Shooting methods for numerical solutions of control problems constrained by linear and nonlinear hyperbolic partial differential equations

by

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A dissertation submitted to the graduate faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY

Major: Applied Mathematics

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2004

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# ABSTRACT

We consider shooting methods for computing approximate solutions of control problems constrained by linear or nonlinear hyperbolic partial differential equations. Optimal control problems and exact controllability problems are both studied, with the latter being approximated by the former with appropriate choices of parameters in the cost functional. The types of equations include linear wave equations, semilinear wave equations, and first order linear hyperbolic equations. The controls considered are either distributed in part of the time-space domain or of the Dirichlet type on the boundary. Each optimal control problem is reformulated as a system of equations that consists of an initial value problem (IVP) for the state equations and a terminal value problem for the adjoint equations. The optimality systems are regarded as a system of an IVP for the state equation and an IVP for the adjoint equations with unknown initial conditions. Then the optimality system is solved by shooting methods, i.e. we attempt to find adjoint initial values such that the adjoint terminal conditions are met. The shooting methods are implemented iteratively and Newton's method is employed to update the adjoint initial values. The convergence of the algorithms are theoretically discussed and numerically verified. Computational experiments are performed extensively for a variety of settings: different types of constraint equations in 1-D or 2-D, distributed or boundary controls, optimal control or exact controllability.

# 1 INTRODUCTION

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In this thesis we study numerical solutions of optimal control problems and exact controllability problems for linear and semilinear hyperbolic partial differential equations defined over the time interval  $[0,T] \subset [0,\infty)$  and on a bounded,  $C^2$  (or convex) spatial domain  $\Omega \subset \mathbb{R}^d$ , d = 1 or 2 or 3. The optimal control problems for the controls being either distributed in part of the time-space domain or of the Dirichlet type on the boundary are reformulated as a system of equations (an optimality system) that consists of an initial value problem for the underlying (linear or semilinear) hyperbolic partial differential equations and a terminal value problem for the adjoint hyperbolic partial differential equations by applying Lagrange multipliers. We develop shooting algorithms to solve the optimality system as follows : The optimality systems are regarded as a system of an IVP for the state equation and an IVP for the adjoint equations with unknown initial conditions. Then the optimality system is solved by shooting methods, i.e. we attempt to find adjoint initial values such that the adjoint terminal conditions are met. The shooting methods are implemented iteratively and Newton's method is employed to update the adjoint initial values.

Let target functions  $W \in L^2(\Omega)$ ,  $Z \in L^2(\Omega)$  or  $\in H^{-1}(\Omega)$ ,  $U \in L^2((0,T) \times \Omega)$  and an initial condition  $w \in L^2(\Omega)$ ,  $z \in L^2(\Omega)$  or  $\in H^{-1}(\Omega)$  be given. Let  $f \in L^2((0,T) \times \Omega)$ denote the distributed control and  $g \in [L^2(0,T)]^2$  denote the boundary control. We wish to find a control f or g that drives the states u and  $u_t$  to W, Z at time T and u to Uin  $(0,T) \times \Omega$ .

In Chapter 2 we consider an optimal control approach with distributed controls

defined on spatial domain  $\Omega$  for solving the exact controllability problems for one and two-dimensional linear and semilinear wave equations defined on a time interval (0,T)and a bounded spatial domain  $\Omega$ . Precisely we consider the following optimal control problem: minimize the cost functional

$$\mathcal{J}(u,f) = \frac{\alpha}{2} \int_0^T \int_\Omega K(u) \, d\mathbf{x} \, dt + \frac{\beta}{2} \int_\Omega \Phi_1(u(T,\mathbf{x})) \, d\mathbf{x} + \frac{\gamma}{2} \int_\Omega \Phi_2(u_t(T,\mathbf{x})) \, d\mathbf{x} + \frac{1}{2} \int_0^T \int_\Omega |f|^2 \, d\mathbf{x} \, dt \tag{1.1}$$

(where  $\alpha, \beta, \gamma$  are positive constants) subject to the wave equation

$$u_{tt} - \Delta u + \Psi(u) = f \qquad \text{in } (0, T) \times \Omega, \tag{1.2}$$

with the homogeneous boundary condition

$$u|_{\partial\Omega} = 0, \qquad (t, \mathbf{x}) \in (0, T) \times \partial\Omega.$$
 (1.3)

and the initial conditions

$$u(0, \mathbf{x}) = w(\mathbf{x}), \qquad u_t(0, \mathbf{x}) = z(\mathbf{x}) \qquad \mathbf{x} \in \Omega.$$
 (1.4)

We first develop the shooting algorithms for 1D and 2D distributed control problems, and then simulate with known smooth solution and generic examples both linear and semilinear cases in 1D and 2D.

In Chapter 3 we consider an optimal control approach with local distributed controls defined on spatial subdomain  $\Omega_1(\subseteq \Omega)$  for solving the exact controllability problems for one-dimensional linear and semilinear wave equations defined on a time interval (0, T)and a bounded spatial domain  $\Omega$ . Precisely we consider the following optimal control problem: minimize the cost functional

$$\mathcal{J}(u,f) = \frac{\alpha}{2} \int_0^T \int_\Omega K(u) \, d\mathbf{x} \, dt + \frac{\beta}{2} \int_\Omega \Phi_1(u(T,\mathbf{x})) \, d\mathbf{x} + \frac{\gamma}{2} \int_\Omega \Phi_2(u_t(T,\mathbf{x})) \, d\mathbf{x} + \frac{1}{2} \int_0^T \int_{\Omega_1} |f|^2 \, d\mathbf{x} \, dt = \frac{\alpha}{2} \int_0^T \int_\Omega K(u) \, d\mathbf{x} \, dt + \frac{\beta}{2} \int_\Omega \Phi_1(u(T,\mathbf{x})) \, d\mathbf{x} + \frac{\gamma}{2} \int_\Omega \Phi_2(u_t(T,\mathbf{x})) \, d\mathbf{x} + \frac{1}{2} \int_0^T \int_\Omega |\chi_{\Omega_1} f|^2 \, d\mathbf{x} \, dt$$

$$(1.5)$$

subject to the wave equation

$$u_{tt} - \Delta u + \Psi(u) = \chi_{\Omega_1} f \qquad \text{in } (0, T) \times \Omega, \tag{1.6}$$

with the homogeneous boundary condition

$$u|_{\partial\Omega} = 0, \qquad (t, \mathbf{x}) \in (0, T) \times \partial\Omega.$$
 (1.7)

and the initial conditions

$$u(0, \mathbf{x}) = w(\mathbf{x}), \qquad u_t(0, \mathbf{x}) = z(\mathbf{x}) \qquad \mathbf{x} \in \Omega.$$
 (1.8)

The shooting algorithms are applied to exact controllability problems with local distributed controls in the examples of known smooth solution and generic examples for both linear and semilinear cases.

In Chapter 4 we consider an optimal boundary control approach for solving the exact boundary control problem for one-dimensional linear or semilinear wave equations defined on a time interval (0,T) and spatial interval (0,X). The exact boundary control problem we consider is to seek a boundary control  $\mathbf{g} = (g_L, g_R) \in \mathbf{L}^2(\mathbf{0}, \mathbf{T}) \subset [L^2(0,T)]^2$ 

and a corresponding state u such that the following system of equations hold:

$$u_{tt} - u_{xx} + f(u) = V \quad \text{in } Q \equiv (0, T) \times (0, X) ,$$
  

$$u|_{t=0} = u_0 \quad \text{and} \quad u_t|_{t=0} = u_1 \quad \text{in } (0, X) ,$$
  

$$u|_{t=T} = W \quad \text{and} \quad u_t|_{t=T} = Z \quad \text{in } (0, X) ,$$
  

$$u|_{x=0} = g_L \text{ and } u|_{x=1} = g_R \quad \text{in } (0, T) ,$$
  
(1.9)

where  $u_0$  and  $u_1$  are given initial conditions defined on (0, X),  $W \in L^2(0, X)$  and  $Z \in H^{-1}(0, X)$  are prescribed terminal conditions, V is a given function defined on  $(0, T) \times (0, X)$ , f is a given function defined on  $\mathbb{R}$ , and  $\mathbf{g} = (g_L, g_R) \in [L^2(0, T)]^2$  is the boundary control. In this chapter we attempt to solve the exact controllability problems by an optimal control approach. Precisely, we consider the following optimal control problem: minimize the cost functional

$$\mathcal{J}_{0}(u,\mathbf{g}) = \frac{\sigma}{2} \int_{0}^{1} |u(T,x) - W(x)|^{2} dx + \frac{\tau}{2} \int_{0}^{1} |u_{t}(T,x) - Z(x)|^{2} dx + \frac{1}{2} \int_{0}^{1} (|g_{L}|^{2} + |g_{R}|^{2}) dt$$
(1.10)

subject to

$$\begin{aligned} u_{tt} - u_{xx} + f(u) &= V \quad \text{in } Q \equiv (0, T) \times (0, 1) \\ u_{t=0} &= u_0 \quad \text{and} \quad u_t|_{t=0} = u_1 \quad \text{in } (0, 1) \\ u_{x=0} &= g_L \text{ and } u_{x=1} = g_R \quad \text{in } (0, T) . \end{aligned}$$
 (1.11)

The shooting algorithms for solving the optimal control problem will be described for the slightly more general functional

$$\mathcal{J}(u,\mathbf{g}) = \frac{\alpha}{2} \int_0^T \int_0^1 |u - U|^2 \, dx \, dt + \frac{\sigma}{2} \int_0^1 |u(T,x) - W(x)|^2 \, dx + \frac{\tau}{2} \int_0^1 |u_t(T,x) - Z(x)|^2 \, dx + \frac{1}{2} \int_0^1 (|g_L|^2 + |g_R|^2) \, dt$$
(1.12)

where the term involving (u - U) reflects our desire to match the candidate state u with a given U in the entire domain Q. Our computational experiments of the proposed numerical methods will be performed exclusively for the case of  $\alpha = 0$ .

In Chapter 5 the linear optimal control problems we study are to minimize the cost functional

$$\mathcal{J}(u,f) = \frac{\alpha}{2} \int_0^T \int_{\Omega} |u - U|^2 \, dx \, dt + \frac{\beta}{2} \int_{\Omega} |u(x,T) - W(x)|^2 \, dx + \frac{1}{2} \int_0^T \int_{\Omega} |f|^2 \, dx \, dt.$$

subject to first order linear hyperbolic equation

$$u_t + au_x = f(x, t), \qquad \text{in } (0, T) \times \Omega, \tag{1.13}$$

with the boundary condition

$$u(t,0) = z(t), \qquad t \in (0,T),$$
 (1.14)

and the initial condition

$$u(0,x) = w(x), \qquad \mathbf{x} \in \Omega. \tag{1.15}$$

We particularly test some examples of distributed optimal control problems with known smooth solution and generic initial and boundary data and with a > 0.

Since this thesis covers several topics, the literature and new contributions of the thesis will be discussed in the context of each chapter.

# 2 Shooting Methods for Numerical Solutions of Distributed Optimal Control Problems Constrained by Linear and Nonlinear Wave Equations

Numerical solutions of distributed optimal Dirichlet control problems for linear and semilinear wave equations are studied. The optimal control problem is reformulated as a system of equations (an optimality system) that consists of an initial value problem for the underlying (linear or semilinear) wave equation and a terminal value problem for the adjoint wave equation. The discretized optimality system is solved by a shooting method. The convergence properties of the numerical shooting method in the context of exact controllability are illustrated through computational experiments.

# 2.1 Distributed optimal control problems for the wave equations

We will study numerical methods for optimal control and controllability problems associated with the linear and nonlinear wave equations. We are particularly interested in investigating the relevancy and applicability of high performance computing (HPC) for these problems.

As an prototype example of optimal control problems for the wave equations we consider the following distributed optimal control problem:

choose a control f and a corresponding u such that the pair (f, u) minimizes the cost

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functional

$$\mathcal{J}(u,f) = \frac{\alpha}{2} \int_0^T \int_\Omega K(u) \, d\mathbf{x} \, dt + \frac{\beta}{2} \int_\Omega \Phi_1(u(T,\mathbf{x})) \, d\mathbf{x} + \frac{\gamma}{2} \int_\Omega \Phi_2(u_t(T,\mathbf{x})) \, d\mathbf{x} + \frac{1}{2} \int_0^T \int_\Omega |f|^2 \, d\mathbf{x} \, dt$$

$$(2.1.1)$$

subject to the wave equation

$$u_{tt} - \Delta u + \Psi(u) = f \quad \text{in } (0, T) \times \Omega,$$
  

$$u|_{\partial\Omega} = 0, \quad u(0, \mathbf{x}) = w(\mathbf{x}), \quad u_t(0, \mathbf{x}) = z(\mathbf{x}).$$
(2.1.2)

Here  $\Omega$  is a bounded spatial domain in  $\mathbb{R}^d$  (d = 1 or 2 or 3) with a boundary  $\partial\Omega$ ; u is dubbed the state, and g is the distributed control. Also, K,  $\Phi$  and  $\Psi$  are  $C^1$  mappings (for instance, we may choose  $K(u) = (u - U)^2$ ,  $\Psi(u) = 0$ ,  $\Psi(u) = u^3 - u$  and  $\Psi(u) = e^u$ ,  $\Phi_1(u) = (u(T, \mathbf{x}) - W)^2$ ,  $\Phi_2(u) = (u_t(T, \mathbf{x}) - Z)^2$ , where U, W, Z is a target function.) Using Lagrange multiplier rules one finds the following optimality system of equations that the optimal solution (f, u) must satisfy:

$$\begin{split} u_{tt} - \Delta u + \Psi(u) &= f & \text{in } (0, T) \times \Omega \\ u|_{\partial\Omega} &= 0, \qquad u(0, \mathbf{x}) = w(\mathbf{x}), \qquad u_t(0, \mathbf{x}) = z(\mathbf{x}) ; \\ \xi_{tt} - \Delta \xi + [\Psi'(u)]^* \xi &= \frac{\alpha}{2} K'(u) \text{ in } Q \\ \xi|_{\partial\Omega} &= 0, \qquad \xi(T, \mathbf{x}) = \frac{\gamma}{2} \Phi'_2(u_t(T, \mathbf{x})), \qquad \xi_t(T, \mathbf{x}) = -\frac{\beta}{2} \Phi'_1(u(T, \mathbf{x})) ; \\ f + \xi &= 0 \text{ in } Q . \end{split}$$

This system may be simplified as

$$u_{tt} - \Delta u + \Psi(u) = -\xi \quad \text{in } (0, T) \times \Omega$$
  

$$u|_{\partial\Omega} = 0, \quad u(0, \mathbf{x}) = w(\mathbf{x}), \quad u_t(0, \mathbf{x}) = z(\mathbf{x});$$
  

$$\xi_{tt} - \Delta \xi + [\Psi'(u)]^* \xi = \frac{\alpha}{2} K'(u) \quad \text{in } (0, T) \times \Omega$$
  

$$\xi|_{\partial\Omega} = 0, \quad \xi(T, \mathbf{x}) = \frac{\gamma}{2} \Phi'_2(u_t(T, \mathbf{x})), \quad \xi_t(T, \mathbf{x}) = -\frac{\beta}{2} \Phi'_1(u(T, \mathbf{x})).$$
  
(2.1.3)

Such control problems are classical ones in the control theory literature; see, e.g., [5] for the linear case and [6] for the nonlinear case regarding the existence of optimal solutions as well as the existence of a Lagrange multiplier  $\xi$  satisfying the optimality system of equations. However, numerical methods for finding discrete (e.g., finite element and/or finite difference) solutions of the optimality system are largely limited to gradient type methods which are sequential in nature and generally require many iterations for convergence. The optimality system involves boundary conditions at t = 0 and t = T and thus cannot be solved by marching in time. Direct solutions of the discrete optimality system, of course, are bound to be expensive computationally in 2 or 3 spatial dimensions since the problem is (d + 1) dimensional (where d is the spatial dimensions.)

The computational algorithms we propose here are based on shooting methods for two-point boundary value problems for ordinary differential equations (ODEs); see, e.g., [1, 2, 3, 4]. The algorithms we propose are well suited for implementations on a parallel computing platform such as a massive cluster of cheap processors.

# 2.2 The solution of the exact controllability problem as the limit of optimal control solutions

The exact distributed control problem we consider is to seek a distributed control  $f \in L^2((0,T) \times \Omega)$  and a corresponding state u such that the following system of equations hold:

$$u_{tt} - \Delta u + \Psi(u) = f \quad \text{in } Q \equiv (0, T) \times \Omega ,$$
  

$$u|_{t=0} = w \quad \text{and} \quad u_t|_{t=0} = z \quad \text{in } \Omega ,$$
  

$$u|_{t=T} = W \quad \text{and} \quad u_t|_{t=T} = Z \quad \text{in } \Omega ,$$
  

$$u|_{\partial\Omega} = 0 \quad \text{in } (0, T) .$$
  
(2.2.4)

Under suitable assumptions on f and through the use of Lagrange multiplier rules,

the corresponding optimal control problem:

minimize 
$$(2.1.1)$$
 with respect to the control f subject to  $(2.1.2)$ .  $(2.2.5)$ 

In this section we establish the equivalence between the limit of optimal solutions and the minimum distributed  $L^2$  norm exact controller. We will show that if  $\alpha \to \infty$ ,  $\beta \to \infty$ and  $\gamma \to \infty$ , then the corresponding optimal solution  $(\hat{u}_{\alpha\beta\gamma}, \hat{f}_{\alpha\beta\gamma})$  converges weakly to the minimum distributed  $L^2$  norm solution of the exact distributed controllability problem (2.2.4). The same is also true in the discrete case.

**Theorem 2.2.1.** Assume that the exact distributed controllability problem (2.2.4) admits a unique minimum distributed  $L^2$  norm solution  $(u_{ex}, f_{ex})$ . Assume that for every  $(\alpha, \beta, \gamma) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$  (where  $\mathbb{R}_+$  is the set of all positive real numbers,) there exists a solution  $(u_{\alpha\beta\gamma}, f_{\alpha\beta\gamma})$  to the optimal control problem (2.2.5). Then

$$\|f_{\alpha\beta\gamma}\|_{L^2(Q)} \le \|f_{\text{ex}}\|_{L^2(Q)} \qquad \forall (\alpha, \beta, \gamma) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+.$$
(2.2.6)

Assume, in addition, that for a sequence  $\{(\alpha_n, \beta_n, \gamma_n)\}$  satisfying  $\alpha_n \to \infty$ ,  $\beta_n \to \infty$ and  $\gamma_n \to \infty$ ,

$$u_{\alpha_n\beta_n\gamma_n} \rightharpoonup \overline{u} \text{ in } L^2(Q) \quad and \quad \Psi(u_{\alpha_n\beta_n\gamma_n}) \rightharpoonup \Psi(\overline{u}) \text{ in } L^2(0,T; [H^2(\Omega) \cap H^1_0(\Omega)]^*).$$

$$(2.2.7)$$

Then

$$f_{\alpha_n\beta_n\gamma_n} \rightharpoonup f_{\text{ex}} \text{ in } L^2(Q) \text{ and } u_{\alpha_n\beta_n\gamma_n} \rightharpoonup u_{\text{ex}} \text{ in } L^2(Q) \text{ as } n \to \infty.$$
 (2.2.8)

Furthermore, if (2.2.7) holds for every sequence  $\{(\alpha_n, \beta_n, \gamma_n)\}$  satisfying  $\alpha_n \to \infty$ ,  $\beta_n \to \infty$ 

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 $\infty$  and  $\gamma_n \rightarrow \infty$ , then

$$f_{\alpha\beta\gamma} \rightharpoonup f_{\text{ex}} \text{ in } L^2(Q) \text{ and } u_{\alpha\beta\gamma} \rightharpoonup u_{\text{ex}} \text{ in } L^2(Q) \text{ as } \alpha, \beta, \gamma \rightarrow \infty.$$
 (2.2.9)

*Proof.* Since  $(u_{\alpha\beta\gamma}, f_{\alpha\beta\gamma})$  is an optimal solution, we have that

$$\frac{\alpha}{2} \|u_{\alpha\beta\gamma}(T) - U\|_{L^2(Q)} + \frac{\beta}{2} \|u_{\alpha\beta\gamma}(T) - W\|_{L^2(\Omega)} + \frac{\gamma}{2} \|\partial_t u_{\alpha\beta\gamma}(T) - Z\|_{H^{-1}(\Omega)} + \frac{1}{2} \|f_{\alpha\beta\gamma}\|_{L^2(Q)} = \mathcal{J}(u_{\alpha\beta\gamma}, f_{\alpha\beta\gamma}) \le \mathcal{J}(u_{\mathrm{ex}}, f_{\mathrm{ex}}) = \frac{1}{2} \|f_{\mathrm{ex}}\|_{L^2(Q)}$$

so that (2.2.6) holds,

$$u_{\alpha\beta\gamma}|_{t=T} \to W \text{ in } L^2(\Omega) \text{ and } (\partial_t u_{\alpha\beta\gamma})|_{t=T} \to Z \text{ in } H^{-1}(\Omega) \text{ as } \alpha, \beta, \gamma \to \infty.$$
 (2.2.10)

Let  $\{(\alpha_n, \beta_n, \gamma_n)\}$  be the sequence in (2.2.7). Estimate (2.2.6) implies that a subsequence of  $\{(\alpha_n, \beta_n, \gamma_n)\}$ , denoted by the same, satisfies

$$f_{\alpha_n\beta_n\gamma_n} \rightharpoonup \overline{f} \text{ in } L^2(Q) \quad \text{and} \quad \|\overline{f}\|_{L^2(Q)} \le \|f_{\mathrm{ex}}\|_{L^2(Q)}.$$
 (2.2.11)

 $(u_{\alpha\beta\gamma}, f_{\alpha\beta\gamma})$  satisfies the initial value problem in the weak form:

$$\int_{0}^{T} \int_{\Omega} u_{\alpha\beta\gamma}(v_{tt} - v_{xx}) \, dx \, dt + \int_{0}^{T} \int_{\Omega} [\Psi(u_{\alpha\beta\gamma}) - f_{\alpha\beta\gamma}] v \, dx \, dt$$
$$+ \int_{\Omega} (v\partial_{t}u_{\alpha\beta\gamma})|_{t=T} \, dx - \int_{\Omega} v|_{t=0} z \, dx - \int_{\Omega} (u_{\alpha\beta\gamma}\partial_{t}v)|_{t=T} \, dx \qquad (2.2.12)$$
$$+ \int_{\Omega} (w\partial_{t}v)|_{t=0} \, dx = 0 \qquad \forall v \in C^{2}([0,T]; H^{2}(\Omega) \cap H^{1}_{0}(\Omega))$$

Passing to the limit in (2.2.12) as  $\alpha, \beta, \gamma \to \infty$  and using relations (2.2.10) and (2.2.11)

we obtain:

$$\begin{split} &\int_0^T \int_\Omega \overline{u} (v_{tt} - v_{xx}) \, dx \, dt + \int_0^T \int_\Omega [\Psi(\overline{u}) - \overline{f}] v \, dx \, dt \\ &+ \int_\Omega v|_{t=T} Z(x) \, dx - \int_\Omega v|_{t=0} z \, dx - \int_\Omega W(x) (\partial_t v)|_{t=T} \, dx \\ &+ \int_\Omega (w \partial_t v)|_{t=0} \, dx = 0 \qquad \forall v \in C^2([0, T]; H^2(\Omega) \cap H^1_0(\Omega)) \end{split}$$

The last relation and (2.2.11) imply that  $(\overline{u}, \overline{f})$  is a minimum boundary  $L^2$  norm solution to the exact control problem (2.2.4). Hence,  $\overline{u} = u_{ex}$  and  $\overline{f} = f_{ex}$  so that (2.2.8) and (2.2.9) follows from (2.2.7) and (2.2.11).

**Remark 2.2.2.** If the wave equation is linear, i.e.,  $\Psi = 0$ , then assumption (2.2.7) is redundant and (2.2.9) is guaranteed to hold. Indeed, (2.2.12) implies the boundedness of  $\{\|u_{\alpha\beta\gamma}\|_{L^2(Q)}\}$  which in turn yields (2.2.7). The uniqueness of a solution for the linear wave equation implies (2.2.7) holds for an arbitrary sequence  $\{(\alpha_n, \beta_n, \gamma_n)\}$ .

