72-19,982

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ECONOMIC DECISION MODELS UNDER LINEAR METHODS OF DECENTRALIZATION.

Iowa State University, Ph.D., 1972
Economics, general

Economic decision models under linear

methods of decentralization

by

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A Dissertation Submitted to the

Graduate Faculty in Partial Fulfillment of

The Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Major Subject: Economics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa

1972

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CHAPTER I. INTRODUCTION

The proposition that an economic system will function best when decisions are made in a decentralized manner has a long history in economic doctrine. Adam Smith's concept of the "invisible hand" involves just such a proposition. Walras' concept of tatônnements embodies such a proposition stated in a mathematical form. Walras outlines an algorithmic process in which at each iteration decentralized decision-makers respond to a set of prices which are in turn revised according to the amount of excess demand when those responses are aggregated. It was Walras' purpose to show that after successive iterations an equilibrium solution would be reached. Such an equilibrium will be associated with an optimum under the usual assumptions of convexity of preferences and production processes (Arrow and Hurwicz, 1960, p. 35).

The discussion of market socialism was concerned with the possibility of using a decentralized market mechanism to obtain an economic optimum in a socialist economy rather than a capitalist economy (Hayek, 1956; Lange and Taylor, 1938). More recently, a number of different mathematical models have been formulated with the specific aim of effecting a certain degree of decentralization in the decision process. The models presented below are basically in this tradition. Some of these models have been concerned with the use of transfer or internal prices in decentralizing decisions in business firms, e.g., Arrow (1959) and Baumol and Fabian (1964). Others have been directed toward decentralizing planning decisions in a national economy, e.g., Malinvaud (1967), Kornai and Liptak (1965), and Aoki (1970). To the extent that these models are presented in an abstract mathematical form they may be considered generally applicable to different types and sizes of economic institutions (Arrow and Hurwicz, 1960, p. 34). In applying such models to a large public institution or to a national economy, however, there is the very basic difficulty of specifying a satisfactory objective function or of circumventing the need for such a function.

The preceding survey is not intended to be comprehensive. The literature on decentralization is vast, and a comprehensive survey would constitute a study in itself. Rather, it is intended that the brief introductory survey will help to identify the position of the models in this study relative to the literature on decentralization in general. Specific references directly related to each of the models presented in this study will be discussed below.

Three related but essentially different types of resource allocation models are presented here. The first type is associated with a large economic system in which highly integrated subunits are linked together by a relatively small number of constraints. It is assumed that the technological relations and resource constraints can be specified by the model and, more crucially, that a set of coefficients indicating the relative value of each activity is available. This type of model will be analyzed in the framework of a decomposable linear program. The contribution to this first type of model includes extensions of the pricing and allocation rules with the intention of making the decentralized solution process more efficient. The theoretical considerations are discussed in Chapter II, and the computational results of an illustrative numerical model are presented in Chapter III.

The second and third types of models presented below are assumed to apply to situations in which the coefficients indicating the relative value of each activity are not readily available. Chapter IV contains a discussion of different alternatives which might be employed when a set of relative prices is not available so that the vector of outputs can be collapsed into a meaningful scalar value. One alternative presented is the possibility of specifying a vector of output goals and employing goal programming to obtain a solution which is optimal relative to the goals. A second alternative discussed is the possibility of computing output vectors which satisfy the less ambitious criterion of efficiency. A method is presented by which it is possible to compute all efficient extreme points which are adjacent to a given efficient extreme point.

The second general type of model is introduced in Chapter V. It is specifically concerned with making resource allocation decisions in a university where relative prices for activities are difficult to obtain. The use of goal programming and efficient output computations are discussed and are applied to an illustrative numerical model. The possibility of effecting a multi-level decentralization through goal programming is also discussed.

The third type of model is outlined in Chapter VI and is essentially theoretical in nature. It assumes a production system in a general equilibrium setting and is concerned with the relation between efficient production and a very specific type of noncompetitive price setting. In this context the relative prices are variables in the system. The possibility of decentralizing decisions when the noncompetitive price setting is present is discussed.

Each of the models in the study is linear and can be written as a linear programming problem. Thus the usual assumption of constant returns to scale and divisibility must be made. A fairly complete comparison of marginal analysis and linear programming with respect to the theory of the firm is given in Naylor (1966). An interesting theoretical study of the effects of indivisibilities can be found in Frank (1969). The models are also essentially static and deterministic. They could be expanded to include certain types of dynamic elements without basic difficulties. This could be accomplished by the well known procedure of defining products produced in different time periods to be different products and including constraints which effectively link the time periods together. Certain types of uncertainty could also be handled by known methods such as chance constrained programming or stochastic linear programming.

By using only linear models which are static and deterministic a number of important difficulties such as increasing returns to scale, more complicated types of externalities, adjustments over time, and the treatment of stochastic elements have been avoided. Such difficulties are avoided only at a high cost to the richness of the model; however, the retention of linearity has the compensating advantage that computation is possible for relatively disaggregated models with many variables. Treating the types of difficulties listed above in any very sophisticated manner requires the inclusion of many nonlinearities with the result that computation, where it is possible, is a very expensive process unless the model is highly aggregated.

CHAPTER II. TWO DECOMPOSITION PROCEDURES

The Basic Models

This chapter and the next will focus on optimal decision-making in a large economic system in which highly integrated subcomponents can be identified. The subcomponents are linked together by relatively few variables so that a satisfactory decentralized decision process greatly economizes the amount of information which the highest level decision-makers need.

We will assume that the system can be satisfactorily approximated by a linear model. The discussion will center around two different specific formulations of linear programming models each of which can be decomposed into a group of smaller linear programming models appropriately linked together.

Model (1) is the type discussed by Kornai and Liptak (1965) in their article on two level planning and by Sengupta (1970) in an article on the active approach:

$\max \sum_{j=1}^{n} c^{j\prime} x^j$, where c^j and x^j are $n^j \times 1$ vectors representing respectively the direct returns and levels of activities of the j^{th} subunit, (1)

subject to

$\sum_{j=1}^{n} A^j x^j \leq b$, where A^j is the $m \times n^j$ matrix of activities of the j^{th} subunit and b represents the vector of resources available,

$$x^j \geq 0, \quad j = 1 \text{ to } n\ .$$

In this case the resource vector b can be decomposed or allocated such that $\sum_{j=1}^{n} u^j \leq b$ and for each such decomposition there is an associated set of n subproblems with constraint sets:

$A^j x^j \leq u^j$, $x^j \geq 0$, where u^j is an allocation of resources to the j^{th} subproblem.

Model (2) is the type discussed by Dantzig and Wolfe (Dantzig, 1959; Dantzig and Wolfe, 1961) in their articles on the decomposition principle and by Charnes and Cooper (1961) as a class of coupled models:

$$\max \sum_{j=1}^{n} c^{j'} x^j, \text{ where } c^j \text{ and } x^j \text{ are defined as in (1)}, \qquad (2)$$

subject to

$\sum_{j=1}^{n} A^j x^j \leq b$, where the A^j matrices and the vector b have as many rows as there are central resources,

$D^j x^j \leq b^j$, $j = 1$ to n, where the matrix D and the vector b^j have as many rows as there are resources specific to the j^{th} subunit,

$x^j \geq 0$, $\quad j = 1$ to n.

In this case each decomposition or allocation of resources is associated with a set of n subproblems with constraint sets:

$$A^j x^j \leq u^j$$
$$D^j x^j \leq b^j$$
$$x^j \geq 0 \ .$$

The basic difference between these two models in economic terms is that in (1) all resources are viewed as being central resources allocable to the subunit; while in (2) certain resources specified by b^j are viewed as an essential part of the subunit and other resources specified by b are viewed as central allocable resources. Such elements specific to the jth subunit might result from an immovable resource such as a plant or from a natural resource associated with a particular subunit and its geographical location. While in general we will refer to elements of b and b^j as quantities of resources, it should be noted that (as well as representing quantities of goods) they may also represent the amount of services available during the period from a stock of fixed capital or may even represent a capacity level imposed on certain activities by institutional regulations. Institutionally imposed capacity levels on certain activities would be necessary if, for example, a corporation were forced to keep its sales of a product below a certain market share so as not to face an antitrust suit, or if a public utility were forced by law to maintain a certain level of specific services even if a loss were incurred. The capacity level on certain activities might also refer to the maximum amount of a particular polluting material which is allowed without violating a certain clean air or water standard. In this case the resource being allocated would be the right to produce a certain quantity of the polluting material. The final shadow price for such a constraint would then be interpreted as the cost which should be levied against the subunits for each unit of the polluting material produced.

It should be pointed out that while all the constraints are written as less than or equal to inequalities, no generality is lost. Minimum output requirements are represented by using negative b_i values and negative a_{ij} or d_{ij} coefficients to represent output per unit of activity. Cases where the assumption of free disposal is not acceptable (i.e., equality constraints) can be represented by two inequality restrictions.

Actually in mathematical terms the structure of model (2) can be considered a special case of model (1) (Kornai and Liptak, 1965). All that is required is that the A matrix of (1) have the following structure:

$$\begin{bmatrix} A^{o1} & A^{o2} & \dots & A^{on} \\ A^{11} & 0 & \dots & 0 \\ 0 & A^{22} & & \\ \vdots & & & \vdots \\ 0 & & & A^{nn} \end{bmatrix} . \tag{3}$$

Then let $A^{oj} = A^j$, $j = 1$ to n and let $A^{ij} = D^j$, $i = j = 1$ to n. Even though (2) can be treated as a special case of (1), model (1) is considered separately because the difficulty of finding a solution is closely related to the number of central resources.

Linear models with matrix structures such as (3) are of general importance because they result not only from situations where a number of subunits are linked together in a single time period, but also in dynamic models where the linking is between time periods (Orchard-Hays, 1968, p. 260). Even if the system being modeled is not such that subunits or time periods are identifiable it still may be possible to rearrange the matrix into the block-angular form (Weil, 1968; Weil and Kattler, 1969).

To facilitate the solution process the original models (1) and (2) are each decomposed into a central or restricted master problem and a group of n subproblems. The central problem obtained from (1) is (4):

$$\max_{\lambda^{jk}} \sum_{j=1}^{n} \sum_{k \varepsilon K^j} \lambda^{jk} p^{jk}, \text{ where } p^{jk} = c^{j'} x^{jk} \text{ and } \lambda^{jk} \text{ is a scalar,}$$

$$\text{subject to } \sum_{j=1}^{n} \sum_{k \varepsilon K^j} \lambda^{jk} q^{jk} \leq b, \text{ where } q^{jk} = A^j x^{jk} \tag{4}$$

$$\lambda^{jk} \geq 0, \text{ all } j,k .$$

The jth subproblem corresponding to the central problem (4) and an allocation u^j of central resources is (5):

$$\max_{x^j} \; (c^{j\prime} - \Pi^{k\prime} A^j) x^j, \text{ where } \Pi^k \text{ is a vector of shadow prices from (4)}^{1}$$

subject to (5)

$$A^j \, x^j \leq u^{jk}$$

$$x^j \geq 0.$$

An optimal solution for the jth subproblem (5), given u^{jk} and Π^k, will be designated as x^{jk} and the optimal dual values will be designated as v^{jk}.

The central problem obtained from (2) is (6):

$$\max_{\lambda^{jk}} \sum_{j=1}^{n} \sum_{k\varepsilon K^j} \lambda^{jk} \, p^{jk}, \text{ where } p^{jk} = c^{j\prime} \, x^{jk}$$

$$\text{subject to } \sum_{j=1}^{n} \sum_{k\varepsilon K^j} \lambda^{jk} \, q^{jk} \leq b, \text{ where } q^{jk} = A^j \, x^{jk} \quad (6)$$

$$\sum_{k\varepsilon K^j} \lambda^{jk} = 1, \; j = 1 \text{ to } n$$

$$\lambda^{jk} \geq 0, \text{ all } j,k \; .$$

The shadow prices for the vector b and for the convexity constraints of (6) are designated as Π^k and y^{jk} respectively. The jth subproblem corresponding to the central problem (6)

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[1]Note that the Π^k vector will always be non-negative since the constraints on central resources in (4) are all "less than" inequalities.

and an allocation of central resources u^{jk} is (7):

$$\max_{x^j} \quad (c^{j'} - \Pi^{k'} A^j)x^j$$

$$\text{subject to} \quad A^j x^j \leq u^{jk} \tag{7}$$

$$D^j x^j \leq b^j$$

$$x^j \geq 0.$$

An optimal solution for the jth subproblem (7) given u^{jk} and Π^k will be designated as x^{jk} and the optimal dual values for u^{jk} and b^j will be designated as v^{jk} and w^{jk} respectively.

The Dantzig-Wolfe Decomposition Algorithm

A quick summary of the Dantzig-Wolfe decomposition algorithm can easily be accomplished using central problem (6) and subproblems (7), with the assumption that the allocation of central resources u^{jk} is so large as to never be constraining to any subproblems. Assume also that the original model (2) has an optimal solution.

If a set of vectors q^{jk} and their associated scalars p^{jk} are available such that (6) has a feasible solution, then the algorithm proceeds in the following steps.

Step 1. Solve (6) obtaining optimal values λ^{jk} and the optimal dual values Π^k and y^{jk}.

Step 2. Substitute Π^k of step 1 into the n subproblems

(7) and obtain optimal solutions[2] x^{jk}, $j = 1$ to n and their associated $z^{jk} = (c^{j'} - \Pi^{k'} A^j)x^{jk}$.

Step 3. Pick $\delta^{ok} = \max_j (z^{jk} - y^{jk})$. If $\delta^{ok} \leq 0$, then an optimal solution to the original model (2) is given by $x^{j*} = \sum_{k \epsilon K^j} \lambda^{jk*} x^{jk}$, $j = 1$ to n. If $\delta^{ok} > 0$, then for j such that $z^{jk} - y^{jk}$ is a maximum compute $q^{jk} = A^j x^{jk}$ and $p^{jk} = c^{j'} x^{jk}$ and use the resulting q^{jk} and p^{jk} to augment problem (6). Then set $k = k+1$ and return to step 1.

If the required initial feasible solution is not available for (6) then artificial vectors can be introduced and an initial feasible solution may be obtained by a phase 1 procedure following the steps just given (Dantzig, 1963, p. 454).

The decomposition type algorithms are important purely as computational techniques for very large scale problems since they allow the large problem to be broken up into a number of smaller problems which can be solved sequentially. This facilitates the solution of problems too large for existing computer capacity but has more general implications for computational costs since marginal computational costs

[2]The jth subproblem cannot be infeasible if the original problem has an optimal solution as assumed. If the jth subproblem is unbounded then a slight variation in the procedure is sufficient to solve the difficulty (Dantzig, 1963, p. 453).

are an increasing function of the problem size. It has been noted that for dynamic problems the number of rows and columns grows in proportion to the number of time periods involved while the computational effort grows about in proportion to the cube of the number of time periods involved (Dantzig, 1970, p. 51). A similar statement would be true with respect to the number of subunits linked crossectionally as will be considered here.

Since Dantzig and Wolfe first applied the decomposition principle to linear programming the principle has been applied to a number of specific nonlinear programming problems such as quadratic, convex, and geometric programming problems (Hass, 1969; Charnes, Fiacco and Littlechild, 1966; Zangwill, 1967; Zener, 1964). While the decomposition algorithm for linear programming has not yet become an important method of computation in actual practice, it continues to receive much attention as is evident from the sections on large scale programming in Dantzig (1968) and Kuhn (1970). Orchard-Hays has stated that "...decomposition is the only really promising extension to mathematical programming for large and complicated models" (Orchard-Hays, 1968, p. 240).

Decomposition and Decentralized Decisions

The intention here is to focus not on decomposition as a purely computational device so much as on the possibility of using decomposition to effect an optimal decentralized decision process within a large economic system. Such a system could be a large corporation composed of different plants, a national economy composed of different sectors, or, where prices for outputs are available, a public institution composed of different departments. From its very inception the importance of decomposition for decentralizing of decisions has been apparent. Dantzig wrote a short dialogue in which decision-makers at two different levels used the decomposition principle to solve a small transportation problem by sending only specific quantity and shadow price information to each other (Dantzig, 1963, p. 456). Examples of references which discuss the use of the decomposition principle for decentralization of decision-making in national planning, multiplant firms, and multidepartment public institutions include (Malinvaud, 1967; Kornai, 1969; Gale, 1960, p. 85; Whinston, 1964; 1966; Fox, McCamley, and Plessner, 1967).

Noting the structure of (6) and (7) and the information which is communicated between the central and subproblems in the steps of the Dantzig-Wolfe algorithm, it is obvious that the information relevant to the solution is largely decentralized. The decision-maker for the jth subunit needs to know the productive activities (i.e., the matrices A^j and D^j), the profit per unit of activity (i.e., the vector c^j), and the specific resource levels or capacities (i.e., the vector b^j) which constrain his jth subunit. The only information he needs concerning the other subunits is contained in the vectors of shadow prices, Π^k, for the central resources, b. This sequence of price vectors is the only information which need be passed from the central to the subdecision-maker during the solution process. After the optimal solution has been obtained, the optimal weights, λ^{jk*}'s, must also be sent to the subunit.

The central decision-maker needs virtually no information about the production techniques of the subunits. He needs only to know the amounts of central resources available along with the specific proposals he receives from subunits. The proposed vectors of central resource use and/or production (i.e., the q^{jk} vectors) along with the amount of profit realizable from each proposal (i.e., the p^{jk} values) are the only pieces of information which need to flow from the subunits to the center to accomplish the desired solution.

The solution is essentially obtained by charging (paying) the subunits for the quantities of central resources which they use (produce) according to an imputed price, Π^k. This imputed price is varied from proposal to proposal to reflect the relative scarcity of the central resources and when the final central solution is obtained Π^k provides an equilibrium imputed price for the system (Gale, 1960, p. 91). However, this equilibrium price vector alone is not necessarily sufficient to insure that the subunit will be led to produce quantities consistent with the overall optimum. The breakdown occurs if some subunits have alternative suboptima for the equilibrium price vector (Charnes, Clower, and Kortanek, 1967, p. 299). To insure consistency, in this case, additional information such as the optimal weights, λ^{jk*}'s, or specific allocations of central resources must be transmitted to the subunit.

The major problem which is of concern here is that the original Dantzig-Wolfe algorithm has been found to converge quite slowly and to require too many major iterations to be effective as an actual decentralized planning mechanism (Beale, Hughes, and Small, 1965, p. 14; Kornai, 1969, p. 153). The following quote indicates the difficulty: "At each major iteration price imputations are computed by the central unit and a price vector is delegated separately to each division. However, not all divisions may take action with respect to these price vectors. It is possible for ninety-nine of one hundred divisions to be economically idle while for division one iterations will proceed through a long sequence of imputed prices $\Pi_1{}^T$, $\Pi_2{}^T$,... . For an ordinary linear programming problem, solution techniques of the type described may involve thousands of iterations, which in terms of real life message time may be utterly impractical" (Charnes, Clower, and Kortanek, 1967, p. 296).

The reason for this can partially be traced to the fact that the imputed prices in iterations are likely to be extremely poor indicators of the relative scarcity of the central resources. Early imputed price vectors are likely to bear little resemblance to the final imputed price vector given by the dual of the central resource vector at the final optimal solution. Especially damaging is the fact that even the most important central resources will have zero shadow prices at different stages giving the subunit no incentive to make proposals which economize in the use of that resource or proposals which would produce quantities of the resource for other firms. Given such unrealistic prices, without any constraints on central resources, the subunits are almost certain to return proposals which grossly over-use and over-produce certain central resources while grossly under-using and under-producing others and as a result contribute little toward obtaining an optimal solution.

The above propositions are supported by experience from a specific application of decomposition to a problem of planning a production and investment schedule for a group of oil fields. It was felt that the main reason for the slow convergence of the original Dantzig-Wolfe algorithm could be traced to the fact that "the Π-values on the common rows oscillated wildly from one major iteration to another." And it was found that a considerable savings in the number of major iterations can be obtained if it is possible to "generate Π-values at the start of the problem that are quite like the Π-values at the final optimum" (Beale, Hughes and Small, 1965, pp. 14-15).

The methods outlined below are proposed with the belief that they may prove to be more useful for decentralized economic planning than the original Dantzig-Wolfe algorithm and the existing similar methods. The goal is to obtain a solution with significantly fewer major iterations by a process that requires that somewhat more information be passed between the different levels at each major iteration and increases the number of rows in subproblems.

The process combines a pricing mechanism with a resource allocation mechanism. It has been suggested that such a procedure would be closer to methods used in actual practice and would provide more useable theoretical conclusions (Malinvaud, 1967, p. 207). In the procedure outlined not only the shadow price vector, Π, but also the vector of quantity allocations u^j will be passed from the center to the jth subunit at each iteration. The purpose of the quantity allocations is to compensate for the poor price imputations which must be used during early iterations, especially the zero values. Furthermore the quantity allocations make it possible to obtain vital dual price information which can be sent to the center along with the new quantity proposal and used by the center to make better reallocations at any later iteration. Economically the prices sent to the center can be viewed as the amount the subunit would bid for an additional unit of resource it uses as an input or the opportunity cost of the last unit of an output quota it must fulfill.

In a planning model applied to the Hungarian economy with a structure identical to (2) except that the D^j matrices were decomposed making a three level hierarchy, Kornai suggested using central resource shadow prices from the sub-units for reallocating central resources in a heuristic manner (Kornai, 1969, p. 156). Such shadow prices from a model structured like (2) have also been used in two decomposition algorithms which obtain a solution by making successive re-allocations of the central resource vector but do not make use of shadow prices obtained from the center (Abadie and Sakarovitch, 1970).

The subunits will vary in their profitability according to relative differences in the amount of return per unit of activity (i.e., c^j values) and due to their efficiency in the production and use of central resources (determined by the elements of A^j). In the final solution subunits which are, in general, more profitable will be most important. They will use and produce the greatest portion of central resources. This would indicate that an iteration process which could identify the more profitable firms at an early stage should be able to arrive at a solution more efficiently by investing most of its effort in obtaining new revised proposals from those profitable firms. A criterion will be proposed by which the subunits can be identified according to profitability or potential profitability at each major iteration so that different actions can be taken toward the subunits.

Important properties of the Dantzig-Wolfe algorithm which are retained in the solution processes given here include the availability of a feasible solution at any iteration and the fact that the objective function increases monotonically with each iteration. This means that while the effort of an additional iteration can be expected to yield an improvement in the solution, if the transactions cost of exchanging information, computing, and waiting for a solution exceeds the expected improvement, then the process can be terminated with a useable solution available. The importance of these properties for actual applications has been noted in the literature (Martos and Kornai, 1965, p. 184). Furthermore it is possible to compute indices which aid in determining the potential for improvement from further iterations and in special cases an upper bound for the objective function value can easily be obtained. Both types of information are important in deciding whether to terminate the iterations before a final optimum has been reached.

Solution Process for Model (1)

An outline of the proposed process for obtaining a solution to model (1) in a decentralized manner is given below, followed by a discussion of the mathematical and economic rationale for the process. After that an outline and discussion of the proposed solution process for model (2) will be given.

Before presenting the precise steps of the process, a rough summary of the economic meaning of the steps will be given. In step 1 the central decision-makers take the proposed vectors of input and/or output quantities which have been obtained from the subunits and form a vector of total input and/or output quantities for the whole system using a weighted sum of the proposal vectors. The weighted sum is formed in such a way that total direct returns to the system are maximized among those weighted sums which satisfy the total resource constraints for the system. The optimal weighted sum is obtained by a linear programming problem which also provides shadow price values for the resources. These shadow prices have the well known value of the marginal product interpretation. They indicate the value of an additional unit of each resource or alternatively the loss which would be incurred by a one unit decrease in any resource.[3]

The new allocation vectors are determined in step 2. These allocation vectors indicate the maximum amount of input the subunit can use and/or the minimum amount of output it must produce. The objective is to make the allocations so that each resource will be used and/or produced most efficiently. Thus the new allocation vectors are obtained by modifying those proposal vectors which are, at that stage, the most economic, i.e., proposals which break even when resources are valued according to the central shadow prices obtained in step 1. The specific breakeven proposals to be modified are determined by identifying maximum positive deviations between the price imputed to a specific resource by the subunit and the central shadow price for that resource. The maximum positive price deviation identifies the subunit which has a high potential for profitably using an additional allocation of the particular resource and using it more profitably than it would be used in the weighted sum solution of step 1. The new allocations then include a large

[3]This interpretation may be valid only for a change in each resource less than a certain amount. The amount can be determined by a range analysis. If the optimal solution happens to be degenerate it may be that the interpretation will not hold even for a very small change in the resource level since any change could cause the optimal basis to become infeasible.

increase in the resource with the maximum price deviation and a small increase in other quantities of the proposal to allow the subunit some flexibility for reshuffling its input and output configuration.

