A dualized Kaczmarz algorithm in Hilbert and Banach space

by

Anna Christine Aboud

A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Mathematics

Program of Study Committee:
Eric Weber, Major Professor
Jennifer Newman
Justin Peters
Paul Sacks
Pablo Raúl Stinga

The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this dissertation. The Graduate College will ensure this dissertation is globally accessible and will not permit alterations after a degree is conferred.

Iowa State University

Ames, Iowa

2019

Copyright © Anna Christine Aboud, 2019. All rights reserved.

DEDICATION

To my loving husband, Zackary, who has tirelessly accompanied me through every step of this process.

TABLE OF CONTENTS

LIST OF TABLES v

LIST OF FIGURES vi

ACKNOWLEDGEMENTS vii

ABSTRACT viii

CHAPTER 1. OVERVIEW 1

CHAPTER 2. REVIEW OF LITERATURE 10

2.1 Frame Theory 10

2.2 Characterization of Effective Sequences 16

2.2.1 The Kaczmarz Theorem 17

2.2.2 The Kwapień and Mycielski Theorem 17

2.2.3 The Haller and Szwarc Theorem 20

2.2.4 Generating Effective Sequences from Bessel Sequences 21

2.3 The Kwapień and Mycielski Theorems in Banach Spaces 22

2.4 Almost Effective Sequences 26

CHAPTER 3. EFFECTIVE PAIRS IN HILBERT SPACE 27

3.1 Dual Algorithm in Hilbert Space 27

3.2 Pair Properties 30

3.3 Relation by a Positive, Invertible T 32

3.4 Improving Almost Effective Sequences 38

3.5 Examples 42

CHAPTER 4. EFFECTIVE PAIRS IN BANACH SPACE 45

4.1 Introduction 45

4.2 Terminology 47

4.3 Duality 52

4.3.1 $\{g_n\}$ and $\{\psi_n\}$ Duality 53

4.3.2 $\{\tilde{g}_n\}$ and $\{\phi_n\}$ Duality 57

4.3.3 $\{g_n\}$ and $\{\tilde{g}_n\}$ Duality 60

4.4 An Informative Example 68

4.4.1 A Theorem in Finite Dimensions 82

4.4.2 The Haller and Szwarc Equivalences 85

4.5 A Banach Space Example 86

BIBLIOGRAPHY 88

APPENDIX A. EXAMPLE 90

APPENDIX B. CODE 93

LIST OF TABLES

Table 4.1 Duality Relationships 46

LIST OF FIGURES

Figure 1.1 The Kaczmarz Algorithm in $\mathbb{R}^2$ 2

Figure 1.2 The Dual Kaczmarz Algorithm in $\mathbb{R}^2$ - Convergence 4

Figure 1.3 The Dual Kaczmarz Algorithm in $\mathbb{R}^2$ - No Convergence 4

ACKNOWLEDGEMENTS

I would like to express my deep gratitude to my major professor, Eric Weber, for his consistent guidance and excellent project ideas. I am also indebted to my 2018 Early Graduate Research group. Thank you for providing an incredibly positive mathematical collaboration as well as many valuable content contributions to Chapter 3.

Mathematical effort comprises only part of what is needed to successfully complete a doctoral degree. Without the incredible belief and support of all of my family—parents, sisters, and husband—I would not have completed this process. I thank you all. I would also like to acknowledge our huskies, Balder and Freya, for keeping me active and joy filled during the final months of writing and researching.

Most importantly, I thank God. *Soli Deo gloria*!

ABSTRACT

The Kaczmarz algorithm is a versatile and computationally efficient method of reconstructing vectors in a Hilbert space using inner products against a sequence $\{e_n\}$. If the algorithm successfully reconstructs any vector in the space, we say that $\{e_n\}$ is an effective sequence. Kwapień and Mycielski provide a twofold criterion for sequences to be effective. Expanding on these results, Haller and Szwarc present an extensive list of conditions equivalent to $\{e_n\}$ being effective.

Within the context of a Hilbert space, we develop a dualized version of the Kaczmarz algorithm which is naturally suited for extension to a Banach space. We provide necessary and sufficient conditions for one part of the Kwapień-Mycielski criterion to be weakly satisfied, ensuring the dense weak convergence of the dualized algorithm in both Banach and Hilbert space. Furthermore, we show that the equivalences of Haller and Szwarc fail in the dualized context, instead separating into two sets of equivalent conditions. We conclude with a presentation of convergence conditions for periodic sequences in a finite-dimensional Banach space.

CHAPTER 1. OVERVIEW

For centuries mathematicians have occupied themselves with the question of solving systems of linear equations. Although there are many effective methods for solving systems of reasonable size, many rely on some form of matrix inversion. In general, inverting matrices is computationally expensive, and the algorithms become impractical after a matrix attains a large enough size. Algorithms which circumvent inversion typically make strong assumptions on the associated coefficient matrix, limiting applicability and yielding inaccurate results when applied on an inappropriate matrix. In the age of big data, there has been a renewed interest in algorithms which require minimal computational power and can also be successfully applied to a broad class of coefficient matrices. In 1937, Stefan Kaczmarz developed an algorithm for solving a linear system using projections onto successive hyperplanes ([Kac37]). Figure 1.1 illustrates an application of the Kaczmarz algorithm in $\mathbb{R}^2$, and is a convenient concrete representation to keep in mind while working in more general spaces. The algorithm is formally constructed as follows:

Definition 1. Let $\{e_n\}$ be a sequence of unit vectors in a separable Hilbert space, $\mathcal{H}$. Define the sequence of approximations $\{x_n\}$ according to the *Kaczmarz algorithm*, where

$$\begin{aligned} x_0 &= \langle x, e_0 \rangle e_0 \\ x_n &= x_{n-1} + \langle x, e_n \rangle e_n. \end{aligned} \tag{1.1}$$

If $||x_n - x|| \to 0$ for every $x \in \mathcal{H}$, then we say that the sequence $\{e_n\}$ is *effective*.

Consider a linear system represented by $Ax = b$, where A is a matrix and x and b are vectors. The Kaczmarz algorithm cycles through the rows of A, using each row to build a better solution approximation. Kaczmarz showed that, if the rows of A (periodized to construct the sequence $\{e_n\}$) form a spanning set in the ambient space, then the algorithm will converge to the solution of the system (provided it is consistent).

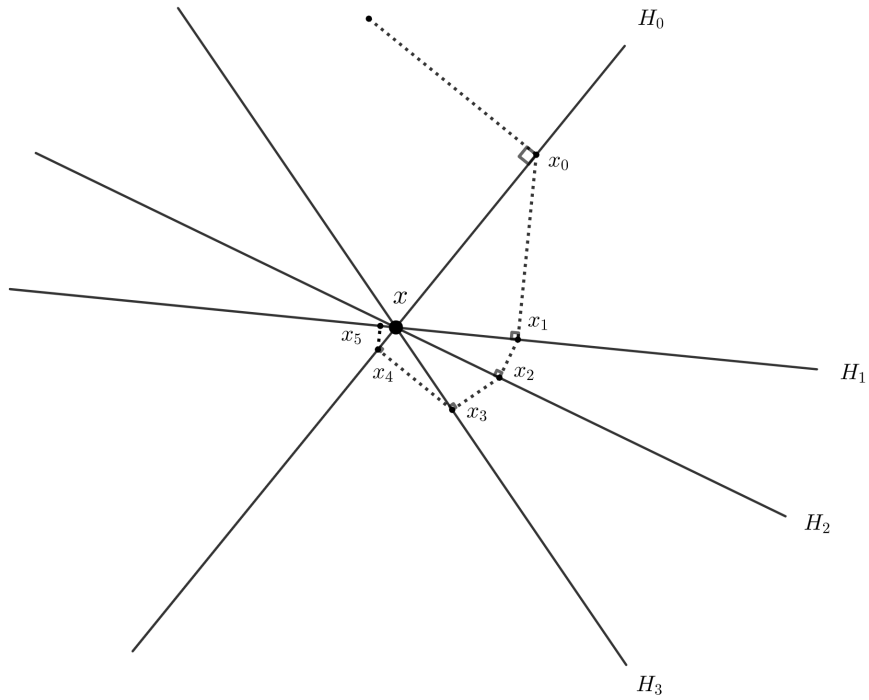

Figure 1.1 The Kaczmarz Algorithm in $\mathbb{R}^2$

Kaczmarz was naturally considering relatively small, finite-dimensional systems. His algorithm escaped largely unnoticed for more than forty years, after which it was adopted for its convenient applications in computerized tomography and digital signal processing [GBH70, Nat01]. It was used extensively under the name of ART, or the Algebraic Reconstruction Technique [GBH70]. About twenty-five years ago, the algorithm enjoyed renewed attention as people realized its particular utility in solving large systems. Since then, it has been employed in phase retrieval [JG17, TV18], optimization [NWS14], and learning theory [KM01]. The versatility of the Kaczmarz algorithm results from its ability to solve a system while only "seeing" one row of the measurement matrix A at a time, resulting in low memory usage. In 2001, Kwapień and Mycielski extended the theory of the Kaczmarz algorithm to infinite dimensions, providing a characterization for the entire class of effective sequences (sequences which provide convergence in the Kaczmarz algorithm) [KM01]. This character-

ization was given in terms of a second sequence, $\{h_n\}$, which together with $\{e_n\}$ provided a resolution of the identity. A few years later, Haller and Szwarc provided yet another characterization in terms of Gram-like infinite matrices [HS05]. Since this time, there have been a multitude of derivations of the original algorithm, each designed for a slightly different context. In the absence of a consistent system, the Extended Kaczmarz algorithm converges to the least squares solution of the system in question [EHL81, Tan71]. Various randomization methods have also been applied to accelerate convergence rates [NT14, NZZ15, SV09]. Recently, variations which lend themselves to distributed data sets have also been developed [KRS15, HKW19]. In this thesis, we will concern ourselves primarily with the dualized variation introduced by Kwapień and Mycielski in [KM01] and expanded by Aboud, Curl, Harding, Vaughan, and Weber in [ACH+19]. The motivation for this derivation will be elucidated shortly. We present the definition in its most general context of a Banach space, although a nontrivial portion of the discussion will be spent working in the context of a Hilbert space. For the reader's benefit, two applications of the dual Kaczmarz algorithm in $\mathbb{R}^2$ are provided in Figures 1.2 and 1.3. Even in the simple case of $\mathbb{R}^2$, it is immediately apparent that nonorthogonal projections complicate questions of convergence considerably.

Definition 2. Let X be a Banach space. Let $\{\phi_n\}$ and $\{\psi_n\}$ be linearly dense sequences in X^* and X, respectively, where $\phi_n(\psi_n) = 1$ for all $n \in \mathbb{N}_0$. Define the sequence of approximations $\{x_n\}$ according to the *dual Kaczmarz algorithm*, where

$$\begin{aligned} x_0 &= \phi_0(x)\psi_0 \\ x_n &= x_{n-1} + \phi_n(x - x_{n-1})\psi_n. \end{aligned} \tag{1.2}$$

If $||x_n - x||_X \to 0$ for every $x \in X$, then we say that $\{(\phi_n, \psi_n)\}$ is an *effective pair*. If $u(x_n) \to u(x)$ for every $x \in X$, $u \in X^*$, then we say that $\{(\phi_n, \psi_n)\}$ is a *weakly effective pair*.

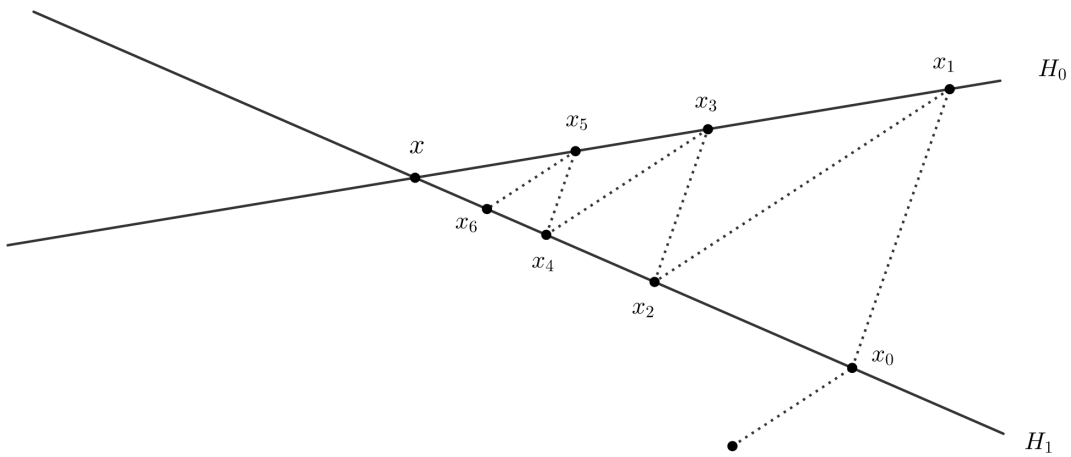

Figure 1.2 The Dual Kaczmarz Algorithm in $\mathbb{R}^2$ - Convergence

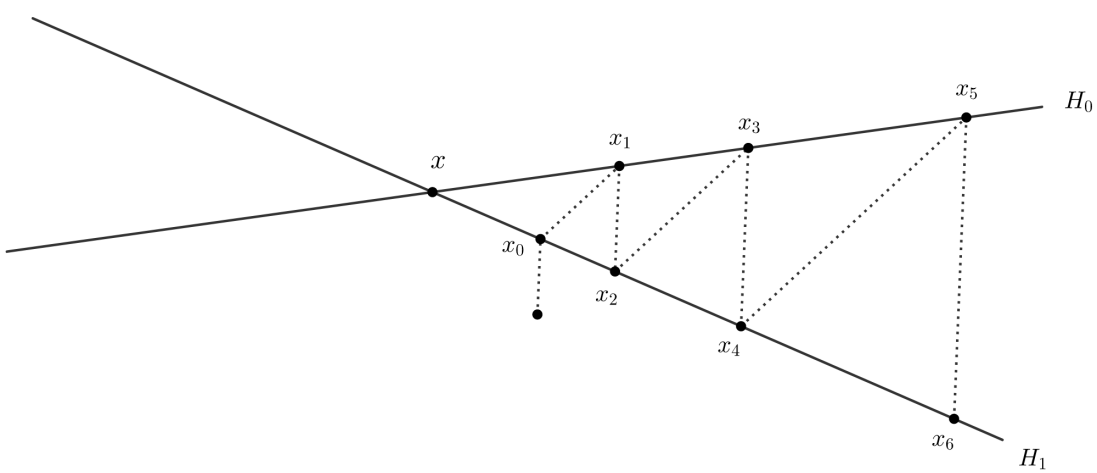

Figure 1.3 The Dual Kaczmarz Algorithm in $\mathbb{R}^2$ - No Convergence

Remarks on Notation. In [KM01], Kwapień and Mycielski say that $\{(\phi_n, \psi_n)\}$ is effective if the approximations in (1.2) converge in norm. Throughout this thesis, we will often choose to call $\{(\phi_n, \psi_n)\}$ fulfilling this condition an effective *pair*. We do this to draw attention to the contextual distinctions between the standard and dualized algorithms. We also point out that the ordering in the notation is significant. I.e, saying $\{(\phi_n, \psi_n)\}$ is an effective pair is not equivalent to saying $\{(\psi_n, \phi_n)\}$ is an effective pair (see Definitions 13 and 14).

The motivation for this dualized variation is twofold. First, as a class, effective sequences are intolerant to perturbation. This causes issues for spatial and temporal (spatiotemporal) data sets, which experience unavoidable perturbation within the measurement matrix, A. The reader should recall that an orthonormal basis is a common method for reconstructing vectors in a Hilbert space using inner products. Specifically, if $\{e_n\}$ is an orthonormal basis of a Hilbert space $\mathcal{H}$, any vector in $\mathcal{H}$ can be reconstructed using $x = \sum_{n=0}^{\infty} \langle x, e_n \rangle e_n$. Although they initially appear convenient, in practice orthonormal bases are irremediably rigid. If even one $\langle x, e_i \rangle$ measurement is lost or corrupt, the information it encodes cannot be regained. It is this brittle structure which historically prompted the study of frames. Informally, frames can be thought of as bases with extra elements. This overrepresentation allows vectors to be correctly reconstructed from two sequences $\{f_n\}$ and $\{g_n\}$ (a frame and its dual frame) as $x = \sum_{n=0}^{\infty} \langle x, g_n \rangle f_n$, even when some of the involved measurements are lost. Furthermore, frames can be perturbed within certain tolerances without ceasing to be frames [CC97, Chr95, Chr99]. We can view this progression from basis to frame as a loosening of structure, using two sequences to analyze and synthesize, rather than using one sequence to perform both duties. Using this progression as motivation, we applied a similar process in the context of the Kaczmarz algorithm. Instead of using one sequence, $\{e_n\}$, as in Definition 1, we used two different sequences, $\{\phi_n\}$ and $\{\psi_n\}$ (as shown in Definition 2), to analyze and synthesize, respectively.

The second motivation for the dualized structure was inspired by a need for an algorithm that is better suited for applications in a Banach space setting. When solving a linear

system, there are some cases where a least absolute deviations solution is preferable to a least squares solution (suppose the system is sparse). Working within a Banach space allows us to apply methods which would lead to these types of solutions, providing an alternate method to various programming systems in place currently. As currently implemented, however, the algorithm functions within a Hilbert, but not a general Banach space setting. It can only provide a least squares solution. To adapt the algorithm for a Banach space, we set aside the inner product in favor of the space of bounded linear functionals. Implicit in this construction is the need for two different sequences to synthesize and analyze (one in the Banach space, and one in its dual space). This is naturally analogous to Definition 2 as opposed to Definition 1, and provided impetus for beginning to investigate the behavior of two different sequences in a Hilbert space context.

In [KM01], Kwapień and Mycielski began to explore what the algorithm would look like in the context of a Banach Space. They presented two conditions which together provide necessary and sufficient conditions for convergence. These conditions hold in the most general context of a Banach space, and thus provide conditions for convergence in either the dual or standard Kaczmarz algorithm (the Kaczmarz algorithm is simply the dual Kaczmarz algorithm with $X = \mathcal{H}$ and $\phi_n = \psi_n$). We shall refer to these two requirements as the Uniformly Bounded (UB) and Densely Effective (DE) conditions.

Proposition (Kwapień and Mycielski)**.** *Let X be a Banach space. Assume that $\{\psi_n\} \subseteq X$ and $\{\phi_n\} \subseteq X^*$ are linearly dense in X and X^*, respectively. Define $P_n : X \to X$ such that $P_n(x) = x - \phi_n(x)\psi_n$. The sequence $\{(\phi_n, \psi_n)\}$ is effective if and only if there exists a constant $C > 0$ such that*

$$\|P_n P_{n-1} \cdots P_0\| \leq C \text{ for all } n \in \mathbb{N} \text{ and} \tag{UB}$$

$$\lim_{n\to\infty} P_n P_{n-1} \cdots P_0(x) = 0 \text{ for all } x \text{ in a dense subset of } X. \tag{DE}$$

Intuitively, condition (DE) gives convergence of the algorithm on a dense subset, and condition (UB) provides an operator bound which allows the convergence of the algorithm

to be extended to the entire space. Although conditions (UB) and (DE) seem closely related, they may occur independently (see Appendix A) and must be proven using independent means. Much of this thesis is devoted to the pursuit of the (DE) condition in the Banach space setting.

In the context of a single sequence in a Hilbert space, $\phi_n = \psi_n$, and the P_n are orthogonal projections with $\|P_n\| \leq 1$ for all n. Consequently, condition (UB) is attained "for free" and the pursuit of effectivity is reduced to proving that (DE) holds. On the other hand, when applying the dualized algorithm (in either a Hilbert or Banach space), the involved projections are no longer orthogonal, making condition (UB) much more elusive. Using the Banach Steinhaus Theorem, we show that we can actually loosen condition (UB) slightly, requiring only a pointwise bound (see Corollary 1). However, achieving even this weakened condition is formidable in practice.

Our goal in this thesis is characterization of the contexts in which pairs of sequences are effective, first in a Hilbert space, and then in the more general setting of a Banach space. As previously discussed, this requires satisfying both conditions (UB) and (DE). To begin our exploration of effective pairs, we work in the more restrictive context of a Hilbert space. Exploiting the existing work on effective sequences in a Hilbert space, we first consider only sequences related by a positive, invertible, operator T. This led to a fruitful exploration of various properties of effective pairs, yielding characterization theorems and shedding light on the strong connections between the Kaczmarz algorithm and frame theory. Specifically, we prove the following theorem.

Theorem (Aboud, Curl, Harding, Vaughan, Weber)**.** *Suppose that $\{\phi_n\}$ and $\{\psi_n\}$ are linearly dense sequences in $\mathcal{H}$, whose respective auxiliary sequences are $\{g_n\}$ and $\{\tilde{g}_n\}$ as defined in (3.2) and (3.5). Assume $\langle \phi_n, \psi_n \rangle = 1$ for all $n \in \mathbb{N}_0$ and suppose there exists a positive, invertible $T \in \mathcal{B}(\mathcal{H})$ such that $T\phi_n = \psi_n$ for all $n \in \mathbb{N}_0$. The following are then equivalent:*

(*i*) *U is a partial isometry, where U is given by equation* (3.12)*.*

(ii) $\{g_n\}$ *and* $\{\tilde{g}_n\}$ *are canonical dual frames.*

(iii) $\{(\phi_n, \psi_n)\}$ *is a symmetric effective pair.*

Moreover, if any of these conditions hold, then T^{-1} *is the frame operator for* $\{g_n\}$.

Analogous to the characterization results for effective sequences from Kwapień and Mycielski and Haller and Szwarc, we provide a characterization for effective pairs (those related by T) both in terms of reconstruction by auxiliary sequences, as well as matrix behavior. These results are proven by connecting the behavior of the pair of sequences to a related effective sequence. This relation allows us to completely address both (UB) and (DE). In finite dimensions, we are also able to categorize when such an operator T exists; this is precisely when the mixed Gramian matrix is positive.

There is a weaker class of sequences, called almost effective sequences, which, instead of giving convergence in the Kaczmarz algorithm, provide a bound for the limit of the approximation sequence [CT13]. Combining this idea with that of an effective pair, we design an augmented Kaczmarz algorithm which does give convergence to the solution. This is significant because, given the same information as the original scenario (the sequence of inner products, $\{\langle x, e_n\rangle\}$), instead of reaching within a certain tolerance of the solution, we are able to reconstruct the solution itself. Although not a central tenet of our work in this thesis, the augmented algorithm is a fruitful diversion that once again evidences the value of a dualized approach.

After achieving characterization in a Hilbert space with sequences related by an appropriate operator, we move to the more general case of a Banach space and its dual. In this case, we are able to show sufficient and necessary matrix conditions for meeting condition (DE) weakly.

Theorem. *Let X be a Banach space. Suppose $\{(\phi_n, \psi_n)\} \subseteq X^* \times X$ with $\phi_n(\psi_n) = 1$ for all n. Let $\Phi \subseteq X^*$ and $\Psi \subseteq X$ be the linear span of $\{\phi_n\}$ and $\{\psi_n\}$, respectively, and $M, \tilde{M}, U, \tilde{U}$ be as defined in (4.5) and (4.7). Then the following are equivalent.*

1. $\langle \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)\delta_j, \delta_i \rangle_{\ell^2} = 0$ *for all* $i, j \in \mathbb{N}_0$.

2. $\{(\phi_n, \psi_n)\}$ *is weakly effective on* $\Phi \times \Psi$.

3. $\{g_n\}$ *and* $\{\psi_n\}$ *form a weak resolution of the identity on* $\Phi \times \Psi$.

Analogous to the Hilbert space result by Haller and Szwarc, we also obtain a matrix condition for the two auxiliary sequences to form a resolution of the identity (see Theorem 6). Strangely enough, the duality of the auxiliary sequences does not imply effectivity of the sequences involved, pointing again to complexities introduced when dealing with non-orthogonal projections. This prompted us to conduct a more rigorous investigation of the three possible dualities involved in the effective pair context, as well as the interactions of these dualities with condition (DE). We provide an informative example to illuminate many of these relationships, and show that, by imposing various constraints upon the involved sequences, we are able to attain condition (UB). Most notably, we are able to determine when a periodic pair of sequences in finite dimensions is effective by calculating the spectral radius of the associated (nonorthogonal) projection operators.

Theorem. *Let X be a finite-dimensional Banach space. Suppose $\{(\phi_n, \psi_n)\} \subseteq X^* \times X$ are k-periodic sequences with $\phi_n(\psi_n) = 1$ for all n, where $\{\psi_n\}$ and $\{\phi_n\}$ are linearly dense in X and X^*, respectively. If $\rho(P_{k-1}P_{k-2}\cdots P_0) < 1$, then $\{(\phi_n, \psi_n)\}$ is effective. If $\rho(P_{k-1}P_{k-2}\cdots P_0) > 1$, then $\{(\phi_n, \psi_n)\}$ is not effective.*

In practice, this result is useful as many applications of solving linear systems involve periodic sequences in finite dimensions. Necessary and sufficient conditions to fulfill (UB) in an infinite-dimensional Banach Space remain open.

CHAPTER 2. REVIEW OF LITERATURE

2.1 Frame Theory

Frames provide a useful alternative to bases in many contexts. Moreover, they have been broadly studied, providing a convenient and large body of established knowledge. Their nontrivial role in the convergence of the various Kaczmarz algorithms makes them salient to this thesis. In this section we provide a brief overview of the aspects of frame theory relevant to and necessary for this discussion.

Definition 3. A sequence of vectors $\{f_n\}$ in a Hilbert space $\mathcal{H}$ is a *frame* if there exist positive constants A and B such that

$$A\|x\|^2 \leq \sum_{n=0}^{\infty} |\langle x, f_n\rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{H}. \tag{2.1}$$

A frame is *tight* if $A = B$ and *Parseval* if $A = B = 1$. A frame is a *Riesz basis* if it ceases to be a frame if any of its elements are removed. The sequence $\{f_n\}$ is *Bessel* if the positive constant B exists in Equation (2.1).

When working with frames, we often speak of the *analysis operator* and the *synthesis operator*. The *analysis operator* associated with the sequence $\{f_n\}$ is the map $\theta_f : \mathcal{H} \to c(\mathbb{N}_0)$ given by

$$\theta_f(x) = \{\langle x, f_n\rangle\}_{n=0}^{\infty}, \tag{2.2}$$

where $c(\mathbb{N}_0)$ is the space of sequences on $\mathbb{N}_0$. When $\{f_n\}$ is Bessel, the operator θ_f is bounded from $\mathcal{H}$ into $\ell^2(\mathbb{N}_0)$. However, this condition is not always assumed. Let $l(\mathbb{N}_0)$ denote the subspace of sequences $\{c_n\}_{n=0}^{\infty} \in c(\mathbb{N}_0)$ for which $\sum_{n=0}^{\infty} c_n f_n$ converges. The *synthesis operator* associated with $\{f_n\}_{n=0}^{\infty}$ is the map $\theta_f^* : l(\mathbb{N}_0) \to \mathcal{H}$ given by

$$\theta_f^*(\{c_n\}_{n=0}^{\infty}) = \sum_{n=0}^{\infty} c_n f_n. \tag{2.3}$$

When $\{f_n\}$ is Bessel, we may replace $l(\mathbb{N}_0)$ by $\ell^2(\mathbb{N}_0)$ and then, as the notation suggests, the synthesis operator is the Hilbert space adjoint of the analysis operator. We define the *frame operator* for a Bessel sequence $\{f_n\}$ by $S_f = \theta_f^* \theta_f : \mathcal{H} \to \mathcal{H}$, where

$$S_f x = \sum_{n=0}^{\infty} \langle x, f_n \rangle f_n. \tag{2.4}$$

When the collection $\{f_n\}$ is clear from the context, we will omit the subscripts on S, θ, and θ^*.

Lemma 1. *If $\{f_n\}$ is a frame, then the frame operator $S : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ is bounded, self-adjoint, positive, and invertible.*

Proof. S is bounded as it is the product of two bounded operators. As $S^* = (\theta^*\theta)^* = \theta^*\theta = S$, the frame operator is clearly self-adjoint. Note that

$$\begin{aligned} \langle Sx, x \rangle &= \left\langle \sum_{n=0}^{\infty} \langle x, f_n \rangle f_n, x \right\rangle \\ &= \sum_{n=0}^{\infty} \langle x, f_n \rangle \langle f_n, x \rangle . \end{aligned}$$

Using this, Equation (2.1) can be rewritten as

$$\langle Ax, x \rangle \leq \langle Sx, x \rangle \leq \langle Bx, x \rangle \tag{2.5}$$

from which we infer that S is positive, as $\langle Ax, x \rangle = A\langle x, x \rangle \geq 0$ for all $x \in \mathcal{H}$. Because S is self-adjoint, $\langle Sx, x \rangle$ is real for every x, and we may write equation (2.5) as $AI \leq S \leq BI$. Manipulating algebraically, we achieve

$$\begin{aligned} AI \leq & S \leq BI \\ -BI \leq & -S \leq -AI \\ -I \leq & -\frac{S}{B} \leq -\frac{A}{B} I \\ 0 \leq & I - \frac{S}{B} \leq I - \frac{A}{B} I \\ 0 \leq & I - \frac{S}{B} \leq \left(\frac{B-A}{B} \right) I. \end{aligned}$$

We use this to show

$$\begin{aligned}\left\|I-\frac{S}{B}\right\| &= \sup_{\|x\|=1}\left|\left\langle\left(I-\frac{S}{B}\right)x,x\right\rangle\right| \\ &\leq \sup_{\|x\|=1}\left|\left\langle\left(\frac{B-A}{B}\right)x,x\right\rangle\right| \\ &\leq \left(\frac{B-A}{B}\right)\sup_{\|x\|=1}|\langle x,x\rangle| \\ &\leq \frac{B-A}{B} \\ &< 1.\end{aligned}$$

By Neumann's Theorem, we know that $\frac{1}{B}S$, and thus S, is invertible. □

When $\{f_n\}$ is a Parseval frame, we can write the frame inequality in Definition 2.1 as

$$\langle Sx,x\rangle = \langle x,x\rangle.$$

Because this is true for all $x \in \mathcal{H}$, we conclude that $S = I$ if and only if $\{f_n\}$ is a Parseval frame. Stated differently, if $\{f_n\}$ is a Parseval frame, then we have the following reconstruction property:

$$x = \sum_{n=0}^{\infty}\langle x,f_n\rangle f_n. \tag{2.6}$$

The reconstruction in (2.6) is often referred to as a resolution of the identity; we will use this nomenclature periodically. If $\{f_n\}$ is a Parseval frame, it forms a resolution of the identity with itself. Once might ask if the same resolution could be achieved with two different sequences, $\{f_n\}$ and $\{g_n\}$, where $\theta_f^*\theta_g = I$. This is exactly the phenomenon demonstrated by dual frames.

