in S} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ij}^m \\ &= 1 . \end{aligned}$$

The last equality is true since for every $m \geq 1$ it can be shown that the appropriate summations can be interchanged to give

$$\begin{aligned} \sum_{j \in S} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ij}^m \\ &= \sum_{j \in S} \sum_{k \in S} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ik}^{m-1} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{kj} \\ &= \sum_{k \in S} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ik}^{m-1} \sum_{n=1}^{\infty} \int_s^t h_n(u) du \sum_{j \in S} (A_n)_{kj} \\ &= \sum_{k \in S} \left(\sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right)_{ik}^{m-1} \sum_{n=1}^{\infty} \int_s^t h_n(u) du \cdot 0 \\ &= 0 . \quad \blacksquare \end{aligned}$$

It was convenient in Theorem IV.1 to assume that each $h_n(t)\|A_n\|$ is a bounded function in order to assure the convergence of the series

$$Q(t) = \sum_{n=1}^{\infty} h_n(t)A_n. \text{ When only a finite sum is considered,}$$

$Q(t) = \sum_{n=1}^N h_n(t)A_n$, convergence is no longer a concern. In this case, it is not necessary to assume each $h_n(t)$ is bounded. For example, for constant causative chains, where $Q(t) = tC + Q$, $h_1(t) = t$ is not a bounded function. Theorem III.6 gives the constant causative result corresponding to that just proved in Theorem IV.1. Theorem III.6 can be extended to the general case of finite sums, $Q(t) = \sum_{n=1}^N h_n(t)A_n$, where the $h_n(t)$'s need not be bounded. In this case, the transition matrix is

$$P(s,t) = \exp \left\{ \sum_{n=1}^N \int_s^t h_n(u) du A_n \right\}.$$

Since it has been assumed that $\|A_n\| < \infty$ for each n , let $a_n > \|A_n\|$. Using these a_n 's, it is possible to write the transition matrices or $X(t)$ in terms of associated discrete-time, homogeneous chains. For $n \geq 1$, let

$$\bar{P}_n = I + (1/a_n)A_n.$$

\bar{P}_n defined in this way is a stochastic matrix and can be considered as the transition matrix of a discrete-time, homogeneous chain. Because a_n was chosen as a strict bound on the elements of A_n , the terms on

the main diagonal of \bar{P}_n , are positive. Thus, \bar{P}_n represents an aperiodic chain.

With this definition of the \bar{P}_n 's, one can write

$$\begin{aligned} P(s,t) &= \exp \left\{ \sum_{n=1}^{\infty} \int_s^t h_n(u) du A_n \right\} \\ &= \exp \left\{ \sum_{n=1}^{\infty} \int_s^t h_n(u) du a_n (\bar{P}_n - I) \right\} \\ &= \exp \left\{ - \sum_{n=1}^{\infty} a_n \int_s^t h_n(u) du \right\} \exp \left\{ \sum_{n=1}^{\infty} a_n \int_s^t h_n(u) du \bar{P}_n \right\}. \quad (41) \end{aligned}$$

An extension of Theorem III.10, concerning irreducibility, even to the case where $Q(t)$ is defined by the finite sum $\sum_{n=1}^N h_n(t) A_n$, seems of little value since any applications of such a theorem would involve the products $\bar{P}_1 \bar{P}_2 \dots \bar{P}_N$, which are computationally unwieldy. Because of this, the only results on irreducibility which are extended to the case of $Q(t) = \sum_{n=1}^{\infty} h_n(t) A_n$ are Corollaries III.11 and III.13.

Theorem IV.2:

$X(t)$ is irreducible if there exists some $n_0 > 1$ such that \bar{P}_{n_0} is irreducible and for every $s > 0$ there exists a $t > s$ such that

$$\int_s^t h_{n_0}(u) du > 0.$$

Proof:

Fix $i, j \in S$ and $s > 0$. From (41), for any $t > s > 0$

$$\begin{aligned}
 p_{ij}(s, t) &= \exp \left\{ - \sum_{n=1}^{\infty} a_n \int_s^t h_n(u) du \right\} \exp \left\{ \sum_{n=1}^{\infty} a_n \int_s^t h_n(u) du \bar{P}_n \right\}_{ij} \\
 &= \exp \left\{ - \sum_{n=1}^{\infty} a_n \int_s^t h_n(u) du \right\} \sum_{k \in S} \left(\exp \left\{ \sum_{\substack{n=1 \\ n \neq n_0}}^{\infty} a_n \int_s^t h_n(u) du \bar{P}_n \right\}_{ik} \right. \\
 &\quad \left. \cdot \exp \left\{ a_{n_0} \int_s^t h_{n_0}(u) du \bar{P}_{n_0} \right\}_{kj} \right) \\
 &= \exp \left\{ - \sum_{n=1}^{\infty} a_n \int_s^t h_n(u) du \right\} \\
 &\quad \cdot \sum_{k \in S} \left(\delta_{ik} + \sum_{m=1}^{\infty} \frac{1}{m!} \left(\sum_{\substack{n=1 \\ n \neq n_0}}^{\infty} a_n \int_s^t h_n(u) du \bar{P}_n \right)^m_{ik} \right) \\
 &\quad \cdot \left(\delta_{kj} + \sum_{m=1}^{\infty} \frac{1}{m!} (a_{n_0} \int_s^t h_{n_0}(u) du)^m (\bar{P}_{n_0})^m_{kj} \right).
 \end{aligned}$$

Since \bar{P}_{n_0} is irreducible there exists some M such that $(\bar{P}_{n_0})^m_{ij} > 0$

for any $m > M$. Choose T such that $\int_s^T h_{n_0}(u) du > 0$. Then,

$$\begin{aligned}
 p_{ij}(s, T) &> \exp \left\{ - \sum_{n=1}^{\infty} a_n \int_s^T h_n(u) du \right\} \\
 &\quad \cdot (\delta_{ii} + 0) \left(0 + a_{n_0}^M \left(\int_s^T h_{n_0}(u) du \right)^M (\bar{P}_{n_0})^M_{ij} \right) > 0. \quad \blacksquare
 \end{aligned}$$

Theorem IV.3:

If there exists a finite set $W = \{n_1, n_2, \dots, n_m\}$ of positive integers such that $\frac{1}{m} \sum_{n \in W} \bar{P}_n$ is the transition matrix of an irreducible Markov chain and if for every $s > 0$ there exist numbers

$t_n = t_n(s)$ such that $\int_s^{t_n} h_n(u) du > 0$ for each $n \in W$, then $X(t)$ is

irreducible.

Proof:

Fix $s > 0$ and $i, j \in S$.

