Moment method in distributed control theory

by

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ABSTRACT

The aim of this thesis is to present and solve moment problems which arise from controllability questions posed on the wave equation and the Mead-Markus model of a beam.

CHAPTER 1. Introduction

In this thesis, we try to examine the controllability of mild solutions of scalar control systems of the following form:

$$\dot{x}(t) = \mathcal{A}x(t) + bu(t) \tag{1.1}$$

$$x(0) = x_0. \tag{1.2}$$

Controllability of Equation (1.1) can be equivalently stated as a moment problem of the following form.

$$< f, f_i >= c_i, \ \ i = 1, 2, 3... \tag{1.3}$$

where f_i's are given functions belonging to a certain Hilbert or Banach space and $\{c_i\}$ is a given sequence. The controllability of Equation (1.1) is equivalent to the existence of a function f which solves (1.3).

The methods used to solve (1.3) often depends on the type of the PDE that describes the distributed parameter system and often involves techniques from functional analysis. Our main interest is in characterizing the set of all states that are exactly controllable for the Mead-Markus model of a sandwich beam. For motivational purposes, the wave equation with and without damping is considered first.

The thesis is organized as follows. In section 1.1, semigroup theory is briefly discussed and the notion of controllability is introduced. In chapter 2 and 3, the wave equation with and without damping is analyzed for exact controllability for motivational purposes. Finally, in chapter 4 the Mead-Markus model of a sandwiched beam (without damping) is analyzed for exact controllability.

1.1 Preliminaries

In this section we briefly describe a few elements of semigroup theory. A detailed treatment of the same can be found in (A. Pazy).

Definition 1.1.1 *Let X be a Banach space. A one parameter family $T(t) : X \to X, 0 \leq t < \infty$, of bounded linear operators is a semigroup on X if*

1. *$T(0) = I$ (I is the identity operator) on X*

2. *$T(t+s) = T(t)(s)$ $\forall t, s \geq 0$.*

Definition 1.1.2 *A semigroup $T(t), 0 \leq t \leq \infty$, of bounded linear operators is a strongly continuous semigroup if*

$$Lim_{t \downarrow 0} T(t)x = x, \forall x \in X. \tag{1.4}$$

A strongly continuous semigroup is sometimes called as a C_0 semigroup.

Definition 1.1.3 *$\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $\{T(t)\}_{t \geq 0}$ if*

$$\mathcal{A}x = Lim_{t \downarrow 0} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt}|_{t=0} \tag{1.5}$$

$$D(\mathcal{A}) = \{x \in X : Lim_{t \downarrow 0} \frac{T(t)x - x}{t} \in X\}.$$

We consider systems whose state evolves according to an equation given by the following:

$$\dot{x}(t) = \mathcal{A}x(t) + f(t) \tag{1.6}$$

$$x(0) = x_0, \tag{1.7}$$

where $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup on X, $f(t) \in L^2([0,T], X)$ and $x_0 \in X$.In Equation (1.6) X is referred to as the state space, $x(t)$ is the state of the system and x_0 is the initial condition.

There are conditions on the resolvent of the operator $\mathcal{A}$ which are sufficient for $\mathcal{A}$ to be the infinitesimal generator of a C_0 semigroup $\{T(t)\}_{t \geq 0}$ of bounded operators (e.g Hille-Yoshida theorem (see (A. Pazy))).

Definition 1.1.4 *A mild solution in* $[0, T]$ *for Equation (1.6) is defined as:*

$$x(t) = T_t x_0 + \int_0^t T_{t-s} f(s) ds, \forall t \in [0, T], \tag{1.8}$$

where $\{T(t)\}_{t \geq 0}$ is the semigroup generated by $\mathcal{A}$. The condition $f(t) \in L^1([0, T], X)$ guarantees the continuity of the mild solution defined above (i.e $x(t) \in C([0, T], X)$)(see (A. Pazy)).

In this thesis we look at scalar control systems described as below.

$$\dot{x}(t) = \mathcal{A}x(t) + bu(t) \tag{1.9}$$

$$x(0) = x_0 \in X,$$

for which the mild solution is given by

$$x(t) = T_t x_0 + \int_0^t T_{t-s} bu(s) ds, \forall t \in [0, T]. \tag{1.10}$$

We define the control to state map as follows.

$$\Phi_t(u) = \int_0^t T_{t-s} bu(s) ds. \tag{1.11}$$

If $b \in X$ then Φ_t is a continuous map from $L^2(0, T)$ to X and hence $||\Phi_t(u(.))||_X \leq C||u(.)||_{L^2(0,t)}$. However, in some cases, the mapping Φ_t may remain continuous even though $b \notin X$.If this is the case, we say that b is admissible. More precisely,

Definition 1.1.5 *b is an admissible input for Equation (1.9) if* $\forall t > 0$, $\exists\ C > 0$ *such that*

$$||\Phi_t(u(.))||_X \leq C||u(.)||_{L^2(0,t)}. \tag{1.12}$$

Also Equation (1.10) remains valid for such an admissible input. Hence well-posedness is equivalent to the continuity of the control to state map. Equation (1.12) guarantees that the mild solution $x(t) \in C([0, T], X)$.

If $b \notin X$, we might still be able to guarantee the well-posedness of Equation (1.9). A necessary and sufficient condition for well-posedness is given by the Carleson measure criterion.

Definition 1.1.6 *The system of equations given by (1.9) with b admissible, is said to be exactly controllable in X in time T, if there exists a control input* $u(t) \in L^2(0,T)$ *which can steer the system from any initial state* $x(0)$ *to any final state* $x(T) \in X$ *in a finite time interval* $(0,T)$.

In definition (1.1.6) *steering* means that the system evolves from any initial condition $x_0 \in X$ to any final condition $x_T \in X$ in a finite time interval $[0,T]$ according to Equation (1.8), maintaining $x(t) \in C([0,T],X)$.

Definition 1.1.7 *The system of equations given by (1.9) with b admissible, is said to be approximately controllable in time T if* $\forall\epsilon > 0$, *there exists a control input which can steer the system from the initial state* x_0 *to within* ϵ *distance of any final state* $x_T \in X$.

CHAPTER 2. Wave equation (without damping)

The following are equations representing the propagation of a wave subjected to a distributed force $f(x)u(t)$ with pinned boundary conditions and prescribed initial conditions.

$$w_{tt} - w_{xx} = f(x)u(t) \tag{2.1}$$

$$w(0,t) = w(\pi,t) = 0 \tag{2.2}$$

$$w(x,0) = w^0(x), w_t(x,0) = w^1(x), \tag{2.3}$$

Let $A = -\frac{\partial^2}{\partial x^2}$, then A has the following eigenvalue-eigenvector pairs $A : (k^2, \sin(kx))$ where $k \in N$. The eigenvectors satisfy the boundary conditions.

$$A(\sin(kx)) = -\frac{\partial^2(\sin(kx))}{\partial x^2} = k^2 \sin(kx) \tag{2.4}$$

$$\sin(0) = \sin(k\pi) = 0$$

Let

$$\begin{pmatrix} w_1(t,x) \\ w_2(t,x) \end{pmatrix} = \begin{pmatrix} w(t,x) \\ w_t(t,x) \end{pmatrix}. \tag{2.5}$$

The arguments (t,x) will be omitted from now on in the above notation for simplicity.

Then (2.1)-(2.3) can be rewritten using equation (2.5) and (2.4) as

$$\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \mathcal{A}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(x) \end{pmatrix} u(t) \tag{2.6}$$

$$\begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix} = \begin{pmatrix} w^0(x) \\ w^1(x) \end{pmatrix} \in \mathcal{D}(\mathcal{A}), \tag{2.7}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) = (H^2 \cap H_0^1) \times H_0^1, \tag{2.8}$$

and the following definitions hold:

$$H^1 = \{\varphi \in L^2(0,\pi) : \varphi \in AC[0,\pi], \varphi_x \in L^2(0,\pi)\}$$

$$H^2 = \{\varphi \in H^1 : \varphi_x \in H^1\}$$

$$H_0^1 = \{\varphi \in L^2(0,\pi) : \varphi \in AC[0,\pi], \varphi_x \in L^2(0,\pi), \varphi(0) = \varphi(\pi) = 0\}.$$

Also $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ where $\mathcal{H} = H_0^1 \times L^2(0,\pi)$ is a Hilbert space with an inner-product defined as follows:

$$\left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} h \\ l \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_0^\pi (f'\bar{h}' + g\bar{l})dx.$$

2.1 Eigenvalues and eigenvectors of $\mathcal{A}$

Now the eigenvalues and eigenvectors of $\mathcal{A}$ are evaluated.

$$\begin{aligned} \mathcal{A}w &= \lambda w \\ \Leftrightarrow \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &= \lambda \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ \Leftrightarrow w_2 = \lambda w_1, -Aw_1 &= \lambda w_2 \\ \Leftrightarrow Aw_1 &= -\lambda^2 w_1. \end{aligned}$$

But we already know the eigenvalues and eigenvectors of A from (2.4) namely A : $(k^2, \sin(kx))$ where $k \in N$.

Hence the eigenvalues and eigenvectors of $\mathcal{A}$ are

$$(\lambda_k, \phi_k) = \left(ik, \frac{1}{\sqrt{\pi}} \begin{pmatrix} \frac{1}{ik}\sin(kx) \\ \sin(kx) \end{pmatrix} \right), k = \pm1, \pm2, \pm3..., \tag{2.9}$$

after introducing a normalizing factor of $\frac{1}{\sqrt{\pi}}$. It can be easily checked that

$$\left\langle \frac{1}{\sqrt{\pi}} \begin{pmatrix} \frac{1}{ik}\sin(kx) \\ \sin(kx) \end{pmatrix}, \frac{1}{\sqrt{\pi}} \begin{pmatrix} \frac{1}{ij}\sin(jx) \\ \sin(jx) \end{pmatrix} \right\rangle_{\mathcal{H}} = 0, k \neq j,$$

and

$$\left\langle \frac{1}{\sqrt{\pi}} \begin{pmatrix} \frac{1}{ik}\sin(kx) \\ \sin(kx) \end{pmatrix}, \frac{1}{\sqrt{\pi}} \begin{pmatrix} \frac{1}{ik}\sin(kx) \\ \sin(kx) \end{pmatrix} \right\rangle_{\mathcal{H}} = 1.$$

Define $Z^* = Z - \{0\}$. Hence $\{\phi_k\}_{k \in Z^*}$ is an orthonormal set of eigenvectors of $\mathcal{A}$ in $\mathcal{D}(\mathcal{A})$.

2.2 Completeness of $\{\phi_k\}_{k \in Z^*}$ in $\mathcal{H}$

Let

$$\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{H}$$

such that

$$\left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \frac{1}{\sqrt{\pi}} \begin{pmatrix} \frac{1}{ik}\sin(kx) \\ \sin(kx) \end{pmatrix} \right\rangle_{\mathcal{H}} = 0, \ \forall k \in Z^*.$$

$$\Rightarrow \frac{1}{\sqrt{\pi}} \int_0^{\pi} (f'(x)\cos(kx) + g(x)\sin(kx))dx = 0, \ \forall k \in Z^*. \tag{2.10}$$

Now we subtract the equations corresponding to $+k$ and $-k$ to get the following:

$$\Rightarrow \int_0^{\pi} (g(x)\sin(kx))dx = 0, \ \forall k \in N,$$

which, in view of (2.10) implies the following:

$$\Rightarrow \int_0^{\pi} (f'(x)\cos(kx))dx = 0, \ \forall k \in N.$$

Finally we integrate the above equation by parts and use the boundary conditions to obtain the following:

$$\Rightarrow \int_0^{\pi} (f(x)\sin(kx))dx = 0, \ \forall k \in N. \tag{2.11}$$

Since $\{\sin(kx)\}_{k\in N}$ is an complete sequence in $L^2(0,\pi)$, we have,

$$\Rightarrow \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore $\{\phi_k\}_{k\in Z^*}$ is complete and hence an orthonormal basis of $\mathcal{H}$.