- Theorem 2.2.3. Assume that
- i) for every (α, β, γ) ∈ ℝ<sub>+</sub> × ℝ<sub>+</sub> × ℝ<sub>+</sub> there exists a solution (u<sub>αβγ</sub>, f<sub>αβγ</sub>) to the optimal control problem (2.2.5);
- ii) the limit terminal conditions hold:

$$u_{\alpha\beta\gamma}|_{t=T} \to W \text{ in } L^2(\Omega) \text{ and } (\partial_t u_{\alpha\beta\gamma})|_{t=T} \to Z \text{ in } H^{-1}(\Omega) \text{ as } \alpha, \beta, \gamma \to \infty;$$

$$(2.2.13)$$

iii) the optimal solution  $(u_{\alpha\beta\gamma}, f_{\alpha\beta\gamma})$  satisfies the weak limit conditions as  $\alpha, \beta, \gamma \to \infty$ :

$$f_{\alpha\beta\gamma} \rightarrow \overline{f} \text{ in } L^2(Q), \quad u_{\alpha\beta\gamma} \rightarrow \overline{u} \text{ in } L^2(Q), \quad (2.2.14)$$

and

$$\Psi(u_{\alpha\beta\gamma}) \rightharpoonup \Psi(\overline{u}) \text{ in } L^2(0,T; [H^2(\Omega) \cap H^1_0(\Omega)]^*)$$
(2.2.15)

for some 
$$\overline{f} \in L^2(Q)$$
 and  $\overline{u} \in L^2(Q)$ .

Then  $(\overline{u}, \overline{f})$  is a solution to the exact boundary controllability problem (2.2.4) with  $\overline{f}$  satisfying the minimum boundary  $L^2$  norm property. Furthermore, if the solution to (2.2.4) admits a unique solution  $(u_{ex}, f_{ex})$ , then

$$f_{\alpha\beta\gamma} \rightharpoonup f_{\text{ex}} \text{ in } L^2(Q) \text{ and } u_{\alpha\beta\gamma} \rightharpoonup u_{\text{ex}} \text{ in } L^2(Q) \text{ as } \alpha, \beta, \gamma \rightarrow \infty.$$
 (2.2.16)

*Proof.*  $(u_{\alpha\beta\gamma}, f_{\alpha\beta\gamma})$  satisfies (2.2.12). Passing to the limit in that equation as  $\alpha, \beta, \gamma \to \infty$ and using relations (2.2.13), (2.2.14) and (2.2.15) we obtain:

$$\begin{split} &\int_0^T \int_{\Omega} \overline{u} (v_{tt} - v_{xx}) \, dx \, dt + \int_0^T \int_{\Omega} [\Psi(\overline{u}) - \overline{f}] v \, dx \, dt \\ &+ \int_{\Omega} v|_{t=T} Z(x) \, dx - \int_{\Omega} v|_{t=0} z \, dx - \int_{\Omega} W(x) (\partial_t v)|_{t=T} \, dx \\ &+ \int_{\Omega} (w \partial_t v)|_{t=0} \, dx = 0 \qquad [\forall v \in C^2([0,T]; H^2(\Omega) \cap H^1_0(\Omega)) \, . \end{split}$$

This implies that  $(\overline{u}, \overline{f})$  is a solution to the exact boundary controllability problem (2.2.4).

To prove that  $\overline{f}$  satisfies the minimum boundary  $L^2$  norm property, we proceeds as follows. Let  $(u_{\text{ex}}, f_{\text{ex}})$  denotes a exact minimum boundary  $L^2$  norm solution to the controllability problem (2.2.4). Since  $(u_{\alpha\beta\gamma}, f_{\alpha\beta\gamma})$  is an optimal solution, we have that

$$\frac{\alpha}{2} \|u_{\alpha\beta\gamma} - U\|_{L^2(Q)}^2 + \frac{\beta}{2} \|u_{\alpha\beta\gamma} - W\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \|\partial_t u_{\alpha\beta\gamma} - Z\|_{H^{-1}(\Omega)}^2 + \frac{1}{2} \|f_{\alpha\beta\gamma}\|_{L^2(Q)}^2 = \mathcal{J}(u_{\alpha\beta\gamma}, f_{\alpha\beta\gamma}) \le \mathcal{J}(u_{\mathrm{ex}}, f_{\mathrm{ex}}) = \frac{1}{2} \|f_{\mathrm{ex}}\|_{L^2(Q)}^2.$$

so that

$$\|f_{\alpha\beta\gamma}\|_{L^{2}(Q)}^{2} \leq \|f_{\mathrm{ex}}\|_{L^{2}(Q)}^{2}.$$

Passing to the limit in the last estimate we obtain

$$\|\overline{f}\|_{L^{2}(Q)}^{2} \leq \|f_{\text{ex}}\|_{L^{2}(Q)}^{2}.$$
(2.2.17)

Hence we conclude that  $(\overline{u}, \overline{f})$  is a minimum boundary  $L^2$  norm solution to the exact boundary controllability problem (2.2.4).

Furthermore, if the exact controllability problem (2.2.4) admits a unique minimum boundary  $L^2$  norm solution  $(u_{\text{ex}}, f_{\text{ex}})$ , then  $(\overline{u}, \overline{f}) = (u_{\text{ex}}, f_{\text{ex}})$  and (2.2.16) follows from assumption (2.2.14).

**Remark 2.2.4.** If the wave equation is linear, i.e.,  $\Psi = 0$ , then assumptions i) and (2.2.15) are redundant.

**Remark 2.2.5.** Assumptions ii) and iii) hold if  $f_{\alpha\beta\gamma}$  and  $u_{\alpha\beta\gamma}$  converges pointwise as  $\alpha, \beta, \gamma \to \infty$ .

**Remark 2.2.6.** A practical implication of Theorem 2.2.3 is that one can prove the exact controllability for semilinear wave equations by examining the behavior of a sequence of optimal solutions (recall that exact controllability was proved only for some special classes of semilinear wave equations.) If we have found a sequence of optimal control solutions  $\{(u_{\alpha_n\beta_n\gamma_n}, f_{\alpha_n\beta_n\gamma_n})\}$  where  $\alpha_n, \beta_n, \gamma_n \to \infty$  and this sequence appears to satisfy the convergence assumptions ii) and iii), then we can confidently conclude that the underlying semilinear wave equation is exactly controllable and the optimal solution  $(u_{\alpha_n\beta_n\gamma_n}, f_{\alpha_n\beta_n\gamma_n})$  when n is large provides a good approximation to the minimum boundary  $L^2$  norm exact controller  $(u_{ex}, f_{ex})$ .

# 2.3 Shooting methods for 1D control problems

The basic idea for a shooting method is to convert the solution of a two-point boundary value problem into that of an initial value problem (IVP). The IVP corresponding to the optimality system (4.3.18) is described by

$$u_{tt} - u_{xx} + \Psi(u) = -\xi \quad \text{in } (0, T) \times \Omega,$$
  

$$u_{|\partial\Omega} = 0, \quad u(0, x) = w(x), \quad u_t(0, x) = z(x);$$
  

$$\xi_{tt} - \xi_{xx} + [\Psi'(u)]^* \xi = \frac{\alpha}{2} K'(u) \quad \text{in } (0, T) \times \Omega,$$
  

$$\xi_{|\partial\Omega} = 0, \quad \xi(0, x) = \omega(x), \quad \xi_t(0, x) = \theta(x),$$
  
(2.3.18)

with unknown initial values  $\omega$  and  $\theta$ . Then the goal is to choose  $\omega$  and  $\theta$  such that the solution  $(u, \xi)$  of the IVP (4.3.19) satisfies the terminal conditions

$$\xi(T,x) = \frac{\gamma}{2}\Phi'_2(u_t(T,x))$$
 and  $\xi_t(T,x) = -\frac{\beta}{2}\Phi'_1(u(T,x))$ . (2.3.19)

A shooting method for solving (4.3.18) consists of the following main steps:

choose initial guesses  $\omega$  and  $\theta$ ;

for  $m = 1, 2, \dots, M$ solve for  $(u, \xi)$  from the IVP (4.3.18) update  $\omega$  and  $\theta$ .

A criterion for updating  $(\omega, \theta)$  can be derived from the terminal conditions (4.3.20). A method for solving the nonlinear system (4.3.20) (as a system for the unknowns  $\omega$  and  $\theta$ ) will yield an updating formula and here we invoke the well-known Newton's method to do so. Also, a discrete version of the algorithm must be used in actual implementations. For the ease of exposition we describe in details a Newton's-method-based shooting algorithm with finite difference discretizations in one space dimension. Algorithms in higher dimensions and their implementations will be briefly discussed subsequently and will form an prominent part of the proposed research.

### 2.3.1 Algorithm for 1D control problems

In one dimension, we discretize the spatial interval [0, X] into  $0 = x_0 < x_1 < x_2 < x_3 < \cdots < x_{I+1} = X$  with a uniform spacing h = X/(I+1) and we divide the time horizon [0, T] into  $0 = t_1 < t_2 < t_3 < \cdots < t_N = T$  with a uniform time step length  $\delta = T/(N-1)$ . We use the central differencing scheme to approximate the initial value problem (4.3.19):

$$u_{i}^{1} = w_{i}, \qquad u_{i}^{2} = w_{i} + \delta z_{i}, \qquad \xi_{i}^{1} = \omega_{i}, \qquad \xi_{i}^{2} = \xi_{i}^{1} + \delta \theta_{i}, \qquad i = 1, 2, \cdots, I;$$

$$u_{i}^{n+1} = -u_{i}^{n-1} + \lambda u_{i-1}^{n} + 2(1-\lambda)u_{i}^{n} + \lambda u_{i+1}^{n} \\ -\delta^{2}\xi_{i}^{n} - \delta^{2}\Psi(u_{i}^{n}), \qquad i = 1, 2, \cdots, I,$$

$$\xi_{i}^{n+1} = -\xi_{i}^{n-1} + \lambda\xi_{i-1}^{n} + 2(1-\lambda)\xi_{i}^{n} + \lambda\xi_{i+1}^{n} \\ + \delta^{2}\frac{\alpha}{2}K(u_{i}^{n}) - \delta^{2}[\Psi'(u_{i}^{n})]^{*}\xi_{i}^{n}, \qquad i = 1, 2, \cdots, I;$$
(2.3.20)

where  $\lambda = (\delta/h)^2$  (we also use the convention that  $u_0^n = \xi_0^n = u_{I+1}^n = \xi_{I+1}^n = 0$ .) The gists of a discrete shooting method are to regard the discrete terminal conditions

$$F_{2i-1} \equiv \frac{\xi_i^N - \xi_i^{N-1}}{\delta} + \frac{\beta}{2} \Phi_1'(u_i^N) = 0, \qquad F_{2i} \equiv \xi_i^N - \frac{\gamma}{2} \Phi_2'(\frac{u_i^N - u_i^{N-1}}{\delta}) = 0, \quad (2.3.21)$$
$$i = 1, 2, \cdots, I.$$

By denoting

$$q_{ij}^{n} = q_{ij}^{n}(\omega_{1}, \theta_{1}, \omega_{2}, \theta_{2}, \cdots, \omega_{I}, \theta_{I}) = \frac{\partial u_{i}^{n}}{\partial \omega_{j}}, \qquad r_{ij}^{n} = r_{ij}^{n}(\omega_{1}, \theta_{1}, \omega_{2}, \theta_{2}, \cdots, \omega_{I}, \theta_{I}) = \frac{\partial u_{i}^{n}}{\partial \theta_{j}},$$
$$\rho_{ij}^{n} = \rho_{ij}^{n}(\omega_{1}, \theta_{1}, \omega_{2}, \theta_{2}, \cdots, \omega_{I}, \theta_{I}) = \frac{\partial \xi_{i}^{n}}{\partial \omega_{j}}, \qquad \tau_{ij}^{n} = \tau_{ij}^{n}(\omega_{1}, \theta_{1}, \omega_{2}, \theta_{2}, \cdots, \omega_{I}, \theta_{I}) = \frac{\partial \xi_{i}^{n}}{\partial \theta_{j}},$$

we may write Newton's iteration formula as

$$(\omega_1^{\text{new}}, \theta_1^{\text{new}}, \omega_2^{\text{new}}, \theta_2^{\text{new}}, \cdots, \omega_I^{\text{new}}, \theta_I^{\text{new}})^T = (\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_I, \theta_I)^T - [F'(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_I, \theta_I)]^{-1} F(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_I, \theta_I)$$

where the vector F and Jacobian matrix  $J=F^\prime$  are defined by

$$\begin{split} F_{2i-1} &= \frac{\xi_i^N - \xi_i^{N-1}}{\delta} + \frac{\beta}{2} \Phi_1'(u_i^N) \,, \qquad F_{2i} = \xi_i^N - \frac{\gamma}{2} \Phi_2'(\frac{u_i^N - u_i^{N-1}}{\delta}) \,, \\ J_{2i-1,2j-1} &= \frac{\rho_{ij}^N - \rho_{ij}^{N-1}}{\delta} + \frac{\beta}{2} \Phi_1''(u_i^N) q_{ij}^N \,, \qquad J_{2i-1,2j} = \frac{\tau_{ij}^N - \tau_{ij}^{N-1}}{\delta} + \frac{\beta}{2} \Phi_1''(u_i^N) r_{ij}^N \,, \\ J_{2i,2j-1} &= \rho_{ij}^N - \frac{\gamma}{2\delta} \Phi_2''(\frac{u_i^N - u_i^{N-1}}{\delta}) (q_{ij}^N - q_{ij}^{N-1}) \,, \\ J_{2i,2j} &= \tau_{ij}^N - \frac{\gamma}{2\delta} \Phi_2''(\frac{u_i^N - u_i^{N-1}}{\delta}) (r_{ij}^N - r_{ij}^{N-1}) \,. \end{split}$$

Moreover, by differentiating (4.4.21) with respect to  $\omega_j$  and  $\theta_j$  we obtain the equations for determining  $q_{ij}$ ,  $r_{ij}$ ,  $\rho_{ij}$  and  $\tau_{ij}$ :

$$\begin{split} q_{ij}^{1} &= 0, \qquad q_{ij}^{2} = 0, \qquad r_{ij}^{1} = 0, \qquad r_{ij}^{2} = 0, \\ \rho_{ij}^{1} &= \delta_{ij}, \qquad \rho_{ij}^{2} = \delta_{ij}, \qquad \tau_{ij}^{1} = 0, \qquad \tau_{ij}^{2} = \delta\delta_{ij}, \\ q_{ij}^{n+1} &= -q_{ij}^{n-1} + \lambda q_{i-1,j}^{n} + 2(1-\lambda)q_{ij}^{n} + \lambda q_{i+1,j}^{n} \\ &\quad -\delta^{2}\rho_{ij}^{n} - \delta^{2}\Psi'(u_{i}^{n})q_{ij}^{n}, \qquad i, j = 1, 2, \cdots, I, \\ r_{ij}^{n+1} &= -r_{ij}^{n-1} + \lambda r_{i-1,j}^{n} + 2(1-\lambda)r_{ij}^{n} + \lambda r_{i+1,j}^{n} \\ &\quad -\delta^{2}\tau_{ij}^{n} - \delta^{2}\Psi'(u_{i}^{n})r_{ij}^{n}, \qquad i, j = 1, 2, \cdots, I, \\ \rho_{ij}^{n+1} &= -\rho_{ij}^{n-1} + \lambda \rho_{i-1,j}^{n} + 2(1-\lambda)\rho_{ij}^{n} + \lambda \rho_{i+1,j}^{n} \\ &\quad +\delta^{2}\frac{\alpha}{2}K'(u_{i}^{n})q_{ij}^{N} - \delta^{2}[\Psi'(u_{i}^{n})]^{*}\rho_{i}^{n} - \delta^{2}[\Psi''(u_{i}^{n})q_{i}^{N}]^{*}\xi_{i}^{n}, \qquad i, j = 1, 2, \cdots, I; \\ \tau_{ij}^{n+1} &= -\tau_{ij}^{n-1} + \lambda \tau_{i-1,j}^{n} + 2(1-\lambda)\tau_{ij}^{n} + \lambda \tau_{i+1,j}^{n} \\ &\quad +\delta^{2}\frac{\alpha}{2}K'(u_{i}^{n})r_{ij}^{N} - \delta^{2}[\Psi'(u_{i}^{n})]^{*}\rho_{i}^{n} - \delta^{2}[\Psi''(u_{i}^{n})r_{i}^{N}]^{*}\xi_{i}^{n}, \qquad i, j = 1, 2, \cdots, I; \end{split}$$

$$(2.3.22)$$

where  $\delta_{ij}$  is the Chronecker delta. Thus, we have the following Newton's-method-based shooting algorithm:

Algorithm 1 - Newton method based shooting algorithm with Euler discretizations for distributed optimal control problems

choose initial guesses  $\omega_i$  and  $\theta_i$ ,  $i = 1, 2, \cdots, I$ ; % set initial conditions for u and  $\xi$ for  $i = 1, 2, \dots, I$  $u_i^1 = w_i, \quad u_i^2 = w_i + \delta z_i,$  $\xi_i^1 = \omega_i, \quad \xi_i^2 = \xi_i^1 + \delta\theta_i;$ % set initial conditions for  $q_{ij}, r_{ij}, \rho_{ij}, \tau_{ij}$ 

for  $j = 1, 2, \dots, I$ for  $i = 1, 2, \cdots, I$  $q_{ij}^1 = 0, \qquad q_{ij}^2 = 0, \qquad r_{ij}^1 = 0, \qquad r_{ij}^2 = 0,$  $\rho_{ij}^1 = 0, \qquad \rho_{ij}^2 = 0, \qquad \tau_{ij}^1 = 0, \qquad \tau_{ij}^2 = 0;$  $\rho_{ij}^1 = 1, \qquad \rho_{ij}^2 = 1, \qquad \tau_{jj}^2 = \delta,$ 

% Newton iterations

for  $m = 1, 2, \cdots, M$ % solve for  $(u,\xi)$ for  $n = 2, 3, \dots, N - 1$  $u_{i}^{n+1} = -u_{i}^{n-1} + \lambda u_{i-1}^{n} + 2(1-\lambda)u_{i}^{n} + \lambda u_{i+1}^{n} - \delta^{2}\xi_{i}^{n} - \delta^{2}\Psi(u_{i}^{n}),$  $\xi_i^{n+1} = -\xi_i^{n-1} + \lambda \xi_{i-1}^n + 2(1-\lambda)\xi_i^n + \lambda \xi_{i+1}^n + \delta^2 \frac{\alpha}{2} K(u_i^n) - \delta^2 [\Psi'(u_i^n)]^* \xi_i^n;$ % solve for  $q, r, \rho, \tau$ for  $j = 1, 2, \cdots, I$ for  $n = 2, 3, \dots, N - 1$ for  $i = 2, \cdots, N-1$  $q_{ij}^{n+1} = -q_{ij}^{n-1} + \lambda q_{i-1,j}^n + 2(1-\lambda)q_{ij}^n + \lambda q_{i+1,j}^n$ 

# $$\begin{split} \delta^2 \rho_{ij}^n &- \delta^2 \Psi'(u_i^n) q_{ij}^n, \\ r_{ij}^{n+1} &= -r_{ij}^{n-1} + \lambda r_{i-1,j}^n + 2(1-\lambda) r_{ij}^n + \lambda r_{i+1,j}^n \\ &- \delta^2 \tau_{ij}^n - \delta^2 \Psi'(u_i^n) r_{ij}^n, \\ \rho_{ij}^{n+1} &= -\rho_{ij}^{n-1} + \lambda \rho_{i-1,j}^n + 2(1-\lambda) \rho_{ij}^n + \lambda \rho_{i+1,j}^n \\ &+ \delta^2 \frac{\alpha}{2} K'(u_i^n) q_{ij}^N - \delta^2 [\Psi'(u_i^n)]^* \rho_i^n - \delta^2 [\Psi''(u_i^n) q_i^N]^* \xi_i^n, \\ \tau_{ij}^{n+1} &= -\tau_{ij}^{n-1} + \lambda \tau_{i-1,j}^n + 2(1-\lambda) \tau_{ij}^n + \lambda \tau_{i+1,j}^n \\ &+ \delta^2 \frac{\alpha}{2} K'(u_i^n) r_{ij}^N - \delta^2 [\Psi'(u_i^n)]^* \rho_i^n - \delta^2 [\Psi''(u_i^n) r_i^N]^* \xi_i^n; \end{split}$$

% (we need to build into the algorithm the following:

$$q_0^n = r_0^n = \rho_0^n = \tau_0^n = 0,$$
  
$$q_{I+1}^n = r_{I+1}^n = \rho_{I+1}^n = \tau_{I+1}^n = 0.)$$

% evaluate F and F'

for 
$$i = 1, 2, \cdots, I$$
  

$$F_{2i-1} = \frac{\xi_i^N - \xi_i^{N-1}}{\delta} + \frac{\beta}{2} \Phi'(u_i^N), \qquad F_{2i} = \xi_i^N;$$
for  $j = 1, 2, \cdots, I$   

$$J_{2i-1,2j-1} = \frac{\rho_{ij}^N - \rho_{ij}^{N-1}}{\delta} + \frac{\beta}{2} \Phi''(u_i^N) q_{ij}^N,$$

$$J_{2i-1,2j} = \frac{\tau_{ij}^N - \tau_{ij}^{N-1}}{\delta} + \frac{\beta}{2} \Phi''(u_i^N) r_{ij}^N,$$

$$J_{2i,2j-1} = \rho_{ij}^N - \frac{\gamma}{2\delta} \Phi_2''(\frac{u_i^N - u_i^{N-1}}{\delta})(q_{ij}^N - q_{ij}^{N-1});$$

$$J_{2i,2j} = \tau_{ij}^N - \frac{\gamma}{2\delta} \Phi_2''(\frac{u_i^N - u_i^{N-1}}{\delta})(r_{ij}^N - r_{ij}^{N-1});$$

solve  $J\mathbf{c} = -F$  by Gaussian eliminations;

for  $i = 1, 2, \cdots, I$ 

 $\omega_i^{\text{new}} = \omega_i + c_{2i-1},$ 

$$\theta_i^{\text{new}} = \theta_i + c_{2i};$$

if  $\max_i |\omega_i^{\text{new}} - \omega_i| + \max_i |\theta_i^{\text{new}} - \theta_i| < \text{tol, stop;}$ 

otherwise, reset  $\omega_i = \omega_i^{\text{new}}$  and  $\theta_i = \theta_i^{\text{new}}$ ,  $i = 1, 2, \cdots, I$ ;

## 2.3.2 1D Computational Results

We consider examples of the following types : Seek the pair (u, f) that minimizes the cost functional

$$\mathcal{J}(u,f) = \frac{\alpha}{2} \int_0^T \int_{\Omega} |u - U|^2 \, dx \, dt + \frac{\beta}{2} \int_{\Omega} |u(T,x) - W(x)|^2 \, dx + \frac{1}{2} \int_0^T \int_{\Omega} |f|^2 \, dx \, dt$$

subject to the wave equation

$$u_{tt} - u_{xx} + \Psi(u) = f \quad \text{in } (0, T) \times \Omega,$$
  

$$u_{\partial\Omega} = 0, \quad u(0, x) = g(x), \quad u_t(0, x) = h(x).$$
(2.3.23)

Example 2.3.1. (linear case)  $\Psi(u) = 0, T = 1, \Omega = [0, 1]$ .

For given target functions,

$$W(x) = \sin(2\pi x)\sin(2\pi T), \quad U(t,x) = \sin(2\pi x)\sin(2\pi t), \quad (2.3.24)$$

it can be verified that the exact optimal solution u and a corresponding Lagrange multiplier  $\xi$  are determined by (4.3.18).

$$u(t,x) = \sin(2\pi x)\sin(2\pi t), \quad \xi(t,x) = 0. \tag{2.3.25}$$



Figure 2.1 Optimal solution u and target W, U for  $\Delta t = 1/40$  and  $\Delta x = 1/20$   $\therefore$  optimal solution u(t, x) -: target functions W(x), U(t, x)  $\circ:$  exact optimal solution  $\alpha = 1000$ ,  $\beta = 1000$ 



Figure 2.2 Optimal solution u and target W, U for  $\Delta t = 1/40$  and  $\Delta x = 1/20$  :optimal solution u(t, x) -: target functions W(x), U(t, x) o: exact optimal solution  $\alpha = 1000000$ ,  $\beta = 1000000$ 

Example 2.3.2. (linear case)  $\Psi(u) = 0, T = 1, \Omega = [0, 1].$ 

For given target functions, W(x) = 1, U(t, x) = 1.