By hypothesis, there exist $t_n = t_n(s)$ such that $\int_s^{t_n} h_n(u) du > 0$ for each $n \in W$. Since $h_n(t) > 0$ for each n , $\int_s^t h_n(u) du$ is increasing in t . Thus, letting $T > \max_{n \in W} \{t_n\}$ it follows that

$\int_s^T h_n(u) du > 0$ for each $n \in W$. Suppose $a_{n_0} \int_s^T h_{n_0}(u) du =$

$\min_{n \in W} \left\{ \int_s^T h_n(u) du \right\} > 0$. By (41)

$$\begin{aligned}
 p_{ij}(s, T) &= \exp \left\{ - \sum_{n=1}^{\infty} a_n \int_s^T h_n(u) du \right\} \\
 &\quad \cdot \exp \left\{ \sum_{n \notin W} a_n \int_s^T h_n(u) du \bar{P}_n + \sum_{n \in W} a_n \int_s^T h_n(u) du \bar{P}_n \right\}_{ij} \\
 &= \exp \left\{ - \sum_{n=1}^{\infty} a_n \int_s^T h_n(u) du \right\} \\
 &\quad \cdot \exp \left\{ \sum_{n \notin W} a_n \int_s^T h_n(u) du \bar{P}_n \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{n \in W} \left(a_n \int_s^T h_n(u) du \bar{P}_n - a_{n_0} \int_s^T h_{n_0}(u) du \bar{P}_n \right) \\
& + \sum_{n \in W} a_{n_0} \int_s^T h_{n_0}(u) du \bar{P}_n \Big\}_{ij} \\
= & \exp \left\{ - \sum_{n=1}^{\infty} a_n \int_s^T h_n(u) du \right\} \\
& \cdot \sum_{k \in S} \exp \left\{ \sum_{n \notin W} a_n \int_s^T h_n(u) du \bar{P}_n \right. \\
& \left. + \sum_{n \in W} \left(a_n \int_s^T h_n(u) du - a_{n_0} \int_s^T h_{n_0}(u) du \right) \bar{P}_n \right\}_{ik} \\
& \cdot \exp \left\{ \sum_{n \in W} a_{n_0} \int_s^T h_{n_0}(u) du \bar{P}_n \right\}_{kj} \\
> & \exp \left\{ - \sum_{n=1}^{\infty} a_n \int_s^T h_n(u) du \right\} \\
& \cdot \sum_{k \in S} \delta_{ik} \exp \left\{ a_{n_0} \int_s^T h_{n_0}(u) du \sum_{n \in W} \bar{P}_n \right\}_{kj} \\
= & \exp \left\{ - \sum_{n=1}^{\infty} a_n \int_s^T h_n(u) du \right\} \\
& \cdot \left\{ \delta_{ij} + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \left(a_{n_0} \int_s^T h_{n_0}(u) du \right)^{\ell} \left(\sum_{n \in W} \bar{P}_n \right)_{ij}^{\ell} \right\}.
\end{aligned}$$

Since $\frac{1}{m} \sum_{n \in W} \bar{P}_n$ is irreducible, there exists a positive integer L

such that $\left(\sum_{n \in W} \bar{P}_n \right)_{ij}^L > 0$. Thus,

$$p_{ij}(s, T) > \exp \left\{ - \sum_{n=1}^{\infty} a_n \int_s^T h_n(u) du \right\} \\ \cdot \frac{1}{L!} \left(a_{n_0} \int_s^T h_{n_0}(u) du \right)^L \left(\sum_{n \in W} \bar{P}_n \right)_{ij}^L > 0 . \quad \blacksquare$$

Because of the assumption that the \bar{P}_n 's commute, the long-run distributions of each \bar{P}_n which is ergodic are identical by Lemma I.22. Since $P(s, t)$ also commutes with the \bar{P}_n 's, when it is ergodic it also has the same long-run distribution as any of the ergodic \bar{P}_n 's. The following theorems explore the conditions needed for $P(s, t)$ to be ergodic. Recall that for any matrix A , it is assumed that $A^0 = I = (\delta_{ij})_{ij \in S}$, the identity matrix

Lemma IV.4:

Suppose that for each $n \geq 1$, P_n is a stochastic matrix and $f_n(s, t)$ is nonnegative for $t \geq s \geq 0$. If $\sum_{n=1}^{\infty} f_n(s, t) < \infty$ for $t \geq s \geq 0$, then $P(s, t) = \exp \left\{ - \sum_{n=1}^{\infty} f_n(s, t) \right\} \exp \left\{ \sum_{n=1}^{\infty} f_n(s, t) P_n \right\}$ is stochastic.

Proof:

For any $i, j \in S$

$$p_{ij}(s, t) = \exp \left\{ - \sum_{n=1}^{\infty} f_n(s, t) \right\} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{n=1}^{\infty} f_n(s, t) P_n \right)_{ij}^m > 0 .$$

Next, it is shown inductively that for any $i \in S$ and $m \geq 1$

$$\sum_{j \in S} \left(\sum_{n=1}^{\infty} f_n(s, t) P_n \right)_{ij}^m = \left(\sum_{n=1}^{\infty} f_n(s, t) \right)^m.$$

For $m = 1$

$$\sum_{j \in S} \sum_{n=1}^{\infty} (f_n(s, t) P_n)_{ij}^1 = \sum_{n=1}^{\infty} f_n(s, t) \sum_{j \in S} (P_n)_{ij} = \sum_{n=1}^{\infty} f_n(s, t)$$

since each P_n is stochastic. Suppose the induction hypothesis holds for m .

$$\begin{aligned} & \sum_{j \in S} \left(\sum_{n=1}^{\infty} f_n(s, t) P_n \right)_{ij}^{m+1} \\ &= \sum_{j \in S} \sum_{k \in S} \left(\sum_{n=1}^{\infty} f_n(s, t) P_n \right)_{ik}^m \left(\sum_{n=1}^{\infty} f_n(s, t) P_n \right)_{kj} \\ &= \sum_{k \in S} \left(\sum_{n=1}^{\infty} f_n(s, t) P_n \right)_{ik}^m \left(\sum_{n=1}^{\infty} f_n(s, t) \sum_{j \in S} (P_n)_{kj} \right) \\ &= \sum_{k \in S} \left(\sum_{n=1}^{\infty} f_n(s, t) P_n \right)_{ik}^m \sum_{n=1}^{\infty} f_n(s, t) \cdot 1 \\ &= \left(\sum_{n=1}^{\infty} f_n(s, t) \right)^{m+1}. \end{aligned}$$

Therefore, for any $i \in S$

$$\begin{aligned}
\sum_{j \in S} p_{ij}(s,t) &= \sum_{j \in S} \exp \left\{ - \sum_{n=1}^{\infty} f_n(s,t) \right\} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{n=1}^{\infty} f_n(s,t) P_n \right)_{ij}^m \\
&= \exp \left\{ - \sum_{n=1}^{\infty} f_n(s,t) \right\} \left(1 + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{j \in S} \left(\sum_{n=1}^{\infty} f_n(s,t) P_n \right)_{ij}^m \right) \\
&= \exp \left\{ - \sum_{n=1}^{\infty} f_n(s,t) \right\} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{n=1}^{\infty} f_n(s,t) \right)^m \\
&= 1. \quad \blacksquare
\end{aligned}$$

Theorem IV.5:

For $n \geq 1$, suppose each P_n is a stochastic matrix and $f_n(s,t) \geq 0$. Assume further that $\sum_{n=1}^{\infty} f_n(s,t) < \infty$ and the P_k 's commute. Let $P(s,t) = \exp \left\{ - \sum_{n=1}^{\infty} f_n(s,t) \right\} \exp \left\{ \sum_{n=1}^{\infty} f_n(s,t) P_n \right\}$. If the discrete-time, homogeneous chain with transition matrix P_1 is ergodic with limit π and if for any $s \geq 0$ $\lim_{t \rightarrow \infty} f_1(s,t) = \infty$ then $P(s,t)$ is ergodic with limit π .