In step 3 the subunits form a new proposal such that the resource allocation which they received is satisfied and such that the value of production activities is a maximum given they are charged for inputs and credited for outputs according to the central shadow prices obtained in step 1. If a new proposal is found which is more profitable than the existing proposals it is sent to the center to be included in a new weighted sum solution for the whole system. The subunit linear programming problem provides a vector of dual variables corresponding to the new proposal vector. These dual variables indicate the value (cost), over and above the central price at which the subunit was charged (credited), of each additional unit of the resource used (produced). Thus the sum of the central shadow price vector and the dual vector indicate prices at which the subunit would demand additional units of input and/or supply additional units of output. This vector sum provides a subunit price imputation for the quantities in the new proposal and is the vector used in step 2 for making new allocations. Step 4 merely indicates the conditions under which it is known that the system has reached an equilibrium and an overall optimum.

The specific steps of the process will now be given. At the kth major iteration a set of q^{jk} vectors will be available such that the central problem (4) has a feasible solution.[4] The problem of obtaining an initial feasible central solution will be discussed below. The central problem must have a bounded optimal solution given the assumption that the

[4]To aid the reader, model (1), the related central problem (4), and jth subproblem (5) are repeated below:

$$\max \sum_{j=1}^{n} c^{j'} x^j$$

$$\text{subject to } \sum_{j=1}^{n} A^j x^j \leq b \qquad\qquad (1)$$
$$x^j \geq 0,\ j = 1 \text{ to } n$$

$$\max_{\lambda^{jk}} \sum_{j=1}^{n} \sum_{k\varepsilon K^j} \lambda^{jk} p^{jk}, \text{ where } p^{jk} = c^{j'} x^{jk}$$

$$\text{subject to } \sum_{j=1}^{n} \sum_{k\varepsilon K^j} \lambda^{jk} q^{jk} \leq b, \text{ where } q^{jk} = A^j x^{jk} \quad (4)$$

$$\lambda^{jk} \geq 0, \text{ all } j,k$$

$$\max_{x^j} (c^{j'} - \Pi^{k'} A^j) x^j, \text{ where } \Pi^k \text{ is a vector of shadow prices from (1)}$$

(5)

$$\text{subject to } A^j x^j \leq u^{jk}, \text{ where } u^{jk} \text{ is the jth subunit's allocation of resources at the kth iteration.}$$
$$x^j \geq 0$$

The optimal primal and dual vectors for (5) are x^{jk} and v^{jk} respectively.

original problem has an optimal solution. Then at the kth iteration, the following steps are taken.

Step 1. Solve problem (4) obtaining the optimal primal values λ^{jk*}, $k \varepsilon K^j$, $j = 1$ to n and the corresponding dual values $\Pi_i^{\,k}$, $i = 1$ to m.

Step 2. Let $Q = \{q^{jk} | \lambda^{jk*} \; 0\}$ and $V = \{\overline{v}^{jk} | \lambda^{jk*} > 0\}$. Then the proposals in Q when evaluated at central price imputations Π^k just break even, and V is the set of subunit imputed prices for the same proposals. For each resource find that $q^{jk} \varepsilon Q$ such that the difference between $v_i^{\,jk}$ and $\Pi_i^{\,k}$ is a maximum. If the maximum difference is positive then that jth subunit is given an allocation u^{jk} equal to $q^{jk} + \delta |q^{jk}|$,[5] $0 < \delta < 1$, except for the addition of a large positive increment to the ith element. Periodically, (i.e., not necessarily every iteration) all subunits not receiving an allocation from the above rule are allocated $u^{jk} = q^{jk}$ such that $q^{jk} \varepsilon Q$, or if some unit has no $q^{jk} \varepsilon Q$ then u^{jk} is allocated such that $u^{jk} > 0$ and $u_i^{\,jk} \geq \max\limits_{k \varepsilon K^j} q^{jk}$.

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[5]The term $|q^{jk}|$ indicates a vector in which each element is the absolute value of the element in q^{jk}.

Step 3. Subunits solve (5) for all allocations received using the central shadow prices Π^k obtained in step 1. If $(c^{j'}-\Pi^{k'}A^j)x^{jk} > 0$, then a new profitable proposal has been obtained and the central problem is augmented by $q^{jk} = A^j x^{jk}$ and $p^{jk}=c^{j'}x^{jk}$. The corresponding subunit price imputations are obtained by setting $\bar{v}_i{}^{jk}=v_i{}^{jk}+\Pi_i{}^k$, unless $v_i{}^{jk}=q_i{}^{jk}=0$ for some resource i, in which case $\bar{v}_i{}^{jk}$ is set equal to 0. Then return to step 1. If no profitable proposals are obtained go to step 4.

Step 4. Make an allocation u^{jk}, such that $u^{jk} > 0$ and $u_i{}^{jk} \geq \max_{k \epsilon K^j} q_i{}^{jk}$, to all subunits not receiving such a strictly positive allocation from step 2. Then go to step 3. If no profitable proposals are obtained then an optimal solution for 1 is given by $x^{j*} = \sum_{k \epsilon K^j} \lambda^{jk*} x^{jk}$, j = 1 to n.

Feasible Solutions for
Model (1)

Following the process just given, the solution to (4) in step 1 will always correspond to a feasible solution to (1), since by subsitution of $q^{jk}=A^j x^{jk}$ and $p^{jk}=c^{j'}x^{jk}$ we get

$$\sum_{j=1}^{n} \sum_{k \epsilon K^j} \lambda^{jk} c^{j'} x^{jk} = \sum_{j=1}^{n} c^{j'} \left(\sum_{k \epsilon K^j} \lambda^{jk} x^{jk} \right)$$

and

$$b \geq \sum_{j=1}^{n} \sum_{k \in K^j} \lambda^{jk} A^j x^{jk} = \sum_{j=1}^{n} A^j \left(\sum_{k \in K^j} \lambda^{jk} x^{jk} \right) .$$

Since each λ^{jk} and x^{jk} is non-negative $(\sum_{k \in K^j} \lambda^{jk} x^{jk})$, $j = 1$ to n, is non-negative and is a feasible solution to (1) with an objective function value equal to that of (4). This implies that a feasible solution for model (1) will be available even if the process is terminated before a final optimal solution for model (1) is obtained.

Reallocation of Resources

The importance of the set Q in step 2 is that it contains those proposals made by the subunits which can be considered to be the most profitable, given the available knowledge at the kth iteration. The difficulty involved in making the division between more and less profitable subunits involves the fact that profitability is defined in relation to the total value of central resources used and/or produced, and the equilibrium price imputation for these central resources is known only after the final iteration. However, the vector of dual variables Π^k obtained from the kth central problem does give the price imputation of central resources if they were to be used according to that kth central problem solution. This is easily seen from the following string of equalities, the left hand term being the value of central resources using the price vector Π^k, the right hand term being the objective function value of (1) at the kth iteration:

$$\Pi^{k\prime} b = \sum_{j=1}^{n} \sum_{k \varepsilon K^j} \lambda^{jk*} p^{jk} = \sum_{j=1}^{n} c^{j\prime} \left(\sum_{k \varepsilon K^j} \lambda^{jk*} x^{jk} \right).$$

The first equality follows from the duality relation; the second from the definition of p^{jk}.

Since Π^k is the price imputation of central resources when used in the best plan obtained at that stage of iteration, the jth subunit will be said to be profitable or to break even if for some $k \varepsilon K^j$, $p^{jk} - \Pi^{k\prime} q^{jk} = 0$ and said to be unprofitable if $p^{jk} - \Pi^{k\prime} q^{jk} < 0$ for all $k \varepsilon K^j$. The direct return from the jkth proposal is given by p^{jk}, and for the proposal to break even this must be as large as the net value of central resources used and/or produced. Any output of a central resource i will show up as a negative q^{jk}, in the proposal; so the subunit is credited for the value of outputs in the proposal and charged for the value of inputs. Since $p^{jk} - \Pi^{k\prime} q^{jk}$ is nothing more than the "$c_j - z_j$" value of the jkth column of problem (4) it is very simple to determine which proposals are unprofitable and which break even. Any proposal having a strictly positive λ^{jk*} will break even, and the set of such proposals is defined as Q in step 2. The proposals contained in Q will, of course, change from iteration to iteration.

Reallocation of Resources and the Use of Subunit Price Imputations

The subunits which are to be given a new allocation of central resources along with the price vector Π^k and asked to make a new proposal are chosen from the subunits having a proposal contained in Q, while only at periodic iterations will all subunits, including those not having a proposal contained in Q, be asked to submit a new proposal. Except for the periodic allocations to all subunits, the new allocations, u^{jk}, will be basically modifications of the q^{jk} vectors contained in Q.

It is in choosing the $q^{jk} \varepsilon Q$ to be modified that the importance of the price information obtained from subunits enters the process. The set V is defined in step 2 to include the subunit price imputations for proposals contained in Q. The important thing to note is that for each vector of quantities $q^{jk} \varepsilon Q$ there are available two different vectors of imputed prices, Π^k and $\bar{v}^{jk}$. The first is the price imputation of central resources relative to the central problem; the second the price imputation of the quantities of central resources q^{jk} relative to the jth subunit. If there are large differences between these imputed values for particular resources, then when the subunit is issued the new prices Π^k it will have an incentive to change the original proposal, q^{jk}.

To the extent that the new allocation vector and the technological structure of the subunits permit, the subunit will have an incentive to subsitute relatively less costly inputs for relatively more costly ones and relatively more valuable outputs for relatively less valuable ones.

A major objective of decentralizing the decision process is to obtain a solution without forcing central decision-makers to have direct knowledge of the technological structure of the subunits. Thus they will not be in a position to know the degree to which substitution will be permitted by the technology. Making the new allocation, however, is a key decision which is made at the center. The allocation vector will never directly restrain a subunit from using less of a resource than it is allocated or from producing more of a resource than the allocation vector requires. The converse, however, is not true. The allocation vector can restrain additional use of inputs and decreases in outputs.

The rule for making new allocations is intended to identify those subunits which have a high potential for changing the relative mix of proposed quantities in such a way as to obtain new proposals which are profitable. The rule in step 2 proposes that this be accomplished by finding for each resource the highest subunit price imputation for those proposals contained in Q. If this highest subunit imputation exceeds the corresponding new central price then the subunit which made the original proposal has the potential to profitably increase its use or decrease its output of that resource. The addition of a large positive increment to the allocation of that particular resource permits the subunit to make such a change in a new proposal. The term "potential" is used because given the information available at the center it cannot be determined whether or not the technology of the subunit will be such that a profitable substitution of relative quantities can be accomplished.

If a large proportion of the resources are produced by some of the subunits, then it would be reasonable to add to the rules in step 2 by also identifying proposals in Q having the largest negative deviation between $\bar{v}_i^{jk}$ and Π_i^k. This would imply a large incentive to decrease the use of or increase the output of that ith resource. In this case there would be no need to add a large positive increment in making the allocation.

The addition of $\delta|q^{jk}|$, $0 < \delta < 1$ to the original proposal, q^{jk}, also needs explanation. It essentially insures that the original jkth solution will be feasible with additional slack to allow relatively small indirect adjustments in the original proposal which may be necessary so that the subunit can take advantage of the large allocation of that particular resource which was valued so highly in the jkth solution. The specific value of δ may be chosen arbitrarily.

Subunit Solutions and New Profitable Proposals

Step 3 begins with subunits solving (5) for all allocations received, using the central shadow prices Π^k. Note that because of the way in which the new allocations are made some subunits may receive more than one allocation and must obtain a solution for each.

Since each allocation to the subunit is less restrictive than some previous proposal, the process assures that a feasible solution exists for each subunit under its new allocation. Making the usual assumption that any output requires a positive input of at least one of the central resources, the subunit problem will be bounded. It follows then that an optimal solution will always exist for the subunit problems.

The rule for deciding whether the subunit's new proposal is profitable is a simple application of the "c_j-z_j" criterion of the simplex method. Given the definition of p^{jk} and q^{jk} we have:

$$(c^{j'}-\Pi^k A^j)x^{jk} = p^{jk}-\Pi^k q^{hk}. \tag{8}$$

The right hand side of the equality is the "c_j-z_j" value which would be obtained for column q^{jk} if it were to be considered a nonbasic column of the kth central problem. Thus $(c^{j'}-\Pi^k A^j)x^{jk} > 0$ implies that the new proposal q^{jk} would increase[6] the value of the central problem in the next iteration if it were included. A zero or negative value for the same quantity indicates that the new proposal would not by itself contribute to the central problem and thus is not included.

Calculation and Meaning of Subunit Price Imputations

Finally step 3 indicates how to compute subunit price imputations, $\bar{v}^{jk}$, for each of the profitable proposals. The vector $\bar{v}^{jk}$ is simply the sum of the dual values, v^{jk}, for (5) and the imputed prices from the center, Π^k. The only exception to this rule is when both v_i^{jk} and q_i^{jk} are zero for some resource i, in which case $\bar{v}_i^{jk}$ is set equal to zero even if Π_i^k is strictly positive.

The economic meaning of the elements of $\bar{v}^{jk}$ is quite simple. If q_i^{jk} is positive so that the ith resource is an input, then Π_i^{jk} is the price which was imposed by the center and v_i^{jk} is the amount the subunit would be willing to pay, over and above Π_i^{jk}, per additional unit of resource i. Therefore the sum $\bar{v}_i^{jk}$ is the total price which the jth subunit would be willing to pay per additional unit of

[6] If the current central problem solution is degenerate, then, of course, an increase cannot be assured.

resource i.[7] If $q_i^{jk}<0$, then the ith resource is produced by subunit j. The subunit is paid Π_i^{jk} per unit of product produced and v_i^{jk} represents the opportunity cost, over and above the return of Π_i^{jk} to the subunit, of being forced to produce the last unit of resource i. In this case, the sum $\bar{v}_i^{jk}$ represents the price which the firm would need to be paid to just cover the opportunity cost of the last unit. In the special case when v_i^{jk} is zero and none of the ith resource is used or produced by the subunit, $\bar{v}_i^{jk}$ is set equal to zero since the resource has no value to the subunit even though the central price imputation for the ith resource is positive.

Next we will indicate the sense in which it is meaningful to interpret and use the $\bar{v}^{jk}$ vectors as they are interpreted and used in the above discussion. First, we will show that the value imputed to the quantities in the vector q^{jk} using the prices in the vector $\bar{v}^{jk}$ is just sufficient to exhaust the direct return from that proposal, i.e., that $\bar{v}^{jk'}q^{jk}=p^{jk}$. For the jth problem (5) the Kuhn-Tucker conditions give us,

$$[v^{jk'}A^j-(c^{j'}-\Pi^{k'}A^j)]x^{jk} = 0 \tag{9}$$

which can be rearranged as,

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[7]This decomposition of the price imputation into the sum of the parts, one imposed from the outside and one obtained from the system, is similar to the problem studied by Nikaidô (1964) in which the imputation imposed from outside was assumed to be a result of monopolistic power.

$$(v^{jk}+\Pi^k)'A^j x^{jk} - c^{j'}x^{jk} = 0$$

and given the definition of $\bar{v}^{jk}$, q^{jk}, and p^{jk}, the result follows:

$$\bar{v}^{jk'}q^{jk} = p^{jk} \ .$$

Note that it is exactly the three vectors of information in this expression which are passed to the central problem.

The imputed price vector $\bar{v}^{jk}$ is not generally valid for quantity configurations differing from q^{jk} by significant amounts; however, since the process uses $\bar{v}^{jk}$ only in conjunction with q^{jk} this is not a problem. A more crucial consideration for the process is that the price $\bar{v}^{jk}$ for the ith resource is not necessarily independent of the price level for other resources even at the quantity configuration q^{jk}.

If the new allocation is a modification of a $q^{jk} \varepsilon Q$, then the subunit can at least obtain a proposal that will break even, since the original proposal is still feasible and breaks even. But the allocation was specifically chosen so that the differences between the elements of the new price vector and the price vector associated with the original proposal were relatively large. If these price differences are large enough that the original proposal is not optimal then the subunit will be able to find a proposal which is strictly profitable as desired.

Final Optimality Conditions

When none of the new proposals from step 3 are profitable then step 4 indicates that a strictly positive allocation should be made to all subunits not receiving a strictly positive allocation from step 2. To insure that the allocation will be feasible it can be made so that it is larger than a previous proposal. If a profitable solution is obtained the process proceeds as before. If no profitable proposals are obtained then an optimal solution has been obtained. It is shown below that x^{j*}, j=1 to n, given in step 4 will be optimal for model (1).

Expression (10) follows from the constraints of (4) and the definitions of q^{jk} and x^{j*}:

$$b \geq \sum_{j=1}^{n} \sum_{k \varepsilon K^j} \lambda^{jk*} q^{jk} = \sum_{j=1}^{n} A^j \sum_{k \varepsilon K^j} \lambda^{jk*} x^{jk} = \sum_{j=1}^{n} A^j x^{j*} \qquad (10)$$

Expression (11) obviously follows from (10):

$$\sum_{j=1}^{n} A^j x^{j*} \leq b \;. \qquad (11)$$

Since the λ^{jk*} values are an optimal solution to (4) we get (12) and (13):

$$\Pi^{k'} (b - \sum_{j=1}^{n} A^j x^{j*}) = 0, \text{ by duality, and} \qquad (12)$$

$$\sum_{k\varepsilon K^j} \lambda^{jk*}p^{jk} = \Pi^{k'} \sum_{k\varepsilon K^j} \lambda^{jk*}q^{jk}, \tag{13}$$

since

$$\lambda^{jk*} > 0 \rightarrow p^{jk} = \Pi^{k'}q^{jk}, \text{ all } j,k.$$

Using the definitions of p^{jk}, q^{jk}, and x^{j*}, (14) follows from (13):

$$(c^{j'}-\Pi^{k'}A^j)x^{j*} = 0, \quad j = 1 \text{ to } n. \tag{14}$$

Given the allocation made in step 4 the vector x^{j*} will be feasible in the jth problem (5). Then from (14) and the fact that no subunit could return a strictly profitable proposal, it follows that x^{j*} must also be optimal. The dual objective function will be $v^{jk*'}u^{jk}$ and it must also equal zero. But $v^{jk*'}u^{jk}=0$ together with $u^{jk}>0$ implies $v^{jk*'}=0$. Therefore from the constraints to the dual of the jth subproblem we get (15):

$$v^{jk*'}A^j=0 \geq c^{j'}-\Pi^{k'}A^j \text{ or } \Pi^{k'}A^j \geq c^{j'}, \; j = 1 \text{ to } n. \tag{15}$$

Finally we note that (11), (12), (14), and (15) are the Kuhn-Tucker conditions for the original problem (1). The fact that the x^{j*} and Π^k values satisfy these conditions is sufficient to prove that they provide an optimal solution to (1) as asserted.

Solution Process for Model (2)

An outline of the proposed process for obtaining a solution to model (2) in a decentralized manner is given below, followed by a discussion of the mathematical and economic rationale for the steps. The rough summary of the economic meaning of the steps for solving model (1) also applies to the steps for model (2) with a few exceptions. The exceptions result from the fact that in model (2) certain resources are identified with specific subunits and impose restraints on those subunits. Thus in step 1 when the center forms an overall solution from the proposals available it must be concerned that not only central resources restrictions are fulfilled but also that the subunit specific resource restrictions are not violated. By forming the overall solution from weighted averages or convex combinations of each subunit's proposals, the center can be assured that the subunit specific constraints will not be violated. There is a dual variable associated with each convexity constraint in the central problem. For each subunit such a dual variable exists and, in economic terms, it is equal to the difference between the direct returns and the net imputed value of central resources in the weighted average solution for that subunit. Thus in step 3 a new proposal is profitable if the difference between direct returns and the net imputed value of central resources in the new proposal is larger than the dual value for the convexity constraint.

As in the solution process for model (1), step 2 is concerned with reallocation of central resources. The reallocation vector is a modification of one of the proposals having a strictly positive weight in the central solution. The modification is made taking into account the difference between the subunit and central imputed prices for resources. Step 4, as before, indicates the conditions under which the central decision-makers can be assured that they have reached an overall optimum for (2).

The steps for the solution of model (2),[8] will now be

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[8]To aid the reader, model (2), the related central problem (6), and jth subproblem (7) are repeated below:

$$\max \sum_{j=1}^{n} c^{j\prime} x^j$$

$$\text{subject to } \sum_{j=1}^{n} A^j x^j \leq b \qquad (2)$$

$$D^j x^j \leq b^j,\ j = 1 \text{ to } n \qquad x^j \geq 0,\ j = 1 \text{ to } n$$

$$\max_{\lambda^{jk}} \sum_{j=1}^{n} \sum_{k\varepsilon K^j} \lambda^{jk} p^{jk}, \text{ where } p^{jk} = c^{j\prime} x^{jk}$$

$$\text{subject to } \sum_{j=1}^{n} \sum_{k\varepsilon K^j} \lambda^{jk} q^{jk} \leq b, \text{ where } q^{jk} = A^j x^{jk} \qquad (6)$$

$$\sum_{k\varepsilon K^j} \lambda^{jk} = 1,\ j = 1 \text{ to } n \quad \lambda^{jk} \geq 0, \text{ all } j,k$$

$$\max_{x^j} (c^{j\prime} - \Pi^{k\prime} A^j) x^j$$

$$\text{subject to } A^j x^j \leq u^{jk},\ D^j x^j \leq b^j,\ x^j \geq 0. \qquad (7)$$

The optimal primal vector for (7) is x^{jk}. The optimal dual vectors for (7) are v^{jk} and w^{jk}.

outlined. At the kth major iteration a set of q^{jk} vectors will be available such that the central problem (6) has a feasible solution. The central problem must have a bounded optimal solution given the assumption that the original problem has an optimal solution. Then at the kth iteration the following steps should be taken.

Step 1. Solve problem (6) obtaining the optimal primal values λ^{jk*}, $k \varepsilon K^j$, $j = 1$ to n and the corresponding dual values Π^k and y^{jk}, $j = 1$ to n.

Step 2. Let $Q = \{q^{jk} | \lambda^{jk*} > 0\}$ and $V = \{\bar{v}^{jk} | \lambda^{jk*} > 0\}$. For each resource find that $q^{jk} \varepsilon Q$ such that the difference between $\bar{v}_i^{jk}$ and Π_i^k is a maximum. If the maximum difference is positive then that jth subunit is given an allocation u^{jk} equal to $q^{jk} + \delta |q^{jk}|$, $0 < \delta < 1$ except for the addition of a large positive increment to the ith element. Periodically (i.e., not necessarily every major iteration) all subunits not receiving an allocation from the above rule are allocated $u^{jk} = q^{jk}$ such that $q^{jk} \varepsilon Q$.

Step 3. Subunits solve (7) for allocations received using the central shadow prices Π^k, obtained in step 1. If $(c^{j'} - \Pi^{k'} A^j) x^{jk} > y^{jk}$, then a new profitable proposal has been obtained and the central problem is augmented by $q^{jk} = A^j x^{jk}$ and

$p^{jk} = c^{j'}x^{jk}$. The corresponding subunit price imputations are obtained by setting $\bar{v}_i^{jk} = v_i^{jk} + \Pi_i^{jk}$, unless $v_i^{jk} = q_i^{jk} = 0$ for some i, in which case v_i^{jk} is set equal to 0. Then return to step 1. If no profitable proposals are obtained go to step 4.