Definition 4. Suppose $\{f_n\}$ and $\{g_n\}$ are frames in $\mathcal{H}$. If for every $x \in \mathcal{H}$,

$$x = \sum_{n=0}^{\infty}\langle x,f_n\rangle g_n, \tag{2.7}$$

then we say that $\{g_n\}$ is a *dual frame* for $\{f_n\}$.

Weaker relationships, with no assumptions about $\{f_n\}$ or $\{g_n\}$ being frames, are given as follows.

Definition 5. Suppose $\{f_n\}, \{g_n\} \subseteq \mathcal{H}$. If for every $x \in \mathcal{H}$,

$$x = \sum_{n=0}^{\infty} \langle x, f_n \rangle g_n,$$

then we say that $\{g_n\}$ and $\{f_n\}$ form a *resolution of the identity.* Alternatively, we say that $\{g_n\}$ and $\{f_n\}$ are *dual.*

Definition 6. Suppose $\{f_n\}, \{g_n\} \subseteq \mathcal{H}$. If for every $x, y \in \mathcal{H}$, we have

$$\langle x, y \rangle = \sum_{n=0}^{\infty} \langle x, f_n \rangle \langle g_n, y \rangle,$$

then we say that $\{g_n\}$ and $\{f_n\}$ form a *weak resolution of the identity.* Alternatively, we say that $\{g_n\}$ and $\{f_n\}$ are *weakly dual.*

Although they may have more, all frames have at least one dual frame, namely the canonical dual frame. For a frame $\{f_n\}$, the canonical dual frame is given by $\{S^{-1} f_n\}$, where S is the frame operator for $\{f_n\}$.

Lemma 2. *Let $\{f_n\}$ be a frame with frame operator S and frame bounds A and B. Then $\{S^{-1} f_n\}$ is a frame and $\{S^{-1} f_n\}$ and $\{f_n\}$ are dual frames.*

Proof. Let $x \in \mathcal{H}$. Because $\{f_n\}$ is a frame, we have that $\|S^{-1}x\|^2 \leq \frac{1}{A} \sum_{n=0}^{\infty} |\langle S^{-1}x, f_n \rangle|^2$. Consider:

$$\begin{aligned}
\|x\|^2 &= \|SS^{-1}x\|^2 \\
&\leq \|S\|^2 \|S^{-1}x\|^2 \\
&\leq \|S\|^2 \frac{1}{A} \sum_{n=0}^{\infty} |\langle S^{-1}x, f_n \rangle|^2 \\
&\leq \frac{B}{A} \|S\|^2 \|S^{-1}x\|^2 \\
&\leq \frac{B}{A} \|S\|^2 \|S^{-1}\|^2 \|x\|^2.
\end{aligned}$$

Multiplying appropriate parts of the inequality by $\frac{A}{\|S\|^2}$, we obtain

$$\frac{A}{\|S\|^2}\|x\|^2 \leq \sum_{n=0}^{\infty} |\langle x, S^{-1}f_n\rangle|^2 \leq B\|S^{-1}\|^2\|x\|^2 \tag{2.8}$$

and conclude that $\{S^{-1}f_n\}$ is a frame.

Furthermore, note that

$$x = SS^{-1}x = \sum_{n=0}^{\infty}\langle S^{-1}x, f_n\rangle f_n = \sum_{n=0}^{\infty}\langle x, S^{-1}f_n\rangle f_n$$
$$x = S^{-1}Sx = S^{-1}\left(\sum_{n=0}^{\infty}\langle x, f_n\rangle f_n\right) = \sum_{n=0}^{\infty}\langle x, f_n\rangle S^{-1}f_n$$

from which we infer that $\{f_n\}$ and $\{S^{-1}f_n\}$ are dual frames. □

As shown by Casazza in [Cas00], the canonical dual frame is the unique frame which is related to the original frame by an invertible operator.

Lemma 3 (Casazza)**.** *If $\{f_n\}$ is a frame and $\{g_n\}$ is a dual frame, then $\{g_n\}$ is the canonical dual frame if and only if there exists an invertible operator T such that $Tf_n = g_n$ for all $n \in \mathbb{N}_0$ In this case, $T = S^{-1}$, where S is the frame operator of $\{f_n\}$.*

Proof. The forward direction of the lemma follows by definition. For the backward direction, suppose that $\{f_n\}$ and $\{g_n\}$ are dual frames and there exists an invertible operator T such that $g_n = Tf_n$ for all $n \in \mathbb{N}_0$. Because $\{f_n\}$ and $\{g_n\}$ are dual frames, for any $x \in \mathcal{H}$, we have

$$x = \sum_{n=0}^{\infty}\langle x, g_n\rangle f_n = \sum_{n=0}^{\infty}\langle x, Tf_n\rangle f_n = \sum_{n=0}^{\infty}\langle T^*x, f_n\rangle f_n = S(T^*x) \tag{2.9}$$

where S is the frame operator of $\{f_n\}$. As S is invertible and self-adjoint, so is S^{-1}, and (2.9) implies $T^*x = S^{-*}x$ for all $x \in \mathcal{H}$. It follows that for all $x, y \in \mathcal{H}$,

$$\langle T^*x, y\rangle = \langle S^{-*}x, y\rangle \Rightarrow \langle x, Ty\rangle = \langle x, S^{-1}y\rangle \Rightarrow Ty = S^{-1}y \text{ for all } y \in \mathcal{H} \Rightarrow T = S^{-1}.$$

We conclude that $\{f_n\}$ and $\{g_n\}$ are canonical dual frames by definition. □

Note that a Parseval frame is its own canonical dual. Furthermore, notice that for frames $\{f_n\}$ and $\{g_n\}$, the dual frame relationship is equivalent to $\theta_g^*\theta_f = I$. As previously mentioned, although $\{f_n\}$ will always have the canonical dual frame, it could have other dual frames as well. Indeed, $\{f_n\}$ will have as many dual frames as θ_f has left inverses. By taking the adjoint of both sides of $\theta_g^*\theta_f = I$, we attain $\theta_f^*\theta_g = I$, which tells us that the dual frame condition is always symmetric. That is, if $\{g_n\}$ is a dual frame to $\{f_n\}$, then $\{f_n\}$ is a dual frame to $\{g_n\}$.

If the frame $\{f_n\}$ is a Riesz basis, then it has only the canonical dual frame $\{g_n\}$, which will also be a Riesz basis. Moreover, it can be shown that $\langle f_m, g_n\rangle = \delta_{m,n}$ for all $m, n \in \mathbb{N}_0$, leading us to refer to $\{f_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$ as *biorthogonal Riesz bases*. To see this, we consider an alternative definition of a Riesz basis as the image of an orthonormal basis $\{e_n\}$ under some bounded, bijective operator, U [Chr03]. Let $\{f_n\}$ be a Riesz basis where $f_n = Ue_n$ and $g_n = U^{-*}e_n$ for all $n \in \mathbb{N}_0$. Choose $x \in \mathcal{H}$ and derive

$$\begin{aligned}
x &= UU^{-1}x \\
&= U\left(\sum_{n=0}^{\infty}\langle U^{-1}x, e_n\rangle e_n\right) \\
&= U\left(\sum_{n=0}^{\infty}\langle x, U^{-*}e_n\rangle e_n\right) \\
&= \sum_{n=0}^{\infty}\langle x, U^{-*}e_n\rangle Ue_n \\
&= \sum_{n=0}^{\infty}\langle x, g_n\rangle f_n
\end{aligned}$$

from which we see that $\{g_n\}$ and $\{f_n\}$ are dual. Since U^{-*} is bounded and bijective, $\{g_n\}$ is a Riesz basis by definition and $\{g_n\}$ and $\{f_n\}$ are dual frames. Because

$$\langle f_n, g_k\rangle = \langle Ue_n, U^{-*}e_k\rangle = \langle U^{-1}Ue_n, e_k\rangle = \langle e_n, e_k\rangle = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases},$$

$\{f_n\}$ and $\{g_n\}$ are biorthogonal.

Another common tool from frame theory will be the Gramian matrix. For a sequence $\{f_n\}$, the Gramian is defined by

$$\theta_f \theta_f^* = (\langle f_i, f_j \rangle)_{i,j \in \mathbb{N}_0} .$$

For our purposes, we will also need the mixed Gramian matrix for two sequences, $\{f_n\}$ and $\{g_n\}$, defined by

$$\theta_f \theta_g^* = (\langle g_i, f_j \rangle)_{i,j \in \mathbb{N}_0} .$$

Note that these matrices may be finite or infinite-dimensional, depending upon the involved sequences. Thus, in general, they do not define a bounded operator on $\ell^2(\mathbb{N}_0)$. We still wish, however, to capture a notion of positivity and will say that an infinite matrix T is positive if every principal submatrix is positive. That is, if for all $n \in \mathbb{N}_0$, the operator

$$T_n = \begin{pmatrix} t_{00} & t_{01} & \dots & t_{0n} & 0 & \dots \\ t_{10} & t_{11} & \dots & t_{1n} & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ t_{n0} & t_{n1} & \dots & t_{nn} & 0 & \\ 0 & 0 & \dots & 0 & 0 & \ddots \\ \vdots & \vdots & & & \ddots & \ddots \end{pmatrix}$$

satisfies $\langle T_n u, u \rangle \geq 0$ for every $u \in \ell^2(\mathbb{N}_0)$. Under this definition the (standard) Gramian matrix will always be positive.

2.2 Characterization of Effective Sequences

Dual frames afford vector reconstruction from inner product measurements with a particular sequence. This is useful in the case of digital signal processing, as well as solving linear systems. As previously mentioned, the redundancy of frames also offers an advantage over orthonormal bases. Another useful method of vector recovery is given by the class of sequences which cause the Kaczmarz algorithm to converge, namely, the effective sequences in Definition 1. The iterative nature of the Kaczmarz algorithm provides effective sequences

a distinct computational advantage over frames. In general, however, they have been far less studied. Nonetheless, each of the three authors (or sets of authors) Kaczmarz, Kwapień and Mycielski (K-M), and Haller and Szwarc (H-S) worked towards the categorization of effective sequences. All approached from a different angle: Kaczmarz worked in finite dimensions, K-M used an auxiliary sequence, borrowing from frame theory, and H-S turned to infinite matrices. Throughout this thesis, we will take inspiration from each of these approaches. As we present the results of these authors, we ask the reader to attend to how conditions (UB) and (DE) are being satisfied in each, even if they are not explicitly mentioned.

2.2.1 The Kaczmarz Theorem

We will call a sequence $\{e_n\}$ *k-periodic* if there exists some $k \in \mathbb{N}_0$ such that $e_n = e_{n+k}$ for all $n \in \mathbb{N}_0$. In [Kac37], Kaczmarz proved the following regarding periodic sequences:

Theorem (Kaczmarz)**.** *Suppose $\{e_n\}$ is a k-periodic sequence of unit vectors in a finite-dimensional Hilbert space $\mathcal{H}_N$. If $\{e_0, \cdots, e_{k-1}\}$ spans $\mathcal{H}_N$, then $\{e_n\}$ is effective.*

Because we are working within a Hilbert space, condition (UB) is fulfilled automatically due to the orthogonal projections in the algorithm. Using a contradiction argument, one can show that the $\lim_{n\to\infty} (P_{k-1}P_{k-2}\cdots P_0)^n = 0$ pointwise on a dense set.

2.2.2 The Kwapień and Mycielski Theorem

Several decades later, Kwapień and Mycielski extended the theory into infinite dimensions. Specifically, they provided a characterization based on a sequence that would be the natural candidate for a "dual" to $\{e_n\}$, called the auxiliary sequence [KM01]. This sequence and its variations will play a central role in this thesis.

Definition 7. The *auxiliary sequence*, $\{h_n\}$ to a sequence $\{e_n\} \subseteq \mathcal{H}$ is given by

$$\begin{aligned} h_0 &= e_0 \\ h_n &= e_n - \sum_{k=0}^{n-1} \langle e_n, h_k \rangle e_k. \end{aligned} \tag{2.10}$$

We first notice that if $\{x_n\}$ is the sequence of approximations given in (1.1), then

$$x_n = \sum_{k=0}^{n} \langle x, h_k \rangle e_k. \tag{2.11}$$

This can be shown by induction. By definition, $x_0 = \langle x, e_0 \rangle e_0$. Note that

$$\begin{aligned} x_1 &= x_0 + \langle x - x_0, e_1 \rangle e_1 \\ &= \langle x, e_0 \rangle e_0 + \langle x - \langle x, e_0 \rangle e_0, e_1 \rangle e_1 \\ &= \langle x, e_0 \rangle e_0 + \left(\langle x, e_1 \rangle - \langle x, \langle e_1, e_0 \rangle e_0 \rangle \right) e_1 \\ &= \langle x, e_0 \rangle e_0 + \langle x, e_1 - \langle e_1, e_0 \rangle e_0 \rangle e_1 \\ &= \sum_{k=0}^{1} \langle x, h_k \rangle e_k \end{aligned}$$

where the last step is because $h_0 = e_0$ and $h_1 = e_1 - \langle e_1, h_0 \rangle e_0 = e_1 - \langle e_1, e_0 \rangle e_0$.

Assume there is some $N > 1$ such that (2.11) holds for all $n \leq N$. Derive

$$\begin{aligned} x_{N+1} &= x_N + \langle x - x_N, e_{N+1} \rangle e_{N+1} \\ &= \sum_{k=0}^{N} \langle x, h_k \rangle e_k + \left\langle x - \sum_{k=0}^{N} \langle x, h_k \rangle e_k, e_{N+1} \right\rangle e_{N+1} \\ &= \sum_{k=0}^{N} \langle x, h_k \rangle e_k + \left(\langle x, e_{N+1} \rangle - \left\langle \sum_{k=0}^{N} \langle x, h_k \rangle e_k, e_{N+1} \right\rangle \right) e_{N+1} \\ &= \sum_{k=0}^{N} \langle x, h_k \rangle e_k + \left(\langle x, e_{N+1} \rangle - \sum_{k=0}^{N} \langle x, h_k \rangle \langle e_k, e_{N+1} \rangle \right) e_{N+1} \\ &= \sum_{k=0}^{N} \langle x, h_k \rangle e_k + \left(\langle x, e_{N+1} \rangle - \langle x, \sum_{k=0}^{N} \langle e_{N+1}, e_k \rangle h_k \rangle \right) e_{N+1} \\ &= \sum_{k=0}^{N} \langle x, h_k \rangle e_k + \left\langle x, e_{N+1} - \sum_{k=0}^{N} \langle e_{N+1}, e_k \rangle h_k \right\rangle e_{N+1} \\ &= \sum_{k=0}^{N+1} \langle x, h_k \rangle e_k, \end{aligned}$$

where the last step is because $h_{N+1} = e_{N+1} - \sum_{k=0}^{N} \langle e_{N+1}, e_k \rangle h_k$. Equation (2.11) holds for all $n \in N_0$ by induction. It is clear by the definition of an effective sequence that $x = \sum_{n=0}^{\infty} \langle x, h_k \rangle e_k$ if and only if $\|x_n - x\| \to 0$ if and only if $\{e_n\}$ is effective. This representation of x_n will provide us with another approach to categorizing effective sequences.

Theorem (Kwapień and Mycielski)**.** *Suppose that $\{e_n\}$ is a linearly dense sequence of unit vectors in a Hilbert space $\mathcal{H}$. $\{e_n\}$ is effective if any only if the auxiliary sequence $\{h_n\}$ defined by (2.10) is a Parseval frame.*

Proof. Let $x \in \mathcal{H}$ and define $P_n x = x - \langle x, e_n \rangle e_n$. Condition (UB) is automatically satisfied by virtue of the projections P_n being orthogonal.

By (2.11), we know that

$$x - x_{n-1} = x - x_n + \langle x, h_n \rangle e_n. \tag{2.12}$$

Because $x - x_n$ is orthogonal to e_n, we can use (2.12) to write

$$\|x\|^2 - \|x - x_0\|^2 = |\langle x, h_0 \rangle|^2$$
$$\|x - x_{n-1}\|^2 - \|x - x_n\|^2 = |\langle x, h_n \rangle|^2, n \geq 1.$$

Summing up the equations with respect to n, we obtain

$$\|x\|^2 - \lim_{n \to \infty} \|x_n - x\|^2 = \sum_{n=0}^{\infty} |\langle x, h_n \rangle|^2. \tag{2.13}$$

We conclude that $\{e_n\}$ is effective if and only if $\{h_n\}$ is a Parseval frame. □

Notice that (2.13) is actually a stronger result than is necessary, as we need only show $\{e_n\}$ effective on the space of its finite linear combinations to attain (DE).

There is a smaller class of sequences which has proved interesting in the context of the Kaczmarz algorithm, providing connections with relevant topics in measure theory.

Definition 8. A sequence $\{e_n\}$ is *stationary* if $\langle e_{k+m}, e_{\ell+m} \rangle = \langle e_k, e_\ell \rangle$ for any $k, \ell, m \in \mathbb{N}$.

Theorem (Kwapień and Mycielski)**.** *Let $\{e_n\}$ be a stationary sequence of unit vectors which is linearly dense in a Hilbert space $\mathcal{H}$. $\{e_n\}$ is effective if and only if its spectral measure either coincides with the normalized Lebesgue measure or is singular with respect to Lebesgue measure.*

We omit the proof of the previous theorem, but encourage the reader to reference [KM01] for more details, and the work of Herr in [Her16] for a more step-by-step progression. The proof involves significant work in complex analysis and the Hardy space on the unit disk. As a result of this theorem, in [Her16], Herr provided a sufficient condition for a sequence $\{e_n\}$ of exponentials to be effective in $L^2(\mu)$. Namely, he showed that if μ is a singular Borel probability measure on $[0,1)$, $\{e_n\}$ will be effective. This yields a Fourier series with Fourier coefficients for any function $f \in L^2(\mu)$.

2.2.3 The Haller and Szwarc Theorem

In frame theory, the Gramian matrix often provides insight into the behavior of a sequence. in [HS05] Haller and Szwarc constructed a lower triangular version of the Gram matrix for the sequence $\{e_n\}$ and worked with its algebraic inverse. After their pattern, we define

$$I + N = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \langle e_1, e_0\rangle & 1 & 0 & 0 & \cdots \\ \langle e_2, e_0\rangle & \langle e_2, e_1\rangle & 1 & 0 & \cdots \\ \langle e_3, e_0\rangle & \langle e_3, e_1\rangle & \langle e_3, e_2\rangle & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2.14}$$

and let $I + V$ be the algebraic inverse of $I + N$. (Under the assumption that the sequence $\{e_n\}$ consists of unit vectors, each principal submatrix is clearly invertible).

Theorem (Haller and Swzarc)**.** *Let $\{e_n\}$ be a linearly dense sequence of unit vectors in a Hilbert space $\mathcal{H}$. Then $\{e_n\}$ is effective if and only if V is a partial isometry.*

In the proof of their theorem, Haller and Szwarc directly and indirectly show that a variety of properties are equivalent. We list these properties, as they foreshadow our discussion of the Banach space setting in Chapter 4.

a. $\{e_n\}$ is effective.

b. $\{h_n\}$ and $\{e_n\}$ are dual.

c. $\langle N^*(VN^* + N^* + I)\delta_j, \delta_i\rangle = 0$ for all $i, j \in \mathbb{N}_0$.

d. $\{h_n\}$ is a Parseval frame (I.e., $\{h_n\}$ is dual to itself).

e. $\langle (NV^*VN^* - NN^*)\delta_j, \delta_i\rangle = 0$ for all $i, j \in \mathbb{N}_0$.

f. V is a partial isometry (V^*V is a projection).

In a more general context of a Banach space, these conditions are not necessarily equivalent. Specifically, none of properties (d), (e), or (f), are equivalent to (a), (b), or (c). There are two features unique to the standard Kaczmarz algorithm within a Hilbert Space which provide exactly the additional machinery needed to tie these properties together. First, the involved projections are orthogonal, naturally providing the (UB) bound. Second, the matrix $I + N^* + N$ is simply the Gramian matrix of $\{e_n\}$, which is always positive. In [HS05], Haller and Szwarc show that this positivity implies that the matrix V is a contraction, which is a powerful tool in the proof.

2.2.4 Generating Effective Sequences from Bessel Sequences

Theorem (Szwarc)**.** *For any normalized Bessel sequence $\{h_n\}$, there exists an effective sequence $\{e_n\}$ of unit vectors with auxiliary sequence $\{h_n\}$.*

In a later work, Szwarc went on to provide a method for generating effective sequences [Szw07]. Specifically, he showed that any normalized Bessel sequence (a Bessel sequence $\{h_n\}$ is said to be normalized if $\|h_0\| = 1$), can be used to generate an effective sequence.

Phrased differently, he showed that any normalized Bessel sequence can be obtained through the Kaczmarz algorithm. In [Szw07], Szwarc provides two proofs for this result. The first speaks to existence, and the second, more involved, proof provides an explicit construction for the sequence $\{e_n\}$.

2.3 The Kwapień and Mycielski Theorems in Banach Spaces

Kwapień and Mycielski also sought to extend the concept of an effective sequence to a Banach space setting, devising the (UB) and (DE) conditions. We restate their result here and also provide a proof.

Proposition (Kwapień and Mycielski)**.** *Assume that $\{\psi_n\}$ is linearly dense in a Banach space X and that $\{\phi_n\}$ is linearly dense in X^*. Define $P_n : X \to X$ such that $P_n(x) = x - \phi_n(x)\psi_n$. The sequence $\{(\phi_n, \psi_n)\}$ is effective if and only if there exists a constant $C > 0$ such that*

$$\|P_n P_{n-1} \cdots P_0\| \leq C \text{ for all } n \in \mathbb{N}. \tag{UB}$$

$$\lim_{n\to\infty} P_n P_{n-1} \cdots P_0(x) = 0 \text{ for all } x \text{ in a dense subset of } X. \tag{DE}$$

Proof. Assume that $\{(\phi_n, \psi_n)\}$ is effective. Note that

$$x - x_n = P_n P_{n-1} \cdots P_0 x \tag{2.15}$$

for all $x \in X$, where the x_n are as in (1.2). Because $\{(\phi_n, \psi_n)\}$ is effective, we know that $\|x - x_n\| \to 0$ for all $x \in X$ from which we infer that

$$\|P_n P_{n-1} \cdots P_0 x - 0\| \to 0 \text{ for all } x \in X \tag{2.16}$$

and thus (DE) holds.

Choose $\varepsilon > 0$. By (2.16), there is some $N \in \mathbb{N}_0$ such that for all $n > N$, $\|P_n P_{n-1} \cdots P_0 x - 0\| < \varepsilon$. Let

$$B = \max\{\|P_0 x\|, \|P_1 P_0 x\|, \|P_2 P_1 P_0 x\|, \ldots, \|P_{N-1} P_{N-2} \cdots P_0 x\|, \varepsilon\}.$$

From this, we conclude that for any fixed $x \in X$

$$\sup_n \|P_n P_{n-1} \cdots P_0 x\| \leq B < \infty. \tag{2.17}$$

By Banach Steinhaus, we conclude that that

$$\sup_{n, \|x\|=1} \|P_n P_{n-1} \cdots P_0 x\| = \sup_n \|P_n P_{n-1} \cdots P_0\| < \infty.$$

I.e., there exists some $C > 0$ such that $\|P_n P_{n-1} \cdots P_0\| \leq C$ and (UB) holds.

Conversely, suppose (UB) and (DE), and let $D \subseteq X$ be the dense subset on which (DE) holds. Choose $x \in X$ and a sequence $\{x_\ell\} \subseteq D$ such that $\|x_\ell - x\| \to 0$.

Let $\varepsilon > 0$, and choose $\ell \in \mathbb{N}$ such that $\|x_\ell - x\| < \frac{\varepsilon}{2C}$. As $x_\ell \in D$, by (DE) there exists $N \in \mathbb{N}$ such that for all $n > N$, $\|P_n P_{n-1} \cdots P_0 x_\ell\| < \frac{\varepsilon}{2}$. Let $n > N$. Using the triangle inequality and the fact that $\|P_n P_{n-1} \cdots P_0\| \leq C$ for all $n \in \mathbb{N}$, we derive

$$\begin{aligned}
\|x - x_n\| &= \|P_n P_{n-1} \cdots P_0 x\| \\
&= \|P_n P_{n-1} \cdots P_0 (x - x_\ell + x_\ell)\| \\
&= \|P_n P_{n-1} \cdots P_0 (x - x_\ell) + P_n P_{n-1} \cdots P_0 x_\ell\| \\
&\leq \|P_n P_{n-1} \cdots P_0 (x - x_\ell)\| + \|P_n P_{n-1} \cdots P_0 x_\ell\| \\
&< C\|(x - x_\ell)\| + \frac{\varepsilon}{2} \\
&< C \cdot \frac{\varepsilon}{2C} + \frac{\varepsilon}{2} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

We conclude that $\{(\phi_n, \psi_n)\}$ is effective. □

Because a Hilbert space is a Banach space, and the effective sequence case is simply the special case of a pair where $\phi_n = \psi_n$, we see that the (UB) and (DE) condition approach is valid for the standard and dualized Kaczmarz algorithms in both a Hilbert and Banach space. We note that, using the Banach Steinhaus Theorem, the (UB) condition can be weakened slightly to a pointwise bound (PB). Formally,

Corollary 1. *Assume that $\{\psi_n\}$ is linearly dense in a Banach space X and that $\{\phi_n\}$ is linearly dense in X^*. Define $P_n : X \to X$ such that $P_n(x) = x - \phi_n(x)\psi_n$. The sequence $\{(\phi_n, \psi_n)\}$ is effective if and only if for every $x \in X$, there exists a constant $C_x > 0$ such that*

$$\|P_n P_{n-1} \cdots P_0 x\| \leq C_x \text{ for all } n \in \mathbb{N}. \tag{PB}$$

and

$$\lim_{n\to\infty} P_n P_{n-1} \cdots P_0(x) = 0 \text{ for all } x \text{ in a dense subset of } X. \tag{DE}$$

We will present a final corollary to the Kwapień-Mycielski proposition which will more closely align with the results we achieve in Chapter 4. Specifically, it relates to the relaxed notion of weak effectivity.

Corollary 2. *Assume that $\{\psi_n\}$ is linearly dense in a Banach space X and that $\{\phi_n\}$ is linearly dense in X^*. Define $P_n : X \to X$ such that $P_n(x) = x - \phi_n(x)\psi_n$. The sequence $\{(\phi_n, \psi_n)\}$ is weakly effective if and only if for every $x \in X$, there exists a constant $C_x > 0$ such that*

$$\|P_n P_{n-1} \cdots P_0 x\| \leq C_x \text{ for all } n \in \mathbb{N} \tag{PB}$$

and

$$\lim_{n\to\infty} u(P_n P_{n-1} \cdots P_0 x) = 0 \text{ for all } x \text{ in a dense subset of } X \text{ and } u \in X^*. \tag{WDE}$$

Throughout the remainder of the thesis we will often speak of the weakened conditions of (PB) and (WDE) rather than the former (UB) and (DE).

In [KM01], Kwapień and Mycielski show that, under the appropriate hypotheses, an effective sequence in a Hilbert space can be used to generate a sequence which is effective in a dense subset of a Banach space.

Theorem (Kwapień and Mycielski)**.** *Let $i : \mathcal{H} \to X$ be a bounded linear transformation from a Hilbert space into a Banach space with a dense image. Assume that $\{e_n\}$ is an effective sequence in $\mathcal{H}$ and $\{(\phi_n, \psi_n)\}$ is a sequence in $X^* \times X$ such that $i(e_n) = \psi_n$ and $i^*(\phi_n) = e_n$ ($i^* : X^* \to \mathcal{H}$ is the adjoint operator). Then in the Banach space X, the sequence $\{(\phi_n, \psi_n)\}$ is effective in $i(\mathcal{H})$.*

Later in [KM01], Kwapień and Mycielski improve this result by using a sequence of unit vectors (not necessarily effective) in a Hilbert space to generate a pair of sequences $\{(\phi_n, \psi_n)\}$, given the existence of a map $i : \mathcal{H} \to X$ with particular properties. Defining a nonlinear version of the Kaczmarz approximations, $\{x_n\}$, their algorithm reaches within ε of any $x \in X$. They also provide an upper bound on the number of iterations needed before the approximations will remain unchanged. They achieve this by defining another inductive process

$$x_0 = 0$$

$$g_0^\varepsilon = 0$$

$$x_n^\varepsilon = x_{n-1}^\varepsilon + g_n^\varepsilon(x) e_n$$

where $g_n^\varepsilon = \phi_n(x - x_{n-1}^\varepsilon)$ if $|\phi_n(x - x_{n-1}^\varepsilon)| > \varepsilon$ and otherwise $g_n^\varepsilon(x) = 0$. The nonlinear approximations K_n^ε are defined by $K_n^\varepsilon(x) = x_n^\varepsilon = \sum_{k=1}^n g_k^\varepsilon(x) e_k$.