Note that by Lemma I.22 all the P_n 's have the same invariant π .

Proof:

Since the P_n 's commute, for fixed i and j in S ,

$$p_{ij}(s,t) = \exp \left\{ - \sum_{n=1}^{\infty} f_n(s,t) \right\} \exp \left\{ \sum_{n=1}^{\infty} f_n(s,t) P_n \right\}_{ij}$$

$$\begin{aligned}
&= \exp \left\{ - \sum_{n=1}^{\infty} f_n(s,t) \right\} \sum_{k \in S} \left[\exp \{ f_1(s,t) P_1 \}_{ik} \right. \\
&\quad \cdot \left. \exp \sum_{n=2}^{\infty} \{ f_n(s,t) P_n \}_{kj} \right] \\
&= \exp \{ -f_1(s,t) \} \sum_{k \in S} \left[\sum_{m=0}^{\infty} \frac{1}{m!} (f_1(s,t))^m (P_1)_{ik}^m \right. \\
&\quad \cdot \left. \exp \left\{ - \sum_{n=2}^{\infty} f_n(s,t) \right\} \exp \left\{ \sum_{n=2}^{\infty} f_n(s,t) P_n \right\}_{kj} \right].
\end{aligned}$$

Let $R(s,t) = \exp \left\{ - \sum_{n=2}^{\infty} f_n(s,t) \right\} \exp \left\{ \sum_{n=2}^{\infty} f_n(s,t) P_n \right\}$. By Lemma IV.4, $R(s,t)$ is stochastic. Since each P_n commutes with P_1 , $R(s,t)$ commutes with P_1 . By Lemma I.22, $\pi R(s,t) = \pi$. Therefore,

$$\begin{aligned}
&\exp \{ -f_1(s,t) \} \sum_{k \in S} \left(\sum_{m=0}^{\infty} \frac{1}{m!} (f_1(s,t))^m ((P_1)_{ik}^m - \pi_k) \right) r_{kj}(s,t) \\
&= \exp \{ -f_1(s,t) \} \left\{ \sum_{k \in S} \left(\sum_{m=0}^{\infty} \frac{(f_1(s,t))^m}{m!} (P_1)_{ik}^m \right) r_{kj}(s,t) \right. \\
&\quad \left. - \sum_{k \in S} \sum_{m=0}^{\infty} \frac{(f_1(s,t))^m}{m!} \pi_k r_{kj}(s,t) \right\} \\
&= p_{ij}(s,t) - \exp \{ -f_1(s,t) \} \exp \{ f_1(s,t) \} \sum_{k \in S} \pi_k r_{kj}(s,t) \\
&= p_{ij}(s,t) - \pi_j.
\end{aligned}$$

Choose $\varepsilon > 0$. By Theorem I.5, there exists an N such that for $m \geq N$

$$\sum_{k \in S} |(P_1)_{ik}^m - \pi_k| < \frac{\varepsilon}{2}.$$

Thus,

$$\begin{aligned} & |p_{ij}(s, t) - \pi_j| \\ & \leq \exp\{-f_1(s, t)\} \sum_{k \in S} \left(\sum_{m=0}^{\infty} \frac{1}{m!} (f_1(s, t))^m |(P_1)_{ik}^m - \pi_k| \right) \cdot 1 \\ & = \exp\{-f_1(s, t)\} \left\{ \sum_{m=0}^{N-1} \frac{1}{m!} (f_1(s, t))^m \sum_{k \in S} |(P_1)_{ik}^m - \pi_k| \right. \\ & \quad \left. + \sum_{m=N}^{\infty} \frac{1}{m!} (f_1(s, t))^m \sum_{k \in S} |(P_1)_{ik}^m - \pi_k| \right\} \\ & \leq \exp\{-f_1(s, t)\} \left\{ 2 \sum_{m=0}^{N-1} \frac{1}{m!} (f_1(s, t))^m + \frac{\varepsilon}{2} \sum_{m=N}^{\infty} \frac{1}{m!} (f_1(s, t))^m \right\} \\ & \leq 2 \sum_{m=0}^{N-1} \exp\{-f_1(s, t)\} \frac{1}{m!} (f_1(s, t))^m \\ & \quad + \exp\{-f_1(s, t)\} \sum_{m=0}^{\infty} \frac{1}{m!} (f_1(s, t))^m \frac{\varepsilon}{2} \\ & = 2 \sum_{m=0}^{N-1} \exp\{-f_1(s, t)\} \frac{1}{m!} (f_1(s, t))^m + \frac{\varepsilon}{2}. \end{aligned}$$

Since $\lim_{t \rightarrow \infty} f_1(s, t) = \infty$, for $0 \leq m \leq N-1$, one can find a T such that for $t > T$

$$\exp\{-f_1(s,t)\} \frac{1}{m!} (f_1(s,t))^m < \frac{\varepsilon}{4N}.$$

Therefore, for $t > T$

$$|p_{ij}(s,t) - \pi_j| \leq 2 \sum_{m=0}^{N-1} \frac{\varepsilon}{4N} + \frac{\varepsilon}{2} = \varepsilon. \quad \blacksquare$$

$$\text{In Theorem IV.5, } P(s,t) = \exp\left\{-\sum_{n=1}^{\infty} f_n(s,t)\right\} \exp\left\{\sum_{n=1}^{\infty} f_n(s,t)P_n\right\}.$$

A Markov chain $X(t)$ with intensity matrix $Q(t) = \sum_{n=1}^{\infty} h_n(t)A_n$ has, as given in (41), a transition matrix

$$P(s,t) = \exp\left\{-\sum_{n=1}^{\infty} a_n \int_s^t h_n(u)du\right\} \exp\left\{\sum_{n=1}^{\infty} a_n \int_s^t h_n(u)du \bar{P}_n\right\}.$$

Thus, Corollary IV.6 follows immediately from Theorem IV.5.

Corollary IV.6:

Suppose there exists an $n_0 > 1$ such that $\bar{P}_{n_0} = I + (1/a_{n_0})A_{n_0}$ is ergodic with limit π and suppose for any $s > 0$

$$\lim_{t \rightarrow \infty} \int_s^t h_{n_0}(u)du = \infty. \text{ Then, } X(t) \text{ is ergodic with limit } \pi.$$

Even if no \bar{P}_{n_0} is ergodic, it may still be possible to use a combination of the \bar{P}_n 's to deduce the ergodicity of $X(t)$.

Corollary IV.7:

Suppose there exists a finite set $W = \{n_1, n_2, \dots, n_m\}$ such that the chain which has transition matrix $\frac{1}{m} \left(\sum_{n \in W} \right) \bar{P}_n$ is ergodic with limit

π . If for each $s \geq 0$ and each $n \in W$ $\int_s^\infty h_n(u) du = \infty$, then $X(t)$ is ergodic with limit π .