Step 4. Allocate all subunits a vector u^{jk} large enough so as not to be constraining, then return to step 3. If no profitable proposals are obtained then an optimal solution for (2) is given by

$$x^{j*} = \sum_{k \varepsilon K^j} \lambda^{jk*} x^{jk}, \; j = 1 \text{ to } n.$$

Feasible Solutions for Model (2)

The basic difference in the structure of (1) discussed above and (2) is that (2) assumes a certain vector of resources, b^j, is specifically identified with the jth subunit. The first effect of this different structure appears in the additional constraints of problem (6) solved in step 1.

These convexity constraints of (6) force the weights on the proposals of the jth subunit to sum to one. They are necessary to insure that the central solution to (6) will not violate the subunit constraints $D^j x^j \leq b^j$. A solution to (6) will always correspond to a feasible solution to model (2) since by substitution of $q^{jk} = A^j x^{jk}$ and $p^{jk} = c^{j'} x^{jk}$ we get:

$$b \geq \sum_{j=1}^{n} \sum_{k\varepsilon K^j} \lambda^{jk} A^j x^{jk} = \sum_{j=1}^{n} A^j (\sum_{k\varepsilon K^j} \lambda^{jk} x^{jk}), \tag{16}$$

which shows that $\sum_{k\varepsilon K^j} \lambda^{jk} x^{jk}$, j = 1 to n, is feasible for the central constraints of model (2), and (17) shows that the subunit constraints of model (2) are also satisfied.

$$D^j x^{jk} \leq b^j, \text{ all j and k} \tag{17}$$

$$\lambda^{jk} D^j x^{jk} \leq \lambda^{jk} b^j, \ \lambda^{jk} \geq 0, \text{ all j and k}$$

$$D^j \sum_{k\varepsilon K^j} \lambda^{jk} x^{jk} \leq b^j \sum_{k\varepsilon K^j} \lambda^{jk} = b^j(1), \text{ all j.}$$

In the special case when $b^j \geq 0$, the jth convexity constraint of (6) may be changed from an equality to a less than or equal constraint (i.e., $\sum_{k\varepsilon K^j} \lambda^{jk} \leq 1$). In that special case expression (17) would be rewritten as (18):

$$D^j \sum_{k\varepsilon K^j} \lambda^{jk} x^{jk} \leq b^j \sum_{k\varepsilon K^j} \lambda^{jk} \leq b^j, \text{ all j.} \tag{18}$$

Reallocation of Resources

The definition of the sets Q and V in step 2 are the same as in the former process; however, the economic interpretations of the proposals contained in Q must be modified. The "$c_j - z_j$" value for each proposal in the central problem is given by $p^{jk} - (\Pi^{k'} q^{jk} + y^{jk})$ and for $\lambda^{jk*} > 0$ we know that this value will equal zero. Therefore it follows that (19) will hold for any proposal contained in Q:

$$p^{jk}-\Pi^{k'}q^{jk} = y^{jk} \quad . \tag{19}$$

The value of y^{jk} which is the dual variable for the jth convexity constraint can be interpreted economically as the difference between the direct returns of the jkth proposal contained in Q and the net value of central resources used, where the resources are priced at the current central imputed prices Π^k. From (19) it is obvious that the proposals in Q cannot be given a straightforward interpretation as being breakeven proposals as they were in the former procedure. The set of proposals in Q were, however, chosen from all available proposals and used to form weighted average proposals which would maximize the central problem's objective function. It is for this reason that the proposals contained in Q are used in making new allocations.

The rule in step 2 for making new allocations when there is a maximum difference between $\bar{v}_i^{jk}$ and Π_i^k is the same as in the previous procedure and the explanation given there applies. As in the previous procedure it would be reasonable to add to the rule in step 2 so that proposals in Q associated with large negative deviations between $\bar{v}_i^{jk}$ and Π_i^k would also be used as the basis for new allocations at each iteration. Subunits not having proposals associated with large absolute deviations between $\bar{v}_i^{jk}$ and Π_i^k indicate less potential for returning profitable new proposals and thus it may be more efficient not to obtain new proposals from these subunits at every iteration.

New Profitable Proposals

The criterion for determining whether or not a new proposal is profitable differs from that of the previous procedure but is easily explained. As was noted above, the "$c_j - z_j$" value for the jkth proposal in the central problem is given by $p^{jk} - (\Pi^{k'} q^{jk} + y^{jk})$ and thus any new proposal such that $p^{jk} - (\Pi^{k'} q^{jk} + y^{jk}) > 0$ would increase the value of the central problem in the next iteration and must be considered profitable. But given the definition of p^{jk} and q^{jk}, $p^{jk} - (\Pi^{k'} q^{jk} + y^{jk}) > 0$ implies $(c^{j'} - \Pi^{k'} q^{jk}) x^{jk} > y^{jk}$ which is the criterion given in step 3. Since the original jkth solution will remain feasible under the new allocation, the subunit will be able to obtain an objective function value equal to y^{jk} using that original solution. Any improvement upon the original solution will result in a profitable new proposal. If the allocations were not determined from $q^{jk} \epsilon Q$ the above statement would not hold in general. The procedure for obtaining $\bar{v}^{jk}$ values is identical to that in step 3 of the previous procedure and so no further discussion will be given here.

Final Optimality Conditions

When none of the allocations from step 2 provide profitable new proposals in step 3, the rule in step 4 indicates that all subunits should be given a new allocation large enough so as not to be constraining. If new profitable proposals are obtained from these allocations they are included in the central problem as usual. If no profitable proposals are obtained then an optimal solution has been obtained. Proof that the result is optimal follows by noting that when subunits are unconstrained in the use or production of central resources the process here is identical to the original Dantzig-Wolfe algorithm and therefore the same optimality criterion applies.

The optimal solution will be reached in a finite number of steps if the central problem (6) is nondegenerate at each iteration (or if appropriate perturbation methods are employed). This result follows from the fact that the Dantzig-Wolfe algorithm terminates in a finite number of steps (Dantzig, 1963, p. 452) and the fact that the solution process given above is identical to the Dantzig-Wolfe algorithm once step 4 has been reached.

Computing an Upper Bound

Step 4 is also important for evaluating the opportunity cost of terminating the process before an optimal solution has been obtained. This is because after all subunits have obtained solutions for which all central resource restraints are slack, it is possible to compute an upper bound for the final overall optimum. If the existing solution is near enough to the upper bound further iterations may not be worthwhile. The decision to terminate before reaching an optimal solution would be exercised if the cost of information involved in making additional major iterations were greater than the deviation between the existing solution and the upper bound. It should be noted that while the upper bound given below is guaranteed to be not less than the final optimal solution there is no guarantee that it will be close to the final solution, nor is there any guarantee that successive upper bounds will each get closer to the final optimal value.

The theorem is due to Dantzig (1963, p. 452); however, the proof is approached in a different manner. If max z is the final optimal value for model (2), z^{k*} the existing central problem value, and x^{jk} the solution for subunit j corresponding to Π^k, and there are no constraining limits on central resources, then (20) follows:

$$\max z \leq z^{k*} + \sum_{j=1}^{n} [(c^{j'} - \Pi^{k'} A^{j}) x^{jk} - y^{jk}]. \qquad (20)$$

First note that when there are no constraining limits on central resources, then the dual constraints of the jth subunit can be written as:

$$D^{j'} w^{jk} + A^{j'} \Pi^{k} \geq c^{j'}, \; j = 1 \text{ to } n. \qquad (21)$$

But (21) when taken for j = 1 to n is exactly the set of constraints for the dual to (2). Therefore, Π^k and w^{jk}, j = 1 to n constitute a feasible solution to the dual of (2) and by a well known lemma (Gale, 1960, p. 10) the value of any feasible dual solution is always at least as large as the value of the optimal primal solution. Thus we have (22) which gives the required result:

$$\max z \leq \Pi^{k'} b + \sum_{j=1}^{n} b^{j'} w^{jk} = \Pi^{k'} b + \sum_{j=1}^{n} (c^{j'} - \Pi^{k} A^{j}) x^{jk} \qquad (22)$$

$$= \Pi' b + \sum_{j=1}^{n} y^{jk} + \sum_{j=1}^{n} [(c^{j'} - \Pi^{k'} A^{j}) x^{jk} - y^{jk}]$$

$$= z^{k*} + \sum_{j=1}^{n} [(c^{j'} - \Pi^{k'} A^{j}) x^{jk} - y^{jk}].$$

The first equality is due to the duality relation in the subproblems; the second results from adding and subtracting $\sum_{j=1}^{n} y^{jk}$, and the third is due to the duality relation in the central problem.

Convexity Constraints and Scale of Proposals

The presence of the convexity constraints in (6) makes the scale of the proposals as well as the relative mix of central resources important. In the former procedure only the mix was important. Since the λ^{jk} weights were not constrained (except of course to be non-negative) the central problem could use a particular proposal at any desired scale. Under this procedure, even when it is known that $b^j \geq 0$, the weights must be constrained such that $\sum_{k \varepsilon K} j \lambda^{jk} \leq 1$ implying that proposals may be scaled down but not scaled up. The scaling problem is even more crucial if b^j is not non-negative so the weights must be constrained such that $\sum_{k \varepsilon K^j} \lambda^{jk} = 1$. This forces the central problem to use, in its solution, proposals from the jth subunit even if all of the proposals from that subunit use central resources less profitably than proposals from other subunits. For this reason it is important that the central problem be augmented not only with proposals appropriately scaled upward from more efficient subunits but also with proposals appropriately scaled downward from less efficient subunits.

Alternatives for Steps 2 and 3

Before continuing the discussion, alternatives to step 2 and step 3 will be given and discussed. These alternative steps 2 and 3 decrease the amount of information obtained and stored at the center by replacing a vector of price imputations with a scalar of imputed total value. The scalar r^{jk} provides an index indicating the degree to which profitable new proposals can be expected to involve, on the average, additions to or subtractions from the original proposal. The new allocations are very simply related to existing proposals, involving, at the most, proportional additions to each element.

Alternative Step 2. Let $Q = \{q^{jk} | \lambda^{jk} \ 0\}$. Then for each $q^{jk} \varepsilon Q$ compute $r^{jk} = \dfrac{t^{jk} - \Pi^{k'} |q^{jk}|}{\Pi^{k'} |q^{jk}|}$. For each j take that r^{jk} having the largest absolute value and use the associated q^{jk} in making a new allocation such that $u^{jk} = q^{jk}$, if $r^{jk} < 0$ or $u^{jk} = q^{jk} + \delta r^{jk} |q^{jk}|$, if $r^{jk} > 0$.

Alternative Step 3. Subunits solve (7) for allocations received using the central shadow prices Π^k obtained in step 1. If $(c^{j'} - \Pi^k A^j) x^{jk} > y^{jk}$ then a new profitable proposal has been obtained and the central problem is augmented by $q^{jk} = A^j x^{jk}$ and $p^{jk} = c^{j'} x^{jk}$. The corresponding total imputed value of central resources used and produced is given by $t^{jk} = (v^{jk} + \Pi^k)' |q^{jk}|$. Then return to step 1.

If no profitable proposals are obtained go to step 4.

Using alternative step 3, a single scalar number t^{jk} replaces the vector, $\bar{v}^{jk}$, of information indicating subunit imputed prices for the proposed quantities. This decrease in the amount of information to be communicated and stored will be significant unless the number of central resources is quite small. The value index t^{jk} is obtained by taking the inner product of the $\bar{v}^{jk}$ vector with the absolute value of the vector of quantities q^{jk}. This gives the total imputed value of central resources used and produced in the jkth proposal from the subunit's view point.

The index t^{jk} is used in step 2 in making new allocations. Letting $\bar{v}^{jk}$ be defined as before we obtain the equality:

$$t^{jk}-\Pi^{k\prime}|q^{jk}| = (\bar{v}^{jk}-\Pi^{k})'|q^{jk}| \tag{23}$$

and thus, the numerator of r^{jk} can be interpreted either as the difference between the imputed total value of q^{jk} at subunit prices compared with central prices or as a sum of the deviations between the price imputations weighted by the corresponding quantities of resources in the proposal. The latter interpretation will be employed in explaining the use of r^{jk}.

If $\bar{v}_i{}^{jk}-\Pi_i{}^{k}>0$ and $q_i{}^{jk}>0$, then the ith resource is used in the proposal and is imputed a larger value by the subunit than by the center. If $\bar{v}_i{}^{jk}-\Pi_i{}^{k}>0$ and $q^{jk}<0$ the resource is produced and the opportunity cost of its production from the subunit's view point is greater than its imputed value at the center. In either case a positive deviation implies that, other things being the same, a new proposal should have a larger ith element. A negative deviation has the opposite interpretation, that use of the resource should be decreased or production of the resource increased. In either case this implies a smaller ith element in a new proposal.

Since the numerator of r^{jk} is a sum of these price deviations weighted by the quantities involved, a large positive sum would indicate that the greatest potential for profitable changes involves increases in the q^{jk} values. A large negative sum would, of course, indicate the converse. To obtain r^{jk} the sum of weighted price deviations is divided by $\Pi^{k'}|q^{jk}|$ which will always be non-negative[9] and which is the total value imputed by the center to the quantities in the proposal. It can be considered a relevant measure of the scale of the proposal. The division normalizes r^{jk} for the scale of the proposal.

A large positive r^{jk} indicates that the subunit has a potential to make relatively large profitable changes in the original proposal which will on the average involve addi-

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[9] In the unlikely case that $\Pi^{k'}|q^{jk}| = 0$ one could divide by t^{jk}. Note that in the special case when $q^{jk}>0$ the value for $\Pi^{k'}|q^{jk}|$ is easily obtained by taking the difference between p^{jk} and y^{jk}.

tions to the original proposal elements. Therefore a new allocation must be made such that the additions will be feasible. Alternative step 2 would make the additions be proportional to the original proposal and scaled according to the relative size of r^{jk}.

Similarly, a large negative r^{jk} indicates a potential for profitable changes which will, on the average, involve subtractions from the original proposal elements. Since the allocation vectors impose only upper bounds, it is sufficient in this case to make the new allocation equal to the original proposal.

Finally, a r^{jk} with a small absolute value may result either because the deviations between imputed subunit and central prices are relatively small or because they tend to cancel each other out. In the first instance we would say that the unit has a low potential for making profitable changes in the original proposal; in the second case that profitable changes could be expected to involve a reshuffling of the elements with some being increased, others decreased.

In the final analysis the efficiency of using the index r^{jk} cannot be determined without applying it to a number of large scale numerical problems and comparing the results with existing solution methods. This has not been done.

Initial Feasible Solutions

The solution processes for both models (1) and (2) assumed that a set of vectors was available such that the central problem had a feasible solution. The problem of obtaining such an initial feasible set may itself involve a significant amount of computation and information exchange. The economic problem is one of finding an initial plan that is at least internally consistent.

If the initial allocations made to subunits are such that $\sum_{j=1}^{n} u^{jo} \leq b$ and if these initial allocations all turn out to be feasible for the subunits then the resulting proposals will provide a set of q^{jk} such that a feasible solution will also exist for the relevant problem ((4) or (6) depending upon which solution process is being initiated). In this case there is no difficulty since a consistent solution is immediately available. Even in this case, it may be efficient to obtain more than one proposal from each subunit by instructing the subunit to use various parameterized values for the price vector Π^{o}, thus allowing the central problem a wider choice of proposals from which to form the initial solution.

However, the initial allocations are not derived from existing proposals as are the allocations in step 2 of each procedure, and therefore there is no guarantee that all of the initial allocations will be feasible. If a subunit does receive an infeasible initial allocation then it can solve a two step problem. First, by adding artificial variables to the central resource constraints as in (24),

$$A^j x^j - Is^j \leq u^{jo}, \; s^j \geq 0 \tag{24}$$

and minimizing a weighted sum of these variables, the subunit can find a vector of additions to u^{jo} just sufficient for feasibility. Second, the subunit can maximize the value of the standard objective function, subject to an additional constraint that the weighted sum of artificial variables not be larger than the value obtained in the first minimizing problem. Then, as usual, the new proposal will be $q^{jk}=A^j x^{jk}$.

When subunits are forced to add to their initial allocations there is then no guarantee that the central problem will be able to form a feasible solution from the proposals which are returned. If necessary, the central problem can add a vector of artificial variables in the constraints and add large penalties in the objective function for positive values of the artificial variables. The artificial variables measure the amount of inconsistency in the central problem. They indicate for each central resource the total decrease in use and/or increase in production necessary to meet the constraints imposed on the central resources by the vector b. The large penalties associated with any artificial variable in the optimal basis of the central problem will be reflected in a large shadow price for the corresponding central resource. This large Π_i^k value will have the desired effect of calling forth increased production or decreased use of each central resource for which there is a deficit in the existing central solution. Thus, from this point of the process on, the vector Π^k can be used as outlined in the four iteration steps. Furthermore, feasible proposals are available so that the allocation rules in step 2 of each procedure can be used, eliminating any need for artificial variables in the subunits after the initial allocation. The iteration process will drive all the central problem artificial variables to zero and proceed on to an optimal solution.

CHAPTER III. NUMERICAL CALCULATIONS FOR A DECOMPOSITION MODEL

In this chapter the calculations for a small illustrative decomposition model are presented and discussed. The solution process employed is that given in Chapter II for model (2). This model is intended to help clarify the solution process outlined in Chapter II by use of an explicit example. Furthermore it is instructive to note the interaction of the price and quantity variables in an actual example as the iterations proceed.

The calculations presented here are not intended to provide a test of the efficiency of the solution process outlined in Chapter II. A test of the degree to which the process is able to decrease the number of major iterations would require that the process presented here be applied to a number of large scale problems and the results compared with those obtained by applying the original Dantzig-Wolfe algorithm. Such a test should be a next step; however, the necessary computer routine for such large scale problems was not available.

First, the characteristics of the numerical model will be indicated. Next, the method of obtaining the initial solution is outlined. This is followed by a presentation of the numerical values of the key variables throughout the iteration process. An upper bound is computed and the final optimal solution is presented. Finally the behavior of the dual variables is discussed. The chapter is concluded by an appraisal of decomposition and its use in economics.

Characteristics of the Numerical Model

The illustrative numerical model actually employed is given in Table 1. The model consists of 33 restrictions and 37 activities; however, because of its structure it can be decomposed into a central problem like (6) and three subunit problems like (7). The central problem has six rows--one for each central resource and one convexity constraint for each subunit. The subunit problems are all relatively small with the largest having thirteen activities, eleven subunit-specific rows, and three central restrictions rows.

The zero values in the first two elements of the central resource vector b imply that there are none of these two resources available at the center; however, the negative values in activity 9 of subunit 1 and in activity 7 of subunit 2 imply that these activities can produce resources 1 and 2 respectively. These two resources are used as inputs in every subunit by several activities and the constraints force production to be at least as large as use. The third central resource is available up to 220 units at the center. It is not produced by any of the subunits but is used by each in several different activities.

Table 1.[a] Decomposable model

	Row No.	c_j^1's	Subunit 1 Activities												
			5.00	3.75	2.75	1.50	1.50	1.50	0	0	0	0	0	0	0
Central Restric-tions	1					10	10	10			-24				
	2					3	3	3							
	3							5.6					7.4	7.4	
Subunit 1 Restrictions	4		3	1			-2								
	5		1	1	1								-1/3	-1/3	-1
	6		3000	1500	750		5500						3600	3600	11,000
	7								-35	-30					
	8					3	3	3				-6.5			
	9								1	0.5			-6	-4	
	10										1	1	-2	-4	
	11							-10		0.5					
	12												1	1	1
	13					1									

[a]Throughout Table 1, blank positions in the matrices represent zero elements.

Table 1 (Continued)

	Row No.	C_j^2's	Subunit 2 Activities										
			2.10	2.40	1.75	1.75	0	0	0	0	0	0	0
Central Restrictions	1				3	3							
	2				10	10			-15				
	3					5.2					8.5	8.5	
Subunit 2 Restrictions	14		1	1							-0.15	-0.15	-1
	15		1000	2600							1500	1500	10,000
	16						-30	-25					
	17				3	3				-7			
	18						1	0.6			-8.5	-6.0	
	19								1	1	-.15	-4.0	
	20					-10		0.4					
	21										1	1	1
	22				1								

Table 1 (Continued)

	Row No.	C_j^3's	Subunit 3 Activities												
			5.40	4.75	3.00	2.00	2.00	2.00	0	0	0	0	0	0	0
Central Restric-tions	1					4	4	4							
	2														
	3							5.5					9.0	9.0	
Subunit 3 Restrictions	23		3	2											
	24		1	1	1		-2						-0.25	-0.25	-1
	25		3000	1750	500		5400						3000	3000	12,000
	26								-40	-35					
	27					9	9				-18				
	28					3	3	9				-7.5			
	29							3	1	0.7			-7	-5	
	30										1	1	-2	-4	
	31							-12		0.3					
	32													1	1
	33					1									

Table 1 (Continued)

	Row Number	Subunit Specific Restrictions b^j vectors	Central Restrictions b vector
Central Restrictions	1		$\leq$ 0
	2		$\leq$ 0
	3		$\leq$ 220
Subunit 1 Restrictions	4	$\leq$ 0	
	5	$\leq$ 0	
	6	$\leq$ 40,000	
	7	$\leq$ -1,850	
	8	$\leq$ 0	
	9	$\leq$ 0	
	10	$\leq$ 0	
	11	$\leq$ 0	
	12	$\leq$ 5.5	
	13	$\leq$ 2.0	
Subunit 2 Restrictions	14	$\leq$ 0	
	15	$\leq$ 20,000	
	16	$\leq$ -2,750	
	17	$\leq$ 0	
	18	$\leq$ 0	
	19	$\leq$ 0	
	20	$\leq$ 0	
	21	$\leq$ 8.5	
	22	$\leq$ 1.0	

Table 1 (Continued)

	Row Number	Subunit Specific Restrictions b^j vectors	Central Restrictions b vector
	23	$\leq$ 0	
Subunit 3	24	$\leq$ 0	
Restrictions	25	$\leq$ 45,000	
	26	$\leq$ -2,250	
	27	$\leq$ - 210	
	28	$\leq$ 0	
	29	$\leq$ 0	
	30	$\leq$ 0	
	31	$\leq$ 0	
	32	$\leq$ 7.5	
	33	$\leq$ 3.0	

The negative elements in the b^j vectors imply that $x^j=0$ will not be a feasible solution for any of the subunits or that each subunit is forced to operate at least some of its activities at a strictly positive level.

Initiating the Process

The solution process used in solving this illustrative problem is essentially that outlined in the original four steps for solving problems with subunit-specific constraints. The process was initiated by making two allocations to each of the subunits and obtaining a proposal for each allocation using central resource prices of zero, i.e., $\Pi=0$. The numerical allocations used are given in Tables 3-5 as u_i values for k equal to 00 and 0. Assuming that nothing was known about the central resource needs of the subunits, the allocations were made by taking the approximate average amount of central resource per subunit available in vector b and, in the first case, subtracting 20 units from each element and, in the second case, adding 20 units to each element. This was an arbitrary choice but did provide in the first case a total allocation less than b and in the second a total allocation greater than b.