Figure 2.3 Optimal solution u and target W, U for  $\Delta t = 1/100$  and  $\Delta x = 1/50$  : optimal solution u(t, x) -: target functions W(x), U(t, x)  $\alpha = 1000000$ ,  $\beta = 1000000$ 

**Example 2.3.3.** (nonlinear case)  $\Psi(u) = u^3 - u$ , T = 1,  $\Omega = [0, 1]$ . For given target functions,

$$W(x) = \sin \pi x \cos \pi T$$

$$U(t, x) = \sin \pi x \cos \pi t.$$
(2.3.26)



Figure 2.4 Optimal solution u and target W, U for  $\Delta t = 1/100$  and  $\Delta x = 1/50$  : optimal solution u(t, x) -: target functions W(x), U(t, x)  $\alpha = 10000$ ,  $\beta = 10000$ 

**Example 2.3.4.** (nonlinear case)  $\Psi(u) = e^{u}$ , T = 1,  $\Omega = [0, 1]$ .

For given target functions,

$$W(x) = \sin \pi x \cos T$$

$$U(t, x) = \sin \pi x \cos t.$$
(2.3.27)



Figure 2.5 Optimal solution u and target W, U for  $\Delta t = 1/100$  and  $\Delta x = 1/50$  : optimal solution u(t, x) -: target functions W(x), U(t, x)  $\alpha = 1000$ ,  $\beta = 1000$ 

**Example 2.3.5.** (Sine-Gordon equation)  $\Psi(u) = \sin u$ , T = 1,  $\Omega = [0, 1]$ .

For given target functions,

$$W(x) = \sin \pi x \cos T$$

$$U(t, x) = \sin \pi x \cos t.$$
(2.3.28)



Figure 2.6 Optimal solution u and target W, U for  $\Delta t = 1/100$  and  $\Delta x = 1/50$  : optimal solution u(t, x) -: target functions W(x), U(t, x)  $\alpha = 10000$ ,  $\beta = 10000$ 

# 2.4 Shooting methods in 2D control problems

The basic idea for a shooting method in 2D is exactly the same as in 1D. In two dimensional case, we replace  $u_{xx}$  by  $\Delta u$ . Lagrange multiplier rules provide the same optimality system of equations as in the previous section except the replacement part. Thus the IVP corresponding to the optimality system (4.3.18) is described by

$$u_{tt} - \Delta u + \Psi(u) = -\xi \quad \text{in } (0, T) \times \Omega,$$
  

$$u|_{\partial\Omega} = 0, \quad u(0, \mathbf{x}) = w(\mathbf{x}), \quad u_t(0, \mathbf{x}) = z(\mathbf{x});$$
  

$$\xi_{tt} - \Delta \xi + [\Psi'(u)]^* \xi = \frac{\alpha}{2} K'(u) \quad \text{in } (0, T) \times \Omega,$$
  

$$\xi|_{\partial\Omega} = 0, \quad \xi(0, \mathbf{x}) = \omega(\mathbf{x}), \quad \xi_t(0, \mathbf{x}) = \theta(\mathbf{x}),$$
  
(2.4.29)

with unknown initial values  $\omega$  and  $\theta$ . Then the goal is to choose  $\omega$  and  $\theta$  such that the solution  $(u,\xi)$  of the IVP (2.4.29) satisfies the terminal conditions

$$\xi(T, \mathbf{x}) = \frac{\gamma}{2} \Phi'_2(u_t(T, \mathbf{x}))$$
 and  $\xi_t(T, \mathbf{x}) = -\frac{\beta}{2} \Phi'_1(u(T, \mathbf{x}))$ . (2.4.30)

# 2.4.1 Algorithm for 2D control problems

In two dimension, we discretize the spatial interval [0, X], [0, Y] into  $0 = x_0 < x_1 < x_2 < x_3 < \cdots < x_{I+1} = X$ ,  $0 = y_0 < y_1 < y_2 < y_3 < \cdots < y_{J+1} = Y$  with a uniform spacing  $h_x = X/(I+1)$ ,  $h_y = Y/(J+1)$  respectively, and we divide the time horizon [0, T] into  $0 = t_1 < t_2 < t_3 < \cdots < t_N = T$  with a uniform time step length  $\delta = T/(N-1)$ . We use the central difference scheme to approximate the initial value problem (2.4.29): For  $i = 1, 2, \cdots, I$ ,  $j = 1, 2, \cdots, J$ ,

$$\begin{aligned} u_{ij}^{1} &= w_{ij}, \quad u_{ij}^{2} &= w_{ij} + \delta z_{ij}, \quad \xi_{ij}^{1} &= \omega_{ij}, \quad \xi_{ij}^{2} &= \xi_{ij}^{1} + \delta \theta_{ij}; \\ u_{ij}^{n+1} &= -u_{ij}^{n-1} + \lambda_{x} (u_{i-1,j}^{n} + u_{i+1,j}^{n}) + 2(1 - \lambda_{x} - \lambda_{y}) u_{ij}^{n} + \lambda_{y} (u_{i,j-1}^{n} + u_{i,j+1}^{n}) \\ &\quad - \delta^{2} \xi_{ij}^{n} - \delta^{2} \Psi(u_{ij}^{n}), \end{aligned}$$
(2.4.31)  
$$\xi_{ij}^{n+1} &= -\xi_{ij}^{n-1} + \lambda_{x} (\xi_{i-1,j}^{n} + \xi_{i+1,j}^{n}) + 2(1 - \lambda_{x} - \lambda_{y}) \xi_{ij}^{n} + \lambda_{y} (\xi_{i,j-1}^{n} + \xi_{i,j+1}^{n}) \\ &\quad + \delta^{2} \frac{\alpha}{2} K(u_{ij}^{n}) - \delta^{2} [\Psi'(u_{ij}^{n})]^{*} \xi_{ij}^{n}, \end{aligned}$$

where  $\lambda_x = (\delta/h_x)^2$ ,  $\lambda_y = (\delta/h_y)^2$  (we also use the convention that  $u_{ij}^n = \xi_{ij}^n = 0$  if i = 0or I + 1 or j = 0 or j = J + 1.) The gists of a discrete shooting method are to regard the discrete terminal conditions: For  $i = 1, 2, \dots, I$ ,  $j = 1, 2, \dots, J$ ,

$$F_{2(ij)^*-1} \equiv \frac{\xi_{ij}^N - \xi_{ij}^{N-1}}{\delta} + \frac{\beta}{2} \Phi_1'(u_{ij}^N) = 0, \quad F_{2(ij)^*} \equiv \xi_{ij}^N - \frac{\gamma}{2} \Phi_2'(\frac{u_{ij}^N - u_{ij}^{N-1}}{\delta}) = 0, \quad (2.4.32)$$

where  $(ij)^*$  is a reordering of the nodes with respect to X, Y except boundary points.

Let  $\omega(\mathbf{x}) \equiv \{\omega_1, \omega_2, \cdots, \omega_{IJ^*}\} = \{\omega_{11}, \omega_{21}, \cdots, \omega_{IJ}\}$ , and  $\theta(\mathbf{x}) \equiv \{\theta_1, \theta_2, \cdots, \theta_{IJ^*}\} = \{\theta_{11}, \theta_{21}, \cdots, \theta_{IJ}\}$  where  $IJ^* = I \cdot J$ . By denoting

$$q_{(ij)^*k}^n = q_{(ij)^*k}^n(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_{IJ^*}, \theta_{IJ^*}) = \frac{\partial u_{ij}^n}{\partial \omega_k},$$
  

$$r_{(ij)^*k}^n = r_{(ij)^*k}^n(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_{IJ^*}, \theta_{IJ^*}) = \frac{\partial u_{ij}^n}{\partial \theta_k},$$
  

$$\rho_{(ij)^*k}^n = \rho_{(ij)^*k}^n(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_{IJ^*}, \theta_{IJ^*}) = \frac{\partial \xi_{ij}^n}{\partial \omega_k},$$
  

$$\tau_{(ij)^*k}^n = \tau_{(ij)^*k}^n(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_{IJ^*}, \theta_{IJ^*}) = \frac{\partial \xi_{ij}^n}{\partial \theta_k},$$

we may write Newton's iteration formula as

$$(\omega_1^{\text{new}}, \theta_1^{\text{new}}, \omega_2^{\text{new}}, \theta_2^{\text{new}}, \cdots, \omega_{IJ^*}^{\text{new}}, \theta_{IJ^*}^{\text{new}})^T = (\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_{IJ^*}, \theta_{IJ^*})^T - [F'(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_{IJ^*}, \theta_{IJ^*})]^{-1}F(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_{IJ^*}, \theta_{IJ^*})$$

where the vector F and Jacobian matrix J = F' are defined by

$$F_{2(ij)^{*}-1} \equiv \frac{\xi_{ij}^{N} - \xi_{ij}^{N-1}}{\delta} + \frac{\beta}{2} \Phi_{1}'(u_{ij}^{N}), \qquad F_{2(ij)^{*}} \equiv \xi_{ij}^{N} - \frac{\gamma}{2} \Phi_{2}'(\frac{u_{ij}^{N} - u_{ij}^{N-1}}{\delta}),$$

$$J_{2(ij)^{*}-1,2k-1} = \frac{\rho_{(ij)_{k}^{*}}^{N} - \rho_{(ij)_{k}^{*}}^{N-1}}{\delta} + \frac{\beta}{2} \Phi''(u_{ij}^{N})q_{(ij)_{k}^{*}}^{N},$$

$$J_{2(ij)^{*}-1,2k} = \frac{\tau_{(ij)_{k}^{*}}^{N} - \tau_{(ij)_{k}^{*}}^{N-1}}{\delta} + \frac{\beta}{2} \Phi''(u_{ij}^{N})r_{(ij)_{k}^{*}}^{N},$$

$$J_{2(ij)^{*},2k-1} = \rho_{(ij)_{k}^{*}}^{N} - \frac{\gamma}{2\delta} \Phi_{2}''(\frac{u_{ij}^{N} - u_{ij}^{N-1}}{\delta})(q_{(ij)_{k}^{*}}^{N} - q_{(ij)_{k}^{*}}^{N-1}),$$

$$J_{2(ij)^{*},2k} = \tau_{(ij)_{k}^{*}}^{N} - \frac{\gamma}{2\delta} \Phi_{2}''(\frac{u_{ij}^{N} - u_{ij}^{N-1}}{\delta})(r_{(ij)_{k}^{*}}^{N} - r_{(ij)_{k}^{*}}^{N-1}).$$

Moreover, by differentiating (2.4.31) with respect to  $\omega_k$  and  $\theta_k$  we obtain the equations for determining  $q_{(ij)^*k}$ ,  $r_{(ij)^*k}$ ,  $\rho_{(ij)^*k}$  and  $\tau_{(ij)^*k}$ :

$$\begin{split} q^{1}_{(ij)^{*k}} &= 0, \quad q^{2}_{(ij)^{*k}} = 0, \quad r^{1}_{(ij)^{*k}} = 0, \quad r^{2}_{(ij)^{*k}} = 0, \\ \rho^{1}_{(ij)^{*k}} &= \delta_{(ij)^{*k}}, \quad \rho^{2}_{(ij)^{*k}} = \delta_{(ij)^{*k}}, \quad i, j = 1, 2, \cdots, I, J; \\ \tau^{1}_{(ij)^{*k}} &= 0, \quad \tau^{2}_{(ij)^{*k}} = \delta\delta_{(ij)^{*k}}, \\ q^{n+1}_{(ij)^{*k}} &= -q^{n-1}_{(ij)^{*k}} + \lambda_{x} (q^{n}_{(i-1,j)^{*k}} + q^{n}_{(i+1,j)^{*k}}) + 2(1 - \lambda_{x} - \lambda_{y})q^{n}_{(ij)^{*k}} \\ &+ \lambda_{y} (q^{n}_{(i,j-1)^{*k}} + q^{n}_{(i,j+1)^{*k}}) - \delta^{2}\rho^{n}_{(ij)^{*k}} \\ &- \delta^{2} \Psi'(u^{n}_{ij})q^{n}_{(ij)^{*k}}, \quad i, j = 1, 2, \cdots, I, J, \\ \tau^{n+1}_{(ij)^{*k}} &= -r^{n-1}_{(ij)^{*k}} + \lambda_{x} (r^{n}_{(i-1,j)^{*k}} + r^{n}_{(i+1,j)^{*k}}) + 2(1 - \lambda_{x} - \lambda_{y})r^{n}_{(ij)^{*k}} \\ &+ \lambda_{y} (r^{n}_{(i,j-1)^{*k}} + r^{n}_{(i,j+1)^{*k}}) - \delta^{2}\tau^{n}_{(ij)^{*k}} \\ &- \delta^{2} \Psi'(u^{n}_{ij})r^{n}_{(ij)^{*k}}, \quad i, j = 1, 2, \cdots, I, J, \\ \rho^{n+1}_{(ij)^{*k}} &= -\rho^{n-1}_{(ij)^{*k}} + \lambda_{x} (\rho^{n}_{(i-1,j)^{*k}} + \rho^{n}_{(i+1,j)^{*k}}) + 2(1 - \lambda_{x} - \lambda_{y})\rho^{n}_{(ij)^{*k}} \\ &+ \lambda_{y} (\rho^{n}_{(i,j-1)^{*k}} + \lambda_{x} (\rho^{n}_{(i-1,j)^{*k}} + \rho^{n}_{(i+1,j)^{*k}}) + 2(1 - \lambda_{x} - \lambda_{y})\rho^{n}_{(ij)^{*k}} \\ &- \delta^{2} [\Psi'(u^{n}_{ij})q^{N}_{(ij)^{*k}}]^{*} \xi^{n}_{ij}, \quad i, j = 1, 2, \cdots, I, J; \\ \tau^{n+1}_{(ij)^{*k}} &= -\tau^{n-1}_{(ij)^{*k}} + \lambda_{x} (\tau^{n}_{(i-1,j)^{*k}} + \tau^{n}_{(i+1,j)^{*k}}) + 2(1 - \lambda_{x} - \lambda_{y})\tau^{n}_{(ij)^{*k}} \\ &+ \lambda_{y} (\tau^{n}_{(i,j-1)^{*k}} + \lambda_{x} (\tau^{n}_{(i-1,j)^{*k}} + \tau^{n}_{(i+1,j)^{*k}}) + 2(1 - \lambda_{x} - \lambda_{y})\tau^{n}_{(ij)^{*k}} \\ &+ \delta^{2} [\Psi''(u^{n}_{ij})q^{N}_{(ij)^{*k}}]^{*} \xi^{n}_{ij}, \quad i, j = 1, 2, \cdots, I, J; \\ \tau^{n+1}_{(ij)^{*k}} &= -\tau^{n-1}_{(ij)^{*k}} + \lambda_{x} (\tau^{n}_{(i-1,j)^{*k}} + \tau^{n}_{(i+1,j)^{*k}}) + \delta^{2} \frac{\alpha}{2} K'(u^{n}_{ij})r^{N}_{(ij)^{*k}} - \delta^{2} [\Psi'(u^{n}_{ij})]^{*} \rho^{n}_{(ij)^{*k}} \\ &+ \lambda_{y} (\tau^{n}_{(i,j-1)^{*k}} + \tau^{n}_{(i,j+1)^{*k}}) + \delta^{2} \frac{\alpha}{2} K'(u^{n}_{ij})r^{N}_{ij^{*k}} - \delta^{2} [\Psi'(u^{n}_{ij})]^{*} \rho^{n}_{(ij)^{*k}} \\ &- \delta^{2} [\Psi'(u^{n}_{ij})r^{N}_{ij^{*k}}]^{*} \xi^{n}_{ij}, \quad i, j = 1, 2, \cdots, I, J; \end{cases}$$

where  $\delta_{ij}$  is the Chronecker delta. Thus, we have the following Newton's-method-based shooting algorithm:

Algorithm 2 – Newton method based shooting algorithm with Euler discretizations for distributed optimal control problems

choose initial guesses  $\omega_{ij}$  and  $\theta_{ij}$ ,  $i, j = 1, 2, \cdots, I, J$ ;

% set initial conditions for u and  $\xi$ 

for 
$$i, j = 1, 2, \dots, I, J$$
  
 $u_{ij}^1 = w_{ij}, \quad u_{ij}^2 = w_{ij} + \delta z_{ij},$   
 $\xi_{ij}^1 = \omega_{ij}, \quad \xi_{ij}^2 = \xi_{ij}^1 + \delta \theta_{ij};$ 

% – set initial conditions for  $q_{(ij)^*k}, r_{(ij)^*k}, \rho_{(ij)^*k}, \tau_{(ij)^*k}$ 

for 
$$k = 1, 2, \cdots, (IJ)^*$$
  
for  $i = 1, 2, \cdots, I$   
for  $j = 1, 2, \cdots, J$   
 $q_{(ij)^*k}^1 = 0, \qquad q_{(ij)^*k}^2 = 0, \qquad r_{(ij)^*k}^1 = 0, \qquad r_{(ij)^*k}^2 = 0,$   
 $\rho_{(ij)^*k}^1 = 0, \qquad \rho_{(ij)^*k}^2 = 0, \qquad \tau_{(ij)^*k}^1 = 0, \qquad \tau_{(ij)^*k}^2 = 0;$   
 $\rho_{kk}^1 = 1, \qquad \rho_{(ij)^*k}^2 = 0, \qquad \tau_{kk}^2 = \delta,$ 

% Newton iterations

for  $m = 1, 2, \cdots, M$ 

% solve for  $(u,\xi)$ 

for 
$$n = 2, 3, \dots, N-1$$
  

$$\begin{aligned} u_{ij}^{n+1} &= -u_{ij}^{n-1} + \lambda_x (u_{i-1,j}^n + u_{i+1,j}^n) + 2(1 - \lambda_x - \lambda_y) u_{ij}^n \\ &+ \lambda_y (u_{i,j-1}^n + u_{i,j+1}^n), \\ \xi_{ij}^{n+1} &= -\xi_{ij}^{n-1} + \lambda_x (\xi_{i-1,j}^n + \xi_{i+1,j}^n) + 2(1 - \lambda_x - \lambda_y) \xi_{ij}^n + \lambda_y (\xi_{i,j-1}^n + \xi_{i,j+1}^n) \\ &+ \delta^2 \frac{\alpha}{2} K(u_{ij}^n) - \delta^2 [\Psi'(u_{ij}^n)]^* \xi_{ij}^n; \end{aligned}$$

% solve for  $q, r, \rho, \tau$ 

# for $j = 1, 2, \dots, I$ for $n = 2, 3, \dots, N-1$ for $i = 2, \dots, N-1$ $q_{(ij)^*k}^{n+1} = -q_{(ij)^*k}^{n-1} + \lambda_x (q_{(i-1,j)^*k}^n + q_{(i+1,j)^*k}^n)$ $+2(1 - \lambda_x - \lambda_y)q_{(ij)^*k}^n + \lambda_y (q_{(i,j-1)^*k}^n + q_{(i,j+1)^*k}^n)$ $-\delta^2 \rho_{(ij)^*k}^n - \delta^2 \Psi'(u_{ij}^n) q_{(ij)^*k}^n,$

$$\begin{split} r_{(ij)^*k}^{n+1} &= -r_{(ij)^*k}^{n-1} + \lambda_x (r_{(i-1,j)^*k}^n + r_{(i+1,j)^*k}^n) \\ &+ 2(1 - \lambda_x - \lambda_y) r_{(ij)^*k}^n + \lambda_y (r_{(i,j-1)^*k}^n + r_{(i,j+1)^*k}^n) \\ &- \delta^2 \tau_{(ij)^*k}^n - \delta^2 \Psi'(u_{ij}^n) r_{(ij)^*k}^n, \\ \rho_{(ij)^*k}^{n+1} &= -\rho_{(ij)}^{n-1} + \lambda_x (\rho_{(i-1,j)^*k}^n + \rho_{(i+1,j)^*k}^n) \\ &+ 2(1 - \lambda_x - \lambda_y) \rho_{(ij)^*k}^n + \lambda_y (\rho_{(i,j-1)^*k}^n + \rho_{(i,j+1)^*k}^n) \\ &+ \delta^2 \frac{\alpha}{2} K'(u_{ij}^n) q_{(ij)^*k}^N - \delta^2 [\Psi'(u_i^n)]^* \rho_{ij}^n - \delta^2 [\Psi''(u_i^n) q_{ij}^N]^* \xi_{ij}^n \\ \tau_{(ij)^*k}^{n+1} &= -\tau_{(ij)^*k}^{n-1} + \lambda_x (\tau_{(i-1,j)^*k}^n + \tau_{(i+1,j)^*k}^n) \\ &+ 2(1 - \lambda_x - \lambda_y) \tau_{(ij)^*k}^n + \lambda_y (\tau_{(i,j-1)^*k}^n + \tau_{(i,j+1)^*k}^n) \\ &+ \delta^2 \frac{\alpha}{2} K'(u_{ij}^n) r_{(ij)^*k}^N - \delta^2 [\Psi'(u_{ij}^n)]^* \rho_{ij}^n - \delta^2 [\Psi''(u_{ij}^n) r_i^N]^* \xi_{ij}^n \end{split}$$

% (we need to build into the algorithm the following:

$$\begin{split} q_0^n &= r_0^n = \rho_0^n = \tau_0^n = 0, \\ q_{I+1}^n &= r_{I+1}^n = \rho_{I+1}^n = \tau_{I+1}^n = 0.) \end{split}$$

% evaluate F and F'

for 
$$i, j = 1, 2, \cdots, I, J$$
  

$$F_{2(ij)^{*}-1} \equiv \frac{\xi_{ij}^{N} - \xi_{ij}^{N-1}}{\delta} + \frac{\beta}{2} \Phi'(u_{ij}^{N}), \qquad F_{2(ij)^{*}} \equiv \xi_{ij}^{N};$$
for  $j = 1, 2, \cdots, I$   

$$J_{2(ij)^{*}-1,2k-1} = \frac{\rho_{(ij)^{*}k}^{N} - \rho_{(ij)^{*}k}^{N-1}}{\delta} + \frac{\beta}{2} \Phi_{1}''(u_{ij}^{N}) q_{(ij)^{*}k}^{N},$$

$$J_{2(ij)^{*}-1,2k} = \frac{\tau_{(ij)^{*}k}^{N} - \tau_{(ij)^{*}k}^{N-1}}{\delta} + \frac{\beta}{2} \Phi_{1}''(u_{ij}^{N}) r_{(ij)^{*}k}^{N},$$

$$J_{2(ij)^{*},2k-1} = \rho_{(ij)^{*}k}^{N} - \frac{\gamma}{2\delta} \Phi_{2}''(\frac{u_{ij}^{N} - u_{ij}^{N-1}}{\delta})(q_{(ij)^{*}k}^{N} - q_{(ij)^{*}k}^{N-1}),$$

$$J_{2(ij)^{*},2k} = \tau_{(ij)^{*}k}^{N} - \frac{\gamma}{2\delta} \Phi_{2}''(\frac{u_{ij}^{N} - u_{ij}^{N-1}}{\delta})(r_{(ij)^{*}k}^{N} - r_{(ij)^{*}k}^{N-1});$$

solve  $J\mathbf{c} = -F$  by Gaussian eliminations;

for 
$$i, j = 1, 2, \cdots, I, J$$
  

$$\omega_{ij}^{\text{new}} = \omega_{ij} + c_{2(ij)^*-1},$$

$$\theta_{ij}^{\text{new}} = \theta_{ij} + c_{2(ij)^*};$$

if  $\max_{ij} |\omega_{ij}^{\text{new}} - \omega_{ij}| + \max_{ij} |\theta_{ij}^{\text{new}} - \theta_{ij}| < \text{tol, stop;}$ 

otherwise, reset  $\omega_{ij} = \omega_{ij}^{\text{new}}$  and  $\theta_{ij} = \theta_{ij}^{\text{new}}$ ,  $i, j = 1, 2, \cdots, I, J$ ;
#### 2.4.2 2D computational results

For the following examples, we provide several graphs with fixed time and y-coordinates in order to observe the computational results easily, so called snap-shot. For fixed time, we present the three graphs with y-coordinates 0.25, 0.5, and 0.75 from left to right.