Proof:

Let $f(s, t) = \min_{n \in W} \left\{ a_n \int_s^t h_n(u) du \right\}$, then $mf(s, t) \geq 0$ and

$\lim_{t \rightarrow \infty} mf(s, t) = \infty$ for each s . Define

$$f_n(s, t) = \begin{cases} a_n \int_s^t h_n(u) du - f(s, t) & \text{for } n \in W \\ a_n \int_s^t h_n(u) du & \text{for } n \notin W. \end{cases}$$

Then, $f_n(s, t) \geq 0$ and $\sum_{n=1}^\infty a_n \int_s^t h_n(u) du = \sum_{i=1}^m f(s, t) + \sum_{n=1}^\infty f_n(s, t)$.

Using (41),

$$\begin{aligned} P(s, t) &= \exp \left\{ - \sum_{n=1}^\infty a_n \int_s^t h_n(u) du \right\} \\ &\quad \cdot \exp \left\{ \sum_{n=1}^\infty a_n \int_s^t h_n(u) du \bar{P}_n \right\} \\ &= \exp \left\{ - \sum_{i=1}^m f(s, t) - \sum_{n=1}^\infty f_n(s, t) \right\} \\ &\quad \cdot \exp \left\{ \sum_{n \in W} f(s, t) \bar{P}_n + \sum_{n=1}^\infty f_n(s, t) \bar{P}_n \right\} \\ &= \exp \left\{ -mf(s, t) - \sum_{n=1}^\infty f_n(s, t) \right\} \\ &\quad \cdot \exp \left\{ mf(s, t) \left(\frac{1}{m} \sum_{n \in W} \bar{P}_n \right) + \sum_{n=1}^\infty f_n(s, t) \bar{P}_n \right\} \end{aligned}$$

Since the P_n 's commute with each other, they commute with $\frac{1}{m} \sum_{n \in W} P_n$. Therefore, the hypotheses of Theorem IV.5 hold and the corollary follows. ■

This corollary can be applied to the constant causative chains of Chapter III to see that whenever $\frac{1}{2}(\bar{P} + \bar{R})$ is ergodic the constant causative chain is also ergodic. Example III.12 shows that it is possible that neither \bar{P} nor \bar{R} is ergodic, but $\frac{1}{2}(\bar{P} + \bar{R})$ is ergodic.

For constant causative chains, one simply has $\int_s^t h_1(u) du = \frac{1}{2}(t^2 - s^2)$ and $\int_s^t h_2(u) du = (t-s)$ and so the condition that $\lim_{t \rightarrow \infty} \int_s^t h_n(u) du = \infty$ for every $s > 0$ is satisfied for such chains. This condition is quite important for ergodicity, as shown by the following theorem.

Theorem IV.8:

Suppose $X(t)$ has intensity matrix given by the finite sum $Q(t) = \sum_{k=1}^n h_k(t) A_k$. If for each $k = 1, 2, \dots, n$, $\int_0^\infty h_k(u) du < \infty$, then $X(t)$ is not ergodic.

Proof:

It will be demonstrated that for any $0 < \epsilon < 1$, one can find an S^* such that for $t > s > S^*$

$$|p_{ij}(s, t) - \delta_{ij}| < \epsilon.$$

Since $P(s, t)$ approaches the identity matrix as s increases, the chain cannot be ergodic.

The proof is by induction on n . Choose $0 < \varepsilon < 1$. For $n=1$, (41) gives

$$\begin{aligned} p_{ij}(s,t) &= \exp\left[-a \int_s^t h(u)du\right] \exp\left[a \int_s^t h(u)du\right] p_{ij} \\ &= \exp\left[-a \int_s^t h(u)du\right] \\ &\quad \cdot \left(\delta_{ij} + \sum_{m=1}^{\infty} \frac{(a \int_s^t h(u)du)^m}{m!} p_{ij}^{(m)} \right). \end{aligned} \quad (42)$$

Let $\eta = \frac{\ln(1+\varepsilon)}{a}$. Since $\int_0^{\infty} h(u)du < \infty$ and $h(t) \geq 0$ there exists an S^* such that for $t \geq s \geq S^*$

$$0 \geq \int_s^t h(u)du \leq \int_s^{\infty} h(u)du \leq \eta.$$

This implies that

$$\begin{aligned} 1 &= \exp\{a \cdot 0\} \leq \exp\left\{a \int_s^t h(u)du\right\} \leq \exp\{a\eta\} = \exp\{\ln(1+\varepsilon)\} \\ &= 1 + \varepsilon \end{aligned}$$

and

$$\begin{aligned} 1 - \varepsilon &\leq 1 - \varepsilon \frac{1}{1+\varepsilon} = \frac{1}{1+\varepsilon} = \exp\{-\ln(1+\varepsilon)\} = \exp\{-a\eta\} \\ &\leq \exp\left\{-a \int_s^t h(u)du\right\} \leq \exp\{-a \cdot 0\} = 1. \end{aligned}$$

Thus, from (42)

$$\begin{aligned}
 \delta_{ij} - \varepsilon &\leq \delta_{ij}(1-\varepsilon) \\
 &\leq (1-\varepsilon)\left(\delta_{ij} + \sum_{m=1}^{\infty} \frac{a^m}{m!} \left(\int_s^t h(u)du\right)^m \bar{p}_{ij}^{(m)}\right) \\
 &\leq \exp\left\{-a \int_s^t h(u)du\right\} \left(\delta_{ij} + \sum_{m=1}^{\infty} \frac{a^m}{m!} \left(\int_s^t h(u)du\right)^m \bar{p}_{ij}^{(m)}\right) \\
 &= p_{ij}(s,t) \\
 &\leq 1 \cdot \left(\delta_{ij} + \sum_{m=1}^{\infty} \left(a \int_s^t h(u)du\right)^m \frac{1}{m!}\right) \\
 &\leq \delta_{ij} + \sum_{m=1}^{\infty} \frac{(a\eta)^m}{m!} = \delta_{ij} + \exp\{a\eta\} - 1 \\
 &\leq \delta_{ij} + 1 + \varepsilon - 1 \\
 &= \delta_{ij} + \varepsilon.
 \end{aligned}$$

Thus, for $n = 1$ $|p_{ij}(s,t) - \delta_{ij}| \leq \varepsilon$ whenever $t > s > S^*$.

Suppose the induction hypothesis is true for $n \leq N$. Choose $0 < \varepsilon < 1$. For $N+1$,

$$\begin{aligned}
 p_{ij}(s,t) &= \exp \left\{ - \sum_{k=1}^{N+1} a_k \int_s^t h_k(u)du \right\} \\
 &\quad \cdot \exp \left\{ \sum_{k=1}^{N+1} a_k \int_s^t h_k(u)du \bar{p}_k \right\}_{ij}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{\ell \in S} \exp \left\{ - \sum_{k=1}^N a_k \int_s^t h_k(u) du \right\} \exp \left\{ \sum_{k=1}^N a_k \int_s^t h_k(u) du \bar{P}_k \right\}_{i\ell} \\
&\quad \cdot \exp \left\{ -a_{N+1} \int_s^t h_{N+1}(u) du \right\} \exp \left\{ a_{N+1} \int_s^t h_{N+1}(u) du \bar{P}_{N+1} \right\}_{\ell j} .
\end{aligned}$$