The allocation vectors for k=0 proved to be feasible for each of the subunits; however, the allocations for k=00 were infeasible for each subunit, forcing the subunits to

Table 2. Central problem values and profitability criterion

Major Iteration	k	1	2	3	4	5	6	7
Central Prob. Value Upper bound	z^k	32.282	50.277	53.512	60.414	60.517	60.772	60.803
Convexity Duals	y^{1k}	26.525	-.555	-9.822	11.209	6.009	3.022	2.155
	y^{2k}	.796	-17.889	-34.711				
	y^{3k}	4.961	-5.264	-17.692	13.162	6.402	2.382	1.252
Subprob. Values	z^{1k}	213.000	6.400	-8.397	11.381	6.440	3.022	2.155
	z^{2k}	.796	-14.448	-33.227	-.354	-.147	-13.648	-16.644
	z^{3k}	6.574	-5.174	-12.451	13.782	7.640	2.414	1.252

Table 3. Proposal, dual, and allocation values for subunit 1

Major Iteration	k	00	0	1	2	3
Direct Returns	p	10.324	19.329	9.662	20.745	20.765
Proposals	q_1	-20.000	20.000	-251.013	-152.173	21.000
	q_2	9.250	20.000	9.250	23.318	23.405
	q_3	55.294	55.294	57.967	57.967	55.294
Included in Central Model		yes	yes	yes	yes	yes
Central Solutions with $\lambda^{ik}>0$		1	2,3	2	3,4,5	4,5,6
Subunit Duals	v_1	0.000	0.000	0.000	0.000	0.000
	v_2	1.167	.500	0.000	0.000	0.000
	v_3	0.000	0.000	0.000	0.000	0.000
Central Duals	Π_1	0.000	0.000	.810	.038	.000
	Π_2	0.000	0.000	.000	.027	.003
	Π_3	0.000	0.000	.000	.336	.526
Allocation Derived from:		-	-	$q^{00}+.1\vert q^{00}\vert$	$q^0+.1\vert q^0\vert$	$q^0+.05\vert q^2\vert$
Allocation Vectors	u_1	-20.000	20.000	-18.000	22.000	-144.564
	u_2	-20.000	20.000	10,000.	10,000.	10,000.
	u_3	55.000	95.000	60.824	60.824	58,059

4	5	6	7
21.286	20.764	20.764	20.764
-144.564	-82.229	78.018	78.018
24.401	23.405	23.405	23.405
59.988	55.294	55.294	55.294
yes	yes	no	no
-	6,7	-	-
.023	0.000	0.000	0.000
0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000
.003	.004	.000	.000
.010	.004	.001	.000
.168	.264	.320	.337
$q^2+.05\lvert q^2\rvert$	-	-	-
-144.564	10,000.	10,000.	10,000.
10,000.	10,000.	10,000.	10,000.
60.865	10,000.	10,000.	10,000.

Table 4. Proposal, dual, and allocation values for subunit 2

Major Iteration	k	00	0	1	2
Direct Returns	p	8.885	15.329	8.885	13.260
Proposals	q_1	9.986	19.731	9.986	16.200
	q_2	-136.566	20.000	-136.566	-141.174
	q_3	89.559	95.000	89.559	88.880
Included in Central Model		yes	yes	no	yes
Central Solutions with $\lambda^{2k}>0$		1,2	2,3,4,5,6,7	-	4,5,6,7
Subunit Duals	v_1	1.050	0.000	0.000	0.000
	v_2	0.0000	0.000	.-16	0.000
	v_3	0.0000	.337	1.953	0.000
Central Duals	Π_1	0.000	0.000	.810	.000
	Π_2	0.000	0.000	.000	.003
	Π_3	0.000	0.000	.000	.526
Allocation Derived from		-	-	q^{00}	q^0
Allocation Vectors	u_1	-20.000	20.000	9.986	19.731
	u_2	-20.000	20.000	-136.566	20.000
	u_3	55.000	95.000	89.559	95.000

3	4	5	6	7
13.268	14.601	80.643	51.377	13.268
16.200	18.647	133.000	81.529	16.200
-85.992	-134.115	443.333	271.675	54.000
88.880	96.607	297.583	202.118	80.880
yes	yes	yes	yes	no
4,5,6,7	-	6	-	-
0.000	0.000	0.000	0.000	0.000
0.000	.018	0.000	0.000	0.000
0.000	.080	0.000	0.000	0.000
.000	.003	.004	.000	.000
.003	.010	.004	.001	.000
.526	.168	.264	.320	.337
q^0	$q^2+.05\|q^2\|$	-	-	-
19.731	10,000.	10,000.	10,000.	10,000.
20.000	-134.115	10,000.	10,000.	10,000.
95.000	96.607	10,000.	10,000.	10,000.

Table 5. Proposals, duals, and allocation values for sub-unit 3

Major Iteration	k	00	0	1	2
Direct Returns	p	6.696	21.162	18.560	18.520
Proposals	q_1	2.143	20.000	14.796	14.796
	q_2	0.000	0.000	0.000	0.000
	q_3	70.446	76.377	69.222	68.835
Included in Central Model		yes	yes	yes	yes
Central Solutions with $\lambda^{2k}>0$		1	1,2	2	3
Subunit Duals	v_1	1.063	.500	0.000	.425
	v_2	0.000	0.000	0.000	0.000
	v_3	0.000	0.000	0.000	0.000
Central Duals	Π_1	0.000	0.000	.810	.038
	Π_2	0.000	0.000	.000	.027
	Π_3	0.000	0.000	.000	.336
Allocation Derived from:		-	-	$q^{00}+.1\|q^{00}\|$	q^1
Allocation Vectors	u_1	-20.000	20.000	10,000.	14.796
	u_2	-20.000	20.000	0.000	0.000
	u_3	55.000	95.000	77.491	69.222

3	4	5	6	7
25.573	26.698	34.199	25.578	25.578
30.093	31.598	48.148	30.117	30.117
0.000	0.000	0.000	0.000	0
72.277	76.510	100.016	72.282	72.282
yes	yes	yes	yes	no
4,5,6	5	-	7	-
0.000	.224	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000
.499	0.000	0.000	0.000	0.000
.000	.003	.004	.000	.000
.003	.010	.004	.001	.000
.526	.168	.264	.320	.337
$q^2+.05\|q^2\|$	$q^3+.05\|q^3\|$	-	-	-
10,000.	31.598	10,000.	10,000.	10,000.
0.000	0.000	10,000.	10,000.	10,000.
72.277	10,000.	10,000.	10,000.	10,000.

solve a two step problem using artificial vectors in the constraints as outlined in (24). The initial proposals are given in Tables 3-5 as the q_i values corresponding to k=00 and k=0. The degree to which the allocation vectors were not feasible is evident from the difference between elements of the proposal vector q^{j00} and corresponding elements of the allocation vector u^{j00}. For example u^{100} set an output quota for subunit 1 of at least 20 units of resource 2, but the proposal which was returned, q^{100}, indicates that, while the allocation values for resources 1 and 3 could be met, the quota for resource 2 could not. Rather than the production of 20 units, the proposal called for the use of 9.250 units of resource 2. The fact that the proposal vectors q^{j0} were all at least as small as the allocation vectors u^{j0} confirms that each of the allocations for k=0 was feasible. The dual values for central resources are recorded in the v_i rows of Tables 3-5.

Major Iterations of the Process

The major iteration, k=1, begins by placing the initial proposals in the central problem (6) and obtaining an optimal solution. For this example, the initial proposals were such that the central problem was feasible so that artificial vectors were not necessary. The optimal value of the central objective function at k=1 is given in Table 2, as z^k, along with the optimal dual values for the convexity constraints. Tables 3-5 each give the central resource shadow prices obtained from (6) at each iteration and also list the iterations in which each proposal appeared with a strictly positive weight in the central solution. For example, in the first iteration the set of proposals with a strictly positive λ^{jk} (i.e., the set Q) is given by:

$$Q = \{q^{100}, q^{200}, q^{300}, q^{30}\}. \tag{25}$$

Thus at iteration k=1, four of the six available proposals are used to obtain a central solution value of 32.282 and such that central resource 1 has the only positive shadow price of .810.

After the central solution has been obtained, a new allocation must be made. The subunit price imputations, $\bar{v}_i^{jk}$'s, used in making new allocations are obtained from Tables 3-5 by summing, for each subunit, the kth values for v_i and Π_i. Then the set of $\bar{v}^{jk}$ vectors corresponding to proposals contained in Q is given by the set V and at the first iteration V is given by:

$$V = \{\bar{v}^{100}, \bar{v}^{200}, \bar{v}^{300}, \bar{v}^{30}\}. \tag{26}$$

The maximum subunit price imputations in V for central resources 1 and 2 are found to be respectively $\bar{v}_1^{300} = 1.063$ and $\bar{v}_2^{200} = 1.167$. All proposals impute a zero value to central resource 3. The differences between the maximum subunit imputation and the central imputation for resources 1 and 2 are both positive and are equal to .251 and 1.062 respectively. Thus, subunit 3 is given a very large allocation of resource 1, and subunit 1 a very large allocation of resource 2. The other elements of the allocation vectors for subunits 1 and 3 are obtained by modifying the elements of proposals q^{100} and q^{300}. The modification involves adding to the original element an amount equal to .1 of the absolute value of that element. This .1 represents the δ value of step 2 in the procedure given in Chapter II. The new allocation for subunit 2 is chosen from the proposals for subunit 2 contained in Q which in this case leaves only one choice of q^{00}.

Tables 6-8 show how the subunit objective function coefficients are affected by the central price vector Π^k. Only those coefficients which are affected by Π^k are shown. Any activity which does not produce or consume any of the central resources will not be affected by the Π^k values.

The first major iteration is completed when each of the subunits has obtained optimal solutions relative to the new allocations and new central prices. The resulting optimal objective function values (where the values in Tables 6-8 are used as coefficients) are recorded in the z^{jk} rows of Table 2. These are the values which must be compared with the y^{jk} values

Table 6. Elements from vector $c^{1'} - \Pi^{k'} A^{1}$ which are a function of Π^{k}

Major Iteration	k	Column 4	5	6	9	11	12
	00 and 0	1.500	1.500	1.500	0.000	0.000	0.000
	1	-6.601	-6.601	-6.601	19.442	0.000	0.000
	2	1.040	1.040	-.842	.910	-2.487	-2.487
	3	1.491	1.491	-1.455	0.000	-3.893	-3.893
	4	1.444	1.444	.505	.065	-1.241	-1.241
	5	1.446	1.446	-.030	.100	-1.950	-1.950
	6	1.497	1.497	-.297	0.000	-2.371	-2.371
	7	1.500	1.500	-.385	0.000	-2.490	-2.490

Table 7. Elements from vector $(c^{2'} - \Pi^{k'} A^2)$ which are a function of Π^k

Major Iteration	k	Column	3	4	7	9	10
	00 and 0		1.750	1.750	0.000	0.000	0.000
	1		-.680	-.680	0.000	0.000	0.000
	2		1.365	-.382	.407	-2.857	-2.857
	3		1.719	-1.016	.046	-4.472	-4.472
	4		1.645	.773	.145	-1.425	-1.425
	5		1.697	.326	.061	-2.240	-2.240
	6		1.741	.074	.014	-2.724	-2.724
	7		1.750	-.000	0.000	-2.861	-2.861

Table 8. Elements from vector $(c^{3'} - \Pi^{k'} A^3)$ which are a function of Π^k

Major Iteration	k	Column 4	5	6	11	12
	00 and 0	2.000	2.000	2.000	0.000	0.000
	1	-1.240	-1.240	-1.240	0.000	0.000
	2	1.848	1.848	-.000	-3.025	-3.025
	3	2.000	2.000	-.893	-4.735	-4.735
	4	1.989	1.989	1.067	-1.509	-1.509
	5	1.983	1.983	.534	-2.372	-2.372
	6	2.000	2.000	.237	-2.884	-2.884
	7	2.000	2.000	.149	-3.029	-3.029

to determine whether or not the new proposal is profitable and therefore used to augment the central problem for the next iteration. At the first iteration subunits 1 and 3 did return profitable proposals; subunit 2 did not.

The new proposals show that subunit 3 did propose a large increase in the use of central resource 1 relative to proposal q^{300}. The potential for such an increase had been evident from the subunit price imputations, and specific allowance had been made in the allocation vector. For subunit 1 specific allowance had been made for a large increase in the use of resource 2; however, the new proposal contains a large increase in the production of resource 1 and no increase in the use of resource 2, indicating that the negative difference between v_1^{100} and Π_1^{1} was more important than the positive difference between $\bar{v}_2^{100}$ and Π_2^{1}.

After the new profitable proposals are augmented to the central problem the second major iteration is initiated by obtaining an optimal solution to the central problem. Note that both of the proposals augmented to the central problem had strictly positive weights in the central solution at iteration k=2. The value of the central problem at iteration 2 was 50.227 as compared with 32.282 at iteration 1.

Computing an Upper Bound

The iteration process was continued up to the fifth iteration in the manner just described except that δ was decreased from .1 to .05 for iterations 3 and 4. The results are recorded in Tables 2-8. The new proposals obtained from the fourth iteration resulted in only a very small increase in the central value of the fifth iteration. This seemed to be an obvious place to obtain an upper bound for the final value. The allocations made at iteration 5 were all vectors with very large positive numbers so as to be unconstraining, since the upper bound formula is valid only under such an allocation. Once the z^{jk} values were obtained an upper bound of 71.913 was computed. The relevant calculations are given in expression (26) using the formula given in (20):

$$\max z \leq z^{5*} + \sum_{j=1}^{3} [c^{j'} - \Pi^{5'} A^{j}) x^{j5} - y^{j5}] = 60.51728 \qquad (26)$$

$$+ (6.44012-6.00853)+(-.14660+9.87286)$$

$$+ (7.63954-6.40187) \approx 71.913 \ .$$

At this point the decision-maker could consider terminating the solution process. He has obtained a solution value of 60.517 and knows that additional iterations will obtain an optimal solution greater than the current solution but not greater than the upper bound of 71.913. The potential gain must be compared with the information cost and waiting cost associated with further iterations.

Final Optimal Solution

In the example the process was continued until an optimal solution was obtained. The central solution at iteration 6 again showed only a small increase, so unconstraining allocations were again made to each of the subunits and a new upper bound of 62.292 was computed. At this point the decision-maker would know his current solution was less than 2 units from the optimum.

At iteration 7 none of the $z^{j7}-y^{j7}$ were strictly positive indicating that no profitable new proposals had been obtained and satisfying the criterion for a final optimal solution. The value of the optimal solution is 60.803. The optimal weights in the central problem at iteration 7 were:

$$\lambda^{15*} = 1,\ \lambda^{20*} = .57897,\ \lambda^{23*} = .42103,\ \lambda^{36*} = 1, \quad (27)$$

and

$$\lambda^{jk*} = 0 \text{ for all other } j \text{ and } k.$$

Using these weights and the formula in step 4 of Chapter II an optimal solution for the original problem can be obtained. Using the weights in (27) the vectors of central resources used and/or produced by each of the subunits is given in (28):

$$q^{1*} = \sum_{k\varepsilon K^1} \lambda^{1k*} q^{1k} = (1) \begin{bmatrix} -82.229 \\ 23.405 \\ 55.294 \end{bmatrix} \tag{28}$$

$$q^{2*} = \sum_{k\varepsilon K^2} \lambda^{2k*} q^{2k} = (.57897) \begin{bmatrix} 19.731 \\ 20.000 \\ 95.000 \end{bmatrix}$$

$$+ (.42103) \begin{bmatrix} 16.200 \\ -85.992 \\ 88.880 \end{bmatrix} = \begin{bmatrix} 18.244 \\ 24.626 \\ 92.423 \end{bmatrix}$$

$$q^{3*} = \sum_{k\varepsilon K^3} \lambda^{3k*} q^{3k} = (1) \begin{bmatrix} 30.117 \\ 0.000 \\ 72.282 \end{bmatrix} .$$

The vectors in (28) show an excess of production over use for the first two central resources. This excess supply is reflected in the final shadow prices of zero for the first two resources. Only resource 3 is used to the limit of availability, and it has a positive shadow price of .337.

Behavior and Interpretation of Dual Variables

The elements in the Π^k vectors fluctuate rather wildly throughout the iteration process as is to be expected. In fact, Π^1 gives a positive shadow price to only the first resource which is a surplus resource in the final solution and gives a zero price to resource 3 which in the end is the only constraining resource. The effect of the Π^k fluctuations on the value of individual activities in the subunits can be observed in Tables 6-8.

The y^{jk} values for the final central problem, in this case y^{j7} values, provide a measure of the net contribution of each subunit to the final solution. For example, y^{17} = 2.155 is equal to the difference between direct returns and the net value of central resources for the optimal solution of subunit 1. The calculations are shown in (29):

$$p^{15}-\Pi^{7'}q^{1*} = 20.764-.33654(55.294) = 2.155. \qquad (29)$$

The fact that y^{27} equals -16.643 implies that subunit 2 makes a negative contribution to the final solution; the value of direct returns are not sufficient to cover the value of resource 3 used. Subunit 2 is forced to operate (even though it is incurring a net loss) by the third subspecific constraint which places a lower bound on the weighted sum of the levels of activities 5 and 6.

An Appraisal of Decomposition

In concluding this chapter a few general comments will be made concerning the importance of the decomposition principle for economics. First of all, it would seem that its promise as a computational method for really large scale models is of sufficient importance to warrant further study. However, we are most interested here in appraising its importance as a method for characterizing multilevel decentralized decision processes. For purposes of discussion it is important to distinguish between the attempt to characterize a planning procedure where the models are explicitly employed to obtain an optimal plan and the attempt to characterize a real market adjustment process. The discussion of decomposition in Chapter II and this chapter has essentially envisioned an explicit planning procedure; however, the possibility of employing a decomposition type algorithm to simulate and analyze a real market adjustment process is intriguing. Given that a model could be constructed which essentially captured the production and exchange alternatives open to participants in a specific market economy, it is still not clear that the simplex type adjustment rule of the algorithm would adequately simulate an actual market adjustment process. For example, a gradient type adjustment rule might be more adequate. Furthermore, the decomposition algorithms discussed here are essentially *tâtonnement* adjustment processes in which no trading or production actually takes place until the final equilibrium has been reached. Adequate representation of actual market adjustments might require specification of a non-*tâtonnement* process in which trading and production take place with each new vector of prices, even though they are not equilibrium prices (Quirk and Saposnik, 1968, p. 191). Even given the possible difficulties just mentioned, the algorithm might be a useful method for gaining further insights into actual market adjustment processes.

An important question concerns the relation of market prices and equilibrium shadow prices. The price systems are important elements of the effective decentralization in both an actual market and in the decision processes outlined above. A property of shadow prices which makes any close relation between them and market prices quite suspect is the fact that, for specific models, a zero shadow price is often obtained for a resource which would never have a zero price in the market. It can be argued, however, that "unrealistic" zero shadow prices are a result of a model which has failed to capture the degree of resource substitutibility present in the market rather than any inherent deficiency in the shadow price concept. An absolute surplus of a particular resource and the corresponding zero shadow price may result if, for example, the model does not include activities which allow resources to be transferred between subunits, locations, or time periods, even though such transfers are possible alternatives in the economic system being modeled. A similar result can occur if the model fails to include all of the alternative input mixes possible for producing a particular output. If sufficient resource substitutibility is included in the model, zero shadow price values will not occur.

Another difficulty encountered, especially when using linear models, is the possibility that the equilibrium shadow price vector will not be unique. In this case it is not at all clear which particular vector should be used for comparing with market prices.

In the absence of the problem of nonuniqueness, and assuming that a model can be specified which embodies all the relevant economic alternatives, it does seem quite relevant to compare equilibrium shadow prices with actual market prices. If the shadow price corresponds to a constraint on the use of the resource then it is essentially a point on a derived demand function for that resource. If the market price is less than the shadow price it will be profitable to purchase additional units of the resource and vice versa. If the shadow price corresponds to a constraint imposing a minimum output of the resource, then it can be considered a point on the supply function and again comparison with the market price is relevant.

The most relevant question concerning the usefulness of decomposable models such as those outlined here for planning purposes would seem to involve a comparison of the expected gain from more nearly optimal decisions and the information cost of building and manipulating the necessary models. The major aim of the decomposable model allowing a decentralization of decision-making is to decrease the information cost, but for a large scale organization the cost would still be considerable. It seems quite likely, however, that for some organizations the gains could be expected to exceed the information costs.

CHAPTER IV. THE VECTOR MAXIMIZATION PROBLEM

In the preceding chapters we have taken as part of the data the price or value of resources used and goods produced. The possibility of formulating the problem of resource allocation and production decisions as that of maximizing a scalar value is entirely dependent upon the availability of such prices or measures of relative value. It is possible to collapse a vector of commodities into a meaningful scalar value only when the relative prices or rates at which one commodity should substitute for another are well established. For a decision-maker in a firm which produces only a small portion of the commodities of the larger economic system the assumption that prices are given as data is quite realistic.

In the next two chapters we will deal with two models in which the price vector cannot be taken as given. The first model involves a public institution which does not necessarily follow a policy of profit or sales maximization and which may not sell its output in a well defined market or may not have an easily measurable output. In such a case one is faced with determining a "reasonable" price system through some type of estimation procedure or with making allocation and production decisions without the aid of prices. We will focus on a university as a specific example of such a public institution. The second model involves a production model in a general equilibrium system which by its very nature must take prices as variables rather than as data. The discussion concerning this model will be theoretical in nature as opposed to the operational decision-making approach taken for the university planning model.

In order to make resource allocation and production decisions which are in some sense optimal for the models considered in this chapter we are forced to consider certain aspects of consumption theory. Up to this point we have been able to discuss optimal resource and production decisions independently of consumption considerations by assuming a given price system.

The basic problem with which we must deal is that of determining how we should rank alternative possible bundles of commodities, or how we should choose from alternative bundles, that bundle or group of bundles which is in some sense "the best." The most common approach in consumption theory has been to begin by assuming for each consumer the existence of an ordinal utility function defined over the commodity space. Such a function, $U(y)$, will, for that consumer represent a complete preference ordering of commodity vectors or bundles such that for any two bundles, y^1 and y^2, either $U(y^1)>U(y^2)$, $U(y^1)=U(y^2)$, or $U(y^1)<U(y^2)$ implies respectively that y^1 is preferred to y^2, y^1 is indifferent to

y^2, or y^2 is preferred to y^1. The problem of determining "the best" commodity bundle for a particular consumer then reduces to that of finding a vector which maximizes the utility function. However, once we consider an economic production system in which the resulting commodity bundle affects the utility of a large number of consumers, as in the case of a public institution or a general equilibrium model, we are immediately faced with the most difficult problems of welfare economics. Choices must be made between different commodity bundles which will increase the utility of one group of consumers at the expense of another. Actually in mathematical terminology the problem is that of defining a vector optimum where the elements of the vector are levels of utility for individual consumers. Two approaches have been followed in attempting to solve or circumvent the problem. One approach has been to attempt to extend the utility function concept in such a way that a new function represents a complete preference ordering for the whole group of consumers. The problem of choosing an optimal vector then reduces to maximizing this new function subject to production and resource constraints. The other approach has been to essentially retreat from the problem and be satisfied with the less ambitious concept of a partial ordering present in the concepts of Paretian optimum and efficient production. The first approach has a long history which can be traced at least to a paper by Bergson (1938), in which he proposed a social welfare function. The whole discussion of inter-personnel comparisons of utility and community indifference curves is, of course, closely related to the problem; see (Mishan, 1964). Arrow's (Arrow, 1963) important contribution concerning the possibility of a social welfare function is not encouraging with respect to the fruitfulness of this approach. Arrow poses the possibility of constructing, from individual orderings, a social ordering of social states consistent with certain conditions which are thought to be "reasonable." He proves that it is not possible to construct a social ordering consistent with those "reasonable" conditions. It follows that construction of such a social preference ordering is possible only if some different and probably less "reasonable" set of conditions is accepted.

Economists working in the general area of quantitative economic policy have essentially followed the approach of choosing that set of policy variables which maximized a welfare function. They have, in general, circumvented the problem of constructing a social welfare function from individual utility functions by taking as their welfare function the function representing the preference ordering of the policy-maker involved (Fox, Sengupta, and Thorbecke, 1966, p. 448). The method by which such a policy-maker derives his position is, or course, a very crucial question in itself. However, whether the policy maker's position is the result of some voting procedure, of negotiating procedures among ruling coalitions, or of personal dictatorial power, once his preference function is given, the procedure to be followed is unchanged.