2.3 Semigroup solution

Our aim is to solve (2.1-2.3), which is equivalent to solving (2.6-2.7). Let

$$\tilde{w}(t,x) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \sum_{k\in Z^*} w_k(t)\phi_k(x),$$

where $\phi_k(x)$ are the eigenvectors of $\mathcal{A}$ and $w_k(t)$ are the Fourier coefficients of $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ with respect to the orthonormal basis $\{\phi_k\}_{k\in Z^*}$. Also let

$$\tilde{w}(0,x) = \begin{pmatrix} w^0(x) \\ w^1(x) \end{pmatrix} = \sum_{k\in Z^*} w_k(0)\phi_k(x).$$

By taking Fourier coefficients of mild solution Equation (1.10) with respect to $\{\phi_k\}_{k\in Z^*}$ we get the following equivalent mild solution in $l^2(Z^*)$:

$$w_k(t) = e^{ikt}w_k(0) + \int_0^t e^{ik(t-s)} f_k u(s)ds, \ \forall k \in Z^*, \tag{2.12}$$

where

$$\begin{pmatrix} 0 \\ f(x) \end{pmatrix} = \sum_k f_k\phi_k(x), f_k = \frac{1}{\sqrt{\pi}} \int_0^{\pi} f(x)\sin(kx)dx, k \in Z^*.$$

Since $\mathcal{H}$ is a Hilbert space, it is isometrically isomorphic to $l^2(Z^*)$(since the eigenfunctions $\{\phi_k\}_{k\in Z^*}$ form an orthonormal basis for $\mathcal{H}$). Hence we consider $\mathcal{A}$ as an equivalent infinite diagonal matrix $\mathbf{A}$ with respect to the standard basis in $l^2(Z^*)$, consisting of the eigenvalues $\{\lambda_k\}_{k\in Z^*}$ on the main diagonal. Controllability of the states $\tilde{w}(.,x) \in \mathcal{H}$ is equivalent to the controllability of the Fourier coefficients $\{w_k(t)\}_{k\in Z^*} \in l^2(Z^*)$ of $\tilde{w}(.,x)$ with respect to $\{\phi_k\}_{k\in Z^*}$. Hence the original control problem is transformed from $\mathcal{H}$ into an equivalent problem on $l^2(Z^*)$. $\mathbf{A}$ can be shown to be the infinitesimal generator of a C_0 semigroup in $l^2(Z^*)$ by checking the sufficient conditions of the Hille-Yosida theorem. We show this calculation for the Mead-Markus beam model later on.

Equation (2.12) is the mild solution of

$$\dot{x}(t) = \mathbf{A}x(t) + bu(t), \tag{2.13}$$

where $\mathbf{A}$ is the generator of a diagonal semigroup $(T_t)_{t\geq 0} : l^2(Z^*) \to l^2(Z^*)$ defined by

$$(T_t x)_k = e^{ikt} x_k, \ \forall k \in Z^*, \tag{2.14}$$

and $b = \{f_k\} \in l^2(Z^*)$. Next we check the well-posedness of Equation (2.13) in $l^2(Z^*)$.

$$\begin{aligned}
|| \int_0^t T_{t-s} f_k u(s) ds ||_{l^2}^2 &= || \int_0^t e^{ik(t-s)} f_k u(s) ds ||_{l^2}^2 \\
&= \sum_{k=-\infty}^{\infty} |f_k|^2 | \int_0^t e^{ik(t-s)} u(s) ds|^2 \\
&\leq t ||f_k||_{l^2}^2 ||u(.)||_{L^2(0,t)}^2 .
\end{aligned}$$

Hence Equation (2.12) is well-posed in $l^2(Z^*)$.

2.4 The moment problem and its solution

The controllability of the undamped wave equation reduces to a moment problem as follows. Equation (2.12) can be rewritten in the following form:

$$\int_0^T e^{ik\tau} \tilde{u}(\tau) d\tau = \frac{c_k}{f_k}, \ \forall k \in Z^*, \tag{2.15}$$

where $c_k = w_k(T) - e^{ikT}w_k(0)$ and T is the final time instant and $\tilde{u}(t) = u(T-t)$. If we are looking for a control in $L^2(0, 2\pi)$, i.e if $T = 2\pi$, then equation (2.15) can be rewritten as

$$\langle \tilde{u}(t), e^{-ikt} \rangle_{L^2(0,2\pi)} = \frac{c_k}{f_k}, \ \forall k \in Z^*. \tag{2.16}$$

Equation (2.16) has a solution namely

$$u(2\pi - t) = \tilde{u}(t) = \sum_{k \in Z^*} \frac{2\pi c_k}{f_k} e^{-ikt}, \tag{2.17}$$

provided $\frac{c_k}{f_k} \in l^2(Z^*)$. Hence we have found a control input $u(t) = \tilde{u}(T-t)$ which solves our moment problem in $T = 2\pi$. If the control time satisfies $T > 2\pi$ then we can still solve the moment problem (2.15) since we can use the zero control until $t = T - 2\pi$ and then apply the previous control $u(t)$ from $t = T - 2\pi$ to $t = T$. If $T < 2\pi$ then there are states which cannot be reached in the state space even if $\frac{c_k}{f_k} \in l^2(Z^*)$, from the following argument. Let $T = 2\pi - \epsilon$ and let $\mathcal{M}$ be the moment operator defined by

$$\mathcal{M} : L^2(0, 2\pi) \longrightarrow l^2(Z)$$
$$\mathcal{M}(u) = \{< u(t), e^{ikt} >_{L^2(0,2\pi)}\}_{k \in Z}.$$

Then the range of $\mathcal{M}$ is the whole of $l^2(Z)$. Let $\mathcal{C} = \{u(t) \in L^2(0, 2\pi) : supp(u) = (0, 2\pi - \epsilon)\}$ i.e $\mathcal{C}$ is the set of all controls which which become zero after time $T - \epsilon$. Then $\mathcal{M}(\mathcal{C}) \subset l^2(Z)$ i.e the range of $\mathcal{M}$ is a proper subset of $l^2(Z)$, namely the space of all Fourier coefficents of $u(t) \in \mathcal{C}$. Hence if we choose a sequence $\{e_k\}_{k \in Z} \in l^2(Z)$ which is the Fourier coefficient of an element in $\mathcal{C}^{\perp} = \{u(t) \in L^2(0, 2\pi) : supp(u) = (2\pi - \epsilon, 2\pi)\}$ then we can never find a control $u(t) \in \mathcal{C}$ such that $\mathcal{M}(u) = \{e_k\}_{k \in Z}$. Hence we have the following theorem:

Theorem 2.4.1 *The moment problem (2.15) has a solution $\tilde{u}(t) \in L^2(0, T)$ provided $T \geq 2\pi$ and $\frac{c_k}{f_k} \in l^2(Z^*)$. If $T < 2\pi$ then there does not exist a solution for all data $\frac{c_k}{f_k} \in l^2(Z^*)$.*

2.5 Controllability results

We investigate the approximate controllability of the system of equations given by (2.1),(2.2) and (2.3).

Theorem 2.5.1 *The system of Equations given by (2.1),(2.2) and (2.3) is approximately controllable in time $T = 2\pi$ in the state space $\mathcal{H}$, if and only if $f_k \neq 0,\ \forall k \in Z^*$.*

Sketch of proof: Since ϕ_k's form an orthonormal basis for $\mathcal{H}$, $\forall \epsilon > 0\ \exists N$ such that $||\tilde{w}(T,x) - \sum_{k=-N}^{k=N} w_k(T)\phi_k(x)|| < \epsilon$. Hence by fixing an ϵ and choosing $N(\epsilon)$, the problem reduces to finding $\tilde{u}(t) = u(T-t)$ in

$$\langle \tilde{u}(t), e^{-ikt} \rangle_{L^2(0,2\pi)} = \frac{\tilde{c}_k}{f_k},\ \forall k = (-N...N) - \{0\}, \tag{2.18}$$

where $\tilde{c}_k = w_k(T)$. If $T = 2\pi$ and $\tilde{f}_k \neq 0,\ \forall k \in N$ then Equation (2.18) is solved by the finite linear combination $u(2\pi - t) = \tilde{u}(t) = \sum_{k=-N,k\neq 0}^{k=N} \frac{2\pi\tilde{c}_k}{f_k} e^{-ikt}$. On the other hand, if $f_k = 0$ for some k, then there is no solution to Equation (2.18). $\square$

We define a subspace $\mathcal{C} \subseteq \mathcal{H}$ given by $\mathcal{C} = \{\tilde{w}(.,x) \in \mathcal{H} : \frac{<\tilde{w}(.,x),\phi_k(x)>_{\mathcal{H}}}{f_k} \in l^2(Z^*)\}$. Since the moment problem (2.15) can be solved for states $\tilde{w}(t,.) \in \mathcal{H}$ whose Fourier coefficients satisfy $\frac{c_k}{f_k} \in l^2(Z^*)$, we have that any initial state $\tilde{w}(0,.) \in \mathcal{C}$ can be driven to any final state $\tilde{w}(T,.) \in \mathcal{C}$ provided $T \geq 2\pi$. Hence we have the following theorem.

Theorem 2.5.2 *Let $\mathcal{C} = \{\tilde{w}(.,x) \in \mathcal{H} : \frac{<\tilde{w}(.,x),\phi_k(x)>_{\mathcal{H}}}{f_k} \in l^2(Z^*)\}$, where $f_k = \frac{1}{\sqrt{\pi}}\int_0^\pi f(x)\sin(kx)dx$, $k \in Z^*$. Then if $T \geq 2\pi$, then $\forall x(0), x(T) \in \mathcal{C}, \exists$ a control input $u(t) \in L^2(0,T)$ such that the system evolves from $x(0) \in \mathcal{C}$ to $x(T) \in \mathcal{C}$ in time T.*

Remark: In Theorem 2.5.2, we are not guaranteed to remain inside $\mathcal{C}$ while moving from $\tilde{w}(0,.) \in \mathcal{C}$ to $\tilde{w}(T,.) \in \mathcal{C}$. Hence Theorem 2.5.2 is not actually an exact controllability result. Although we were not able to obtain an exact controllability result in the entire space $\mathcal{H}$, Theorem 2.5.2 is somewhat better than Theorem 2.5.1 because we were able to characterize the subspace of states which are reachable from an initial state $\tilde{w}(0,.) \in \mathcal{C}$.

CHAPTER 3. Wave equation (with damping)

In this chapter, the wave equation is considered with damping factor $\alpha > 0$ as follows:

$$w_{tt} - w_{xx} + \alpha w_t = f(x)u(t) \tag{3.1}$$

$$w_x(0,t) = 0, w(\pi,t) = 0, u(t) \in L^2(0,\pi) \tag{3.2}$$

$$w(x,0) = w^0(x), w_t(x,0) = w^1(x), \tag{3.3}$$

Also note the change in boundary conditions $w_x(0,t)$ as compared to the condition $w(0,t)$ used in Equation (2.2).

Let $A = -\frac{\partial^2}{\partial x^2}$, then A has the following eigenvalue-eigenvector pair $A : ((k - \frac{1}{2})^2, \cos((k-\frac{1}{2})x)$ where $k \in N$. The eigenvectors satisfy the boundary conditions.

$$\begin{aligned} A(\cos((k-\frac{1}{2})x)) = -\frac{\partial^2(\cos((k-\frac{1}{2})x))}{\partial x^2} &= (k+\frac{1}{2})^2\cos((k-\frac{1}{2})x) \\ \frac{\partial}{\partial x}\cos((k-\frac{1}{2})x)|_{x=0} = -(k-\frac{1}{2})\sin(0)) &= 0 \\ \cos((k-\frac{1}{2})\pi) &= 0. \end{aligned}$$

Let

$$\begin{pmatrix} w_1(t,x) \\ w_2(t,x) \end{pmatrix} = \begin{pmatrix} w(t,x) \\ w_t(t,x) \end{pmatrix}$$

The arguments (t,x) will be omitted from now on in the above equations as usual.

$$\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \mathcal{A}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(x) \end{pmatrix} u(t) \tag{3.4}$$

$$\begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix} = \begin{pmatrix} w^0(x) \\ w^1(x) \end{pmatrix} \in \mathcal{D}(\mathcal{A})$$

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & -\alpha \end{pmatrix}, \mathcal{D}(\mathcal{A}) = \Gamma \times H^1_*, \tag{3.5}$$

where,

$$H^1 = \{\varphi \in L^2(0,\pi) : \varphi \in AC[0,\pi], \varphi_x \in L^2(0,\pi)\}$$
$$H^2 = \{\varphi \in H^1 : \varphi_x \in H^1\}$$
$$H^1_* = \{\varphi \in L^2(0,\pi) : \varphi \in AC[0,\pi], \varphi_x \in L^2(0,\pi), \varphi(\pi) = 0\}$$
$$\Gamma = \{\phi \in H^2 \cap H^1_* : \phi_x(0) = 0\}.$$

Also $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to H^1_* \times L^2(0,\pi)$ where $\mathcal{H} = H^1_* \times L^2(0,\pi)$ is a Hilbert space with an inner-product defined as follows:

$$\left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} h \\ l \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_0^\pi (f'\bar{h}' + g\bar{l})dx.$$

3.1 Eigenvalues and eigenvectors of $\mathcal{A}$

Now the eigenvalues and eigenvectors of $\mathcal{A}$ are evaluated.

$$\mathcal{A}w = \lambda w$$
$$\Leftrightarrow \begin{pmatrix} 0 & I \\ -A & -\alpha \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \lambda \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$
$$\Leftrightarrow w_2 = \lambda w_1, -Aw_1 = (\alpha + \lambda)w_2$$
$$\Leftrightarrow Aw_1 = -(\alpha + \lambda)\lambda w_1.$$

But we already know the eigenvalues and eigenvectors of A from (3.4) namely A : $((k-\frac{1}{2})^2, \cos((k-\frac{1}{2})x)$ where $k \in N$. In our discussion let us assume that $\alpha < \frac{1}{2}$. The eigenvalues and eigenvectors of $\mathcal{A}$ are calculated below.