By the induction hypothesis, one can choose S^* so that for $t \geq s \geq S^*$

both

$$\left| \exp \left\{ - \sum_{k=1}^N a_k \int_s^t h_k(u) du \right\} \exp \left\{ \sum_{k=1}^N a_k \int_s^t h_k(u) du \bar{P}_k \right\}_{ij} - \delta_{ij} \right| < \frac{\varepsilon}{2}$$

and

$$\left| \exp \left\{ -a_{N+1} \int_s^t h_{N+1}(u) du \right\} \exp \left\{ a_{N+1} \int_s^t h_{N+1}(u) du \bar{P}_{N+1} \right\}_{ij} - \delta_{ij} \right| < \frac{\varepsilon}{2} .$$

From Lemma IV.4, $\exp \left\{ - \sum_{k=1}^N a_k \int_s^t h_k(u) du \right\} \exp \left\{ \sum_{k=1}^N a_k \int_s^t h_k(u) du \bar{P}_k \right\}$

is stochastic so

$$\delta_{ij} - \varepsilon = (\delta_{ij} - \varepsilon/2) - \varepsilon/2$$

$$\begin{aligned}
&\leq \exp \left\{ - \sum_{k=1}^N a_k \int_s^t h_k(u) du \right\} \exp \left\{ \sum_{k=1}^N a_k \int_s^t h_k(u) du \bar{P}_k \right\}_{ij} - \frac{\varepsilon}{2} \\
&= \sum_{\ell \in S} \exp \left\{ - \sum_{k=1}^N a_k \int_s^t h_k(u) du \right\} \\
&\quad \cdot \exp \left\{ \sum_{k=1}^N a_k \int_s^t h_k(u) du \bar{P}_k \right\}_{i\ell} (\delta_{\ell j} - \varepsilon/2)
\end{aligned}$$

$$\begin{aligned}
& \leq \sum_{\ell \in S} \exp \left\{ - \sum_{k=1}^N a_k \int_s^t h_k(u) du \right\} \exp \left\{ \sum_{k=1}^N a_k \int_s^t h_k(u) du \bar{P}_k \right\}_{i\ell} \\
& \quad \cdot \exp \left\{ -a_{N+1} \int_s^t h_{N+1}(u) du \right\} \exp \left\{ a_{N+1} \int_s^t h_{N+1}(u) \bar{P}_{N+1} \right\}_{\ell j} \\
& = p_{ij}(s, t)
\end{aligned}$$

$$\begin{aligned}
& \leq \sum_{\ell \in S} \exp \left\{ - \sum_{k=1}^N a_k \int_s^t h_k(u) du \right\} \\
& \quad \cdot \exp \left\{ \sum_{k=1}^N a_k \int_s^t h_k(u) du \bar{P}_k \right\}_{i\ell} (\delta_{\ell j} + \varepsilon/2) \\
& = \exp \left\{ - \sum_{k=1}^N a_k \int_s^t h_k(u) du \right\} \exp \left\{ \sum_{k=1}^N a_k \int_s^t h_k(u) du \bar{P}_k \right\}_{ij} + \frac{\varepsilon}{2} \\
& \leq \delta_{ij} + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
& \leq \delta_{ij} + \varepsilon .
\end{aligned}$$

Thus, for $t \geq s \geq S^*$

$$|p_{ij}(s, t) - \delta_{ij}| < \varepsilon .$$

Clearly $P(s, t)$ is not ergodic since it approaches the identity matrix as s increases. ■

The preceding theorems investigate the ergodicity of $X(t)$. One can also relate the strong ergodicity of the nonhomogeneous, continuous-time chain defined by $Q(t) = \sum_{n=1}^{\infty} h_n(t) A_n$ to that of the corresponding homogeneous, discrete-time \bar{P}_n 's.

Theorem IV.9:

Suppose there exists an n_0 such that $\bar{P}_{n_0} = I + (1/a_{n_0})A_{n_0}$ is strongly ergodic with the constant stochastic matrix L as its limit. If $\int_0^\infty h_{n_0}(u)du = \infty$, then $X(t)$ is strongly ergodic with limit L .

Proof:

Without loss of generality assume $n_0 = 1$. According to Lemma IV.4, $\exp \left\{ -\sum_{n=2}^{\infty} a_n \int_s^t h_n(u)du \right\} \exp \left\{ \sum_{n=2}^{\infty} a_n \int_s^t h_n(u)du \bar{P}_n \right\}$ is stochastic and therefore has norm 1. Thus, (41) gives

$$\begin{aligned}
 \|P(s,t) - L\| &= \left\| \exp \left\{ -\sum_{n=1}^{\infty} a_n \int_s^t h_n(u)du \right\} \exp \left\{ \sum_{n=2}^{\infty} a_n \int_s^t h_n(u)du \bar{P}_n \right\} \right. \\
 &\quad \cdot \exp \left\{ a_1 \int_s^t h_1(u)du (\bar{P}_1 - L) \right\} \Big\| \\
 &= \exp \left\{ -a_1 \int_s^t h_1(u)du \right\} \left\| \exp \left\{ a_1 \int_s^t h_1(u)du (\bar{P}_1 - L) \right\} \right\| \\
 &\leq \exp \left\{ -a_1 \int_s^t h_1(u)du \right\} \sum_{m=0}^{\infty} \frac{1}{m!} (a_1 \int_s^t h_1(u)du)^m \|(\bar{P}_1 - L)^m\|.
 \end{aligned}$$

By Lemma I.11, $(\bar{P}_1)^m L = L$ for any $m \geq 1$ and thus

$(\bar{P}_1 - L)^m = (\bar{P}_1)^m - L$ for $m \geq 1$. Thus,

$$\|P(s,t) - L\|$$

$$\leq \exp\left\{-a_1 \int_s^t h_1(u) du\right\} \cdot \left(\|L\| + \sum_{m=1}^{\infty} \frac{1}{m!} \left(a_1 \int_s^t h_1(u) du\right)^m \|(\bar{P}_1)^m - L\|\right).$$

By the strong ergodicity of \bar{P}_1 , there is an M such that for $m \geq M$

$$\|(\bar{P}_1)^m - L\| < \frac{\varepsilon}{2}.$$

Thus,

$$\|P(s,t) - L\|$$

$$\begin{aligned} &\leq \exp\left\{-a_1 \int_s^t h_1(u) du\right\} \left(1 + \sum_{m=1}^{M-1} \frac{1}{m!} \left(a_1 \int_s^t h_1(u) du\right)^m \right. \\ &\quad \left. + \sum_{m=M}^{\infty} \frac{1}{m!} \left(a_1 \int_s^t h_1(u) du\right)^m \frac{\varepsilon}{2} \right) \\ &\leq 2 \sum_{m=0}^{M-1} \exp\left\{-a_1 \int_s^t h_1(u) du\right\} \frac{1}{m!} \left(a_1 \int_s^t h_1(u) du\right)^m + \frac{\varepsilon}{2}. \end{aligned}$$

By hypothesis, $\lim_{t \rightarrow \infty} \int_s^t h_1(u) du = \infty$. Thus, one can choose T such that for $t \geq T$ and $0 \leq m \leq M-1$,

$$\exp\left\{-a_1 \int_s^t h_1(u) du\right\} \frac{1}{m!} \left(a_1 \int_s^t h_1(u) du\right)^m < \frac{\varepsilon}{4M}.$$

Thus, for $t \geq T$

$$\|P(s,t) - L\|$$

$$< 2 \sum_{m=0}^{M-1} \frac{\varepsilon}{4M} + \frac{\varepsilon}{2} = \varepsilon. \quad \blacksquare$$

Theorem IV.9 assumes the strong ergodicity of one of the \bar{P}_n 's to show $X(t)$ is strongly ergodic. If no \bar{P}_n is strongly ergodic it may be possible, as in Corollary IV.7, to consider instead a finite sum of the \bar{P}_n 's.