Example 2.4.1. (Linear case)  $\Psi(u) = 0, T = 1, \Omega = [0, 1] \times [0, 1].$ 

For given target functions,

$$W(x,y) = x(x-1)y(y-1)\cos 1, \quad U(t,x,y) = x(x-1)y(y-1)\cos t.$$
 (2.4.34)



Figure 2.7 Optimal solution u and target W, U for  $\triangle t = 1/36$ ,  $\triangle x = 1/16$ and  $\triangle y = 1/16$   $\therefore$  optimal solution u(t, x) -: target functions W(x), U(t, x)  $\alpha = 100$ ,  $\beta = 100$ 



Figure 2.8 Optimal solution u and targets W, U for  $\Delta t = 1/36$ ,  $\Delta x = 1/16$ and  $\Delta y = 1/16$  : optimal solution u(t, x) -: target functions W(x), U(t, x)  $\alpha = 10000$ ,  $\beta = 10000$ 

Example 2.4.2. (Nonlinear case)  $\Psi(u) = u^3 - u$ , T = 1,  $\Omega = [0, 1] \times [0, 1]$ .

For given target functions,

$$W(x,y) = \cos \pi \sin \pi x \sin \pi y$$
  

$$U(t,x,y) = \cos \pi t \sin \pi x \sin \pi y.$$
(2.4.35)



Figure 2.9 Optimal solution u and target W, U for  $\Delta t = 1/36$ ,  $\Delta x = 1/16$ and  $\Delta y = 1/16$  : optimal solution u(t, x) -: target functions W(x), U(t, x)  $\alpha = 1000$ ,  $\beta = 1000$ 

# 3 Shooting Methods for Numerical Solutions of Exact Controllability Problems Constrained by Linear and Nonlinear Wave Equations with Local Distributed Controls

Numerical solutions of exact controllability problems for linear and semilinear wave equations with local distributed controls are studied. In chapter 1, we introduced the optimality system of equations and the corresponding algorithm for shooting method. For the nonhomogeneous term in the state equation, we multiply a characteristic function which will render the problems local controllability cases. The algorithm for these problems is a simple modification of the algorithm in chapter 2, and so we will skip the section of the algorithm. The convergence properties of the numerical shooting method in the context of exact controllability are illustrated through computational experiments.

### 3.1 Exact controllability problems for the wave equations

We will study numerical methods for optimal control and exact controllability problems with local distributed controls associated with the linear and nonlinear wave equations. As before, our concern is to investigate the relevancy and applicability of high performance computing (HPC) for these problems.

As an prototype example of optimal control problems for the wave equations with local distributed controls, we consider the following distributed optimal control problem:

choose a control f and a corresponding u such that the pair (f, u) minimizes the cost

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functional

$$\mathcal{J}(u,f) = \frac{\alpha}{2} \int_0^T \int_\Omega K(u) \, d\mathbf{x} \, dt + \frac{\beta}{2} \int_\Omega \Phi_1(u(T,\mathbf{x})) \, d\mathbf{x} + \frac{\gamma}{2} \int_\Omega \Phi_2(u_t(T,\mathbf{x})) \, d\mathbf{x} + \frac{1}{2} \int_0^T \int_{\Omega_1} |f|^2 \, d\mathbf{x} \, dt = \frac{\alpha}{2} \int_0^T \int_\Omega K(u) \, d\mathbf{x} \, dt + \frac{\beta}{2} \int_\Omega \Phi_1(u(T,\mathbf{x})) \, d\mathbf{x} + \frac{\gamma}{2} \int_\Omega \Phi_2(u_t(T,\mathbf{x})) \, d\mathbf{x} + \frac{1}{2} \int_0^T \int_\Omega |\chi_{\Omega_1} f|^2 \, d\mathbf{x} \, dt$$
(3.1.1)

subject to the wave equation

$$u_{tt} - \Delta u + \Psi(u) = \chi_{\Omega_1} f \quad \text{in } (0, T) \times \Omega,$$
  

$$u_{\partial\Omega} = 0, \quad u(0, \mathbf{x}) = w(\mathbf{x}), \quad u_t(0, \mathbf{x}) = z(\mathbf{x}).$$
(3.1.2)

Here  $\Omega$  is a bounded spatial domain in  $\mathbb{R}^d$  (d = 1 or 2 or 3) with a boundary  $\partial\Omega$  and  $\Omega_1 \subseteq \Omega$ ; u is dubbed the state, and g is the distributed control. Also, K,  $\Phi$  and  $\Psi$  are  $C^1$  mappings (for instance, we may choose  $K(u) = (u - U)^2$ ,  $\Psi(u) = 0$ ,  $\Psi(u) = u^3 - u$ and  $\Psi(u) = \sin u$ ,  $\Phi_1(u) = (u(T, \mathbf{x}) - W)^2$ ,  $\Phi_2(u) = (u_t(T, \mathbf{x}) - Z)^2$ , where U, W, Z is a target function.) Using Lagrange multiplier rules one finds the following optimality system of equations that the optimal solution (f, u) must satisfy:

$$\begin{split} u_{tt} - \Delta u + \Psi(u) &= \chi_{\Omega_1} f \quad \text{in } (0, T) \times \Omega \\ u|_{\partial\Omega} &= 0, \quad u(0, \mathbf{x}) = w(\mathbf{x}), \quad u_t(0, \mathbf{x}) = z(\mathbf{x}) \,; \\ \xi_{tt} - \Delta \xi + [\Psi'(u)]^* \xi &= \frac{\alpha}{2} K'(u) \text{ in } Q \\ \xi|_{\partial\Omega} &= 0, \quad \xi(T, \mathbf{x}) = \frac{\gamma}{2} \Phi'_2(u_t(T, \mathbf{x})), \quad \xi_t(T, \mathbf{x}) = -\frac{\beta}{2} \Phi'_1(u(T, \mathbf{x})) \,; \\ \chi_{\Omega_1} f + \chi_{\Omega_1} \xi = 0 \text{ in } Q \,. \end{split}$$

This system may be simplified as

$$u_{tt} - \Delta u + \Psi(u) = -\chi_{\Omega_{1}}\xi \quad \text{in } (0, T) \times \Omega$$
  

$$u|_{\partial\Omega} = 0, \quad u(0, \mathbf{x}) = w(\mathbf{x}), \quad u_{t}(0, \mathbf{x}) = z(\mathbf{x});$$
  

$$\xi_{tt} - \Delta \xi + [\Psi'(u)]^{*}\xi = \frac{\alpha}{2}K'(u) \quad \text{in } (0, T) \times \Omega$$
  

$$\xi|_{\partial\Omega} = 0, \quad \xi(T, \mathbf{x}) = \frac{\gamma}{2}\Phi'_{2}(u_{t}(T, \mathbf{x})), \quad \xi_{t}(T, \mathbf{x}) = -\frac{\beta}{2}\Phi'_{1}(u(T, \mathbf{x})).$$
  
(3.1.3)

Such control problems are classical ones in the control theory literature; see, e.g., [5] for the linear case and [6] for the nonlinear case regarding the existence of optimal solutions as well as the existence of a Lagrange multiplier  $\xi$  satisfying the optimality system of equations. However, numerical methods for finding discrete (e.g., finite element and/or finite difference) solutions of the optimality system are largely limited to gradient type methods which are sequential in nature and generally require many iterations for convergence. The optimality system involves boundary conditions at t = 0 and t = T and thus cannot be solved by marching in time. Direct solutions of the discrete optimality system, of course, are bound to be expensive computationally in 2 or 3 spatial dimensions since the problem is (d + 1) dimensional (where d is the spatial dimensions.)

# 3.2 The solution of the exact local controllability problem as the limit of optimal control solutions

The exact distributed local control problem we consider is to seek a distributed local control  $f \in L^2((0,T) \times \Omega_1)$  where  $\Omega_1 \subsetneq \Omega$  and a corresponding state u such that the following system of equations hold:

$$u_{tt} - \Delta u + \Psi(u) = \chi_{\Omega_1} f \quad \text{in } Q \equiv (0, T) \times \Omega ,$$
  

$$u|_{t=0} = w \quad \text{and} \quad u_t|_{t=0} = z \quad \text{in } \Omega ,$$
  

$$u|_{t=T} = W \quad \text{and} \quad u_t|_{t=T} = Z \quad \text{in } \Omega ,$$
  

$$u|_{\partial\Omega} = 0 \quad \text{in } (0, T) .$$
  
(3.2.4)

Under suitable assumptions on f and through the use of Lagrange multiplier rules, the corresponding optimal local control problem:

minimize (3.1.1) with respect to the control f subject to (3.1.2). (3.2.5)

In this section we establish the equivalence between the limit of optimal solutions and the minimum distributed  $L^2$  norm exact local controller. We will show that if  $\alpha \to \infty$ ,  $\beta \to \infty$  and  $\gamma \to \infty$ , then the corresponding optimal solution  $(\hat{u}_{\alpha\beta\gamma}, \hat{f}_{\alpha\beta\gamma})$ converges weakly to the minimum distributed  $L^2$  norm solution of the exact distributed local controllability problem (3.2.4). The same is also true in the discrete case. Let  $Q_1 = (0, T) \times \Omega_1$ .

**Theorem 3.2.1.** Assume that the exact distributed local controllability problem (3.2.4) admits a unique minimum distributed  $L^2$  norm solution  $(u_{\text{ex}}, f_{\text{ex}})$ . Assume that for every  $(\alpha, \beta, \gamma) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$  (where  $\mathbb{R}_+$  is the set of all positive real numbers,) there exists a solution  $(u_{\alpha\beta\gamma}, f_{\alpha\beta\gamma})$  to the optimal local control problem (3.2.5). Then

$$\|f_{\alpha\beta\gamma}\|_{L^2(Q_1)} \le \|f_{\mathrm{ex}}\|_{L^2(Q_1)} \qquad \forall (\alpha, \beta, \gamma) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ . \tag{3.2.6}$$

Assume, in addition, that for a sequence  $\{(\alpha_n, \beta_n, \gamma_n)\}$  satisfying  $\alpha_n \to \infty$ ,  $\beta_n \to \infty$ 

and  $\gamma_n \to \infty$ ,

$$u_{\alpha_n\beta_n\gamma_n} \rightharpoonup \overline{u} \text{ in } L^2(Q) \quad and \quad \Psi(u_{\alpha_n\beta_n\gamma_n}) \rightharpoonup \Psi(\overline{u}) \text{ in } L^2(0,T; [H^2(\Omega) \cap H^1_0(\Omega)]^*).$$
(3.2.7)

Then

$$f_{\alpha_n\beta_n\gamma_n} \rightharpoonup f_{\text{ex}} \text{ in } L^2(Q_1) \text{ and } u_{\alpha_n\beta_n\gamma_n} \rightharpoonup u_{\text{ex}} \text{ in } L^2(Q) \text{ as } n \to \infty.$$
 (3.2.8)

Furthermore, if (3.2.7) holds for every sequence  $\{(\alpha_n, \beta_n, \gamma_n)\}$  satisfying  $\alpha_n \to \infty$ ,  $\beta_n \to \infty$  and  $\gamma_n \to \infty$ , then

$$f_{\alpha\beta\gamma} \rightharpoonup f_{\text{ex}} \text{ in } L^2(Q_1) \text{ and } u_{\alpha\beta\gamma} \rightharpoonup u_{\text{ex}} \text{ in } L^2(Q) \text{ as } \alpha, \beta, \gamma \rightarrow \infty.$$
 (3.2.9)

*Proof.* Since  $(u_{\alpha\beta\gamma}, f_{\alpha\beta\gamma})$  is an optimal solution, we have that

$$\frac{\alpha}{2} \|u_{\alpha\beta\gamma}(T) - U\|_{L^2(Q)} + \frac{\beta}{2} \|u_{\alpha\beta\gamma}(T) - W\|_{L^2(\Omega)} + \frac{\gamma}{2} \|\partial_t u_{\alpha\beta\gamma}(T) - Z\|_{H^{-1}(\Omega)} + \frac{1}{2} \|f_{\alpha\beta\gamma}\|_{L^2(Q_1)} = \mathcal{J}(u_{\alpha\beta\gamma}, f_{\alpha\beta\gamma}) \le \mathcal{J}(u_{\mathrm{ex}}, f_{\mathrm{ex}}) = \frac{1}{2} \|f_{\mathrm{ex}}\|_{L^2(Q_1)}$$

so that (3.2.6) holds,

$$u_{\alpha\beta\gamma}|_{t=T} \to W \text{ in } L^2(\Omega) \text{ and } (\partial_t u_{\alpha\beta\gamma})|_{t=T} \to Z \text{ in } H^{-1}(\Omega) \text{ as } \alpha, \beta, \gamma \to \infty.$$
 (3.2.10)

Let  $\{(\alpha_n, \beta_n, \gamma_n)\}$  be the sequence in (3.2.7). Estimate (3.2.6) implies that a subsequence of  $\{(\alpha_n, \beta_n, \gamma_n)\}$ , denoted by the same, satisfies

$$f_{\alpha_n \beta_n \gamma_n} \rightharpoonup \overline{f} \text{ in } L^2(Q_1) \quad \text{and} \quad \|\overline{f}\|_{L^2(Q_1)} \le \|f_{\text{ex}}\|_{L^2(Q_1)}.$$
 (3.2.11)

 $(u_{\alpha\beta\gamma}, f_{\alpha\beta\gamma})$  satisfies the initial value problem in the weak form:

$$\int_{0}^{T} \int_{\Omega} u_{\alpha\beta\gamma}(v_{tt} - v_{xx}) \, dx \, dt + \int_{0}^{T} \int_{\Omega} [\Psi(u_{\alpha\beta\gamma}) - \chi_{\Omega_{1}} f_{\alpha\beta\gamma}] v \, dx \, dt$$
$$+ \int_{\Omega} (v\partial_{t}u_{\alpha\beta\gamma})|_{t=T} \, dx - \int_{\Omega} v|_{t=0} z \, dx - \int_{\Omega} (u_{\alpha\beta\gamma}\partial_{t}v)|_{t=T} \, dx \qquad (3.2.12)$$
$$+ \int_{\Omega} (w\partial_{t}v)|_{t=0} \, dx = 0 \qquad \forall v \in C^{2}([0,T]; H^{2}(\Omega) \cap H^{1}_{0}(\Omega))$$

Passing to the limit in (3.2.12) as  $\alpha, \beta, \gamma \to \infty$  and using relations (3.2.10) and (3.2.11) we obtain:

$$\begin{split} &\int_0^T \int_{\Omega} \overline{u}(v_{tt} - v_{xx}) \, dx \, dt + \int_0^T \int_{\Omega} [\Psi(\overline{u}) - \chi_{\Omega_1} \overline{f}] v \, dx \, dt \\ &+ \int_{\Omega} v|_{t=T} Z(x) \, dx - \int_{\Omega} v|_{t=0} z \, dx - \int_{\Omega} W(x) (\partial_t v)|_{t=T} \, dx \\ &+ \int_{\Omega} (w \partial_t v)|_{t=0} \, dx = 0 \qquad \forall v \in C^2([0,T]; H^2(\Omega) \cap H^1_0(\Omega)) \, . \end{split}$$

The last relation and (3.2.11) imply that  $(\overline{u}, \overline{f})$  is a minimum boundary  $L^2$  norm solution to the exact control problem (3.2.4). Hence,  $\overline{u} = u_{ex}$  and  $\overline{f} = f_{ex}$  so that (3.2.8) and (3.2.9) follows from (3.2.7) and (3.2.11).

**Remark 3.2.2.** If the wave equation is linear, i.e.,  $\Psi = 0$ , then assumption (3.2.7) is redundant and (3.2.9) is guaranteed to hold. Indeed, (3.2.12) implies the boundedness of  $\{\|u_{\alpha\beta\gamma}\|_{L^2(Q)}\}$  which in turn yields (3.2.7). The uniqueness of a solution for the linear wave equation implies (3.2.7) holds for an arbitrary sequence  $\{(\alpha_n, \beta_n, \gamma_n)\}$ .

**Theorem 3.2.3.** Assume that

i) for every (α, β, γ) ∈ ℝ<sub>+</sub> × ℝ<sub>+</sub> × ℝ<sub>+</sub> there exists a solution (u<sub>αβγ</sub>, f<sub>αβγ</sub>) to the optimal control problem (3.2.5);

ii) the limit terminal conditions hold:

$$u_{\alpha\beta\gamma}|_{t=T} \to W \text{ in } L^2(\Omega) \text{ and } (\partial_t u_{\alpha\beta\gamma})|_{t=T} \to Z \text{ in } H^{-1}(\Omega) \text{ as } \alpha, \beta, \gamma \to \infty;$$
  
(3.2.13)

iii) the optimal solution  $(u_{\alpha\beta\gamma}, f_{\alpha\beta\gamma})$  satisfies the weak limit conditions as  $\alpha, \beta, \gamma \to \infty$ :

$$f_{\alpha\beta\gamma} \rightharpoonup \overline{f} \ in \ L^2(Q_1) , \quad u_{\alpha\beta\gamma} \rightharpoonup \overline{u} \ in \ L^2(Q) ,$$
 (3.2.14)

and

$$\Psi(u_{\alpha\beta\gamma}) \to \Psi(\overline{u}) \text{ in } L^2(0,T; [H^2(\Omega) \cap H^1_0(\Omega)]^*)$$
(3.2.15)

for some  $\overline{f} \in L^2(Q_1)$  and  $\overline{u} \in L^2(Q)$ .

Then  $(\overline{u}, \overline{f})$  is a solution to the exact boundary controllability problem (3.2.4) with  $\overline{f}$  satisfying the minimum boundary  $L^2$  norm property. Furthermore, if the solution to (3.2.4) admits a unique solution  $(u_{\text{ex}}, f_{\text{ex}})$ , then

$$f_{\alpha\beta\gamma} \rightarrow f_{\text{ex}} \text{ in } L^2(Q_1) \text{ and } u_{\alpha\beta\gamma} \rightarrow u_{\text{ex}} \text{ in } L^2(Q) \text{ as } \alpha, \beta, \gamma \rightarrow \infty.$$
 (3.2.16)

*Proof.*  $(u_{\alpha\beta\gamma}, f_{\alpha\beta\gamma})$  satisfies (3.2.12). Passing to the limit in that equation as  $\alpha, \beta, \gamma \to \infty$ and using relations (3.2.13), (3.2.14) and (3.2.15) we obtain:

$$\int_0^T \int_\Omega \overline{u}(v_{tt} - v_{xx}) \, dx \, dt + \int_0^T \int_\Omega [\Psi(\overline{u}) - \chi_{\Omega_1} \overline{f}] v \, dx \, dt$$
$$+ \int_\Omega v|_{t=T} Z(x) \, dx - \int_\Omega v|_{t=0} z \, dx - \int_\Omega W(x)(\partial_t v)|_{t=T} \, dx$$
$$+ \int_\Omega (w \partial_t v)|_{t=0} \, dx = 0 \qquad \forall v \in C^2([0,T]; H^2(\Omega) \cap H^1_0(\Omega)) \, dx$$

This implies that  $(\overline{u}, \overline{f})$  is a solution to the exact boundary controllability problem (3.2.4).

To prove that  $\overline{f}$  satisfies the minimum boundary  $L^2$  norm property, we proceeds as follows. Let  $(u_{\text{ex}}, f_{\text{ex}})$  denotes a exact minimum boundary  $L^2$  norm solution to the controllability problem (3.2.4). Since  $(u_{\alpha\beta\gamma}, f_{\alpha\beta\gamma})$  is an optimal solution, we have that

$$\frac{\alpha}{2} \|u_{\alpha\beta\gamma} - U\|_{L^{2}(Q)}^{2} + \frac{\beta}{2} \|u_{\alpha\beta\gamma} - W\|_{L^{2}(\Omega)}^{2} + \frac{\gamma}{2} \|\partial_{t}u_{\alpha\beta\gamma} - Z\|_{H^{-1}(\Omega)}^{2} + \frac{1}{2} \|f_{\alpha\beta\gamma}\|_{L^{2}(Q_{1})}^{2} = \mathcal{J}(u_{\alpha\beta\gamma}, f_{\alpha\beta\gamma}) \leq \mathcal{J}(u_{\text{ex}}, f_{\text{ex}}) = \frac{1}{2} \|f_{\text{ex}}\|_{L^{2}(Q_{1})}^{2}$$

so that

$$\|f_{\alpha\beta\gamma}\|_{L^2(Q_1)}^2 \le \|f_{\mathrm{ex}}\|_{L^2(Q_1)}^2.$$

Passing to the limit in the last estimate we obtain

$$\|\overline{f}\|_{L^{2}(Q_{1})}^{2} \leq \|f_{\mathrm{ex}}\|_{L^{2}(Q_{1})}^{2}.$$
(3.2.17)

Hence we conclude that  $(\overline{u}, \overline{f})$  is a minimum boundary  $L^2$  norm solution to the exact boundary controllability problem (3.2.4).

Furthermore, if the exact controllability problem (3.2.4) admits a unique minimum boundary  $L^2$  norm solution  $(u_{ex}, f_{ex})$ , then  $(\overline{u}, \overline{f}) = (u_{ex}, f_{ex})$  and (3.2.16) follows from assumption (3.2.14).

**Remark 3.2.4.** If the wave equation is linear, i.e.,  $\Psi = 0$ , then assumptions i) and (3.2.15) are redundant.

**Remark 3.2.5.** Assumptions ii) and iii) hold if  $f_{\alpha\beta\gamma}$  and  $u_{\alpha\beta\gamma}$  converges pointwise as  $\alpha, \beta, \gamma \to \infty$ .

**Remark 3.2.6.** A practical implication of Theorem 3.2.3 is that one can prove the exact controllability for semilinear wave equations by examining the behavior of a sequence of optimal solutions (recall that exact controllability was proved only for some special classes of semilinear wave equations.) If we have found a sequence of optimal

control solutions  $\{(u_{\alpha_n\beta_n\gamma_n}, f_{\alpha_n\beta_n\gamma_n})\}$  where  $\alpha_n, \beta_n, \gamma_n \to \infty$  and this sequence appears to satisfy the convergence assumptions ii) and iii), then we can confidently conclude that the underlying semilinear wave equation is exactly controllable and the optimal solution  $(u_{\alpha_n\beta_n\gamma_n}, f_{\alpha_n\beta_n\gamma_n})$  when n is large provides a good approximation to the minimum boundary  $L^2$  norm exact controller  $(u_{ex}, f_{ex})$ .

## 3.3 Computational Results

We consider examples of the following types : Seek the pair (u, f) that minimizes

$$\mathcal{J}(u,f) = \frac{\alpha}{2} \int_0^T \int_\Omega |u - U|^2 \, dx \, dt + \frac{\beta}{2} \int_\Omega |u(T,x) - W(x)|^2 \, dx + \frac{\gamma}{2} \int_\Omega |u_t(T,x) - Z(x)|^2 \, dx + \frac{1}{2} \int_0^T \int_{\Omega_1} |f|^2 \, dx \, dt$$
(3.3.18)

subject to the wave equation

$$u_{tt} - u_{xx} + \Psi(u) = \chi_{\Omega_1} f \quad \text{in } (0, T) \times \Omega,$$
  

$$u|_{\partial\Omega} = 0, \qquad (3.3.19)$$
  

$$u(0, x) = g(x), \quad u_t(0, x) = h(x),$$

where  $\Omega_1 \subseteq \Omega$ .

