Inverse Sturm-Liouville problems using multiple spectra

by

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ABSTRACT

An eigenvalue problem for a Sturm-Liouville differential operator containing a parameter function and being studied on a given domain is a model for the infinitesimal, vertical vibration of a string of negligible mass, with the ends subject to various constraints. The parameter function of the Sturm-Liouville operator encodes information about the string (its density), and the eigenvalues of the same operator are the squares of the natural frequencies of oscillation of the string. In an inverse Sturm-Liouville problem one has knowledge about the spectral data of the operator and tries to recover the parameter function of the same operator. This thesis deals with the recovery of the parameter function of a Sturm-Liouville operator from knowledge of three sets of eigenvalues. The recovery is achieved theoretically and numerically in two different situations: a) when the three sets correspond respectively to the vibration of the whole string fixed only at the end points, and the vibrations of each individual piece obtained by fixing the string at an interior node; b) when the three sets correspond respectively to the vibration of the whole string fixed only at the end points, and the vibrations of each individual piece obtained by attaching the string at an interior node to a spring with a known stiffness constant. Situations when existence or uniqueness of the parameter function is lost are also presented.

CHAPTER 1. INTRODUCTION

1.1 A short history of direct and inverse Sturm-Liouville problems

In general, by an inverse spectral problem one means: 'infer properties of a body from its natural frequencies of vibration'. Such problems occur

- in determining the structure of the earth from the vibrations induced by the earthquakes,
- in designing structures, such as the space station, where certain resonant frequencies are to be avoided.

In one-space dimension, direct and inverse spectral problems have been mostly studied for special kinds of second and fourth order differential equations (e.g. $-u'' + q(x)u = \lambda u$, $(a(s)\omega)' + \lambda(a(s)\omega) = 0$, $u'''' + (A(x)u')' + B(x)u = \lambda u$) associated with various boundary conditions. In multi-space dimensions, the eigenvalue problem

$$\begin{cases} -\Delta u + q(x)u = \lambda u, \text{ on a bounded domain } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

for the Schrödinger operator received the most attention.

An important category of direct and inverse spectral problems are the Sturm-Liouville problems. A Sturm-Liouville differential operator with potential function $q \in L^2(a, b)$ has the canonical form

$$L^{(q)}u(x) = -u''(x) + q(x)u(x), \ x \in [a, b].$$

There are several pairs of boundary conditions associated with this differential operator. In the most general form of separated boundary conditions, we write the pair of boundary conditions as:

$$u'(a) - hu(a) = 0 = u'(b) + Hu(b).$$

Here h and H are positive, real numbers and are allowed to be ∞, with the understanding that $h = \infty$ means $u(a) = 0$, and $H = \infty$ means $u(b) = 0$, respectively. The boundary conditions

$$u(a) = 0 = u(b)$$

are called Dirichlet boundary conditions. Other names are associated with other boundary conditions (e.g. Dirichlet-Neumann for $u(a) = 0 = u'(b)$).

By a *direct* Sturm-Liouville problem with Dirichlet boundary conditions, one means the problem of finding the eigenpairs of the Sturm-Liouville differential operator $L^{(q)}$ with domain $D(L^{(q)}) = \{u \in H^2(a,b) | u(a) = 0 = u(b)\}$, knowing its potential function q.

By an *inverse* Sturm-Liouville problem, one means the problem of finding the potential function q from knowledge of the spectral data of the Sturm-Liouville differential operator $L^{(q)}$. The spectral data can take various forms giving rise to various inverse spectral problems. For example, the spectral data can be two sequences of numbers expected to be the Dirichlet eigenvalues and Dirichlet-Neumann eigenvalues of the Sturm-Liouville operator $L^{(q)}$, or the spectral data can be again two sequences of numbers, expected this time to be the Dirichlet eigenvalues, say $\{\lambda_n\}_{n\geq 1}$ and the sequence of L^2 norm of the rescaled Dirichlet eigenfunctions of the Sturm-Liouville operator $L^{(q)}$ corresponding to these eigenvalues (i.e. $\{||\frac{u_n}{u_n'(0)}||_2\}_{n\geq 1}$, where $u_n(x)$ is the Dirichlet eigenfunction corresponding to the Dirichlet eigenvalue λ_n).

Above, we used the notation $H^2(a,b)$ for the Hilbert space

$$\{u \in C^1[a,b] | u'(x) = c + \int_a^x v(s)ds, \text{ for some } c \in \mathbb{R} \text{ and some } v \in L^2(a,b)\}.$$

with the norm:

$$||u||_{H^2(a,b)} = \sqrt{||u||^2_{L^2(a,b)} + ||u'||^2_{L^2(a,b)} + ||u''||^2_{L^2(a,b)}}.$$

Similarly, the Hilbert space $H^1(a,b)$ is defined as

$$H^1(a,b) = \{u \in C[a,b] | u(x) = c + \int_a^x v(s)ds, \text{ for some } c \in \mathbb{R} \text{ and some } v \in L^2(a,b)\}.$$

These two spaces will be used in Chapter 3.

The theory of direct Sturm-Liouville problems started around 1830's in the independent works of Sturm and Liouville. There is however a modern approach: with complex analysis and compact self-adjoint operators theory one can prove the existence of the eigenpairs of the Sturm-Liouville operator and obtain useful characterization of the eigenpairs: countably many eigenvalues, completeness of the set of corresponding eigenfunctions, asymptotic formulas of eigenvalues and eigenfunctions, etc.

The inverse Sturm-Liouville theory originated in a 1929 paper of Ambartsumyan. If $q(x) = 0$ a.e. on $[0,1]$ in the Sturm-Liouville problem

$$\begin{cases} -u''(x) + q(x)u(x) = \lambda u(x), \ x \in (0,1) \\ u'(0) = 0 = u'(1), \end{cases}$$

then all the eigenvalues λ's are the elements of the set $\{(n\pi)^2 | n \geq 0\}$. Ambartsumyan [1] showed the reverse: if $\{(n\pi)^2 | n \geq 0\}$ are all the eigenvalues of the above Sturm-Liouville problem, then $q = 0$ a.e. on $[0,1]$. This led to the natural speculation that one can recover the coefficient function q from only one sequence of eigenvalues. This conjecture turned out to be false. In general, a single spectrum is not enough to uniquely determine the function q. Borg [3] in 1946 gave the first definitive result about uniqueness of coefficient function q (in some papers, this is called potential). Uniqueness results under various hypotheses (i.e. when various kinds of spectral data are known) followed later due to the fundamental paper of Gelfand and Levitan [7], 1951. In the study of Sturm-Liouville problems they introduced the use of the so-called Gelfand-Levitan transformation operator. This is a Volterra type integral operator connecting the solution to one initial value problem for the Sturm-Liouville differential equation with the solution to another initial value problem of similar form. For example, if $v(x)$ solves

$$\begin{cases} -v'' + p(x)v = \lambda v, \ \text{in } (0,1) \\ v(0) = 0 \\ v'(0) = \alpha, \end{cases}$$

and $u(x)$ solves a problem of the same type but with $p(x)$ replaced by $q(x)$, then the two solutions can be related as follows:

$$u(x) = v(x) + \int_0^x K(x,t;p,q)v(t)dt, \ x \in [0,1],$$

where the function $K(x,t;p,q)$ is the solution to a precise hyperbolic boundary value problem (see [13, Theorem 4.18, page 154]). The right hand side of the identity above defines the Gelfand-Levitan transformation operator. In solving the inverse Sturm-Liouville problem, proving the existence of the coefficient function q turned out to be a much more delicate issue than it was proving its uniqueness. It was not until 1968 that a complete resolution of necessary and sufficient conditions on the spectra for the existence of an L^2 potential q giving rise to those spectral values was proved by Levitan. Other proofs of existence can be found in several 1980's papers of Trubowitz and co-workers. Another important question was the following: How well can one numerically reconstruct the potential q when only a finite amount of spectral data is known? Gelfand and Levitan [7], Hochstadt [11], Rundell and Sacks [20], [21] are some of the papers where numerical algorithms for this purpose were developed.

1.2 The work of Pivovarchik 1999

An important reference for this thesis is a 1999 paper of V.N. Pivovarchik [16]. In [16] the author studies an inverse Sturm-Liouville problem using three spectra: given three sequences of real numbers with well designed properties $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$, find a real valued, $L^2(0,a)$ function q such that these three sequences are the Dirichlet eigenvalues corresponding respectively to the following three boundary value problems for a Sturm-Liouville differential equation:

$$\begin{cases} -u'' + q(x)u = \lambda u, \text{ in } (0,a) \\ u(0) = 0 = u(a), \end{cases}$$

$$\begin{cases} -v'' + q(x)v = \mu v, \text{ in } (0,\frac{a}{2}) \\ v(0) = 0 = v(\frac{a}{2}), \end{cases}$$

$$\begin{cases} -w'' + q(x)w = \nu w, \text{ in } (\frac{a}{2},a) \\ w(\frac{a}{2}) = 0 = w(a). \end{cases}$$

Pivovarchik proved existence and uniqueness of the solution to the above inverse three spectra problem, but his approach is rather theoretical and not implementable on a computer.

1.3 The contributions of this thesis

One of the main contributions of this thesis is to give an alternative, simpler and computer-implementable proof for the same inverse spectral problem. The essential difference is that while Pivovarchik [16] constructs two potentials, q_1 defined on $(0, \frac{a}{2})$ and q_2 defined on $(\frac{a}{2}, a)$ using for each potential one pair of sequences of numbers, where the two sequences of each pair are expected to be two sets of eigenvalues, here two potentials are constructed, q_1 defined on $(0, \frac{a}{2})$ and $\tilde{q}_2$ defined also on $(0, \frac{a}{2})$ using for each potential one pair of sequences of numbers, but this time each pair contains sequences that are expected to be one sequence of eigenvalues and one sequence of norming constants in the sense of Pöschel-Trubowitz. Then $\tilde{q}_2$ is reflected about the line $x = \frac{a}{2}$ to produce q_2 on $(\frac{a}{2}, a)$ which is finally pasted together with q_1 to give the complete q on $(0, a)$. The existence part of the proof is a constructive one, suitable for numerical implementation and it only requires notions of the direct Sturm-Liouville problem on a finite interval and some knowledge of entire functions of exponential type and functions of sine type from complex analysis. The uniqueness part of the proof is based only on the uniqueness of a solution-pair to an overdetermined hyperbolic problem.

Some background material on the mathematical modeling of a vibrating string is given in Chapter 2. The existence and uniqueness mentioned above, along with the constructibility of the potential q, are discussed in Chapter 3. Situations where existence or uniqueness of the potential q is lost are presented in Chapter 4. Two generalizations are discussed in Chapter 5: the case of the interval $[0, a]$ broken at an arbitrary interior node a_0 where a boundary condition of the same type as for the main case (i.e. Dirichlet boundary condition) is prescribed, and the case of the interval $[0, a]$ broken at an arbitrary interior node a_0 where a mixed boundary condition (i.e. Robin boundary condition) is prescribed. Complete explanations about implementing the algorithm for reconstructing such a potential from three finite sets of real numbers are presented in Chapter 6 and some numerical results are given in Chapter 7.

Other contributions of this thesis include the proofs of the Auxiliary results in Chapter 3 and Chapter 5, and the proof of Lemma 2 in Appendix.

1.4 A motivation for the inverse three spectra problem

The study of reconstructing a potential q from three sequences of real numbers is motivated by a practical situation: an inaccessible string, fixed at the end-points and of negligible mass, is set into infinitesimal vibrations. To simplify, we assume that the string vibrates only vertically. The first few of its countable many frequencies of oscillation are measured by a special device. Assume next that the string is clamped at its mid-point. Then each half of the string is set into vibrations with its own frequencies. Again, the first few frequencies of oscillation of each half of the string are measured. From these three sets of real numbers obtain information about the string (for example, its density). The potential function q which appears in the Sturm-Liouville problem we are studying contains such information. Some details about how the potential function q of a Sturm-Liouville problem with Dirichlet boundary conditions encodes information about the vibrating string fixed at the end-points are presented in Chapter 2.

Note that in theory three infinite countable sets of real numbers are needed to reconstruct q, but in practice only finite sets are available to us. Even so, numerically we can obtain an approximation of such q.

1.5 Inverse problems on graphs

Another motivation for the stated problem comes from the theory of inverse eigenvalue problems on graphs. Consider a graph G defined by the edges e_j, $j = 1, \ldots, N$ and vertexes v_k, $k = 1, \ldots, M$. On each edge e_j a potential q_j is defined, and vibrations of the edge are specified by u_j defined on e_j. Conditions at the vertexes are of two types. At an external vertex v_k, i.e. a vertex contained by only one edge, a standard boundary condition is specified, such as $u_j(v_k) = 0$ for definiteness. If v_k is a vertex where edges e_j, $j \in J$ meet, then the so-called Kirchhoff conditions must be satisfied,

$$\begin{aligned} &u_{j_1}(v_k) = u_{j_2}(v_k),\ j_1, j_2 \in J \\ &\sum_{j \in J} u'(v_k) = 0,\ k = 1, \ldots, M \end{aligned}$$

where the derivative is in the direction toward the vertex. The eigenvalue problem thus consists in seeking λ such that

$$u_j'' + (\lambda - q_j)u_j = 0, \text{ on } e_j \tag{1.5.1}$$

together with the above boundary and matching conditions.

If we specialize to the simplest nontrivial graph, consisting of 2 edges meeting at one vertex, regard the two edges as being $e_1 = (0, a_0)$, $e_2 = (a_0, a)$, take the boundary conditions at the exterior vertexes 0, a of Dirichlet type, then the matching conditions at a_0 amount to the requirement that u, u' are continuous, where $u = u_j$ on e_j may be regarded as a function on $[0, a]$. It then follows immediately that u is simply an H^2 solution of $u'' + (\lambda - q)u = 0$ on $[0, a]$ where $q = q_j$ on e_j.

Thus, the three spectrum inverse problem may be viewed as that of determining the potential on each edge given the Dirichlet spectrum for each edge ($\{\mu_n, \nu_n\}$) plus the spectrum for the entire graph ($\{\lambda_n\}$). This formulation may clearly be extended to more complicated graphs, in which case the data would always consist of $N + 1$ spectra, one for each edge e_j and one more for the entire graph. Such inverse spectral problems in more complicated special cases have been studied by Pivovarchik [17]. A variety of other inverse eigenvalue problems for graphs have also been considered, e.g. Yurko [24], Belishev [2], Freiling and Yurko [5].

CHAPTER 2. MATHEMATICAL MODELING OF THE VIBRATING STRING FIXED AT THE END POINTS

In this chapter we explain how the vibrating string fixed at the end-points gives rise to a Sturm-Liouville problem with Dirichlet boundary conditions. We also present the physical interpretation of the coefficient function in the Sturm-Liouville differential operator and of the eigenvalues of the same operator.

2.1 The vibrating string and one type of Sturm-Liouville equation

Consider a string fixed at the end-points $\xi = 0$ and $\xi = \xi_0$ with negligible weight compared with the tension in string, which at time $\tau = 0$ is at equilibrium (i.e. it lies along the line $\eta = 0$). We assume that the tension in the string is constant and the string is set into infinitesimal, vertical vibrations from its equilibrium position. See Figure 2.1.

Let $U(\xi, \tau)$ describe the position from equilibrium (line $\eta = 0$) at time $\tau > 0$ of a point-particle on the string which at $\tau = 0$ is at location ξ. Then Newton second law of motion for an infinitesimal element of the string with length $\delta\xi$ takes the form:

$$(\rho(\xi)\delta\xi)\frac{\partial^2 U}{\partial \tau^2}(\xi, \tau) = T\sin\theta(\xi + \delta\xi, \tau) - T\sin\theta(\xi, \tau), \text{ on } \eta\text{-axis}, \tag{2.1.1}$$

since the weight of the string element was neglected, and

$$0 = T\cos\theta(\xi + \delta\xi, \tau) - T\cos\theta(\xi, \tau), \text{ on } \xi\text{-axis}, \tag{2.1.2}$$

since we assumed only vertical vibrations, so there is no horizontal acceleration. Here

$$\frac{\partial^2 U}{\partial \tau^2}(\xi, \tau)$$

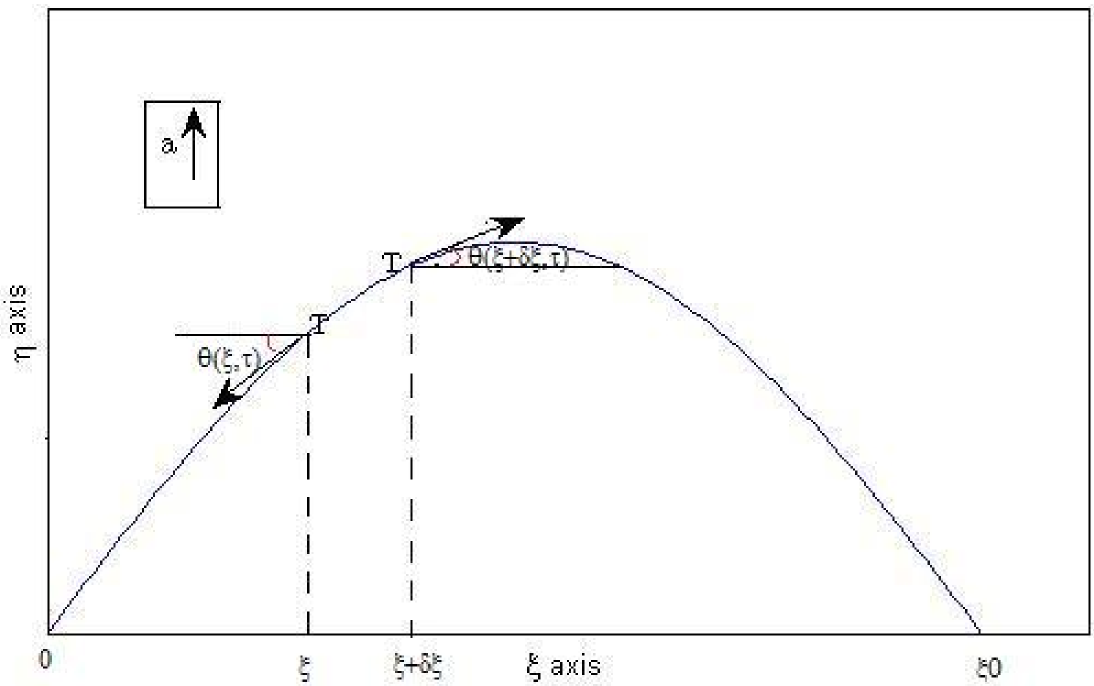

Figure 2.1 The infinitesimal, vertical vibrations of a string fixed at the end-points

is the acceleration of the point particle ξ at time $\tau > 0$, hence the acceleration of the string element $\delta\xi$; T is the tension in string (acting tangentially to the string); and $\theta(\xi + \delta\xi, \tau)$ and $\theta(\xi, \tau)$ are the angles between tension vectors and the horizontal line $\eta = 0$ at time $\tau > 0$ and locations $\xi + \delta\xi$ and ξ respectively (hence these are the angles between the string, or correctly, between the tangent vectors to the string and the horizontal line). Note that the string at time $\tau > 0$ takes the position described by the function

$$\xi \to U(\xi, \tau).$$

Then it follows that

$$\tan\theta(\xi, \tau) = \frac{\partial U}{\partial \xi}(\xi, \tau),$$

$$\tan\theta(\xi + \delta\xi, \tau) = \frac{\partial U}{\partial \xi}(\xi + \delta\xi, \tau).$$

Because the vibrations are small we can write:

$$\sin\theta(\xi+\delta\xi,\tau)\approx\tan\theta(\xi+\delta\xi,\tau),\ \text{and}\ \cos\theta(\xi+\delta\xi,\tau)\approx 1$$

and

$$\sin\theta(\xi,\tau)\approx\tan\theta(\xi,\tau),\ \text{and}\ \cos\theta(\xi,\tau)\approx 1.$$

Thus equation (2.1.2) is trivially satisfied, and equation (2.1.1) becomes

$$\left(\frac{\rho(\xi)}{T}\right)\frac{\partial^2 U}{\partial\tau^2}(\xi,\tau)=\frac{\frac{\partial U}{\partial\xi}(\xi+\delta\xi,\tau)-\frac{\partial U}{\partial\xi}(\xi,\tau)}{\delta\xi}. \tag{2.1.3}$$

Because $\delta\xi$ is an infinitesimal element of the string, equation (2.1.3) becomes:

$$\left(\frac{\rho(\xi)}{T}\right)\frac{\partial^2 U}{\partial\tau^2}(\xi,\tau)=\frac{\partial^2 U}{\partial\xi^2}(\xi,\tau). \tag{2.1.4}$$

Next, try a solution to equation (2.1.4) in the separable variable form:

$$U(\xi,\tau)=\phi(\xi)\psi(\tau). \tag{2.1.5}$$

Insert equation (2.1.5) into equation (2.1.4) and get:

$$\left(\frac{\rho(\xi)}{T}\right)\phi(\xi)\psi''(\tau)=\phi''(\xi)\psi(\tau), \tag{2.1.6}$$

or equivalently

$$\frac{\phi''(\xi)}{\left(\frac{\rho(\xi)}{T}\right)\phi(\xi)}=\frac{\psi''(\tau)}{\psi(\tau)}. \tag{2.1.7}$$

Equation (2.1.7) is satisfied only if its both sides are equal to the same constant.

$$\frac{\phi''(\xi)}{\left(\frac{\rho(\xi)}{T}\right)\phi(\xi)}=\frac{\psi''(\tau)}{\psi(\tau)}=C. \tag{2.1.8}$$

From (2.1.8) we have that

$$\phi''(\xi)=C\frac{\rho(\xi)}{T}\phi(\xi). \tag{2.1.9}$$

Multiplying both sides of equation (2.1.9) by $\bar{\phi}(\xi)$, integrating from $\xi=0$ to $\xi=\xi_0$ and using integration by parts in the integral on the left side, we get:

$$\phi'(\xi)\bar{\phi}(\xi)|_0^{\xi_0}-\int_0^{\xi_0}\phi'(\xi)\bar{\phi}'(\xi)d\xi=C\int_0^{\xi_0}\frac{\rho(\xi)}{T}\phi(\xi)\bar{\phi}(\xi)d\xi. \tag{2.1.10}$$

Using the boundary conditions

$$\phi(0) = 0 = \phi(\xi_0),$$

derivable from (2.1.5) and the fact that the string remains fixed at the end-points at any time, equation (2.1.10) will produce

$$-\int_0^{\xi_0} |\phi'(\xi)|^2 d\xi = C \int_0^{\xi_0} \frac{\rho(\xi)}{T} |\phi(\xi)|^2 d\xi. \tag{2.1.11}$$

Equation (2.1.11) clearly tells that C is a real, negative number, since $\rho(\xi) > 0$ and $T > 0$, as density and tension. Let $\lambda = -C > 0$. Using equation (2.1.8) we obtain:

$$\phi''(\xi) + \lambda \cdot \tilde{\rho}(\xi)\phi(\xi) = 0,\ \xi \in [0, \xi_0] \tag{2.1.12}$$

where $\tilde{\rho}(\xi) = \rho(\xi)/T$, and

$$\psi''(\tau) + \lambda\psi(\tau) = 0,\ \tau \geq 0. \tag{2.1.13}$$

Equation (2.1.13) is easily solvable and we get:

$$\psi(\tau) = C_1 \cos(\sqrt{\lambda}\tau) + C_2 \sin(\sqrt{\lambda}\tau). \tag{2.1.14}$$

Equation (2.1.12) is of Sturm-Liouville type, but not in the canonical form.

2.2 The canonical Sturm-Liouville equation

We shall put (2.1.12) in the canonical form for the reason of an easy mathematical manipulation in the subsequent work and of revealing the connection between the string properties (density $\rho(\xi)$ and tension T) and the potential function $q(x)$ of the Sturm-Liouville operator. We make a change of dependent and independent variables:

$$\xi \leftrightarrow x,\ \text{and } \phi(\xi) \leftrightarrow u(x),$$

by the formula:

$$\phi(\xi) = \frac{u(x)}{v(x)}, \tag{2.2.1}$$

where the functions $x = x(\xi)$ and $v(x)$ are at our disposal to choose. From equation (2.2.1) we calculate:

$$\phi'(\xi) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)} \cdot x'(\xi). \tag{2.2.2}$$

It is wise to choose $x(\xi)$ and $v(x)$ such that simplifications in equation (2.1.12) when coming to replace $\phi(\xi)$ by $u(x)$ occur as much as possible and as soon as possible. Therefore choose:

$$x'(\xi) = v^2(x). \tag{2.2.3}$$

By equation (2.2.3), equation (2.2.2) becomes:

$$\phi'(\xi) = u'(x)v(x) - u(x)v'(x), \tag{2.2.4}$$

from which we further derive

$$\phi''(\xi) = \big(u''(x)v(x) - u(x)v''(x)\big) \cdot x'(\xi). \tag{2.2.5}$$

The reason of choosing a quotient in formula (2.2.1) is now visible: the first order derivative of $u(x)$ will not appear when in equation (2.1.12) we go from $\phi(\xi)$ to $u(x)$. Now use equations (2.1.12), (2.2.1), (2.2.5), and (2.2.3) to write:

$$\left(u''(x) - \frac{v''(x)}{v(x)} \cdot u(x) + \lambda \cdot \frac{\tilde{\rho}(\xi)}{v^4(x)} \cdot u(x)\right) \cdot v^3(x) = 0. \tag{2.2.6}$$

To help us uniquely determine $x(\xi)$ and $v(x)$, in addition to equation (2.2.3), a second equation is now seemingly:

$$v^4(x) = \tilde{\rho}(\xi). \tag{2.2.7}$$

Using equation (2.2.7) in equation (2.2.6), after dividing equation (2.2.6) by $v^3(x)$, and introducing the function

$$q(x) = \frac{v''(x)}{v(x)}, \tag{2.2.8}$$

we obtain the most venerated canonical Sturm-Liouville equation:

$$-u''(x) + q(x)u(x) = \lambda u(x). \tag{2.2.9}$$

We summarize now our choices of $x(\xi)$ and $v(x)$:

$$\begin{cases} x'(\xi) = v^2(x), \text{ by equation (2.2.3)}, \\ v^4(x) = \tilde{\rho}(\xi), \text{ by equation (2.2.7)}. \end{cases}$$

Further manipulation of (2.2.3) and (2.2.7) gives us:

$$x'(\xi) = \sqrt{\tilde{\rho}(\xi)}, \tag{2.2.10}$$

$$v(x) = \sqrt[4]{\tilde{\rho}(\xi)}. \tag{2.2.11}$$

Integrating equation (2.2.10) we obtain:

$$x = x(\xi) = \int_0^{\xi} \sqrt{\tilde{\rho}(\zeta)}d\zeta, \tag{2.2.12}$$

which by inversion gives us $\xi = \xi(x)$, and so

$$v(x) = \sqrt[4]{\tilde{\rho}(\xi(x))}. \tag{2.2.13}$$

Thus we have to solve an ODE for the unknown function $u(x)$ over the interval $[0, a]$, where $x = 0$ corresponds to $\xi = 0$, and $x = a = \int_0^{\xi_0} \sqrt{\tilde{\rho}(\zeta)}d\zeta$, corresponds to $\xi = \xi_0$.

2.3 Relationships between the properties of the string and the parameters in the canonical Sturm-Liouville equation

Now it is clear why we said that the potential function $q(x)$ in equation (2.2.9) encodes information about the string: equations (2.2.8) and (2.2.13). There is one more parameter which appears in equation (2.2.9) and which ought to be paid attention to. This is λ firstly introduced in equation (2.1.12). We now make the following connection: λ appears in equation (2.1.14) which contains the functions sin and cos. These are periodic functions with period 2π. Therefore the function ψ is periodic with period $\tau_0 = 2\pi/\sqrt{\lambda}$. Hence $\sqrt{\lambda} = 2\pi/\tau_0$ has the form of a frequency. Due to this reason we call $\sqrt{\lambda}$, the frequency of oscillation of the string. On the other hand, equation (2.2.9) can be written in the compressed form:

$$L^{(q)}u = \lambda \cdot u, \tag{2.3.1}$$

which means mathematically that λ is an eigenvalue of the Sturm-Liouville operator $L^{(q)}$. We can now conclude that the eigenvalues λ's of the Sturm-Liouville operator $L^{(q)}$ are closely related to the frequencies of oscillation of the string $\sqrt{\lambda}$'s.

2.4 Physical interpretation of the direct and inverse Sturm-Liouville problems with Dirichlet boundary conditions

With the explanations in Section 2.3, the mathematical statement 'obtain information about the potential function $q(x)$ of the Sturm-Liouville operator $L^{(q)}$ from knowledge of the eigenvalues of $L^{(q)}$' has the physical meaning 'obtain information about the density function $\tilde{\rho}(\xi) = \rho(\xi)/T$ of the string from knowledge of the string's frequencies of oscillation'. Also, from this work we learned that, if the string vibrates in one mode (i.e. with frequency $\sqrt{\lambda}$), then by (2.1.5), (2.1.12), and (2.1.14) the position of the string at time τ is given by

$$U(\xi,\tau) = \phi(\xi)\left(C_1\cos(\sqrt{\lambda}\tau) + C_2\sin(\sqrt{\lambda}\tau)\right), \text{ for all } \xi \in [0,\xi_0], \tag{2.4.1}$$

where $\{\lambda, \phi(\xi)\}$ solves (2.1.12), or equivalently $\{\lambda, u(x)\}$ solves (2.3.1), with the changes of variables

$$\begin{cases} \xi \leftrightarrow x \\ \phi(\xi) \leftrightarrow u(x) \end{cases}$$

as described in (2.2.12), (2.2.1), (2.2.13), and $\tilde{\rho}(\xi) = \rho(\xi)/T$. It means that $\{\lambda, u(x)\}$ is an eigenpair of the Sturm-Liouville operator $L^{(q)}$ with domain $\{u \in H^2(0,a) | u(0) = 0 = u(a)\}$. The direct Sturm-Liouville theory insures that the operator $L^{(q)}$ has countable many eigenvalues $\{\lambda_n | n \geq 1\}$, and that the corresponding eigenfunctions $\{u_n(x) | n \geq 1\}$ form a complete, orthogonal set in $L^2(0,a)$. It follows by (2.2.1), (2.2.11), (2.2.10) that $\{\phi_n(\xi) | n \geq 1\}$ form a complete, orthogonal set in the weighted space

$$L^2_{\tilde{\rho}}(0,\xi_0) = \{\phi | \int_0^{\xi_0} |\phi(\xi)|^2 \cdot \tilde{\rho}(\xi) d\xi < \infty\},$$

endowed with the inner-product

$$< \phi, \chi >_{\tilde{\rho}} = \int_0^{\xi_0} \phi(\xi)\chi(\xi) \cdot \tilde{\rho}(\xi) d\xi.$$

Therefore, the position of the string at any time $\tau > 0$ is determined from all frequencies of oscillation $\{\sqrt{\lambda_n} | n \geq 1\}$, by superposing solutions of type (2.4.1) for each $\lambda = \lambda_n$. That is

$$U(\xi,\tau) = \sum_{n=1}^{\infty} \phi_n(\xi)\left(C_{1,n}\cos(\sqrt{\lambda_n}\tau) + C_{2,n}\sin(\sqrt{\lambda_n}\tau)\right), \text{ for all } \xi \in [0,\xi_0]. \tag{2.4.2}$$

The sets of constants $\{C_{1,n}|n \geq 1\}$ and $\{C_{2,n}|n \geq 1\}$ are determined from information about the string at $\tau = 0$ (e.g. $U(\xi, 0)$ and $\frac{\partial U}{\partial \tau}(\xi, 0)$). They will be the generalized Fourier coefficients (up to some constants) of $U(\xi, 0)$ and $\frac{\partial U}{\partial \tau}(\xi, 0)$ respectively, in the basis $\{\phi_n(\xi)|n \geq 1\}$ of $L^2_{\tilde{\rho}}(0, \xi_0)$. Hence, if the properties of the string are known (i.e. the 'weighted' density $\tilde{\rho}(\xi) = \rho(\xi)/T$), then the pairs $\{\phi_n(\xi), \lambda_n\}_{n\geq 1}$ will be known and therefore the position of string can be found at every instant, via equation (2.4.2). This is essentially what the direct Sturm-Liouville theory says. In the inverse problem, one has knowledge not of the position of the string at every instant, but of the spectral data (the frequencies of oscillation and some information of how the string vibrates in all modes of oscillation, i.e. $\{\lambda_n|n \geq 1\}$ and some information about the $\{\phi_n(\xi)|n \geq 1\}$, for example $\{\left(\frac{||\phi_n||_{L_2}}{\phi_n'(0)}\right)^2 |n \geq 1\}$) and wants to find out the density of the string.

CHAPTER 3. EXISTENCE, UNIQUENESS, CONSTRUCTIBILITY OF THE POTENTIAL FUNCTION q

This chapter is devoted mostly to proving existence and uniqueness of a potential function under proper restrictions on spectral data and elaborating the algorithm for reconstructing such a potential. We make the observation that the space l_2^k is needed in this chapter and therefore its definition is included next.

Definition: The space l_2^k is the space of all real sequences $\{\gamma_n\}_{n\geq 1}$ such that

$$\sum_{n=1}^{\infty}(n^k\gamma_n)^2 < \infty.$$

Notation: We shall use in this and next chapters the following notation for the mean value of the function p defined on $[\alpha, \beta]$:

$$[p] = \frac{1}{\beta - \alpha}\int_{\alpha}^{\beta} p(s)ds.$$

3.1 Under which conditions a potential function exists?

The following theorem was proved by Pivovarchik in [16]. However, an alternative proof is presented in Section 3.4, whose main aspects are motivated by the computational method to be developed in Section 3.3.

Theorem 1 *Let $a > 0$ and let M, M_1, M_2 be three real numbers such that*

$$\begin{cases} M_1 \neq M_2 \\ M = \frac{M_1+M_2}{2}. \end{cases} \tag{3.1.1}$$

Let $\{\lambda_n | n \geq 1\}$, $\{\mu_n | n \geq 1\}$, and $\{\nu_n | n \geq 1\}$ be three sequences of real numbers, strictly increasing, such that neither two of them intersect and satisfying the following asymptotic *formulas:*

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2 + M + c_n, \ \text{as } n \to \infty, \text{ where } (c_n)_{n\geq 1} \in l_2^1, \tag{3.1.2}$$

$$\mu_n = \left(\frac{n\pi}{a/2}\right)^2 + M_1 + c_n^1, \ \text{as } n \to \infty, \text{ where } (c_n^1)_{n\geq 1} \in l_2^1, \tag{3.1.3}$$

$$\nu_n = \left(\frac{n\pi}{a/2}\right)^2 + M_2 + c_n^2, \ \text{as } n \to \infty, \text{ where } (c_n^2)_{n\geq 1} \in l_2^1, \tag{3.1.4}$$

and having the following interlacing property:

$$0 < \lambda_1 < \mu_1 \text{ and } 0 < \lambda_1 < \nu_1,$$

and for every $n > 1$ the interval $(\lambda_{n-1}, \lambda_n)$ contains exactly one element of the set

$$\{\mu_m | m \geq 1\} \cup \{\nu_m | m \geq 1\}.$$

Then there exists a real-valued potential $q \in L^2(0,a)$ such that $\{\lambda_n | n \geq 1\}$, $\{\mu_n | n \geq 1\}$ and $\{\nu_n | n \geq 1\}$ are the Dirichlet eigenvalues for the Sturm-Liouville problem with potential q over the intervals $[0,a]$, $[0,\frac{a}{2}]$, $[\frac{a}{2},a]$ respectively.

Observation: All the above hypotheses are needed to prove existence. The second condition of (3.1.1), the asymptotics with the remainder sequences $\{c_n\}_{n\geq 1}$, $\{c_n^1\}_{n\geq 1}$, $\{c_n^2\}_{n\geq 1}$ in l_2 in place of l_2^1, and the interlacing property are necessary conditions for the existence of such a potential (see the discussion in Section 4.1 and respectively Remark 3.1 of Section 3.7).

The proof of Theorem 1 is postponed to Section 3.4.

3.2 Preparatory material

The scope of this section is to give the reader the opportunity to get the intuition of how to construct the potential function needed in Theorem 1. In order to achieve this we assume that such potential exists and gather useful information. Facts from the direct Sturm-Liouville theory pertaining only to this specific three spectra problem will constitute the subject of the following subsections. It will be seen that the construction comes in a natural way. This section is fundamental for developing the numerical algorithm and elaborating the proof of existence and uniqueness of a such potential.

3.2.1 The fundamental set of solution to the canonical Sturm-Liouville differential equation

Consider in general a real-valued function $p \in L^2(\alpha, \beta)$. Denote by $C(\cdot; p, \lambda)$ and $S(\cdot; p, \lambda)$ the $H^2(\alpha, \beta)$ solutions to the canonical Sturm-Liouville ODE:

$$-u''(x) + p(x)u(x) = \lambda u(x), \; x \in (\alpha, \beta), \tag{3.2.1}$$

satisfying the initial conditions:

$$u(\alpha) = 1, \; u'(\alpha) = 0, \tag{3.2.2}$$

and respectively

$$u(\alpha) = 0, \; u'(\alpha) = 1, \tag{3.2.3}$$

These notations are suggestive: if $p = 0$ then the solution to the initial value problem consisting of (3.2.1) and (3.2.2) is

$$\cos(\sqrt{\lambda}(x - \alpha)),$$

and the solution to the initial value problem consisting of (3.2.1) and (3.2.3) is

$$\frac{\sin(\sqrt{\lambda}(x - \alpha))}{\sqrt{\lambda}}.$$

By attaching a subscript 1, or 2 to the functions $C(\cdot; p, \lambda)$ and $S(\cdot; p, \lambda)$ we mean to refer to the solutions to (3.2.1) with initial conditions (3.2.2) and respectively to (3.2.1) with initial conditions (3.2.3) over the first half $[0, \frac{a}{2}]$ and respectively over the second half $[\frac{a}{2}, a]$ of our interval $[0, a]$. Next, we make the observation that $C(\cdot; p, \lambda)$ and $S(\cdot; p, \lambda)$ form a fundamental set of solutions to (3.2.1).

3.2.2 The use of the fundamental set to represent a solution

Next, let q_1 and q_2 be the restrictions of q to the subintervals $[0, \frac{a}{2}]$ and $[\frac{a}{2}, a]$, respectively. Then take the reflection of q_2 over the midpoint $x = \frac{a}{2}$ of the interval $[0, a]$ and obtain

$$\tilde{q}_2(x) = q_2(a - x), \; \text{for } x \in [0, \tfrac{a}{2}].$$

Since $S(\cdot; q, \lambda)$ is a solution to equation (3.2.1) where $p = q$ over the interval $[0, a]$, it will also be a solution to the same ODE over the interval $[\frac{a}{2}, a]$, where q is actually q_2. Therefore it can be written as a linear combination of the fundamental solutions of equation (3.2.1) over the interval $[\frac{a}{2}, a]$, where $p = q|_{[\frac{a}{2},a]} = q_2$:

$$S(x; q, \lambda) = k_1 C_2(x; q_2, \lambda) + k_2 S_2(x; q_2, \lambda), \text{ for all } x \in [\tfrac{a}{2}, a]. \tag{3.2.4}$$

The values of the constants k_1 and k_2 are obtained by taking $x = \frac{a}{2}$ in formula (3.2.4) and using the initial conditions for $C_2(\cdot; q_2, \lambda)$ and $S_2(\cdot; q_2, \lambda)$. Hence we get:

$$S(x; q, \lambda) = S(\frac{a}{2}; q, \lambda) C_2(x; q_2, \lambda) + S'(\frac{a}{2}; q, \lambda) S_2(x; q_2, \lambda), \text{ for all } x \in [\tfrac{a}{2}, a]. \tag{3.2.5}$$

Further we claim that

$$S(x; q, \lambda) = S_1(x; q_1, \lambda), \text{ for all } x \in [0, \tfrac{a}{2}]. \tag{3.2.6}$$

This is so because both $S(\cdot; q, \lambda)$ and $S_1(\cdot; q_1, \lambda)$ satisfy the ODE (3.2.1) on the interval $[0, \frac{a}{2}]$, where $p = q|_{[0,\frac{a}{2}]} = q_1$, and the same initial conditions at $x = 0$. Therefore, equations (3.2.5) and (3.2.6) give us:

$$S(x; q, \lambda) = S_1(\frac{a}{2}; q_1, \lambda) C_2(x; q_2, \lambda) + S_1'(\frac{a}{2}; q_1, \lambda) S_2(x; q_2, \lambda), \text{ for all } x \in [\tfrac{a}{2}, a], \tag{3.2.7}$$

which by taking $x = a$ becomes:

$$S(a; q, \lambda) = S_1(\frac{a}{2}; q_1, \lambda) C_2(a; q_2, \lambda) + S_1'(\frac{a}{2}; q_1, \lambda) S_2(a; q_2, \lambda) \tag{3.2.8}$$

3.2.3 A crucial result

Next we prove a result very useful in the subsequent work, whose generalization is presented in the Appendix.

Looking at the formula (3.2.8), we observe that one term contains $S_1(\frac{a}{2}; q_1, \lambda)$ and the other term contains the derivative $S_1'(\frac{a}{2}; q_1, \lambda)$, both being quantities that correspond to q_1. Not the same thing can be said about the quantities that correspond to q_2. Hence, the idea of casting (3.2.8) into a symmetrical form comes into mind: discover a pair of transformations to replace $C_2(a;q_2,\lambda)$ by a derivative similar to $S_1'(\frac{a}{2};q_1,\lambda)$, and $S_2(a;q_2,\lambda)$ by a function similar to $S_1(\frac{a}{2};q_1,\lambda)$, keeping in mind that the two new quantities must correspond to the same function. This intuition laid down the foundation for Lemma 1.

Lemma 1 *The following hold:*

$$S_2(a;q_2,\lambda) = S_1(\frac{a}{2};\tilde{q}_2,\lambda), \text{ for all } \lambda \in \mathbb{C}, \tag{3.2.9}$$

$$C_2(a;q_2,\lambda) = S_1'(\frac{a}{2};\tilde{q}_2,\lambda), \text{ for all } \lambda \in \mathbb{C}. \tag{3.2.10}$$

Proof of Lemma 1: First we prove formula (3.2.9). Let $\lambda \in \mathbb{C}$ be chosen and fixed. Let us introduce the following notations:

$$\begin{cases} \phi(x) = S_1(x;\tilde{q}_2,\lambda), \text{ for } x \in [0,\frac{a}{2}], \\ \psi(x) = S_2(a-x;q_2,\lambda), \text{ for } x \in [0,\frac{a}{2}]. \end{cases}$$

Since $S_1(\cdot;\tilde{q}_2,\lambda)$ is the solution to the initial value problem consisting of (3.2.1) and (3.2.3) with $p = \tilde{q}_2$ and $[\alpha,\beta] = [0,\frac{a}{2}]$ we can write that

$$-\phi''(x) + \tilde{q}_2(x)\phi(x) = \lambda\phi(x), \text{ for } x \in [0,\tfrac{a}{2}], \tag{3.2.11}$$

and the initial conditions

$$\begin{cases} \phi(0) = 0, \\ \phi'(0) = 1. \end{cases} \tag{3.2.12}$$

Since $S_2(\cdot;q_2,\lambda)$ is the solution to the initial value problem consisting of (3.2.1) and (3.2.3) with $p = q_2$ and $[\alpha,\beta] = [\frac{a}{2},a]$ we can write that

$$-S_2''(\xi;q_2,\lambda) + q_2(\xi)S_2(\xi;q_2,\lambda) = \lambda S_2(\xi;q_2,\lambda), \text{ for } \xi \in [\tfrac{a}{2},a]. \tag{3.2.13}$$

If $x \in [0,\frac{a}{2}]$ then $a-x \in [\frac{a}{2},a]$, and we can replace ξ in equation (3.2.13) by $a-x$ and using our definition of ψ above, we obtain that

$$-\psi''(x) + \tilde{q}_2(x)\psi(x) = \lambda\psi(x), \text{ for } x \in [0,\tfrac{a}{2}], \tag{3.2.14}$$

because

$$\psi''(x) = S_2''(a-x;\tilde{q}_2,\lambda), \text{ for } x \in [0,\tfrac{a}{2}],$$

and

$$\tilde{q}_2(x) = q_2(a-x), \text{ for } x \in [0, \tfrac{a}{2}].$$

The final conditions for ψ are

$$\begin{cases} \psi(\frac{a}{2}) = 0, \\ \psi'(\frac{a}{2}) = -1, \end{cases} \tag{3.2.15}$$

due to our definition of ψ and the initial conditions for $S_2(\cdot; q_2, \lambda)$. Multiplying equation (3.2.11) by $\psi(x)$ and equation (3.2.14) by $\phi(x)$ and subtracting we arrive at:

$$-\phi''(x)\psi(x) + \phi(x)\psi''(x) = 0, \text{ for } x \in [0, \tfrac{a}{2}],$$

which is equivalent to

$$\frac{d}{dx}(W[\phi, \psi](x)) = 0, \text{ for } x \in [0, \tfrac{a}{2}].$$

Here $W[\phi, \psi]$ is the Wronskian of the pair of functions $\{\phi, \psi\}$ defined as follows:

$$W[\phi, \psi](x) = \begin{vmatrix} \phi(x) & \psi(x) \\ \phi'(x) & \psi'(x) \end{vmatrix}.$$

Therefore

$$W[\phi, \psi](x) = const, \text{ for } x \in [0, \tfrac{a}{2}],$$

and hence

$$W[\phi, \psi](0) = W[\phi, \psi](\frac{a}{2}).$$

This means

$$\begin{vmatrix} \phi(0) & \psi(0) \\ \phi'(0) & \psi'(0) \end{vmatrix} = \begin{vmatrix} \phi(\frac{a}{2}) & \psi(\frac{a}{2}) \\ \phi'(\frac{a}{2}) & \psi'(\frac{a}{2}) \end{vmatrix},$$

which by using our definitions of ϕ and ψ and formulas (3.2.12) and (3.2.15) reads

$$\begin{vmatrix} 0 & S_2(a; q_2, \lambda) \\ 1 & -S_2'(a; q_2, \lambda) \end{vmatrix} = \begin{vmatrix} S_1(\frac{a}{2}; \tilde{q}_2, \lambda) & 0 \\ S_1'(\frac{a}{2}; \tilde{q}_2, \lambda) & -1 \end{vmatrix}.$$

Obvious calculations give further the desired formula (3.2.9). Finally, formula (3.2.10) can be proved similarly by introducing the functions:

$$\begin{cases} \phi(x) = S_1(x; \tilde{q}_2, \lambda), & \text{for } x \in [0, \frac{a}{2}], \\ \psi(x) = C_2(a-x; q_2, \lambda), & \text{for } x \in [0, \frac{a}{2}]. \ \square \end{cases}$$

The identities (3.2.8), (3.2.9), and (3.2.10) yield

$$S(a;q,\lambda)=S_1(\frac{a}{2};q_1,\lambda)S_1'(\frac{a}{2};\tilde{q}_2,\lambda)+S_1'(\frac{a}{2};q_1,\lambda)S_1(\frac{a}{2};\tilde{q}_2,\lambda). \tag{3.2.16}$$

Formula (3.2.16) turned out to be of crucial importance in elaborating the proofs of existence and uniqueness, and in developing the algorithm for constructing a such potential function q.

3.2.4 Integral representations with Gelfand-Levitan kernel

Another useful thing is the following general result well known in the direct theory of Sturm-Liouville problems (see [13, Theorem 4.18 and Example 4.19, pages 154-157]): if $p \in L^2(0,\beta)$ is a real valued function and $\lambda \in \mathbb{C}$, then

$$S(x;p,\lambda)=\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}+\int_0^x K(x,t;p)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}dt,\ \text{for } x\in[0,\beta], \tag{3.2.17}$$

where $K(\cdot,\cdot;p)\in C(\bar{\Delta}_0)$, called the Gelfand-Levitan kernel, is the weak solution in the triangle

$$\bar{\Delta}_0=\{(x,t)|0\le t\le x\le\beta\}$$

to the Goursat problem:

$$\begin{cases} W_{xx}(x,t)-W_{tt}(x,t)-p(x)W(x,t)=0, & (x,t)\in\Delta_0,\\ W(x,x)=\frac{1}{2}\int_0^x p(s)ds, & x\in[0,\beta],\\ W(x,0)=0, & x\in[0,\beta]. \end{cases} \tag{3.2.18}$$

Since $p\in L^2(0,\beta)$, it follows that the function

$$f(x)=\frac{1}{2}\int_0^x p(s)ds,\ x\in[0,\beta]$$

is an $H^1(0,\beta)$ function such that $f(0)=0$, and therefore by Theorem 4.15 of [13, page 147], there exists the partial derivative $K_x(\cdot,\cdot;p)$ and so we can differentiate with respect to x in (3.2.17) and using (3.2.18) we obtain:

$$\begin{aligned} S'(x;p,\lambda) &= \cos(\sqrt{\lambda}x)+K(x,x;p)\cdot\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}+\int_0^x K_x(x,t;p)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}dt,\\ &= \cos(\sqrt{\lambda}x)+\left(\frac{1}{2}\int_0^x p(s)ds\right)\cdot\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}\\ &\quad+\int_0^x K_x(x,t;p)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}dt. \end{aligned} \tag{3.2.19}$$

Now take $x = \beta$ in (3.2.17) and in (3.2.19):

$$S(\beta; p, \lambda) = \frac{\sin(\sqrt{\lambda}\beta)}{\sqrt{\lambda}} + \int_0^\beta K(\beta, t; p) \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} dt \tag{3.2.20}$$

and

$$S'(\beta; p, \lambda) = \cos(\sqrt{\lambda}\beta) + \left(\frac{1}{2}\int_0^\beta p(s)ds\right) \cdot \frac{\sin(\sqrt{\lambda}\beta)}{\sqrt{\lambda}} + \int_0^\beta K_x(\beta, t; p) \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} dt. \tag{3.2.21}$$

Formulas (3.2.20) and (3.2.21) are two integral representation formulas with Gelfand-Levitan kernel that we often refer to in the subsequent work.

3.2.5 The eigenvalues and the zeros of the characteristic function of a Sturm-Liouville problem

Another useful general result is that the Dirichlet eigenvalues of the Sturm-Liouville operator $L^{(p)}$ over the interval $[\alpha, \beta]$ are exactly the zeros of the function

$$\lambda \in \mathbb{C} \to S(\beta; p, \lambda),$$

called the characteristic function of the Sturm-Liouville problem with Dirichlet boundary conditions:

$$\begin{cases} -u''(x) + p(x)u(x) = \lambda u(x), \ x \in (\alpha, \beta) \\ u(\alpha) = 0 = u(\beta). \end{cases}$$

This is seen as follows: if λ is such an eigenvalue, then there exists an eigenfunction $u \neq 0$ in $H^2(\alpha, \beta)$ associated with it. That is, u satisfies the ODE (3.2.1), and $u(\alpha) = 0 = u(\beta)$. It follows that $u'(\alpha) \neq 0$ (otherwise, since u satisfies the ODE (3.2.1) and $u(\alpha) = 0$, u would be identically zero, which violates the fact that u is an eigenfunction). So the function $\frac{u}{u'(\alpha)}$ is well defined and satisfies the initial value problem consisting of (3.2.1) and (3.2.3), whose unique solution is $S(\cdot; p, \lambda)$. Hence $S(\beta; p, \lambda) = \frac{u(\beta)}{u'(\alpha)} = 0$. So λ is a zero of $S(\beta; p, \cdot)$. Conversely, if λ is such a zero, then by the fact that $S(\cdot; p, \lambda)$ is the solution to the initial value problem consisting of (3.2.1) and (3.2.3), it follows that $\{\lambda, S(\cdot; p, \lambda)\}$ is a Dirichlet eigenpair of the operator $L^{(p)}$ over the interval $[\alpha, \beta]$.

Also, the Dirichlet eigenvalues of the above mentioned operator form a countable set (see Lemma 4.7(b) in [13, page 136]).

3.2.6 Two important properties of the characteristic function mentioned in Subsection 3.2.5

The function

$$\lambda \in \mathbb{C} \to S(\beta; p, \lambda)$$

mentioned in Subsection 3.2.5 is an entire function of λ (see Analyticity Properties, part (a) in [18, page 10]). Its zeros are simple zeros (see Lemma 4.7(c) in [13, page 136]).

3.2.7 Three sets of identities necessary later

Using the result of Subsection 3.2.5 with $([\alpha,\beta],p)$ replaced by $([0,a],q)$, $\big([0,\frac{a}{2}],q_1\big)$, and respectively $\big([\frac{a}{2},a],q_2\big)$ we obtain:

$$S(a; q, \lambda_n) = 0,\ n \geq 1, \tag{3.2.22}$$

$$S_1(\frac{a}{2}; q_1, \mu_n) = 0,\ n \geq 1, \tag{3.2.23}$$

and

$$S_2(a; q_2, \nu_n) = 0,\ n \geq 1. \tag{3.2.24}$$

Due to (3.2.9), the identities in (3.2.24) become:

$$S_1(\frac{a}{2}; \tilde{q}_2, \nu_n) = 0,\ n \geq 1. \tag{3.2.25}$$

3.2.8 Two more sets of identities necessary later

Using (3.2.22), (3.2.23), and (3.2.25) in (3.2.16), and the fact that

$$\{\mu_n | n \geq 1\} \cap \{\nu_n | n \geq 1\} = \phi, \tag{3.2.26}$$

we obtain two more identities necessary later.

$$S_1'(\frac{a}{2}; q_1, \mu_n) = \frac{S(a; q, \mu_n)}{S_1(\frac{a}{2}; \tilde{q}_2, \mu_n)},\ n \geq 1, \tag{3.2.27}$$

and

$$S_1'(\frac{a}{2}; \tilde{q}_2, \nu_n) = \frac{S(a; q, \nu_n)}{S_1(\frac{a}{2}; q_1, \nu_n)},\ n \geq 1. \tag{3.2.28}$$

Note: The requirement that the sequences $\{\mu_n | n \geq 1\}$ and $\{\nu_n | n \geq 1\}$ do not overlap is needed to insure that the denominators in (3.2.27) and (3.2.28) are nonzero. If, for example

$$S_1(\frac{a}{2}; q_1, \nu_m) = 0,$$

for some $m \geq 1$, then ν_m would had been a Dirichlet eigenvalues of the potential q_1 over the interval $[0, \frac{a}{2}]$ (by the discussion in Subsection 3.2.5), hence $\nu_m \in \{\mu_n | n \geq 1\}$, violating (3.2.26).