The second approach of being satisfied with only a partial ordering certainly has a very long history in economics. The Paretian criterion which is fulfilled when no one can be made better off without someone else being made worse off and the sub-criterion of efficient production which is satisfied when no more of one commodity can be produced without producing less of some other commodity both imply a partial ordering. Such criteria involve a partial ordering since many points are not comparable, as opposed to a complete ordering in which every pair of points is comparable (i.e., one point is preferred to the other or they are indifferent). For example, the difficulty of choosing between efficient points must still be faced and, while we know that for every nonefficient point there exists an efficient point which is preferred, it is not possible on the basis of only efficiency criteria to say that an efficient point is preferred to a nonefficient point having more of at least one commodity. A similar statement is true for the Paretian criterion. Thus such a partial ordering of alternatives simplifies the problem in that only Pareto optimal or efficient points need be considered but the problem of choosing between different Pareto optimal or efficient points must still be faced. The Pareto optimal or efficient point is usually assumed either to be chosen by some type of central planner or to result from some type of market mechanism. Under certain regularity conditions for a general equilibrium model, a price system exists which will sustain any Pareto optimal point as a competitive equilibrium. This price system allows a degree of decentralization of information and decision-making within the system. The models used in this chapter will both be linear activity analysis models. In the special general equilibrium model we will explore the possibility of finding a price vector, consistent with certain noncompetitive pricing behavior, which will sustain an efficient commodity vector and allow a decentralization of information and decision-making. Thus in the general production model we will follow what has just been discussed as the second approach. We will be satisfied with exploring the relationship between efficient production, prices and decentralization, and leave the problem of choosing between efficient points undiscussed. In the university planning model such indeterminancy is not acceptable. Since we are proposing a planning and decision-making model which is to be operational, it must provide a way in which the "best" commodity vector is actually chosen. We will follow the lead of the quantitative economic policy economists and choose the "best" vector from the point of view of the policy-maker or policy-makers. The procedure outlined will indicate how a combination of the approaches discussed above may be used in a university planning model.

Linear Activity Analysis Models

The linear activity analysis model outlined here is essentially taken from Koopmans (1951a; 1957). No attempt is made here to reproduce all the results of activity analysis; only a brief outline of the model and those results central to our interest will be given. The model assumes that there is a vector of primary commodities, y_P, which is available at a rate of not more than η units per time period. These primary commodities may be transformed into vectors of intermediate, y_I, or final commodities, y_F, by a set of linear activities. The technology matrix, A, the columns of which are the input-output coefficients for each activity, can be partitioned in the following way:

$$A = \begin{bmatrix} A_F \\ \hline A_I \\ \hline A_P \end{bmatrix}, \quad \text{where} \quad \begin{array}{l} A_F x = y_F \\ A_I x = y_I \ , \ x \geq 0 \ . \\ A_P x = y_P \end{array}$$

In all cases inputs are designated by negative numbers (e.g., y_P, $\eta \leq 0$) and outputs by positive numbers (e.g., $y_F \geq 0$). Intermediate commodities are outputs of certain activities, inputs to others, and are not desired in themselves so we set $y_I = 0$. Thus the elements of the input matrix, A_P, will all be nonpositive, those of the output matrix of final commodities, A_F, will be non-negative, while the coefficients of A_I include both input and output coefficients and may be positive, negative, or zero. The level of each activity chosen, or bundle of basic activities, is represented by the non-negative vector x. The model may be compactly written as:

$$\begin{bmatrix} A_F \\ A_I \\ A_P \end{bmatrix} x = \begin{bmatrix} y_F \\ y_I \\ y_P \end{bmatrix} \begin{matrix} \geq \\ \geq \\ \geq \end{matrix} \begin{bmatrix} 0 \\ 0 \\ \eta \end{bmatrix} , \quad x,\ y_F \geq 0,\ y_P \leq 0 \ . \tag{30}$$

Any x which is feasible for b (30) is called attainable and defines an attainable bundle of activities. An attainable bundle is said to be efficient if no other attainable bundle produces as much of every final commodity and more of at least one final commodity.

Of interest for our purposes is the relation between efficient bundles and corresponding systems of prices and the use of these price systems for decentralizing decisions. Koopmans has shown that a necessary and sufficient condition for the efficiency of an attainable bundle is that a price system exists such that the value of the output is equal to the value of the inputs, no single activity is profitable, the price vector of each of the final commodities, p_F, is strictly positive, all primary commodities have non-negative prices, and primary commodities which are not used to the limit of their availability have a zero price. This condition may be compactly expressed as:

y_F associated with an attainable x is efficient if and only if there exists a p such that: (31)

1. $p'y = 0$
2. $p'A \leq 0$
3. $p_F \geq 0$

4.a. $p_{pi} \geq 0$, all i

4.b. $p_{pi} = 0$, if $y_{pi} > \eta_i$.

Koopmans used the topological properties of convex cones to obtain the above result; however, (Charnes and Cooper, 1961, p. 310) have shown that the result can be obtained by using linear programming theory. We will make use of the linear programming formulation since it is closely related to the analysis used in the remainder of the dissertation.

The following theorem is found in (Charnes and Cooper, 1961, p. 312). The proof which is short and straightforward is not given below.

The vector $\hat{y}_F$ is efficient if and only if the optimal solution to the following problem is zero:

$$\min\ -e_F'\bar{y}_F \tag{32}$$

subject to

$$\begin{aligned}
A_F x - y_F &= 0\\
A_I x &= 0\\
A_P x - y_P &= 0\\
y_P &\geq \eta\\
y_F - \bar{y}_F &= \hat{y}_F\\
x,\ \bar{y}_F &\geq 0
\end{aligned}$$

where e is a vector with the same dimension as $\bar{y}_F$ and each element equal to one.

A major advantage of this linear programming formulation is that the optimal solution to the dual of (32) provides a price vector which satisfies the conditions of (31) and which can be easily computed. The dual to (32) is given below as (33):

$$\max\ w_P'\eta_P + t_F'\hat{y}_F$$

subject to

$$\begin{aligned}
A_F'u_F + A_I'u_I + A_P'u_P &\leq 0\\
-u_F \qquad\qquad +t_F &= 0\\
-u_P + w_P &= 0\\
-t_F &\leq -e_F\\
w_P &\geq 0.
\end{aligned} \tag{33}$$

It is not difficult to confirm that if we let $p_P = w_P$, $p_I = u_I$, and $p_F = t_F$, then the resulting vector p will satisfy the conditions of (31) for the corresponding efficient vector $\hat{y}_F$.

Also of interest for our purposes is the problem of the existence of a nonzero efficient bundle of activities. Charnes and Cooper (1961, p. 313) have shown that questions of existence can be analyzed by use of a regularized linear programming model which is closely related to the topic of goal programming.

By assuming that no activity can produce at a positive level without using a nonzero amount of at least one input, we are assured that (32) will be bounded from below. If (32) is expanded by the vectors ε_F, ε_I, and ε_P as in (34) below then (34) will always have a feasible solution (e.g., $y_F=\varepsilon_F$, $y_P=\varepsilon_P=\eta$). The fact that (34) is bounded and will always have a feasible solution implies that it will always have an optimal solution. Furthermore since each element of the vector, M, is a so-called "preemptive priority factor"[10] the optimal solution will have all ε_j values equal to zero if there exists such a feasible solution. Such a solution with all $\varepsilon_j=0$ will be feasible and optimal for (32) as well. Problem (34) is written as follows:

$$\min \; -e'\bar{y}_F+M_F'\varepsilon_F+M_I'\varepsilon_I+M_P'\varepsilon_P$$

$$\begin{array}{llllll} \text{Subject to} & A_Fx-y_F & & +\varepsilon_P & & = 0 \\ & A_Ix & & & +\varepsilon_I & = 0 \\ & A_Px & -y_P & & +\varepsilon_P & = 0 \\ & & y_P & & & \geq \eta \\ & y_F & -\bar{y}_F & & & = \hat{y}_F \\ & & x,\ \bar{y}_F, \varepsilon \geq 0. & & & \end{array} \qquad (34)$$

[10]See Ijiri, (1965).

Problem (34) is very useful for computational purposes. First, if and only if the value of the optimal solution to (34) with $\hat{y}_F=0$ is strictly negative then a nonzero efficient point exists and equals the optimal value for $\bar{y}_F$.[11] Second, if the optimal solution for (34) with $\hat{y}_F \geq 0$ has $\varepsilon=0$, then y_F^0 is efficient where y_F^0 is the sum of $\hat{y}_F$ and the optimal value for $\bar{y}_F$. When no feasible solution to (34) exists such that $\varepsilon=0$ then the optimal values for ε immediately indicate minimum changes necessary to obtain a feasible solution.

Problem (34) can also be easily interpreted as a type of goal programming. If the vector of final commodities $\hat{y}_F$ is considered to be a goal to be obtained then the optimal solution to (34) either indicates a bundle of activities which will satisfy that goal or indicates a bundle of activities which will come "closest," in a certain sense, to satisfying that goal. If $\varepsilon=0$ the goal can be satisfied; if $\varepsilon \geq 0$ then the optimal values for ε indicate the minimum infeasibility possible. Specific measures for "closeness" and minimum infeasibility are discussed below.

Goal Programming Models

The discussion immediately following will be directly related to the university decision-making model. We will consider a number of possible variations for the goal

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[11]See (Charnes and Cooper, 1961, p. 317).

programming approach and point out the close relation between these and other types of analysis. The use of goal programming and efficient point computations for a linear activity analysis model of a university will then be discussed. Finally a numerical example with "reasonable" coefficients for a university department will be outlined and corresponding computational results will be given.

It is possible to set different types of goals and to weight or place a priority ordering on the goals. There is a very close relationship between the different types of goals and the concepts of fixed and variable targets formulated by Tinbergen for quantitative economic planning. The priority ordering can be used in cases where a lexicographical preference ordering exists or where decisions are to be made in a decentralized but hierarchical fashion. There is also a close relationship between the goal programming approach and the building of consistency models or models which in some sense minimize inconsistency or infeasibility. The idea of minimizing, in some sense, "organizational slack" will also fit into the general goal programming approach. Finally with respect to the concept of minimizing inconsistency or maximizing slack, different specific measures of the amount of deviation may and must be chosen. We will elaborate on each of these topics below. Most of the goal programming formulations can be found in the book by Ijiri (1965).

We will, in general, identify three different types of goals. First, the goal of attaining consistency (i.e., satisfying as nearly as possible certain equality or inequality relations). Second, the goal of obtaining the largest value possible for given outputs or the smallest for inputs. Third, the goal of obtaining as nearly as possible a given value for certain commodities. Actually the difference between the first and third types is a matter of interpretation rather than of mathematical formulation. The correspondence between the concepts used here and those used by Tinbergen (1955; 1956) is very close.[12] For example, the second and third type of goal mentioned above correspond very closely to the concepts of flexible and fixed targets (Tinbergen, 1956, p. 8). It should further be noted that the vector, x, of activity levels is very similar to the instrument variables in a policy model. Tinbergen's concepts of consistency, boundary conditions, and side conditions (Tinbergen, 1955, p. 15) correspond to the first type of goal above. While Tinbergen's theory of economic policy is primarily directed toward quite aggregative models at the national level, there is no reason why a similar approach cannot be taken for many public institutions facing economic decisions. The following quote from Tinbergen is given in support of this view: "In a narrower sense we may restrict the meaning

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[12] (Contini, 1968, p. 576).

of the term 'economic policy' to the behavior of organized groups, such as trade unions, agricultural or industrial organizations, etc." (Tinbergen, 1955, p. 1). Most of our discussion is in the terminology of goal programming because of the somewhat more general mathematical formulation and the more direct relation to computational methods such as linear programming; however, as indicated the approach corresponds very closely to that of an important group of economists interested in quantitative economic policy.

Examples of the three types of goals outlined above can be given in relation to problem (34). First of all, the goal of minimizing $M'\varepsilon$, for $\varepsilon \geq 0$, is easily seen to be a goal of attaining consistency. If ε can be forced to zero then the relations, $A_F = y_F$, $A_P \geq \eta$, $A_I x = 0$, hold, and the relations between production activities, commodity vectors, and the availability of primary commodities are all consistent. The goal of minimizing $e_F' \bar{y}_F$ corresponds to the second type of goal (i.e., a variable target). The third type of goal would be represented in (34) by addition of a row constraint, $y_{F1} + \gamma^+ - \gamma^- = K$ with $\gamma^+, \gamma^- \geq 0$ and $\gamma^+ \cdot \gamma^- = 0$ and augmenting the objective function by:

$$\min(\gamma^+ + \gamma^-) \ .$$

This corresponds to finding a value for y_{F1} which corresponds as nearly as possible to the constant, K, (i.e., a fixed target value, K, for the variable y_{F1}).

It is very simple to include a weighting system for goals, with larger weights for more important goals and vice-versa. Of special interest is the possibility of using a "preemptive priority"[13] weight such as the M vector in problem (34). The use of such a priority weight is mathematically equivalent to the linear programming computational 'phase I' approach used for driving artificial vectors from the basis to obtain an initial feasible basis if possible. In this case we can give the procedure a meaningful interpretation. The procedure provides a method for solving decision problems when the policy-maker has a lexicographic preference ordering such that certain goals are of such overriding importance that it is preferred that they be fulfilled as far as possible before others are even considered. This is quite important because, while the lexicographic ordering is a very plausible type of preference ordering, it is a well known example of a complete preference ordering which cannot be represented by a continuous utility function.[14] Tinbergen (1956, p. 59) has used this type of priority ordering in his distinction between conditional and unconditional targets where unconditional targets are those

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[13]See (Ijiri, 1965, p. 46).

[14]See (Koopmans, 1957, p. 19).

which should be fulfilled with the highest priority. Also of interest for our purposes is the possibility of using preemptive goals as a device for effective decentralization in an economic system with a hierarchical decision structure. For example, the highest level decision-makers' goals must be fulfilled, as nearly as possible, and only then are the goals of lower level decision-makers considered. We essentially follow Charnes, Clower, and Kortanek (1967) in this approach and will consider it in more detail below with respect to the university decision-making model.

The first type of goal indicated above was that of obtaining consistency. It is not uncommon to find a sharp distinction being made between consistency and optimization models. The comment that it is so difficult to build a reliable consistency model for a given economic system that it is not practical to attempt optimization is not uncommon. Consistency between interrelated variables is very important for a useful planning model and when, as is usually the case, a certain amount of uncertainty is involved in the parameters of the model, then consistency is not easily attained. We suggest that the problem should still be considered an optimization problem with a goal of the first type (i.e., a goal of minimizing inconsistency or of maximizing redundancy so that reliability may be increased in cases where uncertainty is involved). References are common mentioning goals for organizations or firms of maintaining excess or redundant resources in the face of uncertainty or of the need for "organizational slack" (Cyert and March, 1963, p. 36). Contini (1968) has outlined a stochastic approach to goal programming which can be applied when it is known that the goals are related to the decision variables through a system involving normally distributed random errors. The object is to choose the decision or instrument variables such that the probability of the resulting vector of goals or targets lying in a specified region is maximized. The problem can be written as a quadratic programming problem so computation is possible.

The question of how to measure the degree of inconsistency or amount of deviation from goals is not immediately obvious nor is the answer unique. In practice one of two norms is usually chosen, so that the problem is that of minimizing some function of absolute or squared deviations. Both norms are closely related to statistical estimation procedures -- the first to the Chebyshev approximation, the second to least squares estimation. The norm chosen will be important with respect to the computational methods which can be applied. When absolute deviations are chosen the problem may be computed by linear programming methods while use of squared deviations will, if Lagrange multiplier techniques cannot be easily applied, require either the use of quadratic programming or the use of generalized inverses (Ijiri, 1965, p. 30).

It is of interest for us to examine more closely the use of linear programming methods to solve problems when absolute deviations are to be considered. Specifically we will compare Zukhovitskiy and Avdeyeva's (1966, Chapter 5) Chebyshev approximation problem for an inconsistent system of linear equations and the goal programming approach of Charnes and Cooper (1961, p. 215), and Ijiri (1965). Two differences may immediately be mentioned: First, while the criterion of minimizing a weighted sum of the absolute deviations is considered in both cases, only the former authors consider the actual Chebyshev problem of minimizing the maximum absolute deviation or inconsistency. Secondly, in both cases the problem of minimizing the weighted sum of absolute deviations is shown to be equivalent to a linear program but the former authors augment the system of equations by defining one new variable per equation and replacing each equation by two inequalities. The latter authors define two new variables per equation and add the side condition that if one of the pair of variables is positive then the other must be zero. The following examples should help to clarify the close relation of the two approaches as well as the minor differences.

Take for example the following set of equations and inequalities with $\hat{y}_F$ and η vectors of constants:

$$\begin{aligned} A_F x &= \hat{y}_F \\ A_P x &\geq \eta \\ x &\geq 0 \;. \end{aligned} \tag{35}$$

Suppose that no value for x exists such that the system can be satisfied but it is desired to find a solution such that the sum of absolute deviation of $A_F x$ from $\hat{y}_F$ is a minimum. Then the usual approach in goal programming would be to obtain the desired answer by solving the following problem:

$$\min \; e_F'\varepsilon^{+} + e_F'\varepsilon^{-}$$

$$\begin{aligned} \text{subject to } A_F x + I\varepsilon^{+} - I\varepsilon^{-} &= \hat{y}_F \\ A_P x \qquad\qquad &\geq \eta \\ \varepsilon^{+'}\varepsilon &= 0 \\ x, \varepsilon^{+}, \varepsilon^{-} &\geq 0 \;. \end{aligned}$$

Strictly speaking, this is not a linear programming problem because of the nonlinear side condition $\varepsilon^{+'}\varepsilon^{-} = 0$; however, if the remainder of the problem is solved by a usual simplex routine which always maintains a basic solution then the nonlinear condition will be maintained also. This is so because the vector involving ε_i^{+} is a linear combination of the vector involving ε_i^{-} and thus both vectors cannot be in the same basis. The optimal value x^0 would minimize the sum of absolute deviations of $y_F - \hat{y}_F$ where $y_F = A_F x^0$ and the difference between the vectors ε^{-} and ε^{+} would equal the deviations (i.e.,

$y_F - \hat{y}_F = \varepsilon^- - \varepsilon^+$).

Zukhovitskiy and Avdeyeva (1966, Chapter 5) would use the following linear program to obtain the desired solution:

$$\begin{aligned} & \min\ e'\varepsilon^+ \\ \text{subject to}\quad & A_F x + I\varepsilon^+ \geq \hat{y}_F \\ & -A_F x + I\varepsilon^+ \geq -\hat{y}_F \\ & A_P x \geq \eta \\ & x,\ \varepsilon^+ \geq 0\ . \end{aligned} \tag{36}$$

Thus while (35) adds two columns per equation, (36) adds one variable and row per equation but does not require the non-linear condition $\varepsilon^{+'}\varepsilon^- = 0$. In this case ε^+ would give the absolute deviation, $y_F - \hat{y}_F$.

The problem of minimizing the maximum absolute deviation can be obtained from the linear program (37) which is very similar to (36) but smaller in that only a single variable, ε_1^+, need be added rather than a vector of new variables (Zukhovitskiy and Avdeyeva, 1966):

$$\begin{aligned} & \min\ \varepsilon_1^+ \\ \text{subject to}\quad & A_F x + \varepsilon^+ \geq \hat{y}_F, \\ & -A_F x + \varepsilon^+ \geq -\hat{y}_F \\ & A_P x \geq \eta \\ & x,\ \varepsilon_1^+ \geq 0 \end{aligned} \tag{37}$$

where ε^+ is a vector with the same variable, ε_1^+, for every element.

The optimal value for ε_1^+ will give the maximum absolute deviation.

Thus if a vector of desired final commodities, $\hat{y}_F$, which was not attainable were proposed by a policy-maker then any of the last three linear programming problems outlined could be used to find the vector of activities which would come "closest" in a certain sense to producing the desired commodities. The solution would also show the deviations from the desired values.

Computation of Efficient Output Vectors

One procedure for using a model of the type which will be outlined in Chapter V to aid in university decision-making would be to compute all efficient points for the model and then present these to the decision-maker as alternative choices. For a linear activity analysis model the set of efficient points can be represented as convex combinations of a finite number of efficient extreme points. Charnes and Cooper (1961, p. 308) have outlined a computational method for computing all efficient extreme points. An attempt was made to apply this method to the numerical model in Chapter V but was abandoned in favor of a procedure less ambitious computationally but hopefully more promising for the purpose here.

There were two reasons for abandoning the attempt to compute all efficient extreme points. First, even for the small aggregated type of illustrative model in Chapter V it soon became obvious that the number of efficient extreme points would be quite large. Over 80 different efficient extreme points were actually computed for this small model. A large number of efficient extreme points would make it difficult to present the set of efficient points in a manner easily comprehensible to the decision-maker. Furthermore, since only a small subset of the efficient points are likely to be actively considered as alternative choices, the computation of all those efficient extreme points which do not receive active consideration represents wasted computational expense.

The second reason that the attempt to compute all efficient points was abandoned involved difficulties with the method itself[15] and lack of an adequate computer routine.

The alternative procedure is to compute all efficient points adjacent to a particular efficient extreme point which is "close" to the desires of the decision-maker and which has been previously obtained by a goal programming problem. The mathematical procedure for computing all efficient extreme points adjacent to a given point will be outlined below; the rationale for such a procedure will be discussed in Chapter V, along with numerical computations indicating how the procedure would be initiated.

The basic model used for computing efficient points is

[15]See (Gruver, 1970).

structured as the one in Charnes and Cooper (1961, p. 309):

$$
\begin{array}{lll}
 & \max\ c_y' y_F & \\
\text{subject to} & A_F x - I y_F = 0 & \quad (38) \\
 & A_I x \qquad\quad = 0 & \\
 & -A_P x \qquad \leq -\eta_P & \\
 & x \geq 0,\ c_y' > 0\ . &
\end{array}
$$

The vector, $y_F^{\,0}$, is an efficient point if and only if there exists a positive vector, $c_y^{\,0}$, such that the vector, $y_F^{\,0}$, is the optimal solution of (38). Suppose that it has been determined that $y_F^{\,0}$ is efficient and it is desired to compute all efficient points adjacent to $y_F^{\,0}$. Let A be the matrix for (38) including slack vectors; let, B^0,[16] be the matrix of those vectors forming the optimal basis for, $y_F^{\,0}$, and the associated x^0. Let the vector $c' = [c_x' | c_y' | c_s']$ such that c_x, c_y, and c_s correspond respectively to the x, y_F, and slack columns of A. Note that $c_x=0$ and $c_s=0$. Adjacent extreme points can be obtained by exchanging a nonbasic vector, a_k, for basic vector, a_L, in the optimal tableau. However, the adjacent extreme point will be efficient if and only if there exists a $c_y^{\,1}>0$ such that the $y_F^{\,1}$ resulting from the new basis would be an optimal solution to (38). More specifically, there must exist a vector $c_y^{\,+}>0$ such that all the

[16]To simplify the initial discussion assume that the basis matrix, B^0, corresponding to $y_F^{\,0}$ is unique. This assumption will be relaxed below.

$z_j^+ - c_j^+$ values for nonbasic columns remain non-negative except for the $z_j^+ - c_j^+$ value of that vector which is to enter the basis; it must be zero. It is not always simple to determine whether or not such a vector $c_y^+ > 0$ exists for a particular nonbasic vector; that is, to determine whether the adjacent extreme point produced by bringing that vector into the basis will in fact be efficient. Summarizing the above statements more compactly we can say:

Let y_F^0, x^0, be an optimal solution to (38) with B^0 indicating the matrix of basis vectors in the optimal solution and where $c_y = c_y^0 > 0$ and $c_x = 0$; $c_s = 0$. Let y_F^1 be obtained by bringing into the basis, B^0, the vector, a_k, and choosing in the usual manner the vector, a_L, to be dropped. Then y_F^1 will be adjacent to y_F^0 and will be efficient if and only if there exists a price vector $c^+ = c^0 + \phi_c f$ such that $c_y^+ > 0$, $c_x^+ = 0$, $c_s^+ = 0$, and the scalar

$$\phi_c = -\frac{z_k^0 - c_k^0}{f_B B^{0-1} a_k - f_k} = \min_{j \varepsilon J} -\frac{z_j^0 - c_j^0}{f_B B^{0-1} a_j - f_j},$$

where

$$j \varepsilon J \quad \text{if} \quad f_B B^{0-1} a_j - f_j < 0$$

and where the vector $f = [0 | f_y | 0]$ with the subvector f_y specifying the relative changes in c_y. f_B is the vector of elements from f corresponding to basis vectors; f_j is the jth element of f while a_j is the jth column of A .