$$(\lambda_k^{\pm}, \phi_k^{\pm}) = \left(\frac{-\alpha \pm i\sqrt{4\mu_k^2 - \alpha^2}}{2}, \sqrt{\frac{2}{\pi}} \begin{pmatrix} \frac{1}{\lambda_k^{\mp}} \cos(\mu_k x) \\ \cos(\mu_k x) \end{pmatrix} \right),$$

where

$$\mu_k = k - \frac{1}{2}, k \in N.$$

3.2 Riesz Basis properties of $\{\phi_k\}_{k\in N}$

We note the following definitions taken from (R.M. Young):

Definition 3.2.1 *A sequence $\{x_n\}_{n=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is a **Riesz basis** if $x_n = Te_n$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis and $T : \mathcal{H} \to \mathcal{H}$ is a bounded operator such that $T^{-1} : \mathcal{H} \to \mathcal{H}$ is also bounded.*

Furthermore, a Riesz basis $\{x_n\}$ is also a basis for the Hilbert space and possesses a unique complete biorthogonal sequence $\{y_n\}$. To see this, let $\{x_n\}$ be a Reisz basis equivalent to an orthonormal basis $\{e_n\}$. Let T be such that $Te_n = x_n, \ \forall n \in N$. Then $\forall x \in \mathcal{H}$ we have the following unique expansion $x = \sum_{n=1}^{\infty} c_n e_n$. Also since T is bounded and invertible, we have

$$y = Tx = \sum_{n=1}^{\infty} c_n T e_n = \sum_{n=1}^{\infty} c_n x_n, \ \forall y \in \mathcal{H},$$

and the expansion is unique. Since every basis in a Hilbert space possesses a unique complete biorthogonal sequence, we have that every element $x \in \mathcal{H}$ can be uniquely represented in the following equivalent ways:

$$\begin{aligned} x &= \sum_{n=1}^{\infty} < x, x_n > y_n \\ &= \sum_{n=1}^{\infty} < x, y_n > x_n. \end{aligned}$$

Let

$$\theta_k = \begin{pmatrix} \frac{\sqrt{2}}{\mu_k \sqrt{\pi}} \cos(\mu_k x) \\ 0 \end{pmatrix}, k \in N$$

$$\eta_k = \begin{pmatrix} 0 \\ \sqrt{\frac{2}{\pi}}\cos(\mu_k x) \end{pmatrix}, k \in N.$$

Then $\{\phi_k^{\pm}\}_{k\in N}$ is related to θ_k's and η_k's in the following way:

$$\begin{aligned} \phi_k^+ &= \frac{\mu_k \theta_k}{\lambda_k^+} + \eta_k, k \in N \\ \phi_k^- &= \frac{\mu_k \theta_k}{\lambda_k^-} + \eta_k, k \in N, \end{aligned}$$

and

$$\begin{aligned} \theta_k &= \frac{(\phi_k^+ - \phi_k^-)}{\mu_k \left(\frac{1}{\lambda_k^+} - \frac{1}{\lambda_k^-}\right)} \\ \eta_k &= \frac{(\phi_k^+ - \phi_k^-)}{\sqrt{2}}. \end{aligned}$$

$(\{\theta_k\} \bigcup \{\eta_k\})_{k\in N}$ is an orthonormal basis for $\mathcal{H}$ since $\{\eta_k\}_{k\in N}$ is an orthonormal basis for $L^2(0,1)$ and the mapping $\frac{\partial}{\partial x} : H_*^1 \to L^2(0,1)$ is an isometric isomorphism(see (S.W. Hansen) page 49). Define a mapping $T : \mathcal{H} \to \mathcal{H}$ as follows:

$$\begin{aligned} T(\theta_k) &= \phi_k^+ \\ T(\eta_k) &= \phi_k^-, \end{aligned}$$

$\forall k \in N$. Also let

$$u(x) = \sum_{k\in N} a_k \theta_k(x) + \sum_{k\in N} b_k \eta_k(x).$$

Then we have

$$\begin{aligned} Tu &= \sum_{k\in N} \left(\frac{a_k\mu_k}{\lambda_k^+} + \frac{b_k\mu_k}{\lambda_k^-}\right)\theta_k + \sum_{k\in N}(a_k + b_k)\,\eta_k \\ ||Tu||_{\mathcal{H}}^2 &= \sum_{k\in N}\left(\left|\frac{a_k\mu_k}{\lambda_k^+} + \frac{b_k\mu_k}{\lambda_k^-}\right|^2 + |a_k + b_k|^2\right) \end{aligned}$$

$$\leq \sum_{k\in N} \left(2|a_k|^2 + 2|b_k|^2 + 4|a_k||b_k|\right)$$

$$\leq \sum_{k\in N} 4\left(|a_k|^2 + |b_k|^2\right)$$

$$= 4||u||^2_{\mathcal{H}},\ \forall u \in \mathcal{H}$$

$$T^{-1}u = \sum_{k\in N} \left(\frac{a_k}{\mu_k \left(\frac{1}{\lambda_k^+} - \frac{1}{\lambda_k^-}\right)} + \frac{b_k\lambda_k^+}{\lambda_k^+ - \lambda_k^-} \right) \theta_k$$

$$- \left(\frac{a_k}{\mu_k \left(\frac{1}{\lambda_k^+} - \frac{1}{\lambda_k^-}\right)} + \frac{b_k\lambda_k^-}{\lambda_k^+ - \lambda_k^-} \right) \eta_k$$

$$||T^{-1}u||^2_{\mathcal{H}} = \sum_{k\in N} \frac{4\mu_k^2}{4\mu_k^2 - \alpha^2}\left(|a_k|^2 + |b_k|^2 + 2|a_k||b_k|\right)$$

$$\leq \sum_{k\in N} \frac{4\mu_k^2}{4\mu_k^2 - \alpha^2}\left(|a_k|^2 + |b_k|^2\right)$$

$$\leq ||u||^2_{\mathcal{H}}.$$

Hence $\{\phi_k^\pm\}_{k\in N}$ is a Reisz basis for $\mathcal{H}$ and there exists an equivalent norm $||.||_* = ||T^{-1}u||_{\mathcal{H}}$ (see (R.M. Young)) with respect to which $\{\phi_k^\pm\}_{k\in N}$ forms an orthonormal basis.

3.3 Semigroup solution

Similar to the case of wave equation without damping, we have that $\{\phi_k^\pm\}_{k\in N}$ forms an orthonormal basis for the Hilbert space $(\mathcal{H}, ||.||_*)$. Hence we have the following valid expansion:

$$\tilde{w}(.,x) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \sum_{k\in N} w_k(.)\phi_k^\pm(x),\ \forall w(.,x) \in (\mathcal{H}, ||.||_*).$$

Once again we have that $(\mathcal{H}, ||.||_*)$ is isometrically isomorphic to $l^2(N)$. Hence we consider $\mathcal{A}$ as an equivalent infinite diagonal matrix $\mathbf{A}$ with respect to the standard basis in $l^2(Z^*)$, consisting of the eigenvalues $\{\lambda_k^\pm\}_{k\in(N)}$ on the main diagonal.We follow similar steps as in section (2.3), we have the following semigroup solution of Equation (3.4) in $l^2(N)$:

$$w_k^{\pm}(t) = e^{\lambda_k^{\pm} t} w_k(0) + \int_0^t e^{\lambda_k^{\pm}(t-s)} f_k^{\pm} u(s) ds, \ \forall k \in N \tag{3.6}$$

where

$$\begin{pmatrix} 0 \\ f(x) \end{pmatrix} = \sum_k f_k \phi_k(x), \ f_k^{\pm} =< \begin{pmatrix} 0 \\ f(x) \end{pmatrix}, \phi_k^{\pm} >_*, \ \forall k \in N.$$

Equation (3.6) is the mild solution of

$$\dot{x}(t) = \mathbf{A}x(t) + bu(t) \tag{3.7}$$

where $\mathbf{A}$ is the generator of a diagonal semigroup $(T_t)_{t\geq 0} : l^2(N) \to l^2(N)$ defined by

$$(T_t x)_k = e^{\lambda_k^{\pm} t} x_k, \ \forall k \in (N), \tag{3.8}$$

and $b = \{f_k^{\pm}\} \in l^2(N)$. We omit the calculations needed to verify the sufficient conditions of the Hille-Yosida theorem since it is similar to the Mead-Markus case. Also it can be shown that Equation (3.7) is well-posed using the semigroup property and also the facts that $f_k^{\pm} \in l^2(N)$ and $u(t) \in L^2(0,T)$.

3.4 The moment problem and its solution

We rewrite Equation (3.6) as the following moment problem.

$$\int_0^T e^{\lambda_k^{\pm} \tau} \tilde{u}(\tau) d\tau = \frac{c_k^{\pm}}{f_k^{\pm}}, \ \forall k \in N, \tag{3.9}$$

where $c_k^{\pm} = w_k^{\pm}(T) - e^{\lambda_k^{\pm} T} w_k(0)$ and T is the final time instant, $f_k^{\pm}$ are the Fourier coefficients of $\begin{pmatrix} 0 \\ f(x) \end{pmatrix}$ with respect to the orthonormal basis $\{\phi_k^{\pm}\}_{k\in N}$ in $(\mathcal{H}, < . >_*)$

and $\tilde{u}(t) = u(T-t)$. If we are looking for a control in $L^2(0,2\pi)$, i.e if $T = 2\pi$, then equation (3.9) can be rewritten as

$$\left\langle \tilde{u}(t), e^{-\lambda_k^{\pm} t} \right\rangle_{L^2(0,2\pi)} = \frac{c_k^{\pm}}{f_k^{\pm}}, \ \forall k \in N. \tag{3.10}$$

To solve the moment problem given by (3.10) we need to apply results from the theory of non-harmonic Fourier series.

If we are able to show that the system of exponentials given by $\left\{e^{\lambda_k^{\pm} t}\right\}_{k\in N}$ forms a Riesz basis for $L^2(0,2\pi)$ then from Equation (3.6), we can solve the moment problem given by Equation (3.10) by $\tilde{u}(t) = \sum_{n\in N} \frac{c_k^{\pm}}{f_k^{\pm}} y_n(t)$ provided $\left\{\frac{c_k^{\pm}}{f_k^{\pm}}\right\}_{k\in N} \in l^2(N)$, where $(\{y_n(t)\}_{n\in Z}$ is the complete biorthogonal sequence as found in (3.6)).

Theorem 3.4.1 *The sequence* $\left\{e^{\lambda_k^{\pm} t}\right\}_{k\in N}$ *where* $\lambda_k^{\pm} = \frac{-\alpha \pm i\sqrt{4(\mu_k)^2 - \alpha^2}}{2}$, $\mu_k = k - \frac{1}{2}, k \in N$ *is a Riesz basis for* $L^2(0,2\pi)$.

Proof: Let us first redefine our sequence μ_k, $\forall k \in Z$ by simply allowing k to vary over all the integers in the original definition as in the statement of Theorem 3.4.1.

First we note that $\{e^{i\mu_k t}\}_{k\in Z}$ forms a Riesz basis for $L^2(0,2\pi)$. This is true because $\{e^{i\mu_k t}\}_{k\in Z}$ is obtained from the orthonormal basis $\left\{e^{ikt}\right\}_{k\in Z}$ of $L^2(0,2\pi)$ by using the multiplication operator given by $T\{e^{i\mu_k t}\}_{k\in Z} = e^{\frac{-it}{2}}\left\{e^{ikt}\right\}_{k\in Z}$. By a similar argument we have $\left\{e^{\frac{-\alpha}{2} + i\mu_k t}\right\}_{k\in Z}$ is a Riesz basis for $L^2(0,2\pi)$. Next we refer to the following theorem from (R.M. Young), regarding the stability of a Riesz basis.

Theorem 3.4.2 *If the system* $\{e^{i\mu_k t}\}_{k\in Z}$ *is a Riesz basis for* $L^2(0,2\pi)$, *then* $\exists$ *a positive constant* L *with the property that* $\left\{e^{i\lambda_k t}\right\}_{k\in Z}$ *is also a Riesz basis for* $L^2(0,2\pi)$ *whenever* $|\mu_k - \lambda_k| \leq L, \ \forall k \in Z$.