### 3.3.1 Exact controllability problems with linear cases

**Example 3.3.1.** (full domain control)  $\Psi(u) = 0$ , T = 1,  $\Omega = \Omega_1 = [0, 1]$ .

For given target functions,

$$W(x) = 0, \qquad Z(x) = 2\pi \sin(2\pi x),$$
  
(3.3.20)  
$$U(t, x) = \sin(2\pi x) \sin(2\pi t).$$



Figure 3.1 Optimal solution u and target W for  $\Delta t = 1/80$  and  $\Delta x = 1/40$ ,  $\times$ : optimal solution u(T, x) —: target function W(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 1000$ 



Figure 3.2 Optimal solution  $u_t$  and target Z for  $\Delta t = 1/80$  and  $\Delta x = 1/40$  $\cdot, \times:$  optimal solution  $u_t(T, x)$  -: target function Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 100$ 



Figure 3.3 Optimal solution u and target W for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\therefore$  optimal solution u(T, x) -: target function W(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.4 Optimal solution  $u_t$  and target Z for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\therefore$  optimal solution  $u_t(T, x)$  -: target function Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.5 Optimal solution u for  $\triangle t = 1/80$  and  $\triangle x = 1/40$ -: optimal solution u(t, x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 









 $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.8 Optimal solution u for  $\triangle t = 1/400$  and  $\triangle x = 1/400$ -: optimal solution u(t, x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 

Example 3.3.2. (linear case and local domain control)  $\Psi(u) = 0, T = 1, \Omega = [0, 1],$  $\Omega_1 = [0, 0.1] \cup [0.9, 1]$ .







Figure 3.10 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$  $\cdot, \times$ : optimal solution  $u_t(T, x)$  —: target function Z(x) $\alpha = 0$ ,  $\beta = \gamma = 100, 1000$ 



Figure 3.11 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\therefore$  optimal solution u(T, x) -: target function W(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 

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Figure 3.12 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\therefore$  optimal solution  $u_t(T, x)$  -: target function Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.13 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\therefore$  optimal solution u(t, x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 

**Example 3.3.3.** (linear case and local domain control)  $\Psi(u) = 0, T = 1, \Omega = [0, 1],$  $\Omega_1 = [0.45, 0.55]$ .







Figure 3.15 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\cdot$ : optimal solution  $u_t(T, x)$  -: target function Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.16 Optimal solution u for  $\triangle t = 1/80$  and  $\triangle x = 1/40$ : optimal solution u(t, x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 

Example 3.3.4. (linear case and local domain control)  $\Psi(u) = 0, T = 2, \Omega = [0, 1],$  $\Omega_1 = [0.45, 0.55]$ .







Figure 3.18 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\cdot$ : optimal solution  $u_t(T, x)$  -: target function Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.19 Optimal solution u for  $\triangle t = 1/80$  and  $\triangle x = 1/40$   $\therefore$  optimal solution u(t, x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 

Example 3.3.5. (linear case and local domain control)  $\Psi(u) = 0, T = 1, \Omega = [0, 1],$  $\Omega_1 = [0.495, 0.505]$ .







Figure 3.21 Optimal solution u for  $\Delta t = 1/400$  and  $\Delta x = 1/400$   $\cdot$ : optimal solution  $u_t(T, x)$  -: target function Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.22 Optimal solution u for  $\triangle t = 1/400$  and  $\triangle x = 1/400$   $\therefore$  optimal solution u(t, x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 

3.3.2 Exact controllability problems with nonlinear cases

**Example 3.3.6.** (full domain control)  $\Psi(u) = \sin u$ , T = 2,  $\Omega = \Omega_1 = [0, 1]$ . For given target functions,

$$W(x) = \sin(\pi x)\cos(T), \quad Z(x) = -\sin(\pi x)\sin(T),$$
  

$$U(t, x) = \sin(\pi x)\cos(t).$$
(3.3.21)



Figure 3.23 Optimal solution u for  $\triangle t = 1/80$  and  $\triangle x = 1/40$ , ×: optimal solution u(T, x) -: target function W(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 10000$ 



Figure 3.24 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$  $\cdot, \times$ : optimal solution  $u_t(T, x)$  —: target function Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 100$ 



Figure 3.25 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\therefore$  optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.26 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\therefore$  optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.27 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$ -: optimal solution u(t, x) $\alpha = 0, \beta = \gamma = 1000000$ 



Figure 3.28 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$ -: target function U(t, x) $\alpha = 0, \beta = \gamma = 1000000$ 

Example 3.3.7. (local domain control)  $\Psi(u) = \sin u$ , T = 2,  $\Omega = [0, 1]$ ,  $\Omega_1 = [0, 0.1] \cup [0.9, 1]$ .



Figure 3.29 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$ , ×: optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 100000$ 



Figure 3.30 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$  $\cdot, \times$ : optimal solution  $u_t(T, x)$  -: target functions Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 10000$ 



Figure 3.31 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\cdot$ : optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.32 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\therefore$  optimal solution  $u_t(T, x)$  -: target functions Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.33 Optimal solution u for  $\bigtriangleup t=1/80$  and  $\bigtriangleup x=1/40$  –: optimal solution u(t,x)  $\alpha=0$  ,  $\beta=\gamma=1000000$ 

**Example 3.3.8.** (full domain control)  $\Psi(u) = u \ln^2(u^2 + 1), T = 2, \Omega = \Omega_1 = [0, 1]$ . Suppose we have the same target functions as (3.3.21).



Figure 3.34 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$ ,  $\times$ : optimal solution u(T, x) —: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 10000$ 



Figure 3.35 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$ ,  $\times$ : optimal solution  $u_t(T, x)$  —: target functions Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 100$ 



Figure 3.36 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\therefore$  optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.37 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\therefore$  optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$




Example 3.3.9. (local domain control)  $\Psi(u) = u \ln^2(u^2 + 1), T = 2, \Omega = [0, 1], \Omega_1 = [0, 0.1] \cup [0.9, 1]$ .

Suppose we have the same target functions as (3.3.21).



Figure 3.39 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$ , ×: optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 100000$ 



Figure 3.40 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$  $\cdot, \times$ : optimal solution  $u_t(T, x)$  -: target functions Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 10000$ 



Figure 3.41 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\cdot$ : optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.42 Optimal solution u for  $\triangle t = 1/80$  and  $\triangle x = 1/40$   $\therefore$  optimal solution  $u_t(T, x)$  -: target functions Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 





**Example 3.3.10.** (full domain control)  $\Psi(u) = u^3 - u$ , T = 2,  $\Omega = \Omega_1 = [0, 1]$ . For given target functions,

$$W(x) = \sin(2\pi x)\cos(T),$$
  

$$Z(x) = -\sin(2\pi x)\sin(T),$$
  

$$U(t, x) = \sin(2\pi x)\cos(t).$$
  
(3.3.22)



Figure 3.44 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$  $\cdot, \times$ : optimal solution u(T, x) —: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 10000$ 







Figure 3.46 Optimal solution u for  $\triangle t = 1/80$  and  $\triangle x = 1/40$   $\therefore$  optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.47 Optimal solution u for  $\triangle t = 1/80$  and  $\triangle x = 1/40$   $\therefore$  optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 





Figure 3.48 Optimal solution u for  $\triangle t = 1/80$  and  $\triangle x = 1/40$ -: optimal solution u(t, x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 

Example 3.3.11. (local domain control)  $\Psi(u) = u^3 - u$ , T = 2,  $\Omega = [0, 1]$ ,  $\Omega_1 = [0, 0.1] \cup [0.9, 1]$ .

Suppose we have the same target functions as (3.3.22).







Figure 3.50 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$  $\cdot, \times$ : optimal solution  $u_t(T, x)$  —: target functions Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 100$ 



Figure 3.51 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\therefore$  optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.52 Optimal solution u for  $\triangle t = 1/80$  and  $\triangle x = 1/40$   $\therefore$  optimal solution  $u_t(T, x)$  -: target functions Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 





For given target functions,

$$W(x) = \sin(\pi x) \cos(T),$$
  

$$Z(x) = -\sin(\pi x) \sin(T),$$
  

$$U(t, x) = \sin(\pi x) \cos(t).$$
  
(3.3.23)



Figure 3.54 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$  $\cdot, \times:$  optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 10000$ 



Figure 3.55 Optimal solution u for  $\triangle t = 1/80$  and  $\triangle x = 1/40$ ×, : optimal solution  $u_t(T, x)$  -: target functions Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1$ 



Figure 3.56 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\therefore$  optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.57 Optimal solution u for  $\triangle t = 1/80$  and  $\triangle x = 1/40$ : optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 





Example 3.3.13. (local domain control)  $\Psi(u) = e^u$ , T = 2,  $\Omega = [0, 1]$ ,  $\Omega_1 = [0, 0.1] \cup [0.9, 1]$ .

Suppose we have the same target functions as (3.3.23).



Figure 3.59 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$ , ×: optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 100000$ 



Figure 3.60 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$ ,  $\times$ : optimal solution  $u_t(T, x)$  -: target functions Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 100$ 



Figure 3.61 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\therefore$  optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.62 Optimal solution u for  $\Delta t = 1/80$  and  $\Delta x = 1/40$   $\therefore$  optimal solution  $u_t(T, x)$  -: target functions Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 







Example 3.3.14. (full domain control)  $\Psi(u) = e^u$ , T = 3,  $\Omega = \Omega_1 = [0, 1]$ .

Suppose we have the same target functions as (3.3.23).



Figure 3.64 Optimal solution u for  $\Delta t = 1/90$  and  $\Delta x = 1/30$  $\cdot, \times$ : optimal solution u(T, x) —: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 10000$ 



Figure 3.65 Optimal solution u for  $\Delta t = 1/90$  and  $\Delta x = 1/30$  $\cdot, \times:$  optimal solution  $u_t(T, x)$  -: target functions Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 1000$ 



Figure 3.66 Optimal solution u for  $\Delta t = 1/90$  and  $\Delta x = 1/30$   $\therefore$  optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.67 Optimal solution u for  $\triangle t = 1/90$  and  $\triangle x = 1/30$   $\therefore$  optimal solution  $u_t(T, x)$  -: target functions Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 





Example 3.3.15. (local domain control)  $\Psi(u) = e^u$ , T = 3,  $\Omega = [0, 1]$ ,  $\Omega_1 = [0, 0.1] \cup [0.9, 1]$ .

Suppose we have the same target functions as (3.3.23).



Figure 3.69 Optimal solution u for  $\Delta t = 1/90$  and  $\Delta x = 1/30$  $\cdot, \times:$  optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 100000$ 



Figure 3.70 Optimal solution u for  $\Delta t = 1/90$  and  $\Delta x = 1/30$ ,  $\times$ : optimal solution  $u_t(T, x)$  —: target functions Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1, \cdots, 10000$ 



Figure 3.71 Optimal solution u for  $\Delta t = 1/90$  and  $\Delta x = 1/30$   $\therefore$  optimal solution u(T, x) -: target functions W(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.72 Optimal solution u for  $\Delta t = 1/90$  and  $\Delta x = 1/30$   $\cdot$ : optimal solution  $u_t(T, x)$  -: target functions Z(x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 



Figure 3.73 Optimal solution u for  $\Delta t = 1/90$  and  $\Delta x = 1/30$ -: optimal solution u(t, x) $\alpha = 0$ ,  $\beta = \gamma = 1000000$ 

4 Shooting Methods for Numerical Solutions of Exact Boundary Controllability Problems for the 1-D Wave Equation

Numerical solutions of optimal Dirichlet boundary control problems for linear and semilinear wave equations are studied. The optimal control problem is reformulated as a system of equations (an optimality system) that consists of an initial value problem for the underlying (linear or semilinear) wave equation and a terminal value problem for the adjoint wave equation. The discretized optimality system is solved by a shooting method. The convergence properties of the numerical shooting method in the context of exact controllability are illustrated through computational experiments. In the case of the linear wave equation, convergent approximations are obtained for both smooth minimum  $L^2$ -norm Dirichlet control and generic, non-smooth minimum  $L^2$ -norm Dirichlet controls. The cases of certain semilinear wave equations are also tested numerically.

## 4.1 Introduction

In this chapter we consider an optimal boundary control approach for solving the exact boundary control problem for one-dimensional linear or semilinear wave equations defined on a time interval (0, T) and spatial interval (0, X). The exact boundary control problem we consider is to seek a boundary control  $\mathbf{g} = (g_L, g_R) \in \mathbf{L}^2(0, \mathbf{T}) \subset [L^2(0, T)]^2$ 

and a corresponding state u such that the following system of equations hold:

$$u_{tt} - u_{xx} + f(u) = V \quad \text{in } Q \equiv (0, T) \times (0, X) ,$$
  

$$u|_{t=0} = u_0 \quad \text{and} \quad u_t|_{t=0} = u_1 \quad \text{in } (0, X) ,$$
  

$$u|_{t=T} = W \quad \text{and} \quad u_t|_{t=T} = Z \quad \text{in } (0, X) ,$$
  

$$u|_{x=0} = g_L \text{ and } u|_{x=1} = g_R \quad \text{in } (0, T) ,$$
  
(4.1.1)

where  $u_0$  and  $u_1$  are given initial conditions defined on (0, X),  $W \in L^2(0, X)$  and  $Z \in H^{-1}(0, X)$  are prescribed terminal conditions, V is a given function defined on  $(0, T) \times (0, X)$ , f is a given function defined on  $\mathbb{R}$ , and  $\mathbf{g} = (g_L, g_R) \in [L^2(0, T)]^2$  is the boundary control.

It is well known (see, e.g., [15, 16, 18, 19]) that when f = 0 (i.e., the equation is linear) and T is sufficiently large, the exact controllability problem (4.1.1) admits at least one state-control solution pair  $(u, \mathbf{g})$ ; furthermore, the exact controller  $\mathbf{g}$  having minimum boundary  $L^2$  norm is unique. Exact boundary controllability for semilinear wave equations have also been established for certain asymptotically linear or superlinear f; see, e.g., [8, 23, 24].

For the exact boundary controllability problem associated with the linear wave equation there are basically two classes of computational methods in the literature. The first class is HUM-based methods; see, e.g., [10, 13, 15, 17, 22]. The approximate solutions obtained by the HUM-based methods in general do not seem to converge (even in a weak sense) to the exact solutions as the temporal and spatial grid sizes tend to zero. Methods of regularization including Tychonoff regularization and filtering that result in convergent approximations were introduced in those papers on HUM-based methods. The second class of computational methods for boundary controllability of the linear wave equation was those based on the method proposed in [12]. One solves a discrete optimization problem that involves the minimization of the discrete boundary  $L^2$  norm subject to the undetermined linear system of equations formed by the discretization of the wave equation and the initial and terminal conditions. This approach was implemented in [14]. The computational results demonstrated the convergence of the discrete solutions when the exact minimum boundary  $L^2$  norm solution is smooth. In the generic case of a non-smooth exact minimum boundary  $L^2$  norm solution the computational results of [14] exhibited at least a weak  $L^2$  convergence of the discrete solutions.

Although there are well-known theoretical results concerning boundary controllability of *semilinear* wave equations (see, e.g., [8, 23, 24]), little seems to exist in the literature about computational methods for such problems.

In this chapter we attempt to solve the exact controllability problems by an optimal control approach. Precisely, we consider the following optimal control problem: minimize the cost functional

$$\mathcal{J}_{0}(u,\mathbf{g}) = \frac{\sigma}{2} \int_{0}^{1} |u(T,x) - W(x)|^{2} dx + \frac{\tau}{2} \int_{0}^{1} |u_{t}(T,x) - Z(x)|^{2} dx + \frac{1}{2} \int_{0}^{1} (|g_{L}|^{2} + |g_{R}|^{2}) dt$$

$$(4.1.2)$$

subject to

$$\begin{cases} u_{tt} - u_{xx} + f(u) = V & \text{in } Q \equiv (0, T) \times (0, 1) \\ u|_{t=0} = u_0 & \text{and} & u_t|_{t=0} = u_1 & \text{in } (0, 1) \\ u|_{x=0} = g_L & \text{and} & u|_{x=1} = g_R & \text{in } (0, T) . \end{cases}$$

$$(4.1.3)$$

The optimal control problem is converted into an optimality system of equations and this optimality system of equations will be solved by a shooting method.

The optimal control approach of this chapter provides an alternative method to the two classes of methods mentioned in the foregoing for solving the exact controllability problem for the linear wave equations; it also offers a systematic procedure for solving exact controllability problems for the semilinear wave equations. The computational solutions of this chapter obtained by an optimal control approach exhibit behaviors similar to those of the solutions obtained in [14]. Note that an optimal solution exists even when the equation is not exactly controllable. Note also that the solution methods in the literature for optimal control of PDEs can be utilized. and that there are certain intrinsic parallelisms to the algorithms studied in this chapter.

The shooting algorithms for solving the optimal control problem will be described for the slightly more general functional

$$\mathcal{J}(u,\mathbf{g}) = \frac{\alpha}{2} \int_0^T \int_0^1 |u - U|^2 \, dx \, dt + \frac{\sigma}{2} \int_0^1 |u(T,x) - W(x)|^2 \, dx + \frac{\tau}{2} \int_0^1 |u_t(T,x) - Z(x)|^2 \, dx + \frac{1}{2} \int_0^1 (|g_L|^2 + |g_R|^2) \, dt$$
(4.1.4)

where the term involving (u - U) reflects our desire to match the candidate state uwith a given U in the entire domain Q. Our computational experiments of the proposed numerical methods will be performed exclusively for the case of  $\alpha = 0$ .

The rest of this chapter is organized as follows. In Section 4.2 we establish the equivalence between the limit of optimal solutions and the minimum boundary  $L^2$  norm exact controller; this justifies the use of the optimal control approach for solving the exact control problem. In Section 4.3 we formally derive the optimality system of equations for the optimal control problem and discuss the shooting algorithm for solving the optimality system. In Section 4.4 We state the discrete version of the shooting algorithm for solving the discrete optimality system. Finally in Sections 4.5 and 4.6 we present computations of certain concrete controllability problems by the shooting method for solving optimal control problems.

## 4.2 The solution of the exact controllability problem as the limit of optimal control solutions

In this section we establish the equivalence between the limit of optimal solutions and the minimum boundary  $L^2$  norm exact controller. We will show that if  $\alpha = 0, \sigma \to \infty$ and  $\tau \to \infty$ , then the corresponding optimal solution  $(\hat{u}_{\sigma\tau}, \hat{g}_{\sigma\tau})$  converges weakly to the minimum boundary  $L^2$  norm solution of the exact boundary controllability problem (4.1.1). The same is also true in the discrete case.

**Theorem 4.2.1.** Assume that the exact boundary controllability problem (4.1.1) admits a unique minimum boundary  $L^2$  norm solution  $(u_{ex}, \mathbf{g}_{ex})$ . Assume that for every  $(\alpha, \sigma, \tau) \in \{0\} \times \mathbb{R}_+ \times \mathbb{R}_+$  (where  $\mathbb{R}_+$  is the set of all positive real numbers,) there exists a solution  $(u_{\sigma\tau}, \mathbf{g}_{\sigma\tau})$  to the optimal control problem (4.3.17). Then

$$\|\mathbf{g}_{\sigma\tau}\|_{L^{2}(\Sigma)} \leq \|\mathbf{g}_{\mathrm{ex}}\|_{L^{2}(\Sigma)} \qquad \forall (\alpha, \sigma, \tau) \in \{0\} \times \mathbb{R}_{+} \times \mathbb{R}_{+} .$$

$$(4.2.5)$$

Assume, in addition, that for a sequence  $\{(\sigma_n, \tau_n)\}$  satisfying  $\sigma_n \to \infty$  and  $\tau_n \to \infty$ ,

$$u_{\sigma_n\tau_n} \rightharpoonup \overline{u} \text{ in } L^2(Q) \quad and \quad f(u_{\sigma_n\tau_n}) \rightharpoonup f(\overline{u}) \text{ in } L^2(0,T; [H^2(\Omega) \cap H^1_0(\Omega)]^*).$$
 (4.2.6)

Then

$$\mathbf{g}_{\sigma_n \tau_n} \rightharpoonup \mathbf{g}_{\mathbf{ex}} \text{ in } [L^2(0,T)]^2 \text{ and } u_{\sigma_n \tau_n} \rightharpoonup u_{\mathbf{ex}} \text{ in } L^2(Q) \text{ as } n \to \infty.$$
 (4.2.7)

Furthermore, if (4.2.6) holds for every sequence  $\{(\sigma_n, \tau_n)\}$  satisfying  $\sigma_n \to \infty$  and  $\tau_n \to \infty$ , then

$$\mathbf{g}_{\sigma\tau} \rightharpoonup \mathbf{g}_{\mathrm{ex}} \text{ in } [L^2(0,T)]^2 \text{ and } u_{\sigma\tau} \rightharpoonup u_{\mathrm{ex}} \text{ in } L^2(Q) \text{ as } \sigma, \tau \rightarrow \infty.$$
 (4.2.8)

*Proof.* Since  $(u_{\sigma\tau}, \mathbf{g}_{\sigma\tau})$  is an optimal solution, we have that

$$\frac{\sigma}{2} \| u_{\sigma\tau}(T) - W \|_{L^2(0,1)}^2 + \frac{\tau}{2} \| \partial_t u_{\sigma\tau}(T) - Z \|_{H^{-1}(0,1)}^2 + \frac{1}{2} \| \mathbf{g}_{\sigma\tau} \|_{L^2(\Sigma)}^2$$
$$= \mathcal{J}(u_{\sigma\tau}, \mathbf{g}_{\sigma\tau}) \le \mathcal{J}(u_{\mathrm{ex}}, \mathbf{g}_{\mathrm{ex}}) = \frac{1}{2} \| \mathbf{g}_{\mathrm{ex}} \|_{L^2(\Sigma)}^2$$

so that (4.2.5) holds,

$$u_{\sigma\tau}|_{t=T} \to W \text{ in } L^2(0,X) \text{ and } (\partial_t u_{\sigma\tau})|_{t=T} \to Z \text{ in } H^{-1}(0,X) \text{ as } \sigma, \tau \to \infty.$$
 (4.2.9)

Let  $\{(\sigma_n, \tau_n)\}$  be the sequence in (4.2.6). Estimate (4.2.5) implies that a subsequence of  $\{(\sigma_n, \tau_n)\}$ , denoted by the same, satisfies

$$\mathbf{g}_{\sigma_n \tau_n} \rightharpoonup \overline{\mathbf{g}} \text{ in } [L^2(0,T)]^2 \quad \text{and} \quad \|\overline{\mathbf{g}}\|_{\mathbf{L}^2(0,T)} \le \|\mathbf{g}_{\mathbf{ex}}\|_{\mathbf{L}^2(0,T)} \,. \tag{4.2.10}$$

 $(u_{\sigma\tau}, \mathbf{g}_{\sigma\tau})$  satisfies the initial value problem in the weak form:

$$\int_{0}^{T} \int_{0}^{X} u_{\sigma\tau}(v_{tt} - v_{xx}) \, dx \, dt + \int_{0}^{T} \int_{0}^{X} [f(u_{\sigma\tau}) - V] v \, dx \, dt + \int_{0}^{T} \mathbf{g}_{\sigma\tau}|_{x=X}(\partial_{x}v)|_{x=X} \, dt - \int_{0}^{T} \mathbf{g}_{\sigma\tau}|_{x=0}(\partial_{x}v)|_{x=0} \, dt + \int_{0}^{X} (v\partial_{t}u_{\sigma\tau})|_{t=T} \, dx - \int_{0}^{X} v|_{t=0}u_{1} \, dx - \int_{0}^{X} (u_{\sigma\tau}\partial_{t}v)|_{t=T} \, dx + \int_{0}^{X} (u_{0}\partial_{t}v)|_{t=0} \, dx = 0 \qquad \forall v \in C^{2}([0,T]; H^{2} \cap H^{1}_{0}(0,X))$$

$$(4.2.11)$$

where  $\mathbf{g}_{\sigma\tau}|_{x=0}$  denotes the first component of  $\mathbf{g}_{\sigma\tau}$  and  $\mathbf{g}_{\sigma\tau}|_{x=X}$  the second component of  $\mathbf{g}_{\sigma\tau}$ . Passing to the limit in (4.2.11) as  $\sigma, \tau \to \infty$  and using relations (4.2.9) and (4.2.10)

we obtain:

$$\begin{split} &\int_{0}^{T} \int_{0}^{X} \overline{u}(v_{tt} - v_{xx}) \, dx \, dt + \int_{0}^{T} \int_{0}^{X} [f(\overline{u}) - V] v \, dx \, dt + \int_{0}^{T} \overline{g}_{R}(\partial_{x}v)|_{x=X} \, dt \\ &- \int_{0}^{T} \overline{g}_{L}(\partial_{x}v)|_{x=0} \, dt + \int_{0}^{X} v|_{t=T} Z(x) \, dx - \int_{0}^{X} v|_{t=0} u_{1} \, dx - \int_{0}^{X} W(x)(\partial_{t}v)|_{t=T} \, dx \\ &+ \int_{0}^{X} (u_{0}\partial_{t}v)|_{t=0} \, dx = 0 \qquad \forall v \in C^{2}([0,T]; H^{2} \cap H^{1}_{0}(0,X)) \, . \end{split}$$

The last relation and (4.2.10) imply that  $(\overline{u}, \overline{g})$  is a minimum boundary  $L^2$  norm solution to the exact control problem (4.1.1). Hence,  $\overline{u} = u_{ex}$  and  $\overline{g} = g_{ex}$  so that (4.2.7) and (4.2.8) follows from (4.2.6) and (4.2.10).