Theorem (Kwapień and Mycielski)**.** *Let $\{(\phi_n, \psi_n)\}$ be a sequence in $X^* \times X$ such that $\{\psi_n\}$ is linearly dense in X and $\{\phi_n\}$ is norming (i.e., $\|\phi_n\| \leq 1$ for all n and $\limsup_{n\to\infty} |\phi_n(x)| = \|x\|$ for each $x \in X$). Assume that for a Hilbert space $\mathcal{H}$, a sequence $\{e_n\}$ of unit vectors in $\mathcal{H}$, and a linear operator $i : \mathcal{H} \to X$ satisfy $i(e_n) = \psi_n$ and $i^*(\phi_n) = e_n$ for all $n \in \mathbb{N}_0$. Then for each $\varepsilon > 0$ and $x \in X$, $\|x - K_n^\varepsilon(x)\| \leq \varepsilon$ for n sufficiently large and*

$$\#\{n : K_n^\varepsilon(x) \neq K_{n-1}^\varepsilon(x)\} \leq \inf\{2\varepsilon^{-2}\|v\|_{\mathcal{H}}^2 : v \in \mathcal{H}, \|i(v) - x\| \leq \frac{\varepsilon}{4}\}.$$

In Chapter 4, we will present our progress on the Kaczmarz algorithm within a Banach space, proffering nontrivial improvements on both of the previous theorems. Although our

result will only hold on a dense subset of a Banach space, we provide a concrete matrix condition for the effectivity of a pair $\{(\phi_n, \psi_n)\}$, without appealing to any corresponding Hilbert space or map i.

2.4 Almost Effective Sequences

In [CT13], Czaja and Tanis sought to relax the concept of an effective sequence within a Hilbert space, introducing the concept of an almost effective sequence.

Definition 9. A sequence $\{e_n\}$ in $\mathcal{H}$ is *almost effective* if there exists some $0 \leq B < 1$ such that the sequence $\{x_n\}$ in equation (1) satisfies

$$\lim_{n\to\infty} \|x_n - x\|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{H}. \tag{2.18}$$

By Kwapień and Mycielski, we know that the auxiliary sequence of an effective sequence is a Parseval frame. Czaja and Tanis highlight that the class of almost effective sequences is broader than that of effective sequences through the following result.

Theorem (Czaja and Tanis)**.** *Let $0 < A \leq 1$ and $\{e_n\}$ be a linearly dense sequence of unit vectors in $\mathcal{H}$. $\{e_n\}$ is almost effective with bound $0 \leq 1 - A$ if and only if $\{h_n\}$ defined by (2.10) is a frame with bounds A and 1.*

The proof of this result follows immediately from equation (2.13), and provides another succinct connection between the Kaczmarz algorithm and frame theory. Revisiting our previous discussion of the (UB) and (DE) conditions, we notice that, as we still have $P_n x = x - \langle x, e_n\rangle e_n$, condition (UB) is once again satisfied. Although (2.18) holds on the entire space, it does not give the pointwise convergence needed for (DE).

CHAPTER 3. EFFECTIVE PAIRS IN HILBERT SPACE

We begin our exploration of the dualized algorithm by limiting ourselves to the context of a Hilbert space. By examining the behavior of effective pairs in a Hilbert space, we gain the knowledge and tools needed to bridge the gap between the Hilbert and Banach space contexts. To this end, we define the algorithm, derive a variety of properties, present informative examples, and provide a limited characterization of effective pairs. Indulging in a brief diversion, we also use the concept of an effective pair to improve the convergence properties of an almost effective sequence.

3.1 Dual Algorithm in Hilbert Space

While effective sequences are useful in vector recovery, they need not retain their effectivity when subject to perturbation. This was shown by Czaja and Tanis in [CT13] when they proved that a Riesz basis which is not an orthonormal basis cannot be effective (see Remark 5). The counterexample then follows directly from a classic result of Paley and Wiener (see [PW34]), namely that a sufficiently small perturbation of an orthonormal basis, which is necessarily effective, may produce a Riesz basis which is not an orthonormal basis—and hence no longer effective. With the intention of obtaining a more stable structure as well as providing applications in a Banach space context, we introduce a variation on the Kaczmarz algorithm where two sequences work together to achieve reconstruction, in analogy to dual frames. For the reader's convenience, we restate the definition of the dualized algorithm for a Hilbert space.

Definition 10. Let $\{\phi_n\}$ and $\{\psi_n\}$ be two linearly dense sequences in $\mathcal{H}$ such that $\langle \phi_n, \psi_n \rangle = 1$. Given $x \in \mathcal{H}$, we define the *dual Kaczmarz algorithm* applied to x by

$$\begin{aligned} x_0 &= \langle x, \phi_0 \rangle \psi_0, \\ x_n &= x_{n-1} + \langle x - x_{n-1}, \phi_n \rangle \psi_n, \quad n \geq 1. \end{aligned} \tag{3.1}$$

If $\|x - x_n\| \to 0$ for all $x \in \mathcal{H}$, then we say that $\{\phi_n\}$ and $\{\psi_n\}$ form an *effective pair*.

As will be demonstrated in Observation 2, effectivity need not be preserved when ϕ_n and ψ_n are interchanged in the algorithm. Hence, we will call the first sequence $\{\phi_n\}$ the analysis sequence and the second sequence $\{\psi_n\}$ the synthesis sequence, representing the ordering by $\{(\phi_n, \psi_n)\}$. If both $\{(\phi_n, \psi_n)\}$ and $\{(\psi_n, \phi_n)\}$ are effective pairs, we say that the sequences form a *symmetric effective pair*. We note that this is distinct from the dual frame condition which is always symmetric.

Similar to the effective sequence context, we define auxiliary sequences $\{g_n\}$ and $\{\tilde{g}_n\}$ as candidates to provide a resolution of the identity with the sequences $\{\psi_n\}$ and $\{\phi_n\}$, respectively.

Definition 11. The *auxiliary sequence* for $\{(\phi_n, \psi_n)\}$ is given by

$$\begin{aligned} g_0 &= \phi_0, \\ g_n &= \phi_n - \sum_{k=0}^{n-1} \langle \phi_n, \psi_k \rangle g_k, \quad n \geq 1. \end{aligned} \tag{3.2}$$

Proposition 1.

$$x_n = \sum_{k=0}^{n} \langle x, g_k \rangle \psi_k, \quad n \geq 0. \tag{3.3}$$

Furthermore, if $\{(\phi_n, \psi_n)\}$ is an effective pair, then

$$x = \sum_{k=0}^{\infty} \langle x, g_k \rangle \psi_k. \tag{3.4}$$

Proof. It is clear by the definition of x_0 that equation (3.3) holds for $n = 0$. Using (3.1) and (3.2), we calculate

$$\begin{aligned}
x_1 &= x_0 + \langle x - x_0, \phi_1 \rangle \psi_1 \\
&= \langle x, \phi_0 \rangle \psi_0 + \langle x - \langle x, \phi_0 \rangle \psi_0, \phi_1 \rangle \psi_1 \\
&= \langle x, \phi_0 \rangle \psi_0 + \left(\langle x, \phi_1 \rangle - \langle \langle x, \phi_0 \rangle \psi_0, \phi_1 \rangle \right) \psi_1 \\
&= \langle x, \phi_0 \rangle \psi_0 + \langle x, \phi_1 - \langle \phi_1, \psi_0 \rangle \phi_0 \rangle \psi_1 \\
&= \langle x, g_0 \rangle \psi_o + \langle x, g_1 \rangle \psi_1.
\end{aligned}$$

Suppose that there is some $N \in \mathbb{N}_0$ such that (3.3) holds for all $n \leq N$. Using (3.1) and (3.2), we calculate

$$\begin{aligned}
x_{N+1} &= x_N + \langle x - x_N, \phi_{N+1} \rangle \psi_{N+1} \\
&= \sum_{k=0}^{N} \langle x, g_k \rangle \psi_k + \langle x - \sum_{k=0}^{N} \langle x, g_k \rangle \psi_k, \phi_{N+1} \rangle \psi_{N+1} \\
&= \sum_{k=0}^{N} \langle x, g_k \rangle \psi_k + \left(\langle x, \phi_{N+1} \rangle - \langle \sum_{k=0}^{N} \langle x, g_k \rangle \psi_k, \phi_{N+1} \rangle \right) \psi_{N+1} \\
&= \sum_{k=0}^{N} \langle x, g_k \rangle \psi_k + \left(\langle x, \phi_{N+1} \rangle - \langle x, \sum_{k=0}^{N} \langle \phi_{N+1}, \psi_k \rangle g_k \rangle \right) \psi_{N+1} \\
&= \sum_{k=0}^{N} \langle x, g_k \rangle \psi_k + \left\langle x, \phi_{N+1} - \sum_{k=0}^{N} \langle \phi_{N+1}, \psi_k \rangle g_k \right\rangle \psi_{N+1} \\
&= \sum_{k=0}^{N} \langle x, g_k \rangle \psi_k + \langle x, g_{N+1} \rangle \psi_{N+1} \\
&= \sum_{k=0}^{N+1} \langle x, g_k \rangle \psi_k.
\end{aligned}$$

By induction, (3.3) holds for all $\mathbb{N}_0$. It is clear by Definition 11 that (3.4) holds if and only if $\{(\phi_n, \psi_n)\}$ is an effective pair. □

Definition 12. The *auxiliary sequence* for $\{(\psi_n, \phi_n)\}$ is given by

$$\begin{aligned}
\tilde{g}_0 &= \psi_0, \\
\tilde{g}_n &= \psi_n - \sum_{k=0}^{n-1} \langle \psi_n, \phi_k \rangle \tilde{g}_k, \quad n \geq 1.
\end{aligned} \tag{3.5}$$

Proposition 2.

$$x_n = \sum_{k=0}^{n} \langle x, \tilde{g}_k \rangle \phi_k, \quad n \geq 0. \tag{3.6}$$

Furthermore, if $\{(\psi_n, \phi_n)\}$ *is an effective pair, then*

$$x = \sum_{k=0}^{\infty} \langle x, \tilde{g}_k \rangle \phi_k. \tag{3.7}$$

Proof. This is proven similarly to Proposition 1 with the roles of $\{\phi_n\}$ and $\{\psi_n\}$ reversed. □

3.2 Pair Properties

Remark 1. Our notation g_n, $\tilde{g}_n$ suggests duality in the context of frames. Although this is sometimes the case, we also have examples where $\{(\phi_n, \psi_n)\}$ is an effective pair and one or both of $\{g_n\}$ and $\{\tilde{g}_n\}$ fail to be frames. This is shown in Observation 4.

Remark 2. As an effective sequence forms an effective pair with itself, it is natural to ask whether the two sequences in an effective pair are independently effective. Appealing to Schauder bases which are not Riesz bases, we find many examples for which this is not necessarily the case. See Observation 4 for more details.

Remark 3. There are many more effective pairs than there are effective sequences. Indeed, Corollary 3 will demonstrate that any effective sequence and invertible operator can generate an effective pair.

Remark 4. As previously discussed, the concept of an effective pair translates more naturally to the context of a Banach space than that of an effective sequence. For example, a Schauder basis in a Banach space X and its biorthogonal dual in X^* are an effective pair (Observation 4).

Remark 5. Suppose $\{e_n\}$ is a Riesz basis. Let $\{h_n\}$ be the auxiliary sequence for $\{e_n\}$ as defined in (2.10). Then, the pair $\{e_n\}$ and $\{h_n\}$ form a biorthogonal Riesz basis only when

$\{e_n\}$ is an orthonormal basis since $\langle e_0, e_m\rangle = \langle h_0, e_m\rangle$ and for $1 \le n \le m$,

$$\langle e_n, e_m\rangle = \langle h_n, e_m\rangle + \sum_{k=0}^{n-1} \langle e_n, e_k\rangle\langle h_k, e_m\rangle.$$

Therefore the sequence $\{e_n\}$ is effective if and only if it is an orthonormal basis.

We provide a number of elementary results regarding effective pairs, many motivated from analogous results in frame theory.

Theorem 1 (Aboud, Curl, Harding, Vaughan, Weber)**.** *Let $T \in \mathcal{B}(\mathcal{H})$ be invertible. Then $\{(\phi_n, \psi_n)\}$ is an effective pair if and only if $\{\big(T\phi_n, (T^{-1})^*\psi_n\big)\}$ is an effective pair.*

Proof. Suppose that $\{(\phi_n, \psi_n)\}$ is an effective pair. Let $x \in \mathcal{H}$, and attain the sequence of approximations $\{y_n\}$ using equation (3.1) applied to T^*x. Since $\{(\phi_n, \psi_n)\}$ is an effective pair, we know $\|T^*x - y_n\| \to 0$. Next, define the sequence $\{x_n\}$ via the dual Kaczmarz algorithm applied to x using $\{(T\phi_n, (T^{-1})^*\psi_n)\}$, i.e.

$$\begin{aligned} x_0 &= \langle x, T\phi_0\rangle (T^{-1})^*\psi_0, \\ x_n &= x_{n-1} + \langle x - x_{n-1}, T\phi_n\rangle (T^{-1})^*\psi_n, \quad n \ge 1. \end{aligned}$$

Observe that

$$x_0 = \langle x, T\phi_0\rangle (T^{-1})^*\psi_0 = (T^{-1})^* \left(\langle T^*x, \phi_0\rangle \psi_0\right) = (T^{-1})^* y_0.$$

Assume inductively that $x_{n-1} = (T^{-1})^* y_{n-1}$. Then

$$\begin{aligned} x_n &= (T^{-1})^* y_{n-1} + \langle T^*x - T^*x_{n-1}, \phi_n\rangle (T^{-1})^*\psi_n \\ &= (T^{-1})^* \left(y_{n-1} + \langle T^*x - y_{n-1}, \phi_n\rangle \psi_n\right) \\ &= (T^{-1})^* y_n. \end{aligned}$$

Therefore, $x_n = (T^{-1})^* y_n$ for all $n \in \mathbb{N}_0$. As T^{-1} is bounded, we have

$$\begin{aligned} \|x - x_n\| &= \left\|(T^{-1})^* T^*(x - x_n)\right\| \\ &\le \left\|(T^{-1})^*\right\| \|T^*x - T^*x_n\| = \left\|(T^{-1})^*\right\| \|T^*x - y_n\| \to 0, \end{aligned}$$

so that $\{(T\phi_n, (T^{-1})^*\psi_n)\}$ is an effective pair. Conversely, suppose that $\{(T\phi_n, (T^{-1})^*\psi_n)\}$ is an effective pair. Let $S = T^{-1}$. From the above argument, it follows that

$$\{(ST\phi_n, (S^{-1})^*(T^{-1})^*\psi_n)\} = \{(\phi_n, \psi_n)\}$$

is an effective pair. □

Corollary 3 (Aboud, Curl, Harding, Vaughan, Weber)**.** *Let $T \in \mathcal{B}(\mathcal{H})$ be invertible. A linearly dense sequence $\{e_n\}$ of unit vectors is effective if and only if $\{(Te_n, (T^{-1})^*e_n)\}$ is a symmetric effective pair.*

Proof. It is clear that $\{e_n\}$ is effective if and only if $\{(e_n, e_n)\}$ is an effective pair. Applying Theorem 1 with T and $(T^{-1})^*$, we conclude that $\{\big(Te_n, (T^{-1})^*e_n\big)\}$ and $\{\big((T^{-1})^*e_n, Te_n\big)\}$ are both effective pairs if and only if $\{(e_n, e_n)\}$ is an effective pair. □

3.3 Relation by a Positive, Invertible T

Our initial characterization efforts rely upon the existence of a certain operator T satisfying $\psi_n = T\phi_n$. With this in mind, we next present a string of lemmas tied to this condition, each imposing increasingly stringent hypotheses on T. It is assumed in every lemma that $\{\phi_n\}$ and $\{\psi_n\}$ are linearly dense in $\mathcal{H}$ and that $\{g_n\}$ and $\{\tilde{g}_n\}$ are constructed according to (3.2) and (3.5). In these lemmas, as well as in the rest of this thesis, we will reference $T^{\frac{1}{2}}$ as the positive square root of T, when defined.

Lemma 4 (Aboud, Curl, Harding, Vaughan, Weber)**.** *If $T \in \mathcal{B}(\mathcal{H})$ is such that $Tg_n = \tilde{g}_n$ and*

$$\langle \phi_n, \psi_k \rangle = \langle \psi_n, \phi_k \rangle \text{ for all } n, k \in \mathbb{N}_0, \tag{3.8}$$

then $T\phi_n = \psi_n$ for all $n \in \mathbb{N}_0$.

Proof. This is clear for $n = 0$,

$$T\phi_0 = Tg_0 = \tilde{g}_0 = \psi_0.$$

For $n \geq 1$, observe that

$$\psi_n - \sum_{k=0}^{n-1} \langle \psi_n, \phi_k \rangle \tilde{g}_k = \tilde{g}_n = T g_n = T\phi_n - \sum_{k=0}^{n-1} \langle \phi_n, \psi_k \rangle T g_k = T\phi_n - \sum_{k=0}^{n-1} \langle \psi_n, \phi_k \rangle \tilde{g}_k,$$

so $T\phi_n = \psi_n$, as desired. □

Lemma 5 (Aboud, Curl, Harding, Vaughan, Weber)**.** *Suppose $T \in \mathcal{B}(\mathcal{H})$ is positive and such that $T\phi_n = \psi_n$ for all $n \in \mathbb{N}_0$. If we define $\{e_n\}$ by*

$$e_n = T^{\frac{1}{2}} \phi_n, \tag{3.9}$$

then $h_n = T^{\frac{1}{2}} g_n$, where $\{h_n\}$ is the auxiliary sequence to $\{e_n\}$ as constructed in (2.10).

Proof. First note that

$$\langle e_m, e_n \rangle = \langle T^{\frac{1}{2}} \phi_m, T^{\frac{1}{2}} \phi_n \rangle = \langle \phi_m, T\phi_n \rangle = \langle \phi_m, \psi_n \rangle \tag{3.10}$$

for all m, n. Observe that

$$h_0 = e_0 = T^{\frac{1}{2}} \phi_0 = T^{\frac{1}{2}} g_0.$$

Assume inductively that $h_k = T^{\frac{1}{2}} g_k$ for all $0 \leq k < n$. It follows that

$$h_n = e_n - \sum_{k=0}^{n-1} \langle \phi_n, \psi_k \rangle h_k = T^{\frac{1}{2}} \left(\phi_n - \sum_{k=0}^{n-1} \langle \phi_n, \psi_k \rangle g_k \right) = T^{\frac{1}{2}} g_n$$

which concludes the induction. □

Remark 6. Note that in Lemma 5 the sequence $e_n = T^{\frac{1}{2}} \phi_n$ is not necessarily a sequence of unit vectors.

Lemma 6 (Aboud, Curl, Harding, Vaughan, Weber)**.** *Let $T \in \mathcal{B}(\mathcal{H})$ be positive and invertible. If $T\phi_n = \psi_n$ for all $n \in \mathbb{N}_0$, then $T g_n = \tilde{g}_n$ for all $n \in \mathbb{N}_0$.*

Proof. This is clear for $n = 0$,

$$T g_0 = T\phi_0 = \psi_0 = \tilde{g}_0.$$

Assume inductively that $T g_k = \tilde{g}_k$ for all $0 \leq k < n$. Observe that

$$T g_n = T\phi_n - \sum_{k=0}^{n-1} \langle \phi_n, \psi_k \rangle T g_k = \psi_n - \sum_{k=0}^{n-1} \langle T\phi_n, T^{-1}\psi_k \rangle \tilde{g}_k = \psi_n - \sum_{k=0}^{n-1} \langle \psi_n, \phi_k \rangle \tilde{g}_k = \tilde{g}_n$$

Thus, the statement holds for all $n \in \mathbb{N}_0$. □

Armed with the previous lemmas, we seek necessary and sufficient conditions for a pair of sequences to be an effective pair. Most of our results in this area depend upon a positive, invertible operator relating $\{\phi_n\}$ and $\{\psi_n\}$. In finite dimensions, we attain such an operator by exploiting the analysis and synthesis operators associated with the given sequences $\{\phi_n\}$ and $\{\psi_n\}$, as seen in equations (2.2) and (2.3). In infinite dimensions, however, the situation becomes more complex as we are forced to impose various conditions to ensure the existence of such an operator $T \in \mathcal{B}(\mathcal{H})$.

After the pattern of Haller and Szwarc in [HS05], we define the matrix

$$I + M = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \langle \phi_1, \psi_0 \rangle & 1 & 0 & \cdots \\ \langle \phi_2, \psi_0 \rangle & \langle \phi_2, \psi_1 \rangle & 1 & \cdots \\ \langle \phi_3, \psi_0 \rangle & \langle \phi_3, \psi_1 \rangle & \langle \phi_3, \psi_2 \rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.11}$$

and $I + U$ as the algebraic inverse of $I + M$. By this we mean

$$(I + U)(I + M) = (I + M)(I + U) = I. \tag{3.12}$$

Proposition 3 (Aboud, Curl, Harding, Vaughan, Weber)**.** *Suppose that $\{\phi_n\}$ and $\{\psi_n\}$ are linearly dense sequences in $\mathcal{H}$ satisfying $\langle \phi_n, \psi_n \rangle = 1$ for all $n \in \mathbb{N}_0$. Furthermore, suppose that there exists a positive, invertible $T \in \mathcal{B}(\mathcal{H})$ such that $\psi_n = T\phi_n$ for all $n \in \mathbb{N}_0$ and that U is a partial isometry. Then $\{(\phi_n, \psi_n)\}$ is a symmetric effective pair.*

Proof. Define $\{e_n\}$ by equation (3.9), so that $\phi_n = T^{-\frac{1}{2}} e_n$ and $\psi_n = T^{\frac{1}{2}} e_n$. From equation (3.10), we infer that the $\{e_n\}$ are unit vectors and that $M = N$, where M and N are as in (3.11) and (2.14), respectively. Consequently, $U = V$, and $\{e_n\}$ is effective by [HS05, Theorem 1]. By Corollary 3, we conclude that $\{(\phi_n, \psi_n)\}$ is a symmetric effective pair. $\square$

Proposition 4 (Aboud, Curl, Harding, Vaughan, Weber)**.** *Suppose that $\{\phi_n\}$ and $\{\psi_n\}$ are linearly dense sequences in $\mathcal{H}$, whose respective auxiliary sequences are $\{g_n\}$ and $\{\tilde{g}_n\}$, as in (3.2) and (3.5). Suppose that $\langle \phi_n, \psi_n \rangle = 1$ for all $n \in \mathbb{N}_0$, that $\langle \phi_n, \psi_k \rangle = \langle \psi_n, \phi_k \rangle$ for all $n, k \in \mathbb{N}_0$, and that $\{g_n\}$ and $\{\tilde{g}_n\}$ are canonical dual frames. Then $\{(\phi_n, \psi_n)\}$ is a symmetric effective pair.*

Proof. Since $\{g_n\}$ and $\{\tilde{g}_n\}$ are canonical dual frames, we write $\tilde{g}_n = Tg_n$ where T^{-1} is the frame operator for $\{g_n\}$. By Lemma 4, we know that $T\phi_n = \psi_n$ for all $n \in \mathbb{N}_0$. Again define $\{e_n\}$ and $\{h_n\}$ by (3.9) and (2.10), respectively. By Lemma 5, we know that $h_n = T^{\frac{1}{2}} g_n = T^{-\frac{1}{2}} \tilde{g}_n$ for all $n \in \mathbb{N}_0$.

Since $\{g_n\}$ and $\{\tilde{g_n}\}$ are dual frames, we know that $\{T^{\frac{1}{2}} g_n\}$ and $\{T^{-\frac{1}{2}} \tilde{g}_n\}$ are also dual frames. For $x \in \mathcal{H}$, observe that

$$x = \sum_{n=0}^{\infty} \langle x, T^{-\frac{1}{2}} \tilde{g}_n \rangle T^{\frac{1}{2}} g_n = \sum_{n=0}^{\infty} \langle x, h_n \rangle h_n$$

from which it follows that

$$\|x\|^2 = \langle x, x \rangle = \sum_{n=0}^{\infty} \langle x, h_n \rangle \langle h_n, x \rangle = \sum_{n=0}^{\infty} |\langle x, h_n \rangle|^2.$$

Therefore, $\{h_n\}$ is a Parseval frame, so $\{e_n\}$ is effective by [KM01]. Noting that $\phi_n = T^{-\frac{1}{2}} e_n$ and $\psi_n = T^{\frac{1}{2}} e_n$, we conclude that $\{(\phi_n, \psi_n)\}$ is a symmetric effective pair by Corollary 3. □

Theorem 2 (Aboud, Curl, Harding, Vaughan, Weber)**.** *Suppose that $\{\phi_n\}$ and $\{\psi_n\}$ are linearly dense sequences in $\mathcal{H}$, whose respective auxiliary sequences are $\{g_n\}$ and $\{\tilde{g}_n\}$ as in (3.2) and (3.5). Assume $\langle \phi_n, \psi_n \rangle = 1$ for all $n \in \mathbb{N}_0$ and suppose there exists a positive, invertible $T \in \mathcal{B}(\mathcal{H})$ such that $T\phi_n = \psi_n$ for all $n \in \mathbb{N}_0$. The following are then equivalent:*

(i) U is a partial isometry, where U is given by equation (3.12).

(ii) $\{g_n\}$ and $\{\tilde{g}_n\}$ are canonical dual frames.

(iii) $\{(\phi_n, \psi_n)\}$ is a symmetric effective pair.

Moreover, if any of these conditions hold, then T^{-1} is the frame operator for $\{g_n\}$.

Proof. We will show

$$(i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i).$$

Suppose (i) holds. It is immediate from Proposition 3 that $\{(\phi_n, \psi_n)\}$ is a symmetric effective pair.

Suppose (iii) holds. Define the sequences $\{e_n\}$ and $\{h_n\}$ by equations (3.9) and (2.10), respectively. From equation (3.10), we infer that the $\{e_n\}$ are unit vectors. As $e_n = T^{\frac{1}{2}}\phi_n$ and $e_n = T^{-\frac{1}{2}}\psi_n$, we have that $\{e_n\}$ is an effective sequence by Corollary 3, implying that $\{h_n\}$ is a Parsevel frame by [KM01]. By Lemmas 6 and 5, we know that $\tilde{g}_n = Tg_n$ and $h_n = T^{\frac{1}{2}}g_n$. As $g_n = T^{-\frac{1}{2}}h_n$, $\tilde{g}_n = T^{\frac{1}{2}}h_n$, and $\{h_n\}$ is a Parseval frame, we know that $\{g_n\}$ and $\{\tilde{g}_n\}$ are dual frames. Moreover, since $\tilde{g}_n = Tg_n$ for invertible T, we conclude by Lemma 3 that $\{g_n\}$ and $\{\tilde{g}_n\}$ must be canonical dual frames, with frame operator T^{-1}.

Suppose (ii) holds. Since T is self-adjoint, it is straightforward to verify equation (3.8), so by Proposition 4, we infer that $\{(\phi_n, \psi_n)\}$ is a symmetric effective pair. Defining $\{e_n\}$ by equation (3.9) and applying Corollary 3, we see that $\{e_n\}$ is an effective sequence. Appealing to [HS05, Theorem 1], we conclude that the associated matrix V, as defined by equation (2.14), is a partial isometry. By equation (3.10), $U = V$, and we have the desired result. □

For the remainder of this section, we will confine ourselves to finite-dimensional Hilbert spaces, where our characterization effort is aided by the existence of a positive, invertible $T \in \mathcal{B}(\mathcal{H})$ relating the sequences $\{\phi_n\}$ and $\{\psi_n\}$. We present necessary and sufficient conditions for the existence of such a T and then use this result to present a partial characterization of effective pairs in finite dimensions.

Lemma 7 (Aboud, Curl, Harding, Vaughan, Weber)**.** *Suppose $\{\phi_n\}$ and $\{\psi_n\}$ are linearly dense sequences in a finite dimensional Hilbert space $\mathcal{H}_N$. Then there exists a positive, invertible $T \in \mathcal{B}(\mathcal{H}_N)$ such that $T\phi_n = \psi_n$ if and only if $\theta_\psi\theta_\phi^*$ is positive.*

Proof. Suppose that there exists a positive, invertible $T \in \mathcal{B}(\mathcal{H}_N)$ such that $T\phi_n = \psi_n$. First, we show that $\theta_\psi\theta_\phi^*$ is self-adjoint. Let $\{\delta_n\}$ be the canonical orthonormal basis of

$\ell^2(\mathbb{N}_0)$. Observe

$$\theta_\phi\theta_\psi^*\delta_n = \{\langle\psi_n,\phi_k\rangle\}_{k=0}^\infty = \{\langle T\phi_n,\phi_k\rangle\}_{k=0}^\infty = \{\langle\phi_n,T\phi_k\rangle\}_{k=0}^\infty = \{\langle\phi_n,\psi_k\rangle\}_{k=0}^\infty = \theta_\psi\theta_\phi^*\delta_n.$$

From this it immediately follows that $(\theta_\psi\theta_\phi^*)^* = \theta_\phi\theta_\psi^* = \theta_\psi\theta_\phi^*$. Next, we show that $\theta_\psi\theta_\phi^*$ is positive. Observe for any finite sequence $\{c_n\}$ that

$$\begin{aligned}\left\langle \theta_\phi\theta_\psi^*\sum_{j=0}^\infty c_j\delta_j, \sum_{k=0}^\infty c_k\delta_k\right\rangle &= \sum_{j=0}^\infty\sum_{k=0}^\infty c_j\overline{c_k}\langle\theta_\phi\theta_\psi^*\delta_j,\delta_k\rangle = \sum_{j=0}^\infty\sum_{k=0}^\infty c_j\overline{c_k}\langle\psi_j,\phi_k\rangle \\ &= \left\langle\sum_{j=0}^\infty c_j\psi_j, \sum_{k=0}^\infty c_k\phi_k\right\rangle = \left\langle T\sum_{j=0}^\infty c_j\phi_j, \sum_{k=0}^\infty c_k\phi_k\right\rangle \geq 0.\end{aligned}$$

Therefore $\theta_\phi\theta_\psi^*$, and thus $\theta_\psi\theta_\phi^*$, is positive.