Let L be the constant for the Riesz basis $\left\{e^{\frac{-\alpha}{2} + i\mu_k t}\right\}_{k\in Z}$ according to Theorem 3.4.2. We have $\sqrt{\mu_k^2 - \frac{\alpha^2}{4}} = \mu_k\sqrt{1 - \frac{\alpha^2}{4\mu_k^2}} = \mu_k(1 - \frac{1}{2}\frac{\alpha^2}{4\mu_k^2} + \mathcal{O}(k^{-4}))$. From this estimate on μ_k,

$\exists K$ such that $|\lambda_k^{\pm} - (\frac{-\alpha}{2} \pm i\mu_k)| = |\mathcal{O}(k^{-1})| \leq L \;\; \forall k > K$. Now, define a sequence by

$$\tilde{\lambda_k}^{\pm} = \frac{-\alpha}{2} \pm i\mu_k, \;\forall k \leq K, k \in N$$
$$\lambda_k^{\pm}, \;\forall k > K, k \in N.$$

Now we have $|\frac{-\alpha}{2} \pm i\mu_k - \tilde{\lambda_k}^{\pm}| \leq L, \;\forall k \in N$ and hence $\left\{e^{\tilde{\lambda_k}^{\pm} t}\right\}_{k \in N}$ is a Riesz basis for $L^2(0, 2\pi)$. Finally $\left\{e^{\lambda_k^{\pm} t}\right\}_{k \in Z}$ is a Riesz basis for $L^2(0, 2\pi)$ because we can use the following invertible finite dimensional transformation $\forall k < K$.

$$T\{e^{\tilde{\lambda_k}^{\pm}} t\} = e^{i\left(\mp \mu_k \pm \sqrt{4(\mu_k)^2 - \alpha^2}\right)t}\{e^{\tilde{\lambda_k}^{\pm}} t\}, \;\forall k \leq K \tag{3.11}$$
$$T^{-1}\{e^{\lambda_k^{\pm} t}\} = e^{i\left(\pm \mu_k \mp \sqrt{4(\mu_k)^2 - \alpha^2}\right)t}\{e^{\lambda_k^{\pm} t}\}, \;\forall k \leq K, \tag{3.12}$$

and T is defined to be the identity $\forall k > K$. Since T is defined on a basis, it is defined on $L^2(0, 2\pi)$. The real parts of $\lambda_k^{\pm}$ and $\tilde{\lambda_k}^{\pm}$ are both equal to $\frac{-\alpha}{2}$ and $\lambda_k^{\pm} = \tilde{\lambda_k}^{\pm}, \;\forall k > K$. Equations (3.11) and (3.12) indicate the invertibility of the transformation and since it is finite dimensional, the transformation is bounded. Also by construction, we have

$$T\{e^{\tilde{\lambda_k}^{\pm}} t\} = \{e^{\lambda_k^{\pm}} t\}, \;\forall k \in N$$

and hence $\{e^{\lambda_k^{\pm}} t\}_{k \in N}$ is a Riesz basis for $L^2(0, 2\pi)$. $\square$

3.5 Controllability results

By Theorem 3.4.1 we have that the wave equation with damping is exactly controllable in the sense similar to Theorem 2.5.2 in time $T = 2\pi$ provided $\left\{\frac{c_k^{\pm}}{f_k^{\pm}}\right\}_{k \in N} \in l^2(N)$. Hence we have the following theorem:

Theorem 3.5.1 *Let* $\mathcal{C} = \{\tilde{w}(.,x) \in \mathcal{H} : \frac{<\tilde{w}(.,x), \phi_k(x)>_{\mathcal{H}}}{f_k} \in l^2(N)$, *where* $f_k^{\pm} =< \begin{pmatrix} 0 \\ f(x) \end{pmatrix}, \phi_k^{\pm} >_*$, $\forall k \in (N)$. *Then if* $T = 2\pi$, *then* $\forall x(0), x(T) \in \mathcal{C}, \exists$ *a control input* $u(t) \in L^2(0, T)$ *such that the system evolves from* $x(0) \in \mathcal{C}$ *to* $x(T) \in \mathcal{C}$ *in time* T.

3.6 Admissibility and Carleson measure criterion

In this section we briefly describe the concept of admissibility of the (possibly unbounded) sequence $\{f_k^{\pm}\}_{k\in N}$ for Equation (3.6). A detailed discussion of the same can be found in (G. Weiss) and (L.F. Ho and D.L. Russell).

Definition 3.6.1 *The Carleson rectangle is defined for $h > 0$ and $\omega \in \mathcal{R}$ as follows:*

$$R(h,\omega) = \{z \in C : 0 \le Re(z) \le h, |Im(z) - \omega| \le h\} \tag{3.13}$$

Definition 3.6.2 *Let $\{\lambda_k\}_{k\in N}$ be a sequence of complex numbers satisfying*

$$\sup_{k\in N} Re(\lambda_k) = -\sigma < 0.$$

We say that a complex sequence $\{f_k\}_{k\in N}$ satisfies the ***Carleson measure criterion*** *for the sequence $\{\lambda_k\}_{k\in N}$, if for any $h > 0$ and any $\omega \in R$,*

$$\sum_{-\lambda_k \in R(h,\omega)} |f_k|^2 \le Mh, \tag{3.14}$$

where $M > 0$ is independent of h and ω.

Consider the control system,

$$\dot{x}(t) = \boldsymbol{A}x(t) + bu(t),$$

where $\boldsymbol{A}$ is the generator of a diagonal semigroup $(T_t)_{t\ge 0} : l^2(N) \to l^2(N)$ defined by

$$(T_t x)_k = e^{i\lambda_k t} x_k, \ \forall k \in N,$$

and $b = \{b_k\}_{k\in N}$. Formally, the Fourier coefficients of the mild solution are given by

$$w_k(t) = e^{i\lambda_k t} w_k(0) + \int_0^t e^{i\lambda_k^{\pm}(t-s)} b_k u(s) ds, \ \forall k \in N. \tag{3.15}$$

As discussed in the preliminaries, a sufficient condition for Equation (3.15) to be well-posed in $l^2(N)$ is that $b = \{b_k\}_{k\in N}$ is admissible.

We quote the following theorem from (G. Weiss).

Theorem 3.6.1 *Let $\{\lambda_k\}_{k\in N}$ be the eigenvalues of* **A** *satisfying*

$$\sup_{k\in N} Re(\lambda_k) = -\sigma < 0.$$

Then $b = \{b_k\}_{k\in N}$ is admissible if and only if $\{b_k\}_{k\in N}$ satisfies the Carleson measure criterion (3.14).

The concept of admissibility allows us to maintain the well-posedness of Equation (3.7) even if $\{b_k^{\pm}\}_{k\in(N)} \notin l^2(N)$. For example, if we formally allow $f(x) = \delta(x)$, then the Fourier coefficients with respect to $\{\phi_k^{\pm}\}_{k\in N}$ in the innerproduct given by $<,.,>_*$ are calculated below:

$$\delta(x) = \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} \begin{pmatrix} 0 \\ \cos(k-\frac{1}{2})x \end{pmatrix} = \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} \frac{(\phi_k^+ - \phi_k^-)}{\sqrt{2}}$$

$$= \sum_{k=1}^{\infty} \frac{1}{\sqrt{\pi}} \phi_k^+ - \sum_{k=1}^{\infty} \frac{1}{\sqrt{\pi}} \phi_k^-$$

Hence $b_k^{\pm} = \pm\frac{1}{\sqrt{\pi}}$, $\forall k \in N$. Hence let us check the Carleson measure criterion for $\{b_k^{\pm}\}_{k\in N}$ with respect to $\{\lambda_k^{\pm}\}_{k\in N}$ and hence verify that Equation (3.7) is wellposed in $l^2(N)$. Since $\{b_k^{\pm} = \frac{1}{\sqrt{\pi}}\}_{k\in N}$, we have that $|b_k^{\pm}| \le M = \frac{1}{\sqrt{\pi}}$, $\forall k \in N$. Recall that

$$\lambda_k^{\pm} = \frac{-\alpha \pm i\sqrt{4\mu_k^2 - \alpha^2}}{2}. \tag{3.16}$$

We have the following estimate on the imaginary part of $\lambda_k^{\pm}$'s:

$$\sqrt{\mu_k^2 - \frac{\alpha^2}{4}} = \mu_k\sqrt{1 - \frac{\alpha^2}{4\mu_k^2}} = \mu_k(1 - \frac{1}{2}\frac{\alpha^2}{4\mu_k^2} + \mathcal{O}(k^{-4})) = \mu_k + \mathcal{O}(k^{-1}). \tag{3.17}$$

Hence $\exists\ K$ such that $|Im(\lambda_{k+1}^{\pm}) - Im(\lambda_k^{\pm})| \ge 1 - \mathcal{O}(k^{-1}) \ge 0.9,\ \forall k \ge K$. We have the following cases.

1. If $h < \frac{\alpha}{2}$ then $\sum_{-\lambda_k\in R(h,\omega)} |b_k|^2 = 0$ since there are no eigenvalues in this rectangle.

2. If $h \geq \frac{\alpha}{2}$, $\omega > K$, and $h < \omega - K$, then the eigenvalues are separated by a distance of 0.9. Let N be the smallest integer such that $0.9N \geq 2h$. Then we have $\sum_{-\lambda_k \in R(h,\omega)} |b_k|^2 \leq NMh = M_1 h$.

3. If $h \geq \frac{\alpha}{2}$, $\omega < K$ and $h < K - \omega$, then we argue by contradiction. There are at most $2K$ number of eigenvalues in such a rectangle ($2K$ being the number obtained when $\omega = 0$ and $h = K$). Hence the ratio $\frac{\sum_{-\lambda_k \in R(h,\omega)} |b_k|^2}{h}$ can blow up only when h becomes small. But $h \geq \frac{\alpha}{2}$ and hence the ratio can never blow up and hence $\sum_{-\lambda_k \in R(h,\omega)} \leq M_2 h$ for some $M_2 > 0$.

4. Finally we have the case when $h > |\omega - K|$. In this case we split the rectangle into two parts. $R(h,\omega) = R_1(h,\omega) + R_2(h,\omega)$ where $R_1(h,\omega)$ is a rectangle that looks like case 2 and $R_2(h,\omega)$ looks like case 3. We define $M = max(M_1, M_2)$ and hence we have $\sum_{-\lambda_k \in R(h,\omega)} \leq 2Mh$.

Finally $\sum_{-\lambda_k \in R(h,\omega)} \leq 2Mh$ is true for any Carleson rectangle and hence we have verified the admissibility of $\{b_k^{\pm}\}_{k \in N} \in l^2(N)$.

The moment problem given by (3.9) is changed to the following moment problem

$$\int_0^T e^{\lambda_k^{\pm} \tau} \tilde{u}(\tau) d\tau = c_k^{\pm} \left(\pm \frac{1}{\sqrt{\pi}} \right), \ \forall k \in N, \tag{3.18}$$

where $c_k^{\pm} = w_k^{\pm}(T) - e^{i\lambda_k^{\pm} T} w_k(0) \in l^2(N)$ and T is the final time instant. We can follow the same arguments as given in section (3.4) and show that Equation (3.18) is solvable for all initial and final states given by $w_k^{\pm}(0), w_k^{\pm}(T)$ repectively. We conclude this section with the following exact controllability result.

Theorem 3.6.2 *Let* $f(x) = \delta(x)$, *and* $b_k^{\pm} = <\begin{pmatrix} 0 \\ f(x) \end{pmatrix}, \phi_k^{\pm} >_* = \sqrt{\frac{2}{\pi}}$, $\forall k \in (N)$. *Then Equation (3.4) is exactly controllable in* $\mathcal{H}$ *in time* $T = 2\pi$.

Remark: When $f(x) = \delta(x)$, the controllability of (3.5), is equivalent to the controllability of the following problem:

$$w_{tt} - w_{xx} + \alpha w_t = 0 \tag{3.19}$$

$$w_x(0,t) = -u(t), w(\pi, t) = 0, u(t) \in L^2(0,\pi) \tag{3.20}$$

$$w(x,0) = w^0(x), w_t(x,0) = w^1(x), w^{0,1}(x) \in L^2(0,\pi). \tag{3.21}$$

CHAPTER 4. Mead-Markus model of a sandwich beam

In this chapter, boundary control of the Mead Markus model of a sandwich beam is considered. The equations are given below:

$$m\omega_{tt} + A\omega_{xxxx} - B\gamma s_x = 0 \tag{4.1}$$

$$C\gamma s - s_{xx} + B\omega_{xxx} = 0. \tag{4.2}$$

The side conditions are as follows:

1. Homogenous side conditions : $w(0,t) = w(1,t) = 0, s_x(0,t) = s_x(1,t) = 0$.
2. Side conditions involving the control input : $w_{xx}(0,t) = u(t), w_{xx}(1,t) = 0$.
3. Initial conditions : $w(x,0) = w^0(x), w_t(x,0) = w^1(x)$.

The control input $u(t)$ appears as bending moment at one end. We define the following operator and domain for use in subsequent development.

$$D^2 = \frac{\partial^2}{\partial x^2}$$
$$dom(D^2) = \{\phi \in H^2(0,1) : \phi_x(0) = 0, \phi_x(1) = 0\}.$$

Then Equation (4.2) can be rewritten using the above definition of D in the following way:

$$(C\gamma I - D^2)s = -Bw_{xxx}. \tag{4.3}$$

Proposition 4.0.1 *The operator* $R = (C\gamma I - D^2) : dom(R) \to L^2(0,1)$, *where* $dom(R) = dom(D^2) = \{\phi \in H^2(0,1) : \phi_x(0) = 0, \phi_x(1) = 0\}$ *is boundedly invertible.*

Proof: It can be shown that the eigenvalues and eigenvectors of D^2 are given by $(k^2\pi^2, \cos(k\pi x))$ where $k \in N$. We show that there is nothing else in the spectrum by checking the necessary conditions for the existence of a Green's function for the second order boundary value problem $Ru = f, f \in L^2(0,1)$ which shows that R is a boundedly invertible.