Note: While formulas (3.2.27) and (3.2.28) are not standard results in the theory of direct Sturm-Liouville problems, they need to be in this section because they contribute to fulfilling the goal expressed at the beginning of Section 3.2 as it will be seen in Subsections 3.2.11 and 3.2.12.

3.2.9 Asymptotics of eigenvalues and mean values of the potential

Next, since q was assumed to be the real valued potential function in $L^2(0, a)$ having λ_n's, μ_n's and ν_n's as its three sequences of Dirichlet eigenvalues over the intervals $[0, a]$, $[0, \frac{a}{2}]$, and respectively $[\frac{a}{2}, a]$, then by the direct theory of Sturm-Liouville problems we can write:

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2 + \frac{1}{a} \cdot \int_0^a q(s)ds + d_n, \text{ as } n \to \infty, \text{ where } (d_n)_{n\geq 1} \in l_2, \tag{3.2.29}$$

$$\mu_n = \left(\frac{n\pi}{a/2}\right)^2 + \frac{1}{a/2} \cdot \int_0^{\frac{a}{2}} q_1(s)ds + d_n^1, \text{ as } n \to \infty, \text{ where } (d_n^1)_{n\geq 1} \in l_2 \tag{3.2.30}$$

$$\nu_n = \left(\frac{n\pi}{a/2}\right)^2 + \frac{1}{a/2} \cdot \int_0^{\frac{a}{2}} \tilde{q}_2(s)ds + d_n^2, \text{ as } n \to \infty, \text{ where } (d_n^2)_{n\geq 1} \in l_2 \tag{3.2.31}$$

Comparing (3.2.29), (3.2.30), and (3.2.31) with (3.1.2), (3.1.3), and (3.1.4) respectively, and using the fact that all six sequences $(c_n)_{n\geq 1}$, $(c_n^1)_{n\geq 1}$, $(c_n^2)_{n\geq 1}$, $(d_n)_{n\geq 1}$, $(d_n^1)_{n\geq 1}$, $(d_n^2)_{n\geq 1}$ converge to zero because the first three of them are $l_2^1 \subset l_2$ and the last three of them are l_2 sequences, we see what the three means are supposed to be:

$$M = \frac{1}{a} \cdot \int_0^a q(s)ds = [q], \tag{3.2.32}$$

$$M_1 = \frac{1}{a/2} \cdot \int_0^{\frac{a}{2}} q_1(s)ds = [q_1], \tag{3.2.33}$$

$$M_2 = \frac{1}{a/2} \cdot \int_0^{\frac{a}{2}} \tilde{q}_2(s)ds = [\tilde{q}_2] = [q_2]. \tag{3.2.34}$$

3.2.10 Interlacing of the three spectra

Let $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$ be the Dirichlet eigenvalues of the real valued potential $q \in L^2(0,a)$ on the intervals $[0,a]$, $[0,\frac{a}{2}]$, $[\frac{a}{2},a]$, respectively. Denote by $\{\theta_n\}_{n\geq 1}$ the set of $\{\mu_n\}_{n\geq 1} \cup \{\nu_n\}_{n\geq 1}$ arranged increasingly. Then the following interlacing property holds:

$$0 < \lambda_1 < \theta_1 \leq \lambda_2 \leq \theta_2 \ldots \lambda_{n-1} \leq \theta_{n-1} \leq \lambda_n \leq \theta_n \ldots .$$

Furthermore, for each $n > 1$, in the relation $\theta_{n-1} \leq \lambda_n \leq \theta_n$, either all three quantities are different, or else all three are equal. This is discussed in Chapter 4, Section 8 of [6]. If an equality happens, then that is equivalent to one overlap of the three sets $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$. Hence, N relations of the form $\theta_{n-1} = \lambda_n = \theta_n$ mean N overlaps of the three sets.

If the non-overlap of the three sets $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$ is assumed, then an alternative proof of the interlacing property

$$\lambda_1 < \theta_1 < \lambda_2 < \theta_2 < \ldots < \lambda_{n-1} < \theta_{n-1} < \lambda_n < \theta_n < \ldots$$

is shown in Remark 3.1.

3.2.11 Goursat-Cauchy problems

Replacing $([0,\beta],p)$ with $([0,a],q)$ in (3.2.20) and using (3.2.22) we obtain that $K(a,t;q)$ satisfies the system of integral equations:

$$0 = S(a;q,\lambda_n) = \frac{\sin(\sqrt{\lambda_n}a)}{\sqrt{\lambda_n}} + \int_0^a K(a,t;q)\frac{\sin(\sqrt{\lambda_n}t)}{\sqrt{\lambda_n}}dt,\ n \geq 1. \tag{3.2.35}$$

Now from (3.2.18) with $([0,\beta],p)$ replaced by $([0,a],q)$, and using (3.2.32) we have that $K(a,t;q)$ also satisfies the conditions:

$$\begin{cases} K(a,0;q) = 0, \\ K(a,a;q) = \frac{1}{2}\int_0^a q(s)ds = \frac{a\cdot M}{2}. \end{cases} \tag{3.2.36}$$

Formulas (3.2.35) and (3.2.36) are important pieces in reconstructing the potential function q from the three sequences $\{\lambda_n | n \geq 1\}$, $\{\mu_n | n \geq 1\}$, $\{\nu_n | n \geq 1\}$, as it will be seen in Section 3.3. Similarly, one obtains that $K(\frac{a}{2},t;q_1)$ satisfies the system of integral equations:

$$0 = S_1(\frac{a}{2};q_1,\mu_n) = \frac{\sin(\sqrt{\mu_n}\frac{a}{2})}{\sqrt{\mu_n}} + \int_0^{\frac{a}{2}} K(\frac{a}{2},t;q_1)\frac{\sin(\sqrt{\mu_n}t)}{\sqrt{\mu_n}}dt,\ n \geq 1. \tag{3.2.37}$$

and the conditions:

$$\begin{cases} K(\frac{a}{2},0;q_1)=0 \\ K(\frac{a}{2},\frac{a}{2};q_1)=\frac{1}{2}\int_0^{\frac{a}{2}} q_1(s)ds=\frac{a\cdot M_1}{4}, \end{cases} \tag{3.2.38}$$

and $K(\frac{a}{2},t;\tilde{q}_2)$ satisfies the system of integral equations:

$$0=S_1(\frac{a}{2};\tilde{q}_2,\nu_n)=\frac{\sin(\sqrt{\nu_n}\frac{a}{2})}{\sqrt{\nu_n}}+\int_0^{\frac{a}{2}} K(\frac{a}{2},t;\tilde{q}_2)\frac{\sin(\sqrt{\nu_n}t)}{\sqrt{\nu_n}}dt,\ n\geq 1 \tag{3.2.39}$$

and the conditions:

$$\begin{cases} K(\frac{a}{2},0;\tilde{q}_2)=0, \\ K(\frac{a}{2},\frac{a}{2};\tilde{q}_2)=\frac{1}{2}\int_0^{\frac{a}{2}} \tilde{q}_2(s)ds=\frac{a\cdot M_2}{4}. \end{cases} \tag{3.2.40}$$

Next, using (3.2.21) with $([0,\beta],p)$ replaced by $\big([0,\frac{a}{2}],q_1\big)$ and respectively by $\big([0,\frac{a}{2}],\tilde{q}_2\big)$, formulas (3.2.27), (3.2.28), and formulas (3.2.33), (3.2.34) one obtains that $K_x(\frac{a}{2},t;q_1)$ satisfies the system of integral equations:

$$\cos(\sqrt{\mu_n}\frac{a}{2})+\left(\frac{a\cdot M_1}{4}\right)\cdot\frac{\sin(\sqrt{\mu_n}\frac{a}{2})}{\sqrt{\mu_n}}+\int_0^{\frac{a}{2}} K_x(\frac{a}{2},t;q_1)\frac{\sin(\sqrt{\mu_n}t)}{\sqrt{\mu_n}}dt=\frac{S(a;q,\mu_n)}{S_1(\frac{a}{2};\tilde{q}_2,\mu_n)},\ n\geq 1, \tag{3.2.41}$$

and $K_x(\frac{a}{2},t;\tilde{q}_2)$ satisfies the system of integral equations:

$$\cos(\sqrt{\nu_n}\frac{a}{2})+\left(\frac{a\cdot M_2}{4}\right)\cdot\frac{\sin(\sqrt{\nu_n}\frac{a}{2})}{\sqrt{\nu_n}}+\int_0^{\frac{a}{2}} K_x(\frac{a}{2},t;\tilde{q}_2)\frac{\sin(\sqrt{\nu_n}t)}{\sqrt{\nu_n}}dt=\frac{S(a;q,\nu_n)}{S_1(\frac{a}{2};q_1,\nu_n)},\ n\geq 1. \tag{3.2.42}$$

It means that the pairs

$$\{K(\frac{a}{2},t;q_1);K_x(\frac{a}{2},t;q_1)\}\text{ and }\{K(\frac{a}{2},t;\tilde{q}_2);K_x(\frac{a}{2},t;\tilde{q}_2)\}$$

are known pairs of functions, as solutions to the above systems of integral equations. (It will be seen later in this chapter that these systems of 'moment equations' have solutions and the solutions are unique.) This implies that the pairs

$$\{K_x(\frac{a}{2},t;q_1);K_t(\frac{a}{2},t;q_1)\}\text{ and }\{K_x(\frac{a}{2},t;\tilde{q}_2);K_t(\frac{a}{2},t;\tilde{q}_2)\}$$

will also be known pairs of functions. Call them $\{g_1(t); f_1'(t)\}$ and respectively $\{g_2(t); f_2'(t)\}$. All these tell us that $K(x,t;q_1)$ solves the Goursat-Cauchy problem (3.2.43):

$$\begin{cases} W_{xx}(x,t) - W_{tt}(x,t) - q_1(x)W(x,t) = 0, & 0 < t < x < \frac{a}{2} \\ W(x,x) = \frac{1}{2}\int_0^x q_1(s)ds, & 0 \le x \le \frac{a}{2} \\ W(x,0) = 0, & 0 \le x \le \frac{a}{2} \\ W_x(\frac{a}{2},t) = g_1(t), & 0 \le t \le \frac{a}{2} \\ W_t(\frac{a}{2},t) = f_1'(t), & 0 \le t \le \frac{a}{2}, \end{cases} \tag{3.2.43}$$

and $K(x,t;\tilde{q}_2)$ solves the Goursat-Cauchy problem (3.2.44):

$$\begin{cases} W_{xx}(x,t) - W_{tt}(x,t) - \tilde{q}_2(x)W(x,t) = 0, & 0 < t < x < \frac{a}{2} \\ W(x,x) = \frac{1}{2}\int_0^x \tilde{q}_2(s)ds, & 0 \le x \le \frac{a}{2} \\ W(x,0) = 0, & 0 \le x \le \frac{a}{2} \\ W_x(\frac{a}{2},t) = g_2(t), & 0 \le t \le \frac{a}{2} \\ W_t(\frac{a}{2},t) = f_2'(t), & 0 \le t \le \frac{a}{2}. \end{cases} \tag{3.2.44}$$

3.2.12 A non-linear map

The problems (3.2.43) and (3.2.44) are equivalent to saying that q_1 and $\tilde{q}_2$ are solutions to the non-linear equations

$$F(p) = \{g_1(t); f_1'(t)\} \tag{3.2.45}$$

and respectively

$$F(p) = \{g_2(t); f_2'(t)\}, \tag{3.2.46}$$

where the non-linear map F is defined as follows:

$$p \in L^2(0, \frac{a}{2}) \to L^2(0, \frac{a}{2}) \times L^2(0, \frac{a}{2})$$

by

$$F(p) = \{K_x(\frac{a}{2}, t; p), K_t(\frac{a}{2}, t; p)\}, \tag{3.2.47}$$

where $K(x,t;p)$ is the weak solution to the Goursat problem (3.2.18) with $[0,\beta]$ replaced by $[0, \frac{a}{2}]$.

3.3 The algorithm

Our goal is to obtain the Cauchy data $\{g_1(t); f_1'(t)\}$ and $\{g_2(t); f_2'(t)\}$ from the three known sequences of numbers λ_n's, μ_n's, ν_n's and then find q_1 and $\tilde{q}_2$ as solutions to the non-linear equations (3.2.45) and (3.2.46). Let $f(t)$, $f_1(t)$, $f_2(t)$ solve the systems of integral equations:

$$\frac{\sin(\sqrt{\lambda_n}a)}{\sqrt{\lambda_n}} + \int_0^a f(t)\frac{\sin(\sqrt{\lambda_n}t)}{\sqrt{\lambda_n}}dt = 0, \;\; n \geq 1, \tag{3.3.1}$$

$$\frac{\sin(\sqrt{\mu_n}\frac{a}{2})}{\sqrt{\mu_n}} + \int_0^{\frac{a}{2}} f_1(t)\frac{\sin(\sqrt{\mu_n}t)}{\sqrt{\mu_n}}dt = 0, \;\; n \geq 1, \tag{3.3.2}$$

$$\frac{\sin(\sqrt{\nu_n}\frac{a}{2})}{\sqrt{\nu_n}} + \int_0^{\frac{a}{2}} f_2(t)\frac{\sin(\sqrt{\nu_n}t)}{\sqrt{\nu_n}}dt = 0, \;\; n \geq 1, \tag{3.3.3}$$

respectively. By the discussion in Subsection 3.2.11, more precisely (3.2.35), (3.2.37) and (3.2.39), we know that $f(t)$, $f_1(t)$, $f_2(t)$ are expected to be $K(a, t; q)$, $K(\frac{a}{2}, t; q_1)$ and $K(\frac{a}{2}, t; \tilde{q}_2)$. Therefore we can use (3.2.36), (3.2.38), and (3.2.40) in designing their Fourier series representation. The best representations are:

$$f(t) = \left(\frac{M}{2}\right)t + \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi}{a}t\right), \tag{3.3.4}$$

$$f_1(t) = \left(\frac{M_1}{2}\right)t + \sum_{n=1}^{\infty} \alpha_n^{(1)} \sin\left(\frac{n\pi}{a/2}t\right), \tag{3.3.5}$$

$$f_2(t) = \left(\frac{M_2}{2}\right)t + \sum_{n=1}^{\infty} \alpha_n^{(2)} \sin\left(\frac{n\pi}{a/2}t\right). \tag{3.3.6}$$

Note: The integrals

$$\int_0^a \left(\frac{M}{2}\right)t\frac{\sin(\sqrt{\lambda_n}t)}{\sqrt{\lambda_n}}dt,$$

$$\int_0^{\frac{a}{2}} \left(\frac{M_1}{2}\right)t\frac{\sin(\sqrt{\mu_n}t)}{\sqrt{\mu_n}}dt,$$

$$\int_0^{\frac{a}{2}} \left(\frac{M_2}{2}\right)t\frac{\sin(\sqrt{\nu_n}t)}{\sqrt{\nu_n}}dt,$$

are known quantities, therefore the only unknowns in (3.3.1), (3.3.2), and (3.3.3) are the Fourier coefficients α_n's, $\alpha_n^{(1)}$'s, and $\alpha_n^{(2)}$'s, respectively.

Now we make the observation that in practice, one does not solve the infinite systems of integral equations in (3.3.1), (3.3.2), and (3.3.3), since we know only finitely many of λ's, μ's, and ν's. If the first N, N_1, N_2 elements of the sets $\{\lambda_n | n \geq 1\}$, $\{\mu_n | n \geq 1\}$, and respectively $\{\nu_n | n \geq 1\}$ are known, then in (3.3.4), (3.3.5) and (3.3.6) use only the first N, N_1, and respectively N_2 terms of the sums, and in (3.3.1), (3.3.2), and (3.3.3) use only the first N, N_1, and respectively N_2 equations. Each new system of equations becomes a squared linear system to be solved for the coefficients α's, $\alpha^{(1)}$'s and $\alpha^{(2)}$'s respectively. Having $f(t)$, $f_1(t)$ and $f_2(t)$ determined in this way one can construct the functions:

$$S(\lambda) = \frac{\sin(\sqrt{\lambda}a)}{\sqrt{\lambda}} + \int_0^a f(t) \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} dt, \text{ for any } \lambda \in \mathbb{C} \tag{3.3.7}$$

$$S^{(1)}(\lambda) = \frac{\sin(\sqrt{\lambda}\frac{a}{2})}{\sqrt{\lambda}} + \int_0^{\frac{a}{2}} f_1(t) \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} dt, \text{ for any } \lambda \in \mathbb{C} \tag{3.3.8}$$

$$S^{(2)}(\lambda) = \frac{\sin(\sqrt{\lambda}\frac{a}{2})}{\sqrt{\lambda}} + \int_0^{\frac{a}{2}} f_2(t) \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} dt, \text{ for any } \lambda \in \mathbb{C}. \tag{3.3.9}$$

Hence, the quotients

$$\begin{cases} \frac{S(\mu_n)}{S^{(2)}(\mu_n)}, & n \geq 1 \\ \frac{S(\nu_n)}{S^{(1)}(\nu_n)}, & n \geq 1 \end{cases}$$

are well determined. Next, let $g_1(t)$ and $g_2(t)$ solve the systems of integral equations:

$$\cos(\sqrt{\mu_n}\frac{a}{2}) + \left(\frac{a \cdot M_1}{4}\right) \cdot \frac{\sin(\sqrt{\mu_n}\frac{a}{2})}{\sqrt{\mu_n}} + \int_0^{\frac{a}{2}} g_1(t) \frac{\sin(\sqrt{\mu_n}t)}{\sqrt{\mu_n}} dt = \frac{S(\mu_n)}{S^{(2)}(\mu_n)}, \ n \geq 1 \tag{3.3.10}$$

and respectively

$$\cos(\sqrt{\nu_n}\frac{a}{2}) + \left(\frac{a \cdot M_2}{4}\right) \cdot \frac{\sin(\sqrt{\nu_n}\frac{a}{2})}{\sqrt{\nu_n}} + \int_0^{\frac{a}{2}} g_2(t) \frac{\sin(\sqrt{\nu_n}t)}{\sqrt{\nu_n}} dt = \frac{S(\nu_n)}{S^{(1)}(\nu_n)}, \ n \geq 1. \tag{3.3.11}$$

By the discussion in Subsection 3.2.11, more precisely (3.2.41) and (3.2.42), we know that $g_1(t)$, $g_2(t)$ are expected to be $K_x(\frac{a}{2}, t; q_1)$ and $K_x(\frac{a}{2}, t; \tilde{q}_2)$, respectively. Hence we can take advantage of this fact and of the boundary conditions in (3.2.18) with $([0, \beta], p)$ replaced by $\left([0, \frac{a}{2}], q_1\right)$ and by $([0, \beta], \tilde{q}_2)$ respectively, in designing their Fourier series representation. We have the following calculations:

$$K_x(\frac{a}{2}, 0; q_1) = 0,$$

and

$$\begin{aligned} K_x(\frac{a}{2},\frac{a}{2};q_1) &= -K_t(\frac{a}{2},\frac{a}{2};q_1)+\frac{1}{2}\cdot q_1(\frac{a}{2}) \\ &= -K_t(\frac{a}{2},\frac{a}{2};q_1)+\frac{q(\frac{a}{2})}{2}, \text{ since } q_1 = q|_{[0,\frac{a}{2}]} \\ &= -K_t(\frac{a}{2},\frac{a}{2};q_1)+\frac{midq}{2}, \end{aligned}$$

and similarly for $K_x(\frac{a}{2},0;\tilde{q}_2)$ and $K_x(\frac{a}{2},\frac{a}{2};\tilde{q}_2)$, where $midq = q(\frac{a}{2})$. Having in mind that $f_1(t)$ and $f_2(t)$ are expected to be $K(\frac{a}{2},t;q_1)$ and $K(\frac{a}{2},t;\tilde{q}_2)$, we can develop the following plan:

- if $midq = q(\frac{a}{2})$ is known, then obtain first $f_1'(\frac{a}{2})$ and $f_2'(\frac{a}{2})$ (this can be done numerically by applying a centered difference scheme to $f_1(t)$ and $f_2(t)$ on a grid of $[0,\frac{a}{2}]$), then write

$$g_1(t) = \left(\frac{-f_1'(\frac{a}{2})+\frac{midq}{2}}{a/2}\right)t+\sum_{n=1}^{\infty}\beta_n^{(1)}\sin\left(\frac{n\pi}{a/2}t\right), \tag{3.3.12}$$

  and

$$g_2(t) = \left(\frac{-f_2'(\frac{a}{2})+\frac{midq}{2}}{a/2}\right)t+\sum_{n=1}^{\infty}\beta_n^{(2)}\sin\left(\frac{n\pi}{a/2}t\right), \tag{3.3.13}$$

- if $midq = q(\frac{a}{2})$ is not known write:

$$g_1(t) = \sum_{n=1}^{\infty}\beta_n^{(1)}\sin\left(\frac{(n-\frac{1}{2})\pi}{a/2}t\right), \tag{3.3.14}$$

$$g_2(t) = \sum_{n=1}^{\infty}\beta_n^{(2)}\sin\left(\frac{(n-\frac{1}{2})\pi}{a/2}t\right). \tag{3.3.15}$$

Then depending on the case, insert (3.3.12) or (3.3.14) into (3.3.10), and (3.3.13) or (3.3.15) into (3.3.11) to solve for $\beta^{(1)}$'s and $\beta^{(2)}$'s. The same discussion as before applies for numerical implementation (i.e. when only finitely many of λ's, μ's and ν's are available). Finally, the pairs $\{g_1(t), f_1'(t)\}$ and $\{g_2(t), f_2'(t)\}$ are used as Cauchy data to obtain two functions over the interval $[0,\frac{a}{2}]$. Call them q_1 and $\tilde{q}_2$. More precisely, q_1 is the solution to the nonlinear equation

$$F(p) = \{g_1(t), f_1'(t)\},$$

and $\tilde{q}_2$ is the solution to the nonlinear equation

$$F(p) = \{g_2(t), f_2'(t)\},$$

where the mapping F is defined as in (3.2.47). Numerically, both of these equations are solved by a modified Newton method (see [20, page 166]). Also, a complete description of the modified Newton method can be found in Section 6.2. We mention that the successive approximation method (see again [20, page 166]) can also be used to solve the above non-linear equations.

And, as the last things to do, define

$$q_2(x) = \tilde{q}_2(a-x), \text{ for } x \in [\tfrac{a}{2}, a],$$

and then take

$$q = \begin{cases} q_1 & \text{on } [0, \frac{a}{2}] \\ q_2 & \text{on } [\frac{a}{2}, a]. \end{cases}$$

3.4 Proof of Theorem 1

Let $f \in H^2(0,a)$ be the solution to the system of integral equations

$$\int_0^a f(t)\sin(\sqrt{\lambda_n}t)dt = -\sin(\sqrt{\lambda_n}a), \text{ for } n \geq 1, \tag{3.4.1}$$

satisfying the conditions

$$\begin{cases} f(0) = 0 \\ f(a) = \frac{a \cdot M}{2}. \end{cases} \tag{3.4.2}$$

The existence of such a solution is justified as follows: since $\{\lambda_n | n \geq 1\}$ is a strictly increasing sequence of real numbers satisfying (3.1.2), by choosing an arbitrary real valued sequence $\{k_n\}_{n\geq 1}$ such that

$$k_n = \frac{c}{n^2} + \tilde{k}_n, \text{ for all } n \geq 1,$$

with $c \in \mathbb{R}$ and $\{\tilde{k}_n\}_{n\geq 1} \in l_2^2$ we can find a unique real valued function $Q \in H^1(0,a)$ having these two sequences as its sequences of Dirichlet eigenvalues and respectively norming constants, in the sense described in [18, page 59]. The existence and uniqueness of Q in this space is assured by [18, Problem 6(c), page 61]. From the direct Sturm-Liouville theory (see again [18, Problem 6(c), page 61]) it follows that the sequence $\{\lambda_n | n \geq 1\}$ of Dirichlet eigenvalues for the potential function $Q \in H^1(0,a)$ satisfies the asymptotic formula:

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2 + [Q] + \tilde{c}_n, \text{ as } n \to \infty, \tag{3.4.3}$$

for some sequence $(\tilde{c}_n)_{n=1}^{\infty} \in l_2^1$. Comparing (3.4.3) with (3.1.2) we obtain that

$$[Q] = M, \tag{3.4.4}$$

because the sequences $\{c_n\}_{n\geq 1} \in l_2^1 \subset l_2$ and $\{\tilde{c}_n\}_{n\geq 1} \in l_2^1 \subset l_2$ converge to 0, as l_2 sequences.

Also, from the direct Sturm-Liouville theory (see Subsection 3.2.5) we have that

$$S(a; Q, \lambda_n) = 0, \text{ for all } n \geq 1. \tag{3.4.5}$$

Using (3.4.5) and (3.2.20) with $\beta = a$, $p = Q$, $\lambda = \lambda_n$ we get that:

$$\sin(\sqrt{\lambda_n} a) + \int_0^a K(a, t; Q) \sin(\sqrt{\lambda_n} t) dt = 0, \ n \geq 1. \tag{3.4.6}$$

Also, taking $x = a$ in the two boundary conditions of (3.2.18) with $\beta = a$, $p = Q$, and using (3.4.4) we get that

$$\begin{cases} K(a, 0; Q) = 0 \\ K(a, a; Q) = \frac{a \cdot M}{2}. \end{cases} \tag{3.4.7}$$

Equations (3.4.6) and (3.4.7) tell us that $K(a, \cdot; Q)$ is a function which satisfies (3.4.1) and (3.4.2).

The fact that $K(a, \cdot; Q)$ is an $H^2(0, a)$ function follows from the fact that $K(\cdot, \cdot; Q)$ is the weak solution to (3.2.18) with $\beta = a$ and $p = Q$, the fact that $Q \in H^1(0, a)$ and the proof of Theorem 4.15 in [13, pages 147-149]. (Following the proof of Theorem 4.15 in [13, pages 147-149] we observe that the second weak derivatives of $w(\xi, \eta)$ exist due to the fact that the function

$$\xi \to \frac{1}{2} \int_0^{\xi} Q(s) ds$$

is an H^2 function because $Q \in H^1(0, a)$, and hence after the indicated change of variables

$$\begin{cases} (x, t) \leftrightarrow (\xi, \eta) \\ W(x, t) \leftrightarrow w(\xi, \eta) \end{cases}$$

the weak derivative

$$W_{tt}(1, t) = \frac{1}{4} \left(w_{\xi\xi}(\frac{1+t}{2}, \frac{1-t}{2}) - 2w_{\xi\eta}(\frac{1+t}{2}, \frac{1-t}{2}) + w_{\eta\eta}(\frac{1+t}{2}, \frac{1-t}{2}) \right)$$

exists and is an L^2 function.)

Thus, the existence of a solution to (3.4.1) and (3.4.2) in the indicated space is shown. The uniqueness of such a solution follows due to the fact that the set

$$\{\sin(\sqrt{\lambda_n}t) | n \geq 1\}$$

is complete in $L^2(0,a)$. The completeness of this set is proved in Auxiliary result 1.

We also infer:

$$f = K(a, \cdot; Q), \text{ for all } Q \in H^1(0,a) \text{ having } \{\lambda_n | n \geq 1\} \text{ as its Dirichlet eigenvalues.} \tag{3.4.8}$$

Next, let $f_1 \in H^2(0, \frac{a}{2})$ be the solution to the system of integral equations

$$\int_0^{\frac{a}{2}} f_1(t) \sin(\sqrt{\mu_n}t)dt = -\sin(\sqrt{\mu_n}\frac{a}{2}), \text{ for } n \geq 1, \tag{3.4.9}$$

satisfying the conditions:

$$\begin{cases} f_1(0) = 0 \\ f_1(\frac{a}{2}) = \frac{a \cdot M_1}{4}. \end{cases} \tag{3.4.10}$$

The existence and uniqueness of the solution f_1 in $H^2(0, \frac{a}{2})$ are shown analogously to those of f. The completeness in $L^2(0, \frac{a}{2})$ of the set

$$\{\sin(\sqrt{\mu_n}t)\}_{n \geq 1}$$

is proved in Auxiliary result 1. By arguments similar to those for f we also have:

$$f_1 = K(\frac{a}{2}, \cdot; Q_1), \text{ for all } Q_1 \in H^1(0, \tfrac{a}{2}) \text{ having } \{\mu_n | n \geq 1\} \text{ as its Dirichlet eigenvalues.} \tag{3.4.11}$$

We shall use (3.4.11) to show that:

$$\int_0^{\frac{a}{2}} f_1(t) \sin(\sqrt{\nu_n}t)dt + \sin(\sqrt{\nu_n}\frac{a}{2}) \neq 0, \text{ for all } n \geq 1. \tag{3.4.12}$$

Suppose not. Then

$$\int_0^{\frac{a}{2}} f_1(t) \sin(\sqrt{\nu_m}t)dt + \sin(\sqrt{\nu_m}\frac{a}{2}) = 0, \text{ for some } m \geq 1.$$

This along with (3.4.11) and the integral representation of the characteristic function $S_1(\frac{a}{2}; Q_1, \lambda)$ (see formula (3.2.20) with $\beta = \frac{a}{2}$ and $p = Q_1$) imply that

$$\begin{aligned} S_1(\frac{a}{2}; Q_1, \nu_m) &= \frac{\sin(\sqrt{\nu_m}\frac{a}{2})}{\sqrt{\nu_m}} + \int_0^{\frac{a}{2}} K(\frac{a}{2}, t; Q_1)\frac{\sin(\sqrt{\nu_m}t)}{\sqrt{\nu_m}}dt \\ &= \frac{1}{\sqrt{\nu_m}}(\sin(\sqrt{\nu_m}\frac{a}{2}) + \int_0^{\frac{a}{2}} f_1(t)\sin(\sqrt{\nu_m}t)dt) \\ &= 0, \end{aligned}$$

which tells that ν_m is a Dirichlet eigenvalue of the Sturm-Liouville operator with potential $Q_1 \in H^1(0, \frac{a}{2})$. Since the set of all Dirichlet eigenvalues of this operator is $\{\mu_n | n \geq 1\}$ it follows that $\nu_m \in \{\mu_n | n \geq 1\}$, which contradicts the hypothesis $\{\mu_n | n \geq 1\} \cap \{\nu_n | n \geq 1\} = \phi$. Hence, (3.4.12) is shown.

Now, let $f_2 \in H^2(0, \frac{a}{2})$ be the solution to the system of integral equations

$$\int_0^{\frac{a}{2}} f_2(t)\sin(\sqrt{\nu_n}t)dt = -\sin(\sqrt{\nu_n}\frac{a}{2}), \text{ for } n \geq 1, \tag{3.4.13}$$

satisfying the conditions:

$$\begin{cases} f_2(0) = 0 \\ f_2(\frac{a}{2}) = \frac{a \cdot M_2}{4}. \end{cases} \tag{3.4.14}$$

The existence and uniqueness of the solution f_2 in $H^2(0, \frac{a}{2})$ are shown analogously to those of f and f_1. The completeness in $L^2(0, \frac{a}{2})$ of the set

$$\{\sin(\sqrt{\nu_n}t) | n \geq 1\}$$

is proved in Auxiliary result 1. By arguments similar to those for f and f_1 we also have:

$$f_2 = K(\frac{a}{2}, \cdot; \tilde{Q}_2), \text{ for all } \tilde{Q}_2 \in H^1(0, \tfrac{a}{2}) \text{ having } \{\nu_n | n \geq 1\} \text{ as its Dirichlet eigenvalues.} \tag{3.4.15}$$

In a similar way to (3.4.12), formula (3.4.15) is used to show that:

$$\int_0^{\frac{a}{2}} f_2(t)\sin(\sqrt{\mu_n}t)dt + \sin(\sqrt{\mu_n}\frac{a}{2}) \neq 0, \text{ for all } n \geq 1. \tag{3.4.16}$$

Next, define the functions $S(\lambda)$, $S^{(1)}(\lambda)$, and $S^{(2)}(\lambda)$ by the formulas (3.3.7), (3.3.8), and (3.3.9), respectively. Using (3.4.1) and (3.3.7); (3.4.9) and (3.3.8); (3.4.13) and (3.3.9) we obtain that:

$$S(\lambda_n) = 0, \text{ for all } n \geq 1, \tag{3.4.17}$$

$$S^{(1)}(\mu_n) = 0, \text{ for all } n \geq 1, \tag{3.4.18}$$

$$S^{(2)}(\nu_n) = 0, \text{ for all } n \geq 1. \tag{3.4.19}$$

Also from (3.4.12) and (3.4.16) we have that:

$$S^{(1)}(\nu_k) \neq 0, \text{ for all } k \geq 1. \tag{3.4.20}$$

and

$$S^{(2)}(\mu_k) \neq 0, \text{ for all } k \geq 1. \tag{3.4.21}$$

Hence the quotients:

$$\frac{S(\mu_n)}{S^{(2)}(\mu_n)}, \text{ for all } n \geq 1,$$

and

$$\frac{S(\nu_n)}{S^{(1)}(\nu_n)}, \text{ for all } n \geq 1,$$

are well-defined.

Next, construct the following two sequences:

$$k_n^1 = \ln\left((-1)^n \frac{S(\mu_n)}{S^{(2)}(\mu_n)}\right), \ n \geq 1, \tag{3.4.22}$$

and

$$k_n^2 = \ln\left((-1)^n \frac{S(\nu_n)}{S^{(1)}(\nu_n)}\right), \ n \geq 1. \tag{3.4.23}$$

In view of applying Corollary 2 of [18, page 116] to the pairs

$$\{\{\mu_n\}_{n\geq 1}, \{k_n^1\}_{n\geq 1}\} \text{ and } \{\{\nu_n\}_{n\geq 1}, \{k_n^2\}_{n\geq 1}\},$$

we need to prove that $\{k_n^1\}_{n\geq 1}$ and $\{k_n^2\}_{n\geq 1}$ are l_2^1 sequences of real numbers. They are real valued sequences due to the interlacing property of the set $\{\mu_n | n \geq 1\} \cup \{\nu_n | n \geq 1\}$ with the set $\{\lambda_n | n \geq 1\}$, and the asymptotic formulas (3.1.2), (3.1.3), (3.1.4), properties mentioned in the hypotheses of Theorem 1.

A precise explanation of the fact that $\{k_n^1\}_{n\geq 1}$ and $\{k_n^2\}_{n\geq 1}$ are real valued sequences follows now. From (3.3.7) with (3.4.8), from (3.3.9) with (3.4.15), and from (3.2.20) with $([0,\beta],p)$ replaced by $([0,a],Q)$ and respectively by $\left([0,\frac{a}{2}],\tilde{Q}_2\right)$, and from (A.0.10) with $([\alpha,\beta],\hat{p},\{\hat{z}_n\}_{n\geq 1})$ replaced by $([0,a],Q,\{\lambda_n\}_{n\geq 1})$, because λ_n's are the zeros of the function

$$\lambda \in \mathbb{C} \to S(a;Q,\lambda),$$

and respectively by $\left([0,\frac{a}{2}],\tilde{Q}_2,\{\nu_n\}_{n\geq 1}\right)$, because ν_n's are the zeros of the function

$$\lambda \in \mathbb{C} \to S_1(\frac{a}{2};\tilde{Q}_2,\lambda)$$

we obtain:

$$\begin{aligned} \frac{S(\lambda)}{S^{(2)}(\lambda)} &= \frac{S(a;Q,\lambda)}{S_1(\frac{a}{2};\tilde{Q}_2,\lambda)} \\ &= \frac{a\prod_{n=1}^{\infty}\left(\frac{a}{n\pi}\right)^2(\lambda_n-\lambda)}{\frac{a}{2}\prod_{n=1}^{\infty}\left(\frac{a}{2n\pi}\right)^2(\nu_n-\lambda)} \\ &= 8\prod_{n=1}^{\infty}\frac{\lambda_n-\lambda}{\nu_n-\lambda}, \end{aligned} \tag{3.4.24}$$

for any $Q \in H^1(0,a)$ having $\{\lambda_n\}_{n\geq 1}$ as its sequence of Dirichlet eigenvalues, and for any $\tilde{Q}_2 \in H^1(0,\frac{a}{2})$ having $\{\nu_n\}_{n\geq 1}$ as its sequence of Dirichlet eigenvalues. Formula (3.4.24) and the interlacing property of λ_n's, μ_n's, ν_n,s give us, by looking at the sign of each factor of (3.4.24):

$$\frac{S(\mu_1)}{S^{(2)}(\mu_1)} < 0,\ \frac{S(\mu_2)}{S^{(2)}(\mu_2)} > 0,\ \frac{S(\mu_3)}{S^{(2)}(\mu_3)} < 0,\ \ldots,$$

which is equivalent to

$$(-1)^n\frac{S(\mu_n)}{S^{(2)}(\mu_n)} > 0, \text{ for all } n \geq 1.$$

This shows that $\{k_n^1\}_{n\geq 1}$ is a real valued sequence. Similar arguments apply to $\{k_n^2\}_{n\geq 1}$.

Next, we show that

$$\sum_{n=1}^{\infty}(nk_n^1)^2 < \infty \tag{3.4.25}$$

and

$$\sum_{n=1}^{\infty}(nk_n^2)^2 < \infty. \tag{3.4.26}$$

We elaborate only for $\{k_n^1\}_{n\geq 1}$, but similar arguments apply to $\{k_n^2\}_{n\geq 1}$. From the calculations done in Auxiliary result 4, more precisely from formulas (3.6.30), (3.6.32), (3.6.35), (3.6.36), (3.6.38), and (3.6.39) we have the following estimates:

$$
\begin{aligned}
(-1)^n \frac{S(\mu_n)}{S^{(2)}(\mu_n)} &= (-1)^n \frac{w_1}{w_2} \\
&\approx (-1)^n \frac{\frac{\bar{M}_1}{n} - \frac{\bar{M}}{n} + \frac{\varepsilon_{1n}}{n^2} + \left(\frac{a}{2n\pi}\right)^2 \tilde{\omega}(\mu_n)}{(-1)^n \frac{\bar{M}_1}{2n} - (-1)^n \frac{\bar{M}_2}{2n} + \frac{\varepsilon_{2n}}{n^2} + \left(\frac{a}{2n\pi}\right)^2 \tilde{\omega}_2(\mu_n)} \\
&= \frac{\frac{\bar{M}_1}{n} - \frac{\bar{M}}{n} + \frac{a\tilde{c}_n^1}{n^2} + \left(\frac{a}{2n\pi}\right)^2 \tilde{\omega}(\mu_n)}{\frac{\bar{M}_1}{2n} - \frac{\bar{M}_2}{2n} + \frac{\frac{a}{2}\tilde{c}_n^1}{n^2} + (-1)^n \left(\frac{a}{2n\pi}\right)^2 \tilde{\omega}_2(\mu_n)} \\
&= 1 + \left(\frac{\frac{\bar{M}_1}{n} - \frac{\bar{M}}{n} + \frac{a\tilde{c}_n^1}{n^2} + \left(\frac{a}{2n\pi}\right)^2 \tilde{\omega}(\mu_n)}{\frac{\bar{M}_1}{2n} - \frac{\bar{M}_2}{2n} + \frac{\frac{a}{2}\tilde{c}_n^1}{n^2} + (-1)^n \left(\frac{a}{2n\pi}\right)^2 \tilde{\omega}_2(\mu_n)} - 1 \right) \\
&= 1 + \frac{\frac{\bar{M}_1 + \bar{M}_2 - 2\bar{M}}{2n} + \frac{\frac{a}{2}\tilde{c}_n^1}{n^2} + \left(\frac{a}{2n\pi}\right)^2 (\tilde{\omega}(\mu_n) - (-1)^n \tilde{\omega}_2(\mu_n))}{\frac{\bar{M}_1 - \bar{M}_2}{2n} + \frac{\frac{a}{2}\tilde{c}_n^1}{n^2} + (-1)^n \left(\frac{a}{2n\pi}\right)^2 \tilde{\omega}_2(\mu_n)} \\
&= 1 + \frac{\frac{\frac{a}{2}\tilde{c}_n^1}{n^2} + \left(\frac{a}{2n\pi}\right)^2 (\tilde{\omega}(\mu_n) - (-1)^n \tilde{\omega}_2(\mu_n))}{\frac{\bar{M}_1 - \bar{M}_2}{2n} + \frac{\frac{a}{2}\tilde{c}_n^1}{n^2} + (-1)^n \left(\frac{a}{2n\pi}\right)^2 \tilde{\omega}_2(\mu_n)} \\
&\approx 1 + \frac{\frac{\frac{a}{2}\tilde{c}_n^1}{n^2} + \left(\frac{a}{2n\pi}\right)^2 (\tilde{\omega}(\mu_n) - (-1)^n \tilde{\omega}_2(\mu_n))}{\frac{\bar{M}_1 - \bar{M}_2}{2n}} \\
&= 1 + \left(\frac{a}{\bar{M}_1 - \bar{M}_2}\right) \frac{\tilde{c}_n^1}{n} \\
&\quad + \left(\frac{a^2}{2\pi^2(\bar{M}_1 - \bar{M}_2)}\right) \frac{\tilde{\omega}(\mu_n) - (-1)^n \tilde{\omega}_2(\mu_n)}{n}, \text{ as } n \to \infty \quad (3.4.27)
\end{aligned}
$$

using the definitions of ε_{1n} and ε_{2n} right after formula (3.6.29), the definitions of $\bar{M}$, $\bar{M}_1$, $\bar{M}_2$ given in Auxiliary result 4 and the condition (3.1.1). The functions $\tilde{\omega}(\lambda)$ and $\tilde{\omega}_2(\lambda)$ are defined in (3.6.31) and (3.6.33), respectively. Substituting (3.4.27) into (3.4.22), and using the estimate

$$\ln(1 + x) \approx x, \text{ for } x \approx 0,$$

derivable from Taylor series expansion about $x = 0$, we have that

$$
\begin{aligned}
k_n^1 &\approx \ln\left(1 + \left(\frac{a}{\bar{M}_1 - \bar{M}_2}\right) \frac{\tilde{c}_n^1}{n} + \left(\frac{a^2}{2\pi^2(\bar{M}_1 - \bar{M}_2)}\right) \frac{\tilde{\omega}(\mu_n) - (-1)^n \tilde{\omega}_2(\mu_n)}{n}\right) \\
&\approx \left(\frac{a}{\bar{M}_1 - \bar{M}_2}\right) \frac{\tilde{c}_n^1}{n} + \left(\frac{a^2}{2\pi^2(\bar{M}_1 - \bar{M}_2)}\right) \frac{\tilde{\omega}(\mu_n) - (-1)^n \tilde{\omega}_2(\mu_n)}{n}. \quad (3.4.28)
\end{aligned}
$$

Hence, (3.4.28) implies that

$$\begin{aligned}\sum_{n=1}^{\infty}(nk_n^1)^2 &\approx \sum_{n=1}^{\infty}\left(\left(\frac{a}{\bar{M}_1-\bar{M}_2}\right)\tilde{c}_n^1+\left(\frac{a^2}{2\pi^2(\bar{M}_1-\bar{M}_2)}\right)(\tilde{\omega}(\mu_n)-(-1)^n\tilde{\omega}_2(\mu_n))\right)^2\\ &< \infty,\end{aligned}$$

since $\{\tilde{c}_n^1\}_{n\geq 1}\in l_2$ (see right below (3.6.24)), $\{\tilde{\omega}(\mu_n)\}_{n\geq 1}\in l_2$, $\{\tilde{\omega}_2(\mu_n)\}_{n\geq 1}\in l_2$ by (3.6.43) and (3.6.44), and so (3.4.25) is proved.

Due to (3.1.3) and (3.4.25), Corollary 2 of [18, page 116] applies and we obtain a unique real valued potential function $q_1 \in L^2(0,\frac{a}{2})$ having

$$\{\mu_n\}_{n\geq 1} \text{ and } \{k_n^1\}_{n\geq 1}$$

as its sequences of Dirichlet eigenvalues and respectively norming constants in the sense described in [18, pages 50 and 59].

Note that q_1 is only an $L^2(0,\frac{a}{2})$ function, and not an $H^1(0,\frac{a}{2})$ function, because Problem 6(c) in [18, page 61] cannot be applied, due to the fact that even if the sequence $\{\mu_n\}_{n\geq 1}$ satisfies the hypotheses of this statement, the sequence $\{k_n^1\}_{n\geq 1}$ does not. However, the hypotheses of Corollary 2 mentioned above are satisfied by the two sequences, because $l_2^1 \subset l_2$.

Similarly, (3.1.4) and (3.4.26) and Corollary 2 of [18, page 116] imply the existence and uniqueness of a real valued potential function $\tilde{q}_2 \in L^2(0,\frac{a}{2})$ having

$$\{\nu_n\}_{n\geq 1} \text{ and } \{k_n^2\}_{n\geq 1}$$

as its sequences of Dirichlet eigenvalues and respectively norming constants.

It follows using the theory of direct Sturm-Liouville problems (see Subsection 3.2.9, Subsection 3.2.5, and the definition of Pöschel-Trubowitz norming constant in [18, page 59]) that:

$$\mu_n = \left(\frac{2n\pi}{a}\right)^2 + [q_1] + b_n^1, \text{ as } n\to\infty, \text{ where } \{b_n^1\}_{n\geq 1}\in l_2, \tag{3.4.29}$$

$$S_1(\frac{a}{2};q_1,\mu_n) = 0, \text{ for all } n\geq 1, \tag{3.4.30}$$

$$k_n^1 = \ln\left((-1)^n S_1'(\frac{a}{2};q_1,\mu_n)\right), \text{ for all } n\geq 1. \tag{3.4.31}$$

From (3.4.29) and (3.1.3), and because $\{b_n^1\}_{n\geq 1} \in l_2$ and $\{c_n^1\}_{n\geq 1} \in l_2^1 \subset l_2$ are two sequences converging to 0, we have that:

$$[q_1] = M_1. \tag{3.4.32}$$

From (3.4.30) and the integral representation of $S_1(\frac{a}{2}; q_1, \lambda)$ (see (3.2.20) with $\beta = \frac{a}{2}$ and $p = q_1$) we can write (3.2.37).

From (3.4.31), (3.4.22) and the integral representation of $S_1'(\frac{a}{2}; q_1, \lambda)$ (see (3.2.21) with $\beta = \frac{a}{2}$ and $p = q_1$) we get:

$$\cos(\sqrt{\mu_n}\frac{a}{2}) + \left(\frac{a\cdot[q_1]}{4}\right)\frac{\sin(\sqrt{\mu_n}\frac{a}{2})}{\sqrt{\mu_n}} + \int_0^{\frac{a}{2}} K_x(\frac{a}{2}, t; q_1)\frac{\sin(\sqrt{\mu_n}t)}{\sqrt{\mu_n}}dt = \frac{S(\mu_n)}{S^{(2)}(\mu_n)}, \text{ for all } n \geq 1, \tag{3.4.33}$$

where $K(x, t; q_1)$ is the weak solution to (3.2.18) with $\beta = \frac{a}{2}$ and $p = q_1$. Therefore, by Theorem 4.15(c) of [13, page 147] we have that $K(\frac{a}{2}, \cdot; q_1) \in H^1(0, \frac{a}{2})$ and $K_x(\frac{a}{2}, \cdot; q_1) \in L^2(0, \frac{a}{2})$, because q_1 is only an $L^2(0, \frac{a}{2})$ function.

Using the boundary conditions of (3.2.18) with $\beta = \frac{a}{2}$ and $p = q_1$, and (3.4.32) we can write (3.2.38).

From (3.2.37), (3.4.9), and the completeness in $L^2(0, \frac{a}{2})$ of the set $\{\sin(\sqrt{\mu_n}t) | n \geq 1\}$ (see Auxiliary result 1) we have that:

$$K(\frac{a}{2}, \cdot; q_1) = f_1, \tag{3.4.34}$$

and so $K(\frac{a}{2}, \cdot; q_1) \in H^2(0, \frac{a}{2})$ is the solution to (3.4.9), and due to (3.2.38), $K(\frac{a}{2}, \cdot; q_1)$ also satisfies (3.4.10).

From (3.4.33) and (3.4.32) we get that $K_x(\frac{a}{2}, \cdot; q_1)$ is an $L^2(0, \frac{a}{2})$ solution to the system of integral equations:

$$\int_0^{\frac{a}{2}} g_1(t)\sin(\sqrt{\mu_n}t)dt = \sqrt{\mu_n}\left(\frac{S(\mu_n)}{S^{(2)}(\mu_n)} - \cos(\sqrt{\mu_n}\frac{a}{2}) - \left(\frac{a\cdot M_1}{4}\right)\frac{\sin(\sqrt{\mu_n}\frac{a}{2})}{\sqrt{\mu_n}}\right), \text{ for } n \geq 1. \tag{3.4.35}$$

The uniqueness of the solution to (3.4.35) also follows from the completeness in $L^2(0, \frac{a}{2})$ of the set $\{\sin(\sqrt{\mu_n}t) | n \geq 1\}$ (see Auxiliary result 1). Therefore

$$K_x(\frac{a}{2}, \cdot; q_1) = g_1. \tag{3.4.36}$$

Similarly one obtains:

$$[\tilde{q}_2] = M_2, \tag{3.4.37}$$

$$K(\frac{a}{2}, \cdot; \tilde{q}_2) = f_2, \tag{3.4.38}$$

and

$$K_x(\frac{a}{2}, \cdot; \tilde{q}_2) = g_2, \tag{3.4.39}$$

where $g_2 \in L^2(0, \frac{a}{2})$ solves the system of integral equations:

$$\int_0^{\frac{a}{2}} g_2(t) \sin(\sqrt{\nu_n} t) dt = \sqrt{\nu_n} \left(\frac{S(\nu_n)}{S^{(1)}(\nu_n)} - \cos(\sqrt{\nu_n} \frac{a}{2}) - \left(\frac{a \cdot M_2}{4} \right) \frac{\sin(\sqrt{\nu_n} \frac{a}{2})}{\sqrt{\nu_n}} \right), \text{ for } n \geq 1. \tag{3.4.40}$$

From (3.4.34), the integral representation of $S_1(\frac{a}{2}; q_1, \lambda)$ (see (3.2.20) with $\beta = \frac{a}{2}$ and $p = q_1$), and (3.3.8) we get:

$$S_1(\frac{a}{2}; q_1, \lambda) = S^{(1)}(\lambda), \text{ for all } \lambda \in \mathbb{C}. \tag{3.4.41}$$

and similarly:

$$S_1(\frac{a}{2}; \tilde{q}_2, \lambda) = S^{(2)}(\lambda), \text{ for all } \lambda \in \mathbb{C}. \tag{3.4.42}$$

From (3.4.22) and (3.4.31) we get:

$$S_1'(\frac{a}{2}; q_1, \mu_n) = \frac{S(\mu_n)}{S^{(2)}(\mu_n)}, \text{ for all } n \geq 1, \tag{3.4.43}$$

and similarly

$$S_1'(\frac{a}{2}; \tilde{q}_2, \nu_n) = \frac{S(\nu_n)}{S^{(1)}(\nu_n)}, \text{ for all } n \geq 1. \tag{3.4.44}$$

Next, let q_2 be the reflection of $\tilde{q}_2$ about the line $x = \frac{a}{2}$ (i.e. $q_2(x) = \tilde{q}_2(a - x)$, for $x \in [\frac{a}{2}, a]$) and define q by

$$q(x) = \begin{cases} q_1(x), & \text{if } x \in [0, \frac{a}{2}] \\ q_2(x), & \text{if } x \in [\frac{a}{2}, a]. \end{cases} \tag{3.4.45}$$

By an easy verification of the definition of a Dirichlet eigenvalue it follows that

$$\{\mu_n | n \geq 1\}$$

is the set of *all* Dirichlet eigenvalues of q over $[0, \frac{a}{2}]$ (because they are all the Dirichlet eigenvalues of q_1) and that

$$\{\nu_n | n \geq 1\}$$

is the set of *all* Dirichlet eigenvalues of q over $[\frac{a}{2}, a]$ (because they are all the Dirichlet eigenvalues of $\tilde{q}_2$ and hence of q_2).

Also from (3.4.45), (3.4.32), (3.4.37), and (3.1.1) we obtain:

$$\begin{aligned}
[q] &= \frac{1}{a}\int_0^a q(s)ds \\
&= \frac{1}{a}\left(\int_0^{\frac{a}{2}} q_1(s)ds + \int_{\frac{a}{2}}^a q_2(s)ds\right) \\
&= \frac{1}{a}\left(\int_0^{\frac{a}{2}} q_1(s)ds + \int_0^{\frac{a}{2}} \tilde{q}_2(s)ds\right) \\
&= \frac{1}{a}\left(\frac{a}{2}[q_1] + \frac{a}{2}[\tilde{q}_2]\right) \\
&= \frac{M_1 + M_2}{2} \\
&= M.
\end{aligned} \tag{3.4.46}$$

It is only left to show that $\{\lambda_n | n \geq 1\}$ are all the Dirichlet eigenvalues over the whole interval $[0, a]$. If

$$S(a; q, \lambda_n) = 0, \text{ for all } n \geq 1 \tag{3.4.47}$$

is established, then by the theory of direct Sturm-Liouville problems (see Subsection 3.2.5) it follows that

$$\{\lambda_n | n \geq 1\}$$

are Dirichlet eigenvalues of q over $[0, a]$.

To prove that there are no other Dirichlet eigenvalues of q over $[0, a]$ we reason by contradiction. Suppose λ is another Dirichlet eigenvalue. Because the eigenvalues form a strictly increasing sequence or real numbers we have:

$$\lambda_1 < \lambda_2 < \ldots < \lambda_{N-1} < \lambda < \lambda_N < \ldots < \lambda_n < \ldots,$$

for some $N \geq 1$, which tells that $\lambda_N, \lambda_{N+1}, \ldots, \lambda_n, \ldots$ are the

$$N+1, N+2, \ldots, n+1, \ldots$$

Dirichlet eigenvalues of q, and so by the known asymptotic formula of eigenvalues in the direct Sturm-Liouville theory (see Subsection 3.2.9) we can write:

$$\lambda_n = \left(\frac{(n+1)\pi}{a}\right)^2 + [q] + \gamma_{n+1}, \text{ for all } n \geq N, \tag{3.4.48}$$

for some sequence $\{\gamma_n\}_{n\geq 1} \in l_2$ (and not in l_2^1 because $q \in L^2(0,a)$ and not in $H^1(0,a)$). Using (3.4.48), (3.1.2) and (3.4.46) one has:

$$(2n+1)\left(\frac{\pi}{a}\right)^2 = c_n - \gamma_{n+1}, \text{ for all } n \geq N,$$

which leads to a contradiction by letting $n \to \infty$, because $\{c_n\}_{n\geq 1} \in l_2^1 \subset l_2$ and $\{\gamma_n\}_{n\geq 1} \in l_2$ are two sequences converging to 0. Thus $\{\lambda_n | n \geq 1\}$ are *all* the Dirichlet eigenvalues of q over $[0,a]$.