Conversely, suppose that $\theta_\psi\theta_\phi^*$ is positive. As $\mathcal{H}_N$ is finite dimensional, there is some $M \in \mathbb{N}$ such that $\{\phi_n\}_{n=0}^{M-1}$ and $\{\psi_n\}_{n=0}^{M-1}$ both span $\mathcal{H}_N$. It follows that $\{\phi_n\}_{n=0}^{M-1}$ and $\{\psi_n\}_{n=0}^{M-1}$ are frames for $\mathcal{H}_N$, which ensures that the operators

$$\Theta_\phi,\Theta_\psi : \mathcal{H}_N \to \ell^2(\mathbb{Z}_M), \quad \Theta_\phi x = \{\langle x,\phi_n\rangle\}_{n=0}^{M-1}, \quad \Theta_\psi x = \{\langle x,\psi_n\rangle\}_{n=0}^{M-1}$$

$$\Theta_\phi^*,\Theta_\psi^* : \ell^2(\mathbb{Z}_M) \to \mathcal{H}_N, \quad \Theta_\phi^*\{c_n\} = \sum_{n=0}^{M-1} c_n\phi_n, \quad \Theta_\psi^*\{c_n\} = \sum_{n=0}^{M-1} c_n\psi_n$$

are well defined, and that Θ_ϕ^* and Θ_ψ^* are surjective. We know that $\Theta_\psi\Theta_\phi^*$ is positive, as it is a principal submatrix of $\theta_\psi\theta_\phi^*$. We note that $\Theta_\psi\Theta_\phi^* = \Theta_\phi\Theta_\psi^*$ and thus $ran\Theta_\phi = ran\Theta_\psi$. Let $B = ran\Theta_\phi = ran\Theta_\psi$. Then Θ_ϕ^* and Θ_ψ^* are invertible when restricted to B since $B = (ker\Theta_\phi^*)^\perp = (ker\Theta_\psi^*)^\perp$. Let $T : \mathcal{H}_N \to \mathcal{H}_N$ be given by $T = \Theta_\psi^*\big|_B(\Theta_\phi^*\big|_B)^{-1}$.

Let P be the orthogonal projection of $\ell^2(\mathbb{Z}_M)$ onto the closed subspace B. Then the operator $\hat{\Theta}_\phi := P\Theta_\phi$ from $\mathcal{H}_N$ onto B is invertible, and we may write $T = \Theta_\psi^*\big|_B(\hat{\Theta}_\phi \circ \Theta_\phi^*\big|_B)^{-1}\hat{\Theta}_\phi$.

Let $\{\delta_n\}$ be the canonical basis for $\ell^2(\mathbb{Z}_M)$. Note that $\delta_n - P\delta_n \in B^\perp = \ker\Theta_\phi^*$. Thus

$$\phi_n = \Theta_\phi^*\delta_n = \Theta_\phi^*P\delta_n + \Theta_\phi^*(\delta_n - P\delta_n) = \Theta_\phi^*\big|_B P\delta_n$$

and similarly $\psi_n = \Theta_\psi^*\big|_B P\delta_n$. We then have

$$T\phi_n = T\Theta_\phi^*\big|_B P\delta_n = \Theta_\psi^*\big|_B(\hat{\Theta}_\phi \circ \Theta_\phi^*\big|_B)^{-1}(\hat{\Theta}_\phi \circ \Theta_\phi^*\big|_B)P\delta_n = \Theta_\psi^*\big|_B P\delta_n = \psi_n.$$

Note that, for such an operator $\hat{T}$ constructed on a larger collection $\hat{M} \geq M$, $\hat{T}$ must agree with T on a spanning set and, hence, $\hat{T} = T$.

Lastly, for any $x \in \mathcal{H}_N$, there is some $\{c_n\} \in B$ such that $\Theta_\phi^*(\{c_n\}) = x$. Then

$$\langle T^* x, x\rangle = \big\langle (T^*\Theta_\phi^*)(\{c_n\}), \Theta_\phi^*(\{c_n\})\big\rangle = \big\langle \Theta_\phi^*(\{c_n\}), \Theta_\psi^*(\{c_n\})\big\rangle = \big\langle (\Theta_\psi\Theta_\phi^*)\{c_n\}, \{c_n\}\big\rangle \geq 0$$

from which we conclude that T^*, and thus T, is positive. □

Corollary 4 (Aboud, Curl, Harding, Vaughan, Weber)**.** *If $\{\phi_n\}$ and $\{\psi_n\}$ are linearly dense sequences in a finite dimensional Hilbert space $\mathcal{H}_N$ such that $\langle \phi_n, \psi_n\rangle = 1$ for all $n \in \mathbb{N}_0$ and $\theta_\psi\theta_\phi^*$ is positive, then the following are equivalent:*

(i) U is a partial isometry.

(ii) $\{g_n\}$ and $\{\tilde{g}_n\}$ are canonical dual frames.

(iii) $\{(\phi_n, \psi_n)\}$ is a symmetric effective pair.

Proof. This is an immediate consequence of Lemma 7 and Theorem 2. □

At this point, we take a moment to reflect on the (UB) and (DE) conditions throughout this chapter. In the dualized algorithm, as the involved projections are no longer orthogonal, we no longer obtain (UB) automatically. However, by tying the behavior of our pair of sequences to a single effective sequence (through the assumption that $\psi_n = T\phi_n$), we are able to circumvent satisfying (UB) and (DE) directly. Once effectivity has been achieved, it follows by the Kwapień and Mycielski result that (UB) and (DE) hold.

3.4 Improving Almost Effective Sequences

Although almost effective sequences provide more flexibility than effective sequences, they are accompanied by nontrivial disadvantages. Recall the original impetus for our investigation into effective sequences—to reconstruct a vector given its inner products with some linearly dense sequence of unit vectors (whether in the context of signal processing or

solving a linear system). While an effective sequence yields such a reconstruction directly via the Kaczmarz algorithm, an almost effective sequence does not necessarily retain this property. By combining the idea of an almost effective sequence with that of an effective pair, however, we are able to attain approximations based upon the desired inner products.

Similar to Szwarc in [Szw07], who showed that a normalized Bessel sequence generates an effective sequence, we start with a lemma showing that canonical dual frames satisfying the appropriate orthogonality condition generate a symmetric effective pair. This will be an essential tool for our results involving almost effective sequences.

Lemma 8 (Aboud, Curl, Harding, Vaughan, Weber)**.** *Suppose that* $\{g_n\}$ *and* $\{\tilde{g}_n\}$ *are canonical dual frames in a Hilbert space* $\mathcal{H}$ *such that*

$$g_0 \perp \tilde{g}_n \quad \textit{for all } n \in \mathbb{N}.$$

Then there exists a symmetric effective pair $\{(\phi_n, \psi_n)\}$ *with auxiliary sequences* $\{g_n\}$ *and* $\{\tilde{g}_n\}$*, as in* (3.2) *and* (3.5)*.*

Proof. As $\{g_n\}$ and $\{\tilde{g}_n\}$ are canonical dual frames, we have that $S^{-1}g_n = \tilde{g}_n$ where S is the frame operator for $\{g_n\}$. Define a sequence $\{f_n\}$ by $f_n = S^{-\frac{1}{2}}g_n$. Observe that $\{f_n\}$ is a Parseval frame and that $f_0 \perp f_n$ for all $n \in \mathbb{N}$ as

$$\langle f_0, f_n\rangle = \langle S^{-\frac{1}{2}}g_0, S^{-\frac{1}{2}}g_n\rangle = \langle g_0, S^{-1}g_n\rangle = \langle g_0, \tilde{g}_n\rangle = 0.$$

From [Szw07, Theorem 1], we know $\{f_n\}$ is the auxiliary sequence for some effective sequence, say $\{b_n\}$, in $\mathcal{H}$. Define the sequences $\{\phi_n\}$ and $\{\psi_n\}$ by $\phi_n = S^{\frac{1}{2}}b_n$ and $\psi_n = S^{-\frac{1}{2}}b_n$. By Theorem 1, Lemma 5, and Lemma 6, it follows that $\{(\phi_n, \psi_n)\}$ is a symmetric effective pair with auxiliary sequences $\{S^{\frac{1}{2}}f_n\} = \{g_n\}$ and $\{\tilde{g}_n\}$. □

Now that we have a method for generating an effective pair corresponding to certain auxiliary sequences, we use it to produce an effective pair with the same auxiliary sequence $\{h_n\}$ as a given almost effective sequence $\{e_n\}$.

Theorem 3 (Aboud, Curl, Harding, Vaughan, Weber)**.** *Suppose that $\{e_n\}$ is an almost effective sequence in a Hilbert space $\mathcal{H}$ with auxiliary sequence $\{h_n\}$. Then there exists a symmetric effective pair $\{(\phi_n, \psi_n)\}$ with auxiliary sequences $\{g_n\}$ and $\{\tilde{g}_n\}$, as in Equations (3.2) and (3.5), such that*

(i) $h_n = g_n$ for all $n \in \mathbb{N}_0$.

(ii) $\{g_n\}$ and $\{\tilde{g}_n\}$ are canonical dual frames.

Moreover, x can be reconstructed from $\{\langle x, h_n\rangle\}$ by

$$x = \sum_{n=0}^{\infty} \langle x, h_n\rangle \psi_n.$$

Proof. As $\{e_n\}$ is almost effective, its auxiliary sequence $\{h_n\}$ is a frame with Bessel bound $0 < B \leq 1$ [CT13, Theorem 3.1]. Since $\|h_0\|^2 = \|e_0\|^2 = 1$, it follows from the Bessel inequality that $\langle h_0, h_n\rangle = 0$ for all $n \in \mathbb{N}$.

Let S be the frame operator of $\{h_n\}$. Define the canonical dual frames $\{g_n\}$ and $\{\tilde{g}_n\}$, where

$$g_n = h_n, \qquad \tilde{g}_n = S^{-1} h_n.$$

As $h_0 \perp h_n$ for $n \in \mathbb{N}$, we infer that

$$Sh_0 = \sum_{n=0}^{\infty} \langle h_0, h_n\rangle h_n = h_0.$$

For $n \in \mathbb{N}$, we then have

$$\langle g_0, \tilde{g}_n\rangle = \langle h_0, S^{-1} h_n\rangle = \langle S^{-1} h_0, h_n\rangle = \langle h_0, h_n\rangle = 0$$

and

$$\langle \tilde{g}_0, g_n\rangle = \langle S^{-1} h_0, h_n\rangle = \langle h_0, h_n\rangle = 0.$$

By Lemma 8, there are sequences $\{\phi_n\}$ and $\{\psi_n\}$ in $\mathcal{H}$ such that $\{(\phi_n, \psi_n)\}$ is a symmetric effective pair with auxiliary sequences $\{g_n\}$ and $\{\tilde{g}_n\}$.

Furthermore, as $h_n = g_n$, by the reconstruction in equation (3.4) we know that

$$x = \sum_{k=0}^{\infty} \langle x, g_k \rangle \psi_k = \sum_{k=0}^{\infty} \langle x, h_k \rangle \psi_k \quad \text{for all } x \in \mathcal{H}.$$

□

We now have a sequence of approximations to x generated by inner products with the auxiliary sequence of an almost effective sequence. In the following corollary, we introduce another variation on the Kaczmarz algorithm which will allow us to achieve reconstruction based upon the inner products with the almost effective sequence itself.

Corollary 5 (Aboud, Curl, Harding, Vaughan, Weber)**.** *Suppose that $\{e_n\}$ is an almost effective sequence in a Hilbert space $\mathcal{H}$ with auxiliary sequence $\{h_n\}$. Let $\{\psi_n\}$ be as in the conclusion of Theorem 3. For any $x \in \mathcal{H}$, let $\{x_n\}$ be the sequence generated from $\{e_n\}$ as in (1.1). Furthermore, define the sequence $\{y_n\}$ by*

$$\begin{aligned} y_0 &= \langle x, e_0 \rangle \psi_0, \\ y_n &= y_{n-1} + \langle x - x_{n-1}, e_n \rangle \psi_n, \quad n \geq 1. \end{aligned} \tag{3.13}$$

Then, $\lim_{n\to\infty} ||y_n - x|| = 0$.

We call the new formulation in (3.13) the *augmented dual Kaczmarz algorithm.* Note that, as $\{e_n\}$ is merely almost effective, we do not make any assumptions about the convergence of $\{x_n\}$. Indeed, even if $\lim_{n\to\infty} x_n$ exists, it need not be equal to x. However, we use the sequence $\{x_n\}$ as a state variable to gain the sequence of approximations $\{y_n\}$ generated in (3.13).

Proof. From Theorem 3, we know that

$$x = \sum_{k=0}^{\infty} \langle x, h_k \rangle \psi_k \quad \text{for all } x \in \mathcal{H},$$

so it suffices to show that

$$y_n = \sum_{k=0}^{n} \langle x, h_k \rangle \psi_k \quad \text{for all } x \in \mathcal{H}, n \in \mathbb{N}_0. \tag{3.14}$$

This is clear for $n = 0$ as $e_0 = h_0$. Assume inductively that the claim holds for $0 \leq k < n$ and note that

$$\langle x - x_{n-1}, e_n \rangle = \langle x, e_n \rangle - \left\langle \sum_{k=0}^{n-1} \langle x, h_k \rangle e_k, e_n \right\rangle = \left\langle x, e_n - \sum_{k=0}^{n-1} \langle e_n, e_k \rangle h_k \right\rangle = \langle x, h_n \rangle.$$

We then have

$$y_n = y_{n-1} + \langle x - x_{n-1}, e_n \rangle \psi_n = \sum_{k=0}^{n-1} \langle x, h_k \rangle \psi_k + \langle x - x_{n-1}, e_n \rangle \psi_n = \sum_{k=0}^{n} \langle x, h_k \rangle \psi_k$$

and thus equation (3.14) holds. □

3.5 Examples

In this section, we list examples which illuminate some of the interesting characteristics of effective pairs.

Observation 1 (Aboud, Curl, Harding, Vaughan, Weber)**.** *Suppose that $\{\phi_n\}$ and $\{\psi_n\}$ are effective. It is not necessarily true that $\{(\phi_n, \psi_n)\}$ is an effective pair.*

The most straightforward example would be to consider an orthonormal basis $\{\phi_n\}$ and take $\psi_n = -\phi_n$. Clearly $\{\phi_n\}$ and $\{\psi_n\}$ are effective sequences. However, the corresponding dual Kaczmarz algorithm applied to x reproduces $-x$, i.e. $\|x_n - (-x)\| \to 0$, so that $\{(\phi_n, \psi_n)\}$ is not an effective pair. This is immediate from

$$x_n = \sum_{k=0}^{n} \langle -x, \phi_k \rangle \phi_k$$

which follows from equation (3.3).

Observation 2 (Aboud, Curl, Harding, Vaughan, Weber)**.** *There are effective pairs $\{(\phi_n, \psi_n)\}$ satisfying $\langle \phi_n, \psi_n \rangle = 1$ for all $n \in \mathbb{N}_0$ which are not symmetric effective pairs.*

In $\mathbb{R}^2$, consider the periodic sequences $\{\phi_n\}$ and $\{\psi_n\}$ with

$$\left[\begin{array}{c|c|c|c|c|c} \phi_0 & \phi_1 & \phi_2 & \psi_0 & \psi_1 & \psi_2 \end{array}\right] = \left[\begin{array}{c|c|c|c|c|c} 1 & 1 & 0.5 & 1 & 1 & 1.5 \\ -1 & 1 & -0.5 & 0 & 0 & -0.5 \end{array}\right]$$

and $\phi_n = \phi_{(n \mod 3)}$, $\psi_n = \psi_{(n \mod 3)}$. Consider the error sequence $\{\varepsilon_n \,|\, \varepsilon_n = x - x_n\}$ corresponding to the dual Kaczmarz algorithm for $\{(\phi_n, \psi_n)\}$ associated to x where

$$\varepsilon_0 = x - \langle x, \phi_0 \rangle \psi_0$$
$$\varepsilon_n = \varepsilon_{n-1} - \langle \varepsilon_{n-1}, \phi_n \rangle \psi_n, \quad n \geq 1.$$

Then $\{(\phi_n, \psi_n)\}$ is an effective pair if and only if $\varepsilon_n \to 0$. It is simple to show by induction that the error sequence $\{\varepsilon_n\}$ associated to $x = (a, b)$ satisfies

$$\varepsilon_{3k} = \frac{b}{2^k} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \varepsilon_{3k+1} = \frac{b}{2^k} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \varepsilon_{3k+2} = \frac{b}{2^{k+1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad k \geq 0.$$

Therefore, $\{(\phi_n, \psi_n)\}$ is an effective pair. However, $\{(\psi_n, \phi_n)\}$ is not an effective pair for the following reason: Let $\{\varepsilon_n\}$ be the error sequence associated to $x = (0, 4)$. Then, by induction, we find

$$\varepsilon_{3k} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \quad \varepsilon_{3k+1} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \quad \varepsilon_{3k+2} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad k \geq 0.$$

The sequence $\{\varepsilon_n\}$ fails to converge.

Observation 3 (Aboud, Curl, Harding, Vaughan, Weber)**.** *There are symmetric effective pairs $\{(\phi_n, \psi_n)\}$ for which the mixed Grammian operator $\theta_\phi \theta_\psi^*$ is not positive. Furthermore, there are symmetric effective pairs which are not related by an invertible operator, i.e. there does not exist an invertible $T \in \mathcal{B}(\mathcal{H})$ such that $T\phi_n = \psi_n$ for all n.*

In $\mathbb{R}^2$, consider the periodic sequences $\{\phi_n\}$ and $\{\psi_n\}$ with

$$\left[\begin{array}{c|c|c|c|c|c} \phi_0 & \phi_1 & \phi_2 & \psi_0 & \psi_1 & \psi_2 \end{array}\right] = \left[\begin{array}{c|c|c|c|c|c} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

and $\phi_n = \phi_{(n \mod 3)}$, $\psi_n = \psi_{(n \mod 3)}$. As in the previous example, consider the error sequence $\{\varepsilon_n\}$ for the pair $\{(\phi_n, \psi_n)\}$. Since ε_2 is the projection of ε_1 onto the orthogonal complement of ϕ_2, ε_3 is the projection of ε_2 onto the orthogonal complement of ϕ_0, and

$\{\phi_0, \phi_2\}$ form an orthonormal basis, it follows that $\varepsilon_k = (0,0)$ for $k \geq 3$, so $\{(\phi_n, \psi_n)\}$ is an effective pair. Likewise, by the same argument, we observe that $\{(\psi_n, \phi_n)\}$ is an effective pair. The matrix $\theta_\phi \theta_\psi^*$ is not positive since its 3×3 principal submatrix,

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

is not positive. Note that an invertible $T \in \mathcal{B}(\mathcal{H})$ can not possibly map ϕ_n to ψ_n for all $n \in \mathbb{N}_0$ since $\psi_0 = \psi_1$ yet $\phi_0 \neq \phi_1$.

Observation 4 (Aboud, Curl, Harding, Vaughan, Weber)**.** *There are symmetric effective pairs $\{(\phi_n, \psi_n)\}$ where neither $\{\phi_n\}$ nor $\{\psi_n\}$ is effective. Moreover, there are symmetric effective pairs for which their auxiliary sequences do not form (dual) frames.*

Let $\{\phi_n\}$ be a Schauder basis which is not a Riesz basis, and let $\{\psi_n\}$ be its biorthogonal dual basis. We then have the reconstruction property

$$x = \sum_{n=0}^{\infty} \langle x, \phi_n \rangle \psi_n = \sum_{n=0}^{\infty} \langle x, \psi_n \rangle \phi_n.$$

Since the auxiliary sequence of $\{(\phi_n, \psi_n)\}$ is $\{\phi_n\}$ and the auxiliary sequence of $\{(\psi_n, \phi_n)\}$ is $\{\psi_n\}$, it follows that $\{(\phi_n, \psi_n)\}$ is a symmetric effective pair where the auxiliary sequences are not (dual) frames. Moreover, if $\{\phi_n\} \subseteq X^*$ and $\{\psi_n\} \subseteq X$ for a Banach space X, then these sequences form an effective pair as in Definition 10, while also satisfying Definition 2.

CHAPTER 4. EFFECTIVE PAIRS IN BANACH SPACE

4.1 Introduction

The concept of an effective pair is intricately intertwined with that of duality. Most obviously, a sequence $\{e_n\} \subseteq \mathcal{H}$ is effective if and only if the sequence and its auxiliary form a resolution of the identity, i.e., if

$$x = \sum_{n=0}^{\infty} \langle x, h_n \rangle e_n \text{ for all } x \in \mathcal{H}. \tag{4.1}$$

In the context of a Banach Space, a pair of sequences $\{(\phi_n, \psi_n)\} \subseteq X^* \times X$ is effective if and only if the auxiliary sequence $\{g_n\} \subseteq X^*$ and $\{\psi_n\}$ form a resolution of the identity. I.e., if

$$x = \sum_{n=0}^{\infty} g_n(x)\psi_n \text{ for all } x \in X. \tag{4.2}$$

There are analogous statements for the effectivity of $\{(\psi_n, \phi_n)\} \subseteq X \times X^*$. As mentioned in Chapter 2, we will often refer to conditions (4.1) and (4.2) using duality nomenclature. E.g., we will say that $\{(g_n, \psi_n)\}$ are "dual" if (4.2) holds or "weakly dual" if

$$y(x) = \sum_{n=0}^{\infty} g_n(x)y(\psi_n) \text{ for all } x \in X, y \in X^*. \tag{4.3}$$

If (4.3) holds on a dense subset of $X \times X^*$, we will say that $\{(g_n, \psi_n)\}$ are weakly densely dual (WDD). Note that none of the discussed conditions make any assumptions about the involved sequences being frames. For this reason, these sequences are sometimes called "pseudodual" in the literature [LO01]. We will not use this terminology, however.

As we see in the work of both Kwapień and Mycielski and Haller and Szwarc, the effectivity of the sequence $\{e_n\}$ in a Hilbert space is equivalent to the duality of $\{(h_n, e_n)\}$, which is equivalent to the duality of the sequence $\{h_n\}$ with itself (i.e. $\{h_n\}$ being a Parseval frame) [KM01, HS05]. Our work in Chapter 3 shows that a similar phenomenon is exhibited

when restricted to sequences $\{\phi_n\}$ and $\{\psi_n\}$ related by a positive, invertible operator T such that $\psi_n = T\phi_n$ for all $n \in \mathbb{N}_0$. In this case, $\{(\phi_n, \psi_n)\}$ is a symmetric effective pair if and only if $\{(g_n, \psi_n)\}$ and $\{(\tilde{g}_n, \phi_n)\}$ are both dual if and only if $\{g_n\}$ and $\{\tilde{g}_n\}$ are canonical dual frames. In both of these scenarios, the effectivity condition is related to a duality within or between the auxiliary(ies). When we release the constraint that $\psi_n = T\phi_n$, however, these auxiliary relationships break down. Indeed, the duality of the pairs $\{(g_n, \psi_n)\}$, $\{(\tilde{g}_n, \phi_n)\}$, and $\{(g_n, \tilde{g}_n)\}$ may be completely independent conditions. For the reader's benefit, these relationships are summarized in Table 4.1. This representation is not intended to be a precise description (indeed, many of the associated conditions and details are omitted), but rather a heuristic to aid the reader in more fully understanding the relationships involved.

Table 4.1 Duality Relationships

Auxiliary		Sequence/Auxiliary		Effectivity
$\{(h_n, h_n)\}$ Parseval	$\Longleftrightarrow$	$\{(h_n, e_n)\}$ Dual	$\Longleftrightarrow$	$\{e_n\}$ Effective
* $\{(g_n, \tilde{g}_n)\}$ Canonical Dual Frames	$\Longleftrightarrow$	$\{(g_n, \psi_n)\}$ Dual $\Updownarrow$ $\{(\tilde{g}_n, \phi_n)\}$ Dual	$\Longleftrightarrow$	$\{(\phi_n, \psi_n)\}$ Symmetric Effective
$\{(g_n, \tilde{g}_n)\}$ Dual	$\not\Rightarrow$	$\{(g_n, \psi_n)\}$ Dual	$\Longleftrightarrow$	$\{(\phi_n, \psi_n)\}$ Effective
$\{(\tilde{g}_n, g_n)\}$ Dual	$\not\Rightarrow$	$\{(\tilde{g}_n, \phi_n)\}$ Dual	$\Longleftrightarrow$	$\{(\psi_n, \phi_n)\}$ Effective

* In this row, $\psi_n = T\phi_n$, where T is positive, invertible

Because of their independent behaviors, we investigate each of the three possible dualities separately. Specifically, the goal of this chapter is to provide a matrix characterization for the weak dense duality (WDD) of each, after the model of Haller and Szwarc in the effective sequence context [HS05]. Our results in infinite dimensions are restricted to a dense subset as the involved projections are nonorthogonal and so condition (UB) is not necessarily

satisfied. Although we achieve characterization for the weak dense duality of $\{(g_n, \tilde{g}_n)\}$, our proof fails to show that this is equivalent to the weak dense duality of $\{(g_n, \psi_n)\}$ (which is equivalent to the weak effectively of $\{(\phi_n, \psi_n)\}$). Indeed, we use our results to present an example which shows that $\{(g_n, \tilde{g}_n)\}$, $\{(\tilde{g}_n, g_n)\}$, and $\{(g_n, \psi_n)\}$ can be dual when $\{(\tilde{g}_n, \phi_n)\}$ is not. From this example we conclude that the duality of $\{(g_n, \tilde{g}_n)\}$ is not a sufficient condition for effectivity. In order to more quickly determine the effectivity of our examples, we also present a theorem addressing necessary and sufficient conditions for a pair of periodic sequences to be effective in a finite-dimensional Banach space.

4.2 Terminology

We build our terminology for the context of a Banach space, requesting the reader's indulgence as we restate versions of previously stated definitions. There are subtle variations which require careful attention.

Definition 13. Let X be a Banach space. Let $\{\phi_n\}$ and $\{\psi_n\}$ be linearly dense sequences in X^* and X, respectively, where $\phi_n(\psi_n) = 1$ for all $n \in \mathbb{N}_0$. Define the sequence of approximations $\{x_n\}$ according to the *dual Kaczmarz algorithm* for $\{(\phi_n, \psi_n)\} \subseteq X^* \times X$, where

$$\begin{aligned} x_0 &= \phi_0(x)\psi_0 \\ x_n &= x_{n-1} + \phi_n(x - x_{n-1})\psi_n. \end{aligned} \tag{4.4}$$

If $||x_n - x||_X \to 0$ for every $x \in X$, then we say that $\{(\phi_n, \psi_n)\}$ is an *effective pair*. If $u(x_n) \to u(x)$ for every $x \in X$, $u \in X^*$, then we say that $\{(\phi_n, \psi_n)\}$ is a *weakly effective pair*.

To aid our investigations, we define the infinite matrix $I + M$.

$$I + M = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \overline{\phi_1(\psi_0)} & 1 & 0 & 0 & \cdots \\ \overline{\phi_2(\psi_0)} & \overline{\phi_2(\psi_1)} & 1 & 0 & \cdots \\ \overline{\phi_3(\psi_0)} & \overline{\phi_3(\psi_1)} & \overline{\phi_3(\psi_2)} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.5}$$

Let $I+U$ be the algebraic inverse of $I+M$. We will denote the entries of $I+U$ by $(c_{jk})_{j,k\in\mathbb{N}_0}$. Note that $c_{jk} = 1$ when $j = k$, and that $c_{jk} = 0$ when $j < k$.

Our previous version of the dual Kaczmarz algorithm was for $\{(\phi_n, \psi_n)\} \subseteq X^* \times X$. It will also be fruitful to capture the behavior of $\{(\psi_n, \phi_n)\} \in X \times X^*$ as a pair. Though it is not a central focus of our work, this definition will enable necessary terminology to be formulated.

Definition 14. Let X be a Banach space. Let $\{\phi_n\}$ and $\{\psi_n\}$ be linearly dense sequences in X^* and X, respectively, where $\phi_n(\psi_n) = 1$ for all $n \in \mathbb{N}_0$. Define the sequence of approximations $\{u_n\}$ according to the *dual Kaczmarz algorithm* for $\{(\psi_n, \phi_n)\} \subseteq X \times X^*$, where

$$\begin{aligned} u_0 &= u(\psi_0)\phi_0 \\ u_n &= u_{n-1} + ((u - u_{n-1})(\psi_n))\,\phi_n. \end{aligned} \tag{4.6}$$

If $||u_n - u||_{X^*} \to 0$ for any $u \in X^*$, we say that $\{(\psi_n, \phi_n)\}$ is an effective pair. If $u_n(x) \to u(x)$ for any $x \in X$, we say that the pair is weakly effective.