1. The leading coefficient in R is non-zero (namely -1).

2. The boundary conditions $\phi_x(0) = 0, \phi_x(1) = 0$ are independent.

3. The only solution of $Ru = 0$ in $D(R)$ with the boundary conditions $\phi_x(0) = 0, \phi_x(1) = 0$ is $u = 0$.

Hence, a Green's function exists and $J = (C\gamma I - D^2)^{-1}$ exists on $L^2(0,1)$ and is a bounded operator.

Hence Equation (4.3) can be further rewritten as

$$s = -JBD^3w.$$

And hence Equation (4.1) can be rewritten as

$$w_{tt} + Lw = 0 \tag{4.4}$$

$$L = (AD^4 + B^2\gamma DJD^3)\frac{1}{m} \tag{4.5}$$

$$dom(L) = \{\phi \in H^4(0,1) : \phi(0) = \phi(1) = \phi_{xx}(0) = \phi_{xx}(1) = 0\}. \tag{4.6}$$

□

4.1 Eigenvalues and eigenfunctions of J and L

Since $J = (C\gamma I - D^2)^{-1}$, the eigenvalues of J are simply the reciprocal of eigenvalues of $(C\gamma I - D^2)$. Also $dom(C\gamma I - D^2) = dom(D^2) = \{\phi \in H^2(0,1) : \phi_x(0) = 0, \phi_x(1) = 0\}$. The eigenvalues and eigenvectors of J can be easily calculated and shown to be $(\frac{1}{C\gamma+k^2\pi^2}, \cos(k\pi x))$ where $k \in N$. Now we calculate the eigenvalues of L. First, we rewrite L as

$$L = \frac{1}{m} D(A + B^2\gamma J)D^3. \tag{4.7}$$

The eigenvalue-eigenvector pair for the operator $A+B^2\gamma J$ (whose domain is same as that of J), can be shown to be $(A + \frac{B^2\gamma}{C\gamma+k^2\pi^2}, \cos(k\pi x))$ where $k \in N$. Finally, the eigenvalue-eigenvector pairs of L aregiven by $([A + \frac{B^2\gamma}{C\gamma+k^2\pi^2}]\frac{(k\pi)^4}{m}, \sin(k\pi x))$ where $k \in N$. Since $\sin(k\pi x)_{k\in N}$ is a complete sequence in $L^2(0,1)$ we have actually found all eigenfunctions, since any other function in $L^2(0,1)$ (hence in $dom(L)$) can be expressed as a unique linear combination of sines. Now, we take a slight detour in the next section to derive an alternative formulation of Equations following (4.1) .

4.2 Weak formulation

Consider the space of test functions given by:

$$\mathcal{T} = \{\phi \in H^2(0,1) : \phi(0) = \phi(1) = \phi_{xx}(0) = \phi_{xx}(1) = 0\}. \tag{4.8}$$

Now we simply multiply Equation (4.4) by $\phi \in \mathcal{T}$ and integrate by parts.

$$\begin{aligned}
&\int_0^1 (mw_{tt} + AD^4w + B^2\gamma DJD^3w)\phi dx = 0 \qquad (4.9)\\
\Rightarrow &\int_0^1 (mw_{tt}\phi)dx + (Aw_{xxx} + B^2\gamma(Jw_{xxx}))\phi|_0^1 \\
&+ \int_0^1 (Aw_{xx}\phi_{xx} + B^2\gamma(Jw_{xxx})\phi_x)dx \\
&+Au(t)\phi_x(0) = 0.
\end{aligned}$$

We consider the following problem:

$$mw_{tt} + AD^4w + B^2\gamma DJD^3w = -A\delta'(x)u(t), \tag{4.10}$$

with side conditions given by

$$w(0,t) = w(1,t) = 0 \tag{4.11}$$

$$w_{xx}(0,t) = w_{xx}(1,t) = 0. \tag{4.12}$$

We show by doing a similar calculation as above that we end up with the same weak formulation given by (4.10). Here $\delta'(x)$ is the distributional derivative of the Dirac delta with mass at origin. Since point evaluation functionals are continuous on $H^2(0,1)$, $\delta'(x) \in \mathcal{T}'$.

$$\begin{aligned}\int_0^1 (mw_{tt} + AD^4w + B^2\gamma DJD^3w + A\delta'(x)u(t))\phi dx = 0 \\ \Rightarrow \int_0^1 (mw_{tt}\phi)dx + (Aw_{xxx} + B^2\gamma(Jw_{xxx}))\phi|_0^1 \\ + \int_0^1 (Aw_{xx}\phi_{xx} + B^2\gamma(Jw_{xxx})\phi_x)dx \\ + Au(t)\phi_x(0) = 0.\end{aligned}$$

Hence we consider the alternate formulation given above (Equation (4.10) through Equation (4.12)) and try to solve the same.

Let

$$\begin{pmatrix} w_1(t,x) \\ w_2(t,x) \end{pmatrix} = \begin{pmatrix} w(t,x) \\ w_t(t,x) \end{pmatrix}. \tag{4.13}$$

The arguments (t,x) will be omitted from now on in the above notation for simplicity.

Then (4.10) can be rewritten in first order form using equation (4.13) as

$$\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \mathcal{A}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -A\delta'(x) \end{pmatrix} u(t) \tag{4.14}$$

$$\begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix} = \begin{pmatrix} w^0(x) \\ w^1(x) \end{pmatrix} \in \mathcal{D}(\mathcal{A}), \tag{4.15}$$

where

$$\begin{cases} \mathcal{A} = \begin{pmatrix} 0 & I \\ -L & 0 \end{pmatrix}, \\ \mathcal{D}(\mathcal{A}) = (dom(L)) \times H^2_*, \end{cases} \tag{4.16}$$

and

$$H^2_* = \{\phi \in H^2(0,1) : \phi(0) = \phi(1) = 0\} \tag{4.17}$$

$$dom(L) = \{\phi \in H^4(0,1) : \phi(0) = \phi(1) = \phi_{xx}(0) = \phi_{xx}(1) = 0\}, \tag{4.18}$$

and $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{H} = H^2_* \times L^2(0,1)$ where $\mathcal{H}$ is a Hilbert space with the inner product defined as follows: (see (S.W. Hansen) page 43)

$$\left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} h \\ l \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_0^1 (f''\bar{h}'' + g\bar{l})dx. \tag{4.19}$$

4.3 Eigenvalues and eigenfunctions of $\mathcal{A}$

$$\mathcal{A}w = \lambda w$$

$$\Leftrightarrow \begin{pmatrix} 0 & I \\ -L & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \lambda \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\Leftrightarrow w_2 = \lambda w_1, -Lw_1 = \lambda w_2$$

$$\Leftrightarrow Lw_1 = -\lambda^2 w_1.$$

But we already know the eigenvalues and eigenvectors of L from section (4.1) namely $([A + \frac{B^2\gamma}{C\gamma + k^2\pi^2}]\frac{(k\pi)^4}{m}, \sin(k\pi x))$ where $k \in N$.

Hence the eigenvalues and eigenvectors of $\mathcal{A}$ can be shown to be

$$(\lambda_k, \phi_k) = \left(i\mu_k, \begin{pmatrix} \frac{1}{i\mu_k}\sin(k\pi x) \\ \sin(k\pi x) \end{pmatrix} \right), k \in Z^*, \tag{4.20}$$

where

$$\mu_k = \sqrt{\frac{1}{m}[A + \frac{B^2\gamma}{C\gamma + k^2\pi^2}](k\pi)^2}, \ \forall k > 0$$
$$\mu_k = -\mu_{-k}, \ \forall k < 0.$$

4.4 Riesz basis property of $\{\phi_k\}_{k\in Z^*}$

Let

$$\theta_k = \begin{pmatrix} \frac{\sqrt{2}}{k^2\pi^2}\sin(k\pi x) \\ 0 \end{pmatrix}, k \in N$$
$$\eta_k = \begin{pmatrix} 0 \\ \sqrt{2}\sin(k\pi x) \end{pmatrix}, k \in N.$$

Then $\{\phi_k\}_{k\in Z^*}$ is related to θ_k's and η_k's in the following way:

$$\phi_k = \frac{\theta_k}{i\sqrt{\frac{2}{m}\left(A + \frac{B^2\gamma}{C\gamma+k^2\pi^2}\right)}} + \frac{\eta_k}{\sqrt{2}}, k \in N$$
$$\phi_{-k} = \frac{\theta_k}{i\sqrt{\frac{2}{m}\left(A + \frac{B^2\gamma}{C\gamma+k^2\pi^2}\right)}} - \frac{\eta_k}{\sqrt{2}}, k \in N,$$

and

$$\theta_k = \frac{(\phi_k + \phi_{-k})}{\sqrt{2}} i \sqrt{\frac{1}{m}\left(A + \frac{B^2\gamma}{C\gamma + k^2\pi^2}\right)}$$
$$\eta_k = \frac{(\phi_k - \phi_{-k})}{\sqrt{2}}.$$

$(\{\theta_k\}\bigcup\{\eta_k\})_{k\in N}$ is an orthonormal basis for $\mathcal{H}$ since $\{\eta_k\}_{k\in N}$ is an orthonormal basis for $L^2(0,1)$ and the mapping $\frac{\partial^2}{\partial x^2} : H_*^2 \to L^2(0,1)$ is an isometric isomorphism(see (S.W. Hansen) page 49). Define a mapping $T : \mathcal{H} \to \mathcal{H}$ as follows:

$$T(\theta_k) = \phi_k \tag{4.21}$$

$$T(\eta_k) = \phi_{-k}, \tag{4.22}$$

$\forall k \in N$. Also let

$$u(x) = \sum_{k\in N} a_k\theta_k(x) + \sum_{k\in N} b_k\eta_k(x).$$

Then we have,

$$\begin{aligned}
Tu &= \sum_{k\in N}\left(\frac{a_k+b_k}{i\sqrt{\frac{2}{m}\left(A+\left(\frac{B^2\gamma}{C\gamma+k^2\pi^2}\right)\right)}}\right)\theta_k + \sum_{k\in N}\left(\frac{a_k-b_k}{\sqrt{2}}\right)\eta_k \\
||Tu||_{\mathcal{H}}^2 &= \sum_{k\in N}\left(\frac{|a_k+b_k|^2 m}{2\left(A+\left(\frac{B^2\gamma}{C\gamma+k^2\pi^2}\right)\right)} + \frac{|a_k-b_k|^2}{2}\right) \\
&\leq C_1\left(\sum_{k\in N}(|a_k+b_k|)^2 + \sum_{k\in N}(|a_k-b_k|)^2\right) \\
&= C_1\sum_{k\in N}\left(|a_k|^2+|b_k|^2\right) \\
&= C_1||u||_{\mathcal{H}}^2,\ \forall u\in\mathcal{H} \\
T^{-1}u &= \sum_{k\in N}\left(a_k\frac{i}{\sqrt{2}}\sqrt{\frac{1}{m}\left(A+\frac{B^2\gamma}{C\gamma+k^2\pi^2}\right)} + \frac{b_k}{\sqrt{2}}\right)\theta_k \\
&+ \left(a_k\frac{i}{\sqrt{2}}\sqrt{\frac{1}{m}\left(A+\frac{B^2\gamma}{C\gamma+k^2\pi^2}\right)} - \frac{b_k}{\sqrt{2}}\right)\eta_k \\
||T^{-1}u||_{\mathcal{H}}^2 &= \sum_{k\in N}|b_k|^2 + |a_k|^2\frac{1}{m}\left(A+\frac{B^2\gamma}{C\gamma+k^2\pi^2}\right) \\
&\leq \sum_{k\in N} C_2\left(|a_k^2+|b_k|^2\right) \\
&= C_2||u||_{\mathcal{H}}^2,
\end{aligned}$$

where

$$C_1 = max(\frac{m}{2A},\frac{1}{2})$$
$$C_2 = max(\frac{1}{2m}\left[A+\frac{B^2}{C}\right],\frac{1}{2}).$$

Hence $\{\phi_k\}_{k\in Z^*}$ is a Riesz basis and there exists an equivalent norm $||.||_e = ||T^{-1}u||_{\mathcal{H}}$

(R.M. Young) with respect to which $\{\phi_k\}_{k\in Z^*}$ forms an orthonormal basis.