Proof: $y_F^{\,0}$ and $y_F^{\,1}$ are adjacent extreme points because when a new basic feasible solution is obtained by changing one vector according to the usual rule for picking the vector to leave then this is equivalent to moving along an edge of the convex set from the extreme point corresponding to the original basis to an adjacent extreme point corresponding to the new basis (Hadley, 1962, p. 165).

First we will show that existence of a price vector c^+ implies that $y_F^{\,1}$ is efficient. We will show what the $z_j^{\,+}-c_j^{\,+}$ values would be for basis B^0 and vector c^+, (Hadley, 1962, p. 380):

$$z_j^{\,+}-c_j^{\,+}=(c_B^{\,0}+\phi_c f_B)B^{0-1}a_j-c_j^{\,0}-\phi_c f_j=z_j^{\,0}-c_j^{\,0}+\phi_c(f_B B^{0-1}a_j-f_j)$$

where $c_B^{\,0}$ is the vector of c_j's corresponding to basis vectors. Substituting in for ϕ_c we get

$$z_j^{\,+}-c_j^{\,+}=z_j^{\,0}-c_j^{\,0}-\frac{z_k^{\,0}-c_k^{\,0}}{f_B B^{0-1}a_k-f_k}(f_B B^{0-1}a_j-f_j) \begin{cases} = 0, & j = k \\ \geq 0, & j\neq k\ . \end{cases}$$

Next we will note how the $z_j^{\,+}-c_j^{\,+}$ values change when the vector, a_k, is brought into the basis, (Hadley, 1962, p. 110):

$$\hat{z}_j^{\,+}-c_j^{\,+}=z_j^{\,+}-c_j^{\,+}-\frac{x_{Lj}}{x_{Lk}}(z_k^{\,+}-c_k^{\,+}) = z_j^{\,+}-c_j^{\,+},\ \text{all } j$$

since

$$z_k^{\,+}-c_k^{\,+} = 0\ .$$

The scalar x_{Lj} and x_{Lk} are elements of the existing tableau.

Thus all of the nonbasic vectors have $\hat{z}_j^+ - c_j^+ \geq 0$; the new basis is optimal for $c=c^+$ and therefore y_F^1 is efficient.

Now we will show that if y_F^1 is efficient then there exists a c^+. If y_F^1 is efficient then there exists a vector c^1 with subvectors, $c_y^1 > 0$, $c_x^1 = 0$, and $c_s^1 = 0$, such that y_F^1 is an optimal solution to (38). Suppose $c^+ = \lambda c^1 + (1-\lambda) c^0$, $0 \leq \lambda \leq 1$. From the optimality conditions we have

$$c_B^0 B^{0-1} a_j - c_j^0 \geq 0, \text{ all } j$$

$$c_B^1 B^{0-1} a_j - c_j^1 \geq 0, \ j \neq k, \text{ and}$$

$$c_B^1 B^{0-1} a_k - c_k^1 \leq 0.$$

Let

$$z_k^+ - c_k^+ = c_B^+ B^{0-1} a_k - c_k^+ = (\lambda c_B^0 + (1-\lambda) c_B^1) B^{0-1} a_k$$
$$-(\lambda c^0 + (1-\lambda) c^1),$$

then for $\lambda=0$, $z_k^+ - c_k^+ \leq 0$,

and for $\lambda=1$, $z_k^+ - c_k^+ \geq 0$.

Since $z_k^+ - c_k^+$ can be written as a linear function of λ, the above inequalities imply that there exists $0 \leq \lambda^+ \leq 1$ such that $z_k^+ - c_k^+ = 0$. By a similar argument $z_j^+ - c_j^+ \geq 0$ for $0 \leq \lambda \leq 1$, and $j \neq k$. Then $c^+ = \lambda^+ c^0 + (1-\lambda^+) c^1$ is the c^+ vector required. The elements of c_y^+ are strictly positive since λ is non-negative and c_y^0 and c_y^1 are strictly positive; similarly $c_x^+=0$ and $c_s^+=0$. If f_y is set equal to $c_y^+-c_y^0$, then ϕ_c will equal 1 at the minimum over $j \varepsilon J$ since

$$\phi_c = - \frac{z_k^0-c_k^0}{(c_B^+-c_B^0)B^{0-1}a_k-(c_k^+-c_k^0)} = \frac{-(z_k^0-c_k^0)}{-(z_k^0-c_k^0)} = 1$$

and for $j \neq k$, $z_j^+-c_j^+ \geq 0$, thus:

$$- \frac{z_j^0-c_j^0}{(c_B^+-c_B^0)B^{0-1}a_j-(c_j^+-c_j)} = \frac{-(z_j^0-c_j^0)}{-(z_j^0-c_j^0)+(z_j^+-c_j^+)} \geq 1= \phi_c.$$

Therefore if y_F^1 is efficient then there does exist a c^+ defined as above.

From the above proof it is clear that if the required c^+ can be obtained then the new adjacent extreme point will be efficient. We will present three methods for finding the required c^+.

Case 1. If the nonbasic vector, a_k, which is to enter is a vector corresponding to a y_F variable then the required c^+ is easily obtained by letting $f_k=z_k^0-c_k^0$ and $f_j=0$ for $j \neq k$. Then ϕ_c becomes:

$$\phi_c= - \frac{z_k^0-c_k^0}{0B^{0-1}a_k-(z_k^0-c_k^0)} = 1, \text{ since } f_B B^{0-1}a_j-f_j \geq 0,\ j \neq k$$

and

$$c_j^{+}=c_j^{0}+(1)f_j = c_j^{0},\ j \neq k$$
$$= z_k^{0},\ j = k.$$

Case 2. If all the vectors corresponding to y_F variables are in the basis then it may be possible to find a c^+ which will bring in a nonbasic vector a_k by analyzing the effect of varying one of the c_y^0 elements while holding the others unchanged. That is, set $f_{j*}=1$, where c_{j*} corresponds to a y_F variable in the basis, B^0, and set $f_j=0$, $j \neq j^*$. Then finding ϕ_c and which vector will enter simplifies to computing the minimum of a series of quotients of values already available in the tableau. Since $f_j=0$, for $j \neq j^*$:

$$\phi_c = -\frac{z_k^{0}}{x_{j*k}} = \min_{j\varepsilon J} -\frac{z_j^{0}}{x_{j*j}}\ ,\ \text{where } j\varepsilon J \ \text{ if } \ x_{j*j} < 0,$$

$x_{j*j} = B_{j*}^{0-1} a_j$ and where B_{j*}^{0-1} is the row of B^{0-1} corresponding to $j*$.

Then $c^+=c^0+\phi_c f$ will bring a_k into the new basis. By letting $f_{j*}=-1$ and following the same procedure it is possible to determine which a_j will be first to come into the basis when c_{j*}^{0} is decreased rather than increased.

After exhausting the information provided by the above two cases there will be a certain set of the nonbasic vectors for which a c^+ has been found implying that the entrance of that vector into the existing basis will lead to an efficient extreme point. There will usually be a number of nonbasic vectors, however, for which a c^+ has not yet been found. In general, of course, the required c^+ will not exist for all nonbasic vectors, but if all efficient extreme points adjacent to y_F^0 are to be computed then it is imperative that we determine for each and every nonbasic vector whether or not such a c^+ does exist. The procedure in case 3 provides the needed information. It is much more involved computationally but only need be applied to nonbasic vectors for which a c^+ has not been obtained by the procedures in case 1 or case 2.

Case 3. To determine whether a c^+ exists for a specific nonbasic vector, a_k, we can solve the following linear programming problem:

$$\begin{array}{ll} & \min\limits_{c} \; c_B'(B^{0-1}a_k)-c_k \\ \text{subject to} & c_B'(B^{0-1}a_j)-c_j \geq 0, \text{ all } j \text{ including } k \\ & c_y'I \qquad\qquad\quad \geq 1 \\ & c_x=0,\ c_s=0\ . \end{array}$$

If the objective function equals zero at the optimum then the optimal c is the required c^+ vector and thus bringing in the vector a_k will lead to an efficient adjacent extreme point. If the objective function is strictly positive at the optimum, then c^+ does not exist and bringing in the vector a_k will lead to an adjacent point which will not be efficient.

It can easily be shown that when the objective function is zero the optimal c satisfies the conditions for c^+ by letting $\phi_c=1$ and $f=c-c^0$. Then

$$\phi_c = -\frac{z_k^0 - c_k^0}{(c_B - c_B^0)B^{0-1}a_k - (c_k - c_k^0)} = \frac{-(z_k^0 - c_k^0)}{-(z_k^0 - c_k^0) + c_B B^{0-1} a_k - c_k} = 1$$

and 1 is the minimum over $j \varepsilon J$ since for $j \neq k$

$$\frac{-(z_j^0 - c_j^0)}{-(z_j^0 - c_j^0) + c_B B^{0-1} a_j - c_j} \geq 1.$$

A separate optimal solution must be computed each time an a_j vector is checked for the existence of a c^+; however, only the objective function varies from one problem to the next. The original c^0 provides an initial feasible solution and each successive optimal c will be feasible so that iterations for new objective function values can always begin from the existing basis. The linear program will have a lower bound for each c_y element plus as many rows as there were nonbasic vectors in the original problem. Any rows which are strictly non-negative can be dropped.

When the assumption that the basis matrix corresponding to y_F^0 is unique is relaxed, then the process becomes somewhat more involved. Nonuniqueness of the basis may be recognized in two ways. First, if there exists a $z_j - c_j$

value equal to zero for nonbasic vectors, then bringing in that vector will result in the usual alternative optimal solution. If this alternative optimal solution changes the y_F vector from $y_F{}^0$ then an adjacent efficient extreme point has been reached. However, if the new y_F vector is not changed from $y_F{}^0$ and only x_j variables change then a new representation of the same efficient vector, $y_F{}^0$, has been obtained and a new matrix of basis vectors corresponding to $y_F{}^0$ has been obtained.

A second way of recognizing nonuniqueness of the basis matrix corresponding to $y_F{}^0$ arises when degeneracy exists, since even when $z_j - c_j > 0$ it may be possible to bring the vector into the basis at a zero level and obtain a different representation and different basis matrix corresponding to $y_F{}^0$. It will be possible to bring in the vector a_k at the zero level if the ith element of the vector $B^{0-1}a_k$ is not equal to zero where i corresponds to a basic variable equal to zero for the basis B^0. The above cases are closely related to the discussion of determining all optimal solutions in Hadley (1962, p. 166-167).

It is possible for more than one basis matrix to correspond to a particular vector of final outputs, $y_F{}^0$. Therefore, to find all efficient vectors adjacent to $y_F{}^0$ it is necessary to take each nonbasic vector corresponding to each alternative basis and check for the existence of a c^+

satisfying the definition given above. Only when all nonbasic vectors corresponding to each alternative basis have been checked can we be certain that we have located all adjacent efficient extreme points. It is obvious that the existence of alternative basis matrices rapidly increases the necessary computations.

We can summarize the procedure outlined above as follows. First, pick a desired vector of final outputs. Second, run a goal program to obtain the efficient (and therefore feasible) point "closest" to the desired values. Then compute all efficient points adjacent to the efficient point y_F^0 obtained.

To obtain all efficient adjacent points follow the three steps below.

Step 1. Find for each nonbasic vector if the required c^+ exists. Use the procedures of case 1 and case 2 given above where possible. Use the case 3 procedure for all other nonbasic vectors.

Step 2. Compute the new solutions using the c^+ vectors obtained in step 1. If y_F^0 changes, an adjacent efficient extreme y_F has been obtained. If y_F^0 does not change, an alternative basis matrix has been obtained and step 1 must be completed for this alternative basis.

Step 3. Check each alternative basis to see if, due to degeneracy, it is possible to find new alternative bases. If so, then step 1 must be completed for each of these alternative bases.

When it is not possible to find any new alternative basis matrices and when all nonbasic vectors for each existing alternate basis matrix has been checked for the existence of a c^{+} then all adjacent efficient extreme points have been obtained.

CHAPTER V. AN APPROACH TO EFFICIENT DECISION-MAKING IN A UNIVERSITY

Education in general and specifically the graduates and knowledge produced by the university have become increasingly important elements of our social and economic system. The increasing proportion of the population and of the public budget directly committed to the educational process has made imperative the need for a close analysis of the efficiency of the educational process and of its present and future effect on the remainder of the economic system.

Actually the amount of work which has been done in the last few years on the general subject of the economics of education has been quite significant. Blaug (1968, p. 8) has delineated two general classes into which this work may be placed. The first "analyses the economic value of education" and "is concerned with the impact of schooling on labor productivity, occupational mobility, and the distribution of income." The second "analyses the economic aspects of educational systems" and "deals with the internal efficiency of schools and with the relations between the costs of education and methods of financing these costs." Blaug's (1968) book gives important examples of papers falling in the first class. The subject matter of this chapter falls in the second class and is specifically concerned with developing analytical techniques which could aid decision-makers in increasing the internal efficiency of their university. People from a number of different disciplines such as management science, operations research, systems analysis, and economics have worked on the problem or closely related problems (Keeney, Koenig, and Zemach, 1967; Judy, 1969; Weathersby, 1967; Sengupta and Fox, 1970). The approach taken in this chapter is very much in the tradition of other activity analysis type models for university planning investigated by Fox, McCamley, and Plessner (1967) and Plessner, Fox, and Sanyal (1968). The model considered here will be an attempt to generalize the types of objectives or goals to be optimized and to discuss the possibility of a meaningful link of the university with the remainder of the economy through the input vector. While the discussion will be only with respect to a university system we believe that the general approach could be used to aid decision-making in many nonmarket institutions.

We will first identify the environment in which a university operates as well as the structural characteristics and objectives of the university itself which are important for our analysis. Once we have identified these characteristics we will formulate a model which closely approximates these characteristics and a methodology of using the model to make efficient decisions within the university. We will formulate a department level numerical model and present computational results. Finally we will discuss inadequacies of the model and the possibility of extending the model so that stochastic and dynamic elements could be considered.

The University Environment and Structure

A public university is constrained by or linked to the larger economic-political system in a number of important ways. Most of the links found between a multiproduct firm and the larger economic system have counterparts in the links between a university and the larger economic system. The public university, however, has other important links which affect its objectives and feasible actions. If not given precise directives, the public university is at least constrained by the political authority (e.g., the state legislature) in matters such as the setting of tuition levels, the introduction of new programs or the cancellation of existing ones, the addition of new faculty or physical plant, the size of the operating budget, and even in setting the number of students to be admitted. Thus decisions related to pricing policy, product mix, capital expenditure, and level of operation are much more constrained for the university decision-maker than for his counterpart in the multiproduct firm. The political authorities can be considered as higher level decision-makers and one purpose of a university decision model should be to aid decision-making at this level.

Outputs of the University

Even more important than the constraints imposed by the political system on feasible actions, however, is the difficulty of evaluating the outputs of a university. As will be discussed below, the inputs of a university can be measured by market values, but the situation is apparently not so simple in the case of outputs since a well defined market does not exist. For the multiproduct firm a system of prices is available for evaluating the output link between the firm and the remainder of the economy. Such a set of prices is not directly available for university outputs. Thus the vector maximization problem, which was the subject of the last chapter, must be faced. Either a suitable set of prices or relative weights must be estimated, or an approach must be followed which does not require that explicit relative weights be obtained.

In a very broad or aggregative sense we can view the outputs of the university as falling into one of two classes. The production of "educated" persons or of skilled or trained manpower is one output class. The production of new knowledge resulting from the increasingly important research activities undertaken in the university is the second output class. These two outputs, educated persons and new knowledge, are used here in a broad sense and include such often mentioned functions of the university as extension work and service to the community.

With respect to the general output of educated persons we believe that it is possible, with some reservations, to define meaningful measures of output and to estimate a system of prices or relative weights which, at least for some purposes, are useful. With respect to the output of new knowledge we believe that the problem of measurement and relative valuation is inherently more difficult.

The university maintains an extensive record system of course units, credit hours, and grades for each student for the purpose of measuring the type, quantity, and quality of "education" obtained by each student. While it is not difficult to criticize this system of measurement, it is easily argued that an output index using the system is as reasonable as many other output indexes of quite heterogeneous products which are found to be useful.

A significant amount of work has been done with respect to estimating the value of educated manpower. This work falls into Blaug's class which is concerned with analyzing the economic value of education. The results of this work provide the university decision-maker with a partial answer to the problem of evaluating the effect of the output of educated manpower on the larger system. Two different approaches have been followed and we will discuss examples of each as well as the usefulness of the results for a university decision-maker. There does, of course, exist a market for the services of educated manpower. The two different approaches will be discussed with respect to their different implicit assumptions about the shape of the demand curve in this market. We will refer to the first as the rate-of-return approach and the second as the manpower planning approach (for a related discussion see Anderson and Bowman, 1968).

The general idea of the rate-of-return approach is to estimate the increase in lifetime earnings (which is assumed to be closely related to productivity) due to education and express these earnings as a rate-of-return on investment in education (Schultz, 1968). Such calculations can be made for national average data or much more specific situations such as for graduates from a particular university in a particular discipline (Craft and Kaldor, 1968). The concept can either be used to determine a social rate-of-return or a rate-of-return to the particular graduate by using either social costs and benefits (e.g., public subsidies to education would be included in costs and before-tax earnings in benefits) or those costs and benefits relative to the individual. Rather than computing the rate-of-return or cost, it may be of interest to compute the discounted present value of increased benefits due to education. In any case, for the results to be meaningful, an accurate forecast of the salary structure over the relevant time period must be made. Usually it is assumed that the salary structure will remain virtually constant into the relevant future. To use such calculations for decision purposes one must assume that increases in educated manpower will not affect the structure of the salary system. In other words, the assumption of a nearly horizontal demand schedule must be implicitly made.

The manpower planning approach begins by forecasting the number of persons with specific types of training which will be "needed" in the economy over a certain time period. The assumption is made that if that number is not met there will be a shortage; if it is more than met there will be an excess of manpower. In other words the demand schedule is quite steep and the salary structure sticky.

The choice between the rate-of-return and manpower type approach could apparently be made on empirical grounds and is closely related to the amount of substitutability which exists between persons with different training. Arrow and Capron (1968) have discussed the operation of the market for scientists and engineers and have concluded that the market does react to allocate in the short run and modify the supply in the long run.

The university can view its role as that of directly supplying the demand for the services of educated manpower. The university is not a supplier for this market in the usual sense of the term since it does not receive payment directly for the services of its graduates. The flow of educated manpower from a particular university must be viewed as adding to the body of educated manpower (i.e., increasing the stock of human capital) which supplies services to a market which for most purposes must be considered at least national in scope. Thus the decisions of a single university will have, at most, a small effect on the market.

If the university decision-maker uses the results of rate-of-return and manpower planning studies and expands those programs corresponding to high rates-of-return or high manpower needs we would think that he would be contributing to an adjustment process. However, such decentralized action by all universities will not insure that the aggregate adjustment will not be either too large or small. In the one case an accurate feedback of information concerning adjustments in the rate-of-return would be required, in the second a centralized informational system which aggregated the plans of all relevant institutions.

The university can, alternatively, view its role as that of supplying instructional services directly to students who desire those services as a consumption good or as an investment in their own human capital. In practice the instructional services supplied by most universities are heavily subsidized with the price paid directly by students usually accounting for less than one-half the cost. A market does exist for instructional services but since these services are quite differentiated by factors such as geographical location and real and perceived differences in quality a rather wide range of tuition rates (i.e., prices) can and do exist. This leaves open the possibility of using tuition as an instrument variable, at least within a certain range. In general, tuition charges are not differentiated, within the university, according to course of study (even though the cost of instruction in, say, science is much more than in the humanities (see Weathersby, 1967). A certain amount of tuition differentiation does take place through the granting of scholarships and other forms of student aid. Jenny (1968) has noted that the granting of student aid amounts to a type of price discrimination. Some knowledge of the shape of the demand schedule for instructional services would appear to be necessary if tuition rates and student aid are to be used as instrument or control variables. The need for such information is especially important in most private institutions (e.g., see Jenny, p. 275, for a discussion concerning the shape of such a demand schedule for small private colleges).

As we indicated above we believe that the research work just discussed provides a partial answer to the problem of evaluating the effect on the larger economic system of the university's output of educated manpower. We believe that the problem is only partially answered for at least two important reasons. The first involves the consumption aspect of education; the second concerns the presence of externalities.

The university decision-maker can use the results of rate-of-return analysis to place relative values on different instructional activities or he may use the results of manpower analysis research to set fixed goals for certain outputs. These criteria, however, are related almost solely to the effect of education on human capital or productivity; the consumption component of education, which must be relatively significant, is not even considered. If students are free to purchase alternative instructional services, their consumption preferences as well as their choice of investment in human capital should be revealed. While the information about future salary levels necessary to make optimal investment decisions may not be available to students, there would not appear to be any major informational problem with respect to consumption choices. The importance of the consumption component then would appear to be an argument in favor of the university viewing its role as that of supplying instructional services directly to students. For at least two reasons, however, this university role is too narrow.

The first reason is that it is difficult for students to obtain good information with respect to forecasts of the future salary structure. The second involves the importance of externalities present in the educational process (Bowen, 1968, p. 85). A good case can be made for the importance of external effects both in the productive and consumption effects of education. The productivity of highly trained persons must be increased by the possibility of communicating ideas with others possessing similar or supporting training so that the productivity of the group is greater than the sum of individuals working separately. Observe that research personnel often work in teams. The level and type of education possessed by a given individual must enter the utility function of many other individuals as well. For example, in a democratic society a more informed electorate should enter positively in almost everyone's utility function.

As is well known the presence of externalities often interferes with the efficient operation of market mechanisms. The presence of externalities does not preclude the efficient operation as Arrow (1969, p. 57) has shown that by a "reinterpretation of the commodity space, externalities can be regarded as ordinary commodities, and all the formal theory of competitive equilibrium is valid, including its optimality." However, as Arrow points out, the presence of externalities is often associated with two other factors which adversely affect the functioning of the market. The first is the problem of appropriability or the problem of exclusion; the second is the problem of a small number of buyers and sellers.

It is difficult to imagine the possibility of excluding certain persons from the benefits due to the consumption of education by others. While exclusion may be more realistic with respect to productive externalities the number involved in the transaction is likely to be small and the cost of negotiating a price high. As Arrow (1969, p. 58) has noted: "If in addition the costs of bargaining are high, then it may be most efficient to offer the service free."

We can make the following conclusions concerning our discussion of evaluating the university's output of educated manpower. Use of a set of relative values from a rate-of-return type analysis or a set of output targets from a manpower planning analysis does not insure a correct aggregate adjustment if each university follows such a policy independently. Even more important such analysis disregards the importance of the consumption component of education. While the free choice of students in purchases of instruction should reveal consumption preferences accurately, lack of information probably precludes optimal investment in their human capital. Finally the difficulties due to the presence of externalities in education (Weisbrod, 1964) indicate that the market cannot be expected to operate optimally. Thus we feel that such a criterion for evaluating the output of educated manpower provides important guidelines but probably should not completely dominate the preferences of well informed responsible university policy-makers.

As was stated above, the measurement and valuation of new knowledge, the second broad class of university output, is apparently inherently more difficult than that of educated manpower. Research has become an increasingly important activity in the university and in the economy in general. Questions concerning the importance of research, the level at which research activities should be supported, and the method of choosing between alternative projects have received significant attention in the literature (Smith, 1965).

We will discuss two inherent properties of research and new knowledge which indicate that a market mechanism cannot be relied upon to allocate an optimal budget to research activities or to efficiently allocate the benefits of new knowledge. Most important is the fact that knowledge fits almost exactly the definition of a collective or public good (i.e., a good having the property that its use by one individual does not decrease the amount available for use by others). Leontief (1960, p. 4) has very explicitly noted this property of knowledge:

> Not only can the same person make use of an idea, of some specific piece of technical information, over and over again without the slightest danger of exhausting it through wear, but the same idea can serve many users simultaneously, and as the number of customers increases, no one need be getting less of it because the others are getting more.

It is well known that decentralized market mechanisms break down in the presence of public goods. For example, Samuelson (1954, p. 388) has indicated the impossibility of devising a decentralized pricing system which will optimally determine levels of collective consumption.