Note that the convergence demonstrated in $u_n(x) \to u(x)$ is actually convergence in the weak* topology. For a pair of sequences $\{(\psi_n, \phi_n)\} \subseteq X \times X^*$, we define weak effectivity using weak* convergence. For a pair of sequences $\{(\phi_n, \psi_n)\} \subseteq X^* \times X$ we define weak effectivity using the standard weak convergence.

We define the infinite matrix $I + \tilde{M}$ by

$$I + \tilde{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \phi_0(\psi_1) & 1 & 0 & 0 & \cdots \\ \phi_0(\psi_2) & \phi_1(\psi_2) & 1 & 0 & \cdots \\ \phi_0(\psi_3) & \phi_1(\psi_3) & \phi_2(\psi_3) & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.7}$$

Let $I + \tilde{U}$ be the algebraic inverse of $I + \tilde{M}$ with entries $(\tilde{c}_{jk})_{j,k \in \mathbb{N}_0}$. Note that $\tilde{c}_{jk} = 1$ when $j = k$ and $\tilde{c}_{jk} = 0$ when $j < k$.

We define auxiliary sequences which will act as dual sequences to $\{\psi_n\}$ and $\{\phi_n\}$, respectively.

Definition 15. The auxiliary sequence to $\{(\phi_n, \psi_n)\} \subseteq X^* \times X$ is given by

$$g_0 = \phi_0 \tag{4.8}$$

$$g_n = \phi_n - \sum_{k=0}^{n-1} \phi_n(\psi_k) g_k \text{ for } n \geq 1. \tag{4.9}$$

Proposition 5. *For any $x \in X$,*

$$x_n = \sum_{k=0}^{n} g_k(x)\psi_k. \tag{4.10}$$

Furthermore, $\{(\phi_n, \psi_n)\}$ is an effective pair (weakly effective pair) if and only if $x = \sum_{k=0}^{\infty} g_k(x)\psi_k$ for any $x \in X$ ($y(x) = \sum_{k=0}^{\infty} g_k(x) y(\psi_k)$ for any $x \in X$ and $y \in X^*$).

Proof. We know that $g_0 = \phi_0$. According to (4.4), we have

$$\begin{aligned} x_1 &= x_0 + \phi_1(x - x_0)\psi_1 \\ &= \phi_0(x)\psi_0 + \phi_1(x - \phi_0(x)\psi_0)\psi_1 \\ &= \phi_0(x)\psi_0 + \phi_1(x)\psi_1 - \phi_0(x)\phi_1(\psi_0)\psi_1 \\ &= g_0(x)\psi_0 + g_1(x)\psi_1 \\ &= \sum_{k=0}^{1} g_k(x)\psi_k. \end{aligned}$$

Assume there is some $N > 1$ such that $x_n = \sum_{k=0}^{n} g_k(x)\psi_k$ for all $n \leq N$. Consider

$$\begin{aligned}
x_{N+1} &= x_N + \phi_{N+1}(x - x_N)\psi_{N+1} \\
&= \sum_{k=0}^{N} g_k(x)\psi_k + \phi_{N+1}\left(x - \sum_{k=0}^{N} g_k(x)\psi_k\right)\psi_{N+1} \\
&= \sum_{k=0}^{N} g_k(x)\psi_k + \left(\phi_{N+1}(x) - \phi_{N+1}\left(\sum_{k=0}^{N} g_k(x)\psi_k\right)\right)\psi_{N+1} \\
&= \sum_{k=0}^{N} g_k(x)\psi_k + \left(\phi_{N+1}(x) - \sum_{k=0}^{N} \phi_{N+1}(\psi_k)g_k(x)\right)\psi_{N+1} \\
&= \sum_{k=0}^{N} g_k(x)\psi_k + g_{N+1}(x)\psi_{N+1} \\
&= \sum_{k=0}^{N+1} g_k(x)\psi_k.
\end{aligned}$$

We conclude that (4.10) holds by induction. It follows by Definition 13 that $\{(\phi_n, \psi_n)\}$ is an effective pair (weakly effective pair) if and only if $x = \sum_{n=0}^{\infty} g_k(x)\psi_k$ for all $x \in X$ ($y(x) = \sum_{k=0}^{\infty} g_k(x)y(\psi_k)$ for all $x \in X$ and $y \in X^*$). □

We proceed analogously for $\{(\psi_n, \phi_n)\}$, constructing a sequence to serve as a dual to $\{\phi_n\}$.

Definition 16. The auxiliary sequence for $\{(\psi_n, \phi_n)\} \subseteq X \times X^*$ is given by

$$\tilde{g}_0 = \psi_0 \tag{4.11}$$

$$\tilde{g}_n = \psi_n - \sum_{k=0}^{n-1} \phi_k(\psi_n)\tilde{g}_k. \tag{4.12}$$

Proposition 6. *For any $u \in X^*$,*

$$u_n = \sum_{k=0}^{n} u(\tilde{g}_k)\phi_k. \tag{4.13}$$

Furthermore, $\{(\psi_n, \phi_n)\}$ is an effective pair (weakly effective pair) if and only if $u = \sum_{k=0}^{\infty} u(\tilde{g}_k)\phi_k$ for all $u \in X^*$ ($u(x) = \sum_{k=0}^{\infty} u(\tilde{g}_k)\phi_k(x)$ for all $x \in X$, $u \in X^*$).

Proof. From $u_0 = u(\psi_0)\phi_0$, it is clear that $u_0 = \sum_{k=0}^{0} u(\tilde{g}_k)\phi_k$. We now consider the $n = 1$ case.

$$\begin{aligned}
u_1 &= u_0 + (u - u_0)(\psi_1)\phi_1 \\
&= u(\psi_0)\phi_0 + (u - u(\psi_0)\phi_0)(\psi_1)\phi_1 \\
&= u(\psi_0)\phi_0 + (u(\psi_1) - u(\phi_0(\psi_1)\psi_0))\,\phi_1 \\
&= u(\psi_0)\phi_0 + u(\psi_1 - \phi_0(\psi_1)\psi_0)\phi_1 \\
&= u(\tilde{g}_0)\phi_0 + u(\tilde{g}_1)\phi_1 \\
&= \sum_{k=0}^{1} u(\tilde{g}_k)\phi_k.
\end{aligned}$$

Assume there is some $N > 1$ such that $u_n = \sum_{k=0}^{n} u(\tilde{g}_k)\phi_k$ for $n \leq N$. Consider

$$\begin{aligned}
u_{N+1} &= u_N + (u - u_N)(\psi_{N+1})\phi_{N+1} \\
&= \sum_{k=0}^{N} u(\tilde{g}_k)\phi_k + \left(u(\psi_{N+1}) - \sum_{k=0}^{N} u(\tilde{g}_k)\phi_k(\psi_{N+1})\right)\phi_{N+1} \\
&= \sum_{k=0}^{N} u(\tilde{g}_k)\phi_k + \left(u(\psi_{N+1}) - u\left(\sum_{k=0}^{N} \phi_k(\psi_{N+1})\tilde{g}_k\right)\right)\phi_{N+1} \\
&= \sum_{k=0}^{N} u(\tilde{g}_k)\phi_k + \left(u\left(\psi_{n+1} - \sum_{k=0}^{N} \phi_k(\psi_{N+1})\tilde{g}_k\right)\right)\phi_{N+1} \\
&= \sum_{k=0}^{N} \tilde{g}_k(u)\phi_k + u(\tilde{g}_{N+1})\phi_{N+1} \\
&= \sum_{k=0}^{N+1} u(\tilde{g}_k)\phi_k.
\end{aligned}$$

We conclude that (4.13) holds by induction. It follows by Definition 14 that $\{(\psi_n, \phi_n)\}$ is an effective pair (weakly effective pair) if and only if $u = \sum_{n=0}^{\infty} u(\tilde{g}_k)\phi_k$ for all $u \in X^*$ ($u(x) = \sum_{k=0}^{\infty} u(\tilde{g}_k)\phi_k(x)$ for all $x \in X$ and $u \in X^*$). □

There are several helpful interactions between the auxiliary sequences and infinite matrices. To take advantage of these relationships, we must first define the conjugate of a functional. If $u \in X^*$, then we define $\overline{u}(x) = \overline{u(x)}$ for all $x \in X$. With this definition, we

note that

$$(I+M)\cdot\begin{pmatrix}\overline{g}_0\\ \overline{g}_1\\ \overline{g}_2\\ \overline{g}_3\\ \vdots\end{pmatrix}=\begin{pmatrix}1 & 0 & 0 & 0 & \cdots\\ \overline{\phi_1(\psi_0)} & 1 & 0 & 0 & \cdots\\ \overline{\phi_2(\psi_0)} & \overline{\phi_2(\psi_1)} & 1 & 0 & \cdots\\ \overline{\phi_3(\psi_0)} & \overline{\phi_3(\psi_1)} & \overline{\phi_3(\psi_2)} & 1 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\cdot\begin{pmatrix}\overline{g}_0\\ \overline{g}_1\\ \overline{g}_2\\ \overline{g}_3\\ \vdots\end{pmatrix}=\begin{pmatrix}\overline{g}_0\\ \sum_{k=0}^{1}\overline{\phi_1(\psi_k)}\overline{g_k}\\ \sum_{k=0}^{2}\overline{\phi_2(\psi_k)}\overline{g_k}\\ \vdots\sum_{k=0}^{3}\overline{\phi_3(\psi_k)}\overline{g_k}\\ \vdots\end{pmatrix}=\begin{pmatrix}\overline{\phi}_0\\ \overline{\phi}_1\\ \overline{\phi}_2\\ \overline{\phi}_3\\ \vdots\end{pmatrix}.$$

Because $(I+M)(I+U)=I$, we also infer that

$$\overline{g}_n=\sum_{k=0}^{n}c_{nk}\overline{\phi}_k. \tag{4.14}$$

Similarly,

$$(I+\tilde{M})\cdot\begin{pmatrix}\tilde{g}_0\\ \tilde{g}_1\\ \tilde{g}_2\\ \tilde{g}_3\\ \vdots\end{pmatrix}=\begin{pmatrix}1 & 0 & 0 & 0 & \cdots\\ \phi_0(\psi_1) & 1 & 0 & 0 & \cdots\\ \phi_0(\psi_2) & \phi_1(\psi_2) & 1 & 0 & \cdots\\ \phi_0(\psi_3) & \phi_1(\psi_3) & \phi_2(\psi_3) & 1 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\cdot\begin{pmatrix}\tilde{g}_0\\ \tilde{g}_1\\ \tilde{g}_2\\ \tilde{g}_3\\ \vdots\end{pmatrix}=\begin{pmatrix}\tilde{g}_0\\ \sum_{k=0}^{1}\phi_k(\psi_1)\tilde{g}_k\\ \sum_{k=0}^{2}\phi_k(\psi_2)\tilde{g}_k\\ \sum_{k=0}^{3}\phi_k(\psi_3)\tilde{g}_k\\ \vdots\end{pmatrix}=\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\\ \psi_3\\ \vdots\end{pmatrix}.$$

Because $(I+\tilde{M})(I+\tilde{U})=I$, we infer that

$$\tilde{g}_n=\sum_{k=0}^{n}\tilde{c}_{nk}\psi_k. \tag{4.15}$$

4.3 Duality

In our exploration of duality relationships, we will use a variety of operators expressed as infinite matrices. The following sections provide information on the components of these matrices. This is not to be confused with the properties of the matrices viewed as operators on $\ell^2(\mathbb{N})$. Indeed, some of the matrices presented may not even be defined on $\ell^2(\mathbb{N})$. We use the $\ell^2(\mathbb{N})$ inner product as a way of conveniently representing the entries of the matrices in question.

4.3.1 $\{g_n\}$ and $\{\psi_n\}$ Duality

The duality of $\{g_n\}$ and $\{\psi_n\}$, in the sense of (4.2) is equivalent to $\{(\phi_n, \psi_n)\}$ being an effective pair by definition. In the following theorem we present a matrix condition for a weak version of this duality.

Theorem 4. *Let X be a Banach space and $\{\phi_n\}$ and $\{\psi_n\}$ be linearly dense sequences in X^* and X, respectively, where $\phi_n(\psi_n) = 1$ for all $n \in \mathbb{N}_0$. Suppose $\Phi \subseteq X^*$ and $\Psi \subseteq X$ are the linear spans of $\{\phi_n\}$ and $\{\psi_n\}$, respectively. and let $M, \tilde{M}, U, \tilde{U}$ be as defined in (4.5) and (4.7). Then the following are equivalent:*

1. *$\langle \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)\delta_j, \delta_i\rangle_{\ell^2} = 0$ for all $i, j \in \mathbb{N}_0$.*
2. *$\{(\phi_n, \psi_n)\}$ is weakly effective on $\Phi \times \Psi$.*
3. *$\{g_n\}$ and $\{\psi_n\}$ form a weak resolution of the identity on $\Phi \times \Psi$.*

Proof. As Φ and Ψ consist of finite linear combinations, and all of the involved operators are linear, it suffices to consider (2) and (3) on the collections $\{\phi_n\}$ and $\{\psi_n\}$.

We first show that

$$\langle (U\tilde{M}^* + \tilde{M}^* + I)\delta_j, \delta_n\rangle_{\ell^2} = \overline{g_n(\psi_j)}. \tag{4.16}$$

Multiplying matrices, we see that

$$U\tilde{M}^* = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ c_{10} & 0 & 0 & 0 & \cdots \\ c_{20} & c_{21} & 0 & 0 & \cdots \\ c_{30} & c_{31} & c_{32} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} 0 & \overline{\phi_0(\psi_1)} & \overline{\phi_0(\psi_2)} & \overline{\phi_0(\psi_3)} & \cdots \\ 0 & 0 & \overline{\phi_1(\psi_2)} & \overline{\phi_1(\psi_3)} & \cdots \\ 0 & 0 & 0 & \overline{\phi_2(\psi_3)} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & c_{10}\overline{\phi_0(\psi_1)} & c_{10}\overline{\phi_0(\psi_2)} & c_{10}\overline{\phi_0(\psi_3)} & \cdots \\ 0 & c_{20}\overline{\phi_0(\psi_1)} & \sum_{k=0}^{1} c_{2k}\overline{\phi_k(\psi_2)} & \sum_{k=0}^{1} c_{2k}\overline{\phi_k(\psi_3)} & \cdots \\ 0 & c_{30}\overline{\phi_0(\psi_1)} & \sum_{k=0}^{1} c_{3k}\overline{\phi_k(\psi_2)} & \sum_{k=0}^{2} c_{3k}\overline{\phi_k(\psi_3)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$U\tilde{M}^* + \tilde{M}^* + I = \begin{pmatrix} 1 & \overline{\phi_0(\psi_1)} & \overline{\phi_0(\psi_2)} & \overline{\phi_0(\psi_3)} & \cdots \\ 0 & \sum_{k=0}^{1} c_{1k}\overline{\phi_k(\psi_1)} & \sum_{k=0}^{1} c_{1k}\overline{\phi_k(\psi_2)} & \sum_{k=0}^{1} c_{1k}\overline{\phi_k(\psi_3)} & \cdots \\ 0 & c_{20}\overline{\phi_0(\psi_1)} & \sum_{k=0}^{2} c_{2k}\overline{\phi_k(\psi_2)} & \sum_{k=0}^{2} c_{2k}\overline{\phi_k(\psi_3)} & \cdots \\ 0 & c_{30}\overline{\phi_0(\psi_1)} & \sum_{k=0}^{1} c_{3k}\overline{\phi_k(\psi_2)} & \sum_{k=0}^{3} c_{3k}\overline{\phi_k(\psi_3)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that the (n, j) entry of $U\tilde{M}^* + I + \tilde{M}^*$ is given by

$$\begin{cases} 0, \text{ if } n \geq 1, j = 0 \\ \sum_{k=0}^{j-1} c_{nk}\overline{\phi_k(\psi_j)}, \text{ if } n > j, j \geq 1 \\ \sum_{k=0}^{n} c_{nk}\overline{\phi_k(\psi_j)}, \text{ if } n \leq j \end{cases}. \tag{4.17}$$

We now exploit the relationship between $(I + M)$ and $(I + U)$ to more cleanly write the entries of $U\tilde{M}^* + \tilde{M}^* + I$.

$$(I+U)(I+M) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ c_{10} & 1 & 0 & 0 & \cdots \\ c_{20} & c_{21} & 1 & 0 & \cdots \\ c_{30} & c_{31} & c_{32} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \overline{\phi_1(\psi_0)} & 1 & 0 & 0 & \cdots \\ \overline{\phi_2(\psi_0)} & \overline{\phi_2(\psi_1)} & 1 & 0 & \cdots \\ \overline{\phi_3(\psi_0)} & \overline{\phi_3(\psi_1)} & \overline{\phi_3(\psi_2)} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \sum_{k=0}^{1} c_{1k}\overline{\phi_k(\psi_0)} & 1 & 0 & 0 & \cdots \\ \sum_{k=0}^{2} c_{2k}\overline{\phi_k(\psi_0)} & \sum_{k=1}^{2} c_{2k}\overline{\phi_k(\psi_1)} & 1 & 0 & \cdots \\ \sum_{k=0}^{3} c_{3k}\overline{\phi_k(\psi_0)} & \sum_{k=1}^{3} c_{3k}\overline{\phi_k(\psi_1)} & \sum_{k=2}^{3} c_{3k}\overline{\phi_k(\psi_2)} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

From $(I+U)(I+M) = I$, we know that if $n > j$

$$\sum_{k=j}^{n} c_{nk}\overline{\phi_k(\psi_j)} = 0. \tag{4.18}$$

We infer from (4.17) that the (n,j) entry of $U\tilde{M}^* + I + \tilde{M}^*$ is given by

$$\sum_{k=0}^{n} c_{nk}\overline{\phi_k(\psi_j)}.$$

Because $\overline{g_n} = \sum_{k=0}^{n} c_{nk}\overline{\phi_k}$ by (4.14), we see that

$$\overline{g_n}(\psi_j) = \left(\sum_{k=0}^{n} c_{nk}\overline{\phi_k}\right)\psi_j = \sum_{k=0}^{n} c_{nk}\overline{\phi_k(\psi_j)}$$

and conclude that

$$\langle (U\tilde{M}^* + \tilde{M}^* + I)\delta_j, \delta_n\rangle_{\ell^2} = \overline{g_n(\psi_j)}.$$

We can now multiply

$$\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I) = \begin{pmatrix} 0 & \overline{\phi_0(\psi_1)} & \overline{\phi_0(\psi_2)} & \cdots \\ 0 & 0 & \overline{\phi_1(\psi_2)} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} \overline{g_0(\psi_0)} & \overline{g_0(\psi_1)} & \overline{g_0(\psi_2)} & \cdots \\ \overline{g_1(\psi_0)} & \overline{g_1(\psi_1)} & \overline{g_1(\psi_2)} & \cdots \\ \overline{g_2(\psi_0)} & \overline{g_2(\psi_1)} & \overline{g_2(\psi_2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=1}^{\infty} \overline{\phi_0(\psi_k)g_k(\psi_0)} & \sum_{k=1}^{\infty} \overline{\phi_0(\psi_k)g_k(\psi_1)} & \sum_{k=1}^{\infty} \overline{\phi_0(\psi_k)g_k(\psi_2)} & \cdots \\ \sum_{k=2}^{\infty} \overline{\phi_1(\psi_k)g_k(\psi_0)} & \sum_{k=2}^{\infty} \overline{\phi_1(\psi_k)g_k(\psi_1)} & \sum_{k=2}^{\infty} \overline{\phi_1(\psi_k)g_k(\psi_2)} & \cdots \\ \sum_{k=3}^{\infty} \overline{\phi_2(\psi_k)g_k(\psi_0)} & \sum_{k=3}^{\infty} \overline{\phi_2(\psi_k)g_k(\psi_1)} & \sum_{k=3}^{\infty} \overline{\phi_2(\psi_k)g_k(\psi_2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and see that the (i,j) entry of $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)$ is given by $\sum_{k=i+1}^{\infty} \overline{\phi_i(\psi_k)g_k(\psi_j)}$.

Next, notice that

$$\sum_{k=0}^{\infty} \phi_i(\psi_k) g_k(\psi_j) = \sum_{k=0}^{i} \phi_i(\psi_k) g_k(\psi_j) + \sum_{k=i+1}^{\infty} \phi_i(\psi_k) g_k(\psi_j). \tag{4.19}$$

Because $\phi_i = \sum_{k=0}^{i} \phi_i(\psi_k) g_k$ by (4.8), we see that

$$\sum_{k=0}^{i} \phi_i(\psi_k) g_k(\psi_j) = \left(\sum_{k=0}^{i} \phi_i(\psi_k) g_k \right) \psi_j = \phi_i(\psi_j)$$

and from the above derivations, we have

$$\sum_{k=0}^{\infty} g_k(\psi_j) \phi_i(\psi_k) = \phi_i(\psi_j) + \sum_{k=i+1}^{\infty} g_k(\psi_j) \phi_i(\psi_k). \tag{4.20}$$

We note that the last sum in (4.20) is simply the conjugate of the (i, j) entry of $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)$.

Suppose that $\langle \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)\delta_j, \delta_i \rangle = 0$ for all $i, j \in \mathbb{N}_0$. It follows that

$$\sum_{k=i+1}^{\infty} \overline{g_k(\psi_j)\phi_i(\psi_k)} = 0 = \sum_{k=i+1}^{\infty} g_k(\psi_j)\phi_i(\psi_k)$$

(and thus converges) for all $i, j \in \mathbb{N}_0$. By (4.20), we conclude that

$$\sum_{k=0}^{\infty} \phi_i(\psi_k) g_k(\psi_j) = \phi_i(\psi_j)$$

and see that $\{(\phi_n, \psi_n)\}$ is weakly effective on $\Phi \times \Psi$ by definition.

Conversely, suppose that $\{(\phi_n, \psi_n)\}$ is weakly effective on $\Phi \times \Psi$ so that

$$\sum_{k=0}^{\infty} \phi_i(\psi_k) g_k(\psi_j) = \phi_i(\psi_j)$$

for all $i, j \in \mathbb{N}_0$. By (4.20), we know that $\sum_{k=i+1}^{\infty} g_k(\psi_j)\phi_i(\psi_k) = 0 = \sum_{k=i+1}^{\infty} \overline{g_k(\psi_j)\phi_i(\psi_k)}$ (and thus converges) for all $i, j \in \mathbb{N}_0$. It follows that $\langle \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)\delta_j, \delta_i \rangle_{\ell^2} = 0$ for all $i, j \in \mathbb{N}_0$.

(2) and (3) are equivalent by definition as $x_n = \sum_{k=0}^{n} g_k(x)\psi_k$ for all $x \in X$. $\square$

4.3.2 $\{\tilde{g}_n\}$ and $\{\phi_n\}$ Duality

To investigate the duality of $\{\tilde{g}_n\}$ and $\{\phi_n\}$, we proceed in a manner analogous to the previous section, achieving a matrix characterization for weak dense effectively.

Theorem 5. *Let X be a Banach space and $\{\phi_n\}$ and $\{\psi_n\}$ be linearly dense sequences in X^* and X, respectively, where $\phi_n(\psi_n) = 1$ for all $n \in \mathbb{N}_0$. Suppose $\Phi \subseteq X^*$ and $\Psi \subseteq X$ are the linear spans of $\{\phi_n\}$ and $\{\psi_n\}$, respectively. Let $M, \tilde{M}, U, \tilde{U}$ be as defined in (4.5) and (4.7). Then the following are equivalent:*

1. *$\langle M^*(\tilde{U}M^* + M^* + I)\delta_j, \delta_i\rangle_{\ell^2} = 0$ for all $i, j \in \mathbb{N}_0$.*
2. *$\{(\psi_n, \phi_n)\}$ is weakly effective on $\Psi \times \Phi$.*
3. *$\{\tilde{g}_n\}$ and $\{\phi_n\}$ form a weak resolution of the identity on $\Psi \times \Phi$.*

Proof. As Φ and Ψ consist of finite linear combinations, and all of the involved operators are linear, it suffices to consider (2) and (3) on the collections $\{\phi_n\}$ and $\{\psi_n\}$.

We first show that

$$\langle(\tilde{U}M^* + M^* + I)\delta_j, \delta_n\rangle_{\ell^2} = \phi_j(\tilde{g}_n). \tag{4.21}$$

Multiplying matrices, we see that

$$\tilde{U}M^* = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \tilde{c}_{10} & 0 & 0 & 0 & \cdots \\ \tilde{c}_{20} & \tilde{c}_{21} & 0 & 0 & \cdots \\ \tilde{c}_{30} & \tilde{c}_{31} & \tilde{c}_{32} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} 0 & \phi_1(\psi_0) & \phi_2(\psi_0) & \phi_3(\psi_0) & \cdots \\ 0 & 0 & \phi_2(\psi_1) & \phi_3(\psi_1) & \cdots \\ 0 & 0 & 0 & \phi_3(\psi_2) & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & \tilde{c}_{10}\phi_1(\psi_0) & \tilde{c}_{10}\phi_2(\psi_0) & \tilde{c}_{10}\phi_3(\psi_0) & \cdots \\ 0 & \tilde{c}_{20}\phi_1(\psi_0) & \sum_{k=0}^{1}\tilde{c}_{2k}\phi_2(\psi_k) & \sum_{k=0}^{1}\tilde{c}_{2k}\phi_3(\psi_k) & \cdots \\ 0 & \tilde{c}_{30}\phi_1(\psi_0) & \sum_{k=0}^{1}\tilde{c}_{3k}\phi_2(\psi_k) & \sum_{k=0}^{2}\tilde{c}_{3k}\phi_3(\psi_k) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$\tilde{U}M^* + M^* + I = \begin{pmatrix} 1 & \phi_1(\psi_0) & \phi_2(\psi_0) & \phi_3(\psi_0) & \cdots \\ 0 & \sum_{k=0}^{1} \tilde{c}_{1k}\phi_1(\psi_k) & \sum_{k=0}^{1} \tilde{c}_{1k}\phi_2(\psi_k) & \sum_{k=0}^{1} \tilde{c}_{1k}\phi_3(\psi_k) & \cdots \\ 0 & \tilde{c}_{20}\phi_1(\psi_0) & \sum_{k=0}^{2} \tilde{c}_{2k}\phi_2(\psi_k) & \sum_{k=0}^{2} \tilde{c}_{2k}\phi_3(\psi_k) & \cdots \\ 0 & \tilde{c}_{30}\phi_1(\psi_0) & \sum_{k=0}^{1} \tilde{c}_{3k}\phi_2(\psi_k) & \sum_{k=0}^{3} \tilde{c}_{3k}\phi_3(\psi_k) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that the (n, j) entry of $\tilde{U}M^* + M^* + I = \tilde{U}M^* + I + M^*$ is given by

$$\begin{cases} 0, \text{ if } n \geq 1, j = 0 \\ \sum_{k=0}^{j-1} \tilde{c}_{nk}\phi_j(\psi_k), \text{ if } n > j, j \neq 0 \\ \sum_{k=0}^{n} \tilde{c}_{nk}\phi_j(\psi_k), \text{ if } n < j. \end{cases} \tag{4.22}$$

We now use the relationship $(I + \tilde{U})(I + \tilde{M}) = I$ to write

$$I = (I + \tilde{U})(I + \tilde{M}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \tilde{c}_{10} & 1 & 0 & 0 & \cdots \\ \tilde{c}_{20} & \tilde{c}_{21} & 1 & 0 & \cdots \\ \tilde{c}_{30} & \tilde{c}_{31} & \tilde{c}_{32} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \phi_0(\psi_1) & 1 & 0 & 0 & \cdots \\ \phi_0(\psi_2) & \phi_1(\psi_2) & 1 & 0 & \cdots \\ \phi_0(\psi_3) & \phi_1(\psi_3) & \phi_2(\psi_3) & 1 & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \sum_{k=0}^{1} \tilde{c}_{1k}\phi_0(\psi_k) & 1 & 0 & 0 & \cdots \\ \sum_{k=0}^{2} \tilde{c}_{2k}\phi_0(\psi_k) & \sum_{k=1}^{2} \tilde{c}_{2k}\phi_1(\psi_k) & 1 & 0 & \cdots \\ \sum_{k=0}^{3} \tilde{c}_{3k}\phi_0(\psi_k) & \sum_{k=1}^{3} \tilde{c}_{3k}\phi_1(\psi_k) & \sum_{k=2}^{3} \tilde{c}_{3k}\phi_2(\psi_k) & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

and we see that for $n > j$, $\sum_{k=j}^{n} \tilde{c}_{nk}\phi_j(\psi_k) = 0$. This combined with (4.22) yields

$$\langle (\tilde{U}M^* + M^* + I)\delta_j, \delta_n \rangle_{\ell^2} = \sum_{k=0}^{n} \tilde{c}_{nk}\phi_j(\psi_k). \tag{4.23}$$

Using $\tilde{g}_n = \sum_{k=0}^{n} \tilde{c}_{nk}\psi_k$ from (4.15), we derive

$$\begin{aligned}\phi_j(\tilde{g}_n) &= \phi_j\left(\sum_{k=0}^{n} \tilde{c}_{nk}\psi_k\right)\\ &= \sum_{k=0}^{n} \tilde{c}_{nk}\phi_j(\psi_k)\end{aligned}$$

and conclude that

$$\phi_j(\tilde{g}_n) = \langle(\tilde{U}M^* + M^* + I)\delta_j, \delta_n\rangle_{\ell^2}.$$

We may now multiply

$$M^*(\tilde{U}M^* + M^* + I) = \begin{pmatrix} 0 & \phi_1(\psi_0) & \phi_2(\psi_0) & \cdots \\ 0 & 0 & \phi_2(\psi_1) & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} \phi_0(\tilde{g}_0) & \phi_1(\tilde{g}_0) & \phi_2(\tilde{g}_0) & \cdots \\ \phi_0(\tilde{g}_1) & \phi_1(\tilde{g}_1) & \phi_2(\tilde{g}_1) & \cdots \\ \phi_0(\tilde{g}_2) & \phi_1(\tilde{g}_2) & \phi_2(\tilde{g}_2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=1}^{\infty} \phi_k(\psi_0)\phi_0(\tilde{g}_k) & \sum_{k=1}^{\infty} \phi_k(\psi_0)\phi_1(\tilde{g}_k) & \sum_{k=1}^{\infty} \phi_k(\psi_0)\phi_2(\tilde{g}_k) & \cdots \\ \sum_{k=2}^{\infty} \phi_k(\psi_1)\phi_0(\tilde{g}_k) & \sum_{k=2}^{\infty} \phi_k(\psi_1)\phi_1(\tilde{g}_k) & \sum_{k=2}^{\infty} \phi_k(\psi_1)\phi_2(\tilde{g}_k) & \cdots \\ \sum_{k=3}^{\infty} \phi_k(\psi_2)\phi_0(\tilde{g}_k) & \sum_{k=3}^{\infty} \phi_k(\psi_2)\phi_1(\tilde{g}_k) & \sum_{k=3}^{\infty} \phi_k(\psi_2)\phi_2(\tilde{g}_k) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and see that the (i, j) entry of $M^*(\tilde{U}M^* + M^* + I)$ is given by $\sum_{k=i+1}^{\infty} \phi_k(\psi_i)\phi_j(\tilde{g}_k)$.