**Remark 4.2.2.** If the wave equation is linear, i.e., f = 0, then assumption (4.2.6) is redundant and (4.2.8) is guaranteed to hold. Indeed, (4.2.11) implies the boundedness of  $\{||u_{\sigma\tau}||_{L^2(Q)}\}$  which in turn yields (4.2.6). The uniqueness of a solution for the linear wave equation implies (4.2.6) holds for an arbitrary sequence  $\{(\sigma_n, \tau_n)\}$ .

- Theorem 4.2.3. Assume that
- i) for every (α, σ, τ) ∈ {0} × ℝ<sub>+</sub> × ℝ<sub>+</sub> there exists a solution (u<sub>στ</sub>, g<sub>στ</sub>) to the optimal control problem (4.3.17);
- ii) the limit terminal conditions hold:

$$u_{\sigma\tau}|_{t=T} \to W \text{ in } L^{2}(0,X) \text{ and } (\partial_{t}u_{\sigma\tau})|_{t=T} \to Z \text{ in } H^{-1}(0,X) \text{ as } \sigma, \tau \to \infty;$$

$$(4.2.12)$$

iii) the optimal solution  $(u_{\sigma\tau}, \mathbf{g}_{\sigma\tau})$  satisfies the weak limit conditions as  $\sigma, \tau \to \infty$ :

$$\mathbf{g}_{\sigma\tau} \rightarrow \overline{\mathbf{g}} \text{ in } \mathbf{L}^2(0,T), \quad u_{\sigma\tau} \rightarrow \overline{u} \text{ in } L^2(Q), \qquad (4.2.13)$$

and

$$f(u_{\sigma\tau}) \rightharpoonup f(\overline{u}) \text{ in } L^2(0,T; [H^2(\Omega) \cap H^1_0(\Omega)]^*)$$

$$(4.2.14)$$
for some  $\overline{\mathbf{g}} \in \mathbf{L}^2(0,T)$  and  $\overline{u} \in L^2(Q)$ .

Then  $(\overline{u}, \overline{g})$  is a solution to the exact boundary controllability problem (4.1.1) with  $\overline{g}$  satisfying the minimum boundary  $L^2$  norm property. Furthermore, if the solution to (4.1.1) admits a unique solution  $(u_{ex}, g_{ex})$ , then

$$\mathbf{g}_{\sigma\tau} \rightharpoonup \mathbf{g}_{\mathrm{ex}} \text{ in } [L^2(0,T)]^2 \text{ and } u_{\sigma\tau} \rightharpoonup u_{\mathrm{ex}} \text{ in } L^2(Q) \text{ as } \sigma, \tau \rightarrow \infty.$$
 (4.2.15)

*Proof.*  $(u_{\sigma\tau}, \mathbf{g}_{\sigma\tau})$  satisfies (4.2.11). Passing to the limit in that equation as  $\sigma, \tau \to \infty$ and using relations (4.2.12), (4.2.13) and (4.2.14) we obtain:

$$\begin{split} &\int_{0}^{T} \int_{0}^{X} \overline{u}(v_{tt} - v_{xx}) \, dx \, dt + \int_{0}^{T} \int_{0}^{X} [f(\overline{u}) - V] v \, dx \, dt + \int_{0}^{T} \overline{g}_{R}(\partial_{x}v)|_{x=X} \, dt \\ &- \int_{0}^{T} \overline{g}_{L}(\partial_{x}v)|_{x=0} \, dt + \int_{0}^{X} v|_{t=T} Z(x) \, dx - \int_{0}^{X} v|_{t=0} u_{1} \, dx - \int_{0}^{X} W(x)(\partial_{t}v)|_{t=T} \, dx \\ &+ \int_{0}^{X} (u_{0}\partial_{t}v)|_{t=0} \, dx = 0 \qquad \forall v \in C^{2}([0,T]; H^{2} \cap H^{1}_{0}(0,X)) \, . \end{split}$$

This implies that  $(\overline{u}, \overline{g})$  is a solution to the exact boundary controllability problem (4.1.1).

To prove that  $\overline{\mathbf{g}}$  satisfies the minimum boundary  $L^2$  norm property, we proceeds as follows. Let  $(u_{\text{ex}}, \mathbf{g})$  denotes a exact minimum boundary  $L^2$  norm solution to the controllability problem (4.1.1). Since  $(u_{\sigma\tau}, \mathbf{g}_{\sigma\tau})$  is an optimal solution, we have that

$$\frac{\sigma}{2} \| u_{\sigma\tau} - W \|_{L^{2}(0,X)}^{2} + \frac{\tau}{2} \| \partial_{t} u_{\sigma\tau} - Z \|_{H^{-1}(0,X)}^{2} + \frac{1}{2} \| \mathbf{g}_{\sigma\tau} \|_{\mathbf{L}^{2}(0,T)}^{2}$$
$$= \mathcal{J}(u_{\sigma\tau}, \mathbf{g}_{\sigma\tau}) \le \mathcal{J}(u_{\mathrm{ex}}, \mathbf{g}_{\mathrm{ex}}) = \frac{1}{2} \| \mathbf{g}_{\mathrm{ex}} \|_{\mathbf{L}^{2}(0,T)}^{2}$$

so that

 $\|\mathbf{g}_{\sigma\tau}\|_{\mathbf{L}^{2}(0,T)} \leq \|\mathbf{g}_{\mathrm{ex}}\|_{\mathbf{L}^{2}(0,T)}$ 

Passing to the limit in the last estimate we obtain

$$\|\overline{\mathbf{g}}\|_{\mathbf{L}^{2}(0,T)} \le \|\mathbf{g}_{\mathbf{ex}}\|_{\mathbf{L}^{2}(0,T)}.$$
(4.2.16)

Hence we conclude that  $(\overline{u}, \overline{g})$  is a minimum boundary  $L^2$  norm solution to the exact boundary controllability problem (4.1.1).

Furthermore, if the exact controllability problem (4.1.1) admits a unique minimum boundary  $L^2$  norm solution  $(u_{ex}, \mathbf{g}_{ex})$ , then  $(\overline{u}, \overline{\mathbf{g}}) = (u_{ex}, \mathbf{g}_{ex})$  and (4.2.15) follows from assumption (4.2.13).

**Remark 4.2.4.** If the wave equation is linear, i.e., f = 0, then assumptions i) and (4.2.14) are redundant.

**Remark 4.2.5.** Assumptions ii) and iii) hold if  $\mathbf{g}_{\sigma\tau}$  and  $u_{\sigma\tau}$  converges pointwise as  $\sigma, \tau \to \infty$ .

**Remark 4.2.6.** A practical implication of Theorem 4.2.3 is that one can prove the exact controllability for semilinear wave equations by examining the behavior of a sequence of optimal solutions (recall that exact controllability was proved only for some special classes of semilinear wave equations.) If we have found a sequence of optimal control solutions  $\{(u_{\sigma_n\tau_n}, \mathbf{g}_{\sigma_n\tau_n})\}$  where  $\sigma_n, \tau_n \to \infty$  and this sequence appears to satisfy the convergence assumptions ii) and iii), then we can confidently conclude that the underlying semilinear wave equation is exactly controllable and the optimal solution  $(u_{\sigma_n\tau_n}, \mathbf{g}_{\sigma_n\tau_n})$  when n is large provides a good approximation to the minimum boundary  $L^2$  norm exact controller  $(u_{\text{ex}}, \mathbf{g}_{\text{ex}})$ .

## 4.3 An optimality system of equations and a continuous shooting method

Under suitable assumptions on f and through the use of Lagrange multiplier rules, the optimal control problem

minimize (4.1.4) with respect to the control g subject to (4.1.3) (4.3.17)

may be converted into the following system of equations from which an optimal solution may be determined:

$$\begin{split} u_{tt} - u_{xx} + f(u) &= V & \text{in } (0, T) \times (0, X) \,, \\ u|_{x=0} &= g_L \,, \quad u|_{x=1} = g_R \,, \quad u(0, x) = u_0(x) \,, \quad u_t(0, x) = u_1(x) \,, \\ \xi_{tt} - \xi_{xx} + f'(u)\xi &= -\alpha(u - U) & \text{in } (0, T) \times (0, X) \,, \\ \xi|_{x=0} &= 0 \,, \qquad \xi|_{x=1} = 0 \,, \\ \xi(T, x) &= -\tau A^{-1}(u_t(T, x) - Z(x)) \,, \qquad \xi_t(T, x) = -\sigma(u(T, x) - W(x)) \,, \\ g_L &= -\xi_x|_{x=0} \,, \qquad \text{and} \qquad g_R = \xi_x|_{x=1} \,, \end{split}$$

where the elliptic operator  $A : H_0^1(0, X) \to H^{-1}(0, X)$  is defined by  $Av = v_{xx}$  for all  $v \in H_0^1(0, X)$ . By eliminating  $g_L$  and  $g_R$  in the system we arrive at the optimality system

$$u_{tt} - u_{xx} + f(u) = V \quad \text{in } (0, T) \times (0, X),$$

$$u|_{x=0} = -\xi_x|_{x=0}, \quad u|_{x=1} = \xi_x|_{x=1},$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),$$

$$\xi_{tt} - \xi_{xx} + f'(u)\xi = -\alpha(u - U) \quad \text{in } (0, T) \times (0, X),$$

$$\xi|_{x=0} = 0, \quad \xi|_{x=1} = 0,$$

$$\xi(T, x) = \tau(u_t(T, x) - Z(x)), \quad \xi_t(T, x) = -\sigma(u(T, x) - W(x)).$$
(4.3.18)

Derivations and justifications of optimality systems are discussed in [5] for the linear case and in [6] for the semilinear case.

The computational algorithm we propose in this chapter is a shooting method for solving the optimality system of equations. The basic idea for a shooting method is to convert the solution of a initial-terminal value problem into that of a purely initial value problem (IVP); see, e.g., [1] for a discussion of shooting methods for systems of ordinary differential equations. The IVP corresponding to the optimality system (4.3.18) is described by

$$u_{tt} - u_{xx} + f(u) = V \quad \text{in } (0, T) \times (0, X) ,$$
  

$$u_{\partial\Omega} = \frac{\partial \xi}{\partial \nu}\Big|_{\partial\Omega}, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) ;$$
  

$$\xi_{tt} - \xi_{xx} + f'(u)\xi = -\alpha(u - U) \quad \text{in } (0, T) \times (0, X) ,$$
  

$$\xi|_{\partial\Omega} = 0, \quad \xi(0, x) = \omega(x), \quad \xi_t(0, x) = \theta(x) ,$$
  
(4.3.19)

with unknown initial values  $\omega$  and  $\theta$ . Then the goal is to choose  $\omega$  and  $\theta$  such that the solution  $(u, \xi)$  of the IVP (4.3.19) satisfies the terminal conditions

$$F_1(\omega,\theta) \equiv \partial_{xx}\xi(T,x) + \tau(u_t(T,x) - Z(x)) = 0,$$

$$F_2(\omega,\theta) \equiv \xi_t(T,x) + \sigma(u(T,x) - W(x)) = 0.$$
(4.3.20)

A shooting method for solving (4.3.18) can be described by the following iterations:

choose initial guesses  $\omega$  and  $\theta$ ;

for  $iter = 1, 2, \cdots, max_i ter$ 

solve for  $(u,\xi)$  from the IVP (4.3.19)

update  $\omega$  and  $\theta$ .

A criterion for updating  $(\omega, \theta)$  can be derived from the terminal conditions (4.3.20). A method for solving the nonlinear system (4.3.20) (as a system for the unknowns  $\omega$  and  $\theta$ ) will yield an updating formula; for instance, the well-known Newton's method may be invoked.

choose initial guesses 
$$\omega$$
 and  $\theta$ ;

for  $iter = 1, 2, \cdots, max_i ter$ 

solve for  $(u,\xi)$  from the IVP (4.3.19)

update  $\omega$  and  $\theta$ :

$$\begin{split} (\omega^{\text{new}}, \theta^{\text{new}}) &= (\omega, \theta) - [F'(\omega, \theta)]^{-1} F(\omega, \theta); \\ \text{if } F(\omega^{\text{new}}, \theta^{\text{new}}) &= \mathbf{0}, \text{ stop; otherwise, set } (\omega, \theta) = (\omega^{\text{new}}, \theta^{\text{new}}). \end{split}$$

A discussion of Newton's method for an infinite dimensional nonlinear system can be found in many functional analysis textbooks, and for the suitable assumption convergence of Newton iteration for the optimality system is guaranteed.

#### 4.4 The discrete shooting method

The shooting method described in Section (4.3) must be implemented discretely. We discretize the spatial interval [0,1] into  $0 = x_0 < x_1 < x_2 < \cdots < x_I < x_{I+1} = 1$  with a uniform spacing h = 1/(I+1) and we divide the time horizon [0,T] into  $0 = t_1 < t_2 < t_3 < \cdots < t_N = T$  with a uniform time stepping  $\delta = T/(N-1)$ . We use the explicit, central difference scheme to approximate the initial value problem (4.3.19):

$$u_{i}^{1} = (u_{0})_{i}, \quad u_{i}^{2} = (u_{0})_{i} + \delta(u_{1})_{i}, \quad \xi_{i}^{1} = \omega_{i}, \quad \xi_{i}^{2} = \xi_{i}^{1} + \delta\theta_{i}, \quad i = 1, 2, \cdots, I;$$

$$u_{i}^{n+1} = -u_{i}^{n-1} + \lambda u_{i-1}^{n} + 2(1-\lambda)u_{i}^{n} + \lambda u_{i+1}^{n}$$

$$-\delta^{2}f(u_{i}^{n}) + \delta^{2}V(t_{n}, x_{i}), \quad i = 1, 2, \cdots, I,$$

$$\xi_{i}^{n+1} = -\xi_{i}^{n-1} + \lambda\xi_{i-1}^{n} + 2(1-\lambda)\xi_{i}^{n} + \lambda\xi_{i+1}^{n}$$

$$-\delta^{2}f'(u_{i}^{n})\xi_{i}^{n} + \delta^{2}\alpha(u_{i}^{n} - U(t_{n}, x_{i})), \quad i = 1, 2, \cdots, I$$

$$u_{0}^{n+1} = -\frac{\xi_{1}^{n+1} - \xi_{0}^{n+1}}{h}, \quad u_{I+1}^{n+1} = -\frac{\xi_{I+1}^{n+1} - \xi_{I}^{n+1}}{h}.$$

$$(4.4.21)$$

where  $\lambda = (\delta/h)^2$  (we also use the convention that  $\xi_0^n = \xi_{I+1}^n = 0$ .) The gist of a discrete shooting method is to regard the discrete terminal conditions

$$F_{2i-1} \equiv \frac{\xi_i^N - \xi_i^{N-1}}{\delta} + \sigma(u_i^N - W_i) = 0,$$
  

$$F_{2i} \equiv \frac{\xi_{i-1}^N - 2\xi_i^N + \xi_{i+1}^N}{h^2} + \tau(\frac{u_i^N - u_i^{N-1}}{\delta} - Z_i) = 0, \quad i = 1, 2, \cdots, I$$
(4.4.22)

as a system of equations for the unknown initial condition  $\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_m, \theta_m$ .

Similar to the continuous case, the discrete shooting method consists of the following iterations:

choose discrete initial guesses  $\{\omega_i\}_{i=1}^{I}$  and  $\{\theta\}_{i=1}^{I}$ ; for  $iter = 1, 2, \dots, max_i ter$ solve for  $\{(u_i^n, \xi_i^n)\}_{i=1,n=1}^{i=I,n=N}$  from the discrete IVP (4.4.21) update  $\{\omega_i\}_{i=1}^{I}$  and  $\{\theta_i\}_{i=1}^{I}$ .

The initial conditions  $\{\omega_i\}_{i=1}^I$  and  $\{\theta_i\}_{i=1}^I$  are updated by Newton's method applied to the discrete nonlinear system (4.4.22). This requires the calculations of partial derivatives. By denoting

$$q_{ij}^{n} = q_{ij}^{n}(\omega_{1}, \theta_{1}, \omega_{2}, \theta_{2}, \cdots, \omega_{I}, \theta_{I}) = \frac{\partial u_{i}^{n}}{\partial \omega_{j}}, \qquad r_{ij}^{n} = r_{ij}^{n}(\omega_{1}, \theta_{1}, \omega_{2}, \theta_{2}, \cdots, \omega_{I}, \theta_{I}) = \frac{\partial u_{i}^{n}}{\partial \theta_{j}},$$
$$\rho_{ij}^{n} = \rho_{ij}^{n}(\omega_{1}, \theta_{1}, \omega_{2}, \theta_{2}, \cdots, \omega_{I}, \theta_{I}) = \frac{\partial \xi_{i}^{n}}{\partial \omega_{j}}, \qquad \tau_{ij}^{n} = \tau_{ij}^{n}(\omega_{1}, \theta_{1}, \omega_{2}, \theta_{2}, \cdots, \omega_{I}, \theta_{I}) = \frac{\partial \xi_{i}^{n}}{\partial \theta_{j}},$$

we obtain the following Newton's iteration formula:

$$(\omega_1^{\text{new}}, \theta_1^{\text{new}}, \omega_2^{\text{new}}, \theta_2^{\text{new}}, \cdots, \omega_I^{\text{new}}, \theta_I^{\text{new}})^T$$
  
=  $(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_I, \theta_I)^T - [F'(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_I, \theta_I)]^{-1}F(\omega_1, \theta_1, \omega_2, \theta_2, \cdots, \omega_I, \theta_I)$ 

where the vector F and Jacobian matrix J = F' are defined by

$$F_{2i-1} = \frac{\xi_i^N - \xi_i^{N-1}}{\delta} + \sigma(u_i^N - W_i), \qquad F_{2i} = \frac{\xi_{i-1}^N - 2\xi_i^N + \xi_{i+1}^N}{h^2} + \tau(\frac{u_i^N - u_i^{N-1}}{\delta} - Z_i),$$

$$J_{2i-1,2j-1} = \frac{\rho_{ij}^N - \rho_{ij}^{N-1}}{\delta} + \sigma q_{ij}^N, \qquad J_{2i-1,2j} = \frac{\tau_{ij}^N - \tau_{ij}^{N-1}}{\delta} + \sigma r_{ij}^N,$$
  

$$J_{2i,2j-1} = \frac{\rho_{(i-1)j}^N - 2\rho_{ij}^N + \rho_{(i+1)j}^N}{h^2} + \frac{\tau}{\delta} (q_{ij}^N - q_{ij}^{N-1}),$$
  

$$J_{2i,2j} = \frac{\tau_{(i-1)j}^N - 2\tau_{ij}^N + \tau_{(i+1)j}^N}{h^2} + \frac{\tau}{\delta} (r_{ij}^N - r_{ij}^{N-1}).$$

Moreover, by differentiating (4.4.21) with respect to  $\omega_j$  and  $\theta_j$  we obtain the equations for determining  $q_{ij}$ ,  $r_{ij}$ ,  $\rho_{ij}$  and  $\tau_{ij}$ :

$$\begin{split} q_{ij}^{1} &= 0, \qquad q_{ij}^{2} = 0, \qquad r_{ij}^{1} = 0, \qquad r_{ij}^{2} = 0, \\ \rho_{ij}^{1} &= \delta_{ij}, \qquad \rho_{ij}^{2} = \delta_{ij}, \qquad \tau_{ij}^{1} = 0, \qquad \tau_{ij}^{2} = \delta\delta_{ij}, \\ q_{ij}^{n+1} &= -q_{ij}^{n-1} + \lambda q_{i-1,j}^{n} + 2(1-\lambda)q_{ij}^{n} + \lambda q_{i+1,j}^{n} \\ &\quad -\delta^{2}f'(u_{i}^{n})q_{ij}^{n}, \qquad i, j = 1, 2, \cdots, I, \\ r_{ij}^{n+1} &= -r_{ij}^{n-1} + \lambda r_{i-1,j}^{n} + 2(1-\lambda)r_{ij}^{n} + \lambda r_{i+1,j}^{n} \\ &\quad -\delta^{2}f'(u_{i}^{n})r_{ij}^{n}, \qquad i, j = 1, 2, \cdots, I, \\ \rho_{ij}^{n+1} &= -\rho_{ij}^{n-1} + \lambda \rho_{i-1,j}^{n} + 2(1-\lambda)\rho_{ij}^{n} + \lambda \rho_{i+1,j}^{n} \\ &\quad +\delta^{2}\alpha q_{ij}^{N} - \delta^{2}[f'(u_{i}^{n})]\rho_{i}^{n} - \delta^{2}[f''(u_{i}^{n})q_{i}^{N}]\xi_{i}^{n}, \qquad i, j = 1, 2, \cdots, I; \\ \tau_{ij}^{n+1} &= -\tau_{ij}^{n-1} + \lambda \tau_{i-1,j}^{n} + 2(1-\lambda)\tau_{ij}^{n} + \lambda \tau_{i+1,j}^{n} \\ &\quad +\delta^{2}\alpha r_{ij}^{N} - \delta^{2}[f'(u_{i}^{n})]\rho_{i}^{n} - \delta^{2}[f''(u_{i}^{n})r_{i}^{N}]\xi_{i}^{n}, \qquad i, j = 1, 2, \cdots, I; \end{split}$$

where  $\delta_{ij}$  is the Chronecker delta. Thus, we have the following Newton's-method-based shooting algorithm:

Algorithm – Newton method based shooting algorithm with central finite difference approximations of the optimality system

choose initial guesses  $\omega_i$  and  $\theta_i$ ,  $i = 1, 2, \dots, I$ ; % set initial conditions for u and  $\xi$ for  $i = 0, 2, \dots, I + 1$  $u_i^1 = (u_0)_i, \quad u_i^2 = (u_0)_i + \delta(u_1)_i,$  % set initial conditions for  $q_{ij}, r_{ij}, \rho_{ij}, \tau_{ij}$ 

for  $j = 1, 2, \cdots, I$ for  $i = 1, 2, \cdots, I$  $q_{ij}^1 = 0, \quad q_{ij}^2 = 0, \quad r_{ij}^1 = 0, \quad r_{ij}^2 = 0,$  $\rho_{ij}^1 = 0, \quad \rho_{ij}^2 = 0, \quad \tau_{ij}^1 = 0, \quad \tau_{ij}^2 = 0;$  $\rho_{ij}^1 = 1, \quad \rho_{ij}^2 = 1, \quad \tau_{ij}^2 = \delta,$ 