Corollary IV.10:

If there exists a finite set $W = \{n_1, n_2, \dots, n_m\}$ of positive integers such that $\frac{1}{m} \left(\sum_{n \in W} \bar{P}_n \right)$ is strongly ergodic with limit L and for each $n \in W$ $\int_0^\infty h_n(u) du = \infty$, then $X(t)$ is strongly ergodic with limit L .

The proof is similar to that of Corollary IV.7.

The major results from Chapter III have now been extended to this new case where $Q(t)$ is defined by a series of proportional intensity matrices. Chains defined by a slightly different series are now considered.

Because of the difficulty in finding a sequence of matrices which commute, it is tempting to try to replace the sequence of intensity matrices $\{A_n\}_{n=1}^\infty$ by the sequence $\{A^n\}_{n=1}^\infty$, the powers of A . These powers clearly commute with each other. The nonhomogeneous chain might then be defined by $Q(t) = \sum_{n=1}^\infty h_n(t) A^n$. Unfortunately, while A is an intensity matrix, A^n need not be one. $Q(t)$ defined in this way may

not be an intensity matrix and therefore may not define a Markov chain.

The following example is a case where this happens.

Example IV.11:

$$\text{Let } A = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}. \quad A^2 = (-3)A.$$

$$\text{Let } h_n(t) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (1/6)^n & \text{if } n \text{ is odd} \end{cases}.$$

Notice that $h_n(t) \| A^n \| \leq (1/6)^n 4^n = (2/3)^n$.

$$\begin{aligned} Q(t) &= \sum_{n=1}^{\infty} h_n(t) A^n = \sum_{n=0}^{\infty} h_{2n+1}(t) A^{2n+1} + \sum_{n=1}^{\infty} h_{2n}(t) A^{2n} \\ &= 0 + \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{2n} (-3)^{2n-1} A \\ &= \left(-\frac{1}{3}\right) A \sum_{n=1}^{\infty} \left(\left(-\frac{3}{6}\right)^2\right)^n = \left(-\frac{1}{3}\right) A \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n \\ &= \left(-\frac{1}{3}\right) A \left(\frac{1}{1 - 1/4} - 1\right) = \left(-\frac{1}{3}\right) A \left(\frac{4}{3} - \frac{3}{3}\right) = \left(-\frac{1}{9}\right) A \\ &= \begin{pmatrix} \frac{1}{9} & -\frac{1}{9} \\ -\frac{2}{9} & \frac{2}{9} \end{pmatrix} \end{aligned}$$

which is not an intensity matrix.

However, if one also assume $h_n(t) = (h(t))^n$, then one can get an intensity matrix.

Theorem IV.12:

Suppose A is an intensity matrix and $h(t)$ is a nonnegative function. If $h(t)\|A\| < 1$ for every t , then

$$Q(t) = \sum_{n=1}^{\infty} (h(t))^n A^n$$

is an intensity matrix.

Proof:

For fixed t , let

$$D = h(t)A(1 - h(t)\|A\|) + h(t)\|A\|I \quad (43)$$

by the properties of intensity matrices and Corollary I.9,

$$0 \leq h(t)a_{ij} \leq -h(t)a_{ii} \leq \frac{1}{2}h(t)\|A\| < \frac{1}{2}. \quad (44)$$

Thus, since $d_{ii} = h(t)a_{ii}(1 - h(t)\|A\|) + h(t)\|A\|$,

$$-\frac{1}{2}h(t)\|A\|(1 - h(t)\|A\|) + h(t)\|A\| \leq d_{ii} \leq 0 + h(t)\|A\|$$

which implies

$$0 \leq \frac{1}{2}h(t)\|A\| + \frac{1}{2}(h(t)\|A\|)^2 \leq d_{ii} \leq h(t)\|A\|.$$

Since $d_{ij} = h(t)a_{ij}(1 - h(t)\|A\|)$ for $i \neq j$, (44) implies

$$0 \leq d_{ij} \leq \frac{1}{2}h(t)\|A\|(1 - h(t)\|A\|) \leq h(t)\|A\| \quad \text{for } i \neq j.$$

Consider $\sum_{n=1}^{\infty} D^n$. Since

$$\begin{aligned} \|D\| &\leq h(t)\|A\| (1 - h(t)\|A\|) + h(t)\|A\| \\ &= h(t)\|A\| (2 - h(t)\|A\|) < 1, \end{aligned}$$

$\sum_{n=1}^{\infty} D^n$ converges. Also since each element of D is nonnegative, each element of $\sum_{n=1}^{\infty} D^n$ is nonnegative.

Next, it is shown by induction that

$$\sum_{j \in S} (D^n)_{ij} = (h(t))^n \|A\|^n \quad \text{for each } i \in S \text{ and } n \geq 1. \quad (45)$$

For $n = 1$, from (43)

$$\begin{aligned} \sum_{j \in S} d_{ij} &= \sum_{j \in S} (h(t)a_{ij}(1 - h(t)\|A\|) + h(t)\|A\|\delta_{ij}) \\ &= 0 + h(t)\|A\|. \end{aligned}$$

Assume (45) holds for $n \leq N$.

$$\begin{aligned} \sum_{j \in S} (D^{N+1})_{ij} &= \sum_{j \in S} \sum_{k \in S} d_{ik} (D^N)_{kj} = \sum_{k \in S} d_{ik} \sum_{j \in S} (D^N)_{kj} \\ &= \sum_{k \in S} d_{ik} (h(t))^N \|A\|^N = (h(t))^{N+1} \|A\|^{N+1}. \end{aligned}$$

Consider

$$\begin{aligned}
\sum_{n=1}^{\infty} D^n &= \sum_{n=1}^{\infty} (h(t)A(1 - h(t)\|A\|) + h(t)\|A\|I)^n \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^n \binom{n}{k} (h(t))^k A^k (1 - h(t)\|A\|)^k (h(t))^{n-k} \|A\|^{n-k} \\
&= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \binom{n}{k} (h(t))^k A^k (1 - h(t)\|A\|)^k (h(t))^{n-k} \|A\|^{n-k} \\
&\quad + \sum_{n=1}^{\infty} \binom{n}{0} I (h(t))^n \|A\|^n \\
&= \sum_{k=1}^{\infty} \frac{(h(t))^k A^k (1 - h(t)\|A\|)^k}{k!} \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} (h(t))^{n-k} \|A\|^{n-k} \\
&\quad + \left(\frac{1}{1 - h(t)\|A\|} - 1 \right) I \\
&= \sum_{k=1}^{\infty} \frac{(h(t))^k A^k (1 - h(t)\|A\|)^k}{k!} k! (1 - h(t)\|A\|)^{-(k+1)} \\
&\quad + \left(\frac{1 - 1 + h(t)\|A\|}{1 - h(t)\|A\|} \right) I \\
&= \left(\frac{1}{1 - h(t)\|A\|} \right) \sum_{k=1}^{\infty} (h(t))^k A^k + \left(\frac{h(t)\|A\|}{1 - h(t)\|A\|} \right) I .
\end{aligned}$$