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \sum_{k\in N} a_k\theta_k(x) + \sum_{k\in N} b_k\eta_k(x) \tag{4.23}$$

$$\Rightarrow ||u||_e^2 = ||T^{-1}u||_{\mathcal{H}}^2 = \sum_{k\in N} |b_k|^2 + |a_k|^2\frac{1}{m}\left(A + \frac{B^2\gamma}{C\gamma + k^2\pi^2}\right). \tag{4.24}$$

Proposition 4.4.1 *The norm ($||.||_e$) on $\mathcal{H}$ described in Equation (4.24) is the same as the following energy norm described in terms of actual elements of the $\mathcal{H}$(rather than the Fourier coefficients):*

$$|| \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} ||_e^2 = \left(\int_0^1 |u_2|^2 + \frac{1}{m}A|u_1''|^2 - \frac{1}{m}B^2\gamma(\mathcal{P}u_1')\bar{u_1'}dx\right), \tag{4.25}$$

$$\forall \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} \in \mathcal{H}$$

where

$$\mathcal{P} = (C\gamma I - D^2)^{-1}D^2 \tag{4.26}$$

$$dom(\mathcal{P}) = \{\phi \in H^2(0,1) : \phi(0) = 0, \phi(1) = 0\}. \tag{4.27}$$

Proof:The operator $\mathcal{P}$ is is bounded and densely defined and hence can be extended to all of $L^2(0,1)$. Let

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \sum_{k\in N} a_k\theta_k(x) + \sum_{k\in N} b_k\eta_k(x).$$

Then the following calculations can be easily checked.

$$u_1(x) = \sum_{k\in N} a_k\theta_k(x); u_2(x) = \sum_{k\in N} b_k\eta_k(x)$$

$$\int_0^1 |u_2|^2dx = \sum_{k\in N} |b_k|^2 \tag{4.28}$$

$$\int_0^1 A|u_1''|^2dx = \sum_{k\in N} A|a_k|^2 \tag{4.29}$$

$$
\begin{aligned}
\mathcal{P}u_1'(x) &= (C\gamma I - D^2)^{-1}D^2\left[\sum_{k\in N} a_k \begin{pmatrix} \frac{\sqrt{2}}{k\pi}\cos(k\pi x) \\ 0 \end{pmatrix}\right] \\
&= (C\gamma I - D^2)^{-1}\left[\sum_{k\in N} a_k \begin{pmatrix} -\sqrt{2}k\pi\cos(k\pi x) \\ 0 \end{pmatrix}\right] \\
&= \frac{-1}{C\gamma + k^2\pi^2}\left[\sum_{k\in N} a_k \begin{pmatrix} \sqrt{2}k\pi\cos(k\pi x) \\ 0 \end{pmatrix}\right]
\end{aligned}
$$

$$
\Rightarrow -B^2\gamma\int_0^1 \mathcal{P}u_1'\bar{u_1'}dx =
$$

$$
-B^2\gamma\int_0^1 \sum_{k\in N}\frac{-1}{C\gamma + k^2\pi^2}\left[\sum_{k\in N} a_k \begin{pmatrix} \sqrt{2}k\pi\cos(k\pi x) \\ 0 \end{pmatrix}\right]\left[\sum_{k\in N} a_k \begin{pmatrix} \frac{\sqrt{2}}{k\pi}\cos(k\pi x) \\ 0 \end{pmatrix}\right]
$$

$$
= \sum_{k\in N} |a_k|^2 \frac{B^2\gamma}{C\gamma + k^2\pi^2}. \tag{4.30}
$$

Using Equations (4.28,4.29 and 4.30) we have ,

$$
\begin{aligned}
&\left(\int_0^1 |u_2|^2 + \frac{1}{m}A|u_1''|^2 - \frac{1}{m}B^2\gamma(\mathcal{P}u_1')\bar{u_1'}dx\right) \qquad (4.31)\\
&= \sum_{k\in N} |b_k|^2 + |a_k|^2\frac{1}{m}\left(A + \frac{B^2\gamma}{C\gamma + k^2\pi^2}\right).
\end{aligned}
$$

□

Remark: The energy norm ($||.||_e$) described in the Equation (4.25) describes the actual energy of the system(see (R.H. Fabiano and S.W. Hansen)). We call the space ($\mathcal{H}, ||.||_e$) as the energy space.

4.5 Isomorphisms

In this section we describe some isomorphisms which will be used later on.

Let $X = (\mathcal{H}, < . >_{\mathcal{H}})$ be a Hilbert space, $\mathcal{A}$ be the generator of a C_0 semigroup on $\mathcal{H}$ with domain $D(\mathcal{A})$ dense in $\mathcal{H}$. For simplicity, assume $0 \in \rho(\mathcal{A})$ (since this is true in our cases). Then we have the following spaces and isomorphisms.

$$\begin{array}{ccc} (X_1, ||.||_1) & \leftrightarrow & (\mathcal{X}_1, ||.||_1) \\ \downarrow\mathbf{A} & & \downarrow\mathcal{A} \\ (X_0, ||.||_{l^2}) & \leftrightarrow & (\mathcal{X}_0, ||.||_{\mathcal{H}}) \\ \downarrow\widehat{\mathbf{A}} & & \downarrow\widehat{\mathcal{A}} \\ (X_{-1}, ||.||_{-1}) & \leftrightarrow & (\mathcal{X}_{-1}, ||.||_{-1}) \end{array} \tag{4.32}$$

The above diagram has been referred to as "Sobolev towers" and a detailed explanation about the same can be found in (K.J Engel and H. Nagel). The spaces,operators and norms appearing on the right side of Figure (4.32) are defined below.

1. $(\mathcal{X}_1, ||.||_1) = (D(\mathcal{A}), ||.||_1)$ is a Hilbert space with innerproduct defined as

$$< x, y >_1 = < Ax, Ay >_{\mathcal{H}} .$$

2. $D(\mathcal{A}) = \{\phi : \mathcal{A}\phi \in \mathcal{H}\}, ||.||_1 = ||\mathcal{A}x||_{\mathcal{H}}, \ \forall x \in D(\mathcal{A}).$

3. $(\mathcal{X}_0, ||.||_{\mathcal{H}}) = (\mathcal{H}, ||.||_{\mathcal{H}})$ is the given Hilbert space.

4. $\widehat{\mathcal{A}}$ is an extension of $\mathcal{A}$ in the sense described below. We define the following norm on $\mathcal{H}$.

$$||x||_{-1} = ||\mathcal{A}^{-1}x||_{\mathcal{H}}, \ \forall x \in \mathcal{H}.$$

We see from the following derivation that $\mathcal{A}$ is a bounded operator from $(D(\mathcal{A}), ||.||_{\mathcal{H}})$ to $(\mathcal{H}, ||.||_{-1})$.

$$||\mathcal{A}x||_{-1} = ||\mathcal{A}^{-1}\mathcal{A}x||_{\mathcal{H}} = ||x||_{\mathcal{H}}, \ \forall x \in D(\mathcal{A}).$$

Since $D(\mathcal{A})$ is a dense subset of $(\mathcal{H}, ||.||_{\mathcal{H}})$, $\mathcal{A}$ can be extended to an operator $\widehat{\mathcal{A}} : (\mathcal{H}, ||.||_{\mathcal{H}}) \to (\mathcal{H}, ||.||_{-1})$.

5. $(\mathcal{X}_{-1}, ||.||_{-1})$ is the completion of $\mathcal{H}$ with respect to $||.||_{-1}$.

The spaces on the left hand side of Figure (4.32) are spaces of sequences as defined below.

1. $X_1 = \{\{c_k\} : \sum_k |\lambda_k c_k|^2 < \infty\}, ||\{c_k\}||_1 = \sqrt{\sum_k |\lambda_k c_k|^2}$.

2. $X_{-1} = \{\{c_k\} : \sum_k |\frac{c_k}{\lambda_k}|^2 < \infty\}, ||\{c_k\}||_{-1} = \sqrt{\sum_k |\frac{c_k}{\lambda_k}|^2}$.

3. $\mathbf{A}\{c_k\} = \lambda_k c_k$, is the version of $\mathcal{A}$ as applied to sequences in $(X_1, ||.||_1)$.

4. $\widehat{\mathbf{A}}$ is an extension of $\mathbf{A}$ done in a similar way as for $\widehat{\mathcal{A}}$.

Hence the spaces on the left hand side are related to the spaces on the right hand side of Figure (4.32) via the operators $\mathcal{A}$ and $\widehat{\mathcal{A}}$ and the through the spaces $\mathcal{H}$ and l^2, and $\mathcal{H}$ is related to l^2 by the usual Hilbert space isomorphism. The isomorphisms described in Figure (4.32) allow us to view $\mathcal{A}$ equivalently as $\mathbf{A}$ which is the infinitesimal generator of a diagonal semigroup $T(t)$ acting on l^2, instead of the original Hilbert space. Any result proved in the sequence spaces can be translated to the original spaces using the isomorphisms. We use the original symbol $\mathcal{A}$ instead of $\mathbf{A}$ from now on.

4.6 Semigroup formulation

It can be easily checked that $\{\phi_k\}_{k\in Z^*}$ is an orthogonal sequence in the energy norm $(||.||_e)$ and hence after introducing an appropriate normalizing constant is an orthonormal basis for which we use the same symbol $\{\phi_k\}_{k\in Z^*}$. Let

$$\tilde{w}(t,x) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \sum_{k\in Z^*} w_k(t)\phi_k(x) \tag{4.33}$$

be a solution of (4.14) in $\mathcal{D}(\mathcal{A})$ where $\phi_k(x)$ are the eigenvectors of $\mathcal{A}$ found above and $w_k(t)$ are scalar functions of time.

Proposition 4.6.1 $\mathcal{A} : (D(\mathcal{A}), <.>_e) \to (\mathcal{H}, <.>_e)$ *is a skew adjoint operator.*

Proof: By the discussion above, it is enough to consider the operation of $\mathcal{A}$ on sequences.

$$\begin{aligned} < \mathcal{A}\{c_k\}_{k\in Z^*}, \{d_k\}_{k\in Z^*} >_{l^2} &= \sum_{k\in Z^*} c_k\lambda_k\bar{d}_k = \sum_{k\in Z^*} d_k\bar{\lambda}_k\bar{c}_k \\ &= -\sum_{k\in Z^*} d_k\lambda_k\bar{c}_k =< \{c_k\}_{k\in Z^*}, -\mathcal{A}\{d_k\}_{k\in Z^*} >_{l^2} \\ &\qquad \forall\{c_k\},\ \{d_k\} \in l^2(Z^*). \ \square \end{aligned}$$

Also let

$$\tilde{w}(0,x) = \begin{pmatrix} w^0(x) \\ w^1(x) \end{pmatrix} = \sum_{k\in Z^*} w_k(0)\phi_k(x) \tag{4.34}$$

$(\mathcal{H}, <.>_e)$ is isometrically isomorphic to $l^2(Z^*)$ and from the previous section on isomorphisms, we can equivalently consider $\mathcal{A}$ acting on the Fourier coefficients of elements in $(D(\mathcal{A}), <.>_e)$. Substituting (4.33) in (4.14) , we get

$$\tilde{w}_t = \mathcal{A}\tilde{w} + \begin{pmatrix} 0 \\ -A\delta'(x) \end{pmatrix} u(t)$$

$$\Rightarrow \langle \tilde{w}_t, \phi_k(x)\rangle = \langle \mathcal{A}\tilde{w}, \phi_k(x)\rangle + \left\langle \begin{pmatrix} 0 \\ -A\delta'(x) \end{pmatrix} u(t), \phi_k(x) \right\rangle$$

$$\Rightarrow \left\langle \sum_k w_k'(t)\phi_k(x), \phi_k(x) \right\rangle = \left\langle \sum_k w_k(t)(\mathcal{A}\phi_k(x)), \phi_k(x) \right\rangle + \left\langle \sum_k f_k\phi_k(x), \phi_k(x) \right\rangle u(t)$$

$$\Rightarrow w_k'(t) = i\mu_k w_k(t) + f_k u(t),\ \forall k \in Z^*$$

$$\Rightarrow w_k(t) = e^{i\mu_k t}w_k(0) + \int_0^t e^{i\mu_k(t-s)} f_k u(s)ds,\ \forall k \in Z^*,$$

where

$$\begin{pmatrix} 0 \\ -A\delta'(x) \end{pmatrix} = \sum_k f_k\phi_k(x), f_k = \left\langle \begin{pmatrix} 0 \\ -A\delta'(x) \end{pmatrix}, \phi_k(x) \right\rangle_e = -Ak\pi, k \in Z^*. \tag{4.35}$$

Next, we prove two theorems to establish the semigroup property of Equation (4.6).

Theorem 4.6.1 *$\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is the infinitesimal generator of a C_0 semigroup of contractions on $\mathcal{H}$*

Proof: We simply verify the sufficient conditions of the Hille-Yosida theorem (A. Pazy).

1. First we note that $D(\mathcal{A})$ can be identified with a subset of $l^2(Z^*)$, namely $D(\mathcal{A}) = \{\{x_k\} \in l^2(Z - \{0\}) : \{i\mu_k x_k\} \in l^2(Z^*)\}$. This is possible because $\{\phi_k\}_{k \in Z-\{0\}}$ form an orthonormal basis for $(\mathcal{H}, ||.||_e)$. This allows us to $T(t)$ as a diagonal semigroup $T_t(\{x_k\}) = (\{e^{i\mu_k t} x_k\})$, where $\{x_k\} \in l^2(Z^*)$. $D(\mathcal{A}) \supseteq$ {finite linear combinations of the standard basis vectors $e_k \in l^2(Z^*)$} and hence $\mathcal{A}$ is densely defined.