% Newton iterations

for  $m = 1, 2, \cdots, M$ % solve for  $(u, \xi)$ for  $n = 2, 3, \cdots, N-1$   $u_i^{n+1} = -u_i^{n-1} + \lambda u_{i-1}^n + 2(1-\lambda)u_i^n + \lambda u_{i+1}^n - \delta^2 f(u_i^n) + \delta^2 V(t_n, x_i),$   $\xi_i^{n+1} = -\xi_i^{n-1} + \lambda \xi_{i-1}^n + 2(1-\lambda)\xi_i^n + \lambda \xi_{i+1}^n - \delta^2 f'(u_i^n)\xi_i^n$  $+ \delta^2 \alpha (u_i^n - U(t_n, x_i));$ 

$$\begin{split} & \% \quad \text{solve for } q, r, \rho, \tau \\ & \text{for } j = 1, 2, \cdots, I \\ & \text{for } n = 2, 3, \cdots, N-1 \\ & \text{for } i = 2, \cdots, N-1 \\ & q_{ij}^{n+1} = -q_{ij}^{n-1} + \lambda q_{i-1,j}^n + 2(1-\lambda)q_{ij}^n + \lambda q_{i+1,j}^n \\ & -\delta^2 f'(u_i^n)q_{ij}^n, \\ & r_{ij}^{n+1} = -r_{ij}^{n-1} + \lambda r_{i-1,j}^n + 2(1-\lambda)r_{ij}^n + \lambda r_{i+1,j}^n \\ & -\delta^2 f'(u_i^n)r_{ij}^n, \\ & \rho_{ij}^{n+1} = -\rho_{ij}^{n-1} + \lambda \rho_{i-1,j}^n + 2(1-\lambda)\rho_{ij}^n + \lambda \rho_{i+1,j}^n \\ & +\delta^2 \alpha q_{ij}^n - \delta^2 [f'(u_i^n)]\rho_{ij}^n - \delta^2 [f''(u_i^n)q_{ij}^n]\xi_i^n, \\ & \tau_{ij}^{n+1} = -\tau_{ij}^{n-1} + \lambda \tau_{i-1,j}^n + 2(1-\lambda)\tau_{ij}^n + \lambda \tau_{i+1,j}^n \end{split}$$

$$+\delta^2 \alpha r_{ij}^n - \delta^2 [f'(u_i^n)]\rho_{ij}^n - \delta^2 [f''(u_i^n)r_{ij}^n]\xi_i^n;$$

% (we need to build into the algorithm the following:

$$q_{0j}^{n+1} = -\frac{\rho_{1j}^{n+1}}{h}, r_{0j}^n = -\frac{\tau_{1j}^{n+1}}{h}, r_{I+1,j}^n = -\frac{\tau_{Ij}^{n+1}}{h}$$
$$q_{I+1,j}^n = -\frac{\rho_{Ij}^{n+1}}{h}, \rho_{0j}^n = \tau_{0j}^n = \rho_{I+1,j}^n = \tau_{I+1,j}^n = 0.$$

% evaluate F and F'

for 
$$i = 1, 2, \cdots, I$$
  

$$F_{2i-1} = \frac{\xi_i^N - \xi_i^{N-1}}{\delta} + \sigma(u_i^N - W_i),$$

$$F_{2i} = \frac{\xi_{i-1}^N - 2\xi_i^N + \xi_{i+1}^N}{h^2} + \tau(\frac{u_i^N - u_i^{N-1}}{\delta} - Z_i);$$
for  $j = 1, 2, \cdots, I$   

$$J_{2i-1,2j-1} = \frac{\rho_{ij}^N - \rho_{ij}^{N-1}}{\delta} + \sigma q_{ij}^N, \quad J_{2i-1,2j} = \frac{\tau_{ij}^N - \tau_{ij}^{N-1}}{\delta} + \sigma r_{ij}^N$$

$$J_{2i,2j-1} = \frac{\rho_{(i-1)j}^N - 2\rho_{ij}^N + \rho_{(i+1)j}^N}{h^2} + \frac{\tau}{\delta}(q_{ij}^N - q_{ij}^{N-1}),$$

$$J_{2i,2j} = \frac{\tau_{(i-1)j}^N - 2\tau_{ij}^N + \tau_{(i+1)j}^N}{h^2} + \frac{\tau}{\delta}(r_{ij}^N - r_{ij}^{N-1});$$

solve  $J\mathbf{c} = -F$  by Gaussian eliminations;

for  $i = 1, 2, \dots, I$   $\omega_i^{\text{new}} = \omega_i + c_{2i-1}, \quad \theta_i^{\text{new}} = \theta_i + c_{2i};$ if  $\max_i |\omega_i^{\text{new}} - \omega_i| + \max_i |\theta_i^{\text{new}} - \theta_i| < \text{tol, stop};$ otherwise, reset  $\omega_i = \omega_i^{\text{new}}$  and  $\theta_i = \theta_i^{\text{new}}, i = 1, 2, \dots, I;$ 

As in the continuous case, we have the following convergence result for the shooting algorithm which follows from standard convergence results for Newton's method applied to finite dimensional systems of nonlinear equations.

**Remark 4.4.1.** The algorithms we propose are well suited for implementations on a parallel computing platform such as a massive cluster of processors. The shooting algorithms of this chapter can be regarded as a generalization of their counterpart for systems of ODE (see, e.g., [1].) There has been a substantial literature on the parallelization of shooting methods for ODEs [2, 3, 4]; these results will be helpful in parallelizing the shooting algorithms of this chapter.

# 4.5 Computational experiments for controllability of the linear wave equation

We will apply Algorithm 1 to the special case of f = 0, V = 0, W = 0, Z = 0,  $\alpha = 0$ and  $\sigma, \tau >> 1$ . In other words, we will approximate the null controllability problem for the linear wave equation by optimal control problems. We will test our algorithm with a smooth example (i.e., the continuous minimum boundary  $L^2$  norm controller g and the corresponding state u are smooth) and with three generic examples. It was reported in [14] that the discrete minimum boundary  $L^2$  norm controllers converge strongly to the continuous minimum boundary  $L^2$  norm controller for the smooth example and converge weakly in the generic case. The discrete optimal solutions found by Algorithm 1 will exhibit similar behaviors.

#### 4.5.1 An example with known smooth exact solution

A smooth exact solution to the minimum boundary  $L^2$ -norm controllability problem was constructed in [14] by using Fourier series in a way similar to that used in [10]. Suppose that  $Q = (0,7/4) \times (0,1)$  and  $\Sigma = (0,7/4) \times \{0,1\}$ . Let  $\psi_0(t,x) = -\sqrt{2\pi} \cos \pi (t-\frac{1}{4}) \cos 2\pi x$  and

$$\psi_1(t,x) = \left[ 2\sqrt{2}\pi(T-t)\sin\pi(t-\frac{1}{4}) - \frac{10}{3}\sqrt{2}\sin\pi(t-T) \right] \sin\pi x + \sum_{p\geq 3 \text{ and } p \text{ odd}} \frac{4\sqrt{2}}{p^2 - 1} \left[ \frac{3p}{p^2 - 4}\cos\pi(t-\frac{1}{4}) + \sin p\pi(t-T) \right] \sin p\pi x.$$

Then, set the initial conditions

$$u_0(x) = \psi_0(0, x) + \psi_1(0, x)$$
 and  $u_1(x) = \frac{\partial \psi_0}{\partial t}(0, x) + \frac{\partial \psi_1}{\partial t}(0, x)$ . (4.5.24)

The computation of  $u_0$  and  $u_1$  involve the summation of infinite trigonometric series. Figure 5.2 and Figure 4.2 provides plots of  $u_0$  and  $u_1$ , and the exact control and exact solution respectively. Note that initial conditions vanish at the boundary, and due to symmetry, we have  $g_L(t) = u(t,0) = u(t,1) = g_R(t)$ . i.e the controls at two sides of Qare the same. It is worth noting that  $u_0$  is a Lipschitz continuous function but does not belong to  $C^1[0,1]$  and  $u_1$  is a bounded function but does not belong to  $C^0[0,1]$ . For the initial data (4.5.24), it can be shown that  $u(t,x) = \psi_0(t,x) + \psi_1(t,x)$  is the exact solution having minimum boundary  $L^2$ -norm of the controllability problem given by the first three equations in (4.1.1) provided f = 0, V = 0. Let g be the corresponding exact Dirichlet control given by restricting u(t,x) to the lateral sides  $\Sigma$ . i.e g(t) = $(g_L(t), g_R(t)) = (u(t,0), u(t,1))$ , and

$$g_L(t) = g_R(t) = -\sqrt{2\pi} \cos \pi \left(t - \frac{1}{4}\right).$$
 (4.5.25)

For future reference, note that  $||g||_{L^2(\Sigma)} = \sqrt{2\pi^2(\frac{7}{4} + \frac{1}{2\pi})} \approx 6.13882.$ 



Figure 4.1 left -  $u_0$ , right -  $u_1$  given in (4.5.24). h = 1/256.

We apply our numerical method to this example. Computational experiments were carried out for h = 1/16, 1/32, 1/64, 1/128, 1/256, 1/512, and 1/1024 with  $\lambda = 1$  and  $\lambda = 7/8$  respectively, so that the stability condition is satisfied.

The results of our computational experiments are summarized in Table 4.1, where  $g^h$ 



Figure 4.2 left - exact control, right - exact u. with initial data (4.5.24). h = 1/32.

are the computed approximations of the exact solutions g. All norms were calculated by linearly interpolating the nodal values of  $g^h$ . From this table, it seems that  $g^h$  converges to g in the  $L^2(\Sigma)$ -norm at a rate of roughly 1. In order to visualize the convergence of our method as h becomes smaller, in Figure 4.3 we provide plots of the exact solution uand the corresponding computed discrete solutions  $u^h$  for h = 1/256 with  $\lambda = 1$ . Figure 4.4 and 4.5 are plots of the exact solution g and the corresponding computed discrete solutions  $g^h$ , a given function W and approximate solutions  $u^h$ , and a given function Zand approximate solutions  $u^h_t$  for h = 1/16, h = 1/32, h = 1/64, and h = 1/1024.

Table 4.1 Results of computational experiments for the minimum  $L^2(\Sigma)$ -norm case for the examples with initial data (4.5.24).

| h   |                 | 1/16   | 1/32   | 1/64   | 1/128  | 1/256  | 1/512  | 1/1024 |
|---|-----------------|--------|--------|--------|--------|--------|--------|--------|
| $\ g^h\ _{L^2(\Sigma)}$                                 | $\lambda = 1$   | 5.9339 | 6.0294 | 6.0825 | 6.1103 | 6.1244 | 6.1316 | 6.1352 |
| $\ g^h\ _{L^2(\Sigma)}$                                 | $\lambda = 7/8$ | 5.9682 | 6.0468 | 6.0917 | 6.1454 | 6.1262 | 6.1325 | 6.1356 |
| $\frac{\ g^h - g\ _{L^2(\Sigma)}}{\ g\ _{L^2(\Sigma)}}$ | $\lambda = 1$   | 6.93%  | 3.35%  | 1.63%  | 0.79%  | 0.37%  | 0.18%  | 0.09%  |
| $\frac{\ g^h - g\ _{L^2(\Sigma)}}{\ g\ _{L^2(\Sigma)}}$ | $\lambda = 7/8$ | 7.53%  | 4.26%  | 2.88%  | 10.15% | 0.35%  | 0.17%  | 0.08%  |

It seems that our method produces (pointwise) convergent approximations for both  $\lambda = 1$  and  $\lambda = 7/8$  without the need for regularization. This should be contrasted with other methods, e.g., that of [10], for which when  $\lambda < 1$ , regularization was needed in



Figure 4.3 left - approximate control  $u^h$ , right - exact u with initial data (4.5.24). h = 1/256.  $\lambda = 1$ .

order to obtain convergence. Also, the results obtained by our method behave very similarly to those obtained in [14].

## 4.5.2 Generic examples with minimum $L^2(\Sigma)$ -norm boundary control

In the example discussed in Section 4.5.1, the minimum  $L^2(\Sigma)$ -norm control is very smooth. Using our methods, we obtained good approximations for this example without the need for regularization. However, this is not the generic case. In general, even for smooth initial data, the minimum  $L^2(\Sigma)$ -norm Dirichlet control for the controllability problem (4.1.1) will not be smooth. In this section, we illustrate this point and also examine the performance of our method for the generic case.

We choose  $Q = (0,1) \times (0,1)$  in example I and  $Q = (0,7/4) \times (0,1)$  in example II, III, and consider three sets of  $C^{\infty}(\overline{\Omega})$  initial data:

I. 
$$u_0(x) = x(x-1)$$
 and  $u_1(x) = 0$   
II.  $u_0(x) = \sin(\pi x)$  and  $u_1(x) = \pi \sin(\pi x)$  (4.5.26)  
III.  $u_0(x) = e^x$  and  $u_1(x) = xe^x$ .

Note that the initial conditions (I), (II) vanish at the boundary and, that due to symmetry, we have that u(t, 0) = u(t, 1), i.e., the control at the two sides of Q are the same.

For the initial conditions (III), we have that  $u(t, 0) \neq u(t, 1)$ .

First We examine the case  $\lambda = 1$ . In Figure 4.6, 4.8 and 4.10, we show the results for the control for several grid sizes ranging from h = 1/16 to h = 1/1024. The (pointwise) convergence of the approximations is evident. Note that for the initial conditions given in (4.5.26), the minimum  $L^2(\Sigma)$ -norm controls are seemingly piecewise smooth, i.e., they contain jump discontinuities. The pointwise convergence of the approximate control for the case of  $\lambda = 1$  is probably a one-dimensional artifact; it is likely due to the fact that both the space and time variables in the wave equation in one dimension can act as time-like variables.

Further details about the computational results for the examples with initial conditions (I) given in (4.5.26) with  $\lambda = 4/5$  are given in Table 4.2 and Figure 4.7. The convergence in  $L^2(\Sigma)$  of the approximate minimum  $L^2(\Sigma)$ -norm controls  $g^h$  is evident as is the convergence in  $L^2(Q)$  of the approximate solution  $u^h$ ; the rates of convergence are seemingly first order.

Table 4.2 Results of computational experiments for the minimum  $L^2(\Sigma)$ -norm case for Examples I with initial data (4.5.26) and for  $\lambda = 1, 4/5$ .

| h                       |                 | 1/32    | 1/64    | 1/128   | 1/256   | 1/512   | 1/1024  |
|-------------------------|-----------------|---------|---------|---------|---------|---------|---------|
| $\ g^h\ _{L^2(\Sigma)}$ | $\lambda = 1$   | 0.12934 | 0.12908 | 0.12906 | 0.12907 | 0.12908 | 0.12909 |
| $\ g^h\ _{L^2(\Sigma)}$ | $\lambda = 4/5$ | 0.15941 | 0.15269 | 0.14522 | 0.14216 | 0.13907 | 0.13622 |

Computational experiments were also carried out for  $\lambda = 7/8$  for several values of the grid size ranging from h = 1/16 to h = 1/1024. The results are summarized in Table 4.3 and 4.4. In Figures 4.9 and 4.11, we respectively provide, for the two sets of initial conditions (II) and (III), plots of the computed discrete solution  $g^h, u^h, u^h_t$  for the two different values of  $\lambda$  and for different values of the grid size.

From Figures 4.9 and 4.11, we see that the approximate minimum  $L^2(\Sigma)$ -norm Dirichlet controls obtained with values of  $\lambda < 1$  are highly oscillatory. In fact, the frequencies of the oscillations increase with decreasing grid size. However, it seems that the ampli-

Table 4.3 Results of computational experiments for the minimum  $L^2(\Sigma)$ -norm case for Examples II with initial data (4.5.26) and for  $\lambda = 1.7/8$ .

| h   |                 | 1/16   | 1/32   | 1/64   | 1/128  | 1/256  | 1/512  | 1/1024 |
|---|-----------------|--------|--------|--------|--------|--------|--------|--------|
| $\frac{\ g^h\ _{L^2(\Sigma)}}{\ g^h\ _{L^2(\Sigma)}}$ | $\lambda = 1$   | 0.6838 | 0.6388 | 0.6162 | 0.6049 | 0.5992 | 0.5963 | 0.5949 |
| $\frac{\ g^h\ }{\ g^h\ }L^2(\Sigma)$                  | $\lambda = 7/8$ | 0.6734 | 0.6348 | 0.6138 | 0.6039 | 0.5988 | 0.5963 | 0.5949 |

Table 4.4 Results of computational experiments for the minimum  $L^2(\Sigma)$ -norm case for for Examples III with initial data (4.5.26) and for  $\lambda = 1, 7/8$ .

| h                       |                 | 1/16   | 1/32   | 1/64   | 1/128  | 1/256  | 1/512  | 1/1024 |
|-------------------------|-----------------|--------|--------|--------|--------|--------|--------|--------|
| $  g^h  _{L^2(\Sigma)}$ | $\lambda = 1$   | 1.4277 | 1.3187 | 1.2605 | 1.2303 | 1.2149 | 1.2071 | 1.2032 |
| $\ g^h\ _{L^2(\Sigma)}$ | $\lambda = 7/8$ | 1.3932 | 1.3007 | 1.2493 | 1.2252 | 1.2124 | 1.2065 | 1.2028 |

tudes of the oscillations do not increase as the grid size decreases. Furthermore, from the results in Table 4.3 and 4.4, it seems that for  $\lambda < 1$ , the approximate controls  $g^h$ , although oscillatory in nature and nonconvergent in a pointwise sense, converge in an  $L^2(\Sigma)$  sense.

The results of Table 4.2, 4.3, 4.4 and Figures 4.6, 4.7, 4.8, 4.9, 4.10, 4.11 indicate that for the generic case of non-smooth minimum  $L^2(\Sigma)$  controls and for general  $\lambda < 1$ , our method produces convergent (in  $L^2(Q)$  and  $L^2(\Sigma)$ ) approximations without the need of regulatization but the approximations are not in general convergent in a pointwise sense. Of course, approximations that do not converge in a pointwise sense may be of little practical use, even if they converge in a root mean square sense.

## 4.6 Computational experiments for controllability of semilinear wave equations

We will again apply Algorithm 1 to the special case of V = 0, W = 0, Z = 0,  $\alpha = 0$ and  $\sigma, \tau >> 1$ . We will test our algorithm with generic examples. If nonlinear term fsatisfies a certain property such as asymptotically linear or superlinear, then the exact control problem of the system 4.1.1 can be solvable; see, e.g., [8, 23, 24]. In this section, we examine the performance of our method for the asymptotically linear and superlinear cases.

We choose  $Q = (0, 3) \times (0, 1)$  in example I, II, III, and consider three sets of nonlinear term f:

I. 
$$f(u) = \sin u$$
  
II.  $f(u) = u^{3/2}$  (4.6.27)  
III.  $f(u) = \ln(u^2 + 1)$ .

Note that we choose T = 3 for existence of control; see, e.g., [23, 24]. (I) is an example of the asymptotically linear case and (II) is one of the superlinear case. (III) can be considered as either case. In general, we can not expect  $g_L = g_R$  due to the nonlinear terms. We test the case  $\lambda = 1$ . The numerical approximations by Algorithm 1 is convergent in  $L_2$  sense, that is, they have jump discontinuities as well. We will illustrate those through the figures 4.12, 4.13, and 4.14. For the linear cases, the number of iterations of the shooting methods is about 2 or 3, according to the tolerance and the accuracy of the machines we used. However the nonlinear cases are different and we need more iterations than the linear cases. We denote the number of iterations as *count*. It is contained in the tables 4.5, 4.6 and 4.7 with  $L^2(\Sigma)$ -norm of controls  $g^h$ .

Table 4.5 Results of computational experiments for the minimum  $L^2(\Sigma)$ -norm case for Examples I, II, III in (4.6.27) with initial data I in (4.5.26) and for  $\lambda = 1$ .

|     | (10000  m  m (100.20)  m m (101.7) = 1. |               |               |               |               |  |  |  |  |
|-----|---|---------------|---------------|---------------|---------------|--|--|--|--|
|     | h                                       | 1/16          | 1/32          | 1/64          | 1/128         |  |  |  |  |
| Ī   | $\ g^h\ _{L^2(\Sigma)}$                 | 0.08084810765 | 0.08038960736 | 0.08021218880 | 0.08013073451 |  |  |  |  |
|     | count                                   | 17            | 16            | 16            | 17            |  |  |  |  |
| II  | $\ g^h\ _{L^2(\Sigma)}$                 | 0.07346047350 | 0.07314351955 | 0.07307515741 | 0.07306230119 |  |  |  |  |
|     | count                                   | 8             | 8             | 10            | 12            |  |  |  |  |
| III | $\ g^h\ _{L^2(\Sigma)}$                 | 0.07438729446 | 0.07404916115 | 0.07397863882 | 0.07396393744 |  |  |  |  |
|     | count                                   | 6             | 5             | 5             | 5             |  |  |  |  |

| Table 4.6 | Results           | of com     | putational        | experir    | nents  | for th   | e minimum      |
|-----------|-------------------|------------|-------------------|------------|--------|----------|----------------|
|           | $L^2(\Sigma)$ -no | orm case   | for Examp         | les I, II, | III in | (4.6.27) | ) with initial |
|           | data II i         | n (4.5.26) | and for $\lambda$ | = 1.       |        |          |                |
| 7         |                   | 110        | 1 100             |            | - L    | n /      | 1/100          |

|     | h                       | 1/16          | 1/32          | 1/64          | 1/128         |
|-----|-------------------------|---------------|---------------|---------------|---------------|
| I   | $\ g^h\ _{L^2(\Sigma)}$ | 0.45499490909 | 0.43856890841 | 0.43072624972 | 0.42688986194 |
|     | count                   | 12            | 15            | 16            | 14            |
| II  | $\ g^h\ _{L^2(\Sigma)}$ | 0.45794379129 | 0.43786262977 | 0.42823496745 | 0.42350319979 |
|     | count                   | 12            | 12            | 12            | 17            |
| III | $\ g^h\ _{L^2(\Sigma)}$ | 0.45251184147 | 0.43223256110 | 0.42248163895 | 0.41769256787 |
|     | count                   | 6             | 6             | 6             | 6             |

Table 4.7 Results of computational experiments for the minimum  $L^2(\Sigma)$ -norm case for Examples I, II, III in (4.6.27) with initial data III in (4.5.26) and for  $\lambda = 1$ .

|     | h                       | 1/16          | 1/32          | 1/64          | 1/128         |
|-----|-------------------------|---------------|---------------|---------------|---------------|
| I   | $\ g^h\ _{L^2(\Sigma)}$ | 0.94846408635 | 0.86623499117 | 0.82305989083 | 0.80084943961 |
|     | count                   | 11            | 11            | 11            | 11            |
| II  | $  g^h  _{L^2(\Sigma)}$ | 0.99946692390 | 0.90706619363 | 0.85837779589 | 0.83325287647 |
|     | count                   | 13            | 16            | 18            | 20            |
| III | $\ g^h\ _{L^2(\Sigma)}$ | 0.95205362894 | 0.86225101645 | 0.81393537311 | 0.78872084130 |
|     | count                   | 6             | 6             | 6             | 6             |



Figure 4.4 left - approximate control  $g^h$  and g, middle - approximate  $u^h$  and target W, right - approximate  $u^h_t$  and target Z with initial data (4.5.24). h = 1/16, 1/32, 1/64, 1/1024 from top to bottom respectively.  $\lambda = 1$ .



Figure 4.5 left - approximate control  $g^h$  and g, middle - approximate  $u^h$  and target W, right - approximate  $u^h_t$  and target Z with initial data (4.5.24). h = 1/16, 1/32, 1/64, 1/1024 from top to bottom respectively.  $\lambda = 7/8$ .



Figure 4.6 left - approximate control  $g^h$ , middle - approximate  $u^h$  and target W, right - approximate  $u^h_t$  and target Z with initial data (4.5.26-I). h = 1/16, 1/32, 1/64, 1/1024 from top to bottom respectively.  $\lambda = 1$ .



Figure 4.7 left - approximate control  $g^h$ , middle - approximate  $u^h$  and target W, right - approximate  $u^h_t$  and target Z with initial data (4.5.26-I). h = 1/16, 1/32, 1/64, 1/1024 from top to bottom respectively.  $\lambda = 4/5$ .



Figure 4.8 left - approximate control  $g^h$ , middle - approximate  $u^h$  and target W, right - approximate  $u_t^h$  and target Z with initial data (4.5.26-II). h = 1/16, 1/32, 1/64, 1/1024 from top to bottom respectively.  $\lambda = 1$ .



Figure 4.9 left - approximate control  $g^h$ , middle - approximate  $u^h$  and target W, right - approximate  $u^h_t$  and target Z with initial data (4.5.26-II). h = 1/64, 1/128, 1/256, 1/512, 1/1024 from top to bottom respectively.  $\lambda = 7/8$ .



Figure 4.10  $g_L, g_R, u^h(x, T)$ , and  $u_i^h(x, T)$  from left to right with initial data (4.5.26-III). h = 1/16, 1/32, 1/64, 1/1024 from top to bottom respectively.  $\lambda = 1$ .