The second important fact is that research activities by their very nature involve a high degree of uncertainty. Arrow (1969, p. 54) has noted that uncertainty is closely related to the costliness of information necessary for participation in a market. Obtaining information of even a probabilistic nature with respect to the outcome of many research projects will be exceedingly costly. The costliness of information will inhibit the operation of a market mechanism.

The above comments are not meant to indicate that research must be undertaken blindly, but that attempting to institute or define a market mechanism for optimal budget allocation to research, or between research projects, does not appear to be a fruitful exercise. Because of the special nature of research, we feel that a preference ordering resulting from the interaction of policy-makers who, in some general sense reflect society's desires and of researchers with the best information concerning the probable results of various projects is likely to be a better allocation criterion than any type of market prices.

The production of masters theses and doctoral dissertations apparently has elements of both production of new knowledge and educating manpower. Added difficulties are involved in attempting to evaluate outputs in such cases of joint production.

We conclude that the output linkage between the university and the larger economic system in very difficult to evaluate suitably. It is possible to obtain rough estimates of the contribution to national income of students who graduate. It should be of interest to have a decision-model which could provide the solution which would maximize this contribution. However, such an objective is apparently much too narrow to be prescribed in general and a decision-model should be flexible enough to provide solutions for a broader range of objectives. No set of prices or weights is directly available nor apparently possible to estimate from market data, such that production of new knowledge and the consumption component of instruction as well as the productive contribution of graduates are satisfactorily commeasurable.

Thus the vector maximization problem of Chapter IV is present. A decision-model flexible enough to provide solutions which meet as closely as possible multiple goals set by the policy-maker or which can provide the policy-maker with a subset of efficient combinations from which to choose should be useful in pursuing a broad range of objectives.

Inputs of the University

Measurement and evaluation of input links between the university system and the larger economic system is much more straightforward than for output links. The university like a firm buys goods and services through the market system. It purchases new physical plant capacity, supplies, and labor services through the same markets as many multiproduct firms. In the market for faculty personnel the university competes with many private firms as well as other universities.

Student time used in the learning process must be considered an input for most purposes. Even it, however, can be treated as an opportunity cost measured by potential earnings in the labor market. T. W. Schultz (1968, p. 25) says that for the United States "well over half of the costs of higher education consists of income foregone by students."

For decision-making purposes the physical plant and the faculty (due to tenure and institutional hiring practices) must be considered relatively fixed in the short run and essentially involve questions of capacity expansion. The inputs of basic supplies, secretarial and computational services and to a large extent the input of student time are more short run decisions.

The Decentralized Structure of Decisions

The fact that certain university goals and constraints are set outside the university by higher level political authorities was discussed above. The decision structure within the university itself involves a large degree of decentralization. The typical university is composed of colleges which are themselves decomposed into departments corresponding to disciplines of study. This type of decision structure has not gone uncriticized; see, for example, Ackoff (1968), but much can be said in its defense and its prevalence in practice cannot be denied. At each level of the hierarchical structure is a decision-maker whose decisions are closely related to the type of knowledge he possesses. The president must possess general knowledge about the whole university, the dean about his school, and department head detailed and specific knowledge about his department. The decision-maker at each level has the responsibility of making "good" decisions given his knowledge and directives from higher level decision-makers.

This type of multilevel, multigoal structure has been outlined in Sengupta and Fox, (1970, p. 98) and in the following quotation from Plessner, Fox, and Sanyal (1968, p. 256):

> The university president is thus a multiple-goal decision maker. Between the president and the department chairman there may be a number of administrators, such as the vice-president for research, the dean of the graduate college, and the director of the extension service, each of whom contributes to only one of the president's goals. Their actions impose constraints on the department chairman whose decisions must take cognizance of several of the presidential goals.

The directives from higher level decision-makers may be in the form of quotas of inputs and outputs or may be a pricing system which indicates how specific inputs and outputs are to be valued. The type of process described above is very similar to the usual characterization of a decentralized multi-divisional firm.

Decentralization in the university, however, not only involves decisions concerning the best way to fulfill given goals from higher levels, it also involves the actual setting of goals at lower levels. Decisions about what should be taught in a given course and which courses should be offered (i.e., which ideas and bits of knowledge have the "highest value") usually are departmental level decisions. Decisions concerning the mix of courses which should be included in a given major area of study are made at the departmental or college level.

In a decentralized firm each output can usually be closely identified with the revenue it generates for the firm and thus it is relatively easy to collapse the overall objective into a scalar value in terms of money. While it may be possible to roughly estimate the present value of the contribution to national income of a given degree, such a value is not sufficiently disaggregated to aid in many of the important decisions at the departmental level. Decisions must be made involving the relative value of different courses, class sizes, or of staff members performing different tasks. Apparently the preference function of those with the most detailed and specific knowledge of the discipline and department must be relied upon. Thus there is sufficient reason to believe that decision-makers at the departmental level should be responsible for setting certain department goals as well as being responsible for actions of the department which will best fulfill goals or objectives of higher level decision-makers.

The organization of a university is such that more general or aggregative goals set at one level of decision-making must be fulfilled in the best way possible by lower level decision-makers and once this is accomplished all the remaining feasible actions are evaluated according to more specific goals set by the lower level decision-maker. This envisions a kind of lexicographical or preemptive decision structure.

Characteristics of a Useful University Decision Model

The environment and structure of the university requires a useful decision model to have two important characteristics. First, it must be flexible enough to accommodate a wide range of optimization and goal criteria. Second, it must be able to accommodate the decentralized nature of the university and specifically to allow for goal setting by lower level decision-makers.

We will discuss three related approaches for characterizing the objectives of decision-making. The first approach is that of obtaining a general preference function from the policy-maker and maximizing this function subject to the restrictions of the model. Nonlinear functions can be considered; however, unless the function is concave, a global optimum will be very difficult to obtain and except for the special case of quadratic programming, computational expenses will be prohibitive for large models. Tinbergen (1955, p. 2-3) in his discussion of quantitative economic policy has noted the difficulty of actually specifying such a function and that in practice specifying and maximizing such a function will often be passed over and targets will be directly chosen. Some important initial work on the problem of determining such a function has been done by van Eijk and Sandee (1959). If the function is linear then the problem reduces to choosing a set of relative weights which may not be too difficult a problem. However, the implicit assumption of constant marginal value for each commodity involved in the linear function will usually not result in a satisfactory answer unless bounds are placed on certain variables. The setting of such bounds actually amounts to setting a type of target after all.

The second general approach for characterizing objectives is that of attempting to fulfill as closely as possible a vector of goals set by the policy-maker. Lee and Clayton (1969) have discussed the application of goal programming to a university decision model. The goals may be classified according to type, as in Chapter III (i.e., the goal of minimizing inconsistency, the goal of minimizing deviations from fixed targets, and the goal of maximizing flexible target values). Strictly speaking the setting of such goals can be said to specify the policy-maker's preference function; however, we wish to make a distinction between the two for discussion purposes. The first approach is in the tradition of constrained value maximization; the goal approach may be viewed as being more concerned with meeting quantity quotas and insuring consistent functioning of the production system. The goals may be given different weights, some of which may be preemptive priority factors. We would assert that the general idea of goals corresponds more closely to the implicit thought processes of policy-makers than does the idea of a general preference function. In other words we feel that university policy-makers are more likely to be able to easily specify, say, certain quantity quotas which they think should optimally be fulfilled than to be able to specify for every conceivable combination the relative values they would place on commodities.

The third general approach for characterizing objectives in a model is to compute efficient points from which the policy-maker then chooses one. Even if one is not willing to put either absolute or relative values on the outputs of a university system there is still much which can be done to aid in the consideration of alternatives. If it is possible to reach agreement on the identification of outputs and agree that more is preferred to less then we know that only efficient, feasible alternatives need be considered. There may be a very large range of efficient, feasible alternatives which must be compared to each other and from which a unique choice must be made; however by having excluded all infeasible output combinations and all combinations which are dominated by other combinations with as much of every output the range has been greatly narrowed. The policy-maker is thus able to focus on a specific set of output vectors (efficient, feasible vectors) knowing that for any feasible alternatives not in this set there is at least one corresponding vector in the set which is more desirable.

The preference ordering of the policy-maker is effectively revealed when the unique choice is made from these alternative efficient vectors. This means that without encountering the difficult problems of empirically estimating a utility function to represent the preferences of the decision-maker or even determining a system of weights or goals, a solution which is optimal with respect to the decision-maker and the given output possibilities can be obtained. For a linear model the set of efficient points can be computed by systematic use of a linear programming simplex routine. It is necessary to compute only the set of adjacent efficient extreme points, and then the set of efficient points consists of convex combinations of adjacent efficient extreme points.

The difficulty for this third approach is that even for moderately sized models the number of adjacent efficient extreme points may become excessively large for purposes of computation. Even if computation poses no problem it must be possible to present the set of efficient alternatives in a compact enough manner so that choosing between them is not beyond the capability of the policy-maker. Even when the model is small and the commodities are aggregated so that the number of different final commodities considered is small the number of efficient extreme points may exceed a manageable number. In practice, high level decision-makers are usually forced to work with relatively aggregated indexes because of the costliness of gathering and processing highly detailed data at a central point, and for many specific situations it will not be difficult to specify a subregion which would contain all efficient points worthy of consideration. Formally specification of such a subregion could be considered a preemptive goal.

The approach used in the computations of this chapter is to use goal programming to obtain an initial efficient point worthy of consideration, and then compute adjacent efficient points and allow the policy-maker to decide if he wishes to move to any of the efficient points on the line segments between adjacent points. If he decides that he would prefer to move to one of the adjacent points then the process could be repeated until none of the points "around" the point he has chosen are preferred to the point he has chosen.

Of interest to an economist is the fact that each efficient vector is associated with imputed prices for not only the outputs but also the inputs and intermediate commodities. These prices are also computable and are interesting in two respects. First, they provide an intra-model pricing vector which is consistent with the chosen optimal output vector. Secondly, these imputed prices give a link between the educational institution and the remainder of the economy. Most inputs can be given a reasonable market price, and the imputed prices are only relative so they can be normalized such that one of the inputs serves as the numéraire (preferably the budget input measured in dollars). Comparing the market and imputed valuations may then be meaningful. Possible use of these imputed prices for short run adjustments and dynamic considerations will be discussed below.

Specification and Computation for a Department Level Model

The model specified in Table 9 will be used to give numerical examples of the three different approaches for characterizing university objectives. The coefficients are meant to be "reasonable" and the units of measurement and other relevant assumptions are given in Table 10. Discussion of the specification of the model will be divided into three parts concerned with the time period involved, the commodities, and the activities. We assume that some type of university-wide decision process, such as that described in the above section, has been completed and that the department has been allocated certain commodities and directed to pursue certain goals.

The model is essentially static and applies to a nine

Table 9a. Definitions of activity codes for Table 9b

Code	Instruction Activities
XSPI	Small principles section taught by instructor
XLPU	Large principles section taught by undergrad. faculty
XLPG	Large principles section taught by grad. faculty
XI1U	Freshman-Soph. class taught by undergrad. faculty
XI1G	Freshman-Soph. class taught by grad. faculty
XI2U	Jr.-Sr. class taught by undergrad. faculty
XI2G	Jr.-Sr. class taught by grad. faculty
XIM	M.S. course instruction
XID	Ph.D. course instruction
	Research, Dissertation, and Thesis Activities
XRES1	Research activity no. 1
XRES2	Research activity no. 2
XRES3	Research activity no. 3
XRES4	Research activity no. 4
XRES5	Research activity no. 5
XSM	M.S. thesis supervision
XSD	Ph.D. dissertation supervision
	Intermediate Commodity Activities
XTAC	Teaching assistant, course stage
XTAT	Teaching assistant, thesis stage
XINC	Instructor, course stage
XIND	Instructor, dissertation stage
XRAMC	Research assistant, M.S. course stage
XRAMT	Research assistant,M.S. thesis stage
XRADC	Research assistant, Ph.D. course stage
XRADD	Research assistant, Ph.D. dissertation stage
XSEC	Secretarial services
XCOMP	Computational services
	Final Commodity Activities
YUI	Undergrad. instruction
YMI	M.S. level instruction
YDI	Ph.D. level instruction
YMT	M.S. theses
YDD	Ph.D. dissertations
YSRY	Standard research years

Table 9b. Basic model

Code[a]	XSPI	XLPU	XLPG	XI1U	XI1G	XI2U	XI2G	XIM
UI	105.000	840.000	840.000	105.000	105.000	90.000	90.000	
MI								90.000
DI								
MT								
DD								
SRY								
TA		-1.000	-1.000					
INST	-.083							
RAMC								
RAMT								
RADC								
RADD								
SEC	-.020	-.030	-.030	-.020	-.020	-.020	-.020	-.020
COMP								
UF		-.083		-.083		-.083		
GF			-.083		-.083		-.083	-.083
GSMC								-2.500
GSDC								
GSMT								
GSDD								
US12	-2.188	-17.500	-17.500	-2.188	-2.188			
US34						-1.875	-1.875	
BUDG								
GSSO								

[a]Activity and commodity codes are defined in Table 9a and Table 10 respectively.

Table 9b (Continued)

Code[a]	XID	XRES1	XRES2	XRES3	XRES4	XRES5	XSM	XSD
UI								
MI								
DI	90.000							
MT			4.000	4.000		2.000	1.000	
DD				2.000		1.000		1.000
SRY		1.000	2.333	3.667	1.667	3.667		
TA								
INST								
RAMC					-1.000	-1.000		
RAMT			-2.000	-2.000		-1.000		
RADC						-1.000		
RADD				-2.000		-1.000		
SEC	-.020	-.300	-.600	-1.000	-.400	-1.000		
COMP		-.300	-1.000	-1.500	-.600	-1.200	-.150	-.300
UF								
GF	-.083	-1.000	-1.000	-1.000	-1.000	-1.000	-.049	-.098
GSMC								
GSDC	-2.500							
GSMT							-.450	
GSDD								-.900
US12								
US34								
BUDG								
GSSO								

Table 9b (Continued)

Code[a]	XTAC	XTAT	XINC	XIND	XRAMC	XRAMT	XRADC	XRADD
UI								
MI								
DI								
MT								
DD								
SRY								
TA	.500	.500						
INST			.500	.500				
RAMC					.500			
RAMT						.500		
RADC							.500	
RADD								1.000
SEC								
COMP								
UF								
GF								
GSMC	-.500				-.500			
GSDC			-.500				-.500	
GSMT		-.500				-.500		
GSDD				-.500				-1.000
US12								
US34								
BUDG	-2.700	-2.700	-3.600	-3.600	-2.700	-2.700	-3.600	-7.200
GSSO	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000

Table 9b (Continued)

Code[a]	XSEC	XCOMP	YUI	YMI	YDI	YMT	YDD	YSRY	
UI			-1.0						= 0.0
MI				-1.0					= 0.0
DI					-1.0				= 0.0
MT						-1.0			= 0.0
DD							-1.0		= 0.0
SRY								-1.0	= 0.0
TA									= 0.0
INST									= 0.0
RAMC									= 0.0
RAMT									= 0.0
RADC									= 0.0
RADD									= 0.0
SEC	1.0								= 0.0
COMP		1.0							= 0.0
UF									≥ -20.0
GF									≥ -40.0
GSMC									≥ -120.0
GSDC									≥ -60.0
GSMT									≥ -40.0
GSDD									≥ -30.0
US12									≥ -600.0
US34									≥ -375.0
BUDG	-7.2	-1.0							≥ -1000.0
GSSO	-1.0								≥ -260.0

Table 10. Assumptions and codes

Assumptions about class size, student course load, and faculty teaching load consistent with the activities specified	Average class size
Large freshman and sophomore lecture sections	280
Small freshman and sophomore classes	35
Junior and senior level classes	30
M.S. level classes	30
Ph.D. level classes	25

A 3-credit undergrad. course requires $\frac{1}{16}$ = .0624 academic man years per student

A 3-credit grad. course requires $\frac{1}{12}$ = .0833 academic man years per student

Each 3-credit course requires $\frac{1}{12}$ faculty or instructor man year

Large lecture section require $\frac{1}{12}$ faculty man years and 1 teaching assistant man year

Primary Commodities

Type	Code	Units of Measurement
Grad. faculty	GF	Academic man years
Undergrad. faculty	UF	Academic man years
Grad. (Masters course stage)	GSMC	Academic man years
Grad. (Ph.D. course stage)	GSDC	Academic man years
Grad. (Masters thesis stage)	GSMT	Academic man years
Grad. (Ph.D. dissertation stage)	GSDD	Academic man years
Undergrad. (Fresh., Soph.)	US12	Academic man years
Undergrad. (Jr., Sr.)	US34	Academic man years
Budget	BUDG	Thousands of dollars
Offices (Grad. students and secretaries)	GSSO	Number of offices

Table 10 (Continued)

Final Commodities

Type	Code	Unit of Measurement
Undergrad. instruction	UI	Man-credits
Masters level instruction	MI	Man-credits
Ph.D. level instruction	DI	Man-credits
M.S. theses	MT	Number
Ph.D. dissertations	MD	Number
Standard research years	SRY	Standard research man years[a]

Intermediate Commodities

Type	Code	Unit of Measurement
Teaching assistants	TA	Academic man years
Instructors	INST	Academic man years
Research assistants		
(M.S. course level)	RAMC	Academic man years
(M.S. thesis level)	RAMT	Academic man years
Research assistants		
(Ph.D. course level)	RADC	Academic man years
(Ph.D. dissertation level)	RADD	Academic man years
Secretarial assistants	SEC	Academic man years
Computing	COMP	Thousands of dollars

[a]A standard research man year is defined as the amount of research output which would result from one graduate faculty academic man year supported by 1/3 man year of secretarial services and $300 of computation services.

month academic year and the decisions relevant to such a time period. A quarter system is assumed and the model represents an aggregation over these three time periods which for cases of actual planning might not be sufficiently detailed. For example it is assumed that forty man years (the term man year will always refer to academic man year) are available from students at the masters thesis stage, but that fact in itself does not indicate how these forty man years are distributed over the three quarters. To insure that the solution of our model is consistent we must assume that all commodity availabilities are evenly distributed over the time period. If in reality this assumption is not true it would be necessary to disaggregate the model such that a separate submodel would apply to each quarter with the appropriate links between the quarters.

As in the typical activity analysis model the commodities are classified as primary, intermediate, and final. The primary commodities include the budget and physical space as well as man years of faculty and student time. We assume that the vector of available primary commodities is the result of a longer term, university-wide planning process which has allocated a general budget sufficient for faculty and physical plant needs and classroom and office space consistent with the faculty and students. Therefore the budget considered in the model is only that portion allocated for purchasing of supplies and of services of less permanent employees such as graduate student employees and secretarial or clerical personnel. Likewise the physical space considered is the allocation of offices for such less permanent personnel. For many actual situations it is likely that the budget will need to be decomposed into several sub-budgets each designated for specific purposes and that several different classifications of physical space will be necessary. A further disaggregation of faculty man years according to level of ability and field of speciality could easily be made and would probably be necessary for many actual planning purposes.

The intermediate commodities consist of graduate student research and teaching assistants and instructors as well as secretarial and computational services used in the production of the final commodities. As will be noted in the discussion of activities, each of these intermediate commodities can be produced, in a very broad sense of the term, from some combination of primary commodities.

The final commodity vector of the model is highly aggregated. The degree of aggregation used is due, in part, to the need to keep the model to a manageable size for purposes of exposition here; however, a central point of our discussion involves the possibility of using relatively aggregated quantities in the decision process. The aggregated final commodity vector is quite appropriate when higher level goals are considered or when efficient points are computed. Once a solution has been obtained with respect to the aggregated goal or efficiency criterion, one can choose between those more detailed or less aggregative vectors which maintain efficiency or satisfy the aggregate goal.

The final outputs designated in the model are quite obvious choices. A good case could be made for considering experience gained by graduate students through teaching and research as a final commodity since it presumably increases their productivity and can be considered as an investment in their human capital; however the model used here does not include such experience as a final commodity. The choice of units for measuring final commodities is easily criticized but apparently not easily improved upon.

The use of credit hours, theses, and dissertations conforms to well established university measurement systems. For many purposes the measurement of undergraduate instruction only by credit hours is not satisfactory. Salary structure and job qualifications often treat undergraduate degrees as a discrete variable, not divisible by credit hours, and aggregation of credit hours into degrees is complicated by the necessity to consider dropouts.

The difficulty of measuring and evaluating the output of new knowledge and the research activities which produce it has been discussed above. Our approach here defines an arbitrary unit called a standard research year which is the average amount of research produced by one graduate faculty man year supported by one-third man year of secretarial services and \$300. of computational services. Hence the definition is in terms of a specific combination of inputs; however, as will be discussed below, a number of other different input combinations are assumed to produce a certain number of standard man years of research. For many purposes the research commodity would need to be disaggregated. Basic and applied research might be considered separately, or separate components might be considered according to subject matter researched.

Before discussing the activities specified in the model we will note the importance of the restraints imposed by linearity. Specifically the assumption of constant returns to scale and the restraints on input substitutibility will be considered with respect to the model outlined. For the instructional activities which transform intermediate commodities from primary commodities the assumption of constant returns to scale should be adequate since each new unit of activity is essentially a duplication of the former with little interaction. The possibility of a single preparation for more than one section of a given course is an element which could contribute to increasing returns but is not considered here. Research activities are likely to involve some increasing returns to scale since data and techniques used for one project represent a fixed investment of time which may be spread over related research projects. We will assume that for the model considered here constant returns to scale form a sufficiently close approximation to reality. It has been argued that "For the typical college or university, the fixed proportions function is, in fact, utilized by the administrators who behave as though they were faced with just such a function" (Southwick, 1969, p. 169). We should note that those university decisions in which significant returns to scale are most probable are not relevant to this model due to its short term departmental level nature. Decisions concerning capacity expansion of physical plant, computer and library facilities, and administrative staff would be among those considered most likely to involve significant returns to scale.

The amount of restraint on input substitutability in the model is entirely dependent upon the choice of activities to be considered. When the possibility for input substitution exists, determination of optimal input proportions is an important problem.

For example, within a certain range there must certainly be a trade-off between the input of instructional time and student time such that the quantity of a given quality of knowledge gained by the students can be represented by isoquants. For many types of courses it should be feasible to set up controlled experiments and testing techniques so that empirical estimates could be made of such isoquants. Once the isoquants were estimated, the input substitution could be approximately represented in the model by placing a number of alternative production activities in the model for consideration. The mix of student and faculty input time involves the multiple questions of class size, expected course load for students and teachers, and the number of hours students are expected to spend outside the classroom per hour of instruction.[17]

In practice such questions are apparently not considered to involve interrelated decision variables and are often considered not to involve variables at all but fixed parameters. When decisions concerning optimal class size are made and don't involve consideration of available classroom space, they are often posed as an isolated decision where instructional cost is traded off against quality. We would hypothesize that within certain limits quality can be

[17] For a simple but interesting analysis of the interrelation between class size, teachers' salaries, and other variables see Herbert Simon (1967).

maintained while student input time is substituted for faculty input time and that given knowledge of the rate of substitution, the optimal ratio of inputs is an economic consideration involving a price ratio of student and faculty man years.

The activities considered in our model do not allow for large ranges of input substitution. Some possibility for substitution exists in the instruction of large principles courses where teaching assistants are used in combination with a smaller input of faculty time as well as in research where some activities use larger rates of research assistants to graduate faculty than other activities. In general the input proportions used in the model here are chosen to correspond quite closely with existing practices in many departments. Once a solution has been obtained for the type of model used here, it is possible to use the dual variables to determine whether or not a vector representing a new activity or program could be profitably introduced.

The activities specified are highly aggregated. The need for a detailed departmental plan and the type of commodity disaggregation discussed above would require a corresponding disaggregation of activities. The joint production of some of the research activities should be noted. These activities not only produce new knowledge measured by the standard research man year proposed above but also produce masters theses and doctoral dissertations. The coefficients used for research activities are defended only as reasonable approximations of input-output proportions in common use and could certainly be improved upon by extensive study of research programs.