Notice that

$$\sum_{k=0}^{\infty} \phi_k(\psi_i)\phi_j(\tilde{g}_k) = \sum_{k=0}^{i} \phi_k(\psi_i)\phi_j(\tilde{g}_k) + \sum_{k=i+1}^{\infty} \phi_k(\psi_i)\phi_j(\tilde{g}_k). \tag{4.24}$$

Because $\psi_i = \sum_{k=0}^{i} \phi_k(\psi_i)\tilde{g}_k$ by (4.11), we see that

$$\phi_j(\psi_i) = \phi_j\left(\sum_{k=0}^{i} \phi_k(\psi_i)\tilde{g}_k\right) = \sum_{k=0}^{i} \phi_k(\psi_i)\phi_j(\tilde{g}_k)$$

and from the above derivations we have

$$\sum_{k=0}^{\infty} \phi_k(\psi_i)\phi_j(\tilde{g}_k) = \phi_j(\psi_i) + \sum_{k=i+1}^{\infty} \phi_k(\psi_i)\phi_j(\tilde{g}_k). \tag{4.25}$$

We note that the last sum in (4.25) is simply the (i, j) entry of $M^*(\tilde{U}M^* + M^* + I)$.

Suppose that $\langle M^*(\tilde{U}M^* + M^* + I)\delta_j, \delta_i\rangle = 0$ for all $i, j \in \mathbb{N}_0$. It follows that

$$\sum_{k=i+1}^{\infty} \phi_k(\psi_i)\phi_j(\tilde{g}_k) = 0$$

(and thus converges) for all $i, j \in \mathbb{N}_0$. By (4.25), we conclude that

$$\sum_{k=0}^{\infty} \phi_k(\psi_i)\phi_j(\tilde{g}_k) = \phi_j(\psi_i).$$

$\{(\psi_n, \phi_n)\}$ is now weakly effective on $\Psi \times \Phi$ by definition.

Conversely, suppose that $\{(\psi_n, \phi_n)\}$ is weakly effective on $\Psi \times \Phi$. It follows that

$$\sum_{k=0}^{\infty} \phi_k(\psi_i)\phi_j(\tilde{g}_k) = \phi_j(\psi_i)$$

for all $i, j \in \mathbb{N}_0$. By (4.25), we know that $\sum_{k=i+1}^{\infty} \phi_k(\psi_i)\phi_j(\tilde{g}_k) = 0$ (and thus converges) for all $i, j \in \mathbb{N}_0$. It follows that $\langle M^*(\tilde{U}M^* + M^* + I)\delta_j, \delta_i\rangle_{\ell^2} = 0$ for all $i, j \in \mathbb{N}_0$.

(2) and (3) are equivalent by definition as $u_n = \sum_{k=0}^{n} u(\tilde{g}_k)\phi_k$ for all $u \in X^*$. □

4.3.3 $\{g_n\}$ and $\{\tilde{g}_n\}$ Duality

In order to investigate the duality of $\{(g_n, \tilde{g}_n)\}$, we will need to introduce a collection of subspaces inside of $\ell^2(\mathbb{N}_0)$. Let $\mathcal{F}(\mathbb{N}_0) \subseteq \ell^2(\mathbb{N}_0)$ denote the set of all finite linear combinations of the collection $\{\delta_n\}$. Next, let

$$\begin{aligned} \mathcal{H}_0 &= \overline{M^*(\mathcal{F}(\mathbb{N}_0))} \subseteq \ell^2(\mathbb{N}_0) \\ \tilde{\mathcal{H}}_0 &= \overline{\tilde{M}^*(\mathcal{F}(\mathbb{N}_0))} \subseteq \ell^2(\mathbb{N}_0). \end{aligned} \tag{4.26}$$

As $\mathcal{H}_0$ and $\tilde{\mathcal{H}}_0$ are both closed, we have that

$$\mathcal{H}_0 \oplus \mathcal{H}_0^{\perp} = \tilde{\mathcal{H}}_0 \oplus \tilde{\mathcal{H}}_0^{\perp} = \ell^2(\mathbb{N}_0). \tag{4.27}$$

Define U and $\tilde{U}$ as in (4.5) and (4.7), respectively. If U and $\tilde{U}$ are both bounded operators on $\ell^2(\mathbb{N}_0)$, then $U^*\tilde{U}$ will be as well. By (4.26) and (4.27), we may represent $U^*\tilde{U}$ as the matrix of operators

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{4.28}$$

where

$$A : \mathcal{H}_0 \to \tilde{\mathcal{H}}_0, \quad B : \mathcal{H}_0^\perp \to \tilde{\mathcal{H}}_0, \quad C : \mathcal{H}_0 \to \tilde{\mathcal{H}}_0^\perp, \quad D : \mathcal{H}_0^\perp \to \tilde{\mathcal{H}}_0^\perp. \tag{4.29}$$

Theorem 6. *Let X be a Banach space and $\{\phi_n\}$ and $\{\psi_n\}$ be linearly dense sequences in X^* and X, respectively, where $\phi_n(\psi_n) = 1$ for all $n \in \mathbb{N}_0$. Suppose $\Phi \subseteq X^*$ and $\Psi \subseteq X$ are the linear spans of $\{\phi_n\}$ and $\{\psi_n\}$, respectively.. Let $M, \tilde{M}, U, \tilde{U}$ be as defined in (4.5) and (4.7) and let U and $\tilde{U}$ be bounded operators on $\ell^2(\mathbb{N}_0)$. Then the following are equivalent:*

1. *$\langle(\tilde{M}U^*\tilde{U}M^* - \tilde{M}M^*)\delta_j, \delta_i\rangle_{\ell^2} = 0$ for all $i, j \in \mathbb{N}_0$.*

2. *$\{g_n\}$ and $\{\tilde{g}_n\}$ are weakly dual on $\Phi \times \Psi$.*

3. *$A = P_{\tilde{\mathcal{H}}_0}\Big|_{\mathcal{H}_0}$ (the projection on $\tilde{\mathcal{H}}_0$ restricted to $\mathcal{H}_0$).*

We again point out that this situation is markedly different than the effective sequence case, where the duality of the auxiliary sequence (with itself) implies the effectivity of the sequence in question. Although analogous matrix conditions are achieved in our setting, effectivity is not. In [HS05], Haller and Szwarc connect their matrix condition for $NV^*VN^* - NN^*$ to the operator V^*V being an orthogonal projection. We achieve an generalized result from our matrix condition for $\tilde{M}U^*\tilde{U}M^* - \tilde{M}M^*$, in our case showing that a particular component of the operator $U^*\tilde{U}$ is equal to a projection with restricted domain.

Proof. As Φ and Ψ consist of finite linear combinations, and all of the involved operators are linear, it suffices to consider (2) on the collections $\{\phi_n\}$ and $\{\psi_n\}$.

Because M^* and $\tilde{M}^*$ are upper triangular, their columns are elements in $\ell^2(\mathbb{N}_0)$. As $\tilde{U}$ and U are assumed to be bounded operators on $\ell^2(\mathbb{N}_0)$, we also know that $(\tilde{U}M^* + M^* + I)\delta_j \in \ell^2(\mathbb{N}_0)$ and $(U\tilde{M}^* + \tilde{M}^* + I)\delta_i \in \ell^2(\mathbb{N}_0)$ for all $i, j \in \mathbb{N}_0$. We may then use (4.16)

and (4.21) to derive

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_j(\tilde{g}_n)g_n(\psi_i) &= \sum_{n=0}^{\infty}\langle(\tilde{U}M^*+M^*+I)\delta_j,\delta_n\rangle_{\ell^2}\langle\delta_n,(U\tilde{M}^*+\tilde{M}^*+I)\delta_i\rangle_{\ell^2}\\ &= \langle(\tilde{U}M^*+M^*+I)\delta_j,(U\tilde{M}^*+\tilde{M}^*+I)\delta_i\rangle_{\ell^2}\\ &= \langle(U\tilde{M}^*+\tilde{M}^*+I)^*(\tilde{U}M^*+M^*+I)\delta_j,\delta_i\rangle_{\ell^2}.\end{aligned} \tag{4.30}$$

Remarks on Notation. Given an infinite matrix T with entries in $\mathbb{C}$, we use the notation $\langle T\delta_j,\delta_n\rangle_{\ell^2} = \langle\delta_j,T^*\delta_n\rangle_{\ell^2}$ to conveniently manipulate the entries of T. In this case, T^* represents the conjugate transpose of T.

We define the truncations M_n, $\tilde{M}_n$, U_n, and $\tilde{U}_n$ (which are all bounded operators on $\ell_2(\mathbb{N}_0)$ and also strictly lower triangular) as follows:

$$M_n = \begin{pmatrix} 0 & & & & \\ \overline{\phi_1(\psi_0)} & 0 & & & \\ \vdots & \ddots & 0 & & \\ \overline{\phi_n(\psi_0)} & \cdots & \overline{\phi_n(\psi_{n-1})} & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad U_n = \begin{pmatrix} 0 & & & & \\ c_{10} & 0 & & & \\ \vdots & \ddots & 0 & & \\ c_{n0} & \cdots & c_{n,n-1} & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Define $\tilde{M}_n, \tilde{U}_n$ similarly. Because $(I+U_n)(I+M_n) = I$ and $(I+\tilde{U}_n)(I+\tilde{M}_n) = I$, by the definition of $I+M$ and $I+U$ we have

$$\begin{aligned} U_nM_n &= -U_n - M_n = M_nU_n \\ \tilde{U}_n\tilde{M}_n &= -\tilde{U}_n - \tilde{M}_n = \tilde{M}_n\tilde{U}_n. \end{aligned}$$

We now consider the expressions $U_n\tilde{M}_n^* + \tilde{M}_n^* + I$ and $\tilde{U}_nM_n^* + M_n^* + I$ to calculate

$$\begin{aligned}&(U_n\tilde{M}_n^* + \tilde{M}_n^* + I)^*(\tilde{U}_nM_n^* + M_n^* + I)\\ &= (\tilde{M}_nU_n^* + \tilde{M}_n + I)(\tilde{U}_nM_n^* + M_n^* + I)\\ &= \tilde{M}_nU_n^*\tilde{U}_nM_n^* + \tilde{M}_nU_n^*M_n^* + \tilde{M}_nU_n^* + \tilde{M}_n\tilde{U}_nM_n^* + \tilde{M}_nM_n^* + \tilde{M}_n + \tilde{U}_nM_n^* + M_n^* + I\end{aligned}$$

$$
\begin{aligned}
&= \tilde{M}_n U_n^* \tilde{U}_n M_n^* + \tilde{M}_n(-U_n^* - M_n^*) + \tilde{M}_n U_n^* + (-\tilde{M}_n - \tilde{U}_n) M_n^* \\
&+ \tilde{M}_n M_n^* + \tilde{M}_n + \tilde{U}_n M_n^* + M_n^* + I \\
&= \tilde{M}_n U_n^* \tilde{U}_n M_n^* - \tilde{M}_n U_n^* - \tilde{M}_n M_n^* + \tilde{M}_n U_n^* - \tilde{M}_n M_n^* - \tilde{U}_n M_n^* \\
&+ \tilde{M}_n M_n^* + \tilde{M}_n + \tilde{U}_n M_n^* + M_n^* + I \\
&= \tilde{M}_n U_n^* \tilde{U}_n M_n^* - \tilde{M}_n M_n^* + \tilde{M}_n + M_n^* + I.
\end{aligned}
$$

From this we conclude

$$
\begin{aligned}
&\langle (U_n \tilde{M}_n^* + \tilde{M}_n^* + I)^* (\tilde{U}_n M_n^* + M_n^* + I)\delta_j, \delta_i \rangle_{\ell^2} \\
&= \langle (\tilde{M}_n U_n^* \tilde{U}_n M_n^* - \tilde{M}_n M_n^*)\delta_j, \delta_i \rangle_{\ell^2} + \langle (M_n + \tilde{M}_n^* + I)\delta_j, \delta_i \rangle_{\ell^2} \\
&= \langle \tilde{U}_n M_n^* \delta_j, U_n \tilde{M}_n^* \delta_i \rangle_{\ell^2} - \langle \tilde{M}_n M_n^* \delta_j, \delta_i \rangle_{\ell^2} + \langle (\tilde{M}_n + M_n^* + I)\delta_j, \delta_i \rangle_{\ell^2}.
\end{aligned} \tag{4.31}
$$

At this point we wish to take limits as $n \to \infty$. This must be done with care as we only know that U and $\tilde{U}$ are bounded operators. We will exploit our knowledge of the pointwise entries of the operators in question as well as their triangular structures to achieve the desired result. Before this, however, we must establish the strong convergence of the operators $U_n, \tilde{U}_n$.

Lemma 9. *$U_n \to U$, $\tilde{U}_n \to \tilde{U}$, $U_n^* \to U^*$, and $\tilde{U}_n^* \to \tilde{U}^*$ strongly in $\ell^2(\mathbb{N}_0)$.*

Proof. Choose $\{a_k\}$ in $\ell^2(\mathbb{N}_0)$. As U is bounded on $\ell^2(\mathbb{N}_0)$, it follows that there exists some $C > 0$ such that

$$
\|U(\{a_k\})\|^2_{\ell^2(\mathbb{N}_0)} = \sum_{\ell=1}^{\infty} \left| \sum_{k=0}^{\ell-1} c_{\ell k} a_k \right|^2 \leq C \|a_k\|^2_{\ell^2(\mathbb{N}_0)}. \tag{4.32}
$$

I.e., the series $\displaystyle\sum_{\ell=1}^{\infty} \left| \sum_{k=0}^{\ell-1} c_{\ell k} a_k \right|^2$ converges.

Recall that $U_n \to U$ strongly if $\|Ux - U_n x\| \to 0$ as $n \to \infty$ for any $x \in \ell^2(\mathbb{N}_0)$. For $\{a_k\} \in \ell^2(\mathbb{N}_0)$, we see that

$$\begin{aligned}\|U(\{a_k\}) - U_n(\{a_k\})\| &= \left\| \sum_{\ell=1}^{\infty} \left| \sum_{k=0}^{\ell-1} c_{\ell k} a_k \right|^2 - \sum_{\ell=1}^{n} \left| \sum_{k=0}^{\ell-1} c_{\ell k} a_k \right|^2 \right\| \\ &= \left\| \sum_{\ell=n+1}^{\infty} \left| \sum_{k=0}^{\ell-1} c_{\ell k} a_k \right|^2 \right\| \to 0 \text{ as } n \to \infty\end{aligned}$$

as it is the tail of a convergent series. We conclude that $U_n \to U$ strongly. Similarly, $\tilde{U}_n \to \tilde{U}$, $U_n^* \to U^*$, and $\tilde{U}_n^* \to \tilde{U}^*$ strongly in $\ell^2(\mathbb{N}_0)$. □

As M^* and $\tilde{M}^*$ are upper triangular, we know that for $n > j, i$ we have $M_n^* \delta_j = M^* \delta_j$ and $\tilde{M}_n^* \delta_i = \tilde{M}^* \delta_i$, from which we conclude:

$$\lim_{n\to\infty} \langle M_n^* \delta_j, \delta_i \rangle_{\ell^2} = \langle M^* \delta_j, \delta_i \rangle_{\ell^2}$$

$$\lim_{n\to\infty} \langle \tilde{M}_n \delta_j, \delta_i \rangle_{\ell^2} = \lim_{n\to\infty} \langle \delta_j, \tilde{M}_n^* \delta_i \rangle_{\ell^2} = \langle \delta_j, \tilde{M}^* \delta_i \rangle_{\ell^2} = \langle \tilde{M} \delta_j, \delta_i \rangle_{\ell^2}$$

$$\lim_{n\to\infty} \langle \tilde{M}_n M_n^* \delta_j, \delta_i \rangle_{\ell^2} = \lim_{n\to\infty} \langle M_n^* \delta_j, \tilde{M}_n^* \delta_i \rangle_{\ell^2} = \langle M^* \delta_j, \tilde{M}^* \delta_i \rangle_{\ell^2} = \langle \tilde{M} M^* \delta_j, \delta_i \rangle_{\ell^2}.$$

Even without any assumptions of $M, \tilde{M}$ being bounded, by virtue of their upper triangular structure, we know that $M^* \delta_j$, $\tilde{M}^* \delta_i$ are both sequences in $\ell^2(\mathbb{N}_0)$. We use this along with Lemma 9 to carefully justify rest of the the desired convergences.

Choose $i, j \in \mathbb{N}_0$ and $\varepsilon > 0$. As $\{\tilde{U}_n\}$ converges strongly to $\tilde{U}$, it also converges to $\tilde{U}$ in the weak operator topology. Consequently, there is some $N_u \in \mathbb{N}_0$ such that $|\langle \tilde{U}_n M^* \delta_j, \tilde{M}^* \delta_i \rangle_{\ell^2} - \langle \tilde{U} M^* \delta_j, \tilde{M}^* \delta_i \rangle_{\ell^2}| < \varepsilon$, as $M^* \delta_j$ and $\tilde{M}^* \delta_i$ are in $\ell^2(\mathbb{N}_0)$. Let $N > \max\{i, j, N_u\}$ and choose $n > N$. We then have that

$$|\langle \tilde{U}_n M_n^* \delta_j, \tilde{M}_n^* \delta_i \rangle_{\ell^2} - \langle \tilde{U} M^* \delta_j, \tilde{M}^* \delta_i \rangle_{\ell^2}| = |\langle \tilde{U}_n M^* \delta_j, \tilde{M}^* \delta_i \rangle_{\ell^2} - \langle \tilde{U} M^* \delta_j, \tilde{M}^* \delta_i \rangle_{\ell^2}| < \varepsilon$$

and conclude that

$$\lim_{n\to\infty} \langle \tilde{M}_n \tilde{U}_n M_n^* \delta_j, \delta_i \rangle_{\ell^2} = \lim_{n\to\infty} \langle \tilde{U}_n M_n^* \delta_j, \tilde{M}_n^* \delta_i \rangle_{\ell^2} = \langle \tilde{U} M^* \delta_j, \tilde{M}^* \delta_i \rangle_{\ell^2} = \langle \tilde{M} \tilde{U} M^* \delta_j, \delta_i \rangle_{\ell^2}.$$

Because $\{U_n^*\}$ and $\{\tilde{U}_n\}$ converge strongly to U^* and $\tilde{U}$, respectively, we know that $\{U_n^* \tilde{U}_n\}$ converges to $U^* \tilde{U}$ strongly. One can then use arguments analogous to the previous to show

the following:

$$\lim_{n\to\infty} \langle \tilde{M}_n U_n^* M_n^* \delta_j, \delta_i \rangle_{\ell^2} = \langle \tilde{M} U^* M^* \delta_j, \delta_i \rangle_{\ell^2}$$
$$\lim_{n\to\infty} \langle \tilde{M}_n U_n^* \tilde{U}_n M_n^* \delta_j, \delta_i \rangle_{\ell^2} = \langle \tilde{M} U^* \tilde{U} M^* \delta_j, \delta_i \rangle_{\ell^2}.$$

For any $i, j \in \mathbb{N}_0$, we may now evaluate the limit

$$\lim_{n\to\infty} \langle (U_n \tilde{M}_n^* + \tilde{M}_n^* + I)^* (\tilde{U}_n M_n^* + M_n^* + I)\delta_j, \delta_i \rangle_{\ell^2} = \langle (U\tilde{M}^* + \tilde{M}^* + I)^* (\tilde{U} M^* + M^* + I)\delta_j, \delta_i \rangle_{\ell^2}.$$

Finally, we take the limit of both sides of (4.31) as $n \to \infty$ to obtain

$$\begin{aligned} &\langle (U\tilde{M}^* + \tilde{M}^* + I)^* (\tilde{U} M^* + M^* + I)\delta_j, \delta_i \rangle_{\ell^2} \\ &= \langle \tilde{U} M^* \delta_j, U\tilde{M}^* \delta_i \rangle_{\ell^2} - \langle M^* \delta_j, \tilde{M}^* \delta_i \rangle_{\ell^2} + \langle (M^* + \tilde{M} + I)\delta_j, \delta_i \rangle_{\ell^2}. \end{aligned} \tag{4.33}$$

Note that the (i, j) entry of $M^* + \tilde{M} + I$ is simply $\phi_j(\psi_i)$. The equation then becomes:

$$\begin{aligned} &\langle (U\tilde{M}^* + \tilde{M}^* + I)^* (\tilde{U} M^* + M^* + I)\delta_j, \delta_i \rangle_{\ell^2} \\ &= \langle \tilde{U} M^* \delta_j, U\tilde{M}^* \delta_i \rangle_{\ell^2} - \langle M^* \delta_j, \tilde{M}^* \delta_i \rangle_{\ell^2} + \phi_j(\psi_i). \end{aligned} \tag{4.34}$$

Assume that (2) holds. By (4.30), we know that

$$\phi_j(\psi_i) = \sum_{n=0}^{\infty} \phi_j(\tilde{g}_n) g_n(\psi_i) = \langle (U\tilde{M}^* + \tilde{M}^* + I)^* (\tilde{U} M^* + M^* + I)\delta_j, \delta_i \rangle_{\ell^2}.$$

Combining this with (4.34), we obtain

$$\begin{aligned} &\langle \tilde{U} M^* \delta_j, U\tilde{M}^* \delta_i \rangle_{\ell^2} - \langle M^* \delta_j, \tilde{M}^* \delta_i \rangle_{\ell^2} = 0 \text{ for all } i, j \in \mathbb{N}_0 \\ &\Rightarrow \langle (\tilde{M} U^* \tilde{U} M^* - \tilde{M} M^*)\delta_j, \delta_i \rangle_{\ell^2} = 0 \text{ for all } i, j \in \mathbb{N}_0. \end{aligned}$$

Conversely, suppose (1) holds. By (4.34),

$$\phi_j(\psi_i) = \langle (U\tilde{M}^* + \tilde{M}^* + I)^* (\tilde{U} M^* + M^* + I)\delta_j, \delta_i \rangle_{\ell^2} \text{ for all } i, j \in \mathbb{N}_0$$

and by (4.30), we see that

$$\phi_j(\psi_i) = \sum_{n=0}^{\infty} \phi_j(\tilde{g}_n) g_n(\psi_i),$$

as desired.

We will now show that (1) is equivalent to (3). Assume that (1) holds. We begin by proving that

$$\langle U^*\tilde{U}x, y\rangle = \langle x, y\rangle \text{ if } x \in \mathcal{H}_0 \text{ and } y \in \tilde{\mathcal{H}}_0. \tag{4.35}$$

Let $x \in \mathcal{H}_0$ and suppose that $x = M^*w$ where $w = \sum_{i=0}^{n} \alpha_i\delta_i$ for some $\{\alpha_k\} \subseteq \mathbb{C}$ and $n \in \mathbb{N}_0$. Similarly, let $y \in \tilde{\mathcal{H}}_0$ and suppose that $y = \tilde{M}^*z$, where and $z = \sum_{k=0}^{m} \beta_k\delta_k$ for some $\{\beta_k\} \subseteq \mathbb{C}$ and $m \in \mathbb{N}_0$. We derive

$$\begin{aligned}
\langle U^*\tilde{U}x, y\rangle &= \langle U^*\tilde{U}M^*w, \tilde{M}^*z\rangle \\
&= \langle U^*\tilde{U}M^*\sum_{i=0}^{n}\alpha_i\delta_i, \tilde{M}^*\sum_{k=0}^{m}\beta_k\delta_k\rangle \\
&= \langle\sum_{i=0}^{n}\alpha_i U^*\tilde{U}M^*\delta_i, \sum_{k=0}^{m}\beta_k\tilde{M}^*\delta_k\rangle \\
&= \sum_{i=0}^{n}\alpha_i\sum_{k=0}^{m}\overline{\beta}_k\langle U^*\tilde{U}M^*\delta_i, \tilde{M}^*\delta_k\rangle \\
&= \sum_{i=0}^{n}\alpha_i\sum_{k=0}^{m}\overline{\beta}_k\langle M^*\delta_i, \tilde{M}^*\delta_k\rangle \quad \text{by (1)} \\
&= \langle M^*\sum_{i=0}^{n}\alpha_i\delta_i, \tilde{M}^*\sum_{k=0}^{m}\beta_k\delta_k\rangle \\
&= \langle M^*w, \tilde{M}^*z\rangle \\
&= \langle x, y\rangle.
\end{aligned}$$

Write $x = \lim_{n\to\infty} M^*w_n$ and $y = \lim_{k\to\infty} \tilde{M}^*z_k$, where $w_n, z_k \in \mathcal{F}(\mathbb{N}_0)$ for all n, k. As U and $\tilde{U}$ are bounded operators on $\ell^2(\mathbb{N}_0)$, by the work above we have that

$$\begin{aligned}
\langle U^*\tilde{U}x, y\rangle &= \langle U^*\tilde{U}\lim_{n\to\infty} M^*w_n, \lim_{k\to\infty}\tilde{M}^*z_k\rangle \\
&= \lim_{n\to\infty}\lim_{k\to\infty}\langle U^*\tilde{U}M^*w_n, \tilde{M}^*z_k\rangle \\
&= \lim_{n\to\infty}\lim_{k\to\infty}\langle M^*w_n, \tilde{M}^*z_k\rangle \\
&= \langle\lim_{n\to\infty} M^*w_n, \lim_{k\to\infty}\tilde{M}^*z_k\rangle \\
&= \langle x, y\rangle.
\end{aligned}$$

We conclude that (4.35) holds, noting that there are no continuity requirements on $\tilde{M}^*$ or M^* to achieve the above derivations.

Let $x = x_1 + x_2, y = y_1 + y_2 \in \ell^2(\mathbb{N}_0)$ where $x_1 \in \mathcal{H}_0$, $x_2 \in \mathcal{H}_0^\perp$, $y_1 \in \tilde{\mathcal{H}}_0$, and $y_2 \in \tilde{\mathcal{H}}_0^\perp$. Using A, B, C, and D as defined in (4.29), we may then write

$$\left\langle \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle_{\tilde{\mathcal{H}}_0 \oplus \tilde{\mathcal{H}}_0^\perp} = \langle Ax_1 + Bx_2, y_1 \rangle_{\tilde{\mathcal{H}}_0} + \langle Cx_1 + Dx_2, y_2 \rangle_{\tilde{\mathcal{H}}_0^\perp}. \tag{4.36}$$

Suppose that $x \in \mathcal{H}_0$ and $y \in \tilde{\mathcal{H}}_0$ so that $x_2 = y_2 = 0$. Furthermore, write $x_1 = z_1 + z_2$, where $z_1 \in \tilde{\mathcal{H}}_0$ and $z_2 \in \tilde{\mathcal{H}}_0^\perp$. From (4.36) we see that

$$\langle U^* \tilde{U} x_1, y_1 \rangle_{\ell^2} = \langle Ax_1, y_1 \rangle_{\tilde{\mathcal{H}}_0}. \tag{4.37}$$

Now calculate

$$\langle x_1, y_1 \rangle_{\ell^2} = \langle z_1, y_1 \rangle_{\ell^2} + \langle z_2, y_1 \rangle_{\ell^2} = \langle z_1, y_1 \rangle_{\ell^2} = \langle z_1, y_1 \rangle_{\tilde{\mathcal{H}}_0}. \tag{4.38}$$

By (4.35), $\langle x_1, y_1 \rangle_{\ell^2} = \langle U^* \tilde{U} x_1, y_1 \rangle_{\ell^2}$, so we infer from (4.37) and (4.38) that

$$\begin{aligned} \langle Ax_1, y_1 \rangle_{\tilde{\mathcal{H}}_0} &= \langle z_1, y_1 \rangle_{\tilde{\mathcal{H}}_0} \\ \Rightarrow \langle Ax_1 - z_1, y_1 \rangle_{\tilde{\mathcal{H}}_0} &= 0 \end{aligned}$$

for all $x_1 \in \mathcal{H}_0$ and $y_1 \in \tilde{\mathcal{H}}_0$. As $z_1 = P_{\tilde{\mathcal{H}}_0} x_1$ and $x_1 \in \mathcal{H}_0$, we conclude that $A = P_{\tilde{\mathcal{H}}_0}\Big|_{\mathcal{H}_0}$.