$$\text{Thus, } Q(t) = \sum_{k=1}^{\infty} (h(t))^k A^k$$

$$= (1 - h(t)\|A\|) \sum_{n=1}^{\infty} D^n - h(t)\|A\|I .$$

Since the elements of $\sum_{n=1}^{\infty} D^n$ are nonnegative, the off-diagonal elements of $Q(t)$ are nonnegative. By (45), $(D^n)_{ii} \leq (h(t))^n \|A\|^n$. Thus

$$\begin{aligned} q_{ii}(t) &= (1 - h(t)\|A\|) \sum_{n=1}^{\infty} (D^n)_{ii} - h(t)\|A\| \\ &\leq (1 - h(t)\|A\|) \sum_{n=1}^{\infty} (h(t))^n \|A\|^n - h(t)\|A\| \\ &= (1 - h(t)\|A\|) \left(\frac{1}{1 - h(t)\|A\|} - 1 \right) - h(t)\|A\| \\ &= 1 - 1 + h(t)\|A\| - h(t)\|A\| \\ &= 0. \end{aligned}$$

And finally

$$\begin{aligned} \sum_{j \in S} q_{ij}(t) &= \sum_{j \in S} \left((1 - h(t)\|A\|) \sum_{n=1}^{\infty} (D^n)_{ij} - h(t)\|A\| \delta_{ij} \right) \\ &= (1 - h(t)\|A\|) \sum_{n=1}^{\infty} \sum_{j \in S} (D^n)_{ij} - h(t)\|A\| \end{aligned}$$

which by (45) is

$$\begin{aligned} &= (1 - h(t)\|A\|) \sum_{n=1}^{\infty} (h(t))^n \|A\|^n - h(t)\|A\| \\ &= (1 - h(t)\|A\|) \left(\frac{1}{1 - h(t)\|A\|} - 1 \right) - h(t)\|A\| \\ &= 0. \quad \blacksquare \end{aligned}$$

It is possible to show that when $Q(t) = \sum_{n=1}^{\infty} (h(t))^n A^n$ as in Theorem

IV.12, $P(s,t) = \exp \left\{ \sum_{n=1}^{\infty} \int_s^t (h(u))^n du A^n \right\}$. The proof is essentially

the same as that of Theorem IV.1.

One might guess, in light of the results found for

$Q(t) = \sum_{n=1}^{\infty} h_n(t) A_n$, that if $\bar{P} = I + A/a$ is ergodic and $\int_s^{\infty} h(u) du = \infty$

for every $s \geq 0$ then the chain defined by $Q(t) = \sum_{n=1}^{\infty} (h(t))^n A^n$ is

also ergodic. This is in fact the case.

Lemma IV.13: If $Q(t) = \sum_{n=1}^{\infty} (h(t))^n A^n$ where $h(t) \| A \| < 1$ for any

$t \geq 0$ then one can choose $a > 0$ so that for $\bar{P} = I + A/a$

$$P(s,t) = \exp \left\{ - \sum_{k=1}^{\infty} \int_s^t \frac{(ah(u))^k}{(1+ah(u))^{k+1}} du \right\} \\ \cdot \exp \left\{ \sum_{k=1}^{\infty} \int_s^t \frac{(ah(u))^k}{(1+ah(u))^{k+1}} du \bar{P}^k \right\}.$$

Proof:

By Lemma IV.12, $Q(t)$ is an intensity matrix. Since $h(t) \| A \| < 1$, one can choose a so that $\sup_{i \in S} \{ |a_{ii}| \} < a$ and $h(t)a < 1$ for any $t \geq 0$. Let $\bar{P} = I + (1/a)A$ so that $A = a(\bar{P} - I)$. Thus,

$$\begin{aligned}
Q(t) &= \sum_{n=1}^{\infty} (h(t))^n A^n = \sum_{n=1}^{\infty} (h(t))^n (a(\bar{P} - I))^n \\
&= \sum_{n=1}^{\infty} (h(t))^n a^n (\bar{P} - I)^n \\
&= \sum_{n=1}^{\infty} (ah(t))^n \sum_{k=0}^n \binom{n}{k} \bar{P}^k (-I)^{n-k} \\
&= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} (ah(t))^n \frac{n!}{k!(n-k)!} \bar{P}^k (-I)^{n-k} \\
&\quad + \sum_{n=1}^{\infty} (ah(t))^n \binom{n}{0} (-I)^n \\
&= \sum_{k=1}^{\infty} \frac{(ah(t))^k}{k!} \bar{P}^k \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} (-ah(t))^{n-k} \\
&\quad + \sum_{n=1}^{\infty} (-ah(t))^n I \\
&= \sum_{k=1}^{\infty} \frac{(ah(t))^k}{k!} \bar{P}^k k! (1 + ah(t))^{-k-1} \\
&\quad + \left(\frac{1}{1 + ah(t)} - 1 \right) I \\
&= \sum_{k=1}^{\infty} \frac{(ah(t))^k}{(1 + ah(t))^{k+1}} \bar{P}^k + \left(\frac{1 - 1 - ah(t)}{1 + ah(t)} \right) I \\
&= \sum_{k=1}^{\infty} \frac{(ah(t))^k}{(1 + ah(t))^{k+1}} \bar{P}^k - \left(\frac{ah(t)}{1 + ah(t)} \right) I
\end{aligned}$$

But $0 \leq ah(t) < 1$ implies that $1 \leq 1 + ah(t) < 2$ which implies that

$1 \geq \frac{1}{1 + ah(t)} > 1/2$. Thus,

$$1 > ah(t) > \frac{ah(t)}{1 + ah(t)} > \frac{1}{2} ah(t) > 0 .$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(ah(t))^k}{(1 + ah(t))^{k+1}} &= \frac{1}{(1 + ah(t))} \sum_{k=1}^{\infty} \left(\frac{ah(t)}{1 + ah(t)} \right)^k \\ &= \left(\frac{1}{1 + ah(t)} \right) \left(\frac{1}{1 - \frac{ah(t)}{1 + ah(t)}} - 1 \right) \\ &= \left(\frac{1}{1 + ah(t)} \right) \left(\frac{1 + ah(t)}{1 + ah(t) - ah(t)} - 1 \right) \\ &= \left(\frac{1}{1 + ah(t)} \right) (1 + ah(t) - 1) \\ &= \frac{ah(t)}{1 + ah(t)} . \end{aligned}$$

Hence,

$$Q(t) = \sum_{k=1}^{\infty} \frac{(ah(t))^k}{(1 + ah(t))^{k+1}} \bar{P}^k - \sum_{k=1}^{\infty} \frac{(ah(t))^k}{(1 + ah(t))^{k+1}} I .$$