Formula (3.4.47) follows due to (3.4.17), if we show

$$S(a;q,\lambda_n) = S(\lambda_n), \text{ for all } n \geq 1. \tag{3.4.49}$$

We shall prove that

$$S(a;q,\lambda) = S(\lambda), \text{ for all } \lambda \in \mathbb{C}, \tag{3.4.50}$$

from which (3.4.49) is immediate.

The starting point in proving (3.4.50) is to establish that:

$$\begin{cases} S(a;q,\mu_n) = S(\mu_n), & \text{for all } n \geq 1 \\ S(a;q,\nu_n) = S(\nu_n), & \text{for all } n \geq 1. \end{cases} \tag{3.4.51}$$

Since $q \in L^2(0,a)$ constructed by (3.4.45) is now in our hands, we are permitted to write (3.2.16). This formula along with formulas (3.4.41) - (3.4.44), and with (3.4.18) - (3.4.21) yield (3.4.51).

To complete the proof we return now to proving (3.4.50). Let

$$\Delta(\lambda) = S(a;q,\lambda) - S(\lambda), \text{ for all } \lambda \in \mathbb{C}, \tag{3.4.52}$$

and

$$T(\lambda) = \frac{\Delta(\lambda)}{S_1(\frac{a}{2};q_1,\lambda)S_1(\frac{a}{2};\tilde{q}_2,\lambda)}, \text{ for all } \lambda \in \mathbb{C}. \tag{3.4.53}$$

Each of the functions $S(a;q,\lambda)$, $S_1(\frac{a}{2};q_1,\lambda)$, $S_1(\frac{a}{2};\tilde{q}_2,\lambda)$ is an entire function of λ, by the result of Subsection 3.2.6 with the appropriate choices of β and $p \in L^2(0,\beta)$. The function $S(\lambda)$ is an entire function of λ because

$$S(\lambda) = S(a;Q,\lambda),$$

for any $Q \in H^1(0,a)$ having $\{\lambda_n\}_{n\geq 1}$ as its sequence of Dirichlet eigenvalues (this follows from the definition of $S(\lambda)$, see (3.3.7), and formulas (3.4.8) and 3.2.20 with $\beta = a$ and $p = Q$), and hence we can invoke again the result of Subsection 3.2.6 with $\beta = a$ and $p = Q$.

Since we showed previously that all the Dirichlet eigenvalues of the Sturm-Liouville operator $L^{(q_1)}$ over the interval $[0,\frac{a}{2}]$ are $\{\mu_n|n \geq 1\}$, and all the Dirichlet eigenvalues of the Sturm-Liouville operator $L^{(\tilde{q}_2)}$ over the interval $[0,\frac{a}{2}]$ are $\{\nu_n|n \geq 1\}$, it follows by the discussion in Subsection 3.2.5 that the only zeros of the function

$$\lambda \in \mathbb{C} \to S_1(\frac{a}{2};q_1,\lambda)S_1(\frac{a}{2};\tilde{q}_2,\lambda)$$

are $\{\mu_n|n \geq 1\} \cup \{\nu_n|n \geq 1\}$. They are simple zeros by the result mentioned in Subsection 3.2.6. These are also zeros (not necessary all the zeros) of the function

$$\lambda \in \mathbb{C} \to \Delta(\lambda),$$

due to (3.4.52) and (3.4.51). Therefore, each factor of $S_1(\frac{a}{2};q_1,\lambda)S_1(\frac{a}{2};\tilde{q}_2,\lambda)$ cancels a corresponding factor of $\Delta(\lambda)$, and so the function T is an entire function of λ.

Next, each of the functions $S(a;q,\lambda)$ and $S(\lambda)$ has an integral representation (see (3.2.20) with $\beta = a$ and $p = q$, and the definition (3.3.7) of $S(\lambda)$), where the kernels are $K(a,\cdot;q)$ and respectively f, an $H^1(0,a)$ function (by Theorem 4.15(c) of [13, page 147]) and respectively an $H^2(0,a)$ (by the discussion at the beginning of this section). Hence, integration by parts on the right hand sides of these integral representations is allowed (see Auxiliary result 2) and we have:

$$\begin{aligned}
S(a;q,\lambda) &= \frac{\sin(\sqrt{\lambda}a)}{\sqrt{\lambda}} - K(a,t;q)\frac{\cos(\sqrt{\lambda}t)}{\lambda}\Big|_{t=0}^{t=a} + \int_0^a K_t(a,t;q)\frac{\cos(\sqrt{\lambda}t)}{\lambda}dt \\
&= \frac{\sin(\sqrt{\lambda}a)}{\sqrt{\lambda}} - \left(\frac{1}{2}\int_0^a q(s)ds\right)\frac{\cos(\sqrt{\lambda}a)}{\lambda} + \int_0^a K_t(a,t;q)\frac{\cos(\sqrt{\lambda}t)}{\lambda}dt \\
&\quad \text{(by the BCs of (3.2.18))} \\
&= \frac{\sin(\sqrt{\lambda}a)}{\sqrt{\lambda}} - \left(\frac{a\cdot[q]}{2}\right)\frac{\cos(\sqrt{\lambda}a)}{\lambda} + \frac{\omega_1(\lambda)}{\lambda}, \qquad (3.4.54)
\end{aligned}$$

where

$$\omega_1(\lambda) = \int_0^a K_t(a,t;q)\cos(\sqrt{\lambda}t)dt, \qquad (3.4.55)$$

and

$$\begin{aligned}
S(\lambda) &= \frac{\sin(\sqrt{\lambda}a)}{\sqrt{\lambda}} - f(t)\frac{\cos(\sqrt{\lambda}t)}{\lambda}\Big|_{t=0}^{t=a} + \int_0^a f'(t)\frac{\cos(\sqrt{\lambda}t)}{\lambda}dt \\
&= \frac{\sin(\sqrt{\lambda}a)}{\sqrt{\lambda}} - \left(\frac{a\cdot M}{2}\right)\frac{\cos(\sqrt{\lambda}a)}{\lambda} + \int_0^a f'(t)\frac{\cos(\sqrt{\lambda}t)}{\lambda}dt \text{ (by (3.4.2))} \\
&= \frac{\sin(\sqrt{\lambda}a)}{\sqrt{\lambda}} - \left(\frac{a\cdot M}{2}\right)\frac{\cos(\sqrt{\lambda}a)}{\lambda} + \frac{\omega_2(\lambda)}{\lambda}, \qquad (3.4.56)
\end{aligned}$$

where

$$\omega_2(\lambda) = \int_0^a f'(t)\cos(\sqrt{\lambda}t)dt. \qquad (3.4.57)$$

Using (3.4.54), (3.4.56), and (3.4.46) we can write:

$$\begin{cases} S(a;q,\lambda) = \frac{\sin(\sqrt{\lambda}a)}{\sqrt{\lambda}} - \frac{C_0\cos(\sqrt{\lambda}a)}{\lambda} + \frac{\omega_1(\lambda)}{\lambda} \\ S(\lambda) = \frac{\sin(\sqrt{\lambda}a)}{\sqrt{\lambda}} - \frac{C_0\cos(\sqrt{\lambda}a)}{\lambda} + \frac{\omega_2(\lambda)}{\lambda}, \end{cases} \qquad (3.4.58)$$

where

$$C_0 = \frac{a\cdot[q]}{2} = \frac{a\cdot M}{2}.$$

Next we show that

$$\begin{cases} |\omega_1(\lambda)| \le C_1 e^{a|Im(\sqrt{\lambda})|}, & \text{for all } \lambda\in\mathbb{C} \\ |\omega_2(\lambda)| \le C_2 e^{a|Im(\sqrt{\lambda})|}, & \text{for all } \lambda\in\mathbb{C}, \end{cases} \qquad (3.4.59)$$

for some positive constants C_1 and C_2. Since the integrand in (3.4.57) has a simpler notation than that in (3.4.55), we elaborate only for ω_2, the work for ω_1 being similar. Consider F the even extension of $f' \in L^2(0,a)$ to the symmetric interval $[-a,a]$. Hence $F \in L^2(-a,a)$ and the following hold:

$$\begin{aligned}
|\omega_2(\lambda)| &= |\int_0^a f'(t)\cos(\sqrt{\lambda}t)dt| \\
&= \frac{1}{2}|\int_{-a}^a F(t)\cos(\sqrt{\lambda}t)dt|, \text{ because the integrand is an even function} \\
&= \frac{1}{4}|\int_{-a}^a F(t)\left(e^{i\sqrt{\lambda}t}+e^{-i\sqrt{\lambda}t}\right)dt|, \text{ using } \cos w = \tfrac{e^{iw}+e^{-iw}}{2}, \text{ for } w \in \mathbb{C} \\
&\le \frac{1}{4}\left(\int_{-a}^a |F(t)|\cdot|e^{-Im(\sqrt{\lambda})t}e^{iRe(\sqrt{\lambda})t}|dt + \int_{-a}^a |F(t)|\cdot|e^{Im(\sqrt{\lambda})t}e^{-iRe(\sqrt{\lambda})t}|dt\right) \\
&= \frac{1}{4}\left(\int_{-a}^a |F(t)|\cdot e^{-Im(\sqrt{\lambda})t}|e^{iRe(\sqrt{\lambda})t}|dt + \int_{-a}^a |F(t)|\cdot e^{Im(\sqrt{\lambda})t}|e^{-iRe(\sqrt{\lambda})t}|dt\right) \\
&= \frac{1}{4}\left(\int_{-a}^a |F(t)|\cdot e^{-Im(\sqrt{\lambda})t}dt + \int_{-a}^a |F(t)|\cdot e^{Im(\sqrt{\lambda})t}dt\right) \\
&\le \frac{1}{4}||F||_{L^2}\left[\left(\int_{-a}^a e^{-2Im(\sqrt{\lambda})t}dt\right)^{\frac{1}{2}} + \left(\int_{-a}^a e^{2Im(\sqrt{\lambda})t}dt\right)^{\frac{1}{2}}\right], \\
&\quad \text{(by Cauchy-Schwartz inequality)} \\
&\le \frac{1}{4}||F||_{L^2}\left[2\left(\int_{-a}^a e^{2|Im(\sqrt{\lambda})|\cdot|t|}dt\right)^{\frac{1}{2}}\right], \\
&\quad \text{(since } \pm s \le |s|, \text{ for all } s \in \mathbb{R}, \text{ so } e^{\pm s} \le e^{|s|}) \\
&\le \frac{1}{4}||F||_{L^2}\left[2\left(\int_{-a}^a e^{2|Im(\sqrt{\lambda})|\cdot a}dt\right)^{\frac{1}{2}}\right], \text{ (since } t \text{ runs in } [-a,a], \text{ so } |t| \le a) \\
&= \frac{1}{2}||F||_{L^2}\sqrt{2a}e^{a|Im(\sqrt{\lambda})|} \\
&= C_2 e^{a|Im(\sqrt{\lambda})|}. \qquad (3.4.60)
\end{aligned}$$

Next, we have that:

$$S_1(\frac{a}{2};q_1,\lambda) = \frac{a}{2}\prod_{n=1}^{\infty}\frac{\mu_n-\lambda}{\left(\frac{n\pi}{a/2}\right)^2}, \text{ for all } \lambda \in \mathbb{C} \qquad (3.4.61)$$

by using (A.0.10) with $[\alpha,\beta]=[0,\frac{a}{2}]$, $\hat{p}=q_1$, and the zeros $\{\hat{z}_n|n\ge 1\}$ of

$$\hat{S}(\beta;\hat{p},\hat{\lambda}) = S(\frac{a}{2};q_1,\lambda)$$

being $\{\mu_n|n\ge 1\}$ as shown in formula (3.4.30), and

$$S_1(\frac{a}{2};\tilde{q}_2,\lambda) = \frac{a}{2}\prod_{n=1}^{\infty}\frac{\nu_n-\lambda}{\left(\frac{n\pi}{a/2}\right)^2}, \text{ for all } \lambda \in \mathbb{C} \qquad (3.4.62)$$

by (A.0.1), which was also derivable from (A.0.10), and the zeros of $S_1(\frac{a}{2};\tilde{q}_2,\lambda)$ being $\{\nu_n\}_{n\geq 1}$ (by a similar discussion with that for $\{\mu_n\}_{n\geq 1}$). Next, by slightly adjusting formula (6.2) of [4, page 175] we have that:

$$\frac{\sin(\sqrt{\lambda}\frac{a}{2})}{\sqrt{\lambda}} = \frac{a}{2}\prod_{n=1}^{\infty}\frac{\left(\frac{n\pi}{a/2}\right)^2-\lambda}{\left(\frac{n\pi}{a/2}\right)^2}, \text{ for all } \lambda\in\mathbb{C}. \tag{3.4.63}$$

Due to the asymptotic formulas (3.1.3) and (3.1.4), formulas (3.4.61), (3.4.62), (3.4.63), and similar arguments as those in Lemma 2 of [18, page 167] one has:

$$\begin{cases} S_1(\frac{a}{2};q_1,\lambda) = \frac{\sin(\sqrt{\lambda}\frac{a}{2})}{\sqrt{\lambda}}\left(1+\mathcal{O}\left(\frac{\ln m}{m}\right)\right), \text{ uniformly on the circles } |\lambda| = \left(\frac{(m+\frac{1}{2})\pi}{a/2}\right)^2 \\ S_1(\frac{a}{2};\tilde{q}_2,\lambda) = \frac{\sin(\sqrt{\lambda}\frac{a}{2})}{\sqrt{\lambda}}\left(1+\mathcal{O}\left(\frac{\ln m}{m}\right)\right), \text{ uniformly on the circles } |\lambda| = \left(\frac{(m+\frac{1}{2})\pi}{a/2}\right)^2. \end{cases} \tag{3.4.64}$$

Next we show that if $m\in\mathbb{Z}$ and $|\lambda| = \left(\frac{(m+\frac{1}{2})\pi}{a/2}\right)^2$, then

$$|\sqrt{\lambda}\frac{a}{2}-n\pi| \geq \frac{\pi}{4}, \text{ for all } n\in\mathbb{Z}. \tag{3.4.65}$$

Our intention in establishing (3.4.65) is to be able to apply Lemma 4.8 of [13, page 136] with $z=\sqrt{\lambda}\frac{a}{2}$, from which we obtain

$$|\sin(\sqrt{\lambda}\frac{a}{2})| > \frac{1}{4}e^{\frac{a}{2}|Im(\sqrt{\lambda})|}, \text{ for } \lambda \text{ on the circles } |\lambda| = \left(\frac{(m+\frac{1}{2})\pi}{a/2}\right)^2,\ m\in\mathbb{Z}. \tag{3.4.66}$$

Let $m\in\mathbb{Z}$ be chosen and fixed. Let $\lambda\in\mathbb{C}$ be such that $|\lambda| = \left(\frac{(m+\frac{1}{2})\pi}{a/2}\right)^2$. Let $\sqrt{\lambda}$ be either one of the two complex square roots of λ, and let $n\in\mathbb{Z}$ be arbitrary. Then the following calculations hold:

$$
\begin{aligned}
|\sqrt{\lambda}\frac{a}{2}-n\pi| &\geq ||\sqrt{\lambda}\frac{a}{2}|-|n\pi||, \text{ because } |w_1-w_2|\geq ||w_1|-|w_2|| \\
&= ||m+\frac{1}{2}|-|n||\pi \\
&= \begin{cases} |m+\frac{1}{2}-|n||\pi, & \text{if } m=1,2,3,\dots \\ |\frac{1}{2}-|n||\pi, & \text{if } m=0 \\ |-(m+\frac{1}{2})-|n||\pi, & \text{if } m=-1,-2,-3,\dots \end{cases} \\
&= \begin{cases} |(|m|-|n|)+\frac{1}{2}|\pi, & \text{if } m=1,2,3,\dots \\ |\frac{1}{2}-|n||\pi, & \text{if } m=0 \\ |(|m|-|n|)-\frac{1}{2}|\pi, & \text{if } m=-1,-2,-3,\dots \end{cases} \\
&\geq |||m|-|n||-\frac{1}{2}|\pi, \text{ because } |w_1\pm w_2|\geq ||w_1|-|w_2|| \\
&\geq \frac{\pi}{2}. \qquad (3.4.67)
\end{aligned}
$$

The last inequality in (3.4.67) comes from the fact that m, n are integers, so

$$
\begin{cases} ||m|-|n||\geq 1, & \text{if } |n|\neq |m| \\ ||m|-|n||=0, & \text{if } |n|=|m|. \end{cases}
$$

So (3.4.65) immediately follows from (3.4.67), and further (3.4.66) holds.

Using $\lim_{m\to\infty}\frac{\ln m}{m}=0$, and formulas (3.4.64), (3.4.66), we have that:

$$
\begin{cases} |S_1(\frac{a}{2};q_1,\lambda)| = |\frac{\sin(\sqrt{\lambda}\frac{a}{2})}{\sqrt{\lambda}}|\left(1+\mathcal{O}\left(\frac{\ln m}{m}\right)\right) > (\frac{1}{4}\frac{e^{\frac{a}{2}|Im(\sqrt{\lambda})|}}{|\sqrt{\lambda}|})\frac{1}{2}, \\ |S_1(\frac{a}{2};\tilde{q}_2,\lambda)| = |\frac{\sin(\sqrt{\lambda}\frac{a}{2})}{\sqrt{\lambda}}|\left(1+\mathcal{O}\left(\frac{\ln m}{m}\right)\right) > (\frac{1}{4}\frac{e^{\frac{a}{2}|Im(\sqrt{\lambda})|}}{|\sqrt{\lambda}|})\frac{1}{2}, \end{cases} \qquad (3.4.68)
$$

uniformly on the circles $|\lambda|=\left(\frac{(m+\frac{1}{2})\pi}{a/2}\right)^2$, $m\geq m_0$.

Now combine (3.4.52), (3.4.53), (3.4.58), (3.4.59), and (3.4.68) to infer that

$$
\begin{aligned}
|T(\lambda)| &\leq \frac{\frac{|\omega_1(\lambda)|}{|\lambda|}+\frac{|\omega_2(\lambda)|}{|\lambda|}}{|S_1(\frac{a}{2};q_1,\lambda)|\cdot|S_1(\frac{a}{2};\tilde{q}_2,\lambda)|} \\
&\leq \frac{\frac{(C_1+C_2)e^{a|Im(\sqrt{\lambda})|}}{|\lambda|}}{\frac{1}{64}\frac{e^{a|Im(\sqrt{\lambda})|}}{|\lambda|}} \\
&= C, \text{ uniformly on the circles } |\lambda|=\left(\frac{(m+\frac{1}{2})\pi}{a/2}\right)^2,\ m\geq m_0. \qquad (3.4.69)
\end{aligned}
$$

Since T is an entire function of λ, formula (3.4.69) and Maximum Modulus Theorem (see [4, Theorem 1.2, page 128]) imply that

$$\begin{aligned}\max_{|\lambda|\leq r_m} |T(\lambda)| &= \max_{|\lambda|=r_m} |T(\lambda)| \\ &\leq C, \text{ for all } m \geq m_0. \end{aligned} \tag{3.4.70}$$

where $r_m = \left(\frac{(m+\frac{1}{2})\pi}{a/2}\right)^2$. Since T is an entire function, it follows that T is bounded on the compact disk $|\lambda| = r_{m_0}$, and for $\lambda \in \mathbb{C}$ such that $|\lambda| > r_{m_0}$, we can find an $m > m_0$ such that $|\lambda| \leq r_m$ (possible because $r_m \to \infty$), and hence we can apply (3.4.70) and have that $|T(\lambda)| \leq C$. This shows that T is bounded in $\mathbb{C}$, and being an entire function it follows by Liouville Theorem (see [4, Theorem 3.4, page 77]) that

$$T(\lambda) = \tilde{C}, \text{ for all } \lambda \in \mathbb{C}, \tag{3.4.71}$$

for some constant $\tilde{C} \in \mathbb{C}$.

Formula (3.4.71) and similar estimates with those in (3.4.69) give for real values of λ such that $\lambda = \left(\frac{(m+\frac{1}{2})\pi}{a/2}\right)^2$, $m \geq m_0$ that:

$$\begin{aligned}|\tilde{C}| &= |T(\lambda)| \\ &\leq \frac{\frac{|\omega_1(\lambda)|}{|\lambda|} + \frac{|\omega_2(\lambda)|}{|\lambda|}}{|S_1(\frac{a}{2}; q_1, \lambda)| \cdot |S_1(\frac{a}{2}; \tilde{q}_2, \lambda)|} \\ &\leq \frac{\frac{|\omega_1(\lambda)|}{|\lambda|} + \frac{|\omega_2(\lambda)|}{|\lambda|}}{\frac{1}{64}\frac{e^{a|Im(\sqrt{\lambda})|}}{|\lambda|}} \\ &= 64(|\omega_1(\lambda)| + |\omega_2(\lambda)|), \text{ because } Im(\sqrt{\lambda}) = 0, \text{ for those } \lambda\text{'s}. \end{aligned} \tag{3.4.72}$$

Using (3.4.72) with $\sqrt{\lambda} = \pm\frac{(m+\frac{1}{2})\pi}{a/2}$, for $m \geq m_0$, the facts that

$$\lim_{\sqrt{\lambda}\to\pm\infty} \omega_i(\lambda) = 0, \ i = 1, 2 \tag{3.4.73}$$

(see Auxiliary result 3) and (3.4.71) we obtain that

$$T(\lambda) = \tilde{C} = 0, \text{ for } \lambda \in \mathbb{C},$$

and hence by (3.4.53) and (3.4.52), the identity (3.4.50) is proved. $\square$

3.5 Under which conditions a potential function is unique?

Theorem 2 *If $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$ are three sequences of real numbers such that neither two of them have common elements, then there exists at most one potential function having the three sequences as its sequences of Dirichlet eigenvalues over the intervals $[0,a]$, $[0,\frac{a}{2}]$, $[\frac{a}{2},a]$, respectively.*

Observation: We would like to draw the reader's attention that the proof is similar if $\frac{a}{2}$ is replaced by any $a_0 \in (0,a)$. In this case formulas (3.2.16) and (3.5.4) are to be replaced by formula (5.1.9) and its counterpart, respectively. How to derive (5.1.9) is presented in Auxiliary result 8 of Section 5.1 where the inverse three Dirichlet spectra problem for the case of the interval $[0,a]$ broken at an arbitrary point a_0 is analyzed.

Proof of Theorem 2: If there is no potential function, then the statement is proved. Suppose q and q^* are two potential functions having the same Dirichlet eigenvalues, namely $\{\lambda_n | n \geq 1\}$, $\{\mu_n | n \geq 1\}$ and $\{\nu_n | n \geq 1\}$, and let $q_1^* = q^*|_{[0,\frac{a}{2}]}$, $q_2^* = q^*|_{[\frac{a}{2},a]}$, and define

$$\tilde{q}_2^*(x) = q_2^*(a-x), \text{ for } x \in [0, \tfrac{a}{2}].$$

Then by the asymptotic formulas of the three sets of eigenvalues $\{\lambda_n | n \geq 1\}$, $\{\mu_n | n \geq 1\}$, $\{\nu_n | n \geq 1\}$ provided by the direct Sturm-Liouville theory (see formulas (3.2.29) - (3.2.31) of Subsection 3.2.9) one has that:

$$[q] = [q^*], \tag{3.5.1}$$

$$[q_1] = [q_1^*], \tag{3.5.2}$$

$$[\tilde{q}_2] = [q_2] = [q_2^*] = [\tilde{q}_2^*]. \tag{3.5.3}$$

In the same manner formula (3.2.16) was derived we obtain:

$$S(a;q^*,\lambda) = S_1(\frac{a}{2};q_1^*,\lambda)S_1'(\frac{a}{2};\tilde{q}_2^*,\lambda) + S_1'(\frac{a}{2};q_1^*,\lambda)S_1(\frac{a}{2};\tilde{q}_2^*,\lambda). \tag{3.5.4}$$

Since μ_n's are the Dirichlet eigenvalues of both $L^{(q_1)}$ and $L^{(q_1^*)}$ over the interval $[0,\frac{a}{2}]$, it follows by the general result of the theory of direct Sturm-Liouville problems in Subsection 3.2.5 that:

$$S_1(\frac{a}{2};q_1,\mu_n) = 0 = S_1(\frac{a}{2};q_1^*,\mu_n), \text{ for all } n \geq 1. \tag{3.5.5}$$

Taking $\beta = \frac{a}{2}$ and $p = q_1$ and $p = q_1^*$ respectively in (3.2.20), and using (3.5.5) one gets:

$$\begin{cases} \frac{\sin(\sqrt{\mu_n}\frac{a}{2})}{\sqrt{\mu_n}} + \int_0^{\frac{a}{2}} K(\frac{a}{2}, t; q_1) \frac{\sin(\sqrt{\mu_n} t)}{\sqrt{\mu_n}} dt = 0, \text{ for all } n \geq 1 \\ \frac{\sin(\sqrt{\mu_n}\frac{a}{2})}{\sqrt{\mu_n}} + \int_0^{\frac{a}{2}} K(\frac{a}{2}, t; q_1^*) \frac{\sin(\sqrt{\mu_n} t)}{\sqrt{\mu_n}} dt = 0, \text{ for all } n \geq 1, \end{cases} \tag{3.5.6}$$

where $K(x, t; q_1)$ and $K(x, t; q_1^*)$ are the weak solutions to the Goursat problem (3.2.18) with $([0, \beta], p) = \big([0, \frac{a}{2}], q_1\big)$ and respectively $([0, \beta], p) = \big([0, \frac{a}{2}], q_1^*\big)$.

Formula (3.5.6) together with the completeness of the set

$$\{\sin(\sqrt{\mu_n} t)\}_{n \geq 1}$$

in $L^2(0, \frac{a}{2})$ imply that

$$K(\frac{a}{2}, \cdot; q_1) = K(\frac{a}{2}, \cdot; q_1^*), \text{ in } L^2(0, \tfrac{a}{2}). \tag{3.5.7}$$

Note that the completeness of the set

$$\{\sin(\sqrt{\mu_n} t)\}_{n \geq 1}$$

in $L^2(0, \frac{a}{2})$ follows due to the Auxiliary result 1 with the mention that the asymptotic formula for $\{\mu_n\}_{n\geq 1}$ needed in the hypotheses of this result is satisfied because here $\{\mu_n\}_{n\geq 1}$ is the sequence of Dirichlet eigenvalues on $[0, \frac{a}{2}]$ corresponding to q and q^*, (hence of q_1 and q_1^*) and therefore the result of Subsection 3.2.9 applies.

Similarly, one obtains that:

$$K(\frac{a}{2}, \cdot; \tilde{q}_2) = K(\frac{a}{2}, \cdot; \tilde{q}_2^*), \text{ in } L^2(0, \tfrac{a}{2}). \tag{3.5.8}$$

and

$$K(a, \cdot; q) = K(a, \cdot; q^*), \text{ in } L^2(0, a). \tag{3.5.9}$$

Note that in obtaining (3.5.8), the fact that $\{\nu_n\}_{n\geq 1}$ are the zeros of

$$\begin{cases} \lambda \in \mathbb{C} \to S_1(\frac{a}{2}; \tilde{q}_2, \lambda) \\ \lambda \in \mathbb{C} \to S_1(\frac{a}{2}; \tilde{q}_2^*, \lambda) \end{cases}$$

was used. That they are the zeros of these functions is seen as follows: since $\{\nu_n\}_{n\geq 1}$ are the Dirichlet eigenvalues of q and q^* on $[\frac{a}{2}, a]$, hence of q_2 and q_2^*, the result of Subsection 3.2.5 applies and one has

$$S_2(a; q_2, \nu_n) = 0 = S_2(a; q_2^*, \nu_n), \text{ for all } n \geq 1,$$

which in combination with (3.2.9) and its counterpart for q^* yields

$$S_1(\frac{a}{2};\tilde{q}_2,\nu_n) = 0 = S_1(\frac{a}{2};\tilde{q}_2^*,\nu_n),\ \text{for all } n \geq 1. \tag{3.5.10}$$

Furthermore, (3.5.10) and the result of Subsection 3.2.5 tell that $\{\nu_n\}_{n\geq 1}$ are the Dirichlet eigenvalues of $\tilde{q}_2$ and of $\tilde{q}_2^*$. Hence, by the result of Subsection 3.2.9 they have the asymptotic formula needed in the hypotheses of Auxiliary result 1, so the set

$$\{\sin(\sqrt{\nu_n}t)\}_{n\geq 1}$$

is complete in $L^2(0,\frac{a}{2})$.

Using the integral representation of the characteristic functions $S_1(\frac{a}{2};q_1,\lambda)$ and $S_1(\frac{a}{2};q_1^*,\lambda)$; $S_1(\frac{a}{2};\tilde{q}_2,\lambda)$ and $S_1(\frac{a}{2};\tilde{q}_2^*,\lambda)$; $S(a;q,\lambda)$ and $S(a;q^*,\lambda)$ (see (3.2.20)), and formulas (3.5.7), (3.5.8) and (3.5.9) the following are obtained:

$$S_1(\frac{a}{2};q_1,\lambda) = S_1(\frac{a}{2};q_1^*,\lambda),\ \text{for all } \lambda \in \mathbb{C}, \tag{3.5.11}$$

$$S_1(\frac{a}{2};\tilde{q}_2,\lambda) = S_1(\frac{a}{2};\tilde{q}_2^*,\lambda),\ \text{for all } \lambda \in \mathbb{C}, \tag{3.5.12}$$

$$S(a;q,\lambda) = S(a;q^*,\lambda),\ \text{for all } \lambda \in \mathbb{C}. \tag{3.5.13}$$

We make now an important observation: because $\{\mu_n | n \geq 1\}$ are *all* the Dirichlet eigenvalues of $L^{(q_1)}$ and of $L^{(q_1^*)}$ over the interval $[0,\frac{a}{2}]$ (hence, by the general result of Subsection 3.2.5, μ_n's are the *only* zeros of the characteristic functions $S_1(\frac{a}{2};q_1,\cdot)$ and $S_1(\frac{a}{2};q_1^*,\cdot)$), and $\{\nu_n | n \geq 1\}$ are *all* the Dirichlet eigenvalues of $L^{(\tilde{q}_2)}$ and of $L^{(\tilde{q}_2^*)}$ over the interval $[0,\frac{a}{2}]$ (hence, by the same result, ν_n's are the *only* zeros of the characteristic functions $S_1(\frac{a}{2};\tilde{q}_2,\cdot)$ and $S_1(\frac{a}{2};\tilde{q}_2^*,\cdot)$), and because $\{\mu_n | n \geq 1\} \cap \{\nu_n | n \geq 1\} = \phi$ we have that

$$\begin{cases} S_1(\frac{a}{2};q_1,\nu_n) \neq 0,\ \text{for all } n \geq 1, \\ S_1(\frac{a}{2};q_1^*,\nu_n) \neq 0,\ \text{for all } n \geq 1, \\ S_1(\frac{a}{2};\tilde{q}_2,\mu_n) \neq 0,\ \text{for all } n \geq 1, \\ S_1(\frac{a}{2};\tilde{q}_2^*,\mu_n) \neq 0,\ \text{for all } n \geq 1. \end{cases} \tag{3.5.14}$$

Using (3.5.13), (3.5.12), (3.5.5) and the last two relationships of (3.5.14) in (3.2.16) and in (3.5.4) one obtains:

$$\begin{aligned} S_1'(\frac{a}{2};q_1,\mu_n) &= \frac{S(a;q,\mu_n)}{S_1(\frac{a}{2};\tilde{q}_2,\mu_n)} \\ &= \frac{S(a;q^*,\mu_n)}{S_1(\frac{a}{2};\tilde{q}_2^*,\mu_n)} \\ &= S_1'(\frac{a}{2};q_1^*,\mu_n), \text{ for all } n \geq 1. \end{aligned} \tag{3.5.15}$$

Now use the integral representation formula (3.2.21) with $\beta = \frac{a}{2}$ and $p = q_1$ and respectively $p = q_1^*$ and formula (3.5.2) to write:

$$\begin{aligned} S_1'(\frac{a}{2};q_1,\mu_n) &= \cos(\sqrt{\mu_n}\frac{a}{2}) + \left(\frac{1}{2}\int_0^{\frac{a}{2}} q_1(s)ds\right) \cdot \frac{\sin(\sqrt{\mu_n}\frac{a}{2})}{\sqrt{\mu_n}} + \int_0^{\frac{a}{2}} K_x(\frac{a}{2},t;q_1)\frac{\sin(\sqrt{\mu_n}t)}{\sqrt{\mu_n}}dt \\ &= \cos(\sqrt{\mu_n}\frac{a}{2}) + \left(\frac{a[q_1]}{4}\right) \cdot \frac{\sin(\sqrt{\mu_n}\frac{a}{2})}{\sqrt{\mu_n}} + \int_0^{\frac{a}{2}} K_x(\frac{a}{2},t;q_1)\frac{\sin(\sqrt{\mu_n}t)}{\sqrt{\mu_n}}dt \end{aligned} \tag{3.5.16}$$

and

$$\begin{aligned} S_1'(\frac{a}{2};q_1^*,\mu_n) &= \cos(\sqrt{\mu_n}\frac{a}{2}) + \left(\frac{1}{2}\int_0^{\frac{a}{2}} q_1^*(s)ds\right) \cdot \frac{\sin(\sqrt{\mu_n}\frac{a}{2})}{\sqrt{\mu_n}} + \int_0^{\frac{a}{2}} K_x(\frac{a}{2},t;q_1^*)\frac{\sin(\sqrt{\mu_n}t)}{\sqrt{\mu_n}}dt \\ &= \cos(\sqrt{\mu_n}\frac{a}{2}) + \left(\frac{a[q_1^*]}{4}\right) \cdot \frac{\sin(\sqrt{\mu_n}\frac{a}{2})}{\sqrt{\mu_n}} + \int_0^{\frac{a}{2}} K_x(\frac{a}{2},t;q_1^*)\frac{\sin(\sqrt{\mu_n}t)}{\sqrt{\mu_n}}dt. \end{aligned} \tag{3.5.17}$$

Formulas (3.5.16), (3.5.17), along with (3.5.15), (3.5.2), and the completeness of the set

$$\{\sin(\sqrt{\mu_n}\cdot)\}_{n\geq 1}$$

in $L^2(0,\frac{a}{2})$ imply that

$$K_x(\frac{a}{2},\cdot;q_1) = K_x(\frac{a}{2},\cdot;q_1^*), \text{ in } L^2(0,\tfrac{a}{2}). \tag{3.5.18}$$

Introduce the functions:

$$W(x,t) = K(x,t;q_1) - K(x,t;q_1^*)$$

and

$$r(x) = q_1(x) - q_1^*(x).$$

Due to the fact that $K(x,t;q_1)$ and $K(x,t;q_1^*)$ are the solutions to the Goursat problems (3.2.18) with $([0,\beta],p) = \big([0,\frac{a}{2}],q_1\big)$ and respectively $([0,\beta],p) = \big([0,\frac{a}{2}],q_1^*\big)$, and due to equations (3.5.7) and (3.5.18), the pair (W, r) solves:

$$\begin{cases} W_{xx}(x,t) - W_{tt}(x,t) - q_1(x)W(x,t) = K(x,t;q_1^*)r(x), & 0 < t < x < \frac{a}{2} \\ W(x,x) = \frac{1}{2}\int_0^x r(s)ds, & 0 \le x \le \frac{a}{2} \\ W(x,0) = 0, & 0 \le x \le \frac{a}{2} \\ W(\frac{a}{2},\cdot) = 0, & \text{in } L^2(0,\frac{a}{2}) \\ W_x(\frac{a}{2},\cdot) = 0, & \text{in } L^2(0,\frac{a}{2}). \end{cases} \tag{3.5.19}$$

Note: In (3.5.19) the functions $q_1(x)$ and $K(x,t;q_1^*)$ are known, because q and q^* were assumed known potentials with the same three sequences of eigenvalues, and hence q_1, q_1^*, $\tilde{q}_2$, $\tilde{q}_2^*$, and the solutions

$$K(x,t;q_1), K(x,t;q_1^*), K(x,t;\tilde{q}_2), K(x,t;\tilde{q}_2^*)$$

to the Goursat problems (3.2.18) with the appropriate choices of $([0,\beta],p)$ will be known too.

Now we can apply Theorem 4.17(b) of [13, page 152] to conclude that

$$(W,r) = (0,0) \in C(\bar{\Delta}_0) \times L^2(0,\frac{a}{2})$$

is the only solution of (3.5.19). Here $\bar{\Delta}_0 = \{(x,t) | 0 \le t \le x \le \frac{a}{2}\}$. That means that

$$q_1 = q_1^* \text{ in } L^2(0,\tfrac{a}{2}). \tag{3.5.20}$$

A similar work shows that

$$\tilde{q}_2 = \tilde{q}_2^* \text{ in } L^2(0,\tfrac{a}{2}), \tag{3.5.21}$$

which further implies, by the definition of q_2 and q_2^*, that

$$q_2 = q_2^* \text{ in } L^2(0,\tfrac{a}{2}), \tag{3.5.22}$$

Now it is clear from (3.5.20) and (3.5.22) that $q = q^*$. $\square$

3.6 Auxiliary results

The following are needed results in Section 3.4.

Auxiliary result 1 *1. Given the set* $\{\lambda_n\}_{n\geq 1}$ *satisfying (3.1.2), the set* $\{\sin(\sqrt{\lambda_n}t)\}_{n\geq 1}$ *is complete in* $L^2(0,a)$.

2. Given the sets $\{\mu_n\}_{n\geq 1}$ *and* $\{\nu_n\}_{n\geq 1}$ *satisfying (3.1.3) and (3.1.4) respectively, the sets* $\{\sin(\sqrt{\mu_n}t)\}_{n\geq 1}$ *and* $\{\sin(\sqrt{\nu_n}t)\}_{n\geq 1}$ *are complete in* $L^2(0,\frac{a}{2})$.

Proof: We shall prove only assertion 1, the proof of assertion 2 is similar. We start from Kadec's $\frac{1}{4}$-theorem (see [23, Theorem 14, page 42]) by which the set

$$\{e^{i\theta_n s} | n = 0, \pm 1, \pm 2, \ldots\}$$

is a Riesz basis for $L^2(-\pi,\pi)$ (see definition of Riesz basis in [23, page 31]), if for some $L > 0$ the following are satisfied:

$$|\theta_n - n| \leq L < \frac{1}{4}, \text{ for all } n = 0, \pm 1, \pm 2, \ldots. \tag{3.6.1}$$

Since there exists a bijection between $[-\pi,\pi]$ and $[-a,a]$ (here $a > 0$), given by

$$s \in [-\pi,\pi] \leftrightarrow t = \frac{a}{\pi}s \in [-a,a],$$

we obtain after defining $\omega_n = \frac{\pi}{a}\theta_n$ that the set

$$\{e^{i\omega_n t} | n = 0, \pm 1, \pm 2, \ldots\}$$

is a Riesz basis for $L^2(-a,a)$ (use the equivalence (1) $\leftrightarrow$ (3) in [23, Theorem 9, page 32]), if for some $L > 0$ the following are satisfied:

$$|\omega_n - \frac{n\pi}{a}| \leq L < \frac{\pi}{4a}, \text{ for all } n = 0, \pm 1, \pm 2, \ldots. \tag{3.6.2}$$

From (3.1.2) we deduce that:

$$\sqrt{\lambda_n} = \frac{n\pi}{a} + \frac{a}{2n\pi}M + \frac{\tilde{c}_n}{n^2}, \text{ as } n \to \infty, \text{ where } (\tilde{c}_n)_{n\geq 1} \in l_2,$$

which implies that for any choice of $L \in (0, \frac{\pi}{4a})$ there exists a first index N such that

$$|\sqrt{\lambda_n} - \frac{n\pi}{a}| \leq L < \frac{\pi}{4a}, \text{ for all } n = N+1, N+2, \ldots. \tag{3.6.3}$$

Now define

$$\omega_n = \begin{cases} \sqrt{\lambda_n}, & \text{if } n = N+1, N+2, \ldots \\ \frac{n\pi}{a}, & \text{if } n = 0, \pm 1, \pm 2, \ldots, \pm N \\ -\sqrt{\lambda_{-n}}, & \text{if } n = -(N+1), -(N+2), \ldots \end{cases} \tag{3.6.4}$$

Equations (3.6.4) and (3.6.3) imply that (3.6.2) is satisfied, and so by the above mentioned version of Kadec's $\frac{1}{4}$-theorem we have that the set

$$\{e^{i\omega_n t} | n = 0, \pm 1, \pm 2, \ldots\} \text{ is a Riesz basis in } L^2(-a, a). \tag{3.6.5}$$

Using the implication (1) $\rightarrow$ (3) in [23, Theorem 9, page 32], the set is also complete in $L^2(-a,a)$. Next apply repeatedly Theorem 7 of [23, page 129] $2N$ times to replace ω_n by $\pm\sqrt{\lambda_{|n|}}$ for $n = \pm 1, \pm 2, \ldots, \pm N$ and we obtain that the set

$$\{1\} \cup \{e^{\pm i\sqrt{\lambda_n} t} | n = 1, 2, \ldots\} \text{ is complete in } L^2(-a, a). \tag{3.6.6}$$

Next, it follows from this that the set

$$\{\sin(\sqrt{\lambda_n} t) | n = 1, 2, 3 \ldots\} \text{ is complete in } L^2(0, a). \tag{3.6.7}$$

(Here the L^2 spaces are to be understood as the spaces of real-valued functions, squared integrable.) To see (3.6.7), let $f \in L^2(0,a)$ be such that:

$$\int_0^a f(t)\sin(\sqrt{\lambda_n} t)dt = 0, \;\; n = 1, 2, 3, \ldots. \tag{3.6.8}$$

Then consider the odd extension of f to $[-a, a]$ defined by:

$$F(t) = \begin{cases} f(t), & \text{if } 0 \le t \le a \\ -f(-t), & \text{if } -a \le t < 0. \end{cases} \tag{3.6.9}$$

Using (3.6.8) and (3.6.9), and the facts that the integration is over a symmetric interval and that the integrand is an odd or even function we get that:

$$\begin{cases} \int_{-a}^a F(t)dt = 0, & \\ \int_{-a}^a F(t)\cos(\sqrt{\lambda_n} t)dt = 0, & n = 1, 2, 3, \ldots \\ \int_{-a}^a F(t)\sin(\sqrt{\lambda_n} t)dt = 2\int_0^a f(t)\sin(\sqrt{\lambda_n} t)dt = 0, & n = 1, 2, 3, \ldots \end{cases} \tag{3.6.10}$$

The fact that F is a real-valued function and (3.6.10) imply that F is orthogonal to the set $\{1\} \cup \{e^{\pm i\sqrt{\lambda_n}t} | n = 1, 2, 3 \ldots\}$, and so by (3.6.6) we get that $F = 0$ in $L^2(-a, a)$. This along with (3.6.9) imply that $f = 0$ in $L^2(0, a)$. Therefore (3.6.7) is proved. $\square$

Auxiliary result 2 *The following integration by parts formula*

$$\int_\alpha^\beta f(x)g(x)dx = f(\beta)G(\beta) - f(\alpha)G(\alpha) - \int_\alpha^\beta f'(x)G(x)dx$$

holds if $f \in H^1(\alpha, \beta)$ *and* $g \in C[\alpha, \beta]$, *where* G *is an antiderivative of* g.

Proof: Directly from the definition of $H^1(\alpha, \beta)$-space we have that there exists $F \in L^2(\alpha, \beta)$ such that:

$$f(x) = C + \int_\alpha^x F(s)ds, \text{ for all } x \in [\alpha, \beta]. \tag{3.6.11}$$

Since $C[\alpha, \beta]$ is dense in $L^2(\alpha, \beta)$, it follows that there exists a sequence of functions $\{F_n\}_{n\geq 1}$ in $C[\alpha, \beta]$, such that:

$$||F_n - F||_{L^2} \to 0, \text{ as } n \to \infty. \tag{3.6.12}$$

Next, define the sequence $\{f_n\}_{n\geq 1}$ by

$$f_n(x) = C + \int_\alpha^x F_n(s)ds, \text{ for all } x \in [\alpha, \beta]. \tag{3.6.13}$$

Since $\{F_n\}_{n\geq 1} \subset C[\alpha, \beta]$, we have that $\{f_n\}_{n\geq 1} \subset C^1[\alpha, \beta]$. By (3.6.11), (3.6.13), and Cauchy-Schwartz inequality we obtain:

$$\begin{aligned} \sup_{\alpha \leq x \leq \beta} |f_n(x) - f(x)| &\leq \int_\alpha^\beta |F_n(s) - F(s)|ds \\ &\leq \sqrt{\beta - \alpha}||F_n - F||_{L^2}. \end{aligned} \tag{3.6.14}$$

By (3.6.12) and (3.6.14) we have that

$$f_n(x)g(x) \to f(x)g(x), \text{ for all } x \in [\alpha, \beta], \tag{3.6.15}$$

and that

$$\begin{aligned} |f_n(x)g(x)| &\leq |f_n(x) - f(x)| \cdot |g(x)| + |f(x)| \cdot |g(x)| \\ &\leq (||f_n - f||_\infty + ||f||_\infty) \cdot |g(x)| \\ &\leq \left(\sqrt{\beta - \alpha}||F_n - F||_{L^2} + ||f||_\infty\right) \cdot |g(x)| \\ &\leq \tilde{C}|g(x)|, \text{ for all } x \in [\alpha, \beta] \text{ and all } n \geq N, \end{aligned} \tag{3.6.16}$$

hence

$$|f_n(x)g(x)| \leq \max(||f_1||_\infty, \ldots, ||f_{N-1}||_\infty, \tilde{C})|g(x)|, \text{ for all } x \in [\alpha, \beta] \text{ and all } n \geq 1, \quad (3.6.17)$$

where the function $\max(||f_1||_\infty, \ldots, ||f_{N-1}||_\infty, \tilde{C})|g(x)|$ is an $L^1(\alpha, \beta)$ function, because $g \in C[\alpha, \beta]$. Using (3.6.15), (3.6.17) and the Lebesgue Convergence Theorem (see [19, Theorem 16, page 91]) we conclude that

$$\int_\alpha^\beta f_n(x)g(x)dx \to \int_\alpha^\beta f(x)g(x)dx. \quad (3.6.18)$$

We also write:

$$\begin{aligned} \int_\alpha^\beta f_n(x)g(x)dx &= \int_\alpha^\beta f_n(x)G'(x)dx \\ &= f_n(x)G(x)|_{x=\alpha}^{x=\beta} - \int_\alpha^\beta f_n'(x)G(x)dx \\ &= f_n(\beta)G(\beta) - f_n(\alpha)G(\alpha) - \int_\alpha^\beta F_n(x)G(x)dx. \end{aligned} \quad (3.6.19)$$

The last identity in (3.6.19) follows from (3.6.13). Next, observe that

$$\begin{aligned} |\int_\alpha^\beta F_n(x)G(x)dx - \int_\alpha^\beta F(x)G(x)dx| &\leq ||G||_\infty \int_\alpha^\beta |F_n(x) - F(x)|dx \\ &\leq ||G||_\infty \sqrt{\beta - \alpha}||F_n - F||_{L^2}, \end{aligned}$$

which in combination with (3.6.12) leads to:

$$\int_\alpha^\beta F_n(x)G(x)dx \to \int_\alpha^\beta F(x)G(x)dx, \text{ as } n \to \infty \quad (3.6.20)$$

Because of (3.6.14) and (3.6.12) we can write:

$$\begin{cases} f_n(\beta)G(\beta) \to f(\beta)G(\beta), & \text{as } n \to \infty \\ f_n(\alpha)G(\alpha) \to f(\alpha)G(\alpha), & \text{as } n \to \infty. \end{cases} \quad (3.6.21)$$

Combining (3.6.19), (3.6.18), (3.6.20), and (3.6.21) we have that:

$$\begin{aligned} \int_\alpha^\beta f(x)g(x)dx &= fG|_\alpha^\beta - \int_\alpha^\beta F(x)G(x)dx \\ &= fG|_\alpha^\beta - \int_\alpha^\beta f'(x)G(x)dx, \text{ by } (3.6.11) \end{aligned}$$

which is the desired integration by parts formula. □

Auxiliary result 3 *Formula (3.4.73) holds, where* ω_1 *and* ω_2 *are defined in (3.4.55) and (3.4.57).*

Proof: The work is the same for both functions and we shall elaborate only for ω_2 since the integrand has a simpler notation:

$$\begin{aligned}
\omega_2(\lambda) &= \int_0^a f'(t)\cos(\sqrt{\lambda}t)dt \\
&= \frac{1}{2}\int_{-a}^a F(t)\cos(\sqrt{\lambda}t)dt, \\
&= \frac{1}{4}\int_{-a}^a F(t)\left(e^{i\sqrt{\lambda}t}+e^{-i\sqrt{\lambda}t}\right)dt, \text{ since } \cos w = \tfrac{e^{iw}+e^{-iw}}{2} \\
&= \frac{1}{4}\left(\int_{-\infty}^{\infty} F_{ext}(t)e^{i\sqrt{\lambda}t}dt + \int_{-\infty}^{\infty} F_{ext}(t)e^{-i\sqrt{\lambda}t}dt\right) \\
&= \frac{1}{4}(\hat{F}_{ext}(\sqrt{\lambda}) + \hat{F}_{ext}(-\sqrt{\lambda})), \qquad (3.6.22)
\end{aligned}$$

where $F \in L^2(-a,a)$ is the even extension of $f' \in L^2(0,a)$,

$$F_{ext}(t) = \begin{cases} F(t), & \text{for } t \in [-a,a], \\ 0, & \text{for } t \notin [-a,a], \end{cases}$$

and $\hat{F}_{ext}$ is the notation for the Fourier transform of F_{ext}. Since $F \in L^2(-a,a) \subset L^1(-a,a)$, it follows that $F_{ext} \in L^1(\mathbb{R})$, and by the theory of Fourier transform of $L^1(\mathbb{R})$ functions (see Fourier version of Riemann-Lebesgue lemma) we have that

$$\lim_{\xi\to\pm\infty} \hat{F}_{ext}(\xi) = 0. \qquad (3.6.23)$$

Combining (3.6.23) with (3.6.22) we arrive at the second relationship of (3.4.73). □

Auxiliary result 4 *The right hand side of (3.4.35) and the right hand side of (3.4.40) form* l_2 *sequences of real numbers.*

Proof: We elaborate only for (3.4.35), because similar arguments apply to (3.4.40). Equation (3.1.3) allows us to write:

$$\sqrt{\mu_n} = \frac{2n\pi}{a} + \frac{a\cdot M_1}{4n\pi} + \frac{\tilde{c}_n^1}{n^2}, \text{ as } n\to\infty, \qquad (3.6.24)$$

for some sequence of real numbers $\{\tilde{c}_n^1\}_{n\geq 1} \in l_2$. We make now the important observation that the sequence $\{\tilde{c}_n^1\}_{n\geq 1}$ needs only be an l_2 sequence to insure that $\{c_n^1\}_{n\geq 1}$ is an l_2^1 sequence as needed in (3.1.3); this can be seen by squaring the identity (3.6.24) and comparing with (3.1.3); one will get

$$c_n^1 = \frac{4\pi}{a}\frac{\tilde{c}_n^1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right), \text{ as } n \to \infty, \tag{3.6.25}$$

and so both terms in (3.6.25) are l_2^1 sequences provided that $\{\tilde{c}_n^1\}_{n\geq 1} \in l_2$, making $\{c_n^1\}_{n\geq 1}$ an l_2^1 sequence. Then from (3.6.24) the following are derivable:

$$\begin{aligned} \sin(\sqrt{\mu_n}a) &= \sin(\frac{\bar{M}_1}{n} + \frac{\varepsilon_{1n}}{n^2}) \\ &\approx \frac{\bar{M}_1}{n} + \frac{\varepsilon_{1n}}{n^2}, \text{ as } n \to \infty \end{aligned} \tag{3.6.26}$$

$$\begin{aligned} \sin(\sqrt{\mu_n}\frac{a}{2}) &= (-1)^n \sin(\frac{\bar{M}_1}{2n} + \frac{\frac{a}{2}\tilde{c}_n^1}{n^2}) \\ &\approx (-1)^n \frac{\bar{M}_1}{2n} + \frac{\varepsilon_{2n}}{n^2}, \text{ as } n \to \infty \end{aligned} \tag{3.6.27}$$

$$\begin{aligned} \cos(\sqrt{\mu_n}a) &= 1 - 2\sin^2(\sqrt{\mu_n}\frac{a}{2}) \\ &= 1 + \mathcal{O}\left(\frac{1}{n^2}\right), \text{ as } n \to \infty \end{aligned} \tag{3.6.28}$$

$$\begin{aligned} \cos(\sqrt{\mu_n}\frac{a}{2}) &= (-1)^n \cos(\frac{\bar{M}_1}{2n} + \frac{\frac{a}{2}\tilde{c}_n^1}{n^2}) \\ &= (-1)^n \sqrt{\frac{1+\cos(\frac{\bar{M}_1}{n} + \frac{a\tilde{c}_n^1}{n^2})}{2}}, \text{ using } \cos\alpha = \sqrt{\tfrac{1+\cos(2\alpha)}{2}}, \text{ since } \alpha \approx 0 \text{ so } \cos\alpha > 0 \\ &= (-1)^n \sqrt{\frac{1+\cos(\sqrt{\mu_n}a)}{2}}, \text{ by (3.6.24)} \\ &= (-1)^n \sqrt{\frac{1+1+\mathcal{O}\left(\frac{1}{n^2}\right)}{2}}, \text{ by (3.6.28)} \\ &= (-1)^n \sqrt{1+\mathcal{O}\left(\frac{1}{n^2}\right)} \\ &= (-1)^n \left(1+\mathcal{O}\left(\frac{1}{n^2}\right)\right), \text{ as } n \to \infty \end{aligned} \tag{3.6.29}$$

where $\bar{M}_1 = \dfrac{a^2 \cdot M_1}{4\pi}$, and $\varepsilon_{1n} = a\tilde{c}_n^1$ and $\varepsilon_{2n} = (-1)^n \dfrac{a}{2}\tilde{c}_n^1$ define l_2 sequences of real numbers. The last identity in (3.6.29) comes from the approximation

$$\sqrt{1+x} \approx 1 + \frac{x}{2}, \text{ about } x = 0,$$

which is derivable from Taylor series expansion. Next, from (3.3.7) we have the following:

$$\begin{aligned} S(\mu_n) &= \frac{\sin(\sqrt{\mu_n}a)}{\sqrt{\mu_n}} + \int_0^a f(t)\frac{\sin(\sqrt{\mu_n}t)}{\sqrt{\mu_n}}dt \\ &= \frac{\sin(\sqrt{\mu_n}a)}{\sqrt{\mu_n}} - \left(\frac{a\cdot M}{2}\right)\frac{\cos(\sqrt{\mu_n}a)}{\mu_n} \\ &\quad + \int_0^a f'(t)\frac{\cos(\sqrt{\mu_n}t)}{\mu_n}dt, \text{ since } f\in H^1(0,a) \text{ and } (3.4.2) \\ &= \frac{\sin(\sqrt{\mu_n}a)}{\sqrt{\mu_n}} - \left(\frac{a\cdot M}{2}\right)\frac{\cos(\sqrt{\mu_n}a)}{\mu_n} \\ &\quad + f'(a)\frac{\sin(\sqrt{\mu_n}a)}{\mu_n\sqrt{\mu_n}} - \int_0^a f''(t)\frac{\sin(\sqrt{\mu_n}t)}{\mu_n\sqrt{\mu_n}}dt, \text{ since } f\in H^2(0,a) \\ &= \frac{\sin(\sqrt{\mu_n}a)}{\sqrt{\mu_n}} - \left(\frac{a\cdot M}{2}\right)\frac{\cos(\sqrt{\mu_n}a)}{\mu_n} \\ &\quad + f'(a)\frac{\sin(\sqrt{\mu_n}a)}{\mu_n\sqrt{\mu_n}} + \frac{\tilde{\omega}(\mu_n)}{\mu_n\sqrt{\mu_n}}, \end{aligned} \tag{3.6.30}$$

where

$$\tilde{\omega}(\lambda) = -\int_0^a f''(t)\sin(\sqrt{\lambda}t)dt. \tag{3.6.31}$$