Conversely, assume that $A = P_{\tilde{\mathcal{H}}_0}\Big|_{\mathcal{H}_0}$. Let $x = x_1 + x_2$, $y = y_1 + y_2 \in \ell^2(\mathbb{N}_0)$ where $x_1 \in \mathcal{H}_0$, $x_2 \in \mathcal{H}_0^\perp$, $y_1 \in \tilde{\mathcal{H}}_0$, and $y_2 \in \tilde{\mathcal{H}}_0^\perp$. Suppose that $x \in \mathcal{H}_0$ and $y \in \tilde{\mathcal{H}}_0$, so that $x_2 = y_2 = 0$. By (4.36) we derive

$$\begin{aligned} \langle U^* \tilde{U} x, y \rangle_{\ell^2} &= \langle Ax_1, y_1 \rangle_{\tilde{\mathcal{H}}_0} \\ &= \langle Ax_1, y_1 \rangle_{\ell^2} \\ &= \langle P_{\tilde{\mathcal{H}}_0} x_1, y_1 \rangle_{\ell^2} \\ &= \langle x_1, P_{\tilde{\mathcal{H}}_0} y_1 \rangle_{\ell^2} \\ &= \langle x_1, y_1 \rangle_{\ell^2} \\ &= \langle x, y \rangle_{\ell^2}. \end{aligned}$$

and we conclude that for all $x \in \mathcal{H}_0$ and $y \in \tilde{\mathcal{H}}_0$,

$$\langle U^*\tilde{U}x, y\rangle = \langle x, y\rangle \text{ if } x \in \mathcal{H}_0.$$

As $M^*\delta_j \in \mathcal{H}_0$ and $\tilde{M}^*\delta_i \in \tilde{\mathcal{H}}_0$ for any $i, j \in \mathbb{N}_0$, it follows that

$$\begin{aligned} \left\langle U^*\tilde{U}M^*\delta_j, \tilde{M}^*\delta_i \right\rangle &= \left\langle M^*\delta_j, \tilde{M}^*\delta_i \right\rangle \\ \Rightarrow \left\langle \tilde{M}U^*\tilde{U}M^*\delta_j, \delta_i \right\rangle - \left\langle \tilde{M}M^*\delta_j, \delta_i \right\rangle &= 0 \\ \Rightarrow \left\langle \left(\tilde{M}U^*\tilde{U}M^* - \tilde{M}M^*\right)\delta_j, \delta_i \right\rangle &= 0 \end{aligned}$$

for any $i, j \in \mathbb{N}_0$, and the proof is finished. □

Ideally, we would like to release the assumption in Theorem 6 that U and $\tilde{U}$ are bounded. It is possible, however, that it is necessary to at least assume that the product $U^*\tilde{U}$ is bounded. This has been the case in every example and theorem that we have explored thus far. For example, in Chapter 2 we examined the behavior of pairs of sequences where $\psi_n = T\phi_n$ for all n (T positive and invertible). In this case, the mixed Grammian matrix $I + \tilde{M} + M^* = I + \tilde{M}^* + M$ can be shown to be equal to the Grammian matrix for the sequence $e_n = T^{\frac{1}{2}}\phi_n$. Because any Grammian of a single sequence will be positive, by the work in [HS05] we infer that $U = \tilde{U}$ is a contraction and $U^*\tilde{U}$ is bounded. In the following section, we will investigate an effective pair in which $U^*\tilde{U}$ is bounded, but $\tilde{U}$ is not. This example strengthens our hypothesis that the boundedness of $U^*\tilde{U}$ is a necessary condition for effectivity.

4.4 An Informative Example

At the beginning of the chapter, we claimed that the duality of $\{(g_n, \tilde{g}_n)\}$ was not equivalent to the duality of $\{(g_n, \psi_n)\}$ or $\{(\tilde{g}_n, \phi_n)\}$. We substantiate this claim with an example for which exactly two of these dualities holds. We accomplish this using the matrix characterizations developed in the previous sections. Namely, we will provide an example for which for all $i, j \in \mathbb{N}_0$,

(i) $\langle \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)\delta_j, \delta_i\rangle_{\ell^2} = 0$

(ii) $\langle M^*(\tilde{U}M^* + M^* + I)\delta_j, \delta_i\rangle \neq 0$

(iii) $\langle (\tilde{M}U^*\tilde{U}M^* - \tilde{M}M^*)\delta_j, \delta_i\rangle = 0.$

By our previous derivations, this will give the weak dense dualities of both $\{(g_n, \psi_n)\}$ and $\{(g_n, \tilde{g}_n)\}$. Because our example is finite-dimensional, this is equivalent to $\{(g_n, \psi_n)\}$ and $\{(g_n, \tilde{g}_n)\}$ achieving a full resolution of the identity. However, because

$$\langle M^*(\tilde{U}M^* + M^* + I)\delta_j, \delta_i\rangle \neq 0,$$

by Theorem 5 there is no duality of $\{(\tilde{g}_n, \phi_n)\}$, even in some limited sense.

Without further ado, we let $\{\phi_n\}, \{\psi_n\} \subseteq \mathbb{R}^2$, $\phi_n = \phi_{n+3}$, and $\psi_n = \psi_{n+3}$ for all n, where

$$\begin{bmatrix} \phi_0 & \phi_1 & \phi_2 & \psi_0 & \psi_1 & \psi_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & .5 & 1 & 1 & 1.5 \\ -1 & 1 & -.5 & 0 & 0 & -.5 \end{bmatrix}.$$

This example was briefly discussed in Chapter 2, wherein it was shown that $\{(\phi_n, \psi_n)\}$ was effective, but $\{(\psi_n, \phi_n)\}$ was not. Here we show the matrix properties directly.

Proposition 7. $\langle \tilde{M}^*(\tilde{M}^*U + \tilde{M}^* + I)\delta_n, \delta_j\rangle = 0$ *for all* n, j.

Proof. We proceed with several matrix calculations, exploiting the block structure engendered by the periodicity of the involved sequences. If

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ .5 & .5 & 1 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ .5 & .5 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 & -2 \\ 0 & -.5 & 1 \\ 0 & .25 & -.5 \end{pmatrix}, \; C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -.25 & .5 \\ 0 & .125 & -.25 \end{pmatrix},$$

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \; S = \begin{pmatrix} 1 & 1 & .5 \\ 1 & 1 & .5 \\ 2 & 1 & 1 \end{pmatrix}, \; W = \begin{pmatrix} .5 & -.5 & -.5 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \; T = \begin{pmatrix} .5 & -.5 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix},$$

then

$$I+M=\begin{pmatrix} F & 0 & 0 & 0 & \cdots \\ D & F & 0 & 0 & \cdots \\ D & D & F & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad I+U=\begin{pmatrix} F^{-1} & 0 & 0 & 0 & 0 & \cdots \\ B & F^{-1} & 0 & 0 & 0 & \cdots \\ C & B & F^{-1} & 0 & 0 & \cdots \\ \frac{1}{2}C & C & B & F^{-1} & 0 & \cdots \\ \frac{1}{4}C & \frac{1}{2}C & C & B & F^{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.39}$$

$$I+\tilde{M}=\begin{pmatrix} R & 0 & 0 & 0 & \cdots \\ S & R & 0 & 0 & \cdots \\ S & S & R & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad I+\tilde{U}=\begin{pmatrix} R^{-1} & 0 & 0 & 0 & 0 & \cdots \\ W & R^{-1} & 0 & 0 & 0 & \cdots \\ T & W & R^{-1} & 0 & 0 & \cdots \\ T & T & W & R^{-1} & 0 & \cdots \\ T & T & T & W & R^{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.40}$$

It is straightforward to check that $(I+M)(I+U)=I$ and $(I+\tilde{M})(I+\tilde{U})=I$. Let $(I+M)(I+U)_{(n,j)}$ indicate the 3 by 3 block of $(I+M)(I+U)$ in the (n,j) position. It is clear that $(I+M)(I+U)_{(n,j)}$ is equal to zero for $n<j$ and equal to I_3 for $n=j$. We use Sage and induction to calculate the following (Appendix B, Lines 1-21)

$$\begin{aligned}
(I+M)(I+U)_{(n+1,n)} &= DF^{-1}+FB=0 \\
(I+M)(I+U)_{(n+2,n)} &= DF^{-1}+DB+FC=0 \\
(I+M)(I+U)_{(n+3,n)} &= DC-\frac{1}{2}FC=0 \\
(I+M)(I+U)_{(n+k,n)} &= \frac{1}{2^{k-3}}(DC-\frac{1}{2}FC)=0 \text{ for } k>3.
\end{aligned}$$

We now show that $(I+\tilde{M})(I+\tilde{U})=I$. Again it is clear that $(I+\tilde{M})(I+\tilde{U})_{(n,j)}$ is equal to zero if $n<j$ and that $(I+\tilde{M})(I+\tilde{U})_{(n,j)}=I_3$ if $n=j$. Using Sage and induction,

we calculate

$$(I+\tilde{M})(I+\tilde{U})_{(n+1,n)} = SR^{-1} + RW = 0$$

$$(I+\tilde{M})(I+\tilde{U})_{(n+2,n)} = SR^{-1} + SW + RT = 0$$

$$(I+\tilde{M})(I+\tilde{U})_{(n+k,n)} = ST = 0 \text{ for } k \geq 3$$

and conclude that $(I+\tilde{M})(I+\tilde{U}) = I$ (Appendix B, Lines 22-33).

We next calculate the matrix $I + U\tilde{M}^* + \tilde{M}^*$.

$$(I+\tilde{M}^*) + U\tilde{M}^* = \begin{pmatrix} R^* & S^* & S^* & S^* & S^* & \cdots \\ 0 & R^* & S^* & S^* & S^* & \cdots \\ 0 & 0 & R^* & S^* & S^* & \cdots \\ 0 & 0 & 0 & R^* & S^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$+ \begin{pmatrix} F^{-1}-I_3 & 0 & 0 & 0 & 0 & \cdots \\ B & F^{-1}-I_3 & 0 & 0 & 0 & \cdots \\ C & B & F^{-1}-I_3 & 0 & 0 & \cdots \\ \frac{1}{2}C & C & B & F^{-1}-I_3 & 0 & \cdots \\ \frac{1}{4}C & \frac{1}{2}C & C & B & F^{-1}-I_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\cdot \begin{pmatrix} R^*-I_3 & S^* & S^* & S^* & S^* & \cdots \\ 0 & R^*-I_3 & S^* & S^* & S^* & \cdots \\ 0 & 0 & R^*-I_3 & S^* & S^* & \cdots \\ 0 & 0 & 0 & R^*-I_3 & S^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$
= \begin{pmatrix}
R^* + (F^{-1} - I_3)(R^* - I_3) & S^* + (F^{-1} - I_3)S^* & S^* + (F^{-1} - I_3)S^* & S^* + (F^{-1} - I_3)S^* & \cdots \\
B(R^* - I_3) & R^* + BS^* + (F^{-1} - I_3)(R^* - I_3) & S^* + BS^* + (F^{-1} - I_3)S^* & S^* + BS^* + (F^{-1} - I_3)S^* & \cdots \\
C(R^* - I_3) & CS^* + B(R^* - I_3) & R^* + CS^* + BS^* + (F^{-1} - I_3)(R^* - I_3) & S^* + CS^* + BS^* + (F^{-1} - I_3)S^* & \cdots \\
\frac{1}{2}C(R^* - I_3) & \frac{1}{2}CS^* + C(R^* - I_3) & \frac{1}{2}CS^* + CS^* + B(R^* - I_3) & R^* + \frac{1}{2}CS^* + CS^* + BS^* + (F^{-1} - I_3)(R^* - I_3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

Let $\tilde{M}^*(U\tilde{M}^*+\tilde{M}^*+I)_{(n,i)}$ denote the three by three block matrix of $\tilde{M}^*(U\tilde{M}^*+\tilde{M}^*+I)$ in the (n,i) position. To show that every entry of $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)$ is equal to zero, we present a series of lemmas exploiting its block structure. Specifically, we use induction across select diagonals, rows, and columns.

Lemma 10. *(Diagonal)* $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(i,i)} = 0$ *for all* $i \in \mathbb{N}$.

Proof. Using block matrix multiplication combined with the previous derivations, we compute the 3×3 principal submatrix of $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)$.

$$\begin{aligned}
\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(0,0)} &= (R^* - I_3)\left(R^* + (F^{-1} - I_3)(R^* - I_3)\right) + S^*B(R^* - I_3) \\
&+ S^*C(R^* - I_3) + \frac{1}{2}S^*C(R^* - I_3) + \frac{1}{4}S^*C(R^* - I_3) + \cdots \\
&= (R^* - I_3)\left(R^* + (F^{-1} - I_3)(R^* - I_3)\right) + S^*B(R^* - I_3) \\
&+ 2S^*C(R^* - I_3) \\
&= 0,
\end{aligned}$$

where the last line of calculations was completed in Sage (Appendix B, Lines 34-37).

We also calculate

$$\begin{aligned}
\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(1,1)} &= (R^* - I_3)\left(R^* + BS^* + (F^{-1} - I_3)(R^* - I_3)\right) \\
&+ S^*\left(CS^* + B(R^* - I_3)\right) \\
&+ S^*\left(\frac{1}{2}CS^* + C(R^* - I_3)\right) + S^*\left(\frac{1}{4}CS^* + \frac{1}{2}C(R^* - I_3)\right) \\
&+ S^*\left(\frac{1}{8}CS^* + \frac{1}{4}C(R^* - I_3)\right) + S^*\left(\frac{1}{16}CS^* + \frac{1}{8}C(R^* - I_3)\right) \\
&+ \cdots \\
&= (R^* - I_3)\left(R^* + BS^* + (F^{-1} - I_3)(R^* - I_3)\right) + S^*B(R^* - I_3) \\
&+ S^*CS^* + \frac{1}{2}S^*CS^* + \frac{1}{4}S^*CS^* + \frac{1}{8}S^*CS^* + \cdots \\
&+ S^*C(R^* - I_3) + \frac{1}{2}S^*C(R^* - I_3) + \frac{1}{4}S^*C(R^* - I_3) + \cdots \\
&= (R^* - I_3)\left(R^* + BS^* + (F^{-1} - I_3)(R^* - I_3)\right) + S^*B(R^* - I_3) \\
&+ 2S^*CS^* + 2S^*C(R^* - I_3) \\
&= 0,
\end{aligned}$$

where the last line of calculations was completed using Sage (Appendix B, Lines 39-41).

Notice that during both of these calculations we encountered geometric series when looking at the coefficients of the S^*CS^* and $S^*C(R^* - I_3)$ terms. This is a result of the structure of the matrix $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)$. Eventually, for large enough n, every (n, i)

block in the matrix $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)$ is equal to $\frac{1}{2}\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(n-1,i)}$. This provides us with the iterative structure necessary for the blocks in question to be well defined. Furthermore, it allows us to make successful induction arguments concerning the entries of $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)$.

Specifically, we know that for $n \geq 2$,

$$\begin{aligned}\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(n,n)} &= \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(n-1,n-1)} \\ &\quad + \frac{1}{2^{n-1}}(R^* - I_3)CS^* + \frac{1}{2^{n-1}}S^*CS^* \\ &= \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(n-1,n-1)} \\ &\quad + \frac{1}{2^{n-1}}\left((R^* - I_3)CS^* + S^*CS^*\right).\end{aligned} \tag{4.41}$$

Using Sage, we calculate $(R^* - I_3)CS^* + S^*CS^* = 0$ (Appendix B, Lines 42-45). We have already shown that $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(n,n)} = 0$ for $n = 0$ and $n = 1$. Suppose that $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(k,k)} = 0$ for some $k \geq 1$. By (4.41),

$$\begin{aligned}\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(k+1,k+1)} &= \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(k,k)} + \frac{1}{2^k}\left((R^* - I_3)CS^* + S^*CS^*\right) \\ &= 0 + \frac{1}{2^k} \cdot 0 \\ &= 0.\end{aligned}$$

By induction, the lemma is proven. □

Lemma 11. *(Subdiagonal)* $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(i+1,i)} = 0$ *for all* $i \in \mathbb{N}$.

Proof. We calculate

$$\begin{aligned}\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(1,0)} &= (R^* - I_3)B(R^* - I_3) \\ &\quad + S^*C(R^* - I_3) + \frac{1}{2}S^*C(R^* - I_3) + \frac{1}{4}S^*C(R^* - I_3) + \cdots \\ &= (R^* - I_3)B(R^* - I_3) + 2S^*C(R^* - I_3) \\ &= 0,\end{aligned}$$

where the last calculation was performed by Sage (Appendix B, Lines 46-49). Next, derive

$$\begin{aligned}
\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(2,1)} &= (R^* - I_3)\left(CS^* + B(R^* - I_3)\right) + S^*\left(\frac{1}{2}CS^* + C(R^* - I_3)\right) \\
&+ S^*\left(\frac{1}{4}CS^* + \frac{1}{2}C(R^* - I_3)\right) + S^*\left(\frac{1}{8}CS^* + \frac{1}{4}C(R^* - I_3)\right) \\
&+ \cdots \\
&= (R^* - I_3)\left(CS^* + B(R^* - I_3)\right) \\
&+ \frac{1}{2}S^*CS^* + \frac{1}{4}S^*CS^* + \frac{1}{8}S^*CS^* + \cdots \\
&+ S^*C(R^* - I_3) + \frac{1}{2}S^*C(R^* - I_3) + \frac{1}{4}S^*C(R^* - I_3) + \cdots \\
&= (R^* - I_3)\left(CS^* + B(R^* - I_3)\right) + S^*CS^* + 2S^*C(R^* - I_3) \\
&= 0,
\end{aligned}$$

where the last calculation was performed by Sage (Appendix B, Lines 50-53).

We see that $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(n+1,n)} = 0$ holds for $n = 0, 1$. Recall that $(R^* - I_3)CS^* + S^*CS^* = 0$ and note that, for $n \geq 2$,

$$\begin{aligned}
\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(n+1,n)} &= \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(n,n-1)} \\
&+ \frac{1}{2^{n-1}}(R^* - I_3)CS^* + \frac{1}{2^n}S^*CS^* \\
&= \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(n,n-1)} \\
&+ \frac{1}{2^{n-1}}\left((R^* - I_3)CS^* + S^*CS^*\right).
\end{aligned} \tag{4.42}$$

Assume that $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(k+1,k)} = 0$ for some $k \geq 1$. By (4.42),

$$\begin{aligned}
\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(k+2,k+1)} &= \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(k+1,k)} + \frac{1}{2^k}\left((R^* - I_3)CS^* + S^*CS^*\right) \\
&= 0 + \frac{1}{2^k} \cdot 0 \\
&= 0
\end{aligned}$$

and the result follows by induction. □

Having shown the subdiagonal is equal to zero, we will now show that the entire lower triangular portion of the matrix is equal to zero.

Lemma 12. *(Lower Triangle)* $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(i,n)} = 0$, $n < i$.

Proof. We first verify that

$$B(R^* - I_3) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & -.5 & 1 \\ 0 & .25 & -.5 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -.25 & .5 \\ 0 & .125 & -.2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 2C(R^* - I_3).$$

We then see that, for $i > n+1$, $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(i,n)} = \frac{1}{2}\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(i-1,n)}$. By Lemma 11, the result holds. □

We will now use a series of three lemmas to show that every upper triangular entry of the matrix $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)$ is equal to zero.

Lemma 13. *(0th Row)* $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(0,n)} = 0$, $n > 0$.

Proof. We first calculate

$$\begin{aligned}
\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(0,1)} &= (R^* - I_3)\left(S^* + (F^{-1} - I_3)S^*\right) \\
&+ S^*\left(R^* + BS^* + (F^{-1} - I_3)(R^* - I_3)\right) \\
&+ S^*\left(CS^* + B(R^* - I_3)\right) + S^*\left(\frac{1}{2}CS^* + C(R^* - I_3)\right) \\
&+ S^*\left(\frac{1}{4}CS^* + \frac{1}{2}C(R^* - I_3)\right) + S^*\left(\frac{1}{8}CS^* + \frac{1}{4}C(R^* - I_3)\right) \\
&+ \cdots \\
&= (R^* - I_3)\left(S^* + (F^{-1} - I_3)S^*\right) \\
&+ S^*\left(R^* + BS^* + (F^{-1} - I_3)(R^* - I_3)\right) + S^*B(R^* - I_3) \\
&+ S^*CS^* + \frac{1}{2}S^*CS^* + \frac{1}{2}S^*CS^* + \cdots \\
&+ S^*C(R^* - I_3) + \frac{1}{2}S^*C(R^* - I_3) + \frac{1}{4}S^*C(R^* - I_3) + \cdots
\end{aligned}$$

$$
\begin{aligned}
&= (R^* - I_3)\left(S^* + (F^{-1} - I_3)S^*\right) \\
&+ S^*\left(R^* + BS^* + (F^{-1} - I_3)(R^* - I_3)\right) + S^*B(R^* - I_3) \\
&+ 2S^*CS^* + 2S^*C(R^* - I_3) \\
&= 0,
\end{aligned}
$$

where the last calculation was performed using Sage (Appendix B, Lines 54-57).

We see that $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(0,n)} = 0$ holds for $n = 1$ and note that for $n \geq 1$,

$$
\begin{aligned}
\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(0,n+1)} &= \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(0,n)} \\
&\quad + S^*(F^{-1} - I_3)S^* + S^*S^* + S^*BS^* + 2S^*CS^*.
\end{aligned} \tag{4.43}
$$

Assume that $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(0,k)} = 0$ for some $k \geq 1$. Use Sage to calculate $S^*(F^{-1} - I_3)S^* + S^*S^* + S^*BS^* + 2S^*CS^* = 0$ (Appendix B, Lines 106-109). By (4.43),

$$
\begin{aligned}
\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(0,k+1)} &= \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(0,k)} \\
&\quad + S^*(F^{-1} - I_3)S^* + S^*S^* + S^*BS^* + 2S^*CS^* \\
&= 0
\end{aligned}
$$

and the result follows by induction. □

Lemma 14. *(1st Row)* $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(1,n)} = 0$, $n > 1$.

Proof. By Lemma 13, we know that $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(0,n)} = 0$ for all $n > 0$. Note that for all $n > 1$,

$$
\begin{aligned}
\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(1,n)} &= \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(0,n-1)} \\
&\quad + (R^* - I_3)BS^* + 2S^*CS^*.
\end{aligned}
$$

Using Sage to calculate $(R^* - I_3)BS^* + 2S^*CS^* = 0$ (Appendix B, Lines 110-113), we conclude inductively that $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(1,n)} = 0$ for $n > 1$. □

Lemma 15. *(Diagonals in Upper Triangle)* $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(n,n+k)} = 0$, $n \geq 2$, $k \in \mathbb{N}_0$.

Proof. We first note that for $n \geq 2$ and $k \in \mathbb{N}_0$,

$$\begin{aligned}\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(n,n+k)} &= \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(n-1,n+k-1)} \\ &\quad + \frac{1}{2^{n-2}}\left((R^* - I_3)CS^* + S^*CS^*\right).\end{aligned} \tag{4.44}$$

By Lemma 14, we know that $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(n,n+k)} = 0$ holds for $n = 1$. Suppose $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(i,i+k)} = 0$ for some $i \geq 1$. Recall that $(R^* - I_3)CS^* + S^*CS^* = 0$ (Appendix B, Lines 42-45). Thus, by (4.44),

$$\begin{aligned}\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(i+1,i+1+k)} &= \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)_{(i,i+k)} \\ &\quad + \frac{1}{2^{n-2}}\left((R^* - I_3)CS^* + S^*CS^*\right) \\ &= 0.\end{aligned}$$

The result holds by induction. □

We have now shown that every entry of $\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)$ is equal to zero. □

Proposition 8. *There exists some $i, j \in \mathbb{N}_0$ such that $\langle M^*(\tilde{U}M^* + M^* + I)\delta_i, \delta_j\rangle \neq 0$.*

Proof. We first calculate

$$(I + M^*) + \tilde{U}M^* = \begin{pmatrix} F^* & D^* & D^* & D^* & \cdots \\ 0 & F^* & D^* & D^* & \cdots \\ 0 & 0 & F^* & D^* & \cdots \\ 0 & 0 & 0 & F^* & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix} +$$

$$\begin{pmatrix} R^{-1} - I_3 & 0 & 0 & 0 & \cdots \\ W & R^{-1} - I_3 & 0 & 0 & \cdots \\ T & W & R^{-1} - I_3 & 0 & \cdots \\ T & T & W & R^{-1} - I_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} F^* - I_3 & D^* & D^* & D^* & \cdots \\ 0 & F^* - I_3 & D^* & D^* & \cdots \\ 0 & 0 & F^* - I_3 & D^* & \cdots \\ 0 & 0 & 0 & F^* - I_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$
= \begin{pmatrix}
F^* + (R^{-1} - I_3)(F^* - I_3) & D^* + (R^{-1} - I_3)D^* & D^* + (R^{-1} - I_3)D^* & \cdots \\
W(F^* - I_3) & F^* + WD^* + (R^{-1} - I_3)(F^* - I_3) & D^* + WD^* + (R^{-1} - I_3)D^* & \cdots \\
T(F^* - I_3) & TD^* + W(F^* - I_3) & F^* + TD^* + WD^* + (R^{-1} - I_3)(F^* - I_3) & \cdots \\
T(F^* - I_3) & TD^* + T(F^* - I_3) & TD^* + TD^* + W(F^* - I_3) & \cdots \\
\vdots & \vdots & F^* + 2TD^* + WD^* + (R^{-1} - I_3)(F^* - I_3) & \cdots \\
 & & \vdots & \ddots
\end{pmatrix}.
$$

Consider the $(0,0)$ block entry of $M^*(\tilde{U}M^* + M^* + I)$ given by

$$
(F^* - I_3)\left(F^* + (R^{-1} - I_3)(F^* - I_3)\right) + D^*W(F^* - I_3) + D^*T(F^* - I_3) + \cdots \tag{4.45}
$$

where the term $D^*T(F^* - I_3)$ continues infinitely. Using Sage (Appendix B, Lines 62-73), we see that $D^*W(F^* - I_3) = D^*T(F^* - I_3) = 0$ and

$$
(F^* - I_3)\left(F^* + (R^{-1} - I_3)(F^* - I_3)\right) = \begin{pmatrix} 0 & -.5 & 0 \\ 0 & -.5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0. \tag{4.46}
$$

□

Proposition 9. $\langle (MU^*\tilde{U}\tilde{M}^* - M\tilde{M}^*)\delta_i, \delta_j \rangle = 0$ *for all* $i, j \in \mathbb{N}_0$.

Proof. It is clear that $U^*\tilde{U} = I$ implies $\langle (MU^*\tilde{U}\tilde{M}^* - M\tilde{M}^*)\delta_i, \delta_j \rangle = 0$ for all $i, j \in \mathbb{N}_0$. We will show $U^*\tilde{U} = I$ using block matrix multiplication and exploiting the diagonal structure of U^* and $\tilde{U}$.

First consider

$$U^*\tilde{U} = \begin{pmatrix} F^{-*} - I_3 & B^* & C^* & \frac{1}{2}C^* & \frac{1}{4}C^* & \cdots \\ 0 & F^{-*} - I_3 & B^* & C^* & \frac{1}{2}C^* & \cdots \\ 0 & 0 & F^{-*} - I_3 & B^* & C^* & \cdots \\ 0 & 0 & 0 & F^{-*} - I_3 & B^* & \cdots \\ 0 & 0 & 0 & 0 & F^{-*} - I_3 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\cdot \begin{pmatrix} R^{-1} - I_3 & 0 & 0 & 0 & 0 & \cdots \\ W & R^{-1} - I_3 & 0 & 0 & 0 & \cdots \\ T & W & R^{-1} - I_3 & 0 & 0 & \cdots \\ T & T & W & R^{-1} - I_3 & 0 & \cdots \\ T & T & T & W & R^{-1} - I_3 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that $U^*\tilde{U}$ will have diagonal bands of identical block matrices. Specifically, $U^*\tilde{U}_{(n,k)}$ will be equal to $U^*\tilde{U}_{(n+j,k+j)}$ for any $n, k, j \in \mathbb{N}_0$. It suffices then, to show that $U^*\tilde{U}_{(0,k)} = U^*\tilde{U}_{(n,0)} = 0$ and $U^*\tilde{U}_{(0,0)} = I_3$ for $n, k \in \mathbb{N}_0$.

We begin by calculating

$$\begin{aligned} U^*\tilde{U}_{(0,0)} &= (F^{-*} - I_3)(R^{-1} - I_3) + B^*W + C^*T + \frac{1}{2}C^*T + \frac{1}{4}C^*T + \cdots \\ &= (F^{-*} - I_3)(R^{-1} - I_3) + B^*W + 2C^*T \\ &= I_3, \end{aligned}$$

where the last sum was evaluated using Sage (Appendix B, Lines 74-77).

We next consider the 0th block column of $U^*\tilde{U}$. First notice that $U^*\tilde{U}_{(2,0)} = U^*\tilde{U}_{(n,0)}$ for all $n \geq 2$. It is sufficient to calculate

$$\begin{aligned} U^*\tilde{U}_{(1,0)} &= (F^{-*} - I_3)W + B^*T + C^*T + \frac{1}{2}C^*T + \frac{1}{4}C^*T + \cdots \\ &= (F^{-*} - I_3)W + B^*T + 2C^*T \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} U^*\tilde{U}_{(2,0)} &= (F^{-*} - I_3)T + B^*T + C^*T + \frac{1}{2}C^*T + \frac{1}{4}C^*T + \cdots \\ &= (F^{-*} - I_3)T + B^*T + 2C^*T \\ &= 0. \end{aligned}$$

The code for these calculations can be found in Appendix B, Lines 78-85.