2. Next we show that the resolvent of $\mathcal{A}$ contains $(0, \infty)$. Let $\{u_k\}, \{f_k\}, \in l^2(Z^*)\ \forall k \in Z^*$.Let $\mathcal{R}(\lambda; \mathcal{A})$ denote the resolvent operator of $\mathcal{A}$.

$$\mathcal{R}(\lambda; \mathcal{A})\{u_k\} = \{f_k\}, \{f_k\} \in l^2(Z^*)$$
$$\Rightarrow (\mathcal{A} - \lambda I)\{u_k\} = \{f_k\} \Rightarrow u_k = \frac{f_k}{\lambda - \lambda_k}, \ \forall k \in Z - \{0\}, \lambda \in [0, \infty),$$

   where λ_k's are the eigenvalues of $\mathcal{A}$(which are imaginary). We have

$$||u_k||_{l^2}^2 = \sum_{k \in Z^*} \frac{|f_k|^2}{|\lambda - \lambda_k|^2}$$
$$= \sum_{k \in Z^*} \frac{|f_k|^2}{|\lambda|^2 + |\lambda_k|^2} \leq \sum_{k \in Z^*} \frac{|f_k|^2}{|\lambda|^2} < \infty, \ \forall \lambda > 0$$
$$\Rightarrow \{u_k\} \in l^2(Z^*)$$

   Hence we have shown that $||\mathcal{R}(\lambda; \mathcal{A})|| \leq \frac{1}{\lambda}$, $\forall \lambda > 0$, and in fact the resolvent set satisfies $\rho(\mathcal{A}) \supseteq (0, \infty)$.

3. Finally since $\mathcal{A}$ is a linear operator and has a non-empty resolvent, it follows that $\mathcal{A}$ is closed. $\square$

Next we show that $\{f_k\}$ defined in Equation (4.35) forms an admissible input for Equation (4.6) in the space $\mathcal{H}_{-\frac{1}{2}} = (\{d_k\} : \{\frac{d_k}{\sqrt{|\lambda_k|}}\} \in l^2(Z^*))$. We define $\mathcal{A}^{-\frac{1}{2}} : \mathcal{H}_{-\frac{1}{2}} \to l^2(Z^*)$ in the following way:

$$\mathcal{A}^{-\frac{1}{2}}\{c_k\} = \{\frac{c_k}{\sqrt{|\lambda_k|}}\}. \tag{4.36}$$

Then $\mathcal{A}^{-\frac{1}{2}}$ is an isomorphism from $\mathcal{H}_{-\frac{1}{2}}$ into $l^2(Z^*)$,and hence the completion of $\mathcal{H}_{-\frac{1}{2}}$ with respect to the norm defined by $||\{d_k\}||_{\mathcal{H}_{-\frac{1}{2}}} = ||\frac{d_k}{\sqrt{\lambda_k}}||_{l^2(Z^*)}$ is a Hilbert space(denoted by the same symbol $\mathcal{H}_{-\frac{1}{2}}$) .

Theorem 4.6.2 $\{f_k\} = \{-Ak\pi\}$, $\forall k \in Z^*$ *forms an admissible input for Equation (4.6) in* $\mathcal{H}_{-\frac{1}{2}}$.

Proof: Equivalently, we prove that $\mathcal{A}^{-\frac{1}{2}}\{f_k\}$ forms an admissible input for $\mathcal{H}$. This will prove the theorem since $\mathcal{A}^{-\frac{1}{2}} : \mathcal{H}_{-\frac{1}{2}} \to l^2(Z^*)$ is an isomorphism.

We verify that $\{\mathcal{A}^{-\frac{1}{2}}f_k\}_{k\in\{Z^*\}}$ satisfies the Carleson measure criterion as discussed in (G. Weiss). Recall that

$$\lambda_k = i\mu_k, k \in Z^*,$$

where

$$\mu_k = \sqrt{\frac{1}{m}[A + \frac{B^2\gamma}{C\gamma + k^2\pi^2}](k\pi)^2}, \ \forall k > 0$$
$$\mu_k = -\mu_{-k}, \ \forall k < 0,$$

and $f_k = -Ak\pi, k \in Z^*$. Hence $g_k = \mathcal{A}^{-\frac{1}{2}}f_k = \frac{-Ae^{\frac{i\pi}{4}}}{\left(\frac{1}{m}[A+\frac{B^2\gamma}{C\gamma+k^2\pi^2}]\right)^{\frac{1}{4}}}$, $\forall k \in Z^*$. Equation (4.6) can be rewritten in the following way.

$$\mathcal{A}^{-\frac{1}{2}}w_k(t) = \mathcal{A}^{-\frac{1}{2}}e^{i\mu_k t}w_k(0) + \int_0^t e^{i\mu_k(t-s)}g_k u(s)ds, \ \forall k \in Z^* \tag{4.37}$$

If we choose $\{w_k\} \in \mathcal{H}_{-\frac{1}{2}}$, then $\mathcal{A}^{-\frac{1}{2}}\{w_k\} \in l^2(Z^*)$. We define the following rectangle in the complex plane.

$$R(h,\omega) = \{z \in C : 0 \leq Re(z) \leq h, |Im(z) - \omega| \leq h\} \tag{4.38}$$

Then we have

$$\sqrt{\frac{1}{m}[A]}(k\pi)^2 \leq |\lambda_k| \leq \sqrt{\frac{1}{m}[A + \frac{B^2}{C}]}(k\pi)^2$$

which implies λ_k's are clustered around origin and spread out as we go away from the origin on the imaginary axis. We also have the following estimate on $\{g_k\}$ by a direct calculation.

$$\frac{A}{\left(\frac{1}{m}[A + \frac{B^2}{C}]\right)^{\frac{1}{4}}} \leq |g_k| \leq \frac{A}{\left(\frac{1}{m}[A]\right)^{\frac{1}{4}}}, k \in Z^* \tag{4.39}$$

Hence we have

$$\begin{aligned}
\sum_{-\lambda_k \in R(h,\omega)} |g_k|^2 &\leq \sum_{-\lambda_k \in R(h,0)} |g_k|^2 \\
\leq \sum_{-\lambda_k \in R(h,0)} &\left|\frac{-Ae^{\frac{i\pi}{4}}}{\left(\frac{1}{m}[A + \frac{B^2\gamma}{C\gamma + k^2\pi^2}]\right)^{\frac{1}{4}}}\right|^2 \\
&\leq \sum_{-\lambda_k \in R(h,0)} \frac{A^2}{\left(\frac{1}{m}[A]\right)^{\frac{1}{2}}} \\
&= \sum_{-\lambda_k \in R(h,0)} \frac{A^2}{\left(\frac{1}{m}[A]\right)^{\frac{1}{2}}} \\
&\leq \mathcal{O}(h), \ \forall h > 0,
\end{aligned}$$

where in Equation (4.40)we have made use of the fact that the number of eigenvalues in $R(h,0)$ is $\mathcal{O}(\sqrt{h})$. $\square$

Remark(Characterization of $\mathcal{H}_{-\frac{1}{2}}$) In this section we show that

$$\mathcal{H}_{-\frac{1}{2}} \simeq H_0^1 \times H^{-1},$$

where

$$H_0^1 = \{\phi \in H^1[0,1] : \phi' \in L^2(0,1), \phi(0) = \phi(1) = 0\}, H^{-1} = (H_0^1)'. \tag{4.40}$$

To see this, we first note the following equivalent description of $\mathcal{H}_{-\frac{1}{2}}$:

$$\mathcal{H}_{-\frac{1}{2}} = (\{d_k\} : \{\frac{d_k}{k}\} \in l^2(Z^*))$$

We refer back to section (4.4) and the operator T used to prove the Riesz basis property of $\{\phi_k\}_{k \in Z^*}$. Since T was shown to be a boundedly invertible operator, we can equivalently examine the linear combinations of elements from the sequence $(\{\theta_k\} \bigcup \{\eta_k\})_{k \in N}$ with coefficients from $\mathcal{H}_{-\frac{1}{2}}$ where

$$\theta_k = \begin{pmatrix} \frac{\sqrt{2}}{k^2\pi^2} \sin(k\pi x) \\ 0 \end{pmatrix}, k \in N$$

$$\eta_k = \begin{pmatrix} 0 \\ \sqrt{2} \sin(k\pi x) \end{pmatrix}, k \in N.$$

We also note that convergence of $\sum_{k \in N} d_k \theta_k$ with $d_k \in \mathcal{H}_{-\frac{1}{2}}$ is the same as convergence of $\sum_{k \in N} \frac{\sqrt{2}}{k^2\pi^2} d_k \sin(k\pi x)$ and a similar statement for η_k's is true. Hence we have the following alternative description for $\mathcal{H}_{-\frac{1}{2}}$.

$$\mathcal{H}_{-\frac{1}{2}} = \{\{c_k\}_{k \in N} : \{kc_k\} \in l^2(N)\} \times \{\{c_k\}_{k \in N} : \{\frac{c_k}{k}\} \in l^2(N)\}$$

Since $\{\sqrt{2}\sin(k\pi x)\}_{k \in N}\}$ forms an orthonormal basis for $L^2(0,1)$ and point evaluation functionals are continuous in H^1, in terms of actual elements we have,

$$\{\{c_k\}_{k \in N} : \{kc_k\} \in l^2(N)\} \simeq H_0^1 = \{\phi \in L^2(0,1) : \phi_x \in L^2(0,1), \phi(0) = \phi(1) = 0\}.$$

Since $-\frac{\partial^2}{\partial x^2} : H_0^1 \to H^{-1}$ is an isometric isomorphism, we have,

$$\{\{c_k\}_{k \in N} : \{kc_k\} \in l^2(N)\}' = \{\{c_k\}_{k \in N} : \{\frac{c_k}{k}\} \in l^2(N)\} \simeq H^{-1}$$

Hence

$$\mathcal{H}_{-\frac{1}{2}} \simeq H_0^1 \times H^{-1}.$$

The above statement characterizes $\mathcal{H}_{-\frac{1}{2}}$ in terms of the actual elements of the space rather than the Fourier coefficients of its elements. However, the Fourier coefficients allow us to do easy calculations. Since controlling the Fourier coefficients of the states is equivalent to controlling the states themselves, we continue our analysis using the Fourier coefficients.

4.7 The moment problem

In this and the next section we show that the Mead-Markus model of a sandwiched beam given by equations following (4.1) is exactly controllable for all initial and final states $w_k(0), w_k(T) \in \mathcal{H}_{-\frac{1}{2}}$ or equivalently, the controllable space is $H_0^1 \times H^{-1}$. For exact controllability, we want to find $u(t) \in L^2(0,T)$ so that $w_k(0)$ is driven to a specified $w_k(T)$ in finite time. Equation (4.37) can be rewritten in the following form:

$$\int_0^T e^{i\mu_k \tau}\tilde{u}(\tau)d\tau = c_k, \ \forall k \in Z^*, \tag{4.41}$$

where

$$c_k = \frac{\mathcal{A}^{-\frac{1}{2}}\{w_k(T) - e^{i\mu_k T}w_k(0)\}}{\mathcal{A}^{-\frac{1}{2}}f_k}.$$

Using the estimate on $\{g_k\}$ from Equation (4.39), we have

$$\sum_{k\in Z-\{0\}} |c_k|^2 \leq \sum_{k\in Z-\{0\}} \frac{|\mathcal{A}^{-\frac{1}{2}}\{w_k(T) - e^{i\mu_k T}w_k(0)\}|^2}{\left(\frac{A}{\left(\frac{1}{m}[A+\frac{B^2}{C}]\right)^{\frac{1}{4}}}\right)^2} < \infty,$$

and T is the final time instant and $\tilde{u}(t) = u(T-t)$. If we are looking for a control in $L^2(0,T)$, then equation (4.41) can be rewritten as

$$\left\langle \tilde{u}(t), e^{-i\mu_k t}\right\rangle_{L^2(0,T)} = c_k, \ \forall k \in Z^*, \tag{4.42}$$

where $\{c_k\} \in l^2(Z^*)$. Hence the original problem has been transformed in to the moment problem given by Equation (4.42).

4.8 Solution of the moment problem

In order to solve the moment problem given by Equation (4.42) we need the following classical theorem due to Ingham (R.M. Young):

Theorem 4.8.1 ***(Ingham's theorem)*** *Let $\{\mu_k\}_{k\in Z^*}$ be a sequence of real numbers such that*

$$\mu_{k+1} - \mu_k \geq \gamma > 0, \ \forall k \in Z^*, \tag{4.43}$$

Then, for any $T > \frac{2\pi}{\gamma}, \exists C(T,\gamma) > 0, \ni$

$$\frac{1}{C(T,\gamma)} \sum_{k\in Z^*} |a_k|^2 \leq \int_0^T \left| \sum_{k\in Z^*} a_k e^{i\mu_k t} \right|^2 dt \leq C(T,\gamma) \sum_{k\in Z^*} |a_k|^2, \tag{4.44}$$

$\forall \{a_k\}_{k\in Z^*} \in l^2(Z^*)$.