Therefore,

$$\begin{aligned} P(s, t) &= \exp \left\{ \int_s^t Q(u) du \right\} \\ &= \exp \left\{ - \sum_{k=1}^{\infty} \int_s^t \frac{(ah(u))^k}{(1 + ah(u))^{k+1}} du I \right\} \\ &\quad \cdot \exp \left\{ \sum_{k=1}^{\infty} \int_s^t \frac{(ah(u))^k}{(1 + ah(u))^{k+1}} du \bar{P}^k \right\} \end{aligned}$$

$$\begin{aligned}
&= \left[I + \sum_{m=1}^{\infty} \frac{1}{m!} \left(- \sum_{k=1}^{\infty} \int_s^t \frac{(ah(u))^k}{(1+ah(u))^{k+1}} du \right)^m I \right] \\
&\quad \cdot \exp \left\{ \sum_{k=1}^{\infty} \int_s^t \frac{(ah(u))^k}{(1+ah(u))^{k+1}} du \bar{P}^k \right\} \\
&= \left[I + \sum_{m=1}^{\infty} \frac{1}{m!} \left(- \sum_{k=1}^{\infty} \int_s^t \frac{(ah(u))^k}{(1+ah(u))^{k+1}} du \right)^m I \right] \\
&\quad \cdot \exp \left\{ \sum_{k=1}^{\infty} \int_s^t \frac{(ah(u))^k}{(1+ah(u))^{k+1}} du \bar{P}^k \right\} \\
&= I \exp \left\{ - \sum_{k=1}^{\infty} \int_s^t \frac{(ah(u))^k}{(1+ah(u))^{k+1}} du \right\} \\
&\quad \cdot \exp \left\{ \sum_{k=1}^{\infty} \int_s^t \frac{(ah(u))^k}{(1+ah(u))^{k+1}} du \bar{P}^k \right\} \\
&= \exp \left\{ - \sum_{k=1}^{\infty} \int_s^t \frac{(ah(u))^k}{(1+ah(u))^{k+1}} du \right\} \\
&\quad \cdot \exp \left\{ \sum_{k=1}^{\infty} \int_s^t \frac{(ah(u))^k}{(1+ah(u))^{k+1}} du \bar{P}^k \right\} . \quad \blacksquare
\end{aligned}$$

Theorem IV.14:

Suppose $Q(t) = \sum_{n=1}^{\infty} (h(t))^n A^n$ where $h(t) \| A \| < 1$. If \bar{P} (as found in Lemma IV.13) is ergodic and $\int_0^{\infty} h(u) du = \infty$, then the nonhomogeneous chain defined by $Q(t)$ is ergodic.

Proof:

Since \bar{P} is ergodic let $\pi = (\pi_j)_j \in S$ where $\lim_{n \rightarrow \infty} \bar{p}_{ij}^{(n)} = \pi_j$.

$$\text{Let } R(s,t) = \exp \left\{ - \sum_{n=2}^{\infty} \int_s^t \frac{(ah(u))^n}{(1+ah(u))^{n+1}} du \right\} \\ \cdot \exp \left\{ \sum_{n=2}^{\infty} \int_s^t \frac{(ah(u))^n}{(1+ah(u))^{n+1}} du \bar{P}^n \right\}.$$

By Lemma IV.4, $R(s,t)$ is stochastic and since \bar{P}^n commutes with \bar{P} for $n \geq 2$, $R(s,t)$ commutes with \bar{P} . Thus, $\pi R(s,t) = \pi$ by Lemma I.22.

Consider

$$\exp \left\{ - \int_s^t \frac{ah(u)}{(1+ah(u))^2} du \right\} \\ \cdot \left[\sum_{k \in S} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\int_s^t \frac{ah(u)}{(1+ah(u))^2} du \right)^m (p_{ik}^{(m)} - \pi_k) r_{kj}(s,t) \right] \\ = \exp \left\{ - \int_s^t \frac{ah(u)}{(1+ah(u))^2} du \right\} \\ \cdot \left[\sum_{k \in S} \exp \left\{ \int_s^t \frac{ah(u)}{(1+ah(u))^2} du \right\} \bar{P} \right]_{ik} r_{kj}(s,t) \\ - \sum_{k \in S} \exp \left\{ \int_s^t \frac{ah(u)}{(1+ah(u))^2} du \right\} \pi_k r_{kj}(s,t) \Bigg] \\ = p_{ij}(s,t) - \pi_j.$$

Hence,

$$\begin{aligned}
 & |p_{ij}(s, t) - \pi_j| \\
 & \leq \exp \left\{ - \int_s^t \frac{ah(u)}{(1 + ah(u))^2} du \right\} \\
 & \quad \cdot \left[\sum_{k \in S} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\int_s^t \frac{ah(u)}{(1 + ah(u))^2} du \right)^m |p_{ik}^{(m)} - \pi_k| r_{kj}(s, t) \right] \\
 & \leq \exp \left\{ - \int_s^t \frac{ah(u)}{(1 + ah(u))^2} du \right\} \\
 & \quad \cdot \left[\sum_{k \in S} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\int_s^t \frac{ah(u)}{(1 + ah(u))^2} du \right)^m |p_{ik}^{(m)} - \pi_k| \right] 1.
 \end{aligned}$$

Since $\bar{P}^n \rightarrow \pi$, there exists an M such that for $m \geq M$

$$\sum_{k \in S} |p_{ik}(s, t) - \pi_k| < \varepsilon/2$$

by Theorem I.5. Thus,

$$\begin{aligned}
 & |p_{ij}(s, t) - \pi_j| \\
 & \leq \exp \left\{ - \int_s^t \frac{ah(u)}{(1 + ah(u))^2} du \right\} \\
 & \quad \cdot \left[\sum_{m=0}^{M-1} \frac{1}{m!} \left(\int_s^t \frac{ah(u)}{(1 + ah(u))^2} du \right)^m 2 \right. \\
 & \quad \left. + \sum_{m=M}^{\infty} \frac{1}{m!} \left(\int_s^t \frac{ah(u)}{(1 + ah(u))^2} du \right)^m \varepsilon/2 \right]
 \end{aligned}$$

$$\leq 2 \sum_{m=1}^{M-1} \frac{1}{m!} \exp \left\{ - \int_s^t \frac{ah(u)}{(1+ah(u))^2} du \right\} \left(\int_s^t \frac{ah(u)}{(1+ah(u))^2} du \right)^m + \varepsilon/2 .$$

Since $0 \leq ah(u) < 1$, $1 \geq \frac{1}{(1+ah(u))^2} \geq 1/4$. Thus,

$$\int_0^\infty \frac{ah(u)}{(1+ah(u))^2} du \geq 1/4 \int_0^\infty ah(u) du = \infty .$$

Therefore, one can find T such that for $t \geq T$ and

$$m = 0, 1, 2, \dots, M-1$$

$$\exp \left\{ - \int_s^t \frac{ah(u)}{(1+ah(u))^2} du \right\} \frac{1}{m!} \left(\int_s^t \frac{ah(u)}{(1+ah(u))^2} du \right)^m \leq \frac{\varepsilon}{4M} .$$

Thus, for $t \geq T$,

$$\begin{aligned} & |p_{ij}(s, t) - \pi_j| \\ & \leq 2 \sum_{m=0}^{M-1} \frac{\varepsilon}{4M} + \varepsilon/2 \\ & = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon . \quad \blacksquare \end{aligned}$$

These results can be extended to strong ergodicity in the same manner as Theorem IV.9.

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