Finally, we consider the 0th row of $U^*\tilde{U}$. Notice that

$$U^*\tilde{U}_{(0,k)} = \frac{1}{2^{k-1}}\left(2C^*(R^{-1} - I_3) + C^*W + C^*T\right) \text{ for } k \geq 2. \tag{4.47}$$

We use Sage (Appendix B, Lines 86-101) to confirm that

$$\begin{aligned} C^*W &= C^*T \\ C^*(R^{-1} - I_3) &= -C^*T \end{aligned}$$

and conclude from (4.47) that $U^*\tilde{U}_{(0,k)} = 0$ for ≥ 2.

It remains only to note that $B^*(R^{-1} - I_3) = -2C^*T$ (Appendix B, Lines 102-105) and show

$$\begin{aligned} U^*\tilde{U}_{(0,1)} &= B^*(R^{-1} - I_3) + C^*W + \frac{1}{2}C^*T + \frac{1}{4}C^*T + \cdots \\ &= B^*(R^{-1} - I_3) + C^*W + C^*T \\ &= 0. \end{aligned}$$

We see that $U^*\tilde{U} = I$ and thus $\langle (MU^*\tilde{U}\tilde{M}^* - M\tilde{M}^*)\delta_i, \delta_j \rangle = 0$ for all $i, j \in \mathbb{N}_0$.

$\square$

We have now shown all of the desired matrix properties. By Theorems 4 and 5, we know that $\{(\phi_n, \psi_n)\}$ is weakly densely effective and $\{(\psi_n, \phi_n)\}$ is not. As we are working in finite dimensions, we actually know more: $\{(\phi_n, \psi_n)\}$ is effective and $\{(\psi_n, \phi_n)\}$ is not. Incorporating Theorem 6 and Proposition 9, we conclude that $\{(g_n, \tilde{g}_n)\}$ also forms a dual pair. Combining these results, we have produced the promised example for which $\{(g_n, \psi_n)\}$ and $\{(g_n, \tilde{g}_n)\}$ are dual, but $\{(\tilde{g}_n, \phi_n)\}$ is not.

At this point we pause to reflect on the boundedness (or lack thereof) of U, $\tilde{U}$, and $U^*\tilde{U}$. In Proposition 9, we showed that $U^*\tilde{U} = I$, so $U^*\tilde{U}$ is obviously bounded as an operator on $\ell^2(\mathbb{N}_0)$. To examine U and $\tilde{U}$, we rely on their block structures as given in (4.39) and (4.40), respectively. If $n \geq i+2$, then $U_{(n,i)} = \left(\frac{1}{2}\right)^{n-2} C$, where $U_{(n,i)}$ denotes the 3×3 block matrix of U in the (n, i) position, and

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -.25 & .5 \\ 0 & .125 & -.25 \end{pmatrix}.$$

Noting that the entries of $U_{(n,i)}$ approach 0 as n approaches infinity, we conclude by Schur's Lemma that U is a bounded operator on $\ell^2(\mathbb{N}_0)$. On the other hand, note that for $n \geq i+2$, $\tilde{U}_{(n,i)} = T$, where

$$T = \begin{pmatrix} .5 & -.5 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}.$$

As the first two columns of $\tilde{U}$ are not even elements of $\ell^2(\mathbb{N}_0)$, $\tilde{U}$ must be unbounded on $\ell^2(\mathbb{N}_0)$.

4.4.1 A Theorem in Finite Dimensions

The previous example prompted an investigation into the conditions for a pair of periodic sequences in finite dimensions to give convergence in the dual Kaczmarz algorithm. Recall that a pair of sequences will be effective if and only if conditions (UB) and (DE) are satisfied.

In finite dimensions, however, the (DE) condition suffices, as the involved sequences span the entire space (i.e., the dense space in question is the entire space). To fulfill condition (DE), we look to the spectral radius of the product of the appropriate projections.

Theorem 7. *Let X be a finite-dimensional Banach space. Suppose $\{(\phi_n, \psi_n)\} \subseteq X^* \times X$ are k-periodic sequences with $\phi_n(\psi_n) = 1$ for all n, where $\{\psi_n\}$ and $\{\phi_n\}$ are linearly dense in X and X^*, respectively. If $\rho(P_{k-1}P_{k-2}\cdots P_0) < 1$, then $\{(\phi_n, \psi_n)\}$ is effective. If $\rho(P_{k-1}P_{k-2}\cdots P_0) > 1$, then $\{(\phi_n, \psi_n)\}$ is not effective.*

Proof. As we are working in finite dimensions, it suffices to prove that the (DE) condition holds (the dense subset referenced is the entire space). However, we will use (UB) as a tool to obtain (DE). We thus begin by showing that $\rho\left(P_{k-1}P_{k-2}\cdots P_0\right) < 1$ implies the existence of $C > 0$ such that $\|P_nP_{n-1}\cdots P_0\| < C$ for all n.

Let $T = P_{k-1}P_{k-2}\cdots P_0$. By the Jordan normal form theorem, we know there exists an invertible matrix $W \in \mathbb{C}^{k\times k}$ such that $T = WJW^{-1}$, where $J \in \mathbb{C}^{k\times k}$ is the block diagonal Jordan matrix for T. Specifically,

$$J = \begin{pmatrix} J_{m_0}(\lambda_0) & 0 & 0 & \cdots \\ 0 & J_{m_1}(\lambda_1) & 0 & \cdots \\ \vdots & \cdots & \ddots & \cdots \\ 0 & \cdots & 0 & J_{m_{k-1}}(\lambda_{k-1}) \end{pmatrix}$$

where

$$J_{m_j}(\lambda_j) = \begin{pmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 \\ \cdots & \cdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & \cdots & 0 & \lambda_j \end{pmatrix}$$

and $J_{m_j}(\lambda_j) \in \mathbb{C}^{m_j\times m_j}$ and $0 \le j \le k-1$.

Because $T^i = (WJW^{-1})^i = WJ^iW^{-1}$ and J is block diagonal, we have that

$$T^i = WJ^iW^{-1} = W \begin{pmatrix} J^i_{m_0}(\lambda_0) & 0 & 0 & \cdots \\ 0 & J^i_{m_1}(\lambda_1) & 0 & \cdots \\ \vdots & \cdots & \ddots & \cdots \\ 0 & \cdots & 0 & J^i_{m_{k-1}}(\lambda_{k-1}) \end{pmatrix} W^{-1} \tag{4.48}$$

where $\left(J_{m_j}(\lambda_j)\right)^i$ is given by

$$J^i_{m_j}(\lambda_j) = \begin{pmatrix} \lambda_j^i & \binom{i}{1}\lambda_j^{i-1} & \binom{i}{2}\lambda_j^{i-2} & \cdots & \binom{i}{m_j-1}\lambda_j^{i-m_j+1} \\ 0 & \lambda_j^i & \binom{i}{1}\lambda_j^{i-1} & \cdots & \binom{i}{m_j-2}\lambda_j^{i-m_j+2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_j^i & \binom{i}{1}\lambda_j^{i-1} \\ 0 & 0 & \cdots & 0 & \lambda_j^i \end{pmatrix}, \tag{4.49}$$

and $\binom{a}{b} = 0$ if $a < b$ or $b < 0$. If $\rho(T) < 1$, then $|\lambda_j| < 1$ for all j and we see by (4.49) that as $i \to \infty$, every entry of every $J^i_{m_j}(\lambda_j)$ approaches 0. In other words, $\lim_{i\to\infty} J^i = 0$ and we see that

$$\begin{aligned} \lim_{i\to\infty} T^i &= \lim_{i\to\infty} (WJW^{-1})^i \\ &= \lim_{i\to\infty} WJ^iW^{-1} \\ &= W\left(\lim_{i\to\infty} J^i\right)W^{-1} \\ &= 0. \end{aligned}$$

We now have that $\lim_{i\to\infty} \|T^i\| \to 0$, from which we conclude that

$$\lim_{n\to\infty} \|P_nP_{n-1}\cdots P_0\| = \lim_{i\to\infty} \|T^i\| = 0$$

and derive

$$\begin{aligned} \lim_{n\to\infty} \|P_nP_{n-1}\cdots P_0x\| &\le \lim_{n\to\infty} \|P_nP_{n-1}\cdots P_0\|\|x\| \\ &= \|x\| \lim_{n\to\infty} \|P_nP_{n-1}\cdots P_0\| \\ &= 0. \end{aligned}$$

It follows that $\lim_{n\to\infty} \|P_n P_{n-1} \cdots P_0 x\| = 0$ for any $x \in X$. By Kwapień and Mycielski, $\{(\phi_n, \psi_n)\}$ is an effective pair [KM01].

For the last part of the theorem, we note that if $\rho(P_{k-1}P_{k-2}\cdots P_0) = \rho(T) > 1$, then there is some λ_j such that $|\lambda_j| > 1$. In this case, there is some entry of J which diverges to infinity, which means that the Euclidean norm of J^i diverges to infinity as $i \to \infty$. As all matrix norms are equivalent, we conclude that any norm of J^i diverges to infinity as $i \to \infty$. Because W is invertible, the norms of $WJ^iW^{-1} = T^i$ and J^i are equivalent. Consequently, we conclude by Kwapień and Mycielski that $\{(\phi_n, \psi_n)\}$ cannot be effective as

$$\lim_{n\to\infty} \|P_n P_{n-1} \cdots P_0\| = \lim_{i\to\infty} \|T^i\| \to \infty, \tag{4.50}$$

resulting in condition (UB) being violated [KM01]. □

We apply these results to the previous example. For $P_n x = x - \langle x, \phi_n\rangle \psi_n$ and $Q_n x = x - \langle x, \psi_n\rangle \phi_n$, we easily compute $\rho(P_2P_1P_0) = \frac{1}{2}$ and $\rho(Q_2Q_1Q_0) = 2$. By Theorem 7, $\{(\phi_n, \psi_n)\}$ is an effective pair, but $\{(\psi_n, \phi_n)\}$ is not. In other words, $\{(g_n, \psi_n)\}$ forms a dual pair, but $\{(\tilde{g}_n, \phi_n)\}$ does not. Note that the calculations here are significantly less laborious than those completed previously to obtain the same result.

4.4.2 The Haller and Szwarc Equivalences

In Chapter 2, we presented a list of equivalent properties from Haller and Szwarc for effective sequences in a Hilbert Space ([HS05]). We now provide analogous statements for the effective pair context.

a. $\{(\phi_n, \psi_n)\}$ is an effective pair.

b. $\{g_n\}$ and $\{\psi_n\}$ are dual.

c. $\langle \tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)\delta_j, \delta_i\rangle = 0$ for all $i, j \in \mathbb{N}_0$.

d. $\{g_n\}$ and $\{\tilde{g}_n\}$ are dual.

e. $\langle(\tilde{M}U^*\tilde{U}M^* - \tilde{M}M^*)\delta_j, \delta_i\rangle = 0$ for all $i, j \in \mathbb{N}_0$.

f. $U^*\tilde{U}$ has the projection property described in Theorem 6.

Through our work in this chapter, it has become clear that these conditions are not equivalent in the context of the dualized algorithm. It is true that Theorem 4 shows a limited equivalence (the results hold weakly on a dense subset) of (a), (b), and (c), and that Theorem 6 shows a similar equivalence between (d), (e), and (f). However, our finite-dimensional example shows that (b) is not equivalent to (d), creating two groups of equivalences which cannot be equivalent to each other. It is still possible, however, that the group (a), (b), and (c) implies (d), (e), and (f). Obtaining an example disproving this implication is highly desirable.

4.5 A Banach Space Example

We present an example of a weakly densely effective pair in a Banach space which is not a Hilbert space.

Definition 17. Let X be a Banach space. $\{(\phi_n, \psi_n)\} \subseteq X^* \times X$ is *stationary* if $\phi_{n+k}(\psi_{m+k}) = \phi_n(\psi_m)$ for all $n, m, k \in \mathbb{N}_0$.

Let μ be a singular Borel probability measure on $[0, 1]$, $\phi_n(t) = e^{-2\pi int} \in L^2(\mu)$, and $\psi_m(t) = e^{2\pi imt} \in L^2(\mu)$. As both $\{\phi_n\}$ and $\{\psi_m\}$ are stationary sequences of unit vectors and are linearly dense in $L^2(\mu)$, by [KM01], they are both effective sequences in $L^2(\mu)$. By Haller and Szwarc in [HS05], we know that if

$$I + N = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \int_0^1 e^{2\pi it} d\mu & 1 & 0 & 0 & \cdots \\ \int_0^1 e^{4\pi it} d\mu & \int_0^1 e^{2\pi it} d\mu & 1 & 0 & \cdots \\ \int_0^1 e^{6\pi it} d\mu & \int_0^1 e^{4\pi it} d\mu & \int_0^1 e^{2\pi it} d\mu & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and $I + V$ is the algebraic inverse of $I + N$, then V is a partial isometry. Furthermore, $\langle N^*(VN^* + N^* + I)\delta_j, \delta_i\rangle = 0$ for all $i, j \in \mathbb{N}_0$.

Let $p > 2$, and let q be such that $\frac{1}{p}+\frac{1}{q} = 1$. We now consider $\{\phi_n\}$ and $\{\psi_n\}$ as collections within $L^p(\mu)$ and $L^q(\mu)$, respectively. We construct $I+M$ and $I+\tilde{M}$ as defined in (4.5) and (4.7), noticing that $N = M = \tilde{M}$. Consequently, $V = U = \tilde{U}$, where $I+U$ and $I+\tilde{U}$ are the algebraic inverses of $I + M$ and $I + \tilde{M}$, respectively. Because $N^*(VN^* + N^* + I)\delta_j, \delta_i\rangle = 0$ for all $i, j \in \mathbb{N}_0$, we have

$$\tilde{M}^*(U\tilde{M}^* + \tilde{M}^* + I)\delta_j, \delta_i\rangle = 0 \text{ for all } i, j \in \mathbb{N}_0$$

$$M^*(\tilde{U}M^* + M^* + I)\delta_j, \delta_i\rangle = 0 \text{ for all } i, j \in \mathbb{N}_0.$$

By Theorems 4 and 5, we know that $\{(\phi_n, \psi_n)\}$ is weakly densely effective on $L^p(\mu)$ and $\{(\phi_n, \psi_n)\}$ is weakly densely effective on $L^q(\mu)$.

BIBLIOGRAPHY

[ACH+19] Anna Aboud, Emelie Curl, Steven N Harding, Mary Vaughan, and Eric S Weber. The dual Kaczmarz algorithm. *Acta Applicandae Mathematicae*, pages 1–16, 2019.

[Cas00] Peter G Casazza. The art of frame theory. *Taiwanese Journal of Mathematics*, 4(2):129–201, 2000.

[CC97] Peter G Cazassa and Ole Christensen. Perturbation of operators and applications to frame theory. *Journal of Fourier Analysis and Applications*, 3(5):543–557, 1997.

[Chr95] Ole Christensen. Frame perturbations. *Proceedings of the American Mathematical Society*, pages 1217–1220, 1995.

[Chr99] Ole Christensen. Operators with closed range, pseudo-inverses, and perturbation of frames for a subspace. *Canadian Mathematical Bulletin*, 42(1):37–45, 1999.

[Chr03] Ole Christensen. *An introduction to frames and Riesz bases, Applied and Numerical Harmonic Analysis*. 2003.

[CT13] Wojciech Czaja and James H Tanis. Kaczmarz algorithm and frames. *International Journal of Wavelets, Multiresolution and Information Processing*, 11(05):1350036, 2013.

[EHL81] Paulus Petrus Bernardus Eggermont, Gabor T Herman, and Arnold Lent. Iterative algorithms for large partitioned linear systems, with applications to image reconstruction. *Linear algebra and its applications*, 40:37–67, 1981.

[GBH70] Richard Gordon, Robert Bender, and Gabor T Herman. Algebraic reconstruction techniques (ART) for three-dimensional electron microscopy and X-ray photography. *Journal of theoretical Biology*, 29(3):471–481, 1970.

[Her16] John Herr. *Fourier series for singular measures and the Kaczmarz algorithm*. PhD thesis, Iowa State University, 2016.

[HKW19] Chinmay Hegde, Fritz Keinert, and Eric S Weber. A Kaczmarz algorithm for solving tree based distributed systems of equations. *arXiv preprint arXiv:1904.05732*, 2019.

[HS05] Rainis Haller and Ryszard Szwarc. Kaczmarz algorithm in Hilbert space. *Studia Mathematica - STUD MATH*, 169:123–132, 01 2005.

[JG17] Halyun Jeong and C Sinan Güntürk. Convergence of the randomized Kaczmarz method for phase retrieval. *arXiv preprint arXiv:1706.10291*, 2017.

[Kac37] Stefan Kaczmarz. Angenäherte Auflösung von Systemen linearer Gleichungen. *Bulletin International de l'Académie Polonaise des Sciences et des Lettres. Classe des Sciences Mathématiques et Naturelles A*, 35:355–357, 1937.

[KM01] Stanisław Kwapień and Jan Mycielski. On the Kaczmarz algorithm of approximation in infinite-dimensional spaces. *Studia Mathematica*, 1(148):75–86, 2001.

[KRS15] Goutham Kamath, Paritosh Ramanan, and Wen-Zhan Song. Distributed randomized Kaczmarz and applications to seismic imaging in sensor network. In *2015 International Conference on Distributed Computing in Sensor Systems*, pages 169–178. IEEE, 2015.

[LO01] Shidong Li and Hidemitsu Ogawa. Pseudo-duals of frames with applications, 2001.

[Nat01] Frank Natterer. *The mathematics of computerized tomography*. SIAM, 2001.

[NT14] Deanna Needell and Joel A Tropp. Paved with good intentions: analysis of a randomized block Kaczmarz method. *Linear Algebra and its Applications*, 441:199–221, 2014.

[NWS14] Deanna Needell, Rachel Ward, and Nati Srebro. Stochastic gradient descent, weighted sampling, and the randomized Kaczmarz algorithm. In *Advances in Neural Information Processing Systems*, pages 1017–1025, 2014.

[NZZ15] Deanna Needell, Ran Zhao, and Anastasios Zouzias. Randomized block Kaczmarz method with projection for solving least squares. *Linear Algebra and its Applications*, 484:322–343, 2015.

[PW34] Raymond Edward Alan Christopher Paley and Norbert Wiener. *Fourier transforms in the complex domain*, volume 19. American Mathematical Soc., 1934.

[SV09] Thomas Strohmer and Roman Vershynin. A randomized Kaczmarz algorithm with exponential convergence. *Journal of Fourier Analysis and Applications*, 15(2):262, 2009.

[Szw07] Ryszard Szwarc. Kaczmarz algorithm in Hilbert space and tight frames. *Applied and Computational Harmonic Analysis*, 22(3):382–385, 2007.

[Tan71] Kunio Tanabe. Projection method for solving a singular system of linear equations and its applications. *Numerische Mathematik*, 17(3):203–214, 1971.

[TV18] Yan Shuo Tan and Roman Vershynin. Phase retrieval via randomized Kaczmarz: Theoretical guarantees. *Information and Inference: A Journal of the IMA*, 8(1):97–123, 2018.

APPENDIX A. EXAMPLE

Following is a sequence of bounded operators $\{M_n\}$ which converges pointwise on a dense subset of its domain, but is not uniformly bounded with respect to n.

For $f \in L^2[0,1]$, define M_n by $M_n(f) = F_n \cdot f$, where

$$F_n(x) = \begin{cases} n \text{ if } x \in [0, \frac{1}{n}] \\ \\ 1 \text{ if } x \in [\frac{1}{n}, 1]. \end{cases}$$

Let $\mathcal{M} = \bigcup_{\varepsilon>0} \mathcal{M}_\varepsilon$ where $\mathcal{M}_\varepsilon = \{f \in L^2[0,1] : support(f) \subseteq [\varepsilon, 1]\}$. Notice:

1. Each M_n is bounded on $L^2[0,1]$.

   Choose $f \in L^2[0,1]$. Note that $||f||_\infty < \infty$ on $[0,1]$. We then have

$$\begin{aligned} \|M_n f\|^2_{L^2[0,1]} &= \int_0^1 |M_n f(x)|^2 dx \\ &= \int_0^1 |F_n(x) f(x)|^2 dx \\ &= \int_0^1 |F_n(x)|^2 |f(x)|^2 dx \\ &\leq \int_0^1 n^2 |f(x)|^2 dx \\ &= n^2 \int_0^1 |f(x)|^2 dx \\ &= n^2 \|f\|^2_{L^2[0,1]}. \end{aligned}$$

2. $\mathcal{M}$ is dense in $L^2[0,1]$.

   Choose $f \in L^2[0,1]$ and let $\varepsilon > 0$ be given. As f^2 is integrable on $[0,x]$ for any $0 \leq x \leq 1$, the function $F(y) = \int_0^y |f(x)|^2 dx$ is continuous for $0 \leq y \leq 1$. Because $F(0) = 0$, there is some $b > 0$ such that $\int_0^b |f(x)|^2 dx < \varepsilon$.

Consider the function $g \in \mathcal{M}_b \subseteq \mathcal{M}$ defined by

$$g(x) = \begin{cases} f(x), \text{ if } x \in (b, 1] \\ 0, \text{ otherwise.} \end{cases}$$

We see that

$$\begin{aligned} \|f - g\|^2_{L^2[0,1]} &= \int_0^1 |f(x) - g(x)|^2 dx \\ &= \int_0^b |f(x) - g(x)|^2 dx + \int_b^1 |f(x) - g(x)|^2 dx \\ &= \int_0^b |f(x) - g(x)|^2 dx \\ &= \int_0^b |f(x)|^2 dx \\ &< \varepsilon \end{aligned}$$

and we have the desired result.

3. $M_n \to I$ pointwise on $\mathcal{M}$.

   Let $\varepsilon > 0$ and $f \in \mathcal{M}$. This means that there is some $\varepsilon_0 > 0$ such that $f \in \mathcal{M}_{\varepsilon_0}$. Choose N such that for all $n > N$, $\frac{1}{2} \leq \varepsilon_0$. Then, for $n > N$,

$$\begin{aligned} \|M_n f - f\|^2 &= \int_0^1 |M_n f(x) - f(x)|^2 dx \\ &= \int_0^{\varepsilon_0} |M_n f(x) - f(x)|^2 dx + \int_{\varepsilon_0}^1 |M_n f(x) - f(x)|^2 dx \\ &= \int_{\varepsilon_0}^1 |M_n f(x) - f(x)|^2 dx \\ &= 0 \\ &< \epsilon. \end{aligned}$$

4. There is some $f \in L^2[0, 2]$ such that $\|M_n f\| \to \infty$ as $n \to \infty$. I.e. the M_n are not uniformly bounded with respect to n on $L^2[0, 1]$.

   Let $0 < \delta < 1$ and $f(x) = x^{\delta - \frac{1}{2}}$.

We see that

$$\int_0^1 |f(x)|^2 dx = \int_0^1 x^{2\delta-1} dx < \infty$$

as $2\delta_1 - 1 > -1$, and so $f \in L^2[0,1]$.

However,

$$\begin{aligned}
||M_n f||^2_{L^2[0,1]} &= \int_0^1 |M_n f(x)|^2 dx \\
&= \int_0^{\frac{1}{n}} |M_n f(x)|^2 dx + \int_{\frac{1}{n}}^1 |M_n f(x)|^2 dx \\
&= \int_0^{\frac{1}{n}} n^2 x^{2\delta-1} dx + \int_{\frac{1}{n}}^1 x^{2\delta-1} dx \\
&= \left(\frac{n^2 x^{2\delta}}{2\delta}\right)_0^{\frac{1}{n}} + \left(\frac{x^{2\delta}}{2\delta}\right)_{\frac{1}{n}}^1 \\
&= \frac{n^{2-2\delta}}{2\delta} + \frac{1}{2\delta} - \left(\frac{1}{n}\right)^{2\delta} \cdot \frac{1}{2\delta}.
\end{aligned}$$

As $0 < \delta < 1$, $\lim_{n\to\infty} \left(\frac{1}{n}\right)^{2\delta} = 0$ and $\lim_{n\to\infty} \frac{n^{2-2\delta}}{2\delta} = \infty$. We conclude that there is no constant $C > 0$ such that for any $f \in L^2[0,1]$, $\|M_n f\| \leq C\|f\|$ for all n.

APPENDIX B. CODE

```
sage: F=matrix(3,3,[[1, 0, 0],[1, 1, 0],[.5, .5, 1]])
sage: D=matrix(3,3,[[1, 1, 2],[1, 1, 1],[.5, .5, 1]])
sage: B=matrix(3,3,[[0, 0, −2],[0, −.5, 1],[0, .25, −.5]])
sage: C=matrix(3,3,[[0, 0, 0],[0, −.25, .5],[0, .125, −.25]])
sage: R=matrix(3,3,[[1, 0, 0],[1, 1, 0],[2,1, 1]])
sage: S=matrix(3,3,[[1, 1, .5],[1, 1, .5],[2, 1, 1]])
sage: T=matrix(3,3,[[.5, −.5, 0],[0,0, 0],[−1,1,0]])
sage: I=matrix(3,3,[[1, 0, 0],[0, 1, 0],[0, 0, 1]])
sage: W=matrix(3,3,[[.5, −.5, −.5],[0, 0, 0],[−1, 1, 0]])
sage: D∗F^(−1)+F∗B
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: D∗F^(−1)+D∗B+F∗C
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: D∗C−.5∗F∗C
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: S∗R^(−1)+R∗W
[0.000000000000000 0.000000000000000 0.000000000000000]
```

```
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: S*R^(-1)+S*W+R*T
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: S*T
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: (transpose(R)-I)*(transpose(R)+(F^(-1)-I)*(transpose(R)-I))+transpose(S)*B*(
    transpose(R)-I)+2*transpose(S)*C*(transpose(R)-I)
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: (transpose(R)-I)*(transpose(R)+B*transpose(S)+(F^(-1)-I)*(transpose(R)-I))+
    transpose(S)*B*(transpose(R)-I)+2*transpose(S)*C*transpose(S)+2*transpose(S)*C
    *(transpose(R)-I)
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: (transpose(R)-I)*C*transpose(S)+transpose(S)*C*transpose(S)
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: (transpose(R)-I)*B*(transpose(R)-I)+2*transpose(S)*C*(transpose(R)-I)
[0.000000000000000 0.000000000000000 0.000000000000000]
```

```
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: (transpose(R)-I)*(C*transpose(S)+B*(transpose(R)-I))+transpose(S)*C*
    transpose(S)+2*transpose(S)*C*(transpose(R)-I)
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: (transpose(R)-I)*(transpose(S)+(F^(-1)-I)*transpose(S))+transpose(S)*(
    transpose(R)+B*transpose(S)+(F^(-1)-I)*(transpose(R)-I))+transpose(S)*B*(
    transpose(R)-I)+2*transpose(S)*C*transpose(S)+2*transpose(S)*C*(transpose(R)-
    I)
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: (transpose(R)-I)*(transpose(S)+B*transpose(S)+(F^(-1)-I)*transpose(S))+
    transpose(S)*(transpose(R)+C*transpose(S)+B*transpose(S)+(F^(-1)-I)*(
    transpose(R)-I))+transpose(S)*B*(transpose(R)-I)+transpose(S)*C*transpose(S)
    +2*transpose(S)*C*transpose(S)+2*transpose(S)*C*(transpose(R)-I)
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: transpose(D)*W*(transpose(F)-I)
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: transpose(D)*T*(transpose(F)-I)
[0.000000000000000 0.000000000000000 0.000000000000000]
```

```
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: (transpose(F)-I)*(transpose(F)+(R^(-1)-I)*(transpose(F)-I))
[ 0.000000000000000 -0.500000000000000 0.000000000000000]
[ 0.000000000000000 -0.500000000000000 0.000000000000000]
[ 0.000000000000000 0.000000000000000 0.000000000000000]
sage: ((transpose(F))^(-1)-I)*(R^(-1)-I)+transpose(B)*W+2*transpose(C)*T
[ 1.00000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 1.00000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 1.00000000000000]
sage: ((transpose(F))^(-1)-I)*W+transpose(B)*T+2*transpose(C)*T
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: ((transpose(F))^(-1)-I)*T+transpose(B)*T+2*transpose(C)*T
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: transpose(C)*W
[ 0.000000000000000 0.000000000000000 0.000000000000000]
[-0.125000000000000 0.125000000000000 0.000000000000000]
[ 0.250000000000000 -0.250000000000000 0.000000000000000]
sage: transpose(C)*T
[ 0.000000000000000 0.000000000000000 0.000000000000000]
[-0.125000000000000 0.125000000000000 0.000000000000000]
[ 0.250000000000000 -0.250000000000000 0.000000000000000]
sage: transpose(C)*(R^(-1)-I)
```

```
[ 0.000000000000000 0.000000000000000 0.000000000000000]
[ 0.125000000000000 −0.125000000000000 0.000000000000000]
[−0.250000000000000 0.250000000000000 0.000000000000000]
sage: −1∗transpose(C)∗T
[ 0.000000000000000 0.000000000000000 0.000000000000000]
[ 0.125000000000000 −0.125000000000000 0.000000000000000]
[−0.250000000000000 0.250000000000000 0.000000000000000]
sage: transpose(B)∗(R^(−1)−I)
[ 0.000000000000000 0.000000000000000 0.000000000000000]
[ 0.250000000000000 −0.250000000000000 0.000000000000000]
[−0.500000000000000 0.500000000000000 0.000000000000000]
sage: transpose(S)∗(F^(−1)−I)∗transpose(S)+transpose(S)∗transpose(S)+transpose(S)∗B
    ∗transpose(S)+2∗transpose(S)∗C∗transpose(S)
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
sage: (transpose(R)−I)∗B∗transpose(S)+2∗transpose(S)∗C∗transpose(S)
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
[0.000000000000000 0.000000000000000 0.000000000000000]
```