Note that certainly $f\in H^2(0,a)$, by the discussion about (3.4.1) and (3.4.2), and so integration by parts twice in the integral

$$\int_0^a f(t)\frac{\sin(\sqrt{\mu_n}t)}{\sqrt{\mu_n}}dt$$

is permitted (see Auxiliary result 2). Similarly, from (3.3.9) and (3.4.14) one derives that:

$$S^{(2)}(\mu_n) = \frac{\sin(\sqrt{\mu_n}\frac{a}{2})}{\sqrt{\mu_n}} - \left(\frac{a\cdot M_2}{4}\right)\frac{\cos(\sqrt{\mu_n}\frac{a}{2})}{\mu_n} + f_2'(\frac{a}{2})\frac{\sin(\sqrt{\mu_n}\frac{a}{2})}{\mu_n\sqrt{\mu_n}} + \frac{\tilde{\omega}_2(\mu_n)}{\mu_n\sqrt{\mu_n}}, \tag{3.6.32}$$

where

$$\tilde{\omega}_2(\lambda) = -\int_0^{\frac{a}{2}} f_2''(t)\sin(\sqrt{\lambda}t)dt, \tag{3.6.33}$$

where $f_2\in H^2(0,\frac{a}{2})$ as justified in the discussion about (3.4.13) and (3.4.14), thus making the integration by parts twice in the integral

$$\int_0^{\frac{a}{2}} f_2(t)\frac{\sin(\sqrt{\mu_n}t)}{\sqrt{\mu_n}}dt$$

a legal operation (see Auxiliary result 2). Now let α_n stand for the right hand side of equation (3.4.35). Hence, by (3.6.30) and (3.6.32) we have that:

$$\alpha_n = \sqrt{\mu_n}\left(\frac{w_1}{w_2} - \cos(\sqrt{\mu_n}\frac{a}{2}) - w_3\right), \tag{3.6.34}$$

where

$$w_1 = \sin(\sqrt{\mu_n}a) - \left(\frac{a \cdot M}{2}\right)\frac{\cos(\sqrt{\mu_n}a)}{\sqrt{\mu_n}} + f'(a)\frac{\sin(\sqrt{\mu_n}a)}{\mu_n} + \frac{\tilde{\omega}(\mu_n)}{\mu_n}, \tag{3.6.35}$$

$$w_2 = \sin(\sqrt{\mu_n}\frac{a}{2}) - \left(\frac{a \cdot M_2}{4}\right)\frac{\cos(\sqrt{\mu_n}\frac{a}{2})}{\sqrt{\mu_n}} + f_2'(\frac{a}{2})\frac{\sin(\sqrt{\mu_n}\frac{a}{2})}{\mu_n} + \frac{\tilde{\omega}_2(\mu_n)}{\mu_n}, \tag{3.6.36}$$

$$w_3 = \left(\frac{a \cdot M_1}{4}\right)\frac{\sin(\sqrt{\mu_n}\frac{a}{2})}{\sqrt{\mu_n}}. \tag{3.6.37}$$

Using (3.6.24) and the Taylor series expansion of $\dfrac{1}{b+x}$ about $x = 0$, where $b = \dfrac{2n\pi}{a}$ (since $\dfrac{a \cdot M_1}{4n\pi} + \dfrac{\tilde{c}_n^1}{n^2} \to 0$, as $n \to \infty$) one has that

$$\frac{1}{\sqrt{\mu_n}} \approx \frac{a}{2n\pi}, \text{ as } n \to \infty$$

which in combination with (3.6.26) - (3.6.29), and (3.6.35) - (3.6.37) produces the following estimates after ignoring all terms of order $\mathcal{O}\left(\frac{1}{n^3}\right)$, $\mathcal{O}\left(\frac{1}{n^4}\right)$, etc:

$$\begin{aligned} w_1 &\approx \left(\frac{\bar{M}_1}{n} + \frac{\varepsilon_{1n}}{n^2}\right) - \left(\frac{a \cdot M}{2}\right)\frac{a}{2n\pi}\left(1 + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \\ &\quad + f'(a)\left(\frac{a}{2n\pi}\right)^2\left(\frac{\bar{M}_1}{n} + \frac{\varepsilon_{1n}}{n^2}\right) + \left(\frac{a}{2n\pi}\right)^2\tilde{\omega}(\mu_n) \\ &\approx \frac{\bar{M}_1}{n} + \frac{\varepsilon_{1n}}{n^2} - \frac{\bar{M}}{n} + \left(\frac{a}{2n\pi}\right)^2\tilde{\omega}(\mu_n), \\ &= \frac{\bar{M}_1}{n} - \frac{\bar{M}}{n} + \frac{\varepsilon_{1n}}{n^2} + \left(\frac{a}{2n\pi}\right)^2\tilde{\omega}(\mu_n) \end{aligned} \tag{3.6.38}$$

where $\bar{M} = \dfrac{a^2 \cdot M}{4\pi}$,

$$\begin{aligned} w_2 &\approx \left((-1)^n\frac{\bar{M}_1}{2n} + \frac{\varepsilon_{2n}}{n^2}\right) - \left(\frac{a \cdot M_2}{4}\right)\frac{a}{2n\pi}(-1)^n\left(1 + \mathcal{O}\left(\frac{1}{n^2}\right)\right) \\ &\quad + f_2'(\frac{a}{2})\left(\frac{a}{2n\pi}\right)^2\left((-1)^n\frac{\bar{M}_1}{2n} + \frac{\varepsilon_{2n}}{n^2}\right) + \left(\frac{a}{2n\pi}\right)^2\tilde{\omega}_2(\mu_n) \\ &\approx \left((-1)^n\frac{\bar{M}_1}{2n} + \frac{\varepsilon_{2n}}{n^2}\right) - (-1)^n\frac{\bar{M}_2}{2n} + \left(\frac{a}{2n\pi}\right)^2\tilde{\omega}_2(\mu_n) \\ &= (-1)^n\frac{\bar{M}_1}{2n} - (-1)^n\frac{\bar{M}_2}{2n} + \frac{\varepsilon_{2n}}{n^2} + \left(\frac{a}{2n\pi}\right)^2\tilde{\omega}_2(\mu_n), \end{aligned} \tag{3.6.39}$$

where $\bar{M}_2 = \dfrac{a^2 \cdot M_2}{4\pi}$,

$$\begin{aligned} w_3 &\approx \left(\frac{a \cdot M1}{4}\right)\frac{a}{2n\pi}\left((-1)^n\frac{\bar{M}_1}{2n} + \frac{\varepsilon_{2n}}{n^2}\right) \\ &\approx (-1)^n\left(\frac{\bar{M}_1}{2n}\right)^2, \end{aligned} \tag{3.6.40}$$

where $\bar{M}_1 = \dfrac{a^2 \cdot M_1}{4\pi}$. Inserting (3.6.24), (3.6.29), (3.6.38), (3.6.39), (3.6.40) into (3.6.34) and ignoring all terms of order $\mathcal{O}\left(\frac{1}{n^3}\right)$, $\mathcal{O}\left(\frac{1}{n^4}\right)$, etc, one arrives at:

$$\begin{aligned}
\alpha_n \quad &\approx \quad \left(\frac{2n\pi}{a} + \xi_n\right)\left(\frac{\frac{\bar{M}_1}{n} - \frac{\bar{M}}{n} + \frac{\varepsilon_{1n}}{n^2} + \left(\frac{a}{2n\pi}\right)^2 \tilde{\omega}(\mu_n)}{(-1)^n \frac{\bar{M}_1}{2n} - (-1)^n \frac{\bar{M}_2}{2n} + \frac{\varepsilon_{2n}}{n^2} + \left(\frac{a}{2n\pi}\right)^2 \tilde{\omega}_2(\mu_n)}\right) \\
&\quad - \left(\frac{2n\pi}{a} + \xi_n\right)\left((-1)^n\left(1 + \mathcal{O}\left(\frac{1}{n^2}\right)\right) - (-1)^n \left(\frac{\bar{M}_1}{2n}\right)^2\right) \\
&\approx \quad \left(\frac{2n\pi}{a} + \xi_n\right)\left(\frac{\frac{\bar{M}_1}{n} - \frac{\bar{M}}{n} + \frac{\varepsilon_{1n}}{n^2} + \left(\frac{a}{2n\pi}\right)^2 \tilde{\omega}(\mu_n)}{(-1)^n \frac{\bar{M}_1}{2n} - (-1)^n \frac{\bar{M}_2}{2n} + \frac{\varepsilon_{2n}}{n^2} + \left(\frac{a}{2n\pi}\right)^2 \tilde{\omega}_2(\mu_n)}\right) \\
&\quad + \left(\frac{2n\pi}{a} + \xi_n\right)\left(\frac{-\frac{\bar{M}_1}{2n} + \frac{\bar{M}_2}{2n} - (-1)^n \frac{\varepsilon_{2n}}{n^2} - (-1)^n \left(\frac{a}{2n\pi}\right)^2 \tilde{\omega}_2(\mu_n)}{(-1)^n \frac{\bar{M}_1}{2n} - (-1)^n \frac{\bar{M}_2}{2n} + \frac{\varepsilon_{2n}}{n^2} + \left(\frac{a}{2n\pi}\right)^2 \tilde{\omega}_2(\mu_n)}\right) \\
&= \quad \left(\frac{2n\pi}{a} + \xi_n\right)\left(\frac{\frac{\bar{M}_1 + \bar{M}_2 - 2\bar{M}}{2n} + \frac{\varepsilon_{1n} - (-1)^n \varepsilon_{2n}}{n^2} + \left(\frac{a}{2n\pi}\right)^2 \left(\tilde{\omega}(\mu_n) - (-1)^n \tilde{\omega}_2(\mu_n)\right)}{\frac{(-1)^n}{2n}(\bar{M}_1 - \bar{M}_2) + \frac{\varepsilon_{2n}}{n^2} + \left(\frac{a}{2n\pi}\right)^2 \tilde{\omega}_2(\mu_n)}\right) \\
&= \quad \left(\frac{2n\pi}{a} + \xi_n\right)\left(\frac{\frac{\varepsilon_{1n} - (-1)^n \varepsilon_{2n}}{n^2} + \left(\frac{a}{2n\pi}\right)^2 \left(\tilde{\omega}(\mu_n) - (-1)^n \tilde{\omega}_2(\mu_n)\right)}{\frac{(-1)^n}{2n}(\bar{M}_1 - \bar{M}_2) + \frac{\varepsilon_{2n}}{n^2} + \left(\frac{a}{2n\pi}\right)^2 \tilde{\omega}_2(\mu_n)}\right), \qquad (3.6.41)
\end{aligned}$$

where $\xi_n = \dfrac{a \cdot M_1}{4n\pi} + \dfrac{\tilde{c}_n^1}{n^2}$ defines an l_2 sequence of real numbers, since $\{\tilde{c}_n^1\}_{n \geq 1} \in l_2$, and $\bar{M}_1 + \bar{M}_2 = 2\bar{M}$, and $\bar{M}_1 \neq \bar{M}_2$ due to the definitions of $\bar{M}_1$, $\bar{M}_2$, $\bar{M}$ and to the hypotheses of Theorem 1. Next, from (3.6.31) we have that:

$$\begin{aligned}
\tilde{\omega}(\mu_n) \quad &= \quad \int_0^a \left(-f''(t)\right)\left(\sin(\sqrt{\mu_n} t) - \sin(\sqrt{\lambda_{2n}} t)\right) dt + \int_0^a \left(-f''(t)\right) \sin(\sqrt{\lambda_{2n}} t) dt \\
&= \quad \int_0^a 2\left(-f''(t)\right) \cos\left(\frac{\sqrt{\mu_n} + \sqrt{\lambda_{2n}}}{2} t\right) \sin\left(\frac{\sqrt{\mu_n} - \sqrt{\lambda_{2n}}}{2} t\right) dt \\
&\quad + \int_0^a \left(-f''(t)\right) \sin(\sqrt{\lambda_{2n}} t) dt \\
&= \quad \int_0^a 2\left(-f''(t)\right) \cos\left(\frac{\sqrt{\mu_n} + \sqrt{\lambda_{2n}}}{2} t\right) \sin\left(\frac{a(M_1 - M)t}{4n\pi} + \frac{(4\tilde{c}_n^1 - \tilde{c}_{2n})t}{4n^2}\right) dt \\
&\quad + \int_0^a \left(-f''(t)\right) \sin(\sqrt{\lambda_{2n}} t) dt \\
&= \quad \mathcal{O}\left(\frac{1}{n}\right) + \int_0^a \left(-f''(t)\right) \sin(\sqrt{\lambda_{2n}} t) dt, \qquad (3.6.42)
\end{aligned}$$

because

$$\begin{cases} \sqrt{\lambda_n} = \frac{n\pi}{a} + \frac{a \cdot M}{2n\pi} + \frac{\tilde{c}_n}{n^2}, \text{ as } n \to \infty, \text{ where } \{\tilde{c}_n\}_{n\geq 1} \in l_2 \\ \sin\left(\frac{a(M_1 - M)t}{4n\pi} + \frac{(4\tilde{c}_n^1 - \tilde{c}_{2n})t}{4n^2}\right) \approx \frac{a(M_1 - M)t}{4n\pi} + \frac{(4\tilde{c}_n^1 - \tilde{c}_{2n})t}{4n^2} = \mathcal{O}\left(\frac{1}{n}\right), \text{ as } n \to \infty \\ |\cos\left(\frac{\sqrt{\mu_n} + \sqrt{\lambda_{2n}}}{2} t\right)| \leq 1 \\ f \in H^2(0,a), \text{ so } f'' \in L^2(0,a). \end{cases}$$

It is easily seen that the first term in (3.6.42) defines an l_2 sequence, whereas the second term in (3.6.42) defines the subsequence of even-numbered terms in the moment sequence of $-f'' \in L^2(0,a)$ with respect to the set of vectors

$$\{\sin(\sqrt{\lambda_n} t) | n \geq 1\},$$

which is a Riesz basis of $L^2(0,a)$ (see Auxiliary result 6), therefore by Theorem 8 in [23, page 169] the whole sequence is an l_2 sequence. (The definition of the moment sequence of a given vector in a Hilbert space can be found in [23, page 146].) Thus, both terms in (3.6.42) are l_2 sequences, and so

$$\{\tilde{\omega}(\mu_n)\}_{n=1}^{\infty} \in l_2. \tag{3.6.43}$$

Also from (3.6.33) one has that

$$\{\tilde{\omega}_2(\mu_n)\}_{n=1}^{\infty} \in l_2, \tag{3.6.44}$$

because it represents the moment sequence of $-f_2'' \in L^2(0, \frac{a}{2})$ with respect to the set of vectors

$$\{\sin(\sqrt{\mu_n} t) | n \geq 1\}$$

which is a Riesz basis of $L^2(0, \frac{a}{2})$ (see Auxiliary result 5), and so Theorem 8 in [23, page 169] is applicable. Multiplying out in (3.6.41) and using (3.6.43) and (3.6.44) we get:

$$\begin{aligned} \alpha_n \quad \approx \quad & \frac{4\pi}{a(\bar{M}_2 - \bar{M}_1)} \left(\varepsilon_{2n} - (-1)^n \varepsilon_{1n}\right) + \frac{a}{\pi(\bar{M}_2 - \bar{M}_1)} \left(\tilde{\omega}_2(\mu_n) - (-1)^n \tilde{\omega}(\mu_n)\right) \\ & + \xi_n \mathcal{O}\left(\frac{1}{n}\right), \text{ as } n \to \infty, \end{aligned}$$

which clearly shows that $\{\alpha_n\}_{n\geq 1} \in l_2$, because $\{\xi_n\}_{n\geq 1} \in l_2$, and $\{\varepsilon_{in}\}_{n\geq 1} \in l_2$, $i = 1, 2$. Note the importance of having $\bar{M}_1 \neq \bar{M}_2$ in (3.6.41). If $\bar{M}_1 = \bar{M}_2$, then the second factor in (3.6.41) would have been of order $\mathcal{O}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right) = \mathcal{O}(1)$ which would further give that

$$\alpha_n = \mathcal{O}(n), \text{ as } n \to \infty,$$

hence not an l_2 sequence. $\square$

Auxiliary result 5 *Given the sets $\{\mu_n | n \geq 1\}$ and $\{\nu_n | n \geq 1\}$ satisfying (3.1.3) and (3.1.4) respectively, the sets*

$$\{\sin(\sqrt{\mu_n} t) | n \geq 1\} \text{ and } \{\sin(\sqrt{\nu_n} t) | n \geq 1\}$$

are Riesz bases in $L^2(0, \frac{a}{2})$.

Proof: The proof works equally well for an arbitrary $a_0 \in (0, a)$, not only for $a_0 = \frac{a}{2}$, with the specification that $\{\mu_n | n \geq 1\}$ and $\{\nu_n | n \geq 1\}$ must have the right asymptotics, as described in (5.1.3) and (5.1.4). In that situation, the sets

$$\{\sin(\sqrt{\mu_n} t) | n \geq 1\} \text{ and } \{\sin(\sqrt{\tilde{\nu}_n} t) | n \geq 1\}$$

are Riesz bases in $L^2(0, a_0)$, where $\tilde{\nu}_n = \left(\dfrac{a - a_0}{a_0}\right)^2 \nu_n$. We shall make the proof only for the set

$$\{\sin(\sqrt{\mu_n} t) | n \geq 1\},$$

where $a_0 = \dfrac{a}{2}$. The proof for the other set, $\{\sin(\sqrt{\nu_n} t) | n \geq 1\}$ is similar. For this we intend to apply Theorem 10 of [23, page 172], but slightly modified. For the convenience of the reader we state the result here: "If $\{\omega_n\}_{n=-\infty}^{\infty}$ is the set of all zeros of a function of sine type $\dfrac{a}{2}$, then the set

$$\{e^{i\omega_n t}\}_{n=-\infty}^{\infty}$$

is a Riesz basis in $L^2(-\dfrac{a}{2}, \dfrac{a}{2})$". The definition of a Riesz basis can be found in [23, page 31], and the definition of an 'entire function of sine type $\dfrac{a}{2}$' can be obtained by slightly adjusting the definition of an 'entire function of sine type π' found in [23, page 171]. Also the notion of an 'entire function of exponential type' is needed, and this can be found in [23, page 61]. For the convenience of the reader we give here the definition of an entire function of sine type $\frac{a}{2}$. The definition of an entire function of exponential type $\frac{a}{2}$ is included therein:

Definition: An entire function $f : \mathbb{C} \to \mathbb{C}$ is said to be of *sine type* $\frac{a}{2}$ if:

1. f is of exponential type $\frac{a}{2}$ (i.e. there exists a constant $C > 0$, such that $|f(z)| \leq Ce^{\frac{a}{2}|z|}$, for all $z \in \mathbb{C}$);

2. the zeros of f are separated (i.e. there exists a constant $\gamma > 0$ such that $|z_m - z_n| \geq \gamma$, whenever $m \neq n$);

3. there exist constants $A, B, H > 0$, such that

$$Ae^{\frac{a}{2}|Im(z)|} \leq |f(z)| \leq Be^{\frac{a}{2}|Im(z)|}, \text{ for all } z \in \mathbb{C} \text{ with } |Im(z)| \geq H.$$

Our plan is as follows:

- define the function

$$P_1(z) = \frac{\sin(z\frac{a}{2})}{z} + \int_0^{\frac{a}{2}} f_1(t)\frac{\sin(zt)}{z}dt, \text{ for all } z \in \mathbb{C} - \{0\}, \tag{3.6.45}$$

  where $f_1 \in H^2(0, \frac{a}{2})$ is as in (3.4.9) and (3.4.10); Note that formula (3.1.3) was needed to insure the existence and uniqueness of f_1 with the properties just mentioned. These were established by analogy with the properties of f at the beginning of Section 3.4.

- show that $zP_1(z)$ is a function of sine type $\dfrac{a}{2}$;

- show that $\{0\} \cup \{\pm\sqrt{\mu_n} | n \geq 1\}$ are *all* the zeros of $zP_1(z)$;

- show that $\{1\} \cup \{e^{\pm i\sqrt{\mu_n}t} | n \geq 1\}$ is a Riesz basis in $L^2(-\dfrac{a}{2}, \dfrac{a}{2})$;

- show that $\{\sin(\sqrt{\mu_n}t) | n \geq 1\}$ is a Riesz basis in $L^2(0, \dfrac{a}{2})$.

The second item of the above plan is the most difficult one and we leave it at the end. The third item follows immediately from (3.6.45), (3.4.9) and (3.4.11), and the fact that sin is an odd function. Note that (3.4.11) assures that $zP_1(z)$ has no other zeros except $\{0\} \cup \{\pm\sqrt{\mu_n} | n \geq 1\}$. (Assume that $\pm\sqrt{\mu} \notin \{0\} \cup \{\pm\sqrt{\mu_n} | n \geq 1\}$ is a zero of $zP_1(z)$. Hence we get that

$$\begin{aligned}
S(\frac{a}{2}; Q_1, \mu) &= \frac{\sin(\pm\sqrt{\mu}\frac{a}{2})}{\pm\sqrt{\mu}} + \int_0^{\frac{a}{2}} K(\frac{a}{2}, t; Q_1)\frac{\sin(\pm\sqrt{\mu}t)}{\pm\sqrt{\mu}}dt, \text{ by (3.2.20) with } \beta = \tfrac{a}{2},\ p = Q_1 \\
&= \frac{\sin(\pm\sqrt{\mu}\frac{a}{2})}{\pm\sqrt{\mu}} + \int_0^{\frac{a}{2}} f_1(t)\frac{\sin(\pm\sqrt{\mu}t)}{\pm\sqrt{\mu}}dt, \text{ by (3.4.11)} \\
&= P_1(\pm\sqrt{\mu}), \text{ by (3.6.45)} \\
&= 0, \text{ since } \pm\sqrt{\mu} \neq 0 \text{ is a zero of } zP_1(z).
\end{aligned}$$

So μ is a Dirichlet eigenvalue of the Sturm-Liouville operator with potential $Q_1 \in L^2(0, \frac{a}{2})$, and therefore by (3.4.11) must be in the set $\{\mu_n | n \geq 1\}$, which contradicts our assumption.) The fourth item follows from the second and third items and the above mentioned version of Theorem 10 in [23, page 172], where

$$\omega_n = \begin{cases} \sqrt{\mu_n}, & \text{if } n = 1, 2, 3, \ldots \\ 0, & \text{if } n = 0 \\ -\sqrt{\mu_{-n}}, & \text{if } n = -1, -2, -3, \ldots. \end{cases}$$

The fifth item is derived from the fourth item and the equivalence (1) $\leftrightarrow$ (3) of Theorem 9 in [23, page 32] in the following way: since

$$\{e^{i\omega_n t}\}_{n=-\infty}^{\infty} = \{1\} \cup \{e^{\pm i\sqrt{\mu_n}t} | n \geq 1\}$$

is a Riesz basis in $L^2(-\dfrac{a}{2}, \dfrac{a}{2})$, using (1) $\rightarrow$ (3) in [23, Theorem 9, page 32] we have that

$$\{e^{i\omega_n t}\}_{n=-\infty}^{\infty} = \{1\} \cup \{e^{\pm i\sqrt{\mu_n}t} | n \geq 1\} \text{ is complete in } L^2(-\tfrac{a}{2}, \tfrac{a}{2}), \tag{3.6.46}$$

and there exist constants $A > 0$, $B > 0$ such that

$$A \sum_{j=-n}^{n} |\tilde{c}_j|^2 \leq || \sum_{j=-n}^{n} \tilde{c}_j e^{i\omega_j t}||^2 \leq B \sum_{j=-n}^{n} |\tilde{c}_j|^2, \text{ for arbitrary } n \geq 1 \text{ and arbitrary } \tilde{c}_j\text{'s.} \tag{3.6.47}$$

Now, let $h \in L^2(0, \frac{a}{2})$ be such that

$$\int_0^{\frac{a}{2}} h(t) \sin(\sqrt{\mu_n}t)dt = 0, \text{ for all } n \geq 1.$$

It follows from this and by considering the odd extension $H \in L^2(-\frac{a}{2}, \frac{a}{2})$ of h that:

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} H(t)dt = 0, \text{ since the integrand is odd,}$$

$$\begin{aligned} \int_{-\frac{a}{2}}^{\frac{a}{2}} H(t) \sin(\sqrt{\mu_n}t)dt &= 2\int_0^{\frac{a}{2}} H(t) \sin(\sqrt{\mu_n}t)dt, \text{ since the integrand is even} \\ &= 2\int_0^{\frac{a}{2}} h(t) \sin(\sqrt{\mu_n}t)dt, \text{ by the definition of } H \\ &= 0, \text{ for all } n \geq 1, \text{ by the above property of } h, \end{aligned}$$

and

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} H(t)\cos(\sqrt{\mu_n}t)dt = 0, \text{ for all } n \geq 1, \text{ since the integrand is odd.}$$

All these show that

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} H(t)e^{i\omega_n t}dt = 0, \text{ for all } n = 0, \pm 1, \pm 2, \ldots,$$

which in combination with (3.6.46) imply that $H = 0$, and hence $h = 0$. Therefore, we proved that the set

$$\{\sin(\sqrt{\mu_n}t) | n \geq 1\} \text{ is complete in } L^2(0, \tfrac{a}{2}). \tag{3.6.48}$$

Next, let $n \geq 1$ be arbitrary but fixed, and $c_1, c_2, \ldots, c_n$ be arbitrary constants in $\mathbb{C}$. We have the following calculations:

$$\begin{aligned}
||\sum_{j=1}^{n} c_j \sin(\sqrt{\mu_j}t)||^2 &= \sum_{j,k=1}^{n} c_j \bar{c_k} \int_0^{\frac{a}{2}} \sin(\sqrt{\mu_j}t)\sin(\sqrt{\mu_k}t)dt \\
&= \sum_{j,k=1}^{n} c_j \bar{c_k} \frac{1}{2} \int_{-\frac{a}{2}}^{\frac{a}{2}} \sin(\sqrt{\mu_j}t)\sin(\sqrt{\mu_k}t)dt, \text{ since the integrand is even} \\
&= \frac{1}{2} \sum_{j,k=1}^{n} c_j \bar{c_k} \int_{-\frac{a}{2}}^{\frac{a}{2}} \left(\frac{e^{i\sqrt{\mu_j}t} - e^{-i\sqrt{\mu_j}t}}{2i}\right)\left(\frac{e^{i\sqrt{\mu_k}t} - e^{-i\sqrt{\mu_k}t}}{2i}\right) dt \\
&= \frac{1}{8} \sum_{j,k=1}^{n} \langle c_j e^{i\sqrt{\mu_j}t} + (-c_j)e^{-i\sqrt{\mu_j}t}, c_k e^{i\sqrt{\mu_k}t} + (-c_k)e^{-i\sqrt{\mu_k}t} \rangle_{L^2(-\frac{a}{2},\frac{a}{2})} \\
&= \frac{1}{8} \sum_{j,k=1}^{n} \langle \tilde{c}_j e^{i\omega_j t} + \tilde{c}_{-j} e^{i\omega_{-j}t}, \tilde{c}_k e^{i\omega_k t} + \tilde{c}_{-k} e^{i\omega_{-k}t} \rangle_{L^2(-\frac{a}{2},\frac{a}{2})} \\
&= \frac{1}{8} \langle \sum_{j=-n}^{n} \tilde{c}_j e^{i\omega_j t}, \sum_{k=-n}^{n} \tilde{c}_k e^{i\omega_k t} \rangle_{L^2(-\frac{a}{2},\frac{a}{2})} \\
&= \frac{1}{8} || \sum_{j=-n}^{n} \tilde{c}_j e^{i\omega_j t}||^2,
\end{aligned} \tag{3.6.49}$$

where

$$\tilde{c}_j = \begin{cases} c_j, & \text{if } j = 1, 2, 3, \ldots \\ 0, & \text{if } j = 0 \\ -c_{-j}, & \text{if } j = -1, -2, -3, \ldots. \end{cases} \tag{3.6.50}$$

Using (3.6.49), (3.6.50) and (3.6.47) we obtain

$$\frac{A}{4} \sum_{j=1}^{n} |c_j|^2 \leq || \sum_{j=1}^{n} c_j \sin(\sqrt{\mu_j}t)||^2 \leq \frac{B}{4} \sum_{j=1}^{n} |c_j|^2. \tag{3.6.51}$$

Finally, formulas (3.6.48) and (3.6.51), and (3) $\rightarrow$ (1) in [23, Theorem 9, page 32] tell us that the set

$$\{\sin(\sqrt{\mu_n}t) | n \geq 1\}$$

is a Riesz basis in $L^2(0, \frac{a}{2})$. Now we are ready to return to the second item of our plan, namely to proving that $zP_1(z)$ is a function of sine type $\dfrac{a}{2}$. We shall establish the conditions in the definition of such a function (see [23, page 171]). Certainly $zP_1(z)$ is an entire function, and it is of exponential type $\dfrac{a}{2}$ as the following calculations show:

$$\begin{aligned}
|zP_1(z)| &= |\sin(z\frac{a}{2}) + \int_0^{\frac{a}{2}} f_1(t)\sin(zt)dt|, \text{ by (3.6.45)} \\
&\leq |\sin(z\frac{a}{2})| + \int_0^{\frac{a}{2}} |f_1(t)| \cdot |\sin(zt)|dt \\
&\leq |\frac{e^{iz\frac{a}{2}} - e^{-iz\frac{a}{2}}}{2i}| + \int_0^{\frac{a}{2}} |f_1(t)| \cdot |\frac{e^{izt} - e^{-izt}}{2i}|dt \\
&\leq \frac{|e^{iz\frac{a}{2}}| + |e^{-iz\frac{a}{2}}|}{2} + \int_0^{\frac{a}{2}} |f_1(t)| \frac{|e^{izt}| + |e^{-izt}|}{2} dt \\
&= \frac{e^{-\frac{a}{2}Im(z)} + e^{\frac{a}{2}Im(z)}}{2} + \int_0^{\frac{a}{2}} |f_1(t)| \frac{e^{-tIm(z)} + e^{tIm(z)}}{2} dt \\
&\leq e^{\frac{a}{2}|Im(z)|} + \int_0^{\frac{a}{2}} |f_1(t)| e^{|t||Im(z)|} dt, \text{ since } \pm s \leq |s|, \text{ so } e^{\pm s} \leq e^{|s|} \\
&\leq e^{\frac{a}{2}|Im(z)|}\left(1 + \int_0^{\frac{a}{2}} |f_1(t)|dt\right), \text{ since } |t| \leq \tfrac{a}{2}, \text{ so } e^{|t||Im(z)|} \leq e^{\frac{a}{2}|Im(z)|} \\
&\leq e^{\frac{a}{2}|Im(z)|}(1 + \sqrt{\frac{a}{2}}||f_1||_{L^2}), \text{ by Cauchy-Schwartz inequality} \\
&\leq Ce^{\frac{a}{2}|z|}, \text{ for all } z \in \mathbb{C}, \qquad (3.6.52)
\end{aligned}$$

where $C = 1 + \sqrt{\frac{a}{2}}||f_1||_{L^2}$ is a positive constant. From the calculations shown in (3.6.52) we also have:

$$|zP_1(z)| \leq Ce^{\frac{a}{2}|Im(z)|}, \text{ for all } z \in \mathbb{C}. \qquad (3.6.53)$$

Using (3.6.45) we can write:

$$\begin{aligned}
|zP_1(z)| &= |\sin(z\frac{a}{2}) - f_1(t)\frac{\cos(zt)}{z}|_{t=0}^{t=\frac{a}{2}} + \int_0^{\frac{a}{2}} f_1'(t)\frac{\cos(zt)}{z}dt|, \text{ since } f_1 \in H^1(0,\tfrac{a}{2}) \\
&= |\sin(z\frac{a}{2}) - \left(\frac{a\cdot M_1}{4}\right)\frac{\cos(z\frac{a}{2})}{z} + \int_0^{\frac{a}{2}} f_1'(t)\frac{\cos(zt)}{z}dt|, \text{ by (3.4.10)} \\
&\geq ||\sin(z\frac{a}{2})| - |\left(\frac{a\cdot M_1}{4}\right)\frac{\cos(z\frac{a}{2})}{z} - \int_0^{\frac{a}{2}} f_1'(t)\frac{\cos(zt)}{z}dt||, \\
&= ||\sin(z\frac{a}{2})| - \frac{1}{|z|}|\left(\frac{a\cdot M_1}{4}\right)\cos(z\frac{a}{2}) - \int_0^{\frac{a}{2}} f_1'(t)\cos(zt)dt||. \qquad (3.6.54)
\end{aligned}$$

The inequality in (3.6.54) is due to $|w_1 - w_2| \geq ||w_1| - |w_2||$. Next:

$$\begin{aligned}
|\left(\frac{a\cdot M_1}{4}\right)\cos(z\frac{a}{2}) - \int_0^{\frac{a}{2}} f_1'(t)\cos(zt)dt| &\leq \gamma|\frac{e^{iz\frac{a}{2}} + e^{-iz\frac{a}{2}}}{2}| + \int_0^{\frac{a}{2}} |f_1'(t)\frac{e^{izt}+e^{-izt}}{2}|dt \\
&\leq e^{\frac{a}{2}|Im(z)|}\left(\gamma + \sqrt{\frac{a}{2}}||f_1'||_{L^2}\right), \qquad (3.6.55)
\end{aligned}$$

where $\gamma = \dfrac{a|M_1|}{4}$. The inequalities in (3.6.55) were obtained using $|w_1 \mp w_2| \leq |w_1| + |w_2|$, $|e^{\pm iw}| = e^{\mp Im(w)}$, $e^{\pm s} \leq e^{|s|}$, the fact that $t \in [0, \frac{a}{2}]$, and Cauchy-Schwartz inequality for integrals. Lemma 4.8 in [13, page 136] provides us with a comparison between $e^{\frac{a}{2}|Im(z)|}$ and $|\sin(z\frac{a}{2})|$:

$$e^{\frac{a}{2}|Im(z)|} < 4|\sin(z\frac{a}{2})|, \text{ provided } |z\tfrac{a}{2} - n\pi| \geq \tfrac{\pi}{4}, \text{ for all } n \in \mathbb{Z}. \qquad (3.6.56)$$

It means that for $z \in \mathbb{C}$ such that

$$|z - \frac{2n\pi}{a}| \geq \frac{\pi}{2a}, \text{ for all } n \in \mathbb{Z}, \qquad (3.6.57)$$

the inequality (3.6.56) is satisfied. Observe that the set described in (3.6.57) represents the exterior of open disks with centers on the real axis (x-axis) and the same radius, $\dfrac{\pi}{2a}$. The open disks $\{z \in \mathbb{C}||z - \frac{2n\pi}{a}| < \frac{\pi}{2a}\}_{n\in\mathbb{Z}}$ lie within the strip $|Im(z)| < \frac{\pi}{2a}$. Hence, if we choose a positive constant H with

$$H \geq \frac{\pi}{2a} \qquad (3.6.58)$$

then for $z \in \mathbb{C}$ such that $|Im(z)| \geq H$, the inequality (3.6.57) holds, which further implies that (3.6.56) is satisfied, and so using (3.6.55) we can write:

$$\begin{aligned} \frac{1}{|z|}|\left(\frac{a \cdot M_1}{4}\right)\cos(z\frac{a}{2}) - \int_0^{\frac{a}{2}} f_1'(t)\cos(zt)dt| &\leq \frac{1}{|Im(z)|}e^{\frac{a}{2}|Im(z)|}\left(\gamma + \sqrt{\frac{a}{2}}||f_1'||_{L^2}\right) \\ &\leq \frac{1}{H}4|\sin(z\frac{a}{2})\left(\gamma + \sqrt{\frac{a}{2}}||f_1'||_{L^2}\right)| \\ &\leq \frac{1}{2}|\sin(z\frac{a}{2})|, \end{aligned} \tag{3.6.59}$$

if in addition to (3.6.58) we require

$$H \geq 8\left(\gamma + \sqrt{\frac{a}{2}}||f_1'||_{L^2}\right). \tag{3.6.60}$$

It follows using (3.6.58), (3.6.60), (3.6.59), (3.6.54), and (3.6.56) that if we choose

$$H = \max(\frac{\pi}{2a}, 8\left(\gamma + \sqrt{\frac{a}{2}}||f_1'||_{L^2}\right)),$$

then:

$$\begin{aligned} |zP_1(z)| &\geq ||\sin(z\frac{a}{2})| - \frac{1}{|z|}|\left(\frac{a \cdot M_1}{4}\right)\cos(z\frac{a}{2}) - \int_0^{\frac{a}{2}} f_1'(t)\cos(zt)dt|| \\ &= |\sin(z\frac{a}{2})| - \frac{1}{|z|}|\left(\frac{a \cdot M_1}{4}\right)\cos(z\frac{a}{2}) - \int_0^{\frac{a}{2}} f_1'(t)\cos(zt)dt| \\ &\geq |\sin(z\frac{a}{2})| - \frac{1}{2}|\sin(z\frac{a}{2})| \\ &= \frac{1}{2}|\sin(z\frac{a}{2})| \\ &\geq \frac{1}{8}e^{\frac{a}{2}|Im(z)|}, \text{ for all } z \text{ with } |Im(z)| \geq H. \end{aligned} \tag{3.6.61}$$

Finally, formulas (3.6.52), (3.6.53), (3.6.61), and the fact that all the zeros $\{0\} \cup \{\pm\sqrt{\mu_n} | n \geq 1\}$ of $zP_1(z)$ are separated (see definition in [23, page 98]) due to the asymptotic formula (3.1.3) show that all conditions in the definition of a function of sine type $\frac{a}{2}$ are fulfilled. □

Auxiliary result 6 *Given the sequence $\{\lambda_n | n \geq 1\}$ satisfying (3.1.2), the set*

$$\{\sin(\sqrt{\lambda_n}t) | n \geq 1\}$$

is a Riesz basis in $L^2(0, a)$.

Proof: It is similar to the proof of Auxiliary result 5. □

3.7 Remarks

Remark 3.1: The interlacing of the set $\{\mu_n | n \geq 1\} \cup \{\nu_n | n \geq 1\}$ with the set $\{\lambda_n | n \geq 1\}$ in the sense described in the hypotheses of Theorem 1 is a necessary condition for the existence of a potential function q with the properties specified in Theorem 1. This is what the direct three spectra problem guarantees, for if $q \in L^2(0,a)$ has the three sets as its Dirichlet eigenvalues on the intervals $[0,a]$, $[0,\frac{a}{2}]$, $[\frac{a}{2},a]$ respectively, then denoting by q_1 and q_2 the restrictions of q to $[0,\frac{a}{2}]$ and respectively to $[\frac{a}{2},a]$, and by $\tilde{q}_2$ the reflection of q_2 about the line $x = \frac{a}{2}$ one obtains that formula (3.2.16) holds and the pairs

$$\{\{\mu_n\}_{n\geq 1}; \{k_n^1\}_{n\geq 1}\}$$

and

$$\{\{\nu_n\}_{n\geq 1}; \{k_n^2\}_{n\geq 1}\}$$

are well defined and represent the pairs of {eigenvalues; norming constants} corresponding to the potentials q_1 and respectively $\tilde{q}_2$, where

$$\begin{aligned} k_n^1 &= \ln\Big((-1)^n S_1'(\frac{a}{2}; q_1, \mu_n)\Big) \\ &= \ln\left((-1)^n \frac{S(a;q,\mu_n)}{S_1(\frac{a}{2};\tilde{q}_2,\mu_n)}\right), \text{ by (3.2.27)} \end{aligned}$$

and

$$\begin{aligned} k_n^2 &= \ln\Big((-1)^n S_1'(\frac{a}{2}; \tilde{q}_2, \nu_n)\Big) \\ &= \ln\left((-1)^n \frac{S(a;q,\nu_n)}{S_1(\frac{a}{2};q_1,\nu_n)}\right), \text{ by (3.2.28).} \end{aligned}$$

Assuming the interlacing property does not hold, then the proper sign alternation in the ratios

$$\frac{S(a;q,\mu_n)}{S_1(\frac{a}{2};\tilde{q}_2,\mu_n)}, \ n \geq 1$$

and in the ratios

$$\frac{S(a;q,\nu_n)}{S_1(\frac{a}{2};q_1,\nu_n)}, \ n \geq 1$$

will not hold, making k_n^1 and/or k_n^2 undefined. A contradiction! The signs of the above mentioned ratios are determined by using formula (A.0.10) to write the functions

$$\begin{cases} \lambda \in \mathbb{C} \to S(a;q,\lambda), \\ \lambda \in \mathbb{C} \to S_1(\frac{a}{2};q_1,\lambda), \\ \lambda \in \mathbb{C} \to S_1(\frac{a}{2};\tilde{q}_2,\lambda) \end{cases}$$

as infinite products involving their zeros (which are λ_n's, μ_n's, ν_n's respectively) and the length of the intervals $[0,a]$, $[0,\frac{a}{2}]$. These infinite product formulas are:

$$\begin{cases} S(a;q,\lambda) = a\prod_{n=1}^{\infty}\frac{a^2}{(n\pi)^2}(\lambda_n-\lambda), \\ S_1(\frac{a}{2};q_1,\lambda) = \frac{a}{2}\prod_{n=1}^{\infty}\frac{a^2}{(2n\pi)^2}(\mu_n-\lambda), \\ S_1(\frac{a}{2};\tilde{q}_2,\lambda) = \frac{a}{2}\prod_{n=1}^{\infty}\frac{a^2}{(2n\pi)^2}(\nu_n-\lambda). \end{cases}$$

Remark 3.2: An alternative way to prove the existence of solutions to (3.4.35) and to (3.4.40) is to show that the right hand side of (3.4.35) and of (3.4.40) form l_2 sequences of real numbers (see Auxiliary result 4), the sets $\{\sin(\sqrt{\mu_n}t)|n \geq 1\}$ and $\{\sin(\sqrt{\nu_n}t)|n \geq 1\}$ are Riesz bases in $L^2(0,\frac{a}{2})$ (see Auxiliary result 5), and then apply Theorem 8 of [23, page 169] having in mind the definition in [23, page 146] for the moment space of a sequence of vectors in a Hilbert space.

Remark 3.3: We showed in Section 3.4 that the $L^2(0,\frac{a}{2})$ potentials q_1 and $\tilde{q}_2$ exist. But how do we practically (i.e. numerically) construct them? By (3.4.34), (3.4.36) and (3.2.47) we have that:

$$F(q_1) = \{g_1(t), f_1'(t)\} \tag{3.7.1}$$

and from (3.4.38), (3.4.39) and (3.2.47) we get:

$$F(\tilde{q}_2) = \{g_2(t), f_2'(t)\}. \tag{3.7.2}$$

Hence q_1 and $\tilde{q}_2$ are numerically obtained by solving the non-linear equations (3.7.1) and (3.7.2) by a Newton type method (see Subsections 6.1 and 6.2).

CHAPTER 4. NON-EXISTENCE AND NON-UNIQUENESS PHENOMENA

4.1 Non-existence phenomenon

The fact that the remainder-sequences $\{c_n\}_{n\geq 1}$, $\{c_n^1\}_{n\geq 1}$, $\{c_n^2\}_{n\geq 1}$ in formulas (3.1.2) - (3.1.4) are l_2^1 sequences is essential for the proof of Theorem 1 in Section 3.4.

The direct theory of Sturm-Liouville problems tells that if the real valued $q \in L^2(0,a)$ has $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$ as its sequences of Dirichlet eigenvalues on $[0,a]$, $[0,\frac{a}{2}]$, $[\frac{a}{2},a]$ respectively, then formulas (3.1.2) - (3.1.4) hold with $\{c_n\}_{n\geq 1}$, $\{c_n^1\}_{n\geq 1}$, $\{c_n^2\}_{n\geq 1}$ in l_2 (see for example Corollary 1 in [18, page 116]).

However, to guarantee the existence of a real valued $q \in L^2(0,a)$ consistent with the three sequences of λ's, μ's and ν's more is needed in formulas (3.1.2) - (3.1.4) than only having $\{c_n\}_{n\geq 1}$, $\{c_n^1\}_{n\geq 1}$, $\{c_n^2\}_{n\geq 1}$ in l_2. We shall show below that there are sequences $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$ with $\{c_n\}_{n\geq 1}$, $\{c_n^1\}_{n\geq 1}$, $\{c_n^2\}_{n\geq 1}$ in l_2 and $M = \frac{M_1+M_2}{2}$, but with no corresponding q (see Corollary 1 below).

Auxiliary result 7 *Let $q \in L^2(\alpha,\beta)$ be a real valued function and denote by $\{\theta_n\}_{n\geq 1}$ its sequence of Dirichlet eigenvalues. Then the following asymptotic formula holds:*

$$\theta_n = \left(\frac{n\pi}{\beta-\alpha}\right)^2 + \frac{1}{\beta-\alpha}\int_\alpha^\beta q(x)dx - \frac{1}{\beta-\alpha}\int_\alpha^\beta q(x)\cos(2n\pi\frac{x-\alpha}{\beta-\alpha})dx + \mathcal{O}\left(\frac{1}{n}\right),\ \ \textit{as } n\to\infty. \tag{4.1.1}$$

Proof: We start from formula (4.27) of Theorem 4.11 in [13, pages 140-142]: if $\hat{q} \in L^2(0,1)$ is a real valued function having $\{\hat{\theta}_n\}_{n\geq 1}$ as its Dirichlet eigenvalues, then

$$\hat{\theta}_n = (n\pi)^2 + \int_0^1 \hat{q}(s)ds - \int_0^1 \hat{q}(s)\cos(2n\pi s)ds + \mathcal{O}\left(\frac{1}{n}\right),\ \text{as } n\to\infty. \tag{4.1.2}$$

Let $\{\theta_n, u_n(x)\}$ be a Dirichlet eigenpair of q. Then

$$\begin{cases} -u_n''(x) + q(x)u_n(x) = \theta_n u_n(x), \text{ in } (\alpha, \beta) \\ u_n(\alpha) = 0 = u_n(\beta). \end{cases} \tag{4.1.3}$$

Making the change of independent variable $x \leftrightarrow s$ and dependent variable $u_n(x) \leftrightarrow v_n(s)$ by

$$\begin{cases} x = (\beta - \alpha)s + \alpha \\ u_n(x) = v_n(s), \end{cases}$$

the Sturm-Liouville eigenvalue problem (4.1.3) is transformed into:

$$\begin{cases} -v_n''(s) + \hat{q}(s)v_n(s) = \hat{\theta}_n v_n(s), \text{ in } (0, 1) \\ v_n(0) = 0 = v_n(1), \end{cases} \tag{4.1.4}$$

where

$$\begin{cases} \hat{q}(s) = (\beta - \alpha)^2 q(x), \quad \text{with } x = (\beta - \alpha)s + \alpha \\ \hat{\theta}_n = (\beta - \alpha)^2 \theta_n. \end{cases} \tag{4.1.5}$$

Thus (4.1.4) tells that $\{\hat{\theta}_n, v_n(s)\}$ is a Dirichlet eigenpair of $\hat{q} \in L^2(0, 1)$. Hence (4.1.2) applies, and after using (4.1.5) and the right change of variable in the two integrals of (4.1.2) one obtains (4.1.1). $\Box$

Theorem 3 *If the real valued function $q \in L^2(0, a)$ has the Dirichlet spectra $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$ on $[0, a]$, $[0, \frac{a}{2}]$, $[\frac{a}{2}, a]$, respectively then*

$$2\lambda_{2n} - \mu_n - \nu_n = \mathcal{O}\left(\frac{1}{n}\right), \quad \textit{as } n \to \infty. \tag{4.1.6}$$

Proof: Applying Auxiliary result 7 with $([\alpha, \beta], \{\theta_n\}_{n\geq 1})$ replaced by $([0, a], \{\lambda_n\}_{n\geq 1})$, by $\left([0, \frac{a}{2}], \{\mu_n\}_{n\geq 1}\right)$, and by $\left([\frac{a}{2}, a], \{\nu_n\}_{n\geq 1}\right)$ respectively, and using the periodicity of the function cos, formula (4.1.6) is immediate. $\Box$

Corollary 1 *There are sequences*

$$\begin{cases} \lambda_n = (n\pi)^2 + M + c_n, & \textit{as } n \to \infty \\ \mu_n = (2n\pi)^2 + M_1 + c_n^1, & \textit{as } n \to \infty \\ \nu_n = (2n\pi)^2 + M_2 + c_n^2, & \textit{as } n \to \infty \end{cases}$$

with $\{c_n\}_{n\geq 1}$, $\{c_n^1\}_{n\geq 1}$, $\{c_n^2\}_{n\geq 1}$ *in* l_2, *and* $2M = M_1 + M_2$, $M_1 \neq M_2$, *and satisfying the interlacing property stated in Theorem 1, such that no real valued* $q \in L^2(0,1)$ *exists having* $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$ *as its Dirichlet spectra on* $[0,1]$, $[0,\frac{1}{2}]$, $[\frac{1}{2},1]$ *respectively.*

Proof: Let $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$ be such that:

$$\begin{cases} \lambda_n = (n\pi)^2 + \frac{1}{n^{3/4}}, & \text{for } n \geq 1 \\ \mu_n = (2n\pi)^2 + \frac{1}{2} + \frac{1}{n^{3/4}}, & \text{for } n \geq 1 \\ \nu_n = (2n\pi)^2 - \frac{1}{2} + \frac{1}{n^{3/4}}, & \text{for } n \geq 1. \end{cases}$$

We observe that $c_n = c_n^1 = c_n^2 = \frac{1}{n^{3/4}}$ define l_2 sequences, but not l_2^1 sequences (because

$$\sum_{n=1}^{\infty} (nc_n)^2 = \sum_{n=1}^{\infty} \sqrt{n} = \infty).$$

Also $M = 0$, $M_1 = -M_2 = \frac{1}{2}$, so $2M = M_1 + M_2$ and $M_1 \neq M_2$. And we observe that the interlacing of $\{\lambda_n\}_{n\geq 1}$ with $\{\mu_n\}_{n\geq 1} \cup \{\nu_n\}_{n\geq 1}$ as stated in Theorem 1 holds. More precisely:

$$0 < \lambda_1 < \nu_1 < \lambda_2 < \mu_1 < \lambda_3 < \ldots < \nu_n < \lambda_{2n} < \mu_n < \lambda_{2n+1} < \nu_{n+1} < \lambda_{2n+2} < \mu_{n+1} < \ldots.$$

If there were a potential q consistent with the sets of λ's, μ's and ν's, then condition (4.1.6) of Theorem 3 with $a = 1$ would have been satisfied for these three sequences. But it is not, because

$$2\lambda_{2n} - \mu_n - \nu_n = \frac{\sqrt[4]{2} - 2}{n^{3/4}} \neq \mathcal{O}\left(\frac{1}{n}\right), \text{ as } n \to \infty.$$

With this example, Corollary 1 is proved. □

4.2 Non-uniqueness phenomenon

In this section we analyze what happens if the non-overlap hypothesis in Theorem 1 is given up. The claim is that in this situation, the uniqueness of the potential q is lost.

4.2.1 Some previous results on non-uniqueness

Before this analysis is fulfilled we would like to mention that Gesztesy and Simon (1997) have looked into the non-uniqueness phenomenon. In [9, Section 5] the authors discuss the non-uniueness of the coefficient function q when a complete overlap of the spectral data happens for a special kind of inverse three spectra problem. More precisely, the inverse three spectra problem they investigated is: find a real valued function $q \in L^2(0,1)$ such that the given sequences $\{\lambda_n\}_{n\geq 0}$, $\{\mu_n\}_{n\geq 0}$, $\{\nu_n\}_{n\geq 0}$ with prescribed properties are respectively the sequences of eigenvalues for the Sturm-Liouville problems

$$\begin{cases} -u''(x) + q(x)u(x) = \lambda u(x), \text{ in } (0,1) \\ u'(0) - hu(0) = 0 = u'(1) + hu(1), \end{cases}$$

$$\begin{cases} -u''(x) + q(x)u(x) = \mu u(x), \text{ in } (0,\frac{1}{2}) \\ u'(0) - hu(0) = 0 = u(\frac{1}{2}), \end{cases}$$

$$\begin{cases} -u''(x) + q(x)u(x) = \nu u(x), \text{ in } (\frac{1}{2},1) \\ u(\frac{1}{2}) = 0 = u'(1) + hu(1). \end{cases}$$

They showed that the requested q is not unique if the complete overlap

$$\{\mu_n\}_{n\geq 0} = \{\nu_n\}_{n\geq 0}$$

happens. The authors ingeniously constructed (theoretically, only) two coefficient functions q_1, q_2 and the boundary parameter $h \in \mathbb{R}$ such that the same three sequences $\{\lambda_n\}_{n\geq 0}$, $\{\mu_n\}_{n\geq 0}$, $\{\nu_n\}_{n\geq 0}$ correspond simultaneously to the three Sturm-Liouville problems above with $q = q_1$ and $q = q_2$. Their procedure to construct q's worked so nicely due to the symmetry of the boundary conditions at $x = 0$ and $x = 1$. They conjectured only the k-overlap case: if $\{\mu_n\}_{n\geq 0}$ and $\{\nu_n\}_{n\geq 0}$ have k elements in common, then a k-parameter family of q's consistent with the given sets $\{\lambda_n\}_{n\geq 0}$, $\{\mu_n\}_{n\geq 0}$, $\{\nu_n\}_{n\geq 0}$ would exist.