Figure 4.11  $g_L, g_R, u^h(x, T)$ , and  $u_t^h(x, T)$  from left to right with initial data (4.5.26-III). h = 1/64, 1/128, 1/256, 1/512, 1/1024 from top to bottom respectively.  $\lambda = 7/8$ .



Figure 4.12  $g_L$ ,  $g_R$ ,  $u^h(x,T)$ , and  $u^h_t(x,T)$  from left to right with  $f(u) = \sin u$  and initial data (4.5.26-III). h = 1/16, 1/32, 1/64, 1/128 from top to bottom respectively.  $\lambda = 1$ .



Figure 4.13  $g_L$ ,  $g_R$ ,  $u^h(x,T)$ , and  $u^h_t(x,T)$  from left to right with  $f(u) = u^{3/2}$  and initial data (4.5.26-III). h = 1/16, 1/32, 1/64, 1/128 from top to bottom respectively.  $\lambda = 1$ .



Figure 4.14  $g_L$ ,  $g_R$ ,  $u^h(x,T)$ , and  $u^h_t(x,T)$  from left to right with  $f(u) = \ln(u^2 + 1)$  and initial data (4.5.26-III). h = 1/16, 1/32, 1/64, 1/128 from top to bottom respectively.  $\lambda = 1$ .

## 5 Shooting Methods for Numerical Solutions of Distributed Optimal Control Problems Constrained by first order linear hyperbolic equation

## 5.1 Distributed Optimal control problems for first order linear hyperbolic equation

We will study numerical methods for optimal control and controllability problems associated with first order linear hyperbolic equation. We are particularly interested in investigating the relevancy and applicability of high performance computing (HPC) for these problems.

As an prototype example of optimal control problems for first order linear hyperbolic equation we consider the following distributed optimal control problem with a > 0:

**Problem 5.1.1.** Given U, W and fixed time T, find a pair of the optimizer  $(\hat{u}, \hat{f})$  such that

$$u_t + au_x = f(t, x), \quad (t, x) \in \Omega \times (0, T) = Q$$
$$u(0, x) = g(x),$$
$$u(t, 0) = h(t), \qquad x \in \Omega$$
(5.1.1)

minimizing

$$\mathcal{J}(u,f) = \frac{\alpha}{2} \int_0^T \int_{\Omega} |u - U|^2 \, dx \, dt + \frac{\beta}{2} \int_{\Omega} |u(T,x) - W(x)|^2 \, dx + \frac{1}{2} \int_0^T \int_{\Omega} |f|^2 \, dx \, dt.$$

Here  $\Omega = (0, 1)$  is a bounded domain in  $\mathbb{R}^1$ ; u is dubbed the state, and f is the distributed control. Also, U is a target function.

The existence and uniqueness of this problem will be treated later on.

**Problem 5.1.2.** (Existence and Uniqueness) For given  $f \in L^2(Q)$ ,  $w \in L^2(\Omega)$ ,  $z \in L^2(0,T)$ , there is the unique solution of the following;

$$\begin{cases} u_t + au_x = f(t, x), & (t, x) \in Q \\ u(0, x) = w(x), & x \in \Omega = (0, 1) \\ u(t, 0) = z(t), & t \in (0, T). \end{cases}$$
(5.1.2)

Let  $L = \frac{\partial}{\partial t} + a \frac{\partial}{\partial x}$ .

**Definition 5.1.3.** Given  $f \in L^2(Q)$ ,  $w \in L^2(\Omega)$ ,  $z \in L^2(0,T)$ , we say  $u \in L^2(Q)$ is a weak solution of (5.1.2) if  $\int_Q u(L\phi) dx dt = -\int_Q f\phi dx dt - \int_\Omega w(x)\phi(0,x) dx - \int_0^T az(t)\phi(t,0) dt$  for all  $\phi \in H^1(Q)$  with  $\phi(t,x) = 0$ ,  $\phi(t,1) = 0$ .

**Lemma 5.1.4.** For smooth initial and boundary data of (5.1.2), we can obtain the explicit formula.

$$u(t,x) = \begin{cases} \int_0^t f(as+x-at,s)ds + w(x-at) & \text{in } Q_1, \\ \int_0^x \frac{1}{a}f(y,\frac{y}{a}+t-\frac{x}{a})dy + z(t-\frac{x}{a}) & \text{in } Q_2, \end{cases}$$
(5.1.3)

where  $Q_1 = \{(t, x) \in Q | x \ge at\}, Q_2 = \{(t, x) \in Q | x \le at\}.$ 

*Proof.* Use the method of characteristics.

 $\Box$ 

Lemma 5.1.5. For

$$\begin{cases}
Lu = f(t, x), & (t, x) \in Q \\
u(T, x) = 0, & x \in (0, 1) \\
u(t, 1) = 0, & t \in (0, T),
\end{cases}$$
(5.1.4)

we have the following solution:

$$u(t,x) = \begin{cases} \int_{0}^{T-t} -f(as+(1-x)-a(T-t),s)ds & in Q_{1}, \\ \int_{0}^{1-x} -\frac{1}{a}f(y,\frac{y}{a}+(1-t)-\frac{1-x}{a})dy & in Q_{2}. \end{cases}$$
(5.1.5)

Proof. Use the previous lemma and change of variables,  $(t, x) \mapsto (T - t, 1 - x)$ .  $\Box$ Lemma 5.1.6. (A priori estimates) For smooth function  $u_{\epsilon}$ , let  $y_{\epsilon}$  satisfy the following:

$$\begin{cases} L^* y_{\epsilon} = u_{\epsilon}, & in \ Q \\ y_{\epsilon}(T, x) = 0, & x \in (0, 1) \\ y_{\epsilon}(t, 1) = 0, & t \in (0, T). \end{cases}$$
(5.1.6)

Then

$$\| y_{\epsilon} \|_{L^{2}(Q)} \leq C \| u_{\epsilon} \|_{L^{2}(Q)},$$

$$\| y_{\epsilon}(t,0) \|_{L^{2}(0,T)} \leq \frac{1}{a} \| u_{\epsilon} \|_{L^{2}(Q)},$$

$$\| y_{\epsilon}(0,x) \|_{L^{2}(0,1)} \leq \| u_{\epsilon} \|_{L^{2}(Q)}.$$

$$(5.1.7)$$

 $\Box$ 

Proof. Use the previous lemma.

Now let  $f_{\epsilon}, w_{\epsilon}, z_{\epsilon}$  such that  $f_{\epsilon} \longrightarrow f, w_{\epsilon} \longrightarrow w, z_{\epsilon} \longrightarrow z$  in  $L^2$  sense.

Consider the following equations:

$$\begin{cases}
Lu_{\epsilon} = f_{\epsilon}, & \text{in } Q \\
u_{\epsilon}(0, x) = w_{\epsilon}, & x \in (0, 1) \\
u_{\epsilon}(t, 0) = z_{\epsilon}, & t \in (0, T),
\end{cases}$$
(5.1.8)

and

$$\begin{cases} L^* y_{\epsilon} = u_{\epsilon}, & \text{in } Q \\ y_{\epsilon}(T, x) = 0, & x \in (0, 1) \\ y_{\epsilon}(t, 1) = 0, & t \in (0, T). \end{cases}$$
(5.1.9)

Clearly the solutions of above two equations exist uniquely by previous lemmas. Multiply equation (5.1.8) by  $y_{\epsilon}$ . Then integrate both sides over Q, and so

$$(Lu_{\epsilon}, z_{\epsilon})_{L^{2}(Q)} = (f_{\epsilon}, z_{\epsilon})_{L^{2}(Q)}$$

$$\Rightarrow (u_{\epsilon}, L^{*}z_{\epsilon})_{L^{2}(Q)} + (u_{\epsilon}, z_{\epsilon})_{L^{2}(\Omega)}|_{0}^{T} + (u_{\epsilon}, az_{\epsilon})_{L^{2}(0,T)}|_{0}^{1} = (f_{\epsilon}, z_{\epsilon})_{L^{2}(Q)}$$

$$\Rightarrow (u_{\epsilon}, u_{\epsilon})_{L^{2}(Q)} = (w_{\epsilon}(x), z_{\epsilon}(x, 0))_{L^{2}(\Omega)} + (z_{\epsilon}(t), az_{\epsilon}(0, t))_{L^{2}(0,T)} + (f_{\epsilon}, z_{\epsilon})_{L^{2}(Q)}$$

$$\Rightarrow || u_{\epsilon} ||_{L^{2}(Q)}^{2} \leq \left( || w_{\epsilon} ||_{L^{2}(\Omega)}^{2} + \frac{|| u_{\epsilon} ||_{L^{2}(Q)}^{2}}{4} \right) + \left( || z_{\epsilon} ||_{L^{2}(0,T)}^{2} + \frac{|| u_{\epsilon} ||_{L^{2}(Q)}^{2}}{4} \right) \qquad (5.1.10)$$

$$+ \left( || f_{\epsilon} ||_{L^{2}(Q)}^{2} + \frac{|| u_{\epsilon} ||_{L^{2}(Q)}^{2}}{4} \right)$$

$$\Rightarrow || u_{\epsilon} ||_{L^{2}(Q)}^{2} \leq C \left( || w_{\epsilon} ||_{L^{2}(\Omega)}^{2} + || z_{\epsilon} ||_{L^{2}(0,T)}^{2} + || f_{\epsilon} ||_{L^{2}(Q)}^{2} \right)$$

Lemma 5.1.7. ( A priori estimate ) Let  $u_{\epsilon}$  satisfy (5.1.8). Then

$$\| u_{\epsilon} \|_{L^{2}(Q)}^{2} \leq C \left( \| w_{\epsilon} \|_{L^{2}(\Omega)}^{2} + \| z_{\epsilon} \|_{L^{2}(0,T)}^{2} + \| f_{\epsilon} \|_{L^{2}(Q)}^{2} \right).$$

*Proof.* By the above argument.

Note that as  $\epsilon \to 0$ ,  $w_{\epsilon} \to w$ ,  $z_{\epsilon} \to z, f_{\epsilon} \to f$  in  $L^2$ . So for any  $\epsilon_1, \epsilon_2$  and  $\epsilon_3$ ,

$$\| w_{\epsilon} \|_{L^{2}(\Omega)}^{2} \leq \| w \|_{L^{2}(\Omega)}^{2} + \epsilon_{1},$$

$$\| z_{\epsilon} \|_{L^{2}(0,T)}^{2} \leq \| z \|_{L^{2}(0,T)}^{2} + \epsilon_{2},$$

$$\| f_{\epsilon} \|_{L^{2}(Q)}^{2} \leq \| f \|_{L^{2}(Q)}^{2} + \epsilon_{3}$$
(5.1.11)

for small enough  $\epsilon$ . Let  $\tilde{\epsilon} = max(\epsilon_1, \epsilon_2, \epsilon_3)$ . Then

$$\| w_{\epsilon} \|_{L^{2}(\Omega)}^{2} + \| z_{\epsilon} \|_{L^{2}(0,T)}^{2} + \| f_{\epsilon} \|_{L^{2}(Q)}^{2} \le \| w \|_{L^{2}(\Omega)}^{2} + \| z \|_{L^{2}(0,T)}^{2} + \| f \|_{L^{2}(Q)}^{2} + 3\widetilde{\epsilon}.$$

Assume  $\tilde{\epsilon}$  is sufficiently small enough. Then

$$\| w_{\epsilon} \|_{L^{2}(\Omega)}^{2} + \| z_{\epsilon} \|_{L^{2}(0,T)}^{2} + \| f_{\epsilon} \|_{L^{2}(Q)}^{2} \leq C \left( \| w \|_{L^{2}(\Omega)}^{2} + \| z \|_{L^{2}(0,T)}^{2} + \| f \|_{L^{2}(Q)}^{2} \right).$$

Therefore we have the following a priori estimate:

**Lemma 5.1.8.** (A priori estimate ) Let  $u_{\epsilon}$  satisfy (5.1.8). Then for small enough  $\epsilon$ ,

$$|| u_{\epsilon} ||_{L^{2}(Q)}^{2} \leq C \left( || w ||_{L^{2}(\Omega)}^{2} + || z ||_{L^{2}(0,T)}^{2} + || f ||_{L^{2}(Q)}^{2} \right).$$

Note that  $u_{\epsilon}$  can be found by (5.1.3).

**Lemma 5.1.9.** (Existence of weak solution,  $u_{\epsilon}$ ) Any u which can be taken from (5.1.3) is a weak solution of (5.1.2).

*Proof.* Clearly the classical solution is also a weak solution.

### 5.2 An optimality system of equations

Through the use of Lagrange multiplier rules, the optimal control problem (5.1.1) may be converted into the following system of equations from which an optimal solution may be determined:

$$u_{t} + au_{x} = -\xi, \qquad (t, x) \in \Omega \times (0, T)$$

$$u(0, x) = g(x), \qquad x \in \Omega$$

$$u(t, 0) = h(t), \qquad x \in \Omega$$

$$\xi_{t} + a\xi_{x} = -\alpha(u - U), \qquad (t, x) \in \Omega \times (0, T)$$

$$\xi(T, x) = \beta(u(T, x) - W(x)), \quad x \in \Omega$$

$$\xi(t, 1) = 0, \qquad x \in \Omega$$
(5.2.12)

Next, we give a precise definition of an optimal solution, i.e. a minimizer of  $\mathcal{J}_{\beta}(u, f)$ . Let the *admissibility set* be defined by

 $\mathcal{U}_{ad} = \{(u, f) \in L^2(Q)) \times L^2(Q) \text{ such that } (5.2.12) \text{ is satisfied and } \mathcal{J}_\beta(u, f) < \infty \}.$ 

Then  $(\hat{u}, \hat{f})$  is called an optimal solution if

$$\mathcal{J}_{\beta}(\hat{u}, \hat{f}) \leq \mathcal{J}_{\beta}(u, f),$$

for all  $(u, f) \in \mathcal{U}_{ad}$ .

**Proposition 1.** There exists a unique optimal solution  $(\hat{u}, \hat{f}) \in U_{ad}$  for Problem (5.1.1). Proof. (1)(Uniqueness)

Note that every strictly increasing convex function of a convex function is convex. Therefore Uniqueness follows from the convexity of the functional and the admissibility set ,and the linearity of the constraints.

(2)(Existence)

Clearly  $\mathcal{U}_{ad}$  is not empty(from Lemma). Let  $\{(u_n, f_n)\}$  be a minimizing sequence in  $\mathcal{U}_{ad}$ , i.e.

$$\lim_{n\to\infty}\mathcal{J}_{\beta}(u_n,f_n)=\inf_{(u,f)\in\mathcal{U}_{ad}}\mathcal{J}_{\beta}(u,f).$$

Note that from Lemma we have the following priori estimate;

$$\max_{0 \le t \le T} \|u_n(t)\|_{L^2(Q)} \le C \left( \|g\|_{L^2(\Omega)} + \|h\|_{L^2(0,T)} + \|f_n\|_{L^2(Q)} \right).$$

By the definition of  $\mathcal{U}_{ad}$  and this a priori estimate, there exists a subsequence  $(u_{n_j}, f_{n_j})$ such that

$$u_{n_j} \rightharpoonup \hat{u}$$
 weakly in  $L^2(Q)$ ,  
 $f_{n_j} \rightharpoonup \hat{f}$  weakly in  $L^2(Q)$ 

for some  $(\hat{u}, \hat{f}) \in \mathcal{U}_{ad}$ .

Now, by the process of passing to the limit, we have that  $(\hat{u}, \hat{f})$  is a weak solution of (2). Then the fact that the functional  $\mathcal{J}_{\beta}(\cdot, \cdot)$  is weakly lower semi-continuous implies that

$$\inf_{(u,f)\in\mathcal{U}_{ad}}\mathcal{J}_{\beta}(u,f) = \lim_{j\to\infty}\mathcal{J}_{\beta}(u_{n_j},f_{n_j})$$
$$\geq \mathcal{J}_{\beta}(\hat{u},\hat{f}).$$

Hence

$$\inf_{(u,f)\in\mathcal{U}_{ad}}\mathcal{J}_{\beta}(u,f)=\mathcal{J}_{\beta}(\hat{u},\hat{f}),$$

so that  $(\hat{u}, \hat{f})$  is an optimal solution.

Such control problems are classical ones in the control theory literature; see, e.g., [5] for the linear case and [6] for the nonlinear case regarding the existence of optimal

 $\Box$ 

solutions as well as the existence of a Lagrange multiplier  $\xi$  satisfying the optimality system of equations. However, numerical methods for finding discrete (e.g., finite element and/or finite difference) solutions of the optimality system are largely limited to gradient type methods which are sequential in nature and generally require many iterations for convergence. The optimality system involves boundary conditions at t = 0 and t = T and thus cannot be solved by marching in time. Direct solutions of the discrete optimality system, of course, are bound to be expensive computationally in 2 or 3 spatial dimensions since the problem is (d + 1) dimensional (where d is the spatial dimensions.)

The computational algorithms we propose here are based on shooting methods for two-point boundary value problems for ordinary differential equations (ODEs); see, e.g., [1, 2, 3, 4]. The algorithms we propose are well suited for implementations on a parallel computing platform such as a massive cluster of cheap processors.

#### 5.3 Computational Results

We will apply our shooting Algorithm 1 in the previous chapter with slight modification to the special case of a = 1. We will experiment with two generic examples.

**Example 5.3.1.** (full domain control) T = 1,  $\Omega = [0, 1]$ .

For given target functions,

$$W(x) = 1, \qquad U(t, x) = 1.$$
 (5.3.13)

For initial and boundary data,

$$g(x) = x(x-1), \qquad h(t) = 0.$$
 (5.3.14)


Figure 5.1 Optimal solution u and target W, U for  $\Delta t = 1/40$ ,  $\Delta x = 1/20$   $\therefore$  optimal solution u(t, x) -: target functions W(x), U(t, x) $\alpha = \beta = 1, 10, 100, 1000$ 





 $\begin{array}{lll} \mbox{Figure 5.2} & \mbox{Optimal solution $u$ and target $W$,$U$ for $\bigtriangleup t=1/40$, $\bigtriangleup x=1/20$ \\ $\therefore$ optimal solution $u(t,x)$ $-:$ target functions $W(x)$,$U(t,x)$ \\ $\alpha=\beta=1,10,100$, 1000$ \end{array}$ 

## 6 CONCLUSION

In the thesis, we discussed and successfully implemented shooting methods for solving optimal control problems constrained by linear wave equations, semilinear wave equations, and linear conservation laws. The shooting algorithms for optimal control problems were also utilized effectively to find approximate solutions to controllability problems for these equations. Both distributed and boundary controls were treated. The convergence of the algorithms were numerically demonstrated when the true solutions are smooth. Weak convergence of the algorithms were also numerically demonstrated when the true solutions were not smooth.

However, a host of issues still need be addressed in future work; these include other control objectives, a thorough study of parallel implementations and a analysis of computing complexity, the case of high spatial dimensions, rigorous numerical analysis, and generalizations to control other types of equations. A list of concrete topics is as follows:

• Task A. Control problems with other functionals. Instead of minimizing functional (4.1.4) we may consider the minimization of a more general functional

$$\mathcal{J}(u,g) = \frac{\alpha}{2} \int_0^T \int_\Omega K(u) \, d\mathbf{x} \, dt + \frac{\beta}{2} \int_\Omega \Phi_1(u(T,\mathbf{x})) \, d\mathbf{x} + \frac{\gamma}{2} \int_\Omega \Phi_2(u_t(T,\mathbf{x})) \, d\mathbf{x} + \frac{1}{2} \int_0^T \int_\Omega |g|^2 \, d\mathbf{x} \, dt \,, \tag{6.0.1}$$

where the terms involving K(u) and  $\Phi(u)$  model certain physical quantities to be minimized. For instance, to track a target state U one may choose K(u) = |u - u|

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 $U|^{s}/s$  for some s > 1; to stabilize a structure one may choose  $K(u) = |u|^{s}$  for some s > 1; to attain a uniform displacement distribution one may choose  $K(u) = |\nabla u|^{2}$ ; and to reach a desired target  $U_{T}$  at time T one may choose  $\Phi(u(T)) = |u(T) - U_{T}|^{2}$ .

• Task B. Parallel implementations. Observe that the loop in j in Algorithm 1 may be executed in parallel, say on I processors. The loops in n and i are comprised of an explicit time marching scheme for solving a discrete initial value problem and can therefore be performed on low cost processors; in addition, these loops require the storage of only three vectors of dimension I containing the solutions at three time levels (which are updated by time marching). We will implement the algorithm on a cluster of processors to assess its practicality. We will also experiment with ways to improve implementation efficiency when the number of processors available is less than I; this issue will be of particular importance in two and three space dimensions and it is worthwhile to first explore ideas in the case of one space dimension.

We also observe that the loop in i is well suited for implementations on a vector machine. The ideal computing platform for this algorithm is a network of vector machines.

The analysis of computing complexity is of both theoretical and practical importance. The computational time needed to find the solution to the optimality system depend on the number of processors, the organization of the algorithm in making use of all available processors, data communications, and initial guesses (or the number of Newton iterations). We will analyze theoretically the computing complexity and estimate the computing time for finding the optimal solution in various mathematical, algorithmical, and computer architectural settings.

• Task C. Higher spatial dimensions. We may carry out Tasks A and B in higher space dimensions. An advantage of the shooting algorithms is their explicit, time

marching solutions of initial value problems (the large number of such IVPs can be solved in parallel.) A major shortcoming of the shooting algorithms arises from the fact that the matrix in each Newton iteration is a full one, rendering the solution of the corresponding linear system expensive. Computational costs in implementing shooting algorithms in higher dimensions are formidable. Numerical solutions of the underlying control problems in two dimensions pose a real computational challenge and it is even more so in three dimensions.

- Task D. Establishing theoretical convergence rates. In our experimental results, if the wave equation and conservation law are linear (and so will be the optimality system), then the algorithms converge in precisely one iteration. When a nonlinear wave equation is considered, Newton's methods require only a small number of iterations for good initial guesses. We benchmarked the convergence of the algorithms for various linear and nonlinear cases. We will attempt to rigorously establish the convergence rates for those problems. We will also investigate methods for generating good initial guesses; e.g., for an optimal control problem with a tracking type functional we may use the target state to help generate initial guesses for the Lagrange multiplier  $\xi$ .
- Task E. Equations of linear elasticity, nonlinear elasticity, and nonlinear conservation laws. Wave equations given in this thesis are special cases of PDE systems modelling elastic materials and structures. It is of significant practical interest to study optimal control problems for linear and nonlinear elasticity. Nonlinear conservation laws are more useful in applications than linear conservation laws. We will attempt to extend the results of Tasks A–D into numerical solutions of control problems for elasticity and nonlinear conservation laws. We are confident about the successes of research into such problems in one space dimension, and we hope to be able to make tangible progress in higher dimensions as well.

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## ACKNOWLEDGMENTS

I am deeply indebted to my advisor Dr. L. Steven Hou. Without his expert advice and guidance, this dissertation may not have been born. He has provided his almost interminable patience and love to my career and life. His philosophy will remain part of mine.

I am grateful to my committee members for their efforts and contributions to this work : Dr. Oleg Emanouvilov, Dr. Hailiang Liu, Dr. Michael Smiley and Dr. Sung-Yell Song. Special thanks go to Dr. Oleg Emanouvilov who was generous in sharing his knowledge and answering questions related to partial differential equations. I also appreciate Dr. Hailiang Liu who generously provided me many insights into conservation law.

I also would like to express my gratitude to Dr. Seok-Zun Song who has provided me his endless support no matter where I am. Dr. Jerold Mathews, his wife Eleanor Mathews, Eun-Youn, my hometown (Jeju) friends, and Alex's family have showed their love to me, and there are so many other Korean students and visitors who have encouraged me during the school year. I deeply appreciate all of them.

The entire honor goes to my wife beautiful Mi-Young. I want to express my deepest thanks to her for her love, friendship and unfailing support throughout my career in graduate school. My son, Hae-In has been the joy of my life. My parents Hong-Lim, In-Sook and parents-in-law Seok-Yoon, Hee-Jin, their continuous support with deep love and belief in me have kept me from losing my sanity during the entire school year. I do love all of you from bottom of my heart.

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