We define the moment operator $\mathcal{M} : \mathcal{S} \rightarrow l^2(Z^*)$ (where T is the control time to be determined and), as follows:

$$\mathcal{M}(u) = \left\{ \langle u(t), e^{i\mu_k t} \rangle_{L^2(0,T)} \right\}_{k\in Z^*}, \ \forall u \in \mathcal{S}$$
$$\mathcal{S} = \left\{ \overline{Span\{e^{i\mu_k t}\}_{k\in Z^*}} \right\} \subseteq L^2(0,T).$$

The following is a derivation of the $\mathcal{M}^* : l^2(Z^*) \rightarrow \mathcal{S}$, the adjoint of $\mathcal{M}$

$$\begin{aligned}
\langle \mathcal{M}u, \{a_k\} \rangle_{l^2(Z^*)} &= \sum_{k=-\infty}^{\infty} \left(\int_0^T u(\tau) e^{-i\mu_k \tau} d\tau \right) \overline{a_k} \\
&= \int_0^T u(\tau) \overline{\left(\sum_{k=-\infty}^{\infty} a_k e^{i\mu_k \tau} \right)} \\
&= \left\langle u, \left(\sum_{k=-\infty}^{\infty} a_k e^{i\mu_k \tau} \right) \right\rangle_{L^2(0,T)} \\
&= \langle u, \mathcal{M}^* \{a_k\} \rangle_{L^2(0,T)} \ \forall \{a_k\} \in l^2(Z^*), u \in \mathcal{S}.
\end{aligned}$$

Hence $\mathcal{M}^*\{a_k\} = \sum_{k=-\infty}^{\infty} a_k e^{i\mu_k \tau}\ \forall\{a_k\} \in l^2(Z^*)$. Ingham's theorem states that under a certain minimum separation of the eigenvalues ($\{\mu_k\}_{k\in Z^*}$), $\mathcal{M}^* : l^2 \to \mathcal{S}$ is a bounded operator and is also bounded away from $\{0\}$.

We check the separation property needed to apply Ingham's theorem. Recall that

$$\mu_k = \sqrt{\frac{1}{m}[A + \frac{B^2\gamma}{C\gamma + k^2\pi^2}](k\pi)^2},\ \forall k > 0$$
$$\mu_k = -\mu_{-k},\ \forall k < 0$$

$$\begin{aligned}
\mu_{k+1} - \mu_k &= \sqrt{\frac{1}{m}[A + \frac{B^2\gamma}{C\gamma + (k+1)^2\pi^2}]((k+1)\pi)^2} - \sqrt{\frac{1}{m}[A + \frac{B^2\gamma}{C\gamma + k^2\pi^2}](k\pi)^2} \\
&\geq \sqrt{\frac{1}{m}[A + \frac{B^2\gamma}{C\gamma + (k+1)^2\pi^2}]((k-\frac{1}{2})\pi)^2} - \sqrt{\frac{1}{m}[A + \frac{B^2\gamma}{C\gamma + (k+1)^2\pi^2}](k\pi)^2} \\
&\geq \sqrt{\frac{A}{m}}[((k+\frac{1}{2})\pi)^2 - (k\pi)^2] = \sqrt{\frac{A}{m}}(k\pi + \frac{\pi^2}{4}) \geq \sqrt{\frac{A}{m}}\frac{\pi^2}{4},\ \forall k > 0
\end{aligned}$$

A similar calculation is true $\forall k < 0$ and the same lower bound can be proved. Hence, by Ingham's theorem $\mathcal{M}^*$ is a bounded operator and is also bounded away from zero. This implies that $\mathcal{M}$ is onto, i.e $\mathcal{M} : \mathcal{S} \to l^2(Z^*)$ is onto from the theorem stated below.

Theorem 4.8.2 *Let $\mathcal{M} : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. If $\mathcal{M}^* : \mathcal{H}_2 \to \mathcal{H}_1$ is a bounded linear operator and bounded away from zero then $\mathcal{M}$ is onto.*

Proof: From the hypotheses, we have that

$$C_1||x||_{\mathcal{H}_2} \leq ||\mathcal{M}^*x||_{\mathcal{H}_1} \leq C_2||x||_{\mathcal{H}_2}, C_1, C_2 \geq 0$$

Let $y_0 \in \mathcal{H}_2$, $y_0 \neq 0$.

Claim: $\mathcal{M}^*(y_0^\perp)$ is closed in $\mathcal{H}_1$ and $\mathcal{M}^*(y_0^\perp) \neq \mathcal{M}^*(\mathcal{H}_2)$

Reason: Let $y_n \in \mathcal{M}^*(y_0^\perp)$, $y_n \to y$. Then $y_n = \mathcal{M}^*x_n$, $||x_n - x_m|| \leq \frac{1}{C_1}||y_n - y_m|| \to 0$

$\Rightarrow x_n \to x \in y_0^\perp$

$\Rightarrow \mathcal{M}^* x_n \to \mathcal{M}^* x$

$\Rightarrow \mathcal{M}^* x = y.$

Also since $y_0 \notin y_0^\perp$, and $\mathcal{M}$ is one to one we have $\mathcal{M}^*(y_0^\perp) \neq \mathcal{M}^*(\mathcal{H}_2)$.

Finally we claim that $\mathcal{M}$ is onto. Since $\mathcal{M}^*(y_0^\perp)$ is closed, $\exists$ a vector $x \in \mathcal{H}_1$ such that $< x, v >_{\mathcal{H}_1} = 0 \ \forall z \in \mathcal{M}^*(y_0^\perp)$, $< x, w >_{\mathcal{H}_1} \neq 0 \ \forall z \in \mathcal{M}^*(\mathcal{H}_2)$

$\Rightarrow < x, \mathcal{M}^* w >_{\mathcal{H}_1} \neq 0 \ \forall w \in \mathcal{H}_2$

$\Rightarrow < \mathcal{M} x, w >_{\mathcal{H}_1} \neq 0 \ \forall w \in \mathcal{H}_2$

$\Rightarrow \mathcal{M} x \neq 0$

Similarly we have $< \mathcal{M} x, v >_{\mathcal{H}_1} = 0 \ \forall v \in y_0^\perp$.

$\Rightarrow \mathcal{M} x = \alpha y_0$ for some $\alpha \neq 0$.

And hence we have found $x_0 = \frac{1}{\alpha} x$, such that $\mathcal{M} x_0 = y_0$. Since $y_0 \in \mathcal{H}_2$ was arbitrary, we have that $\mathcal{M}$ is onto. $\square$

Hence the moment problem given by Equation (4.41) has a solution for a control time $T > \frac{8}{\pi}\sqrt{\frac{m}{A}}$. Hence we have the following theorem:

Theorem 4.8.3 *The Mead-Markus model of a sandwiched beam given by equations following (4.1) through is exactly controllable for all initial and final states* $(w, w_t) \in H_0^1 \times H^{-1}$ *if the control time satisfies* $T > \frac{8}{\pi}\sqrt{\frac{m}{A}}$.

4.9 Improvement in control time

We show that the control time in Theorem 4.8.3 can be made arbitrarily small. We first define some terminologies state some main theorems associated with sets of complex exponentials (R. Redheffer)which will be used in the proof.

Definition 4.9.1 *A set of complex exponentials* $\{e^{i\lambda_k t}\}$ *is free if no one of them is in the closure of the space spanned by linear combination of others.*

Definition 4.9.2 *A set of complex exponentials $\{e^{i\lambda_k t}\}$ is linked if every one of them is in the closure of the space spanned by others.*

We have the following theorem due to *L.Schwartz*

Theorem 4.9.1 *On a given interval, every set $\{e^{i\lambda_k t}\}$ with distinct λ_k's is either free or linked.*

We have the following Lemma from (S.A. Avdonin)

Lemma 4.9.1 *If $\mathcal{M}$ and $\mathcal{N}$ are closed subspaces in a Hilbert space $\mathcal{H}$ satisfying the following conditions:*

- $\mathcal{M} \bigcap \mathcal{N} = \{0\}$
- *dim*$(\mathcal{N}) < \infty$

Then the restricted projection operators $P_{\mathcal{M}|\mathcal{N}^\perp} : \mathcal{N}^\perp \to \mathcal{M}$ and $P_{\mathcal{N}|\mathcal{M}^\perp} : \mathcal{M}^\perp \to \mathcal{N}$ are isomorphisms onto their respective images.

Now we state and prove the main theorem in this section.

Theorem 4.9.2 *The Mead-Markus model of a sandwiched beam given by equations following (4.1) is exactly controllable for all initial and final states $(w, w_t) \in H_0^1 \times H^{-1}$ in time T, where T is any arbitrary positive number.*

Proof: We first set up things in the framework of Theorem 4.9.1. Recall that exact controllabilty is equivalent to solving the following moment problem.

$$\left\langle \tilde{u}(t), e^{-i\mu_k t} \right\rangle_{L^2(0,T)} = c_k, \ \forall k \in Z^* \tag{4.45}$$

where $\{c_k\} \in l^2(Z^*)$ and

$$\mu_k = \sqrt{\frac{1}{m}[A + \frac{B^2\gamma}{C\gamma + k^2\pi^2}](k\pi)^2}, \ \forall k > 0$$
$$\mu_k = -\mu_{-k}, \ \forall k < 0$$

$$\sqrt{\frac{1}{m}[A]}(k\pi)^2 \leq |\lambda_k| \leq \sqrt{\frac{1}{m}[A+\frac{B^2}{C}]}(k\pi)^2$$

We repeat the minimum gap calculation, and realize that we can make the gap γ arbitrarily bigger by choosing N large enough.

$$\begin{aligned}
\mu_{k+1}-\mu_k &= \sqrt{\frac{1}{m}[A+\frac{B^2\gamma}{C\gamma+(k+1)^2\pi^2}]((k+1)\pi)^2} - \sqrt{\frac{1}{m}[A+\frac{B^2\gamma}{C\gamma+k^2\pi^2}](k\pi)^2} \\
&\geq \sqrt{\frac{1}{m}[A+\frac{B^2\gamma}{C\gamma+(k+1)^2\pi^2}]((k-\frac{1}{2})\pi)^2} - \sqrt{\frac{1}{m}[A+\frac{B^2\gamma}{C\gamma+(k+1)^2\pi^2}](k\pi)^2} \\
&\geq \sqrt{\frac{A}{m}((k+\frac{1}{2})\pi)^2-(k\pi)^2} = \sqrt{\frac{A}{m}(N\pi+\frac{\pi^2}{4})} \geq \sqrt{\frac{A}{m}(N\pi)}, \ \forall |k|>N
\end{aligned}$$

Hence we decompose the moment problem given by Equation (4.45) into two moment problems in the following way:

$$\langle \tilde{u}(t), e^{-i\mu_k t}\rangle_{L^2(0,T)} = c_k, k \in Z-[-N,N] \tag{4.46}$$

$$\langle \tilde{u}(t), e^{-i\mu_k t}\rangle_{L^2(0,T)} = c_k, k \in [-N,N]-\{0\} \tag{4.47}$$

Let $\mathcal{H}=\overline{Span(\{e^{i\mu_k t}\}_{k\in Z^*})}, \mathcal{M}=\overline{Span(\{e^{i\mu_k t}\}_{k\in Z-[-N,N]})}, \mathcal{N}=Span(\{e^{i\mu_k t}\}_{k\in[-N,N]-\{0\}})$. We choose N large enough to make the gap γ arbitrarily large. Then we can solve Equation (4.46) for $u_{\mathcal{M}} \in \mathcal{M} \subsetneq \mathcal{H} = L^2(0,T)$ for arbitrarily small control time T using Ingham's theorem. Using Theorem 4.9.1, we include those exponentials from $\mathcal{N}$ which belong to $\mathcal{M}$ into $\mathcal{M}$ itself until we end up either with the empty set or finitely many exponentials in $\mathcal{N}$. We rename the new spaces using the same symbols $\mathcal{M}$ and $\mathcal{N}$.

- If we end up with empty set for $\mathcal{N}$ then we are done since we have already solved Equation (4.45) for $u_{\mathcal{M}} \in \mathcal{M}$ and $\mathcal{M}=\mathcal{H}$. So $u=u_{\mathcal{M}}$ solves the problem.
- If not, then Equation (4.47) can also be solved for $u_{\mathcal{N}} \in \mathcal{N}$,by using a biorthogonal sequence to $\{e^{i\mu_k t}\}_{k\in[-N,N]}$. Define $l_k : \mathcal{N} \to \mathcal{R}$ in the following way:

$$l_k(e^{i\mu_j t}) = \delta_{kj}, \ \forall k \in [-N,N]-\{0\} \tag{4.48}$$

Then it can be checked that l_k is a bounded linear functional $\forall k \in [-N, N] - \{0\}$. Hence by the Riesz representation theorem $\exists y_k \in \mathcal{N}$ such that

$$l_k(.) = < y_k, . >, \ \forall k \in [-N, N] - \{0\}.$$

Hence $< e^{i\mu_k t}, y_j > = \delta_{kj}$, and $u(t) = \sum_{k\in[-N,N]-\{0\}} c_k y_k$ solves Equation (4.47)

Now we have $\mathcal{M} \bigcap \mathcal{N} = \{0\}$ and $\dim(\mathcal{N}) < \infty$. Hence we can use Lemma (4.9.1) to construct $u = (P_{\mathcal{M}|\mathcal{N}^\perp})^{-1} u_{\mathcal{M}} + (P_{\mathcal{N}|\mathcal{M}^\perp})^{-1} u_{\mathcal{N}}$ which will solve Equation (4.45). $\square$

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