4.2.2 Our results on non-uniqueness

The material to follow in the remaining of this chapter refers only to the non-uniqueness of the solution to the inverse three Dirichlet-spectra problem we have looked into Chapter 3. The proofs we present next are completely different from the proof in [9, Section 5]

Theorem 4 (one overlap) *Let M, M_1, M_2 be three real numbers and $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$ be three sequences of real numbers as in hypotheses of Theorem 1 with the exception* *that*

$$0 < \lambda_1 < \mu_1 = \lambda_2 = \nu_1 < \lambda_3 < \mu_2, \nu_2$$

and having the interlacing property: for every $n \geq 4$ *the interval* $(\lambda_{n-1}, \lambda_n)$ *contains exactly one element of the set*

$$\{\mu_n | n \geq 2\} \cup \{\nu_n | n \geq 2\},$$

and then $\lambda_n \notin \{\mu_n | n \geq 2\} \cup \{\nu_n | n \geq 2\}$*. Then, if there exists a real valued* $L^2(0, a)$ *potential function* q *with Dirichlet sequences of eigenvalues* $\{\lambda_n\}_{n\geq 1}$*,* $\{\mu_n\}_{n\geq 1}$*,* $\{\nu_n\}_{n\geq 1}$ *over the intervals* $[0, a]$*,* $[0, \frac{a}{2}]$*,* $[\frac{a}{2}, a]$ *respectively, the potential* q *depends on only one parameter. In other words, there exists at most a one-parameter family of real valued potentials consistent with the given three sets.*

Observation: The one-overlap $\mu_1 = \lambda_2 = \nu_1$ was chosen for simplicity in exposition, but the one-overlap can be equally well taken as $\mu_k = \lambda_{2k} = \nu_k$, for some $k \geq 1$. We note here that it is more likely to have $\mu_k = \lambda_{2k} = \nu_k$, for some $k \geq 1$ (at least when k is large) than to have any other combination, say $\mu_k = \lambda_p = \nu_l$, for some $k, p, l \geq 1$. This is so because of the leading term in each of the asymptotic formula (3.1.2), (3.1.3), (3.1.4). These formulas in combination with the interlacing property stated in Theorem 1 tell that $\mu_k \in (\lambda_{2k-1}, \lambda_{2k})$ and $\nu_k \in (\lambda_{2k}, \lambda_{2k+1})$ or vice-versa, hence if $\mu_k = \nu_k$, then this forces $\mu_k = \lambda_{2k} = \nu_k$.

Proof of Theorem 4: Construct the functions $f(t)$, $f_1(t)$, $f_2(t)$ and the functions $S(\lambda)$, $S^{(1)}(\lambda)$, $S^{(2)}(\lambda)$ as in the proof of Theorem 1 (see Section 3.4). Then formulas (3.4.17), (3.4.18) and (3.4.19) hold. Also, identical arguments with those in Section 3.4 show that:

$$S^{(1)}(\nu_k) \neq 0, \ S^{(2)}(\mu_k) \neq 0, \text{ for all } k \geq 2. \tag{4.2.1}$$

Due to (3.4.18), (3.4.19) and the one-overlap $\mu_1 = \nu_1$ we have that:

$$S^{(1)}(\nu_1) = 0 = S^{(2)}(\mu_1), \tag{4.2.2}$$

and due to (3.4.17) and the one-overlap $\mu_1 = \lambda_2 = \nu_1$ we have that:

$$S(\mu_1) = 0 = S(\nu_1). \tag{4.2.3}$$

Formulas (4.2.1), (4.2.2) and (4.2.3) tell us that the sequences of quotients

$$\{\frac{S(\mu_n)}{S^{(2)}(\mu_n)}\}_{n\geq 2} \text{ and } \{\frac{S(\nu_n)}{S^{(1)}(\nu_n)}\}_{n\geq 2}$$

are well-defined, but the quotients

$$\frac{S(\mu_1)}{S^{(2)}(\mu_1)} \text{ and } \frac{S(\nu_1)}{S^{(1)}(\nu_1)}$$

are undefined (as $\left[\frac{0}{0}\right]$), and so we can construct the sequences $\{k_n^1\}_{n\geq 1}$ and $\{k_n^2\}_{n\geq 1}$ by choosing arbitrarily k_1^1 and k_1^2 in $\mathbb{R}$ and defining k_n^1 and k_n^2 for $n \geq 2$ as in (3.4.22) and (3.4.23) respectively. Note that at this point we are free to choose two parameters, but we shall argue later in the proof that the two parameters are related, hence leaving us with at most one degree of freedom.

The proof that $\{k_n^1\}_{n\geq 1}$ and $\{k_n^2\}_{n\geq 1}$ are l_2^1 real valued sequences is identical with the one given in Section 3.4). Next Corollary 2 of [18, page 116] applies to the pairs

$$\{\{\mu_n\}_{n\geq 1}, \{k_n^1\}_{n\geq 1}\} \text{ and } \{\{\nu_n\}_{n\geq 1}, \{k_n^2\}_{n\geq 1}\}$$

to yield two real valued functions $q_1 \in L^2(0, \frac{a}{2})$ and $\tilde{q}_2 \in L^2(0, \frac{a}{2})$ respectively, each having the indicated pair as its pair of Dirichlet eigenvalues and Pöschel-Trubowitz norming constants (see [18, pages 50 and 59]).

Then define

$$q(x) = \begin{cases} q_1(x), & \text{in } [0, \frac{a}{2}] \\ \tilde{q}_2(a-x), & \text{in } [\frac{a}{2}, a]. \end{cases}$$

Clearly q has $\{\mu_n\}_{n\geq 1}$ and $\{\nu_n\}_{n\geq 1}$ as its Dirichlet sequences on $[0, \frac{a}{2}]$ and $[\frac{a}{2}, a]$, but it remains under investigation that q has $\{\lambda_n\}_{n\geq 1}$ as its sequences of Dirichlet eigenvalues on $[0, a]$.

If this happens, then a real valued potential consistent with the given data, and depending on two parameters k_1^1 and k_1^2 would exist. Now we argue that in this case, k_1^1 and k_1^2 cannot be independent. Suppose they are independent. Then for a fixed value of k_1^1 we can choose two different values, say k_1^2 and $\bar{k}_1^2$ (otherwise k_1^2 would be uniquely determined by the relationship between k_1^1 and k_1^2). Then, as above q_1 and $\tilde{q}_2^*$ can be constructed with the first Pöschel-Trubowitz norming constant k_1^1 and respectively $\bar{k}_1^2$, and having $\{\mu_n\}_{n\geq 1}$ and $\{\nu_n\}_{n\geq 1}$ as Dirichlet eigenvalues on $[0, \frac{a}{2}]$ respectively. Then, similarly

$$q^*(x) = \begin{cases} q_1(x), & \text{in } [0, \frac{a}{2}] \\ \tilde{q}_2^*(a-x), & \text{in } [\frac{a}{2}, a]. \end{cases}$$

would have $\{\lambda_n\}_{n\geq 1}$ as its Dirichlet eigenvalues on $[0, a]$. But then, Hochstadt and Lieberman result [12] would imply that

$$q = q^*$$

from which it follows that $\tilde{q}_2 = \tilde{q}_2^*$, and so $\tilde{q}_2$ and $\tilde{q}_2^*$ have the same Dirichlet eigenvalues and Pöschel-Trubowitz norming constants. This implies that $k_1^2 = \bar{k}_1^2$. But this contradicts our assumption.

In conclusion, there exists *at most* a one-parameter family of real valued $L^2(0, \frac{a}{2})$ potentials consistent with the given three spectra. $\square$

Theorem 5 (N overlaps) *Let M, M_1, M_2 be three real numbers and $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$ be three sequences of real numbers as in hypotheses of Theorem 1 with the exception that*

$$0 < \lambda_1 < \mu_1 = \lambda_2 = \nu_1 < \ldots < \lambda_{2N-1} < \mu_N = \lambda_{2N} = \nu_N < \lambda_{2N+1} < \mu_{N+1}, \nu_{N+1}$$

and having the interlacing property: for every $n \geq 2N+2$ the interval $(\lambda_{n-1}, \lambda_n)$ contains exactly one element of the set

$$\{\mu_n | n \geq N+1\} \cup \{\nu_n | n \geq N+1\},$$

and then $\lambda_n \notin \{\mu_n | n \geq N+1\} \cup \{\nu_n | n \geq N+1\}$. Then, if there exists a real valued $L^2(0, a)$ potential function q with Dirichlet sequences of eigenvalues $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$ over the intervals $[0, a]$, $[0, \frac{a}{2}]$, $[\frac{a}{2}, a]$ respectively, the potential q depends on N parameters, only.

Observation: The N-overlaps above were chosen for an easy exposition in the proof, but they can equally well be taken as

$$\mu_{k_1} = \lambda_{2k_1} = \nu_{k_1} < \ldots < \mu_{k_N} = \lambda_{2k_N} = \nu_{k_N},$$

for some $1 \leq k_1 < \ldots < k_N$.

Proof of Theorem 5: It is similar to the proof of Theorem 4. The only differences are as follows:

$$\begin{cases} S^{(1)}(\nu_k) \neq 0 \neq S^{(2)}(\mu_k), & \text{for all } k \geq N+1, \\ S^{(1)}(\nu_k) = 0 = S^{(2)}(\mu_k), & \text{for } k = 1, 2, \ldots, N, \\ S(\mu_k) = 0 = S(\nu_k), & \text{for } k = 1, 2, \ldots, N. \end{cases} \tag{4.2.4}$$

Therefore the quotients

$$\frac{S(\mu_n)}{S^{(2)}(\mu_n)}, \text{ and } \frac{S(\nu_n)}{S^{(1)}(\nu_n)} \text{ for } n \geq N+1$$

are well-defined, but the quotients

$$\frac{S(\mu_n)}{S^{(2)}(\mu_n)}, \text{ and } \frac{S(\nu_n)}{S^{(1)}(\nu_n)}, \text{ for } n = 1, 2, \ldots, N$$

are undefined (as $\left[\frac{0}{0}\right]$). Hence we can construct two sequences of numbers $\{k_n^1\}_{n\geq 1}$ and $\{k_n^2\}_{n\geq 1}$ by choosing arbitrarily $k_1^1, \ldots, k_N^1$ and $k_1^2, \ldots, k_N^2$ in $\mathbb{R}$ and then defining k_n^1 and k_n^2, for $n \geq N+1$ by (3.4.22) and (3.4.23), respectively. Then we continue as in the proof of Theorem 4, with the scalars k_1^1 and k_1^2 replaced by the vectors $(k_1^1, \ldots, k_N^1)$ and $(k_1^2, \ldots, k_N^2)$ respectively.

In conclusion, there exists *at most* one N-parameter family of q's consistent with the given three spectra. □

Observation: The dependence of $(k_1^2, \ldots, k_N^2)$ on $(k_1^1, \ldots, k_N^1)$ can be written in the form of a vectorial equation:

$$\Phi(k_1^1, \ldots, k_N^1; k_1^2, \ldots, k_N^2) = 0$$

where Φ takes values in $\mathbb{R}^N$, or in the form of a system of N scalar equations:

$$\begin{cases} \phi_1(k_1^1, \ldots, k_N^1; k_1^2, \ldots, k_N^2) = 0 \\ \vdots \\ \phi_N(k_1^1, \ldots, k_N^1; k_1^2, \ldots, k_N^2) = 0. \end{cases}$$

CHAPTER 5. GENERALIZATIONS

5.1 The case of unequal subintervals

One possible generalization of the problem described in Chapter 3 is to consider the interval $[0, a]$ divided into unequal subintervals $[0, a_0]$, $[a_0, a]$, and maintain the Dirichlet boundary conditions at the end-points of the interval $[0, a]$ and at the interior node a_0. Similar results with the ones in Theorem 1 will be presented, but now for the case of unequal subintervals.

Theorem 6 *Let $a > 0$ and $0 < a_0 < a$. Let M, M_1, M_2 be three real numbers such that*

$$\begin{cases} M_1 \neq \left(\frac{a-a_0}{a_0}\right)^2 M_2 \\ M = \frac{a_0 \cdot M_1 + (a-a_0)\cdot M_2}{a}. \end{cases} \tag{5.1.1}$$

Let $\{\lambda_n | n \geq 1\}$, $\{\mu_n | n \geq 1\}$, and $\{\nu_n | n \geq 1\}$ be three sequences of real numbers, strictly increasing, such that neither two of them intersect, and satisfying the following asymptotic formulas:

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2 + M + c_n, \text{ as } n \to \infty, \text{ where } (c_n)_{n\geq 1} \in l_2^1, \tag{5.1.2}$$

$$\mu_n = \left(\frac{n\pi}{a_0}\right)^2 + M_1 + c_n^1, \text{ as } n \to \infty, \text{ where } (c_n^1)_{n\geq 1} \in l_2^1, \tag{5.1.3}$$

$$\nu_n = \left(\frac{n\pi}{a - a_0}\right)^2 + M_2 + c_n^2, \text{ as } n \to \infty, \text{ where } (c_n^2)_{n\geq 1} \in l_2^1, \tag{5.1.4}$$

and having the following interlacing property:

$$0 < \lambda_1 < \mu_1 \text{ and } 0 < \lambda_1 < \nu_1$$

and for every $n > 1$ the interval $(\lambda_{n-1}, \lambda_n)$ contains exactly one element of the set

$$\{\mu_m | m \geq 1\} \cup \{\nu_m | m \geq 1\}.$$

Then there exists a real-valued potential $q \in L^2(0,a)$ such $\{\lambda_n | n \geq 1\}$, $\{\mu_n | n \geq 1\}$ and $\{\nu_n | n \geq 1\}$ are the Dirichlet eigenvalues for the Sturm-Liouville problem with potential q over the intervals $[0,a]$, $[0,a_0]$, $[a_0,a]$ respectively. The potential function with this property is unique.

Observation: The uniqueness of a potential q having the specified three sets as its sets of Dirichlet eigenvalues, assuming only pairwise disjointness of these three sets was also proved by Gesztesy and Simon (see Theorem 1 and Section 3 in [9]). However, the proof of uniqueness given here is completely different from their proof.

Proof of Theorem 6: Assuming first that such potential function q exists we can write the following three key ingredients:

$$S(a;q,\lambda) = S_1(a_0;q_1,\lambda)C_2(a;q_2,\lambda) + S_1'(a_0;q_1,\lambda)S_2(a;q_2,\lambda), \text{ for all } \lambda \in \mathbb{C}, \tag{5.1.5}$$

$$C_2(a;q_2,\lambda) = S_1'(a_0;\tilde{q}_2,\tilde{\lambda}), \tag{5.1.6}$$

$$S_2(a;q_2,\lambda) = \frac{a-a_0}{a_0} \cdot S_1(a_0;\tilde{q}_2,\tilde{\lambda}), \tag{5.1.7}$$

where $q_1 = q|_{[0,a_0]}$, $q_2 = q|_{[a_0,a]}$, the pairs $\{q_2,\lambda\}$ and $\{\tilde{q}_2,\tilde{\lambda}\}$ are related to each other by the formulas

$$\begin{cases} \xi \in [0,a_0] \leftrightarrow x = a - \left(\frac{a-a_0}{a_0}\right)\xi \in [a_0,a] \\ \tilde{q}_2(\xi) = \left(\frac{a-a_0}{a_0}\right)^2 \cdot q_2(x) \\ \tilde{\lambda} = \left(\frac{a-a_0}{a_0}\right)^2 \cdot \lambda, \end{cases} \tag{5.1.8}$$

$C(\cdot;p,\lambda)$ and $S(\cdot;p,\lambda)$ are the unique solutions to (3.2.1) & (3.2.2) and respectively to (3.2.1) & (3.2.3), the subscripts 1, 2 attached to either one of the previous function are meant to refer to the first subinterval $[0,a_0]$ and to the second subinterval $[a_0,a]$, respectively. The proofs of formulas (5.1.5), (5.1.6) and (5.1.7) are presented in Auxiliary result 8, soon after the proof of Theorem 6. From (5.1.5), (5.1.6) and (5.1.7) we infer that

$$S(a;q,\lambda) = S_1(a_0;q_1,\lambda)S_1'(a_0;\tilde{q}_2,\tilde{\lambda}) + \left(\frac{a-a_0}{a_0}\right) S_1'(a_0;q_1,\lambda)S_1(a_0;\tilde{q}_2,\tilde{\lambda}), \tag{5.1.9}$$

where the same relationship (5.1.8) holds. Having in mind the integral representation of $S(a;q,\lambda)$, $S_1(a_0;q_1,\lambda)$, $S_1(a_0;\tilde{q}_2,\tilde{\lambda})$ (see (3.2.20)), and of $S_1'(a_0;q_1,\lambda)$, $S_1'(a_0;\tilde{q}_2,\tilde{\lambda})$ (see (3.2.21)), formula (5.1.9), the fact that

$$\begin{cases} S(a;q,\lambda_n)=0,\ \ n\geq 1, \\ S_1(a_0;q_1,\mu_n)=0,\ \ n\geq 1, \\ S_1(a_0;\tilde{q}_2,\tilde{\nu}_n)=0,\ \ n\geq 1, \end{cases}$$

and the Goursat problems of type (3.2.18) which the kernels $K(x,t;q)$, $K(x,t;q_1)$, and $K(x,t;\tilde{q}_2)$ satisfy, we can describe a natural way of constructing the potential function q. Solve for $f\in H^2(0,a)$, $f_1\in H_0^2(0,a_0)$, and $f_2\in H_0^2(0,a_0)$ the following systems of integral equations:

$$\frac{\sin(\sqrt{\lambda_n}a)}{\sqrt{\lambda_n}}+\int_0^a f(t)\frac{\sin(\sqrt{\lambda_n}t)}{\sqrt{\lambda_n}}dt=0,\ \ n\geq 1, \tag{5.1.10}$$

$$\frac{\sin(\sqrt{\mu_n}a_0)}{\sqrt{\mu_n}}+\int_0^{a_0} f_1(t)\frac{\sin(\sqrt{\mu_n}t)}{\sqrt{\mu_n}}dt=0,\ \ n\geq 1, \tag{5.1.11}$$

and respectively

$$\frac{\sin(\sqrt{\tilde{\nu}_n}a_0)}{\sqrt{\tilde{\nu}_n}}+\int_0^{a_0} f_2(t)\frac{\sin(\sqrt{\tilde{\nu}_n}t)}{\sqrt{\tilde{\nu}_n}}dt=0,\ \ n\geq 1. \tag{5.1.12}$$

Existence and uniqueness of the solutions to these systems can be shown analogously to those in Section 3.4. The asymptotics (5.1.2), (5.1.3), (5.1.4) of the sequences $\{\lambda_n|n\geq 1\}$, $\{\mu_n|n\geq 1\}$, and $\{\nu_n|n\geq 1\}$ respectively are essential in this regard. The uniqueness of f follows due to the Auxiliary result 1, and uniqueness of f_1 and f_2 follow due to the Auxiliary result 9 presented after the proof of Theorem 6.

Then construct the functions

$$S(\lambda)=\frac{\sin(\sqrt{\lambda}a)}{\sqrt{\lambda}}+\int_0^a f(t)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}dt,\ \text{for all }\lambda\in\mathbb{C} \tag{5.1.13}$$

$$S_1(\lambda)=\frac{\sin(\sqrt{\lambda}a_0)}{\sqrt{\lambda}}+\int_0^{a_0} f_1(t)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}dt,\ \text{for all }\lambda\in\mathbb{C} \tag{5.1.14}$$

$$S_2(\tilde{\lambda})=\frac{\sin(\sqrt{\tilde{\lambda}}a_0)}{\sqrt{\tilde{\lambda}}}+\int_0^{a_0} f_2(t)\frac{\sin(\sqrt{\tilde{\lambda}}t)}{\sqrt{\tilde{\lambda}}}dt,\ \text{for all }\tilde{\lambda}=\left(\tfrac{a-a_0}{a_0}\right)^2\lambda\in\mathbb{C}. \tag{5.1.15}$$

Further the quotients

$$\begin{cases} \frac{S(\mu_n)}{S_2(\tilde{\mu}_n)},\ \ n\geq 1, \\ \frac{S(\nu_n)}{S_1(\nu_n)},\ \ n\geq 1, \end{cases}$$

with $\tilde{\mu}_n = \left(\frac{a-a_0}{a_0}\right)^2 \mu_n$ will be known quantities. An argument similar to the one presented in Section 3.4 shows that

$$\begin{cases} S_2(\tilde{\mu}_n) \neq 0, & \text{for all } n \geq 1 \\ S_1(\nu_n) \neq 0, & \text{for all } n \geq 1. \end{cases}$$

The non-overlap property $\{\mu_n | n \geq 1\} \cap \{\nu_n | n \geq 1\} = \phi$ is also used. Next, solve for $g_1 \in L^2(0, a_0)$ and $g_2 \in L^2(0, a_0)$ the systems of integral equations

$$\cos(\sqrt{\mu_n} a_0) + \left(\frac{a_0 \cdot M_1}{2}\right) \frac{\sin(\sqrt{\mu_n} a_0)}{\sqrt{\mu_n}} + \int_0^{a_0} g_1(t) \frac{\sin(\sqrt{\mu_n} t)}{\sqrt{\mu_n}} dt = \left(\frac{a_0}{a-a_0}\right) \frac{S(\mu_n)}{S_2(\tilde{\mu}_n)}, \; n \geq 1 \tag{5.1.16}$$

and respectively

$$\cos(\sqrt{\tilde{\nu}_n} a_0) + \left(\frac{a-a_0}{a_0}\right) \left(\frac{(a-a_0) \cdot M_2}{2}\right) \frac{\sin(\sqrt{\tilde{\nu}_n} a_0)}{\sqrt{\tilde{\nu}_n}} + \int_0^{a_0} g_2(t) \frac{\sin(\sqrt{\tilde{\nu}_n} t)}{\sqrt{\tilde{\nu}_n}} dt = \frac{S(\nu_n)}{S_1(\nu_n)}, \; n \geq 1 \tag{5.1.17}$$

Similarly, the reader is referred to Section 3.4 for the existence and uniqueness of the solutions g_1 and g_2 to (5.1.16) and (5.1.17), using the same asymptotic formulas (5.1.2), (5.1.3), (5.1.4). Then solve the non-linear equations

$$\begin{cases} F(q_1) = \{g_1(t), f_1'(t)\}, \\ F(\tilde{q}_2) = \{g_2(t), f_2'(t)\}, \end{cases}$$

for q_1 and $\tilde{q}_2$. Here the map F is defined by (3.2.47) with $[0, \beta]$ replaced by $[0, a_0]$. Finally, write

$$q_2(x) = \left(\frac{a_0}{a-a_0}\right)^2 \tilde{q}_2\left(\frac{a_0}{a-a_0}(a-x)\right), \text{ for } x \in [a_0, a],$$

and paste together q_1 and q_2 to get q over the entire interval $[0, a]$. The proof that the potential function q constructed above is the one having as its Dirichlet eigenvalues over the intervals $[0, a]$, $[0, a_0]$, $[a_0, a]$, the three sequences of real numbers $\{\lambda_n | n \geq 1\}$, $\{\mu_n | n \geq 1\}$ and $\{\nu_n | n \geq 1\}$, respectively is similar to the proof in Section 3.4. Also, the proof that q with the properties described in the hypotheses of Theorem 6 is unique goes similarly with the proof in Section 3.5. $\square$

Auxiliary result 8 *Formulas (5.1.5), (5.1.6) and (5.1.7) hold.*

Proof: Formula (5.1.5) follows from the fact that $S(\cdot; q, \lambda)$ is a solution to the equation

$$-u''(x) + q(x)u(x) = \lambda u(x),$$

over the interval $[0, a]$, and so it is a solution to the same equation over the interval $[a_0, a]$ where $C_2(\cdot; q_2, \lambda)$ and $S_2(\cdot; q_2, \lambda)$ form a fundamental set of solutions. Therefore we can write

$$S(x; q, \lambda) = (k_1)C_2(x; q_2, \lambda) + (k_2)S_2(x; q_2, \lambda),\ x \in [a_0, a]. \tag{5.1.18}$$

Taking the derivative with respect to x in (5.1.18) we obtain:

$$S'(x; q, \lambda) = (k_1)C_2'(x; q_2, \lambda) + (k_2)S_2'(x; q_2, \lambda),\ x \in [a_0, a]. \tag{5.1.19}$$

Then taking $x = a_0$ in (5.1.18) and (5.1.19) and using the initial conditions the functions $C_2(\cdot; q_2, \lambda)$ and $S_2(\cdot; q_2, \lambda)$ satisfy we find the constants k_1 and k_2 to be $S(a_0; q, \lambda)$ and $S'(a_0; q, \lambda)$. Easily one can show that

$$S(x; q, \lambda) = S_1(x; q_1, \lambda),\ \text{for } x \in [0, a_0],$$

by using the uniqueness of solution to the initial value problem

$$\begin{cases} -u''(x) + q_1(x)u(x) = \lambda u(x),\ 0 < x < a_0, \\ u(0) = 0, \\ u'(0) = 1 \end{cases}$$

and the fact that $q_1 = q|_{[0,a_0]}$. From these we have that:

$$\begin{cases} k_1 = S_1(a_0; q_1, \lambda) \\ k_2 = S_1'(a_0; q_1, \lambda). \end{cases} \tag{5.1.20}$$

Taking $x = a$ in (5.1.18) and using (5.1.20), formula (5.1.5) is immediate. To prove (5.1.6) and (5.1.7) one can either use ideas similar to those in Lemma 1 or else similar to those in Lemma 2. If for example formula (A.0.10) is used then we can write:

$$S_2(a; q_2, \lambda) = (a - a_0) \prod_{n=1}^{\infty} \left(\frac{a - a_0}{n\pi} \right)^2 (z_n - \lambda), \tag{5.1.21}$$

where z_n's are all the zeros of $S_2(a;q_2,\cdot)$, the characteristic function of the Sturm-Liouville problem

$$\begin{cases} -u''(x)+q_2(x)u(x)=\lambda u(x),\ a_0<x<a, \\ u(a_0)=0, \\ u(a)=0, \end{cases} \tag{5.1.22}$$

and

$$S_1(a_0;\tilde{q}_2,\tilde{\lambda})=(a_0-0)\prod_{n=1}^{\infty}\left(\frac{a_0-0}{n\pi}\right)^2(\tilde{z}_n-\tilde{\lambda}), \tag{5.1.23}$$

where $\tilde{z}_n$'s are all the zeros of $S_1(a_0;\tilde{q}_2,\cdot)$, the characteristic function of the Sturm-Liouville problem

$$\begin{cases} -v''(\xi)+\tilde{q}_2(\xi)v(\xi)=\tilde{\lambda}v(\xi),\ 0<\xi<a_0, \\ v(0)=0, \\ v(a_0)=0. \end{cases} \tag{5.1.24}$$

Remembering that the zeros of the characteristic function $S_2(a;q_2,\cdot)$ are exactly the Dirichlet eigenvalues of the operator $L^{(q_2)}$ over the interval $[a_0,a]$, and the zeros of the characteristic function $S_1(a_0;\tilde{q}_2,\cdot)$ are exactly the Dirichlet eigenvalues of the operator $L^{(\tilde{q}_2)}$ over the interval $[0,a_0]$, and noting that

$$\{\lambda,u(x)\}$$

is a Dirichlet eigenpair of $L^{(q_2)}$ (i.e. $\{\lambda,u(x)\}$ solves (5.1.22)) if and only if

$$\{\tilde{\lambda}=\left(\frac{a-a_0}{a_0}\right)^2\lambda, v(\xi)=u(a-\left(\frac{a-a_0}{a_0}\right)\xi)\}$$

is a Dirichlet eigenpair of $L^{(\tilde{q}_2)}$ (i.e. $\{\tilde{\lambda},v(\xi)\}$ solves (5.1.24)), where (5.1.8) holds, we have that

$$\tilde{z}_n=\left(\frac{a-a_0}{a_0}\right)^2 z_n,\ \text{for all } n\geq 1. \tag{5.1.25}$$

Now (5.1.7) follows immediately from (5.1.21), (5.1.23), (5.1.8) and (5.1.25). Now we turn our attention to proving formula (5.1.6). By the change of variable

$$s\in[0,1]\leftrightarrow x=(a-a_0)s+a_0\in[a_0,a],$$

we observe that $u(x)$ solves the initial value problem

$$\begin{cases} -u''(x) + q_2(x)u(x) = \lambda u(x), \; a_0 < x < a, \\ u(a_0) = 1, \\ u'(a_0) = 0, \end{cases} \tag{5.1.26}$$

if and only if

$$v(s) = u((a - a_0)s + a_0) = u(x) \tag{5.1.27}$$

solves the initial value problem

$$\begin{cases} -v''(s) + p(s)v(s) = \hat{\lambda} v(s), \; 0 < s < 1, \\ v(0) = 1, \\ v'(0) = 0, \end{cases} \tag{5.1.28}$$

where

$$\begin{cases} p(s) = (a - a_0)^2 q_2((a - a_0)s + a_0) = (a - a_0)^2 q_2(x), \\ \hat{\lambda} = (a - a_0)^2 \lambda. \end{cases} \tag{5.1.29}$$

Since the unique solution to (5.1.28) is $C(s; p, \hat{\lambda})$ and the unique solution to (5.1.26) is $C_2(x; q_2, \lambda)$, formula (5.1.27) yields

$$C(s; p, \hat{\lambda}) = C_2(x; q_2, \lambda), \text{ with } x = (a - a_0)s + a_0. \tag{5.1.30}$$

Taking $s = 1$, so $x = a$ in (5.1.30) we have that

$$C_2(a; q_2, \lambda) = C(1; p, \hat{\lambda}), \tag{5.1.31}$$

where (5.1.29) holds. Next, by making the change of variable

$$s \in [0, 1] \leftrightarrow \xi = a_0 s \in [0, a_0]$$

one can show that $u(\xi)$ solves the initial value problem

$$\begin{cases} -u''(\xi) + \tilde{q}_2(\xi)u(\xi) = \tilde{\lambda} u(\xi), \; 0 < \xi < a_0, \\ u(0) = 0, \\ u'(0) = 1, \end{cases} \tag{5.1.32}$$

if and only if

$$v(s) = \frac{u(a_0 s)}{a_0} = \frac{u(\xi)}{a_0} \tag{5.1.33}$$

solves the initial value problem

$$\begin{cases} -v''(s) + \tilde{p}(s)v(s) = \lambda^* v(s), \ 0 < s < 1, \\ v(0) = 0, \\ v'(0) = 1, \end{cases} \tag{5.1.34}$$

where

$$\begin{cases} \tilde{p}(s) = (a_0)^2 \tilde{q}_2(a_0 s) = (a_0)^2 \tilde{q}_2(\xi), \\ \lambda^* = (a_0)^2 \tilde{\lambda}. \end{cases} \tag{5.1.35}$$

Since the unique solution to (5.1.34) is $S(s; \tilde{p}, \lambda^*)$, and the unique solution to (5.1.32) is $S_1(\xi; \tilde{q}_2, \tilde{\lambda})$, formula (5.1.33) reads

$$S(s; \tilde{p}, \lambda^*) = \frac{S_1(\xi; \tilde{q}_2, \tilde{\lambda})}{a_0}, \text{ with } \xi = a_0 s. \tag{5.1.36}$$

Taking the derivative with respect to s in (5.1.36) and then making $s = 1$, so $\xi = a_0$ we obtain

$$S_1'(a_0; \tilde{q}_2, \tilde{\lambda}) = S'(1; \tilde{p}, \lambda^*), \tag{5.1.37}$$

where (5.1.35) holds. Using (5.1.35), (5.1.8), and (5.1.29) one gets:

$$\begin{aligned} \lambda^* &= (a_0)^2 \tilde{\lambda} \\ &= (a - a_0)^2 \lambda \\ &= \hat{\lambda}, \end{aligned} \tag{5.1.38}$$

and

$$\begin{aligned} \tilde{p}(s) &= (a_0)^2 \tilde{q}_2(a_0 s) \\ &= (a - a_0)^2 q_2(a - \left(\frac{a - a_0}{a_0}\right) a_0 s) \\ &= (a - a_0)^2 q_2(a(1 - s) + a_0 s) \\ &= (a - a_0)^2 q_2((a - a_0)(1 - s) + a_0) \\ &= p(1 - s), \ \text{for all } s \in [0, 1]. \end{aligned} \tag{5.1.39}$$

Formulas (5.1.38) and (5.1.39) allow us to use (A.0.12) and write

$$C(1; p, \hat{\lambda}) = S'(1; \tilde{p}, \lambda^*). \tag{5.1.40}$$

Then (5.1.40) along with (5.1.31) and (5.1.37) lead to (5.1.6). □

Auxiliary result 9 *Given the sets $\{\mu_n\}_{n\geq 1}$ and $\{\nu_n\}_{n\geq 1}$ satisfying (5.1.3) and (5.1.4), the sets*

$$\{\sin(\sqrt{\mu_n} t)\}_{n\geq 1} \text{ and } \{\sin(\sqrt{\tilde{\nu}_n} t)\}_{n\geq 1}$$

are complete in $L^2(0, a_0)$. Here $\tilde{\nu}_n = \left(\frac{a - a_0}{a_0}\right)^2 \nu_n$.

Proof: Similar to the proof of Auxiliary result 1. □

The algorithm: The main steps of the algorithm were already outlined in the proof of Theorem 6. The only thing we want to mention here is that for solving numerically the systems (5.1.10), (5.1.11), (5.1.12), the Fourier series representations

$$f(t) = \left(\frac{M}{2}\right) t + \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi}{a} t\right), \; t \in [0, a], \tag{5.1.41}$$

$$f_1(t) = \left(\frac{M_1}{2}\right) t + \sum_{n=1}^{\infty} \alpha_n^{(1)} \sin\left(\frac{n\pi}{a_0} t\right), \; t \in [0, a_0], \tag{5.1.42}$$

$$f_2(t) = \left(\frac{a - a_0}{a_0}\right)^2 \left(\frac{M_2}{2}\right) t + \sum_{n=1}^{\infty} \alpha_n^{(2)} \sin\left(\frac{n\pi}{a_0} t\right), \; t \in [0, a_0], \tag{5.1.43}$$

are used. Inserting them into (5.1.10), (5.1.11) and respectively (5.1.12), three linear systems of equations are obtained to be solved for the Fourier coefficients of f, f_1, and f_2. And similarly, write

$$g_1(t) = \sum_{n=1}^{\infty} \beta_n^{(1)} \sin\left(\frac{(n - \frac{1}{2})\pi}{a_0} t\right), \; t \in [0, a_0] \tag{5.1.44}$$

and

$$g_2(t) = \sum_{n=1}^{\infty} \beta_n^{(2)} \sin\left(\frac{(n - \frac{1}{2})\pi}{a_0} t\right), \; t \in [0, a_0]. \tag{5.1.45}$$

Insert these Fourier series into (5.1.16) and respectively into (5.1.17) to solve for the Fourier coefficients of g_1 and g_2.

5.2 The case of mixed boundary condition at the interior node

Here by a mixed boundary condition at the interior node a_0 of the interval $[0, a]$ we mean a boundary condition of Robin type:

$$u'(a_0) + Hu(a_0) = 0, \tag{5.2.1}$$

where $H \in \mathbb{R}$ is a known constant. This boundary condition would correspond to the practical situation of a string clamped at the interior node to a spring of stiffness not exactly H, but depending on H. Or in other words, the boundary condition (5.2.1) corresponds to a string which set into infinitesimal vibrations will bend at any instant at the interior node proportionally to the string displacement from the equilibrium position at that instant. To see why this is so we may assume that the string is in equilibrium when it lies on the horizontal line. Then from (5.2.1) we have

$$u'(a_0) = -Hu(a_0),$$

which used in (2.2.4) along with the fact that $a_0 = x(\xi_0^*)$ give

$$\phi'(\xi_0^*) = u(a_0)\left(-Hv(a_0) - v'(a_0)\right). \tag{5.2.2}$$

Using next (2.2.1), and $a_0 = x(\xi_0^*)$ in (5.2.2) we obtain

$$\phi'(\xi_0^*) + \left(v'(a_0) + Hv(a_0)\right)v(a_0) \cdot \phi(\xi_0^*) = 0. \tag{5.2.3}$$

One can easily check that starting with (5.2.3) and using (2.2.4) and (2.2.1), formula (5.2.1) is recovered. Therefore, (5.2.1) and (5.2.3) are equivalent, and we are going to manipulate further only (5.2.3). Denote

$$\hat{H} = \left(v'(a_0) + Hv(a_0)\right)v(a_0). \tag{5.2.4}$$

Hence (5.2.3) can be written equivalently

$$\phi'(\xi_0^*) + \hat{H}\phi(\xi_0^*) = 0. \tag{5.2.5}$$

Below we show that the constant $\hat{H}$ contains information about the string at location ξ_0^* only. Expressing v from equation (2.2.7) we have

$$v(x) = \sqrt[4]{\tilde{\rho}(\xi)}. \tag{5.2.6}$$

Recalling that $x = x(\xi)$, taking the derivative with respect to ξ in (5.2.6) and using (2.2.3) and (2.2.7) the following calculations come out:

$$\begin{aligned} v'(x) &= \frac{1}{4\sqrt[4]{\tilde{\rho}^3(\xi)}} \cdot \frac{1}{x'(\xi)} \\ &= \frac{1}{4\sqrt[4]{\tilde{\rho}^3(\xi)}} \cdot \frac{1}{v^2(x)} \\ &= \frac{1}{4\sqrt[4]{\tilde{\rho}^3(\xi)}} \cdot \frac{1}{\sqrt[4]{\tilde{\rho}^2(\xi)}} \\ &= \frac{1}{4\sqrt[4]{\tilde{\rho}^5(\xi)}}. \end{aligned} \tag{5.2.7}$$

With $\xi = \xi_0^*$, so $x = a_0$ in (5.2.7) and in (2.2.7) we obtain the final expession for $\hat{H}$ in (5.2.4):

$$\hat{H} = \frac{1}{4\tilde{\rho}(\xi_0^*)} + H\sqrt{\tilde{\rho}(\xi_0^*)}. \tag{5.2.8}$$

Next, if we multiply (5.2.5) by $\psi(\tau)$ and use (2.1.5) we get:

$$\frac{\partial U}{\partial \xi}(\xi_0^*, \tau) + \hat{H}U(\xi_0^*, \tau) = 0. \tag{5.2.9}$$

Now the interpretation of (5.2.1) is straightforward, because (5.2.1) is equivalent to (5.2.5), and to (5.2.9), and

$$\frac{\partial U}{\partial \xi}(\xi_0^*, \tau)$$

represents the tangent of the angle between horizontal line (i.e. the string at equilibrium) and the tangent line to the curve

$$\xi \in [0, \xi_0] \to U(\xi, \tau)$$

(i.e. the string at time τ) at location $\xi = \xi_0^*$.

The practical situation we want to solve is the recovery of a string (in fact, of its density) from measurements of three sets of frequencies of oscillation: one corresponding to the vibration of the entire string, another two corresponding to the independent vibrations of the first part and of the second part of the string, parts obtained by clamping the string at an interior node to a spring with a known stiffness constant.

Theorem 7 *Let $a > 0$, $0 < a_0 < a$ and H be a real number. Let M, M_1, M_2 be three real numbers such that*

$$\begin{cases} M_1 \neq \left(\frac{a-a_0}{a_0}\right)^2 M_2 \\ M = \frac{a_0 \cdot M_1 + (a-a_0) \cdot M_2}{a}. \end{cases} \tag{5.2.10}$$

Let $\{\lambda_n | n \geq 1\}$, $\{\mu_n | n \geq 1\}$, and $\{\nu_n | n \geq 1\}$ be three sequences of real numbers, strictly increasing, such that neither two of them intersect, and satisfying the following asymptotic formulas:

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2 + M + c_n, \text{ as } n \to \infty, \text{ where } (c_n)_{n\geq 1} \in l_2^1, \tag{5.2.11}$$

$$\mu_n = \left(\frac{(n-\frac{1}{2})\pi}{a_0}\right)^2 + 2\frac{H}{a_0} + M_1 + c_n^1, \text{ as } n \to \infty, \text{ where } (c_n^1)_{n\geq 1} \in l_2^1, \tag{5.2.12}$$

$$\nu_n = \left(\frac{(n-\frac{1}{2})\pi}{a-a_0}\right)^2 - 2\frac{H}{a-a_0} + M_2 + c_n^2, \text{ as } n \to \infty, \text{ where } (c_n^2)_{n\geq 1} \in l_2^1, \tag{5.2.13}$$

and having the following interlacing property:

$$0 < \mu_1 < \lambda_1, \text{ or } 0 < \nu_1 < \lambda_1,$$

and for every $n > 1$ the interval $(\lambda_{n-1}, \lambda_n)$ contains exactly one element of the set

$$\{\mu_m | m \geq 1\} \cup \{\nu_m | m \geq 1\}.$$

Then there exists a real-valued potential $q \in L^2(0,a)$ such $\{\lambda_n | n \geq 1\}$, $\{\mu_n | n \geq 1\}$ and $\{\nu_n | n \geq 1\}$ are the Dirichlet, the Dirichlet-Robin, and the Robin-Dirichlet eigenvalues for the Sturm-Liouville problem with potential function q and impedance parameter H over the intervals $[0,a]$, $[0,a_0]$, $[a_0,a]$ respectively. That is, for each λ_n there exists a solution $u_n(x)$ to the boundary value problem

$$\begin{cases} -u''(x) + q(x)u(x) = \lambda_n u(x), \ 0 < x < a \\ u(0) = 0 \\ u(a) = 0, \end{cases}$$

for each μ_n there exists a solution $v_n(x)$ to the boundary value problem

$$\begin{cases} -u''(x) + q(x)u(x) = \mu_n u(x), \ 0 < x < a_0 \\ u(0) = 0 \\ u'(a_0) + Hu(a_0) = 0, \end{cases}$$

and for each ν_n there exists a solution $w_n(x)$ to the boundary value problem

$$\begin{cases} -u''(x) + q(x)u(x) = \nu_n u(x), \ a_0 < x < a \\ u'(a_0) + Hu(a_0) = 0 \\ u(a) = 0. \end{cases}$$

The potential function with this property is unique.

Observation: The uniqueness of a potential q having these three sets as its sets of eigenvalues in the sense presented above, assuming only pairwise disjointness of these three sets, was also proved by Gesztesy and Simon (see Theorem 2 and Sections 3 and 4 in [9]). However, the proof here is completely different from their proof.

Proof of Theorem 7: Assuming that such potential function q exists we can write formula (5.1.9) derived in the proof of Theorem 6, where the pairs $\{q_2, \lambda\}$ and $\{\tilde{q}_2, \tilde{\lambda}\}$ are related to each other by formula (5.1.8). Here $q_1 = q|_{[0,a_0]}$ and $q_2 = q|_{[a_0,a]}$. We also observe that

$$\{\lambda, u(x)\}$$

solves the boundary value problem

$$\begin{cases} -u''(x) + q_2(x)u(x) = \lambda u(x), \ a_0 < x < a, \\ u'(a_0) + Hu(a_0) = 0, \\ u(a) = 0, \end{cases} \tag{5.2.14}$$

(or equivalently $\{\lambda, u(x)\}$ is a Robin-Dirichlet eigenpair corresponding to the potential function q over the interval $[a_0, a]$, and hence to the potential function q_2) if and only if

$$\{\tilde{\lambda} = \left(\frac{a-a_0}{a_0}\right)^2 \lambda, v(\xi) = u(a - \frac{a-a_0}{a_0}\xi) = u(x)\}$$

solves the boundary value problem

$$\begin{cases} -v''(\xi) + \tilde{q}_2(\xi)v(\xi) = \tilde{\lambda} v(\xi), \ 0 < \xi < a_0, \\ v(0) = 0, \\ v'(a_0) - H\frac{a-a_0}{a_0} v(a_0) = 0 \end{cases} \tag{5.2.15}$$

(or equivalently $\{\tilde{\lambda}, v(\xi)\}$ is a Dirichlet-Robin eigenpair corresponding to the potential function $\tilde{q}_2$ over the interval $[0, a_0]$). Since ν_n's are the Robin-Dirichlet eigenvalues corresponding to the potential function q over the interval $[a_0, a]$ we infer from the above discussion that

$$S_1'(a_0; \tilde{q}_2, \tilde{\nu}_n) - H\frac{a-a_0}{a_0}S_1(a_0; \tilde{q}_2, \tilde{\nu}_n) = 0, \text{ for all } n \geq 1. \tag{5.2.16}$$

Here $\tilde{\nu}_n = \left(\frac{a-a_0}{a_0}\right)^2 \nu_n$. Since μ_n's are the Dirichlet-Robin eigenvalues corresponding to the potential function q over the interval $[0, a_0]$ we also have that

$$S_1'(a_0; q_1, \mu_n) + HS_1(a_0; q_1, \mu_n) = 0, \text{ for all } n \geq 1. \tag{5.2.17}$$

The Dirichlet eigenvalues of the potential function q over the interval $[0, a]$ are λ_n's. So

$$S(a; q, \lambda_n) = 0, \text{ for all } n \geq 1. \tag{5.2.18}$$

Formula (5.1.9) can be transformed to make room for the characteristic function

$$\lambda \to S_1'(a_0; q_1, \lambda) + HS_1(a_0; q_1, \lambda)$$

of the Sturm-Liouville problem with Dirichlet-Robin boundary conditions

$$\begin{cases} -u''(x) + q_1(x)u(x) = \lambda u(x), \ 0 < x < a_0, \\ u(0) = 0, \\ u'(a_0) + Hu(a_0) = 0, \end{cases}$$

and the characteristic function

$$\tilde{\lambda} \to S_1'(a_0; \tilde{q}_2, \tilde{\lambda}) - H\frac{a-a_0}{a_0}S_1(a_0; \tilde{q}_2, \tilde{\lambda})$$

of the Sturm-Liouville problem with Dirichlet-Robin boundary conditions

$$\begin{cases} -v''(\xi) + \tilde{q}_2(\xi)v(\xi) = \tilde{\lambda}v(\xi), \ 0 < \xi < a_0, \\ v(0) = 0, \\ v'(a_0) - H\frac{a-a_0}{a_0}v(a_0) = 0. \end{cases}$$

Here the calculations are:

$$\begin{aligned} S(a;q,\lambda) &= S_1(a_0;q_1,\lambda)S_1'(a_0;\tilde{q}_2,\tilde{\lambda}) + S_1'(a_0;q_1,\lambda)\frac{a-a_0}{a_0}S_1(a_0;\tilde{q}_2,\tilde{\lambda}) \\ &= S_1(a_0;q_1,\lambda)\left(S_1'(a_0;\tilde{q}_2,\tilde{\lambda}) - H\frac{a-a_0}{a_0}S_1(a_0;\tilde{q}_2,\tilde{\lambda})\right) \\ &+ H\frac{a-a_0}{a_0}S_1(a_0;q_1,\lambda)S_1(a_0;\tilde{q}_2,\tilde{\lambda}) \\ &+ \left(S_1'(a_0;q_1,\lambda) + HS_1(a_0;q_1,\lambda)\right)\frac{a-a_0}{a_0}S_1(a_0;\tilde{q}_2,\tilde{\lambda}) \\ &- H\frac{a-a_0}{a_0}S_1(a_0;q_1,\lambda)S_1(a_0;\tilde{q}_2,\tilde{\lambda}). \end{aligned}$$

So we have:

$$\begin{aligned} S(a;q,\lambda) &= S_1(a_0;q_1,\lambda)\left(S_1'(a_0;\tilde{q}_2,\tilde{\lambda}) - H\frac{a-a_0}{a_0}S_1(a_0;\tilde{q}_2,\tilde{\lambda})\right) \\ &+ \left(S_1'(a_0;q_1,\lambda) + HS_1(a_0;q_1,\lambda)\right)\frac{a-a_0}{a_0}S_1(a_0;\tilde{q}_2,\tilde{\lambda}), \end{aligned} \tag{5.2.19}$$

where the same relationship (5.1.8) holds. Next, formulas (5.2.19), (5.2.16), (5.2.17), and (5.2.18), as well as the integral representation of $S(a;q,\lambda)$, $S_1(a_0;q_1,\lambda)$, $S_1(a_0;\tilde{q}_2,\tilde{\lambda})$ (see (3.2.20)), and of $S_1'(a_0;q_1,\lambda)$, $S_1'(a_0;\tilde{q}_2,\tilde{\lambda})$ (see (3.2.21)), and the Goursat problems of type (3.2.18) which the kernels

$$K(x,t;q),\ K(x,t;q_1),\ K(x,t;\tilde{q}_2)$$

satisfy are essential in developing the procedure for constructing such potential function q. This procedure follows now. Find $f \in H^2(0,a)$ which solves the system of integral equations:

$$\frac{\sin(\sqrt{\lambda_n}a)}{\sqrt{\lambda_n}} + \int_0^a f(t)\frac{\sin(\sqrt{\lambda_n}t)}{\sqrt{\lambda_n}}dt = 0,\ \text{for all } n \geq 1 \tag{5.2.20}$$

and satisfies the conditions:

$$\begin{cases} f(0) = 0 \\ f(a) = \frac{a\cdot M}{2}, \end{cases} \tag{5.2.21}$$

find $h_1 \in L^2(0,a_0)$ which solves the system of integral equations:

$$\cos(\sqrt{\mu_n}a_0) + \left(\frac{a_0\cdot M_1}{2} + H\right)\frac{\sin(\sqrt{\mu_n}a_0)}{\sqrt{\mu_n}} + \int_0^{a_0} h_1(t)\frac{\sin(\sqrt{\mu_n}t)}{\sqrt{\mu_n}}dt = 0,\ \text{for all } n \geq 1 \tag{5.2.22}$$

and satisfies the condition:

$$h_1(0) = 0, \tag{5.2.23}$$

and find $h_2 \in L^2(0, a_0)$ which solves the system of integral equations:

$$\cos(\sqrt{\tilde{\nu}_n} a_0) + \frac{a - a_0}{a_0}\left(\frac{(a - a_0) \cdot M_2}{2} - H\right)\frac{\sin(\sqrt{\tilde{\nu}_n} a_0)}{\sqrt{\tilde{\nu}_n}} + \int_0^{a_0} h_2(t)\frac{\sin(\sqrt{\tilde{\nu}_n} t)}{\sqrt{\tilde{\nu}_n}} dt = 0, \text{ for all } n \geq 1 \tag{5.2.24}$$

and satisfies the condition:

$$h_2(0) = 0. \tag{5.2.25}$$

Here $\tilde{\nu}_n = \left(\frac{a-a_0}{a_0}\right)^2 \nu_n$. The uniqueness of f follows due to the fact that the set

$$\{\sin(\sqrt{\lambda_n} t) | n \geq 1\}$$

is complete in $L^2(0, a)$ (see Auxiliary result 1 in Section 3.6), and the uniqueness of h_1 and h_2 follows due to the fact that the sets

$$\{\sin(\sqrt{\mu_n} t) | n \geq 1\} \text{ and } \{\sin(\sqrt{\tilde{\nu}_n} t) | n \geq 1\}$$

are complete in $L^2(0, a_0)$ (see Auxiliary result 10 immediately after the end of this proof). The existence of f, h_1, h_2 can be shown analogously to those of f, f_1, f_2 presented in Section 3.4. Then we can construct the functions

$$S(\lambda) = \frac{\sin(\sqrt{\lambda} a)}{\sqrt{\lambda}} + \int_0^a f(t)\frac{\sin(\sqrt{\lambda} t)}{\sqrt{\lambda}} dt, \tag{5.2.26}$$

$$P_1(\lambda) = \cos(\sqrt{\lambda} a_0) + \left(\frac{a_0 \cdot M_1}{2} + H\right)\frac{\sin(\sqrt{\lambda} a_0)}{\sqrt{\lambda}} + \int_0^{a_0} h_1(t)\frac{\sin(\sqrt{\lambda} t)}{\sqrt{\lambda}} dt, \tag{5.2.27}$$

$$P_2(\tilde{\lambda}) = \cos(\sqrt{\tilde{\lambda}} a_0) + \frac{a - a_0}{a_0}\left(\frac{(a - a_0) \cdot M_2}{2} - H\right)\frac{\sin(\sqrt{\tilde{\lambda}} a_0)}{\sqrt{\tilde{\lambda}}} + \int_0^{a_0} h_2(t)\frac{\sin(\sqrt{\tilde{\lambda}} t)}{\sqrt{\tilde{\lambda}}} dt. \tag{5.2.28}$$

Here $\tilde{\lambda} = \left(\frac{a-a_0}{a_0}\right)^2 \lambda$. Furthermore, the quotients

$$\begin{cases} \frac{S(\mu_n)}{P_2(\tilde{\mu}_n)}, & n \geq 1 \\ \frac{S(\nu_n)}{P_1(\nu_n)}, & n \geq 1 \end{cases}$$

with $\tilde{\mu}_n = \left(\frac{a-a_0}{a_0}\right)^2 \mu_n$ will be known quantities. The fact that

$$\begin{cases} P_2(\tilde{\mu}_n) \neq 0, & \text{for all } n \geq 1 \\ P_1(\nu_n) \neq 0, & \text{for all } n \geq 1, \end{cases}$$

follows using the non-overlap property $\{\mu_n | n \geq 1\} \cap \{\nu_n | n \geq 1\} = \phi$, in a similar way the relationships (3.4.20) and (3.4.21) were proved in Section 3.4. Next, let $f_1 \in H^1(0, a_0)$ and $f_2 \in H^1(0, a_0)$ be the solutions to the system of integral equations

$$\frac{S(\mu_n)}{P_2(\tilde{\mu}_n)} = \frac{\sin(\sqrt{\mu_n} a_0)}{\sqrt{\mu_n}} + \int_0^{a_0} f_1(t) \frac{\sin(\sqrt{\mu_n} t)}{\sqrt{\mu_n}} dt, \ n \geq 1, \tag{5.2.29}$$

and of the system of integral equations

$$\left(\frac{a_0}{a - a_0} \right) \frac{S(\nu_n)}{P_1(\nu_n)} = \frac{\sin(\sqrt{\tilde{\nu}_n} a_0)}{\sqrt{\tilde{\nu}_n}} + \int_0^{a_0} f_2(t) \frac{\sin(\sqrt{\tilde{\nu}_n} t)}{\sqrt{\tilde{\nu}_n}} dt, \ n \geq 1, \tag{5.2.30}$$

respectively. Use the results of [23, chapters 3 and 4] along with formulas (5.2.11), (5.2.12), (5.2.13) to prove existence of such solutions. Uniqueness of these solutions is due to Auxiliary result 10 immediately after the end of this proof. Then construct

$$\begin{cases} g_1(t) = h_1(t) - H f_1(t), & t \in [0, a_0] \\ g_2(t) = h_2(t) + H \frac{a - a_0}{a_0} f_2(t), & t \in [0, a_0]. \end{cases}$$

Then solve the non-linear equations

$$\begin{cases} F(q_1) = \{g_1(t), f_1'(t)\} \\ F(\tilde{q}_2) = \{g_2(t), f_2'(t)\} \end{cases}$$

for q_1 and $\tilde{q}_2$. Here the map

$$p \in L^2(0, a_0) \to F(p) \in L^2(0, a_0) \times L^2(0, a_0)$$

is defined by

$$F(p) = \{K_x(a_0, t; p), K_t(a_0, t; p)\},$$

where $K(x, t; p)$ is the weak solution to the Goursat problem (3.2.18) with $[0, \beta]$ replaced by $[0, a_0]$. Next define

$$q_2(x) = \left(\frac{a_0}{a - a_0} \right)^2 \tilde{q}_2 \left(\frac{a_0}{a - a_0} (a - x) \right), \ x \in [a_0, a],$$

and paste together q_1 and q_2 to obtain q over the entire interval $[0, a]$. The proof that the potential function q this way constructed is the correct one is similar to that in Section 3.4, and the proof of uniqueness goes along the same lines as the one in Section 3.5. □

Auxiliary result 10 *Given the sets* $\{\mu_n | n \geq 1\}$ *and* $\{\nu_n | n \geq 1\}$ *satisfying (5.2.12) and (5.2.13) respectively, the sets*

$$\{\sin(\sqrt{\mu_n} t) | n = 1, 2, ...\} \text{ and } \{\sin(\sqrt{\tilde{\nu}_n} t) | n = 1, 2, ...\},$$

are complete in $L^2(0, a_0)$*. Here* $\tilde{\nu}_n = \left(\frac{a - a_0}{a_0}\right)^2 \nu_n$.

Proof: The proof of completeness is different from the one for their counterpart sets stated in Auxiliary result 9, because μ_n's and ν_n's have different asymptotics. We shall present now the proof of completeness of the set $\{\sin(\sqrt{\mu_n} t) | n = 1, 2, ...\}$ in $L^2(0, a_0)$, only. For the other set, similar arguments apply. From (5.2.12) we deduce that:

$$\sqrt{\mu_n} = \frac{(n - \frac{1}{2})\pi}{a_0} + \frac{H}{(n - \frac{1}{2})\pi} + \frac{a_0}{2(n - \frac{1}{2})\pi} M_1 + \frac{\tilde{c}_n^1}{n^2}, \text{ as } n \to \infty, \text{ where } (\tilde{c}_n^1)_{n \geq 1} \in l_2. \quad (5.2.31)$$

Our purpose is to satisfy hypotheses in Problem 4 of [23, page 122] with $p = 2$, which for the convenience of the reader we shall state it here: if $\{\varepsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that

$$\sum_{n=0}^{\infty} \frac{\varepsilon_n}{n+1} < \infty,$$

then the set $\{e^{i\theta_n s}\}_{n=-\infty}^{\infty}$ will be complete in $L^p(-\pi, \pi)$, $1 < p < \infty$, whenever

$$|\theta_n| \leq |n| + \frac{1}{2p} + \varepsilon_{|n|}, \text{ for } n = 0, \pm 1, \pm 2, \ldots.$$

By the change of variable: $s \in [-\pi, \pi] \leftrightarrow t = \frac{a_0}{\pi} s \in [-a_0, a_0]$, and letting $\omega_n = \frac{\pi}{a_0} \theta_n$, the set

$$\{e^{i\omega_n t}\}_{n=-\infty}^{\infty}$$

will be complete in $L^p(-a_0, a_0)$, $1 < p < \infty$, whenever

$$|\omega_n| \leq |n| \frac{\pi}{a_0} + \frac{\pi}{2pa_0} + \varepsilon_{|n|}, \text{ for } n = 0, \pm 1, \pm 2, \ldots, \quad (5.2.32)$$

for a sequence of $\{\varepsilon_n\}_{n=0}^{\infty}$ as above. From (5.2.31) we have that for some constant $C > 0$, there exists a positive integer N such that:

$$|-\sqrt{\mu_n}| = |\sqrt{\mu_n}| \leq n \frac{\pi}{a_0} + \frac{C}{n}, \text{ for } n = N + 1, N + 2, \ldots \quad (5.2.33)$$

Next take

$$\omega_n = \begin{cases} \sqrt{\mu_n}, & \text{if } n = N+1, N+2, \ldots \\ \frac{n\pi}{a_0}, & \text{if } n = 0, \pm 1, \pm 2, \ldots, \pm N \\ -\sqrt{\mu_{-n}}, & \text{if } n = -(N+1), -(N+2), \ldots, \end{cases} \tag{5.2.34}$$

and

$$\varepsilon_n = \begin{cases} 0, & \text{if } n = 0, 1, 2, \ldots, N \\ \frac{C}{n}, & \text{if } n = N+1, N+2, \ldots. \end{cases} \tag{5.2.35}$$

Then by (5.2.33), (5.2.34) and (5.2.35) we have that $\{\varepsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\sum_{n=0}^{\infty} \frac{\varepsilon_n}{n+1} < \infty$ and $\{\omega_n\}_{n=-\infty}^{\infty}$ satisfies (5.2.32) with $p = 2$ and for this choice of $\{\varepsilon_n\}_{n=0}^{\infty}$. Therefore, by the above mentioned result it follows that the set

$$\{e^{i\omega_n t} | n = 0, \pm 1, \pm 2, \ldots\}$$

is complete in $L^2(-a_0, a_0)$. Then apply repeatedly Theorem 7 of [23, page 129] to replace ω_n by $\pm\sqrt{\mu_{|n|}}$ for $n = \pm 1, \pm 2, \ldots, \pm N$ and get that the set

$$\{1\} \cup \{e^{\pm i\sqrt{\mu_n}t} | n = 1, 2, \ldots\}$$

is complete in $L^2(-a_0, a_0)$. From this and using arguments similar to those for showing (3.6.7), one obtains the desired completeness. □

The algorithm: The main steps were already outlined in the constructive procedure in the proof of Theorem 7. The only thing we want to add is how numerically the equations (5.2.20), (5.2.22), (5.2.24), and the equations (5.2.29), (5.2.30) can be solved. That is, write

$$f(t) = \left(\frac{M}{2}\right) t + \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi}{a} t\right), \; t \in [0, a] \tag{5.2.36}$$

($f(t)$ is expected to be $K^{(q)}(a, t)$, so the first term $\left(\frac{M}{2}\right) t$ in the above expression is due to the fact that $K^{(q)}(a, a) = \frac{1}{2}\int_0^a q(x)dx = \frac{a \cdot M}{2}$), then

$$h_1(t) = \sum_{n=1}^{\infty} \gamma_n^{(1)} \sin\left(\frac{(n - \frac{1}{2})\pi}{a_0} t\right), \; t \in [0, a_0] \tag{5.2.37}$$

($h_1(t)$ is expected to be $K_x^{(q_1)}(a_0,t)+HK^{(q_1)}(a_0,t)$, so the above basis functions were chosen because both $K_x^{(q_1)}(a_0,t)$ and $K^{(q_1)}(a_0,t)$ are zero at $t=0$, but not necessary at $t=a_0$, as these sine functions $\{\sin\left(\frac{(n-\frac{1}{2})\pi}{a_0}t\right)|n\geq 1\}$ are), and

$$h_2(t)=\sum_{n=1}^{\infty}\gamma_n^{(2)}\sin\left(\frac{(n-\frac{1}{2})\pi}{a_0}t\right),\ t\in[0,a_0] \tag{5.2.38}$$

($h_2(t)$ is expected to be $K_x^{(\tilde{q}_2)}(a_0,t)-H\frac{a-a_0}{a_0}K^{(\tilde{q}_2)}(a_0,t)$, so the above basis functions were chosen because both $K_x^{(\tilde{q}_2)}(a_0,t)$ and $K^{(\tilde{q}_2)}(a_0,t)$ are zero at $t=0$, but not necessary at $t=a_0$, as these sine functions $\{\sin\left(\frac{(n-\frac{1}{2})\pi}{a_0}t\right)|n\geq 1\}$ are). Next, insert (5.2.36), (5.2.37), (5.2.38) into the system of integral equations (5.2.20), (5.2.22), (5.2.24), respectively, to solve for the Fourier coefficients of f, h_1, and h_2. Next write the Fourier series

$$f_1(t)=\left(\frac{M_1}{2}\right)t+\sum_{n=1}^{\infty}\alpha_n^{(1)}\sin\left(\frac{n\pi}{a_0}t\right),\ t\in[0,a_0]. \tag{5.2.39}$$

Note: $f_1(t)$ is expected to be $K^{(q_1)}(a_0,t)$, so the first term $\left(\frac{M_1}{2}\right)t$ in the above expression is due to the fact that

$$K^{(q_1)}(a_0,a_0)=\frac{1}{2}\int_0^{a_0}q_1(x)dx=\frac{a_0\cdot M_1}{2}.$$

And write

$$f_2(t)=\left(\frac{a-a_0}{a_0}\right)^2\left(\frac{M_2}{2}\right)t+\sum_{n=1}^{\infty}\alpha_n^{(2)}\sin\left(\frac{n\pi}{a_0}t\right),\ t\in[0,a_0]. \tag{5.2.40}$$

Note: $f_2(t)$ is expected to be $K^{(\tilde{q}_2)}(a_0,t)$, so the first term $\left(\frac{a-a_0}{a_0}\right)^2\left(\frac{M_2}{2}\right)t$ in the above expression is due to the fact that

$$K^{(\tilde{q}_2)}(a_0,a_0)=\frac{1}{2}\int_0^{a_0}\tilde{q}_2(\xi)d\xi=\left(\frac{a-a_0}{a_0}\right)\frac{(a-a_0)\cdot M_2}{2}\ \text{(by (5.1.8))}.$$

Insert (5.2.39) and (5.2.40) into the system of integral equations (5.2.29) and (5.2.30) respectively, to solve for the Fourier coefficients of f_1 and f_2.

Note: In practice, only a finite amount of spectral data is available (i.e. only finitely many λ_n's, μ_n's, ν_n's are known). In this case the series (5.2.36) will only have as many terms as λ_n's are known, and the system of integral equations (5.2.20) will only have as many equations as λ_n's are known. A similar discussion applies to the other four Fourier series (5.2.37), (5.2.38), (5.2.39), (5.2.40) and the other four systems of integral equations (5.2.22), (5.2.24), (5.2.29) and (5.2.30).

CHAPTER 6. NUMERICAL DETAILS

6.1 The forward map

We consider the map F defined by $F(p) = \{W_x(\beta, t; p), W_t(\beta, t; p)\}$, where $W(x, t; p)$ is the weak solution to the Goursat problem (3.2.18). Here we show the numerical details of how to calculate $F(p)$, when p is known, or in other words, how to calculate the Cauchy data

$$\{W_x(\beta, t; p), W_t(\beta, t; p)\},$$

when the potential function p is known. To simplify notation, in the remaining of this section we suppress the dependence on p in $W(x, t; p)$ and simply write $W(x, t)$.

6.1.1 The transformation of the 2'nd order PDE of (3.2.18) into two 1'st order PDEs

By the following change of variables

$$\begin{cases} Z_1 = W_x + W_t + \alpha(x)W, \\ Z_2 = W_x - W_t + \alpha(x)W, \end{cases} \tag{6.1.1}$$

where $\alpha(x)$ is to be chosen conveniently, the 2'nd order PDE of (3.2.18) is transformed equivalently into two 1'st order PDEs:

$$\begin{cases} Z_{1,x} - Z_{1,t} = \alpha(x)Z_2 + (\alpha'(x) - \alpha^2(x) + p(x))W, \\ Z_{2,x} + Z_{2,t} = \alpha(x)Z_1 + (\alpha'(x) - \alpha^2(x) + p(x))W. \end{cases} \tag{6.1.2}$$

Now choose $\alpha(x)$ such that

$$\alpha'(x) - \alpha^2(x) + p(x) = 0, \text{ for } x \in (0, \beta), \tag{6.1.3}$$

because we want to eliminate the dependence on the initial unknown W, and leave only a system of PDEs in the new unknowns Z_1 and Z_2. Hence, we have:

$$\begin{cases} Z_{1,x} - Z_{1,t} = \alpha(x) Z_2, \\ Z_{2,x} + Z_{2,t} = \alpha(x) Z_1. \end{cases} \tag{6.1.4}$$

The two BCs of (3.2.18) imply that

$$\begin{cases} Z_1(x,x) = \frac{1}{2}(p(x) + \alpha(x) r(x)), & \text{for } x \in [0,\beta] \\ Z_1(x,0) + Z_2(x,0) = 0, & \text{for } x \in [0,\beta], \end{cases} \tag{6.1.5}$$

by using (6.1.1). Here $r(x) = \int_0^x p(s)ds$. Conversely, if we start from (6.1.5) and use (6.1.1), we obtain:

$$\begin{cases} \frac{d}{dx}\left(W(x,x) - \frac{1}{2}r(x)\right) + \alpha(x)\left(W(x,x) - \frac{1}{2}r(x)\right) = 0, & \text{for } x \in [0,\beta] \\ \frac{d}{dx}\left(W(x,0)\right) + \alpha(x) W(x,0) = 0, & \text{for } x \in [0,\beta], \end{cases} \tag{6.1.6}$$

since $r'(x) = p(x)$. Denoting

$$\begin{cases} \phi(x) = W(x,x) - \frac{1}{2}r(x) \\ \psi(x) = W(x,0), \end{cases}$$

we see that (6.1.6) says that ϕ and ψ satisfy the (IVP)

$$\begin{cases} y' + \alpha(x) y = 0, & \text{for } x \in [0,\beta] \\ y(0) = W(0,0) = 0 \end{cases}$$

whose unique solution is the trivial solution. Hence

$$\phi(x) = 0 = \psi(x), \text{ for } x \in [0,\beta],$$

or equivalently

$$\begin{cases} W(x,x) = \frac{1}{2}\int_0^x p(s)ds, & \text{for } x \in [0,\beta] \\ W(x,0) = 0, & \text{for } x \in [0,\beta]. \end{cases}$$

All this work shows that solving (3.2.18) is the same as solving

$$\begin{cases} Z_{1,x} - Z_{1,t} = \alpha(x) Z_2, & 0 \le t \le x \le \beta \\ Z_{2,x} + Z_{2,t} = \alpha(x) Z_1, & 0 \le t \le x \le \beta \\ Z_1(x,x) = \frac{1}{2}(p(x) + \alpha(x) r(x)), & \text{for } x \in [0,\beta] \\ Z_1(x,0) + Z_2(x,0) = 0, & \text{for } x \in [0,\beta], \end{cases} \tag{6.1.7}$$

via (6.1.1), where $\alpha(x)$ is as in (6.1.3). We solve numerically (6.1.7) for Z_1 and Z_2 by placing a mesh on the domain $\bar{\Delta}_0 = \{(x,t)|0 \le t \le x \le \beta\}$, where the grid lines are the characteristic curves of the hyperbolic PDE in (3.2.18). These are:

$$\begin{cases} x + t = constant, \\ x - t = constant. \end{cases}$$

Before describing the numerical procedure of solving (6.1.7) for Z_1 and Z_2 on the specified grid points of the domain $\bar{\Delta}_0$ we want to make precise the known quantities in (6.1.7). Since p is assumed known, r will be known too (by its definition $r(x) = \int_0^x p(s)ds$), and α will be uniquely determined from (6.1.3) if an initial or final condition is prescribed. Since our purpose is to obtain $W_x(\beta,t)$ and $W_t(\beta,t)$, equations (6.1.1) suggest that the best choice in order to make the calculations simplest and fastest is to have

$$\alpha(\beta) = 0.$$

However, in some instances (depending on the potential p) requiring $\alpha(\beta) = 0$ entails a solution α to (6.1.3) which blows up at $x = 0$ or before $x = 0$. Therefore, we have to make a "right" choice of $\alpha(\beta) = \alpha_0$, to be guaranteed an well behaved solution α. Then, from this point, the calculations of $W_x(\beta,t)$ and $W_t(\beta,t)$ will go by using (6.1.1) in the following way:

$$\begin{cases} W_x(\beta,t) + \alpha_0 W(\beta,t) = \frac{1}{2}\left(Z_1(\beta,t) + Z_2(\beta,t)\right) \\ W_t(\beta,t) = \frac{1}{2}\left(Z_1(\beta,t) - Z_2(\beta,t)\right). \end{cases} \tag{6.1.8}$$

From the second equation of (6.1.8) we obtain $W_t(\beta,t)$, and then from it we obtain $W(\beta,t)$ using a divided difference scheme. Then plug $W(\beta,t)$ in the first equation of (6.1.8) to get $W_x(\beta,t)$. Thus we just calculated

$$\{W_t(\beta,t); W(\beta,t); W_x(\beta,t)\}.$$

6.1.2 How to choose the final condition for the Riccati equation (6.1.3)

Now we continue with a discussion on what a "right" choice of $\alpha(\beta) = \alpha_0$ means. If it is known that the first Dirichlet eigenvalue of the potential p is strictly positive, then we can argue below that the solution α to (6.1.3) with $\alpha(\beta) = \alpha_0$ properly chosen remains finite. Let $\{\lambda_1, \phi_1(x)\}$ be the first Dirichlet eigenpair of the Sturm-Liouville operator $L^{(p)}$ over the interval $[0, \beta]$. That is:

$$\begin{cases} -\phi_1''(x) + p(x)\phi_1(x) = \lambda_1\phi_1(x), \quad x \in (0, \beta) \\ \phi_1(0) = 0 \\ \phi_1(\beta) = 0. \end{cases} \tag{6.1.9}$$

And let $\psi \in H^2(0, \beta)$ solve the problem:

$$\begin{cases} -\psi''(x) + p(x)\psi(x) = 0, \quad x \in (0, \beta) \\ \psi(\beta) = 0 \\ \psi'(\beta) = -1. \end{cases} \tag{6.1.10}$$

Multiplying the ODE of (6.1.9) by $\psi(x)$ and the ODE of (6.1.10) by $\phi_1(x)$ and subtracting we get:

$$-(\phi_1'\psi - \phi_1\psi')'(x) = \lambda_1\phi_1(x)\psi(x), \; x \in (0, \beta). \tag{6.1.11}$$

Next, we claim that

$$\psi(x) > 0, \; \text{for all } x \in [0, \beta). \tag{6.1.12}$$

We argue by contradiction. Starting from $\psi'(\beta) = -1$ (see (6.1.10)) and using the fact that $\psi \in H^2(0, \beta)$ (so $\psi \in C^1[0, \beta]$) we have that ψ' keeps constant sign (same as $\psi'(\beta)$) in a neighborhood of β. So, there exists $\delta > 0$ such that

$$\psi'(x) < 0, \; \text{for all } x \in (\beta - \delta, \beta).$$

It follows that ψ is a decreasing function on $(\beta - \delta, \beta)$, and hence, using $\psi(\beta) = 0$ (see (6.1.10)) we have that

$$\psi(x) > 0, \; \text{for } x \in (\beta - \delta, \beta). \tag{6.1.13}$$

Since we supposed that ψ does not remain strictly positive on $[0, \beta)$, using (6.1.13), the Intermediate Value Theorem (possible since $\psi \in H^2(0, \beta)$, so continuous on $[0, \beta]$) we find at least one value of x in $[0, \beta - \delta]$ where ψ is zero. Let $x_0 \in [0, \beta - \delta]$ be the closest x-value to

$\beta - \delta$ where ψ is zero. Using this property of x_0, formula (6.1.13) and the Intermediate Value Theorem applied to ψ, we have after reasoning by contradiction that

$$\psi(x) > 0, \text{ for all } x \in (x_0, \beta). \tag{6.1.14}$$

Integrating equation (6.1.11) on $[x_0, \beta]$, using (6.1.9), (6.1.10), and $\psi(x_0) = 0$ (see definition of x_0 above) we obtain:

$$-\phi_1(x_0)\psi'(x_0) = \lambda_1 \int_{x_0}^{\beta} \phi_1(x)\psi(x)dx. \tag{6.1.15}$$

The nodal property of eigenfunctions (a property due to Sturm) known in the direct Sturm-Liouville theory insures that ϕ_1 has no zeros in the open interval $(0, \beta)$, and hence using continuity we have that ϕ_1 keeps constant sign on $(0, \beta)$. Based on this fact we can write that

$$\phi_1(x) = |\phi_1(x)| sgn(\phi_1(x)) = |\phi_1(x)| sgn(\phi_1(x_0)), \text{ for all } x \in (0, \beta). \tag{6.1.16}$$

Using (6.1.14), (6.1.16), and the fact the $\lambda_1 > 0$ in (6.1.15), we obtain that

$$\psi'(x_0) < 0, \tag{6.1.17}$$

in order to maintain the same sign on the left and on the right hand sides of (6.1.15). The continuity of ψ' (see $\psi \in H^2(0, \beta)$) and (6.1.17) imply that there exists $\gamma > 0$ such that

$$\begin{cases} (x_0 - \gamma, x_0 + \gamma) \subset [0, \beta] \\ \psi'(x) < 0, \text{ for all } x \in (x_0 - \gamma, x_0 + \gamma). \end{cases} \tag{6.1.18}$$

From (6.1.18) we have that ψ is a decreasing function on $(x_0 - \gamma, x_0 + \gamma)$, hence

$$\psi(x) < \psi(x_0) = 0, \text{ for all } x \in (x_0, x_0 + \gamma).$$

But this contradicts (6.1.14), since $(x_0, x_0 + \gamma) \subset (x_0, \beta)$. Therefore (6.1.12) holds. Next, due to continuous dependence of solution on initial (or final) conditions the same is true for the solution ψ_ε to the final value problem

$$\begin{cases} -\psi_\varepsilon''(x) + p(x)\psi_\varepsilon(x) = 0, \ x \in (0, \beta) \\ \psi_\varepsilon(\beta) = \varepsilon \\ \psi_\varepsilon'(\beta) = -1, \end{cases} \tag{6.1.19}$$

if $\varepsilon > 0$ is small enough. So $\psi_\varepsilon > 0$ on $[0, \beta]$, and therefore the function defined by

$$\alpha(x) = -\frac{\psi_\varepsilon'(x)}{\psi_\varepsilon(x)}$$

remains finite and satisfies (6.1.3) and $\alpha(\beta) = \frac{1}{\varepsilon}$. Our conclusion is that if the first Dirichlet eigenvalue of the operator $L^{(p)}$ is known to be strictly positive, then assigning a very large positive value to $\alpha(\beta)$ will produce a finite solution to (6.1.3). If the first Dirichlet eigenvalue of the operator $L^{(p)}$ is not strictly positive, then the argument above does no longer apply, and no choice of $\alpha(\beta) = \alpha_0$ guarantees that the solution to (6.1.3) remains finite.

6.1.3 Solving numerically (6.1.7)

Now we can return to solving numerically (6.1.7). Partition the interval $[0, \beta]$ into $N-1$ equal subintervals of length $\delta x = \dfrac{\beta}{N-1}$, by putting the points

$$0 = x_1 < x_2 < \ldots < x_j = (j-1)\delta x < \ldots < x_N = \beta,$$

where N must be chosen an odd number. Then from each point

$$(x, t) = (x_{2j-1}, 0),\ j = 1, 2, \ldots, \tfrac{N+1}{2}$$

draw the characteristic lines

$$\begin{cases} x + t = x_{2j-1} \\ x - t = x_{2j-1}. \end{cases}$$

This way a full grid was generated only on a half of the triangle $\bar{\Delta}_0$, namely that part of $\bar{\Delta}_0$ situated at the left of the line $x + t = x_N = \beta$. To complete the grid on the triangle $\bar{\Delta}_0$ we need to add more points in x-direction. Set

$$\beta = x_N < x_{N+1} < \ldots < x_{2N-1} = 2\beta,$$

where the spacing between the two consecutive points is δx as before. Then from each point

$$(x, t) = (x_{2j-1}, 0),\ j = \tfrac{N+3}{2}, \tfrac{N+5}{2}, \ldots, N$$

draw only the characteristic line

$$x + t = x_{2j-1}.$$

The reason we chose the mesh with grid lines being the characteristic curves of the hyperbolic PDE in (3.2.18) is because of the two first order PDEs in (6.1.7). They take the form of two first order ODEs along the characteristic curves:

$$\begin{cases} \frac{d}{dx} Z_1(x,t) = \alpha(x) Z_2(x,t), & \text{for } (x,t) \in \bar{\Delta}_0 \text{ such that } x+t = \textit{constant} \\ \frac{d}{dx} Z_2(x,t) = \alpha(x) Z_1(x,t), & \text{for } (x,t) \in \bar{\Delta}_0 \text{ such that } x-t = \textit{constant}, \end{cases}$$

and as one can see, these differential equations are easily integrable along the specified lines. Once the mesh was placed on our domain $\bar{\Delta}_0$, we can calculate $\{Z_1, Z_2\}$ on the grid points of the characteristic line $x - t = x_{2j-1}$, then descend to the next characteristic line $x - t = x_{2j+1}$, for each $j = 1, 2, \ldots, \frac{N-1}{2}$. Figure 6.1 is provided, along with the details of calculations:

- on the grid line $x - t = x_1$, (i.e. on the line $t = x$, since $x_1 = 0$):

$$\begin{cases} Z_1(x,x) = \frac{1}{2}(p(x) + \alpha(x) r(x)), & \text{for } x = x_1, x_2, \ldots, x_N \\ Z_2(x_1,x_1) = -Z_1(x_1,x_1), & \text{since } x_1 = 0, \text{ and the 2'nd BC in (6.1.7)} \\ \frac{d}{dx} Z_2(x,x) = \alpha(x) Z_1(x,x), & \end{cases}$$

  which after integrating the last equation above on the intervals $[x_{i-1}, x_i]$, with $i = 2, 3, \ldots, N$ and using the trapezoid rule in the integral on the right hand side become

$$\begin{cases} Z_1(x,x) = \frac{1}{2}(p(x) + \alpha(x) r(x)),\ x = x_1, \ldots, x_N \\ Z_2(x_1,x_1) = -Z_1(x_1,x_1), \\ Z_2(x_i,x_i) = Z_2(x_{i-1},x_{i-1}) + \frac{\delta x}{2} \left(\alpha(x_{i-1}) Z_1(x_{i-1},x_{i-1}) + \alpha(x_i) Z_1(x_i,x_i)\right),\ i = 2 \ldots N \end{cases}$$

- descend from the grid line $x - t = x_{2j-1}$ to the grid line $x - t = x_{2j+1}$, for each $j = 1, 2, \ldots, \frac{N-1}{2}$:

$$\begin{cases} Z_1(x,0) + Z_2(x,0) = 0, \\ \frac{d}{dx} Z_1(x, -x + x_{2m+1}) = \alpha(x) Z_2(x, -x + x_{2m+1}), & m = j, j+1, \ldots, N-j-1 \\ \frac{d}{dx} Z_2(x, x - x_{2j+1}) = \alpha(x) Z_1(x, x - x_{2j+1}), & \end{cases}$$

  which after integrating the last two equations above on appropriate intervals of length

δx and using the trapezoid rule in the integrals on the right hand sides become

$$\begin{cases} Z_1(B) + Z_2(B) = 0 \\ Z_1(B) - Z_1(A) = \frac{1}{2}\delta x \left(\alpha(x_A) Z_2(A) + \alpha(x_B) Z_2(B)\right) \\ Z_1(D) - Z_1(C) = \frac{1}{2}\delta x \left(\alpha(x_C) Z_2(C) + \alpha(x_D) Z_2(D)\right) \\ Z_2(D) - Z_2(E) = \frac{1}{2}\delta x \left(\alpha(x_E) Z_1(E) + \alpha(x_D) Z_1(D)\right), \end{cases} \tag{6.1.20}$$

where

$$\begin{cases} \{A(x_A, t_A)\} = \{(x,t) | x + t = x_{2j+1}\} \cap \{(x,t) | x - t = x_{2j-1}\} \\ \{B(x_B, t_B)\} = \{(x,t) | x + t = x_{2j+1}\} \cap \{(x,t) | x - t = x_{2j+1}\} \\ \{C(x_C, t_C)\} = \{(x,t) | x + t = x_{2m+3}\} \cap \{(x,t) | x - t = x_{2j-1}\} \\ \{D(x_D, t_D)\} = \{(x,t) | x + t = x_{2m+3}\} \cap \{(x,t) | x - t = x_{2j+1}\} \\ \{E(x_E, t_E)\} = \{(x,t) | x + t = x_{2m+1}\} \cap \{(x,t) | x - t = x_{2j+1}\}, \end{cases}$$

(see Figure 6.1 again) with the specification that the locations of the points C, D, E change according to the value of m in the set $\{j, j+1, \ldots, N-j-2\}$.

Since Z_1 and Z_2 are known quantities on the characteristic line $x - t = x_{2j-1}$, it follows that $Z_1(A)$, $Z_2(A)$, and $Z_1(C)$, $Z_2(C)$ are known, and so we can solve the system of the first two linear equations of (6.1.20) for $Z_1(B)$ and $Z_2(B)$, and then move on, step by step, to the cells of the type $\{C, D, E\}$ to calculate $Z_1(D)$ and $Z_2(D)$:

1. the point E takes first the place of the point B, so when calculating $Z_1(D)$ and $Z_2(D)$ using the last two linear equations of (6.1.20), the quantities $Z_1(E)$ and $Z_2(E)$ are already calculated, so known;
2. then the point E moves into the location of the point D and the points D and C advance on the characteristic lines $x - t = x_{2j+1}$ and respectively $x - t = x_{2j-1}$;
3. the process repeats.

In other words, the calculation of Z_1 and Z_2 at the grid points of the new characteristic line $x - t = x_{2j+1}$ progresses step by step, from the boundary line $t = 0$ of our domain $\bar{\Delta}_0$, upward, toward the boundary line $x = x_N = \beta$ of the domain $\bar{\Delta}_0$.

Hence

$$\begin{cases} Z_1(B) + Z_2(B) = 0 \\ Z_1(B) - \frac{\delta x}{2}\alpha(x_B)Z_2(B) = Z_1(A) + \frac{\delta x}{2}\alpha(x_A)Z_2(A), \end{cases}$$

and

$$\begin{cases} Z_1(D) - \frac{\delta x}{2}\alpha(x_D)Z_2(D) = Z_1(C) + \frac{\delta x}{2}\alpha(x_C)Z_2(C) \\ Z_2(D) - \frac{\delta x}{2}\alpha(x_D)Z_1(D) = Z_2(E) + \frac{\delta x}{2}\alpha(x_E)Z_1(E). \end{cases}$$

Thus, going from characteristic line to characteristic line we get to know Z_1 and Z_2 at the grid points of the boundary line $x = \beta$ of our domain $\bar{\Delta}_0$, and so we get to know $\{W_t, W, W_x\}$ at the same points. And this was our purpose.

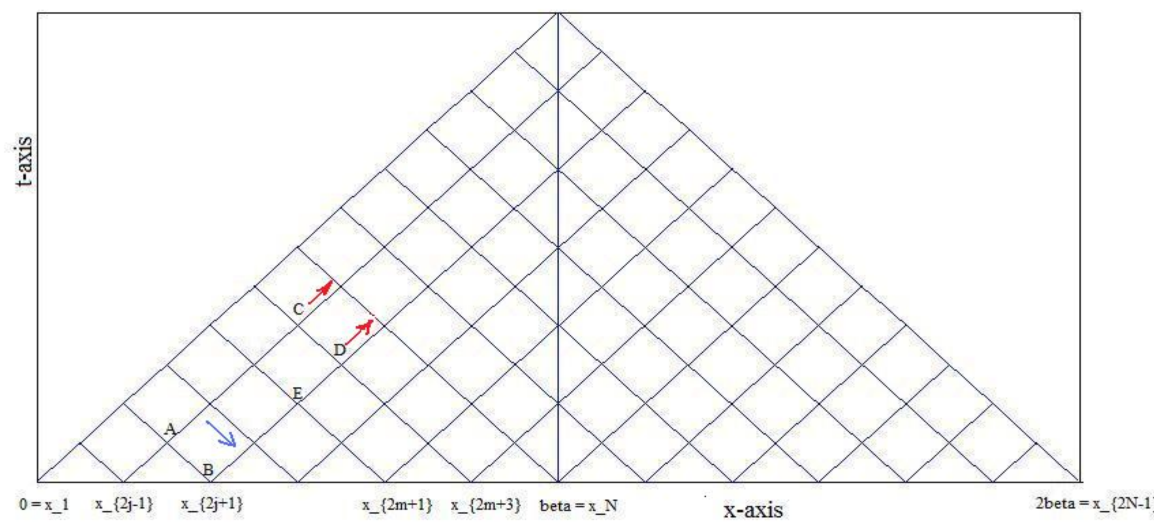

Figure 6.1 The partitioning of our domain of integration

6.1.4 How to improve the accuracy of our numerical method for solving (6.1.7)

To improve accuracy, a Richardson extrapolation scheme can be used. We start from the following estimation formula:

$$E = A(\delta x) + C \cdot (\delta x)^2, \tag{6.1.21}$$

where C is a real constant, E stands for the exact answer and $A(\delta x)$ stands for the approximate answer obtained numerically for a mesh generated by partitioning the interval $[0, \beta]$ into equal subintervals of length δx each. Now place a finner mesh on our domain $\bar{\Delta}_0$, by choosing $\dfrac{\delta x}{2}$ for the length of the subintervals of $[0, \beta]$. Hence for this choice, formula (6.1.21) reads:

$$E = A(\frac{\delta x}{2}) + C \cdot \left(\frac{\delta x}{2}\right)^2. \tag{6.1.22}$$

Next, eliminate the constant C from (6.1.21) and (6.1.22) to produce the Richardson extrapolation scheme:

$$E = \frac{4}{3}A(\frac{\delta x}{2}) - \frac{1}{3}A(\delta x). \tag{6.1.23}$$

How is formula (6.1.23) interpreted in the numerical algorithm? Even if we numerically calculate

$$\{W_t(\beta, t), W(\beta, t), W_x(\beta, t)\}$$

on the grid points t of the boundary line $x = \beta$ of our domain $\bar{\Delta}_0$, the quantities in (6.1.23) are going to mean:

$$\begin{cases} E = \{\bar{W}t, \bar{W}x\} \\ A(\delta x) = \{Wt, Wx\} \\ A(\frac{\delta x}{2}) = \{Vt, Vx\}, \end{cases} \tag{6.1.24}$$

because the order of error for W is different from that for Wt and Wx. So in order for (6.1.21) to hold, E and $A(\delta x)$ must have the meaning in (6.1.24).

In (6.1.24), $\{\bar{W}t, \bar{W}x\}$ is the final, numerical answer, $\{Wt, Wx\}$ represents two column vectors of length $M = \dfrac{N+1}{2}$, meaning the pair $\{W_t(\beta, t), W_x(\beta, t)\}$ evaluated at the column vector $t = (t_1, t_2, \ldots, t_M)'$, which contains the partition points in t-direction determined on the boundary line $x = \beta$ of the domain $\bar{\Delta}_0$ by the mesh corresponding to δx, and $\{Vt, Vx\}$ represents two column vectors of the same length $M = \dfrac{N+1}{2}$, obtained from $\{Ut, Ux\}$ by keeping only those components that correspond to the nodes $t = (t_1, t_2, \ldots, t_M)'$. Here $\{Ut, Ux\}$ is the numerical answer for the mesh corresponding to $\dfrac{\delta x}{2}$, and is seen as the pair $\{W_t(\beta, \tilde{t}), W_x(\beta, \tilde{t})\}$ evaluated at the column vector $\tilde{t} = (\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_{\tilde{M}})'$, which contains the partition points in t-direction determined on the boundary line $x = \beta$ of the domain $\bar{\Delta}_0$ by the mesh corresponding to $\dfrac{\delta x}{2}$.

We would like to mention here that due to the specific of the mesh there are only half as many points in t-direction than there are in x-direction, since the spacing in t-direction is twice as large as it is in x-direction. Hence the components of the vector $\tilde{t}$ with odd index overlap the components of the vector t.

We close this discussion with a few words about calculating M and $\tilde{M}$. As said above, M is the number of equally spaced points in t-direction within the interval $[0, \beta]$ for the mesh corresponding to δx. Since the spacing between two consecutive t's was denoted by δt we have that:

$$\begin{cases} (M-1)\delta t = \beta = (N-1)\delta x, \\ \delta t = 2\delta x, \end{cases}$$

from which we obtain $M = \dfrac{N+1}{2}$. Similarly, one gets $\tilde{M} = \dfrac{\tilde{N}+1}{2} = N$, because for the finner mesh (i.e. the one corresponding to $\dfrac{\delta x}{2}$) the number of points in x-direction within the interval $[0, \beta]$ is $\tilde{N} = N + (N-1) = 2N - 1$.

6.2 The inverse map

We have seen in Remark 3.3 of Section 3.7 that two nonlinear equations (3.7.1) and (3.7.2) have to be solved. Hence our purpose is to invert the map F defined by (3.2.47). Numerically, this can be done by using a Newton or modified Newton method. Note that the interval for these equations is $[0, \frac{a}{2}]$, but similar equations corresponding to the interval $[0, \beta]$ can be solved. For this reason and also because of the algorithm for solving the inverse three Dirichlet spectra for the general break point $a_0 \in (0, a)$ (see Section 5.1) we shall carry on the discussion with the generic interval $[0, \beta]$ in place of $[0, \frac{a}{2}]$. For solving equation (3.7.1), for example the modified Newton method is described below:

$$\begin{cases} \text{choose the initial guess } q^{(0)} \\ 0 - \tilde{F}(q^{(n)}) = D\tilde{F}(0)(q^{(n+1)} - q^{(n)}), \ n \geq 0 \end{cases} \tag{6.2.1}$$

or equivalently

$$\begin{cases} \text{choose the initial guess } q^{(0)} \\ q^{(n+1)} = q^{(n)} - \left(D\tilde{F}(0)\right)^{-1} (\tilde{F}(q^{(n)})), \ n \geq 0 \end{cases} \tag{6.2.2}$$

where

$$\tilde{F}(p) = F(p) - \{g_1(t), f_1'(t)\}. \tag{6.2.3}$$

The reason the modified Newton's method is preferred over the Newton's method is that the inverse operator $\left(D\tilde{F}(0)\right)^{-1}$ of the Frechet derivative $D\tilde{F}(0)$ is easier to calculate than $\left(D\tilde{F}(q^{(n)})\right)^{-1}$, and we even have an analytic formula for $\left(D\tilde{F}(0)\right)^{-1}$:

$$\left(D\tilde{F}(0)\right)^{-1}(\{h_1,h_2\})(x) = 2\left[h_1(2x-\beta)+h_2(2x-\beta)\right],\ x\in[0,\beta], \tag{6.2.4}$$

In (6.2.4) h_1 and h_2 are odd and respectively even $L^2(-\beta,\beta)$ functions, for the reason to be seen next. We observe that knowing the solution to (3.2.18) in the triangle

$$\bar{\Delta}_0 = \{(x,t)|0\le t\le x\le\beta\}$$

is equivalent to knowing the solution to the following homogeneous Goursat problem:

$$\begin{cases} W_{xx}(x,t)-W_{tt}(x,t)-p(x)W(x,t)=0, & 0<|t|<x<\beta \\ W(x,\pm x)=\pm\frac{1}{2}\int_0^x p(s)ds, & 0\le x\le\beta, \end{cases} \tag{6.2.5}$$

in the extended triangle

$$\bar{\Delta} = \{(x,t)|0\le|t|\le x\le\beta\}.$$

This is so because, if $\tilde{K}(x,t;p)\in C(\bar{\Delta})$ is the weak solution to (6.2.5), then so is $-\tilde{K}(x,-t;p)$ and so, by the uniqueness of the solution to (6.2.5) we can deduce that

$$-\tilde{K}(x,-t;p)=\tilde{K}(x,t;p).$$

Hence $\tilde{K}(x,0;p)=0$, which then means that $K(x,t;p)$ defined by

$$K(\cdot,\cdot;p)=\tilde{K}(\cdot,\cdot;p)|_{\bar{\Delta}_0}$$

is the weak solution to (3.2.18) in the triangle $\bar{\Delta}_0$. Conversely, if $K(\cdot,\cdot;p)\in C(\bar{\Delta}_0)$ is the weak solution to (3.2.18) then extending $K(\cdot,\cdot;p)$ to an odd function in variable t generates a function $\tilde{K}(\cdot,\cdot;p)$ which turns out to be the weak solution to (6.2.5) in the extended triangle $\bar{\Delta}$. More precisely, $\tilde{K}(\cdot,\cdot;p)$ is defined as

$$\tilde{K}(x,t;p)=\begin{cases} K(x,t;p), & \text{if } 0\le t\le x\le\beta \\ -K(x,-t;p), & \text{if } 0\le -t\le x\le\beta. \end{cases} \tag{6.2.6}$$

The condition $K(x,0;p)=0$ is used to give consistency to the definition of $\tilde{K}(x,t;p)$ in (6.2.6). Also note that $\tilde{K}(x,x;p)$ is calculated using the first formula in (6.2.6) with $t=x$ because $0\le x\le x\le\beta$, whereas $\tilde{K}(x,-x;p)$ is calculated using the second formula in (6.2.6) with $t=-x$ because $0\le -(-x)\le x\le\beta$. Based on this discussion, for now on, $K(\cdot,\cdot;p)$ is to be regarded as the solution to (6.2.5) in the extended triangle $\bar{\Delta}$. Hence, F is to be regarded as a non-linear map between the spaces $L^2(0,\beta)$ and $L^2(-\beta,\beta)\times L^2(-\beta,\beta)$. From (6.2.3) we have that

$$D\tilde{F}(p)(\delta p) = DF(p)(\delta p). \tag{6.2.7}$$

Therefore $D\tilde{F}(p)$ is a linear operator between the spaces $L^2(0,\beta)$ and $L^2(-\beta,\beta)\times L^2(-\beta,\beta)$. We will show next that the range of this operator is a linear subspace contained in

$$S=\{(h_1,h_2)|h_1,h_2\in L^2(-\beta,\beta), h_1=odd, h_2=even\}.$$

One can show by verifying the definition of Frechet derivative and slightly adapting the proof of Theorem 4.15 of [13, page 147] that

$$DF(p)(\delta p)=\{W_x^{(p,\delta p)}(\beta,\cdot), W_t^{(p,\delta p)}(\beta,\cdot)\}, \tag{6.2.8}$$

where $W^{(p,\delta p)}\in C(\bar{\Delta})$ is the weak solution to the non-homogeneous Goursat problem:

$$\begin{cases} W_{xx}^{(p,\delta p)}(x,t)-W_{tt}^{(p,\delta p)}(x,t)-p(x)W^{(p,\delta p)}(x,t)=\delta p(x)K(x,t;p), & 0<|t|<x<\beta \\ W^{(p,\delta p)}(x,\pm x)=\pm\frac{1}{2}\int_0^x \delta p(s)ds, & 0\le x\le\beta \end{cases} \tag{6.2.9}$$

with $K(\cdot,\cdot;p)\in C(\bar{\Delta})$ the weak solution to the homogeneous Goursat problem (6.2.5) in the extended triangle $\bar{\Delta}=\{(x,t)|0\le|t|\le x\le\beta\}$. By the discussion made right after (6.2.5), the function $K(\cdot,\cdot;p)$ is an odd function in variable t in the extended triangle $\bar{\Delta}$. This entails that the solution $W^{(p,\delta p)}$ to the non-homogeneous Goursat problem (6.2.9) is an odd function in variable t in the same triangle $\bar{\Delta}$. It follows that the functions

$$\begin{cases} W_x^{(p,\delta p)}(\beta,t) \\ W_t^{(p,\delta p)}(\beta,t) \end{cases}$$

are odd and even functions in variable t, belonging to $L^2(-\beta,\beta)$. This along with formulas (6.2.7) and (6.2.8) motivate our claim that the range of the linear operator $D\tilde{F}(p)$ is a subspace contained in S. Hence the reason of requiring in (6.2.4) that h_1 and h_2 be odd and respectively even $L^2(-\beta,\beta)$ functions is now visible.

We can continue now with the derivation of (6.2.4). We see that if $p=0$, then the unique solution to (6.2.5) is $K(\cdot,\cdot;p)=0$, and hence by (6.2.9) we obtain

$$\begin{cases} W^{(0,\delta p)}_{xx}(x,t) - W^{(0,\delta p)}_{tt}(x,t) = 0, & 0<|t|<x<\beta \\ W^{(0,\delta p)}(x,\pm x) = \pm\frac{1}{2}\int_0^x \delta p(s)ds, & 0\le x\le\beta, \end{cases} \tag{6.2.10}$$

which further means

$$\begin{cases} W^{(0,\delta p)}(x,t) = \phi(x+t)+\psi(x-t) \\ \delta p(x) = 2W^{(0,\delta p)}_x(x,x) + 2W^{(0,\delta p)}_t(x,x) \\ \delta p(x) = -2W^{(0,\delta p)}_x(x,-x) + 2W^{(0,\delta p)}_t(x,-x). \end{cases} \tag{6.2.11}$$

From (6.2.11) we have that

$$\begin{cases} \phi'(x) = \frac{1}{4}\delta p(\frac{x}{2}) \\ \psi'(x) = -\frac{1}{4}\delta p(\frac{x}{2}) \\ W^{(0,\delta p)}_x(x,t) = \frac{1}{4}\left(\delta p(\frac{x+t}{2}) - \delta p(\frac{x-t}{2})\right) \\ W^{(0,\delta p)}_t(x,t) = \frac{1}{4}\left(\delta p(\frac{x+t}{2}) + \delta p(\frac{x-t}{2})\right) \end{cases} \tag{6.2.12}$$

Noting that

$$\left(D\tilde{F}(0)\right)^{-1}(\{h_1,h_2\}) = \delta p \tag{6.2.13}$$

means

$$\{h_1,h_2\} = D\tilde{F}(0)(\delta p), \tag{6.2.14}$$

which by (6.2.7), (6.2.8), and (6.2.12) is equivalent to

$$\begin{cases} h_1(t) = \frac{1}{4}\left(\delta p(\frac{\beta+t}{2}) - \delta p(\frac{\beta-t}{2})\right) \\ h_2(t) = \frac{1}{4}\left(\delta p(\frac{\beta+t}{2}) + \delta p(\frac{\beta-t}{2})\right), \end{cases} \tag{6.2.15}$$

we obtain that

$$\begin{cases} \delta p\left(\frac{\beta+t}{2}\right) = 2\left(h_2(t)+h_1(t)\right) \\ \delta p\left(\frac{\beta-t}{2}\right) = 2\left(h_2(t)-h_1(t)\right), \end{cases} \tag{6.2.16}$$

or equivalently

$$\begin{cases} \delta p(x) = 2\left(h_2(2x-\beta)+h_1(2x-\beta)\right) \\ \delta p(x) = 2\left(h_2(\beta-2x)-h_1(\beta-2x)\right). \end{cases} \tag{6.2.17}$$

The two identities in (6.2.17) are the same because h_1 and h_2 are odd and respectively even $L^2(-\beta, \beta)$ functions. Finally, formulas (6.2.13) and (6.2.17) yield the desired formula (6.2.4).

6.3 A variant of the algorithm preceding the proof of Theorem 1

The algorithm described before the proof of Theorem 1 can be slightly modified and in some instances this modified algorithm produces slightly better results. How to make the changes was suggested by the formulas (3.3.4)— (3.3.3), and the asymptotic formulas (3.1.2), (3.1.3), (3.1.4) of λ_n's, μ_n's, ν_n's. For example, formula (3.1.2) tells us that

$$\sqrt{\lambda_n - M} \approx \frac{n\pi}{a},$$

and then

$$\sin(\sqrt{\lambda_n - M} \cdot t) \approx \sin(\frac{n\pi}{a} t),$$

and hence if in (3.3.1) we replace λ_n by $\lambda_n - M$ and use (3.3.4), the matrix associated with the linear system we have to solve for the Fourier coefficients α_n's will be almost diagonal (since $\{\sin\left(\frac{n\pi}{a}t\right) |n \geq 1\}$ is an orthogonal set in $L^2(0, a)$), so easy to invert. The key point in making the things work so nicely is that

$$S(x; p, \lambda) = S(x; p - const, \lambda - const), \text{ for all } x \in [\alpha, \beta], \tag{6.3.1}$$

where $S(\cdot; p, \lambda)$ is the $H^2(\alpha, \beta)$ solution to the initial value problem consisting of (3.2.1) and (3.2.3) and $S(\cdot; p - const, \lambda - const)$ is the $H^2(\alpha, \beta)$ solution to (6.3.2):

$$\begin{cases} -u''(x) + (p(x) - const)u(x) = (\lambda - const)u(x), & \alpha < x < \beta \\ u(\alpha) = 0 \\ u'(\alpha) = 1. \end{cases} \tag{6.3.2}$$

The derivation of (6.3.1) is trivial and from it we also have:

$$S'(x; p, \lambda) = S'(x; p - const, \lambda - const), \text{ for all } x \in [\alpha, \beta], \tag{6.3.3}$$

Since replacing λ_n by $\lambda_n - M$, μ_n by $\mu_n - M_1$, ν_n by $\nu_n - M_2$ makes the linear systems (3.3.1), (3.3.2), (3.3.3) easier to solve, formula (6.3.1) suggests us consider the characteristic functions

$$\begin{cases} \lambda \in \mathbb{C} \to S(a; q - M, \lambda - M), \\ \lambda \in \mathbb{C} \to S_1(\frac{a}{2}; q_1 - M_1, \lambda - M_1), \\ \lambda \in \mathbb{C} \to S_1(\frac{a}{2}; \tilde{q}_2 - M_2, \lambda - M_2) \end{cases}$$

in place of the characteristic functions

$$\begin{cases} \lambda \in \mathbb{C} \to S(a; q, \lambda), \\ \lambda \in \mathbb{C} \to S_1(\frac{a}{2}; q_1, \lambda), \\ \lambda \in \mathbb{C} \to S_1(\frac{a}{2}; \tilde{q}_2, \lambda). \end{cases}$$

In other words, if you subtract a constant from the eigenvalues, then you must subtract the same constant from the potential function, and vice-versa. With this in mind and using (6.3.1) and (6.3.3), formula (3.2.16) is equivalent to

$$\begin{aligned} S(a; q - M, \lambda - M) &= S_1(\frac{a}{2}; q_1 - M_1, \lambda - M_1) S_1'(\frac{a}{2}; \tilde{q}_2 - M_2, \lambda - M_2) \\ &+ S_1'(\frac{a}{2}; q_1 - M_1, \lambda - M_1) S_1(\frac{a}{2}; \tilde{q}_2 - M_2, \lambda - M_2), \\ &\quad \text{for all } \lambda \in \mathbb{C}. \end{aligned} \tag{6.3.4}$$

In designing the modified algorithm, formula (6.3.4) plays the same role formula (3.2.16) played in designing the algorithm described before the proof of Theorem 1. We will also need formulas (3.2.20) and (3.2.21) with the appropriate choices of p and λ. Another useful observation is that we are going to recover the potential functions $q_1 - M_1$ and $\tilde{q}_2 - M_2$, and hence M_1 is to be replaced by

$$\begin{aligned} [q_1 - M_1] &= \frac{1}{a/2} \int_0^{\frac{a}{2}} (q_1(s) - M_1) ds \\ &= \frac{1}{a/2} \int_0^{\frac{a}{2}} q_1(s) ds - M_1 \\ &= 0 \text{ (because } M_1 \text{ is expected to be } [q_1]\text{)}, \end{aligned} \tag{6.3.5}$$

and similarly M_2, M are to be replaced by

$$[\tilde{q}_2 - M_2] = 0 \tag{6.3.6}$$

and respectively by

$$[q - M] = 0, \tag{6.3.7}$$

because M_2 is expected to be $[\tilde{q}_2]$ and M is expected to be $[q]$. Now we can sketch the changes of the algorithm described before the proof of Theorem 1. Since $f(t)$, $f_1(t)$ and $f_2(t)$ are expected to be

$$K(a, t; q - M),\ K(\frac{a}{2}, t; q_1 - M_1),\ K(\frac{a}{2}, t; \tilde{q}_2 - M_2)$$

respectively, and because of (6.3.5), (6.3.6), (6.3.7) and of the two boundary conditions of the Goursat problems (3.2.18) with the appropriate choices of $[0, \beta]$ and p we have that

$$\begin{cases} K(a, 0; q - M) = 0, \\ K(a, a; q - M) = 0, \end{cases}$$

$$\begin{cases} K(\frac{a}{2}, 0; q_1 - M_1) = 0, \\ K(\frac{a}{2}, \frac{a}{2}; q_1 - M_1) = 0, \end{cases}$$

$$\begin{cases} K(\frac{a}{2}, 0; \tilde{q}_2 - M_2) = 0, \\ K(\frac{a}{2}, \frac{a}{2}; \tilde{q}_2 - M_2) = 0. \end{cases}$$

It follows then that the appropriate Fourier series expansions of $f(t)$, $f_1(t)$ and $f_2(t)$ are:

$$f(t) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi}{a} t\right),\ t \in [0, a], \tag{6.3.8}$$

$$f_1(t) = \sum_{n=1}^{\infty} \alpha_n^{(1)} \sin\left(\frac{n\pi}{a/2} t\right),\ t \in [0, \tfrac{a}{2}], \tag{6.3.9}$$

$$f_2(t) = \sum_{n=1}^{\infty} \alpha_n^{(2)} \sin\left(\frac{n\pi}{a/2} t\right),\ t \in [0, \tfrac{a}{2}]. \tag{6.3.10}$$

To solve for α_n's insert (6.3.8) into (3.3.1) but with $\lambda_n - M$ in place of λ_n; to solve for $\alpha_n^{(1)}$'s insert (6.3.9) into (3.3.2) but with $\mu_n - M_1$ in place of μ_n; to solve for $\alpha_n^{(2)}$'s insert (6.3.10) into (3.3.3) but with $\nu_n - M_2$ in place of ν_n. Once these coefficients are known, the functions $f(t)$, $f_1(t)$ and $f_2(t)$ will be known too, and we can proceed in obtaining the functions $S(\lambda)$, $S^{(1)}(\lambda)$ and $S^{(2)}(\lambda)$ by the formulas (3.3.7), (3.3.8) and (3.3.9) respectively. Since μ_n's and

ν_n's are expected to be the Dirichlet eigenvalues of $L^{(q_1)}$ and $L^{(\tilde{q}_2)}$ over $[0, \frac{a}{2}]$ we have by (6.3.1) that

$$S_1(\frac{a}{2}; q_1 - M_1, \mu_n - M_1) = 0, \text{ for all } n \geq 1, \tag{6.3.11}$$

and

$$S_1(\frac{a}{2}; \tilde{q}_2 - M_2, \nu_n - M_2) = 0, \text{ for all } n \geq 1. \tag{6.3.12}$$

Formulas (6.3.11), (6.3.12) in combination with (6.3.4) and (3.2.21), and using (6.3.5), (6.3.6) give:

$$\begin{aligned} \frac{S(a; q - M, \mu_n - M)}{S_1(\frac{a}{2}; \tilde{q}_2 - M_2, \mu_n - M_2)} &= S_1'(\frac{a}{2}; q_1 - M_1, \mu_n - M_1) \text{ (by (6.3.11) and (6.3.4))} \\ &= \cos(\sqrt{\mu_n - M_1}\frac{a}{2}) \\ &\quad + \left(\frac{1}{2}\int_0^{\frac{a}{2}} (q_1(s) - M_1) ds\right) \cdot \frac{\sin(\sqrt{\mu_n - M_1}\frac{a}{2})}{\sqrt{\mu_n - M_1}} \\ &\quad + \int_0^{\frac{a}{2}} g_1(t) \frac{\sin(\sqrt{\mu_n - M_1} \cdot t)}{\sqrt{\mu_n - M_1}} dt \text{ (by (3.2.21))} \\ &= \cos(\sqrt{\mu_n - M_1}\frac{a}{2}) \\ &\quad + \int_0^{\frac{a}{2}} g_1(t) \frac{\sin(\sqrt{\mu_n - M_1} \cdot t)}{\sqrt{\mu_n - M_1}} dt, \text{ (by (6.3.5))}, \end{aligned} \tag{6.3.13}$$

because $g_1(t)$ is expected to be $K(\frac{a}{2}, t; q_1 - M_1)$, and

$$\begin{aligned} \frac{S(a; q - M, \nu_n - M)}{S_1(\frac{a}{2}; q_1 - M_1, \nu_n - M_1)} &= S_1'(\frac{a}{2}; \tilde{q}_2 - M_2, \nu_n - M_2) \text{ (by (6.3.12) and (6.3.4))} \\ &= \cos(\sqrt{\nu_n - M_2}\frac{a}{2}) \\ &\quad + \left(\frac{1}{2}\int_0^{\frac{a}{2}} (\tilde{q}_2(s) - M_2) ds\right) \cdot \frac{\sin(\sqrt{\nu_n - M_2}\frac{a}{2})}{\sqrt{\nu_n - M_2}} \\ &\quad + \int_0^{\frac{a}{2}} g_2(t) \frac{\sin(\sqrt{\nu_n - M_2} \cdot t)}{\sqrt{\nu_n - M_2}} dt \text{ (by (3.2.21))} \\ &= \cos(\sqrt{\nu_n - M_2}\frac{a}{2}) \\ &\quad + \int_0^{\frac{a}{2}} g_2(t) \frac{\sin(\sqrt{\nu_n - M_2} \cdot t)}{\sqrt{\nu_n - M_2}} dt, \text{ (by (6.3.6))}, \end{aligned} \tag{6.3.14}$$

because $g_2(t)$ is expected to be $K(\frac{a}{2}, t; \tilde{q}_2 - M_2)$.

Formulas (6.3.13) and (6.3.14), and the fact that the functions

$$S(\lambda - M),\ S^{(1)}(\lambda - M_1),\ S^{(2)}(\lambda - M_2)$$

are expected to be

$$S(a; q-M, \lambda-M),\ S_1(\frac{a}{2}; q_1-M_1, \lambda-M_1),\ S_1(\frac{a}{2}; \tilde{q}_2-M_2, \lambda-M_2)$$

respectively, suggest us the following continuation of the modified algorithm: write by case, the Fourier series expansion (3.3.14) or (3.3.12) and insert it into the system of integral equations

$$\frac{S(\mu_n-M)}{S^{(2)}(\mu_n-M_2)} = \cos(\sqrt{\mu_n-M_1}\frac{a}{2}) + \int_0^{\frac{a}{2}} g_1(t)\frac{\sin(\sqrt{\mu_n-M_1}\cdot t)}{\sqrt{\mu_n-M_1}}dt,\ n \geq 1 \tag{6.3.15}$$

and find the coefficients $\beta_n^{(1)}$'s. Then write by case, the Fourier series expansion (3.3.15) or (3.3.13) and insert it into the system of integral equations

$$\frac{S(\nu_n-M)}{S^{(1)}(\nu_n-M_1)} = \cos(\sqrt{\nu_n-M_2}\frac{a}{2}) + \int_0^{\frac{a}{2}} g_2(t)\frac{\sin(\sqrt{\nu_n-M_2}\cdot t)}{\sqrt{\nu_n-M_2}}dt,\ n \geq 1 \tag{6.3.16}$$

and find the coefficients $\beta_n^{(2)}$'s. With these coefficients, the functions $g_1(t)$ and $g_2(t)$ are well-determined. Then solve the non-linear equations (3.7.1) and (3.7.2) to obtain $q_1 - M_1$ and $\tilde{q}_2 - M_2$ over the interval $[0, \frac{a}{2}]$. Finally, add back the constants M_1 and M_2 to the functions $q_1 - M_1$ and respectively $\tilde{q}_2 - M_2$ to get q_1 and $\tilde{q}_2$, reflect $\tilde{q}_2$ about the mid-line $x = \frac{a}{2}$ and get q_2 over the interval $[\frac{a}{2}, a]$, then paste together q_1 and q_2 to get q over the whole interval $[0, a]$.

6.4 What are the best numbers of elements of the sequences $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$ for a good reconstruction of the potential function q?

This discussion applies to the sets of $\{\lambda_n | n \geq 1\}$, $\{\mu_n | n \geq 1\}$, $\{\nu_n | n \geq 1\}$ as in hypotheses of Theorem 1 as well as to the same three sets as in hypotheses of Theorem 6 and as in those of Theorem 7. Since Theorem 1 is a particular case of Theorem 6 (but we preferred to give the content and the proof of Theorem 1 before those of Theoerm 6 because many things were easily observed in the simpler case of the interval $[0, a]$ broken at the mid-point $a_0 = \frac{a}{2}$, and because the proof of Theorem 1 offered good suggestions for the two generalizations in Chapter 5), we can restrict our attention to the sets of λ's, μ's, ν's as in Theorem 6 and as in Theorem 7. In the proof of Theorem 6 the quotients

$$\begin{cases} \frac{S(\mu_n)}{S_2(\tilde{\mu}_n)},\ n \geq 1, \\ \frac{S(\nu_n)}{S_1(\nu_n)},\ n \geq 1, \end{cases} \tag{6.4.1}$$

played an important role. The functions S, S_1 and S_2 were obtained from $\{\lambda_n | n \geq 1\}$, $\{\mu_n | n \geq 1\}$, and respectively from $\{\tilde{\nu}_n | n \geq 1\}$ via (5.1.13), (5.1.10); via (5.1.14), (5.1.11); and respectively via (5.1.15), (5.1.12). In practice, only finitely many of $\{\lambda_n | n \geq 1\}$, $\{\mu_n | n \geq 1\}$, and $\{\nu_n | n \geq 1\}$ are available. Suppose that only $\{\lambda_1, \ldots, \lambda_J\}$, $\{\mu_1, \ldots, \mu_{J_1}\}$, and $\{\nu_1, \ldots, \nu_{J_2}\}$ are known. When evaluating $S(\mu_n)$ ($n \in \{1, \ldots, J_1\}$) and $S(\nu_n)$ ($n \in \{1, \ldots, J_2\}$) in (6.4.1) using (5.1.13), the values are good if μ_n, ν_n fall not too far away from $[\lambda_1, \lambda_J]$. This is so because $S(\lambda)$ is constructed now only from $\lambda_1, \ldots, \lambda_J$. Similarly, the calculation of $S_1(\nu_n)$ ($n \in \{1, \ldots, J_2\}$) via (5.1.14) is good if ν_n is not too far away from $[\mu_1, \mu_{J_1}]$, and the calculation of $S_2(\tilde{\mu}_n)$ ($n \in \{1, \ldots, J_1\}$) via (5.1.15) is good if $\tilde{\mu}_n$ is not too far away from $[\tilde{\nu}_1, \tilde{\nu}_{J_2}]$, which is equivalent to saying that μ_n falls not too far away from $[\nu_1, \nu_{J_2}]$. These are so because $S_1(\lambda)$ and $S_2(\tilde{\lambda})$ are constructed now only from $\mu_1, \ldots, \mu_{J_1}$, and respectively from $\tilde{\nu}_1, \ldots \tilde{\nu}_{J_2}$. This amounts to requiring

$$\lambda_J \approx \mu_{J_1} \approx \nu_{J_2}, \tag{6.4.2}$$

due to the fact that λ's, μ's, ν's are three strictly increasing sequences of real numbers. Looking at the leading terms of λ_J, μ_{J_1}, ν_{J_2} (see (5.1.2), (5.1.3), (5.1.4)), formula (6.4.2) translates into

$$\frac{J}{a} \approx \frac{J_1}{a_0} \approx \frac{J_2}{a - a_0}. \tag{6.4.3}$$

Due to the interlacing property of λ's, μ's, ν's stated in hypothesis of Theorem 6, which can be written equivalently: 'between any two consecutive λ's there is exactly one element of the set $\{\mu_n | n \geq 1\} \cup \{\nu_n | n \geq 1\}$', and using (6.4.2) and (6.4.3) we conclude that:

$$\begin{cases} J - 1 = J_1 + J_2 \quad \text{ or } J = J_1 + J_2 \\ \frac{J_1}{J_2} \approx \frac{a_0}{a - a_0}. \end{cases} \tag{6.4.4}$$

Similar arguments apply to the situation in Theorem 7 and we arrive at the conclusion that:

$$\begin{cases} J = J_1 + J_2 \qquad \text{ or } J = J_1 + J_2 - 1 \\ \frac{J_1 - 1/2}{J_2 - 1/2} \approx \frac{a_0}{a - a_0}. \end{cases} \tag{6.4.5}$$

CHAPTER 7. NUMERICAL RESULTS

7.1 Non-overlap situations illustrated by numerical examples

This section is meant to illustrate by several results the effectiveness of the three numerical algorithms described in Chapter 3 and Chapter 5, for the non-overlap case. In order for this to be accomplished we must be able to compare the true answer with the numerical one. This requires that we know the potential function q. Since the input data in the three algorithms are three sets of real numbers (expected to be the three spectra for the Sturm-Liouville differential operator with potential function q and various boundary conditions) we need to have them available to us. This is achieved due to the fact that q is known (an analytical formula we start with) and due to MATSLISE [15], a software which calculates the various types of eigenvalues (i.e. Dirichlet, Neumann, Robin) provided that the potential function is known in an analytical form. Figures 7.1,. . ., 7.10 contain four plots each. Each of the four plots for one example illustrates the graph of the chosen potential function in analytical form and the graph of the potential function obtained by running the Matlab program built up from the numerical algorithm mentioned above for the case of

- Dirichlet boundary condition at the midpoint (upper-left plot),
- Dirichlet boundary condition at an arbitrary interior node (upper-right plot),
- Neumann boundary condition at an arbitrary interior node (lower-left plot),
- Robin boundary condition at an arbitrary interior node (lower-right plot).

For the first five examples continuous potentials were chosen, and for the last five examples discontinuous potentials with one jump were chosen. In the captions of the following figures, the function H is used for the 'heaviside function'.

7.2 A one-overlap situation illustrated by a numerical example

We conclude this chapter with one example of reconstructing the potential function in the case of one overlap. The reader is advised to revisit Subsection 4.2.2 to refresh his (her) memory about the overlap situations, since some information presented there is needed next.

The function to be reconstructed is

$$q(x) = \begin{cases} \frac{2}{(1+x)^2}, & \text{in } [0,1] \\ C, & \text{in } [1,2], \end{cases} \tag{7.2.1}$$

where the constant C is chosen such that $\mu_1 = \nu_1$ (the first Dirichlet eigenvalue on $[0,1]$ and respectively on $[1,2]$). The Dirichlet eigenvalues of $q_1 = q|_{[0,1]}$ ($\{\mu_n\}_{n\geq 1}$) are calculated with MATSLISE [15], and the Dirichlet eigenvalues of $q_2 = q|_{[1,2]}$ ($\{\nu_n\}_{n\geq 1}$) can be calculated analytically. They have the formula:

$$\nu_n = (n\pi)^2 + C, \text{ for all } n \geq 1.$$

After the value of C is calculated by the procedure above, the Dirichlet eigenvalues of q (on $[0,2]$ of course), $\{\lambda_n\}_{n\geq 1}$ are calculated with MATSLISE [15]. It can be justified that only one overlap of $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$ happens.

This example is particularly illustrative because we can analytically calculate the first Pöschel-Trubowitz norming constants corresponding to $[0,1]$ and respectively to $[1,2]$ and use them in our algorithm to reconstruct the potential having $\{\lambda_n\}_{n\geq 1}$, $\{\mu_n\}_{n\geq 1}$, $\{\nu_n\}_{n\geq 1}$ as its three Dirichlet spectra. In our numerical reconstruction we used $(20; 10; 10)$ elements of the above mentioned sets. It will be seen in Figure 7.11 that the numerically constructed potential stays very close to the exact potential q defined in (7.2.1).

The Pöschel-Trubowitz norming constants k_1^1 and k_1^2 are:

$$\begin{cases} k_1^1 = \ln\left((-1)^1 S_1'(1; q_1, \mu_1)\right) = \ln\left(-\cos(\sqrt{\mu_1}) - \frac{\sin(\sqrt{\mu_1})}{\sqrt{\mu_1}}\right), \\ k_1^2 = \ln\left((-1)^1 S_1'(1; \tilde{q}_2, \nu_1)\right) = \ln\left(-(-1)^1\right) = 0. \end{cases}$$

These calculations were possible because a fundamental set of solutions to

$$-u''(x) + \frac{2}{(1+x)^2}u(x) = \lambda u(x), \text{ in } (0,1)$$

was known. This set consists of

$$\begin{cases} \cos(\sqrt{\lambda}x) - \frac{1}{1+x} \cdot \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} \\ \sin(\sqrt{\lambda}x) + \frac{1}{1+x} \cdot \frac{\cos(\sqrt{\lambda}x)}{\sqrt{\lambda}}, \end{cases}$$

so $S_1(x; q_1, \lambda)$ is obtained as a linear combination of them, where the constants are determined from the initial conditions

$$S_1(0; q_1, \lambda) = 0,\ S_1'(0; q_1, \lambda) = 1,$$

and

$$S_1(x; \tilde{q}_2, \lambda) = \frac{\sin(\sqrt{\lambda - C}x)}{\sqrt{\lambda - C}},$$

by a simple calculation, since $S_1(\cdot; \tilde{q}_2, \lambda)$ is the solution to the initial value problem consisting of (3.2.1) and (3.2.3) with $[\alpha, \beta] = [0, 1]$ and $p = \tilde{q}_2$, where $\tilde{q}_2(x) = q_2(2-x) = C$, for $x \in [0, 1]$.

To numerically illustrate the non-uniqueness phenomenon, we should construct a second example of potential, say q^* with the same three Dirichlet spectra as q. Unfortunately, this was not possible, since even if we had chosen arbitrarily the first Pöschel-Trubowitz norming constant k_1^1 corresponding to $[0, 1]$, the other first norming constant k_1^2 corresponding to the second subinterval $[1, 2]$ should have been determined from the relationship between k_1^1 and k_1^2, a relationship we know that exists (see the proof of Theorem 4 in Subsection 4.2.2), but we do not know its form. So we do not know how to choose k_1^2 and thus the construction of the second half piece of q^* was not possible.

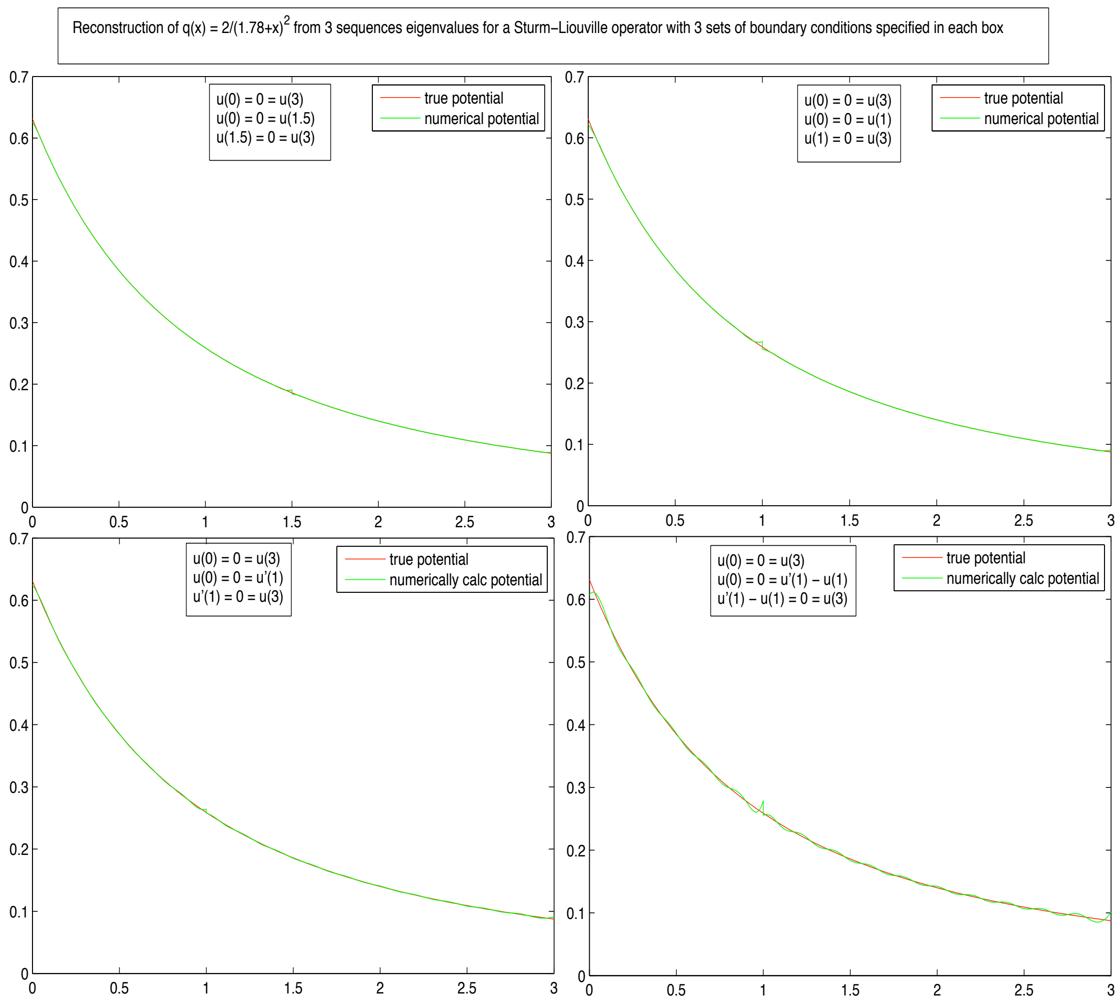

Figure 7.1 $q(x) = 2/(1.78+x)^2$ is being reconstructed from three sequences of eigenvalues for a Sturm-Liouville operator with three sets of boundary conditions, each set being specified in one box

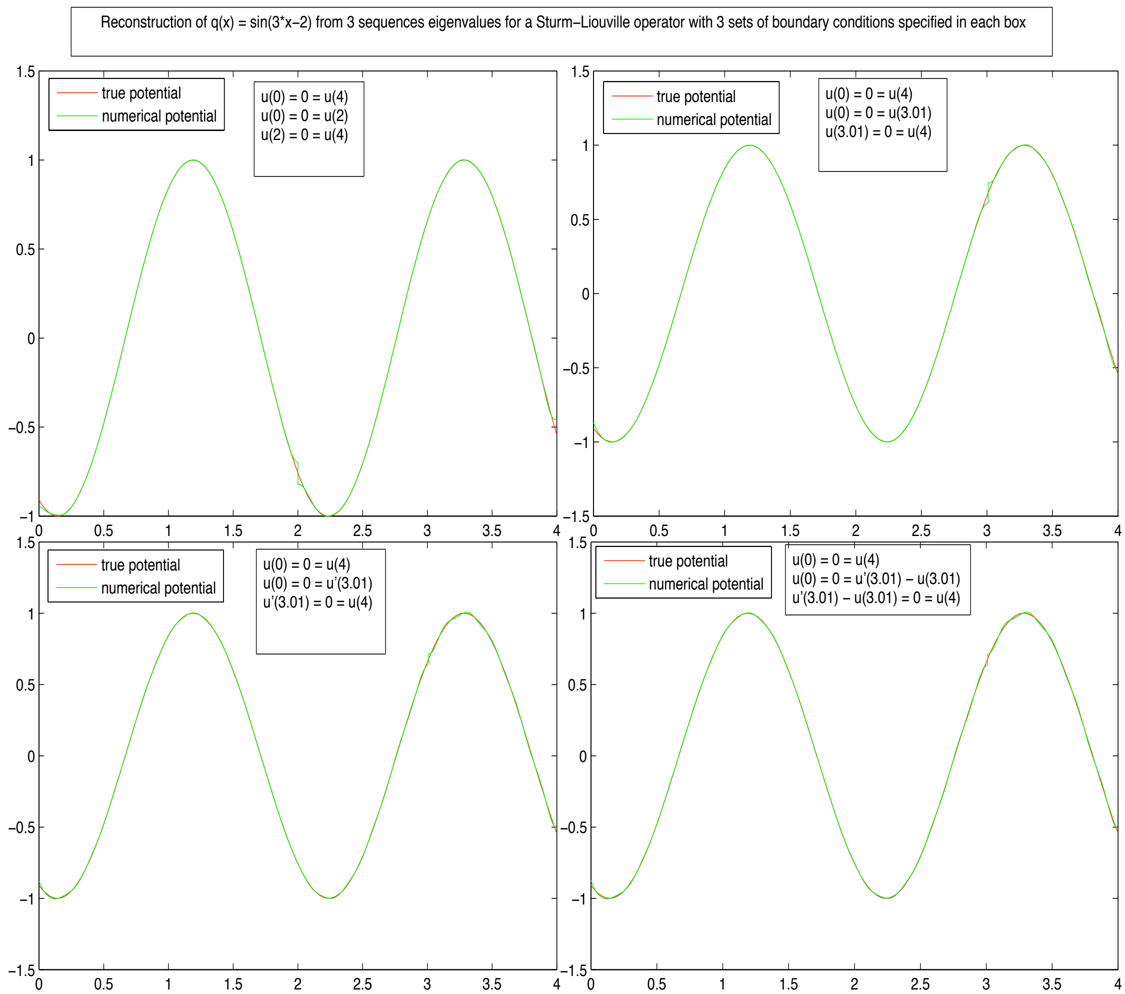

Figure 7.2 $q(x) = \sin(3x - 2)$ is being reconstructed from three sequences of eigenvalues for a Sturm-Liouville operator with three sets of boundary conditions, each set being specified in one box

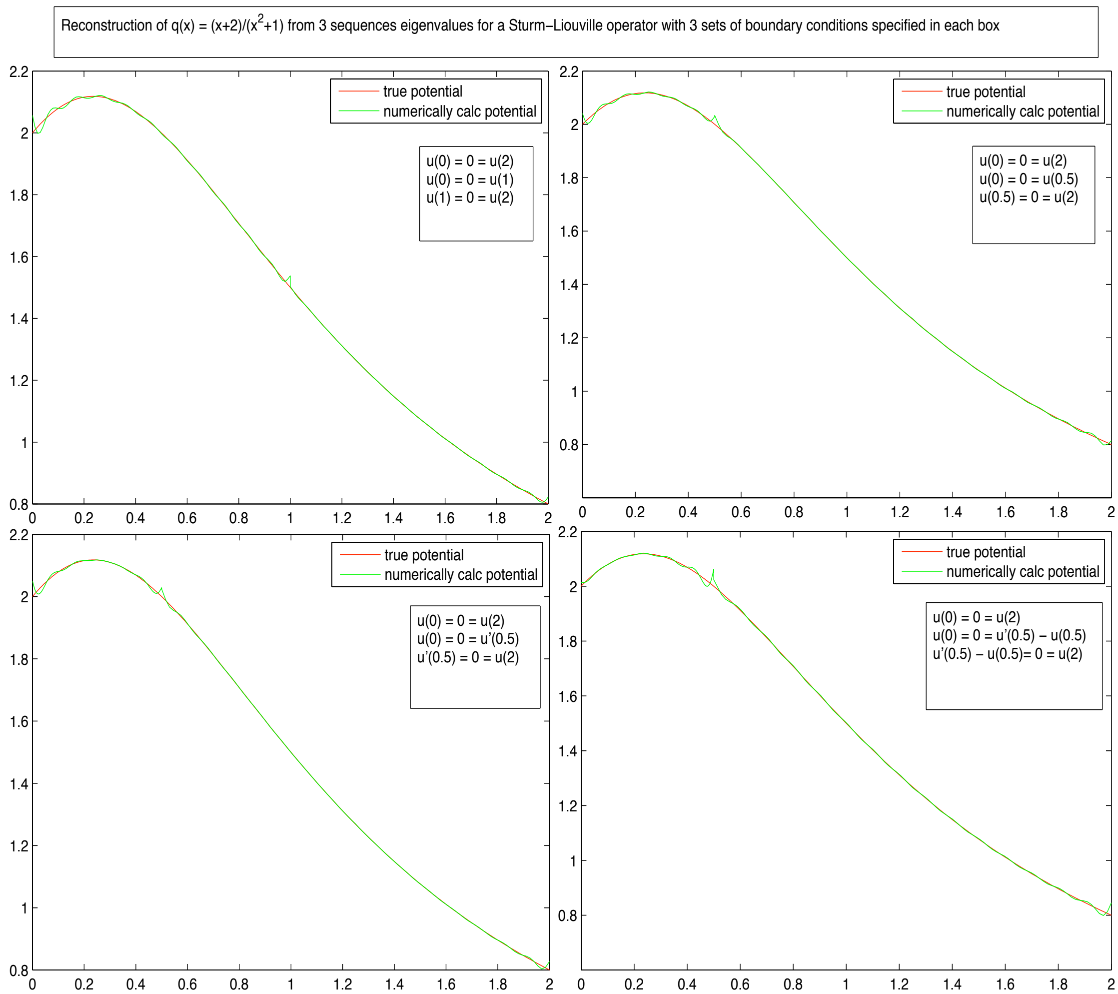

Figure 7.3 $q(x) = (x+2)/(x^2+1)$ is being reconstructed from three sequences of eigenvalues for a Sturm-Liouville operator with three sets of boundary conditions, each set being specified in one box

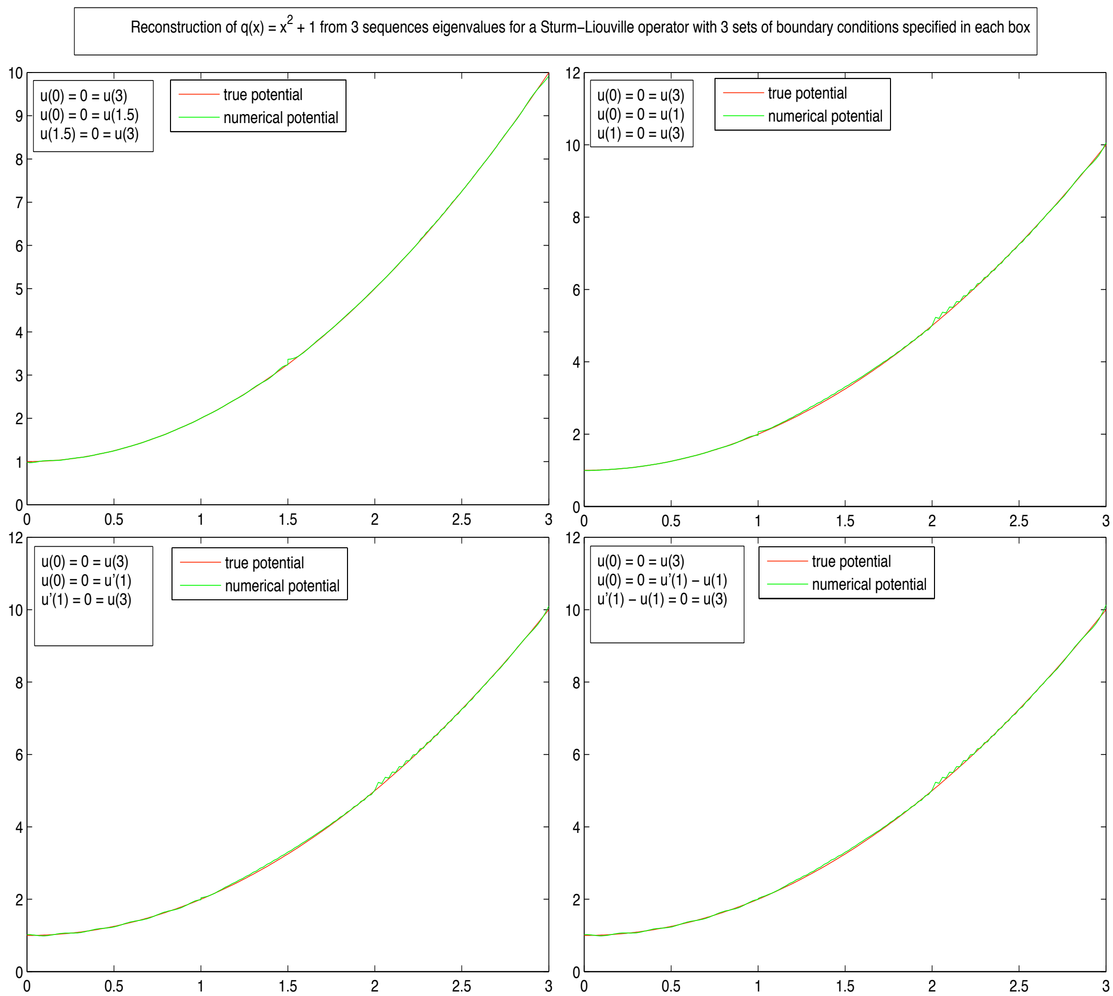

Figure 7.4 $q(x) = (x^2 + 1)$ is being reconstructed from three sequences of eigenvalues for a Sturm-Liouville operator with three sets of boundary conditions, each set being specified in one box

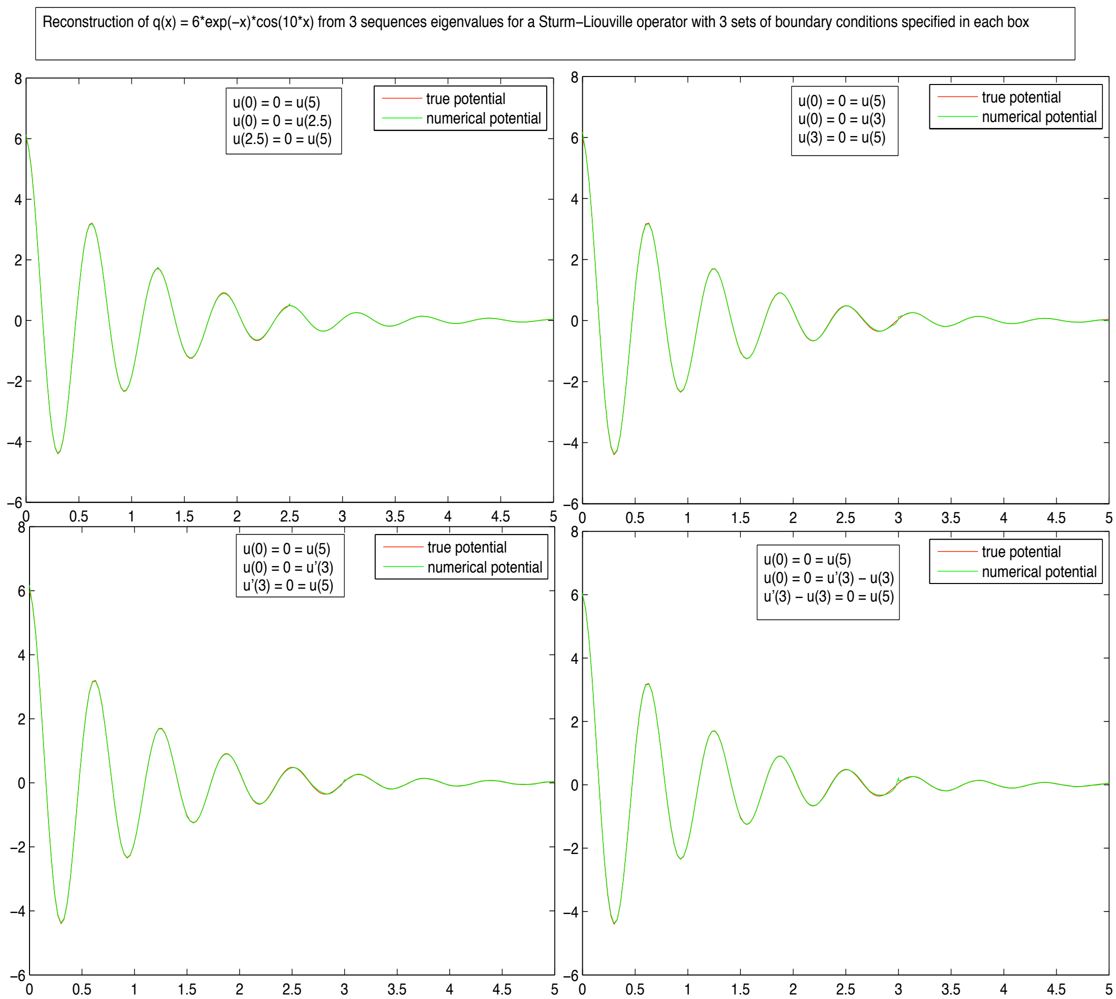

Figure 7.5 $q(x) = 6e^{-x}\cos(10x)$ is being reconstructed from three sequences of eigenvalues for a Sturm-Liouville operator with three sets of boundary conditions, each set being specified in one box

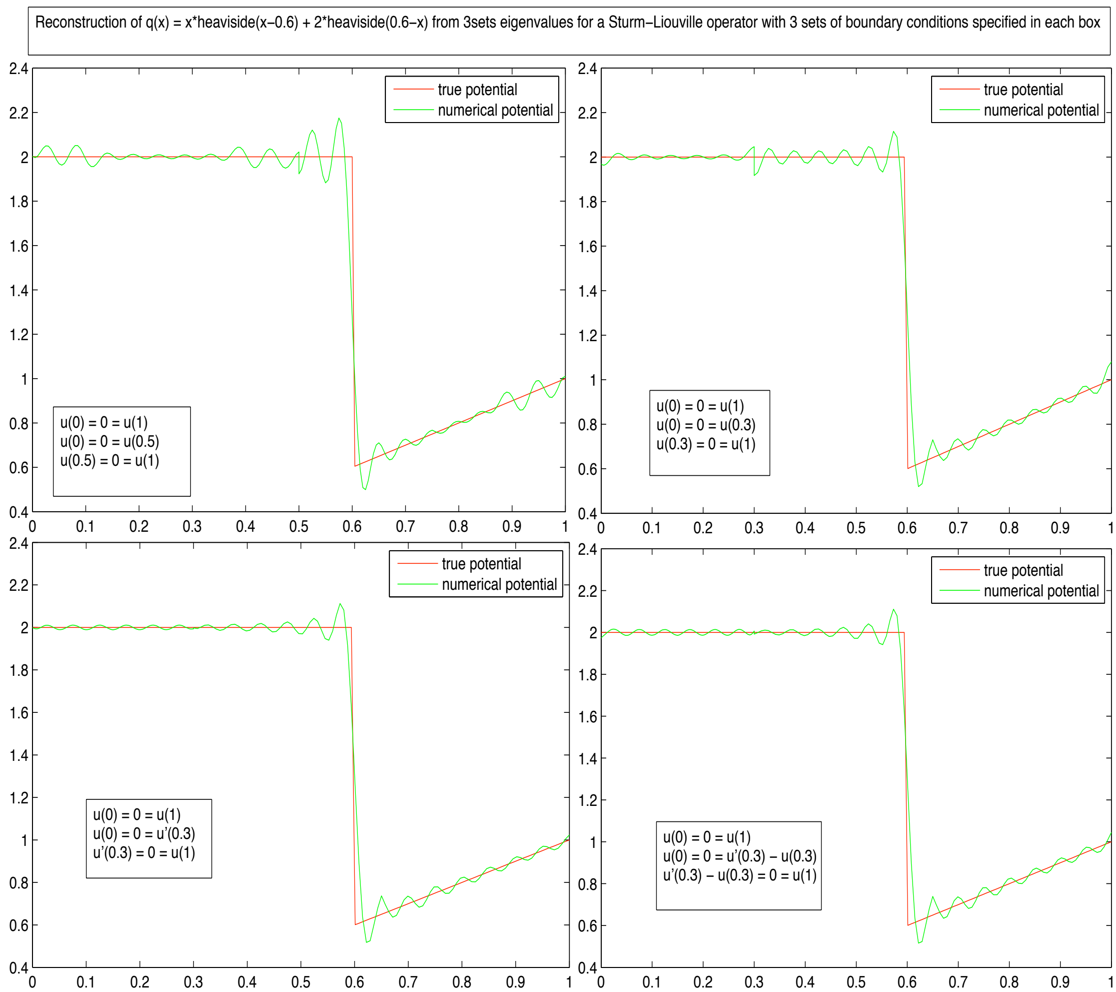

Figure 7.6 $q(x) = 2H(0.6 - x) + xH(x - 0.6)$ is being reconstructed from three sequences of eigenvalues for a Sturm-Liouville operator with three sets of boundary conditions, each set being specified in one box

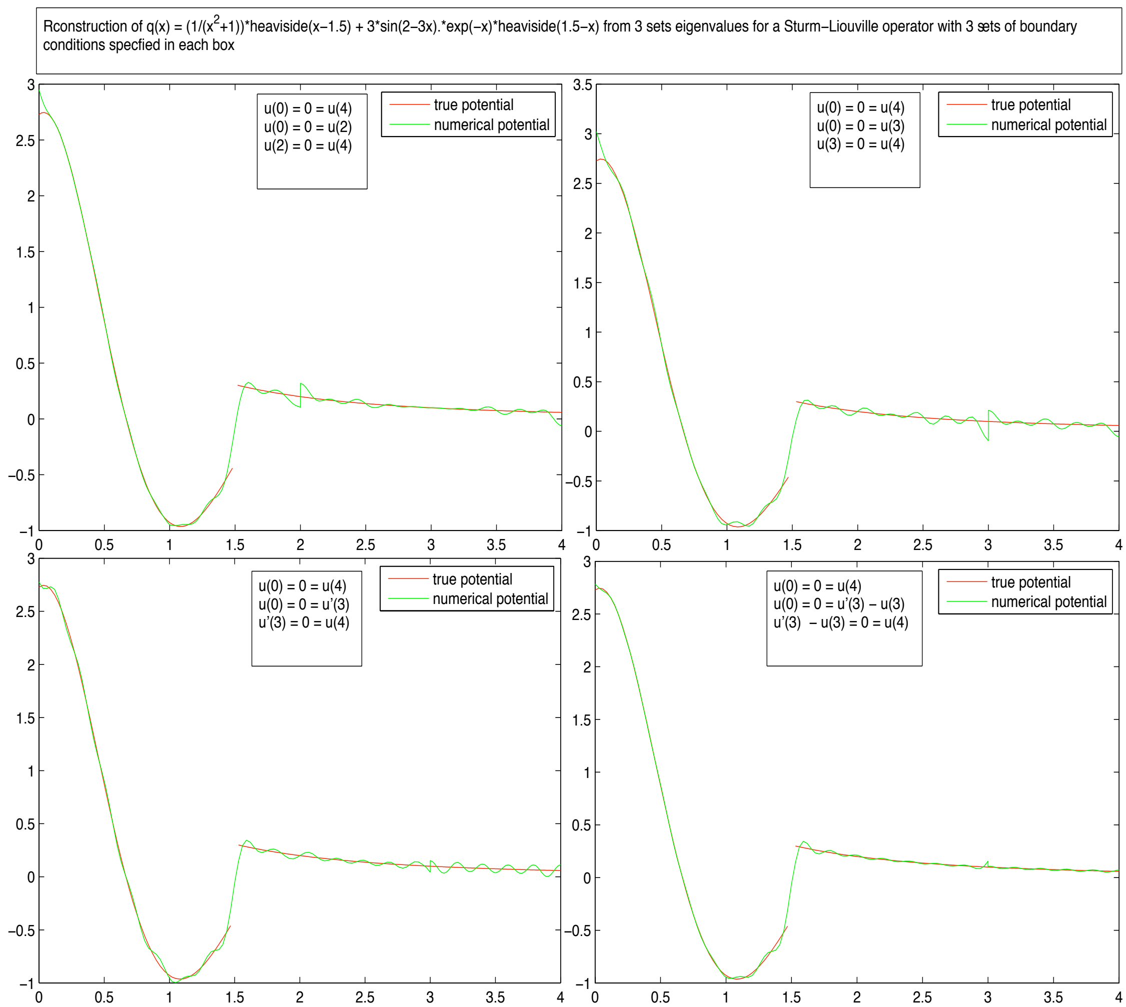

Figure 7.7 $q(x) = 3e^{-x}\sin(2-3x)H(1.5-x) + (1/(x^2+1))H(x-1.5)$ is being reconstructed from three sequences of eigenvalues for a Sturm-Liouville operator with three sets of boundary conditions, each set being specified in one box

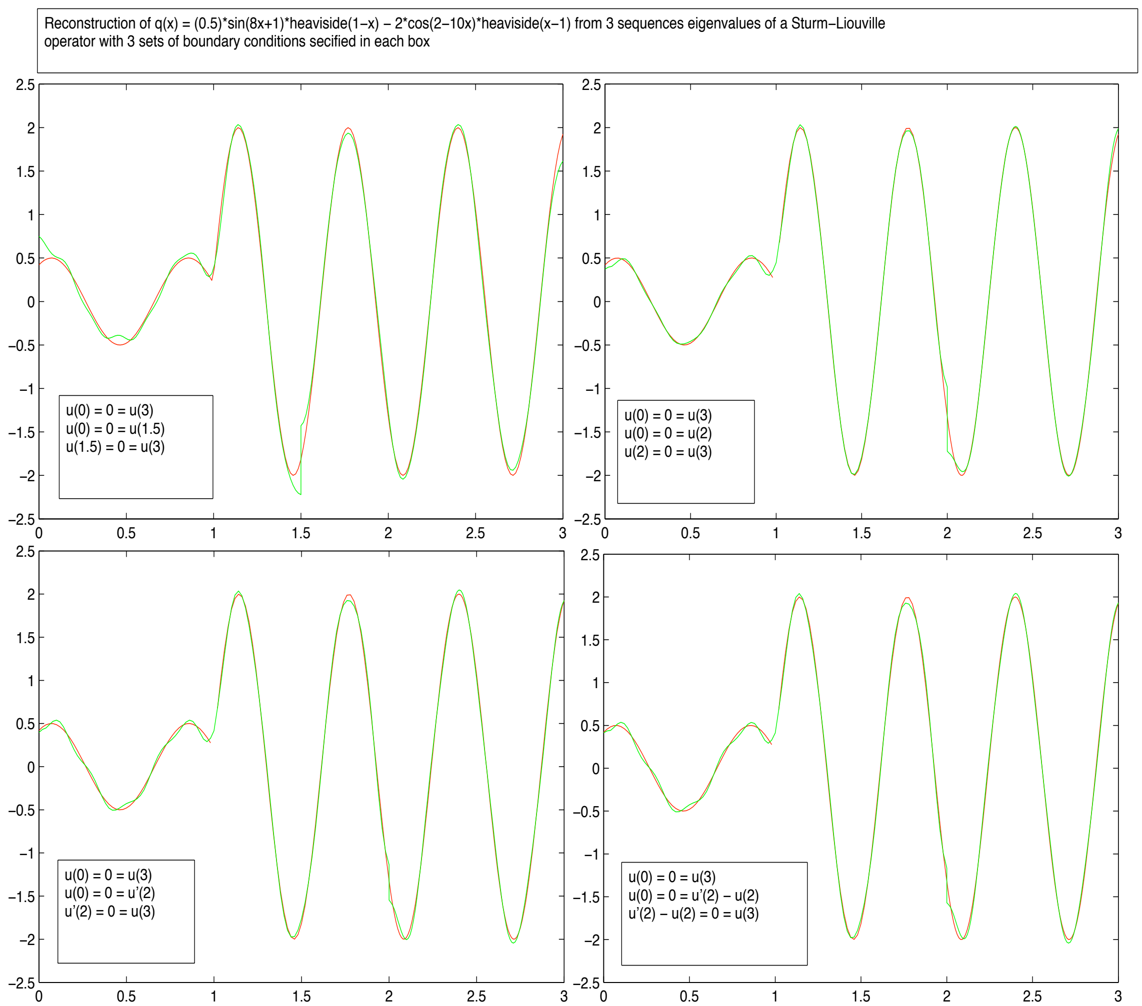

Figure 7.8 $q(x) = (0.5)\sin(8x+1)H(1-x) - 2\cos(2-10x)H(x-1)$ is being reconstructed from three sequences of eigenvalues for a Sturm-Liouville operator with three sets of boundary conditions, each set being specified in one box

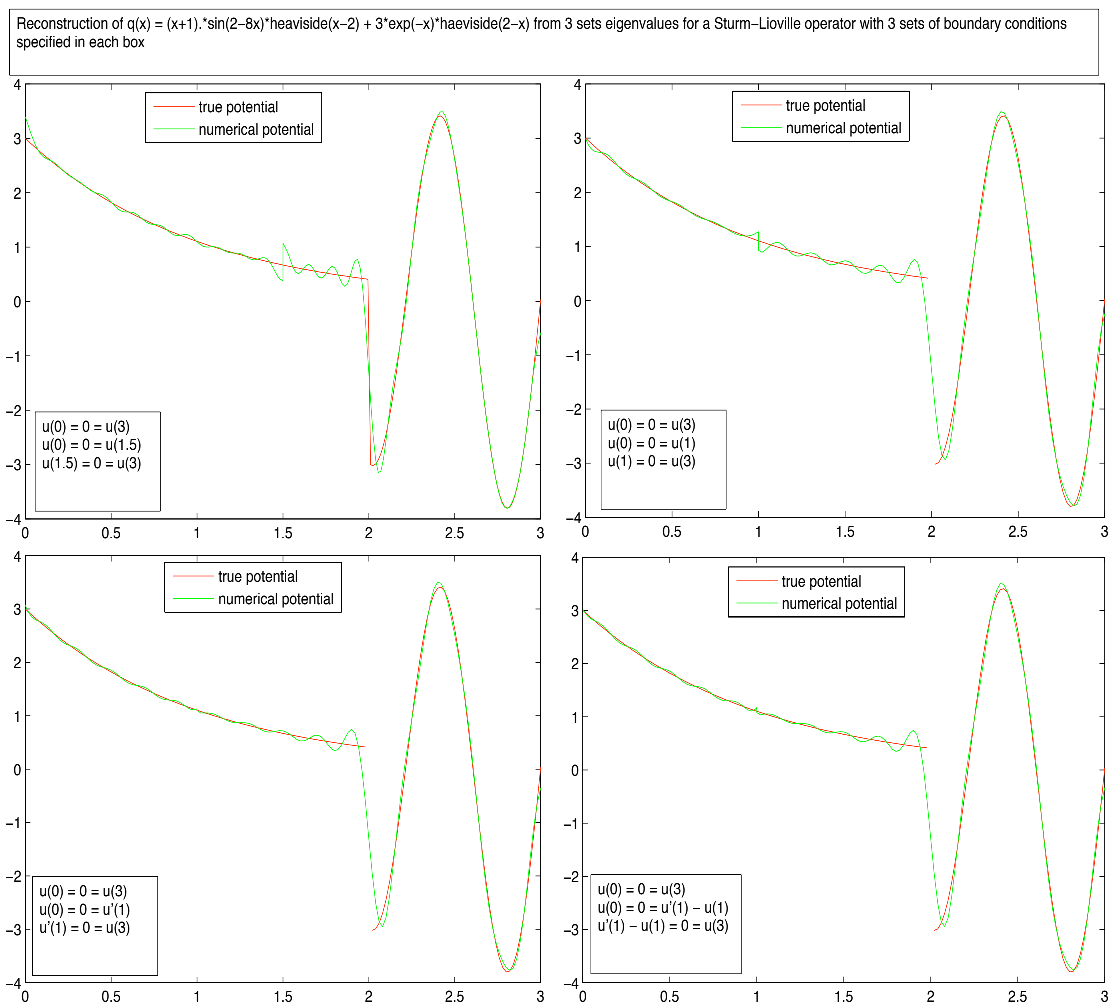

Figure 7.9 $q(x) = 3e^{-x}H(2-x) + (x+1)\sin(2-8x)H(x-2)$ is being reconstructed from three sequences of eigenvalues for a Sturm-Liouville operator with three sets of boundary conditions, each set being specified in one box

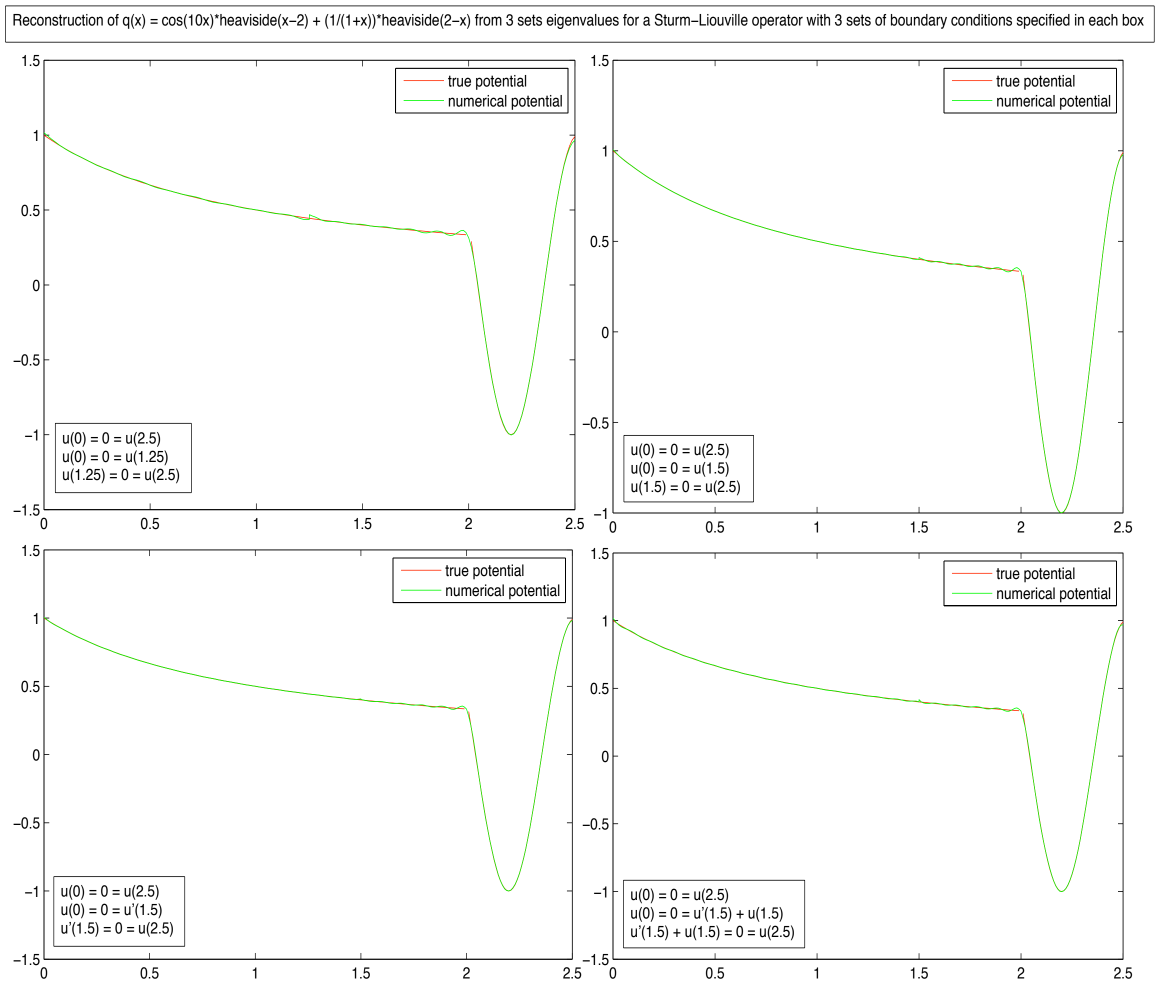

Figure 7.10 $q(x) = (1/(1+x))H(2-x) + \cos(10x)H(x-2)$ is being reconstructed from three sequences of eigenvalues for a Sturm-Liouville operator with three sets of boundary conditions, each set being specified in one box

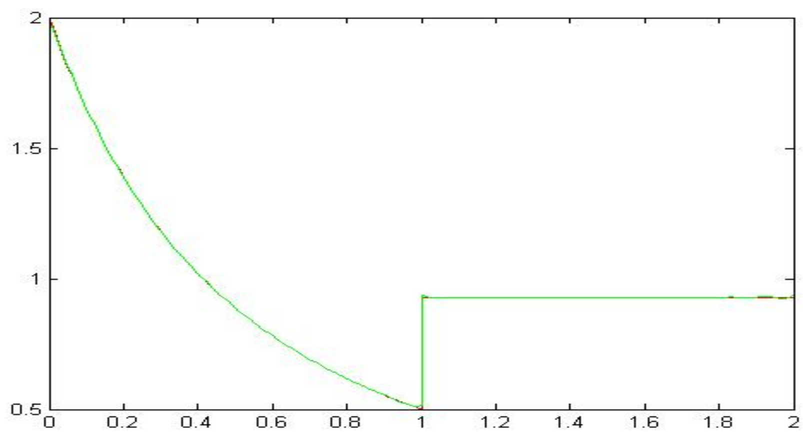

Figure 7.11 $q(x) = 2/(1+x)^2 H(1-x) + CH(x-1)$ is being reconstructed from three sequences of Dirichlet eigenvalues, in the case of one overlap of these sets

APPENDIX A. A GENERALIZATION OF LEMMA 1

This appendix provides a generalization of Lemma 1: the infinite product representations of the four functions presented in Lemma 1. The result is stated as:

Lemma 2 *The following hold:*

1. *the functions*
$$\begin{cases} \lambda \in \mathbb{C} \to S_2(a; q_2, \lambda) \\ \lambda \in \mathbb{C} \to S_1(\frac{a}{2}; \tilde{q}_2, \lambda) \end{cases}$$
have the same zeros, which are countable many. Call them z_n, for $n \geq 1$.

2. *these functions have the same infinite product representation and therefore they are equal to each other:*
$$S_2(a; q_2, \lambda) = \left(\frac{a}{2}\right) \prod_{n=1}^{\infty} \left(\frac{a/2}{n\pi}\right)^2 (z_n - \lambda) = S_1(\frac{a}{2}; \tilde{q}_2, \lambda), \text{ for all } \lambda \in \mathbb{C}, \tag{A.0.1}$$

3. *the functions*
$$\begin{cases} \lambda \in \mathbb{C} \to C_2(a; q_2, \lambda) \\ \lambda \in \mathbb{C} \to S_1'(\frac{a}{2}; \tilde{q}_2, \lambda) \end{cases}$$
have the same zeros, which are countable many. Call them w_n, for $n \geq 1$.

4. *these functions have the same infinite product representation and therefore they are equal to each other:*
$$C_2(a; q_2, \lambda) = \prod_{n=1}^{\infty} \left(\frac{a/2}{(n - \frac{1}{2})\pi}\right)^2 (w_n - \lambda) = S_1'(\frac{a}{2}; \tilde{q}_2, \lambda), \text{ for all } \lambda \in \mathbb{C}. \tag{A.0.2}$$

Proof: First we prove the assertions 1 and 2. The zeros of the functions
$$\begin{cases} \lambda \in \mathbb{C} \to S_2(a; q_2, \lambda), \\ \lambda \in \mathbb{C} \to S_1(\frac{a}{2}; \tilde{q}_2, \lambda) \end{cases}$$

are exactly the Dirichlet eigenvalues of the Sturm-Liouville operators

$$\begin{cases} L^{(q_2)} \text{ with the domain } D(L^{(q_2)}) = \{u \in H^2(\frac{a}{2}, a) | u(\frac{a}{2}) = 0 = u(a)\}, \\ L^{(\tilde{q}_2)} \text{ with the domain } D(L^{(\tilde{q}_2)}) = \{u \in H^2(0, \frac{a}{2}) | u(0) = 0 = u(\frac{a}{2})\}, \end{cases}$$

respectively.

To see this we will argue on the general case: the zeros of the function

$$\lambda \in \mathbb{C} \to S(\beta; p, \lambda)$$

are exactly the Dirichlet eigenvalues of the operator

$$L^{(p)} \text{ with the domain } D(L^{(p)}) = \{u \in H^2(\alpha, \beta) | u(\alpha) = 0 = u(\beta)\}.$$

Then the reader can particularize the interval $[\alpha, \beta]$ and the function p to the interval $[\frac{a}{2}, a]$ and the function q_2, and respectively to the interval $[0, \frac{a}{2}]$ and the function $\tilde{q}_2$. If λ is a zero of the function $S(\beta; p, \cdot)$, then by the fact that $S(\cdot; p, \lambda)$ is the solution to the initial value problem consisting of (3.2.1) and (3.2.3) it follows that $\{S(\cdot; p, \lambda), \lambda\}$ is a Dirichlet eigenpair of the Sturm-Liouville operator $L^{(p)}$ over the interval $[\alpha, \beta]$. Hence, λ is a Dirichlet eigenvalue of the operator $L^{(p)}$. Conversely, if λ is a Dirichlet eigenvalue of the Sturm-Liouville operator $L^{(p)}$ over the interval $[\alpha, \beta]$, then there exists a function $v \neq 0$ which is an associated eigenfunction to λ. That means

$$\begin{cases} -v''(x) + p(x)v(x) = \lambda v(x), \ x \in (\alpha, \beta) \\ v(\alpha) = 0, \\ v(\beta) = 0. \end{cases} \tag{A.0.3}$$

Since $v \neq 0$ on $[\alpha, \beta]$ and the facts that v satisfies the ODE of problem (A.0.3) and $v(\alpha) = 0$ it follows that $v'(\alpha) \neq 0$. Therefore

$$\frac{v}{v'(\alpha)}$$

is well defined and satisfies the initial value problem consisting of (3.2.1) and (3.2.3). But this problem has only one solution, which is $S(\cdot; p, \lambda)$. Therefore

$$\frac{v(x)}{v'(\alpha)} = S(x; p, \lambda), \text{ for all } x \in [\alpha, \beta].$$

This combined with the fact that $v(\beta) = 0$ (see the second boundary condition of problem (A.0.3)) gives us $S(\beta; p, \lambda) = 0$, which means that λ is a zero of the function $S(\beta; p, \cdot)$.
Our next claim is that the operators

$$L^{(q_2)} \text{ with the domain } D(L^{(q_2)}) = \{u \in H^2(\tfrac{a}{2}, a) | u(\tfrac{a}{2}) = 0 = u(a)\}$$

and

$$L^{(\tilde{q}_2)} \text{ with the domain } D(L^{(\tilde{q}_2)}) = \{u \in H^2(0, \tfrac{a}{2}) | u(0) = 0 = u(\tfrac{a}{2})\}$$

have the same eigenvalues. This is easily seen by the fact that $\{u(x), \lambda\}$ is an eigenpair of the operator $L^{(q_2)}$ over the interval $[\frac{a}{2}, a]$ if and only if $\{v(x) = u(a - x), \lambda\}$ is an eigenpair of the operator $L^{(\tilde{q}_2)}$ over the interval $[0, \frac{a}{2}]$.
Using the well known fact from the direct theory of Sturm-Liouville problem that the eigenvalues of the general Sturm-Liouville operator $L^{(p)}$ with the real valued potential function $p \in L^2(\alpha, \beta)$ are countable many (see [13, page 135, Lemma 4.7]), we can now conclude that the functions

$$\begin{cases} \lambda \in \mathbb{C} \to S_2(a; q_2, \lambda), \\ \lambda \in \mathbb{C} \to S_1(\frac{a}{2}; \tilde{q}_2, \lambda) \end{cases}$$

have the same zeros (they are the common Dirichlet eigenvalues of the operators $L^{(q_2)}$ and $L^{(\tilde{q}_2)}$) and these zeros are countable many.

Now we show formula (A.0.1). If the interval in discussion were $[0, 1]$ then the function $S(1; p, \cdot)$ would have had the infinite product representation

$$S(1; p, \lambda) = \prod_{n=1}^{\infty} \frac{z_n - \lambda}{(n\pi)^2}, \text{ for all } \lambda \in \mathbb{C}, \tag{A.0.4}$$

where $\{z_n | n \geq 1\}$ are the zeros of the function

$$\lambda \in \mathbb{C} \to S(1; p, \lambda),$$

or equivalently, the Dirichlet eigenvalues of the operator

$$L^{(p)} \text{ with domain } D(L^{(p)}) = \{u \in H^2(0, 1) | u(0) = 0 = u(1)\}.$$

For the proof of formula (A.0.4) the reader can consult [18, page 39, Theorem 5].

Next we go from the interval $[0,1]$ to the interval $[\alpha,\beta]$ by the change of variables:

$$x \in [0,1] \leftrightarrow \xi = (\beta-\alpha)x + \alpha \in [\alpha,\beta].$$

Introduce the function $\tilde{S}(\xi;p,\lambda)$ by the formula

$$\tilde{S}(\xi;p,\lambda) = S(\frac{\xi-\alpha}{\beta-\alpha};p,\lambda), \text{ for } \xi \in [\alpha,\beta]. \tag{A.0.5}$$

Thus we have

$$\tilde{S}(\xi;p,\lambda) = S(x;p,\lambda),$$

and since $S(\cdot;p,\lambda)$ satisfies the initial value problem consisting of (3.2.1) and (3.2.3) over the interval $[0,1]$, the function $\tilde{S}(\cdot;p,\lambda)$ will satisfy the initial value problem

$$\begin{cases} -\tilde{u}''(\xi) + \hat{p}(\xi)\tilde{u}(\xi) = \hat{\lambda}\tilde{u}(\xi), \ \xi \in (\alpha,\beta) \\ \tilde{u}(\alpha) = 0, \\ \tilde{u}'(\alpha) = \frac{1}{\beta-\alpha}, \end{cases} \tag{A.0.6}$$

where

$$\hat{p}(\xi) = \frac{1}{(\beta-\alpha)^2}p(\frac{\xi-\alpha}{\beta-\alpha}) = \frac{1}{(\beta-\alpha)^2}p(x),$$

and

$$\hat{\lambda} = \frac{\lambda}{(\beta-\alpha)^2}.$$

Next, introduce the function $\hat{S}(\xi;\hat{p},\hat{\lambda})$ by the formula

$$\hat{S}(\xi;\hat{p},\hat{\lambda}) = (\beta-\alpha)\tilde{S}(\xi;p,\lambda), \text{ for } \xi \in [\alpha,\beta]. \tag{A.0.7}$$

Then since $\tilde{S}(\cdot;p,\lambda)$ satisfies the initial value problem (A.0.6), the function $\hat{S}(\cdot;\hat{p},\hat{\lambda})$ will satisfy the initial value problem

$$\begin{cases} -\hat{u}''(\xi) + \hat{p}(\xi)\hat{u}(\xi) = \hat{\lambda}\hat{u}(\xi), \ \xi \in (\alpha,\beta) \\ \hat{u}(\alpha) = 0, \\ \hat{u}'(\alpha) = 1. \end{cases} \tag{A.0.8}$$

We can easily see by (A.0.8) that the zeros of the function

$$\hat{\lambda} \in \mathbb{C} \to \hat{S}(\beta;\hat{p},\hat{\lambda})$$

are exactly the Dirichlet eigenvalues of the Sturm-Liouville operator $L^{(\hat{p})}$ over the interval $[\alpha, \beta]$. So they are countable many. Call them $\hat{z}_n$, for $n \geq 1$. Due to the equivalence between the initial value problems { (3.2.1), (3.2.3)}, (A.0.6), and (A.0.8) (or else due to the fact that $\hat{\lambda}$ is a zero of $\hat{S}(\beta; \hat{p}, \cdot)$ if and only if λ is a zero of $S(1; p, \cdot)$—this is so by formulas (A.0.7), (A.0.5), and the way we defined $\hat{\lambda}$) we can write that

$$\hat{z}_n = \frac{z_n}{(\beta - \alpha)^2}, \tag{A.0.9}$$

where z_n, with $n \geq 1$ are the Dirichlet eigenvalues of the operator $L^{(p)}$ over the interval $[0, 1]$ (or equivalently, the zeros of the function $\lambda \in \mathbb{C} \to S(1; p, \lambda)$).

Now using formulas (A.0.5), (A.0.7), (A.0.4), and (A.0.9), and the way we defined $\hat{\lambda}$ we obtain the infinite product representation of the function

$$\hat{\lambda} \in \mathbb{C} \to \hat{S}(\beta; \hat{p}, \hat{\lambda}) :$$

$$\begin{aligned} \hat{S}(\beta; \hat{p}, \hat{\lambda}) &= (\beta - \alpha)\tilde{S}(\beta; p, \lambda) \\ &= (\beta - \alpha) S(1; p, \lambda) \\ &= (\beta - \alpha) \prod_{n=1}^{\infty} \frac{z_n - \lambda}{(n\pi)^2} \\ &= (\beta - \alpha) \prod_{n=1}^{\infty} \left(\frac{\beta - \alpha}{n\pi} \right)^2 (\hat{z}_n - \hat{\lambda}) \end{aligned} \tag{A.0.10}$$

Now replace in equation (A.0.10) the interval $[\alpha, \beta]$ and the function $\hat{p}$ by the interval $[\frac{a}{2}, a]$ and the function q_2, thus the function $S(\beta; \hat{p}, \cdot)$ becomes the function $S_2(a; q_2, \cdot)$. Then replace in equation (A.0.10) the interval $[\alpha, \beta]$ and the function $\hat{p}$ by the interval $[0, \frac{a}{2}]$ and the function $\tilde{q}_2$, thus the function $S(\beta; \hat{p}, \cdot)$ becomes the function $S_1(\frac{a}{2}; \tilde{q}_2, \cdot)$. Therefore we get the desired formula (A.0.1), because the zeros $\hat{z}_n$'s of the function $\hat{S}(\beta; \hat{p}, \cdot)$ are in each of the two cases the zeros z_n's of the functions $S_2(a; q_2, \cdot)$ and $S_1(\frac{a}{2}; \tilde{q}_2, \cdot)$. (Recall: these latter functions have the same zeros, as we previously showed.)

Now we prove the assertions 3 and 4. We claim that the zeros of the function

$$\lambda \in \mathbb{C} \to C_2(a; q_2, \lambda)$$

are exactly the Neumann-Dirichlet eigenvalues of the Sturm-Liouville operator $L^{(q_2)}$ over the interval $[\frac{a}{2}, a]$, that is they are the eigenvalues of the operator

$$L^{(q_2)} \text{ with the domain } D(L^{(q_2)}) = \{u \in H^2(\tfrac{a}{2}, a) | u'(\tfrac{a}{2}) = 0 = u(a)\}.$$

Let λ be a zero of the function $C_2(a; q_2, \cdot)$. This combined with the fact that $C_2(\cdot; q_2, \lambda)$ is the solution to the initial value problem consisting of (3.2.1) and (3.2.2) where $p = q_2$ and $[\alpha, \beta] = [\frac{a}{2}, a]$ shows that the pair $\{C_2(\cdot; q_2, \lambda), \lambda\}$ is a Neumann-Dirichlet eigenpair of the operator $L^{(q_2)}$ over the interval $[\frac{a}{2}, a]$. So λ is a desired eigenvalue. Conversely, if λ is a Neumann-Dirichlet eigenvalue of the operator $L^{(q_2)}$ over the interval $[\frac{a}{2}, a]$, then there exists a non-zero function $u \in H^2(\frac{a}{2}, a)$ such that

$$\begin{cases} -u''(x) + q_2(x)u(x) = \lambda u(x), \ x \in (\frac{a}{2}, a) \\ u'(\frac{a}{2}) = 0, \\ u(a) = 0. \end{cases} \tag{A.0.11}$$

Hence $u(\frac{a}{2}) \neq 0$ (otherwise, due to the fact that u satisfies the ODE of problem (A.0.11) and the boundary condition at $x = \frac{a}{2}$, u would have been identical zero on $[\frac{a}{2}, a]$ which contradicts the fact that u is an eigenfunction). Therefore, the function

$$\frac{u}{u(\frac{a}{2})}$$

is well defined and satisfies the initial value problem (3.2.1)+ (3.2.2) with $p = q_2$ and $[\alpha, \beta] = [\frac{a}{2}, a]$, whose unique solution is $C_2(\cdot; q_2, \lambda)$. This implies that

$$\frac{u(x)}{u'(\frac{a}{2})} = C_2(x; q_2, \lambda), \text{ for all } x \in [\tfrac{a}{2}, a].$$

From this and the second boundary condition of problem (A.0.11) we obtain that

$$C_2(a; q_2, \lambda) = 0.$$

Thus, λ is a zero of the function $C_2(a; q_2, \cdot)$. Similarly one can show that the zeros of the function

$$\lambda \in \mathbb{C} \to S_1'(\frac{a}{2}; \tilde{q}_2, \lambda)$$

are exactly the Dirichlet-Neumann eigenvalues of the Sturm-Liouville operator $L^{(\tilde{q}_2)}$ over the interval $[0, \frac{a}{2}]$, that is they are the eigenvalues of the operator

$$L^{(\tilde{q}_2)} \text{ with the domain } D(L^{(\tilde{q}_2)}) = \{u \in H^2(0, \tfrac{a}{2}) | u(0) = 0 = u'(\tfrac{a}{2})\}.$$

Now invoking the well known fact from the direct theory of Sturm-Liouville problem that the Dirichlet-Neumann eigenvalues and Neumann-Dirichlet eigenvalues of a Sturm-Liouville operator over the interval $[\alpha, \beta]$ are countable many, we conclude that the two functions $C_2(a; q_2, \cdot)$ and $S_1'(\frac{a}{2}; \tilde{q}_2, \cdot)$ have countable many zeros. Also it is easy to show that $\{u(x), \lambda\}$ is a Neumann-Dirichlet eigenpair of the Sturm-Liouville operator $L^{(q_2)}$ over the interval $[\frac{a}{2}, a]$ if and only if $\{v(x) = u(a - x), \lambda\}$ is a Dirichlet-Neumann eigenpair of the Sturm-Liouville operator $L^{(\tilde{q}_2)}$ over the interval $[0, \frac{a}{2}]$. All these prove so far that the two functions $C_2(a; q_2, \cdot)$ and $S_1'(\frac{a}{2}; \tilde{q}_2, \cdot)$ have the same zeros which are countable many.

The proof of formula (A.0.2) requires much more work. It follows a similar path as the proof of formula (A.0.1), but since there is nothing similar to formula (A.0.4) to quote, we propose two formulas and prove them, thus all the work to derive formula (A.0.2) will follow in depth now.

First we start with the interval $[0, 1]$, then we make all the changes to switch to the interval $[0, a]$.

Here are the two formulas similar to (A.0.4) to be used later:

Claim 1: Let $p \in L^2(0, 1)$ be a real valued function, and $\tilde{p}$ be its reflection about the midpoint $s = \frac{1}{2}$ of the interval $[0, 1]$. That is

$$\tilde{p}(s) = p(1 - s), \text{ for } s \in [0, 1].$$

Then the following hold:

1. the functions

$$\hat{\lambda} \in \mathbb{C} \to C(1; p, \hat{\lambda})$$

   and

$$\hat{\lambda} \in \mathbb{C} \to S'(1; \tilde{p}, \hat{\lambda})$$

   have the same zeros, which are countable many. Call them μ_n, for $n \geq 1$.

2.
$$C(1;p,\hat{\lambda}) = \prod_{n=1}^{\infty} \frac{\mu_n - \hat{\lambda}}{\left((n-\frac{1}{2})\pi\right)^2} = S'(1;\tilde{p},\hat{\lambda}), \text{ for all } \hat{\lambda} \in \mathbb{C}. \tag{A.0.12}$$

proof of Claim 1: The proof of assertion 1 is the same as the one for the functions $C_2(a; q_2, \cdot)$ and $S_1'(\frac{a}{2}; \tilde{q}_2, \cdot)$ and will not be repeated here. The work for the formula (A.0.12) follows. Let $\hat{\lambda} \in \mathbb{C}$ be chosen and fixed. From the direct theory of Sturm-Liouville problem (see [13, page 132, Theorem 4.5]) we know that

$$|C(1;p,\hat{\lambda}) - \cos\sqrt{\hat{\lambda}}| \leq \frac{1}{|\sqrt{\hat{\lambda}}|} \exp\left(|Im\sqrt{\hat{\lambda}}| + \int_0^1 |p(s)|ds\right) \tag{A.0.13}$$

and

$$|S'(1;\tilde{p},\hat{\lambda}) - \cos\sqrt{\hat{\lambda}}| \leq \frac{1}{|\sqrt{\hat{\lambda}}|} \exp\left(|Im\sqrt{\hat{\lambda}}| + \int_0^1 |\tilde{p}(s)|ds\right). \tag{A.0.14}$$

Observing that

$$\int_0^1 |p(s)|ds = \int_0^1 |\tilde{p}(s)|ds \text{ (because } \tilde{p}(s) = p(1-s),\ s \in [0,1])$$

we can denote by

$$C = \exp\left(\int_0^1 |p(s)|ds\right) = \exp\left(\int_0^1 |\tilde{p}(s)|ds\right),$$

and write (A.0.13) and (A.0.14) in the form:

$$|C(1;p,\hat{\lambda}) - \cos\sqrt{\hat{\lambda}}| \leq \frac{C}{|\sqrt{\hat{\lambda}}|} \exp\left(|Im\sqrt{\hat{\lambda}}|\right) \tag{A.0.15}$$

and

$$|S'(1;\tilde{p},\hat{\lambda}) - \cos\sqrt{\hat{\lambda}}| \leq \frac{C}{|\sqrt{\hat{\lambda}}|} \exp\left(|Im\sqrt{\hat{\lambda}}|\right). \tag{A.0.16}$$

It is known from complex analysis theory (see [4, page 176, Exercise 1]) that

$$\cos\sqrt{\hat{\lambda}} = \prod_{n=1}^{\infty} \frac{\left((n-\frac{1}{2})\pi\right)^2 - \hat{\lambda}}{\left((n-\frac{1}{2})\pi\right)^2}. \tag{A.0.17}$$

Roughly, formulas (A.0.15) and (A.0.16) tell us that the functions

$$\begin{cases} \hat{\lambda} \in \mathbb{C} \to C(1;p,\hat{\lambda}) \\ \hat{\lambda} \in \mathbb{C} \to S'(1;\tilde{p},\hat{\lambda}) \end{cases}$$

behave as

$$\hat{\lambda} \in \mathbb{C} \to \cos\sqrt{\hat{\lambda}}.$$

This gives us encouragements that (A.0.12) holds, because the zeros $\{\mu_n | n \geq 1\}$ of $C(1; p, \cdot)$ and $S'(1; \tilde{p}, \cdot)$ behave asymptotically as the zeros $\left\{ \left((n - \frac{1}{2})\pi\right)^2 | n \geq 1 \right\}$ of $\cos\sqrt{\cdot}$. More precisely, due to the fact that μ_n's, the zeros of $C(1; p, \cdot)$, are the Neumann-Dirichlet eigenvalues of the Sturm-Liouville operator $L^{(p)}$ over the interval $[0, 1]$, and the same μ_n's, the zeros of $S'(1; \tilde{p}, \cdot)$, are the Dirichlet-Neumann eigenvalues of the Sturm-Liouville operator $L^{(\tilde{p})}$ over the interval $[0, 1]$, the following asymptotic formula (see [13, pages 144-145, Theorem 4.14]) holds:

$$\begin{aligned} \mu_n &= \left((n - \frac{1}{2})\pi\right)^2 + \int_0^1 p(s)ds + \alpha_n, \\ &= \left((n - \frac{1}{2})\pi\right)^2 + \int_0^1 \tilde{p}(s)ds + \alpha_n, \text{ as } n \to \infty, \end{aligned}$$

since $\tilde{p}(s) = p(1 - s)$, for $s \in [0, 1]$. Above, the sequence $(\alpha_n)_{n \geq 1}$ is an l_2 sequence, and hence converges to 0, so bounded. From this, it follows that

$$\mu_n = \left((n - \frac{1}{2})\pi\right)^2 + \mathcal{O}(1), \text{as } n \to \infty. \tag{A.0.18}$$

A cascade of auxiliary results is coming up (their proofs are postponed until after the end of the proof of Lemma 2), and we shall show how to connect them to arrive at (A.0.12):

- due to (A.0.18), the infinite product

  $$\prod_{n=1}^{\infty} \frac{\mu_n - \hat{\lambda}}{\left((n - \frac{1}{2})\pi\right)^2}$$

  converges to an entire function of $\hat{\lambda}$, whose zeros are exactly $\{\mu_n | n \geq 1\}$. Call it $P(\hat{\lambda})$.

- the function $P(\hat{\lambda})$ satisfies the asymptotic formula

  $$P(\hat{\lambda}) = \cos\sqrt{\hat{\lambda}} \left(1 + \mathcal{O}(\left(\frac{\ln m}{m}\right))\right), \text{ uniformly on the circles } |\hat{\lambda}| = (m\pi)^2,\ m \in \mathbb{Z}, \tag{A.0.19}$$

- for all $z \in \mathbb{C}$ such that $|z - \left(n - \frac{1}{2}\right)\pi| \geq \frac{\pi}{4}$, for all $n \in \mathbb{Z}$ the following is true:

  $$\exp(|Imz|) < 4|\cos z| \tag{A.0.20}$$

Now let $m \in \mathbb{Z}$ be chosen and fixed. Let $\hat{\lambda}$ be chosen on the circle $|\hat{\lambda}| = (m\pi)^2$. Let $z = \sqrt{\hat{\lambda}}$ be either one of the two complex squared roots of $\hat{\lambda}$. Let $n \in \mathbb{Z}$ be arbitrary. Then $|z| = |m|\pi$

and using the well known triangle inequality

$$|z_1 - z_2| \geq ||z_1| - |z_2||,$$

we have the following:

$$\begin{aligned}
|z - \left(n - \frac{1}{2}\right)\pi| &\geq ||z| - |\left(n - \frac{1}{2}\right)\pi|| \\
&= ||m| - |\left(n - \frac{1}{2}\right)||\pi \\
&= \begin{cases} ||m| - \left(n - \frac{1}{2}\right)|\pi, & \text{if } n \geq 1, \\ ||m| + \left(n - \frac{1}{2}\right)|\pi, & \text{if } n \leq 0 \end{cases} \\
&= \begin{cases} |(|m| - n) + \frac{1}{2}|\pi, & \text{if } n \geq 1, \\ |(|m| + n) - \frac{1}{2}|\pi, & \text{if } n \leq 0 \end{cases} \\
&\geq \begin{cases} |||m| - n| - \frac{1}{2}|\pi, & \text{if } n \geq 1, \\ |||m| + n| - \frac{1}{2}|\pi, & \text{if } n \leq 0 \end{cases} \\
&= |||m| - |n|| - \frac{1}{2}|\pi \\
&\geq \frac{\pi}{2}. \qquad \text{(A.0.21)}
\end{aligned}$$

The last inequality in (A.0.21) comes from the fact that m, n are integers, so

$$||m| - |n|| \geq \begin{cases} 1, & \text{if } n \in \mathbb{Z} \text{ is such that } |n| \neq |m| \\ 0, & \text{if } n \in \mathbb{Z} \text{ is such that } |n| = |m|. \end{cases}$$

The calculations in (A.0.21), tell us that for $\hat{\lambda}$ on the circle $|\hat{\lambda}| = (m\pi)^2$, either one of its two complex squared roots

$$z = \sqrt{\hat{\lambda}}$$

is such that

$$|z - \left(n - \frac{1}{2}\right)| \geq \frac{\pi}{2} \geq \frac{\pi}{4}, \text{ for all } n \in \mathbb{Z},$$

so formula (A.0.20) applies, and due to formulas (A.0.15) and (A.0.16) we can write:

$$C(1; p, \hat{\lambda}) = \cos\sqrt{\hat{\lambda}}\left(1 + \mathcal{O}\left(\frac{1}{m}\right)\right), \text{ uniformly on the circles } |\hat{\lambda}| = (m\pi)^2,\ m \in \mathbb{Z}, \qquad \text{(A.0.22)}$$

$$S'(1;\tilde{p},\hat{\lambda}) = \cos\sqrt{\hat{\lambda}}\left(1+\mathcal{O}\left(\frac{1}{m}\right)\right), \text{ uniformly on the circles } |\hat{\lambda}| = (m\pi)^2,\ m\in\mathbb{Z}, \quad \text{(A.0.23)}$$

Formula (A.0.22) together with (A.0.19), and formula (A.0.23) together with (A.0.19) produce:

$$\frac{P(\hat{\lambda})}{C(1;p,\hat{\lambda})} = 1+\mathcal{O}\left(\frac{\ln m}{m}\right), \text{ uniformly on the circles } |\hat{\lambda}| = (m\pi)^2,\ m\geq 1, \qquad \text{(A.0.24)}$$

and

$$\frac{P(\hat{\lambda})}{S'(1;\tilde{p},\hat{\lambda})} = 1+\mathcal{O}\left(\frac{\ln m}{m}\right), \text{ uniformly on the circles } |\hat{\lambda}| = (m\pi)^2,\ m\geq 1. \qquad \text{(A.0.25)}$$

(Note that the functions $C(1;p,\hat{\lambda})$, $S'(1;\tilde{p},\hat{\lambda})$ and $P(\hat{\lambda})$ all have the same zeros, which are μ_n's, therefore the quotients

$$\begin{cases} \frac{P(\hat{\lambda})}{C(1;p,\hat{\lambda})} \\ \frac{P(\hat{\lambda})}{S'(1;\tilde{p},\hat{\lambda})} \end{cases}$$

are well defined and entire functions of $\hat{\lambda}$.)

Formulas (A.0.24) and (A.0.25) give further:

$$\sup_{|\hat{\lambda}|=(m\pi)^2} \left|\frac{P(\hat{\lambda})}{C(1;p,\hat{\lambda})}-1\right| \to 0, \text{ as } m\to\infty$$

and

$$\sup_{|\hat{\lambda}|=(m\pi)^2} \left|\frac{P(\hat{\lambda})}{S'(1;\tilde{p},\hat{\lambda})}-1\right| \to 0, \text{ as } m\to\infty.$$

From these, using the Maximum Modulus Theorem (see [4, Theorem 1.2, page 128]), we obtain

$$\sup_{|\hat{\lambda}|\leq(m\pi)^2} \left|\frac{P(\hat{\lambda})}{C(1;p,\hat{\lambda})}-1\right| = \sup_{|\hat{\lambda}|=(m\pi)^2} \left|\frac{P(\hat{\lambda})}{C(1;p,\hat{\lambda})}-1\right| \to 0, \text{ as } m\to\infty \qquad \text{(A.0.26)}$$

and

$$\sup_{|\hat{\lambda}|\leq(m\pi)^2} \left|\frac{P(\hat{\lambda})}{S'(1;\tilde{p},\hat{\lambda})}-1\right| = \sup_{|\hat{\lambda}|=(m\pi)^2} \left|\frac{P(\hat{\lambda})}{S'(1;\tilde{p},\hat{\lambda})}-1\right| \to 0, \text{ as } m\to\infty. \qquad \text{(A.0.27)}$$

Formula (A.0.26) implies that for a given constant, $\gamma>0$, there exists a first positive index m_γ such that

$$\sup_{|\hat{\lambda}|\leq(m\pi)^2} \left|\frac{P(\hat{\lambda})}{C(1;p,\hat{\lambda})}-1\right| \leq \gamma, \text{ for all } m\geq m_\gamma. \qquad \text{(A.0.28)}$$

The function

$$\hat{\lambda} \in \mathbb{C} \to \frac{P(\hat{\lambda})}{C(1; p, \hat{\lambda})} - 1 \tag{A.0.29}$$

is an entire function, therefore:

$$\left|\frac{P(\hat{\lambda})}{C(1; p, \hat{\lambda})} - 1\right| \leq \delta, \text{ for all } |\hat{\lambda}| \leq (m_\gamma \pi)^2 \tag{A.0.30}$$

If $\hat{\lambda} \in \mathbb{C}$ is arbitrary and fixed, then either

$$|\hat{\lambda}| \leq (m_\gamma \pi)^2,$$

or else

$$(m_\gamma \pi)^2 < |\hat{\lambda}| \leq (m\pi)^2, \text{ for some } m > m_\gamma$$

(possible because $(m\pi)^2 \to \infty$). Hence, either (A.0.30) or (A.0.28) applies and we get that

$$\left|\frac{P(\hat{\lambda})}{C(1; p, \hat{\lambda})} - 1\right| \leq \max(\delta, \gamma).$$

Thus we showed that the entire function (A.0.29) is bounded in $\mathbb{C}$. It follows by Liouville Theorem (see [4, Theorem 3.4, page 77]) that

$$\frac{P(\hat{\lambda})}{C(1; p, \hat{\lambda})} - 1 = C, \text{ for all } \hat{\lambda} \in \mathbb{C},$$

for some constant C, which in combination with (A.0.26) yields

$$\frac{P(\hat{\lambda})}{C(1; p, \hat{\lambda})} - 1 = C = 0, \text{ for all } \hat{\lambda} \in \mathbb{C}, \tag{A.0.31}$$

Similarly, one shows that

$$\frac{P(\hat{\lambda})}{S'(1; \tilde{p}, \hat{\lambda})} - 1 = \hat{C} = 0, \text{ for all } \hat{\lambda} \in \mathbb{C}. \tag{A.0.32}$$

From (A.0.31), (A.0.32) and the definition of $P(\hat{\lambda})$ we get (A.0.12). This ends the proof of *Claim 1.*

Claim 2: The following hold:

$$C_2(x; q_2, \lambda) = C(s; p, \hat{\lambda}), \text{ for } s \in [0,1] \text{ and } x = \tfrac{a}{2}(s+1) \in [\tfrac{a}{2}, a], \tag{A.0.33}$$

where

$$\begin{cases} p(s) = \left(\frac{a}{2}\right)^2 \cdot q_2(x), \text{ with } x = \frac{a}{2}(s+1), \\ \hat{\lambda} = \left(\frac{a}{2}\right)^2 \cdot \lambda, \end{cases} \tag{A.0.34}$$

and

$$S_1'(\tilde{x}; \tilde{q}_2, \lambda) = S'(s; \hat{p}, \hat{\lambda}), \text{ for } s \in [0,1] \text{ and } \tilde{x} = \tfrac{a}{2}s \in [0, \tfrac{a}{2}], \tag{A.0.35}$$

where

$$\begin{cases} \hat{p}(s) = \left(\frac{a}{2}\right)^2 \cdot \tilde{q}_2(\tilde{x}), \text{ with } \tilde{x} = \frac{a}{2}s, \\ \hat{\lambda} = \left(\frac{a}{2}\right)^2 \cdot \lambda. \end{cases} \tag{A.0.36}$$

proof of Claim 2: Making the change of variables

$$s \in [0,1] \to x = \frac{a}{2}(s+1) \in [\frac{a}{2}, a]$$

and

$$v(s) = C_2(\frac{a}{2}(s+1); q_2, \lambda) = C_2(x; q_2, \lambda), \tag{A.0.37}$$

and using the fact that $C_2(x; q_2, \lambda)$ satisfies the initial value problem:

$$\begin{cases} -u''(x) + q_2(x)u(x) = \lambda u(x), \ x \in (\frac{a}{2}, a) \\ u(\frac{a}{2}) = 1, \\ u'(\frac{a}{2}) = 0, \end{cases}$$

we obtain that $v(s)$ satisfies the initial value problem:

$$\begin{cases} -v''(s) + p(s)v(s) = \hat{\lambda}v(s), \ s \in (0,1) \\ v(0) = 1, \\ v'(0) = 0, \end{cases}$$

where $p(s)$ and $\hat{\lambda}$ are as in (A.0.34). Since the latter problem has only one solution, which is $C(s; p, \hat{\lambda})$, it follows that $v(s) = C(s, p, \hat{\lambda})$, which combined with relation (A.0.37) gives exactly formula (A.0.33).

Next, making the change of variables

$$s \in [0,1] \to \tilde{x} = \frac{a}{2}s \in [0, \frac{a}{2}]$$

and

$$v(s) = S_1(\frac{a}{2}s; \tilde{q}_2, \lambda) = S_1(\tilde{x}; \tilde{q}_2, \lambda), \tag{A.0.38}$$

and using the fact that $S_1(\tilde{x}; \tilde{q}_2, \lambda)$ satisfies the initial value problem:

$$\begin{cases} -u''(\tilde{x}) + \tilde{q}_2(\tilde{x})u(\tilde{x}) = \lambda u(\tilde{x}), \ \ \tilde{x} \in (0, \frac{a}{2}) \\ u(0) = 0, \\ u'(0) = 1, \end{cases}$$

we obtain that $v(s)$ satisfies the initial value problem:

$$\begin{cases} -v''(s) + \hat{p}(s)v(s) = \hat{\lambda}v(s), \ \ s \in (0, 1) \\ v(0) = 0, \\ v'(0) = \frac{a}{2}, \end{cases}$$

where $\hat{p}(s)$ and $\hat{\lambda}$ are as in (A.0.36).

Then introduce the function

$$w(s) = \frac{v(s)}{\frac{a}{2}}, \ \text{for } s \in [0, 1]. \tag{A.0.39}$$

It follows that $w(s)$ satisfies the initial value problem:

$$\begin{cases} -w''(s) + \hat{p}(s)w(s) = \hat{\lambda}w(s), \ \ s \in (0, 1) \\ w(0) = 0, \\ w'(0) = 1, \end{cases}$$

whose unique solution is $S(s; \hat{p}, \hat{\lambda})$. Therefore $w(s) = S(s; \hat{p}, \hat{\lambda})$ and then

$$\begin{aligned} S'(s; \hat{p}, \hat{\lambda}) &= w'(s) \\ &= \frac{v'(s)}{\frac{a}{2}}, \ \text{by (A.0.39)} \\ &= S_1'(\tilde{x}; \tilde{q}_2, \lambda), \ \text{by (A.0.38)} \end{aligned}$$

which is actually the identity (A.0.35). This finishes the proof of *Claim 2*.

Using *Claim 1* and *Claim 2* we can complete the proof of formula (A.0.2). Due to the fact that $\tilde{q}_2$ is the reflection of q_2 about the midpoint $x = \frac{a}{2}$ of the interval $[0, a]$, and formulas (A.0.34) and (A.0.36) we can show that $\hat{p}$ is the reflection of p about the midpoint $s = \frac{1}{2}$

of the interval $[0,1]$:

$$\begin{aligned}
\hat{p}(s) &= \left(\frac{a}{2}\right)^2 \cdot \tilde{q}_2(\tilde{x}) \\
&= \left(\frac{a}{2}\right)^2 \cdot \tilde{q}_2(\frac{a}{2}s), \text{ by (A.0.36)} \\
&= \left(\frac{a}{2}\right)^2 \cdot q_2(a - \frac{a}{2}s), \text{ because } \tilde{q}_2(x) = q_2(a-x), \text{ for } x \in [0, \tfrac{a}{2}] \\
&= \left(\frac{a}{2}\right)^2 \cdot q_2(\frac{a}{2} + \frac{a}{2}(1-s)) \\
&= p(1-s), \text{ by (A.0.34)} \qquad \text{(A.0.40)}
\end{aligned}$$

Now take in (A.0.33) $s = 1$, so $x = a$; take in (A.0.35) $s = 1$, so $\tilde{x} = \frac{a}{2}$; and use (A.0.12) and (A.0.40) to write:

$$\begin{aligned}
C_2(a; q_2, \lambda) &= C(1; p, \hat{\lambda}) \\
&= \prod_{n=1}^{\infty} \frac{\mu_n - \hat{\lambda}}{\left((n - \frac{1}{2})\pi\right)^2} \\
&= S'(1; \hat{p}, \hat{\lambda}) \\
&= S_1'(\frac{a}{2}; \tilde{q}_2, \lambda), \qquad \text{(A.0.41)}
\end{aligned}$$

where μ_n's are the zeros of $C(1; p, \cdot)$ and of $S'(1; \hat{p}, \cdot)$. Due to the identities in (A.0.41) and the fact that

$$\hat{\lambda} = \left(\frac{a}{2}\right)^2 \cdot \lambda,$$

the zeros w_n's of $C_2(a; q_2, \cdot)$ and of $S_1'(\frac{a}{2}; \tilde{q}_2, \cdot)$ are related to μ_n's by

$$\mu_n = \left(\frac{a}{2}\right)^2 \cdot\ w_n.$$

Therefore (A.0.41) becomes (A.0.2). $\square$

APPENDIX B. AUXILIARY RESULTS FOR APPENDIX A

Here we re-state and prove the three auxiliary results used to prove (A.0.12) of *Claim 1*.

Auxiliary result 11 *Let* $\mu_n = \left((n-\frac{1}{2})\pi\right)^2 + \mathcal{O}(1),$ *as* $n \to \infty$. *Then the infinite product*

$$\prod_{n=1}^{\infty} \frac{\mu_n - \hat{\lambda}}{\left((n-\frac{1}{2})\pi\right)^2}$$

converges to an entire function of $\hat{\lambda}$, *whose zeros are exactly* $\{\mu_n | n \geq 1\}$. *Call it* $P(\hat{\lambda})$.

Proof: A key ingredient in the proof is the inequality:

$$e^x \geq x + 1, \text{ for all } x \geq 0. \tag{B.0.1}$$

(It can be proved by the fact that the function $f(x) = e^x - x - 1$ has a positive derivative on $[0, \infty)$ and hence it is an increasing function on $[0, \infty)$.) Then for $\hat{\lambda}$ in bounded subsets of $\mathbb{C}$ (i.e. $|\hat{\lambda}| \leq C$) the following hold:

$$\begin{aligned}
|\prod_{n=1}^{\infty} \frac{\mu_n - \hat{\lambda}}{\left((n-\frac{1}{2})\pi\right)^2}| &= |\prod_{n=1}^{\infty}\left(1 + \frac{\mu_n - \hat{\lambda} - \left((n-\frac{1}{2})\pi\right)^2}{\left((n-\frac{1}{2})\pi\right)^2}\right)| \\
&= \prod_{n=1}^{\infty} |1 + \frac{\mu_n - \hat{\lambda} - \left((n-\frac{1}{2})\pi\right)^2}{\left((n-\frac{1}{2})\pi\right)^2}| \\
&\leq \prod_{n=1}^{\infty}\left(1 + \frac{|\mu_n - \hat{\lambda} - \left((n-\frac{1}{2})\pi\right)^2|}{\left((n-\frac{1}{2})\pi\right)^2}\right) \\
&\leq \prod_{n=1}^{\infty}\left(1 + \frac{|\mu_n - \left((n-\frac{1}{2})\pi\right)^2| + |\hat{\lambda}|}{\left((n-\frac{1}{2})\pi\right)^2}\right)
\end{aligned} \tag{B.0.2}$$

Then using (B.0.1) with $x = \frac{|\mu_n - ((n-\frac{1}{2})\pi)^2| + |\hat{\lambda}|}{((n-\frac{1}{2})\pi)^2}$ in (B.0.2) the following are obtained:

$$
\begin{aligned}
\left|\prod_{n=1}^{\infty} \frac{\mu_n - \hat{\lambda}}{\left((n-\frac{1}{2})\pi\right)^2}\right| &\le \prod_{n=1}^{\infty} e^{\frac{|\mu_n - ((n-\frac{1}{2})\pi)^2| + |\hat{\lambda}|}{((n-\frac{1}{2})\pi)^2}} \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{|\mu_n - \left((n-\frac{1}{2})\pi\right)^2| + |\hat{\lambda}|}{\left((n-\frac{1}{2})\pi\right)^2}\right) \\
&\le \exp\left(\sum_{n=1}^{\infty} \frac{C}{\left((n-\frac{1}{2})\pi\right)^2}\right), \text{ because } \mu_n = \left((n-\tfrac{1}{2})\right)^2 + \mathcal{O}(1), \text{ and } |\hat{\lambda}| \le C \\
&\sim \exp\left(\sum_{n=1}^{\infty} \frac{C}{n^2}\right) \\
&< \infty \qquad \text{(B.0.3)}
\end{aligned}
$$

Formula (B.0.3) shows that the infinite product

$$\prod_{n=1}^{\infty} \frac{\mu_n - \hat{\lambda}}{\left((n-\frac{1}{2})\pi\right)^2}$$

converges on bounded subsets of $\mathbb{C}$, and therefore converges to an entire function of $\hat{\lambda}$. Clearly, its zeros are only μ_n's. $\square$

Auxiliary result 12 *The function $P(\hat{\lambda})$ satisfies the asymptotic formula (A.0.19) mentioned in the proof of* Claim 1. *That is*

$$P(\hat{\lambda}) = \cos\sqrt{\hat{\lambda}}\left(1 + \mathcal{O}(\left(\frac{\ln m}{m}\right))\right), \text{ uniformly on the circles } |\hat{\lambda}| = (m\pi)^2,\ m \in \mathbb{Z},$$

Proof: Let $m \in \mathbb{Z}$ be chosen and fixed. Pick any $\hat{\lambda}$ on the circle $|\hat{\lambda}| = (m\pi)^2$. From the definition of $P(\hat{\lambda})$ and formula (A.0.17) we can write:

$$
\begin{aligned}
\frac{P(\hat{\lambda})}{\cos\sqrt{\hat{\lambda}}} &= \prod_{n=1}^{\infty} \frac{\mu_n - \hat{\lambda}}{\left((n-\frac{1}{2})\pi\right)^2 - \hat{\lambda}} \\
&= \prod_{n=1}^{\infty} \left(1 + \frac{\mu_n - \left((n-\frac{1}{2})\pi\right)^2}{\left((n-\frac{1}{2})\pi\right)^2 - \hat{\lambda}}\right) \qquad \text{(B.0.4)}
\end{aligned}
$$

Introduce the notation

$$a_n = \frac{\mu_n - \left((n-\frac{1}{2})\pi\right)^2}{\left((n-\frac{1}{2})\pi\right)^2 - \hat{\lambda}}. \qquad \text{(B.0.5)}$$

It follows:

$$
\begin{aligned}
|a_n| &\leq \frac{C}{|\left((n-\frac{1}{2})\pi\right)^2 - \hat{\lambda}|}, \text{ because } \mu_n = \left((n-\tfrac{1}{2})\pi\right)^2 + \mathcal{O}(1) \\
&\leq \frac{C}{|\left((n-\frac{1}{2})\pi\right)^2 - |\hat{\lambda}||}, \text{ by } |z_1 - z_2| \geq ||z_1| - |z_2|| \\
&\leq \frac{C}{|\left((n-\frac{1}{2})\pi\right)^2 - (m\pi)^2|}, \text{ since } |\hat{\lambda}| = (m\pi)^2.
\end{aligned}
\tag{B.0.6}
$$

The calculations in (B.0.6) mean that:

$$
|a_n| = \begin{cases} \mathcal{O}\left(\frac{1}{|n^2-m^2|}\right), & \text{if } n \neq m \\ \mathcal{O}\left(\frac{1}{m}\right), & \text{if } n = m. \end{cases}
\tag{B.0.7}
$$

Finally, equations (B.0.4), (B.0.5), and (B.0.7), along with Lemma 1 of [18, page 165] yield:

$$
\begin{aligned}
\frac{P(\hat{\lambda})}{\cos\sqrt{\hat{\lambda}}} &= (1+a_m) \cdot \prod_{n\geq 1, n\neq m}^{\infty} (1+a_n) \\
&= \left(1 + \mathcal{O}\left(\frac{1}{m}\right)\right)\left(1 + \mathcal{O}\left(\frac{\ln m}{m}\right)\right) \\
&= 1 + \mathcal{O}\left(\frac{\ln m}{m}\right),
\end{aligned}
$$

from which the desired formula (A.0.19) follows. □

Auxiliary result 13 *For all $z \in \mathbb{C}$ such that*

$$
|z - \left(n - \frac{1}{2}\right)\pi| \geq \frac{\pi}{4}, \ \textit{for all } n \in \mathbb{Z},
$$

formula (A.0.20) mentioned in the proof of Claim 1 *is true. That is*

$$
\exp(|Imz|) < 4|\cos z|.
$$

Proof: We adapt the proof of Lemma 4.8 of [13, page 136]. Let $z \in \mathbb{C}$ be as above. Write

$$
z = x + iy, \text{ where } x, \, y \in \mathbb{R},
$$

and start from the well known complex analysis formula:

$$
\begin{aligned}
\cos z &= \frac{e^{iz} + e^{-iz}}{2} \\
&= \frac{e^{-y+ix} + e^{y-ix}}{2}.
\end{aligned}
\tag{B.0.8}
$$

case 1: $|y| > \frac{\ln 2}{2}$. A chain of equalities and inequalities follows:

$$
\begin{aligned}
\frac{\exp(|Imz|)}{|\cos z|} &= \frac{2e^{|y|}}{|e^{-y+ix}+e^{y-ix}|} \\
&\leq \frac{2e^{|y|}}{||e^{-y+ix}|-|e^{y-ix}||} \text{ (by } |z_1+z_2| \geq ||z_1|-|z_2||) \\
&= \frac{2e^{|y|}}{|e^{-y}-e^{y}|} \\
&= \begin{cases} \frac{2e^{|y|}}{e^{-y}-e^{y}}, \text{ if } y<0 \\ \frac{2e^{|y|}}{e^{y}-e^{-y}}, \text{ if } y>0 \\ \text{(note: } y=0 \text{ cannot happen in } \textit{case 1}) \end{cases} \\
&= \frac{2e^{|y|}}{e^{|y|}-e^{-|y|}} \\
&= \frac{2}{1-e^{-2|y|}} \\
&< 4 \text{ (since } |y| > \tfrac{\ln 2}{2}).
\end{aligned}
$$

Thus we got formula (A.0.20).

case 2: $|y| \leq \frac{\ln 2}{2}$. From the fact that

$$|z-\left(n-\frac{1}{2}\right)\pi| \geq \frac{\pi}{4}, \text{ for all } n \in \mathbb{Z}$$

we write that:

$$\left(x-\left(n-\frac{1}{2}\right)\pi\right)^2 + y^2 = |z-\left(n-\frac{1}{2}\right)\pi|^2 \geq \frac{\pi^2}{16},$$

so

$$
\begin{aligned}
\left(x-\left(n-\frac{1}{2}\right)\pi\right)^2 &\geq \frac{\pi^2}{16} - y^2 \\
&\geq \frac{\pi^2}{16} - \left(\frac{\ln 2}{2}\right)^2, \text{ because } |y| \leq \tfrac{\ln 2}{2} \\
&\geq \frac{\pi^2}{64},
\end{aligned}
$$

hence

$$|x-\left(n-\frac{1}{2}\right)\pi| \geq \frac{\pi}{8}, \text{ for all } n \in \mathbb{Z},$$

from which we obtain that

$$\begin{aligned}
|\cos x| &\geq \cos\left(\frac{\pi}{2}-\frac{\pi}{8}\right) \\
&= \sin\left(\frac{\pi}{8}\right) \\
&= \frac{\sqrt{2-\sqrt{2}}}{2}.
\end{aligned} \tag{B.0.9}$$

To convince the reader that formula (B.0.9) holds, Figure B.1 is provided at the end of the proof. On the other hand, from (B.0.8) we obtain that:

$$\begin{aligned}
|\cos z| &= |\cos x\left(\frac{e^{-y}+e^{y}}{2}\right)+i\sin x\left(\frac{e^{-y}-e^{y}}{2}\right)| \\
&\geq |\cos x|\left(\frac{e^{-y}+e^{y}}{2}\right) \\
&\geq |\cos x|\left(\frac{e^{|y|}+e^{-|y|}}{2}\right).
\end{aligned} \tag{B.0.10}$$

Now using (B.0.9) and (B.0.10) we calculate:

$$\begin{aligned}
\frac{\exp(|Imz|)}{|\cos z|} &\leq \frac{2e^{|y|}}{e^{|y|}+e^{-|y|}}\cdot\frac{1}{|\cos x|},\ \text{by (B.0.10)} \\
&= \frac{2}{1+e^{-2|y|}}\cdot\frac{1}{|\cos x|} \\
&\leq \frac{2}{1+e^{-2|y|}}\cdot\frac{2}{\sqrt{2-\sqrt{2}}},\ \text{by (B.0.9)} \\
&\leq \frac{4}{3}\cdot\frac{2}{\sqrt{2-\sqrt{2}}},\ \text{because } |y|\leq\tfrac{\ln 2}{2} \\
&< 4,
\end{aligned}$$

which is the stated inequality (A.0.20). □

In Figure B.1, the red, thick lines on the x-axis represent part of the set

$$\bigcup_{n\in\mathbb{Z}}\{x\in\mathbb{R}||x-(n-\frac{1}{2})\pi|<\frac{\pi}{8}\}.$$

Hence it is clear that for x not on the red, horizontal lines (i.e. $|x-(n-\frac{1}{2})\pi|\geq\frac{\pi}{8}$) the graph of the function $y=|\cos x|$ lies above the line $y=\cos(\frac{\pi}{2}-\frac{\pi}{8})$, which is the black, thick, horizontal line in Figure B.1. That means that for $x\in\mathbb{R}$ such that

$$|x-(n-\frac{1}{2})\pi|\geq\frac{\pi}{8},\ \text{for all } n\in\mathbb{Z},$$

formula (B.0.9) holds.

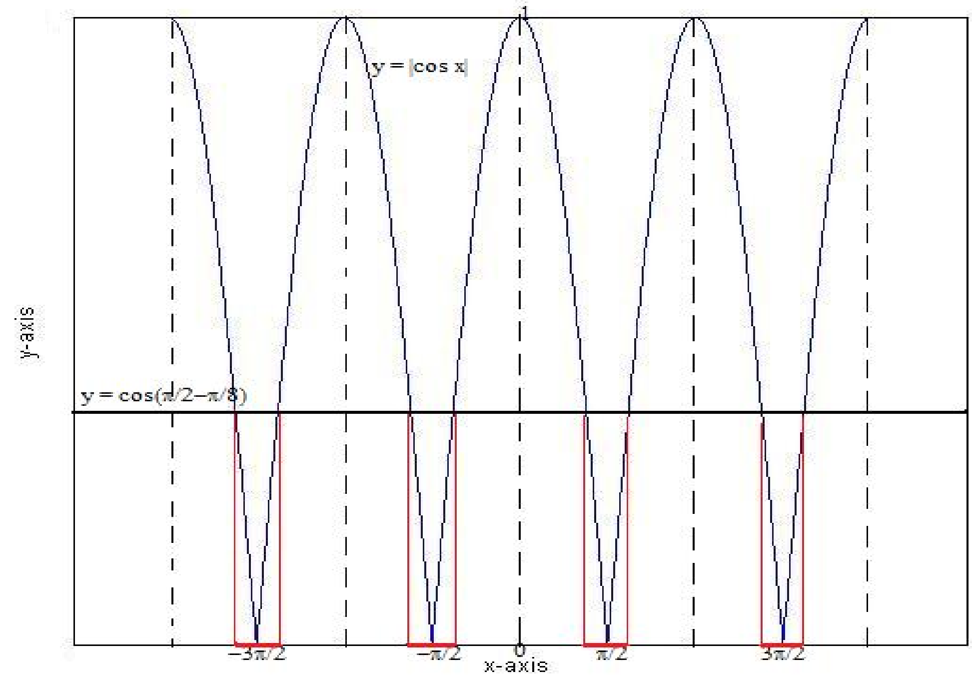

Figure B.1 The graph of the function $y = |\cos x|$

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