The first illustrative computation is for the following problem:

$$\begin{aligned} & \max\ c'_y y_F \\ \text{subject to}\quad & A_F x - I y_F = 0 \\ & A_I x \qquad = 0 \\ & A_p x \qquad \geq \eta \\ & x,\ y_F \qquad \geq 0 \end{aligned} \tag{39}$$

which maximizes a linear function of the final commodities. The vector of coefficients c_y is assumed to represent the preference function of the decision-maker. Table 11 gives the results of the computations for (39). The matrix coefficients and commodity codes used are those in Table 9. The values used for c_y are only example values. The optimal final output values are given as well as the dual values for the primary commodities. A price range is given for each element of y_F. This indicates the range in which a single price can be changed without changing the optimal output levels. For example, the price of a doctoral dissertation could vary between 9.332 and 28.111 without affecting the optimal solution, and the price of undergraduate

Table 11. Problem (39) results

	Final Output y_F	Price Vector c_y	Price Range
YUI	46800	1.000	.039 ∞
YMI	4320	1.500	.873 ∞
YDI	2160	2.000	1.225 ∞
YMT	80	9.000	6.774 30.623
YDD	24.494	24.000	9.332 28.111
YSRY	62.996	28.000	17.983 42.833

	Dual Values
UF	42.668
GF	42.668
GSMC	52.578
GSDC	70.578
GSMT	29.332
GSDD	42.668
US12	46.375
US34	46.104
BUDG	.000
GSSO	.000

instruction, M.S. level instruction, or Ph.D. level instruction could each be increased by any amount, however large, without affecting the optimal y_F. Note that these price ranges are valid only if all other prices are held constant while a single price is changed. Note also that the optimal dual values will not remain unchanged.

Three different examples of goal programming problems were computed. Problems (40) and (41) are closely related to (36) and (37) of Chapter IV. They are problems combining variable goals which should be fulfilled at a minimum level and fixed goals (i.e., a quota of final outputs which should not be over or underfulfilled). Problem (40) minimizes a weighted sum of negative deviations from the variable goals and absolute deviations from the fixed goals $\hat{y}^*_F$:

$$\begin{aligned}
&\min \; c'_\varepsilon \varepsilon_F \\
\text{subject to} \quad & c'_\varepsilon \varepsilon_F < \gamma \\
& A_F x - I\varepsilon_F \leq \hat{y}_F \\
& A^*_F x + I\varepsilon^*_F \geq \hat{y}^*_F \\
& A_I x = 0 \\
& -A_P x \leq -\eta \\
& x,\ \varepsilon_F \geq 0.
\end{aligned} \tag{40}$$

For the computation, M.S. level instruction (YMI) and Ph.D level instruction (YDI) were chosen as commodities with variable minimum goals of 3600 and 1800 man-credits respectively. The other final commodities were considered to have fixed goals the values of which are given in Table 12.

The matrix values used are those of the original model in Table 9. The starred symbols, A^*_F, ε^*_F, $\hat{y}^*_F$ are equivalent to their unstarred counterparts except that the rows corresponding to YMI and YDI have been dropped. They have a row for each of the fixed goals but not the variable goals. In general the elements of c_F are weights for the goals and can be specified according to the relative importance of the goals. For the computations here each element was set equal to one.

Problem (41) minimizes the maximum absolute deviation:

$$\begin{aligned}
&\min \ \varepsilon_1 \\
\text{subject to} \quad &\varepsilon_1 \leq \gamma \qquad\qquad (41)\\
&A_F x - \varepsilon_F \leq \hat{y}_F \\
&A_F^* x + \varepsilon_F^* \geq y_F^* \\
&A_I x = 0 \\
&-A_P x \leq -n \\
&x,\ \varepsilon_1 \geq 0,
\end{aligned}$$

where ε_F and ε_F^* are vectors with the same variable ε_1 for every element.

The starred symbols have the same interpretation as in (40). For variable goals only negative deviations are considered while both positive and negative deviations are considered for fixed goals.

For both problems (40) and (41) γ is initially set as a very large number which will not affect the solution. Once the initial optimal solution has been completed γ is set equal to the optimal value of the objective function of each problem respectively. Then any new objective function can be maximized (i.e., subgoals may be maximized) subject to this constraint which insures that goals are fulfilled or fulfilled as nearly as possible.

For (40) the optimal value of the objective function was 23.874 which resulted from a single nonzero deviation. Only the goal of 100 standard research years could not be fulfilled. The optimal level of final commodities, $y_F=A_Fx^0$, as well as the corresponding levels of production activities, x^0, are given in Table 12. It would, at this stage, be possible to maximize any set of subgoals subject to the constraint that the sum of absolute deviations from the initial goals not exceed the minimum value by setting $\gamma=23.874$ in (40).

For (41) the optimal value of the objective function was 14.336. The optimal level of final commodities and corresponding activity levels are given in Table 12. Note that in this case every commodity deviated from the desired level and all but one, Ph.D. dissertations, deviated by the same amount, 14.336. For maximizing subgoals γ in (41) should be set equal to 14.336.

The different results from the two different criterion (i.e., minimizing the sum of absolute deviations and minimizing the maximum absolute deviation) can be expected to occur in general, the min-max gives a solution in which more of the goals fail to be fulfilled relative to the minimum sum criterion; however, no single deviation will be as large

Table 12. Solutions to problems (40)-(43)

	$\hat{y}_F$	Solution to (40)	Solution to (41)	Solution to (42) to (43)
YUI	46800.000	46800.000	46785.666	46800.000
YMI	3600.000	3600.000	3585.664	3600.000
YDI	1800.000	1800.000	1785.664	1800.000
YMT	40.000	40.000	54.334	80.000
YDD	30.000	30.000	27.168	30.000
YSRY	100.000	76.126	85.664	84.729
XSPI		94.190	144.205	120.048
XLPU		20.000	12.832	0.000
XLPG		0.000	0.000	9.163
XI1U		20.096	27.423	40.096
XI1G		0.000	0.000	40.837
XI2U		200.000	199.841	200.000
XI2G		0.000	0.000	0.000
XIM		40.000	39.841	40.000
XID		20.000	19.841	20.000
XTAC		0.000	0.000	18.326
XTRT		40.000	25.664	0.000
XINC		13.692	18.360	20.000
XIND		2.000	5.664	0.000
XRAMC		40.000	40.796	21.674
XRAMT		40.000	54.336	80.000
XRADC		6.308	2.436	0.000
XRADD		20.000	27.168	30.000
XSEC		28.081	31.370	31.829
XCOMP		31.207	32.929	34.002
XRES1		5.599	1.655	0.000
XRES2		0.000	0.000	5.000
XRES3		8.423	12.975	15.000
XRES4		16.846	19.180	10.837
XRES5		3.154	1.218	0.000
XSM		0.000	0.000	0.000
XSD		10.000	0.000	0.000

as for the minimum sum criterion.

Problem (42) is a series of linear programs which are solved sequentially. The first program (42) will minimize the weighted sum of deviations from minimum goals, $\hat{y}_F$. By minimum goals we mean that we wish to minimize the extent to which goals are underfulfilled but do not care if they are overfulfilled.

The first step is to solve,

$$\begin{aligned} & \min\ c_\varepsilon' \varepsilon_F \\ \text{subject to}\quad & A_F x - \bar{y}_F + \varepsilon_F \geq \hat{y}_F \\ & A_I x = 0 \\ & A_P x \geq \eta \\ & x, \bar{y}_F, \varepsilon_F \geq 0 \end{aligned} \tag{42}$$

and note that the optimal vectors $\varepsilon_F{}^0$ and $\bar{y}_F{}^0$ give the amounts by which the goals are underfulfilled and overfulfilled respectively. The next step is to solve program (43) which will find an efficient vector of final commodities subject to the constraint that the optimal weighted sum of deviations from minimum goals obtained from (42) be maintained. Set $c_\varepsilon' \varepsilon_F{}^0 = g_\varepsilon$. Then solve:

$$\begin{aligned} & \max\ c_y' \bar{y}_F \\ \text{subject to}\quad & \\ & c_F' \varepsilon_F \leq g_\varepsilon \\ & A_F x - \bar{y}_F + \varepsilon_F \geq \hat{y}_F \\ & A_I x = 0 \\ & A_P x \geq \eta \\ & x, \bar{y}_F, \varepsilon_F \geq 0, \text{ and } c_y > 0, \end{aligned} \tag{43}$$

and the efficient vector will be $y_F^E = \hat{y}_F + \bar{y}_F^* - \varepsilon^*$, where $\bar{y}_F^*$ and ε_F^* are optimal vectors from (43). The vector c_y must be strictly positive to insure that y_F^E is efficient. The elements of c_y give the relative weights of additions to $\hat{y}_F$. Finally we can maximize certain subgoals while maintaining efficiency and maintaining minimum deviation from major goals. The final subgoal program may be approached in two different ways. If there is no reason to maintain the specific efficient vector y_F^E obtained from (43), then set $c_y'\bar{y}_F^*=g_y$ and solve the program:

$$\begin{array}{rrl} & \max\ c_x'x & \\ \text{subject to} & c_y'\bar{y}_F & \leq g_y \\ & c_\varepsilon'\varepsilon_F & \leq g_\varepsilon \\ & A_F x-\bar{y}_F+\varepsilon_F & \geq \hat{y}_F \\ & A_I x & = 0 \\ & A_P x & \geq \eta \\ & x, \bar{y}_F,\ \varepsilon_F & \geq 0, \end{array} \tag{44}$$

which will result in an efficient vector of final output, but it will not necessarily be the same vector as y_F^E obtained from (43) since the first constraint in (44) allows trade-offs at the relative rates designated by c_y', and therefore any alternative optimal solution to (43) will also be feasible in (44).

If, on the other hand, it is desired that the specific efficient vector y_F^E obtained from (43) be maintained then the more restrictive program:

$$\begin{aligned} &\max\ c_x'x \\ \text{subject to}\quad & c_\varepsilon'\varepsilon_F \le g_\varepsilon \\ & A_Fx-\bar{y}_F+\varepsilon_F \ge \hat{y}_F^E \\ & A_Ix = 0 \\ & A_Px \ge \eta \\ & x,\ \bar{y}_F,\varepsilon_F \ge 0, \end{aligned} \tag{45}$$

should be solved. It is more restrictive since any feasible solution for (45) is also feasible for (44), but the converse is not true in general.

Programs (42) and (43) were computed using the values for $\hat{y}_F$ given in Table 12; the solutions obtained are also shown. For this particular vector of minimum goals the solution obtained from (42) was not altered by program (43). The elements in the two vectors c_ε and c_y were all set equal to unity for this computation, and since the only goal underfulfilled was research YSRY which deviated by 15.271, the value for $c_\varepsilon'\varepsilon_F{}^0 = g_\varepsilon = 15.271$. The programs (44) and (45) were not computed for $\hat{y}_F$ shown in Table 12; however, the full sequence of programs was computed for a different $\hat{y}_F$ and the results are given in Table 13.

Note that in this case programs (42) and (43) give different vectors and the vector obtained from (42) is clearly not efficient since the other vector is either as large or strictly larger in every component. In this case all of the minimum goals are fulfilled and the optimal solution to (42) is $c_{\varepsilon}'\varepsilon^0=0$. The elements of c_{ε} and c_y were all unity as above, and for the programs (44) and (45) the vector c_x was composed of zeroes except for elements corresponding to XLPU, X12G, and XTAC, each of which were set equal to unity. The optimal value for (42) was $c_y'\bar{y}_F$ = 1117.103. The meaning of the values in c_x in terms of subgoals is that we would like to have as large as possible the number of large principles courses taught by undergraduate faculty, the number of junior, senior courses taught by graduate faculty, and the number of teaching assistants at the course stage. However, none of the higher level objectives are to be sacrificed to increase these subgoals. That is, changes will be allowed in the x vector only if the higher level goals, $\hat{y}_F$, can still be met and only if the final output vector is an efficient one. And (45) requires that the final output vector be not only efficient but the same efficient vector obtained in (43).

For these particular example computations both of the subgoal programs have the same solution. In both cases the vector of final commodities is the same as obtained from (43); however, the x vector differs from the solution to (43). The subgoal programs were successful in increasing two of the subgoal variables, XLPU and X12G, (large principles sections taught by undergraduate faculty and junior, senior level courses taught by graduate faculty). It was not

Table 13. Solutions to problems (42)-(45)

	$\hat{y}_F$	Solution to (42)	Solution to (43)	Solution to (44)	Solution to (45)
YUI	46800.000	46800.000	46800.000	46800.000	46800.000
YMI	3600.000	3600.000	4320.000	4320.000	4320.000
YDI	1800.000	1800.000	2160.000	2160.000	2160.000
YMT	40.000	40.000	77.102	77.102	77.102
YDD	30.000	30.000	30.000	30.000	30.000
YSRY	55.000	55.000	55.001	55.001	55.001
XSPI		5.833	0.000	0.000	0.000
XLPU		0.000	0.000	2.304	2.304
XLPG		0.000	2.304	0.000	0.000
XI1U		40.096	40.096	237.792	237.792
XI1G		228.357	215.756	18.060	18.060
XI2U		200.000	200.000	0.000	0.000
XI2G		0.000	0.000	200.000	200.000
XIM		40.000	48.000	48.000	48.000
XID		20.000	24.000	24.000	24.000
XTAC		0.000	0.000	0.000	0.000
XTAT		0.000	4.608	4.608	4.608
XINC		0.000	0.000	0.000	0.000
XIND		.972	0.000	0.000	0.000
XRAMC		19.999	0.000	0.000	0.000
XRAMT		40.000	60.000	60.000	60.000
XRADC		19.999	0.000	0.000	0.000
XRADD		20.000	30.000	30.000	30.000
XSEC		25.686	25.626	25.626	25.626
XCOMP		22.500	25.065	25.065	25.065
XRES1		0.000	0.000	0.000	0.000
XRES2		0.000	0.000	0.000	0.000
XRES3		5.000	15.000	15.000	15.000
XRES4		0.000	0.000	0.000	0.000
XRES5		10.000	0.000	0.000	0.000
XSM		0.000	17.102	17.101	17.101
XSD		10.000	0.000	0.000	0.000

possible to increase the other subgoal, XTAC (teaching assistants at course stage). The increases in XLPU and X12G were compensated in the model by changes in XLPG, XI1U, XI1G, and XI2U.

Once a solution to (43)(hasbeen obtained the higher level policy-maker who set the goals $\hat{y}_F$ may be interested in knowing the efficient output points "around" the efficient vector y_F^E. By using the procedure outlined in the last part of Chapter IV it is possible to obtain all of the efficient extreme points which are adjacent to y_F^E. By joining these adjacent extreme points by line segments and noting that each point on the segment is an efficient point we can construct a whole set of efficient points "around" y_F^E.

The set of efficient points provides a set from which the decision-maker may choose that point which he most prefers and thus effectively reveal his preferences. On the other hand, the line segments may be viewed as defining the trade-offs which must be made between final commodities when moving from one efficient point to another on the segment.

If problem (38) is solved where c_Y is set equal to the dual values from (43) corresponding to $\hat{y}_F$, then the solution of final output y_F will be the same as obtained in (43). Table 14 gives the values for c_Y, the final output vector obtained from (38) and the dual prices for the primary commodities. Also given are the price ranges which have the same interpretation as those in Table 11 along with the nonbasic vector which will enter the basis when a single price change reaches a limit of the price range. That is, if the value of YDI were dropped to .092, holding all other prices constant, then the vector XINC would enter a new optimal basis. Note that any symbols corresponding to rows in the original model, such as US34, represent, in this context, slack vectors for the corresponding row.

The procedure outlined at the end of Chapter IV could now be employed to compute all efficient extreme points adjacent to the final output vector given in Table 14. First, note that every vector corresponding to y_F variables is in the optimal basis for the solution given in Table 14. Therefore, case 1 of the procedure in Chapter IV cannot be used. Case 2, however, can be used. In fact, the price ranges in Table 14 are computed by the formula given in case 2.

For example, to find the increase in the price of YSRY which will be sufficient to bring in a new vector we compute ϕ_c as in (46):

$$\phi_c = -\frac{z_k^0}{x_{6k}} = \min_{j \varepsilon J} -\frac{z_j^0}{x_{6j}} = -\frac{.623}{-2.33} = .267 . \tag{46}$$

From the tableau given in Table 15 we see that only for j = XRAMC, XRES1, and XRES2 are the values of $x_{6j} < 0$ and for j = XRES2 we get a minimum. Thus by adding $\phi_c = .267$ to 1 (the existing price of YSRY) we obtain the new price vector which will just be sufficient to bring a new vector into the optimal basis and that vector will be XRES2.

Table 14. Efficient point 0 and corresponding prices

	Final Output y_F	Price Vector c_y	Price Range	Vector Entering Basis
			.003	US34
YUI	46800.000	1.000	∞	-
			.055	XTAC
YMI	4320.000	1.000	∞	-
			.092	XINC
YDI	2160.000	1.000	∞	-
			.789	XRES2
YMT	77.102	1.000	1.000	XIND
			2.856	XIND
YDD	30.000	2.856	6.343	XSD
			1.000	XIND
YSRY	55.001	1.000	1.267	XRES2
UF		3.211		
GF		3.211		
GSMC		35.893		
GSDC		35.893		
GSMT		1.873		
GSDD		3.211		
US12		47.878		
US34		47.857		
BUDG		.000		
GSSO		.000		

Table 15. Optimal tableau values for y_F and nonbasic vectors

	YUI	YMI	YDI	YMT	YDD	YSRY	z_j^0
UI	1.000	.000	.000	.000	.000	.000	1.000
MI	.000	1.000	.000	.000	.000	.000	1.000
DI	.000	.000	1.000	.000	.000	.000	1.000
MT	.000	.000	.000	1.000	.000	.000	1.000
DD	.000	.000	.000	.000	1.000	.000	2.856
SRY	.000	.000	.000	.000	.000	1.000	1.000
TA	.000	.000	.000	1.873	.000	.000	1.873
INST	.000	.000	.000	3.211	.000	.000	3.211
RAMC	.000	.000	.000	-1.606	.000	1.833	.228
RAMT	.000	.000	.000	1.873	.000	0	1.873
RADD	.000	.000	.000	-1.478	1.000	1.833	3.211
SEC	.000	.000	.000	.000	.000	.000	.000
COMP	.000	.000	.000	.000	.000	.000	.000
UF	.000	.000	.000	3.211	.000	.000	3.211
GF	.000	.000	.000	3.211	.000	.000	3.211
GSMC	.000	36.000	.000	-.107	.000	.000	35.893
GSDC	.000	.000	36.000	-.107	.000	.000	35.893
GSMT	.000	.000	.000	1.873	.000	.000	1.873
GSDD	.000	.000	.000	-.478	1.000	1.833	3.211
US12	48.000	.000	.000	-.122	.000	.000	47.878
US34	48.000	.000	.000	-.143	.000	.000	47.857
XLPG	.000	.000	.000	.000	.000	.000	.000
XIRG	.000	.000	.000	.000	.000	.000	.000
XIAC	.000	18.000	.000	-.990	.000	.000	17.010
XINC	.000	.000	18.000	-1.659	.000	.000	16.341
XIND	.000	.000	.000	-2.345	.500	0.917	≈0
XRAMC	.000	18.000	.000	.749	.000	-.917	17.833
XRADC	.000	.000	18.000	-.054	.000	.000	17.947
XRES1	.000	.000	.000	3.211	.000	-1.000	2.211
XRES2	.000	.000	.000	2.956	.000	-2.333	.623
XRES4	.000	.000	.000	1.606	.000	.167	1.772
XSD	.000	.000	.000	-1.016	-.100	1.650	.349

Each of the price levels and vectors entering the basis shown in Table 15 may be obtained by analogous calculations. There are six different nonbasic vectors (US34, XTAC, XINC, XRES2, XIND, and XSD) which will enter the basis when elements of c_y are changed one at a time. Each of the resulting six different optimal sets of basis vectors will produce an efficient extreme y_F adjacent to the y_F in Table 14. These six different vectors were computed and are given in Table 16 along with the corresponding prices for final and primary commodities and price ranges for y_F commodities within which the basis will remain optimal.

At this point six adjacent efficient extreme points have been obtained but there remain 20 other nonbasis vectors which might produce an efficient adjacent extreme point if interchanged with one of the basic vectors. To determine whether or not such interchanges would produce efficient points we must use the analysis of case 3 in Chapter IV. That is, we must solve the linear programming problem (47):

$$\begin{aligned} &\min_{c} \ (B^{0-1}a_k)' \ c_B - c_k \\ \text{subject to} \quad &(B^{0-1}a_j)' \ c_B - c_j \geq 0, \text{ all } j \text{ including } k \\ &Ic_y \geq 1 \\ &c_x = 0, \ c_s = 0. \end{aligned} \tag{47}$$

This will determine whether a c^+ exists for each specific vector a_k. The jth column of the optimal tableau gives the

Table 16. Efficient points adjacent to point 0 and corresponding prices

	1			2			3		
	Final Output	Price Vector	Price Range	Final Output	Price Vector	Price Range	Final Vector	Price Vector	Price Range
			.002			.003			.003
YUI	45077.263	.003	.003	46800.	1.000	∞	46800.000	1.00000	∞
			.055			.055			.055
YMI	4320.000	1.000	∞	4320.000	1.000	∞	4320.000	1.00000	∞
			.092			.069			.092
YDI	2160.000	1.000	∞	2104.449	.092	.092	2160.000	1.00000	∞
			.996			1.000			.000
YMT	82.222	1.000	1.083	82.222	1.000	1.001	69.834	1.00000	1.000
			2.838			2.856			2.367
YDD	30.000	2.856	6.377	30.000	2.856	6.344	30.000	2.85623	10.750
			.990			1.000			1.267
YSRY	55.001	1.000	1.263	55.001	1.000	1.267	60.737	1.26706	2.460
UF		3.198			3.211			3.211	
GF		3.198			3.211			3.211	
GSMC		35.893			35.893			35.893	
GSDC		35.893			3.211			35.893	
GSMT		1.874			1.873			1.873	
GSDD		3.217			3.211			3.701	
US12		.020			47.878			47.878	
US34		.000			47.878			47.878	
BUDG		.000			.000			.000	
GSSO		.000			.000			.000	

Table 16 (Continued)

	4			5			6		
	Final Output	Price Vector	Price Range	Final Output	Price Vector	Price Range	Final Output	Price Vector	Price Range
YUI	46800.000	1.000	.003 ∞	46800.000	1.000	.003 ∞	46800.000	1.000	.003 ∞
YMI	4226.879	.055	.041 .055	4320.000	1.000	.055 ∞	4320.000	1.000	.055 ∞
YDI	2160.000	1.000	.092 ∞	2160.000	1.000	.092 ∞	2160.000	1.000	.092 ∞
YMI	82.222	1.000	1.000 1.003	82.477	1.000	1.000 3.187	83.478	1.000	1.000 3.808
YDD	30.000	2.856	2.856 6.344	28.854	2.856	1.748 2.856	30.628	6.343	6.343 8.367
YSRY	55.001	1.000	1.000 1.267	52.899	1.000	.735 1.000	44.693	1.000	.773 1.000
UF		3.211			3.211			3.211	
GF		3.211			3.211			3.211	
GSMC		1.872			35.893			35.893	
GSDC		35.893			35.893			35.893	
GSMT		1.873			1.873			1.873	
GSDD		3.212			3.211			6.698	
US12		47.878			47.878			47.878	
US34		47.857			47.878			47.878	
BUDG		.000			.000			.000	
GSSO		.000			.000			.000	

coefficients for $(B^{0-1}a_j)$. Since only objective function values of c_y are non-zero the only rows of the tableau which need be considered are those corresponding to y_F (i.e., those values given in the columns of Table 15). And those columns of the tableau (rows of Table 15) containing only non-negative values need not be included since their inner product with the strictly positive c_y will always be non-negative. Thus the above linear program simplifies to:

$$\begin{aligned} &\min \quad t_k c_y \\ \text{subject to} \quad &Tc_y \geq 0 \\ &Ic_y \geq 1, \end{aligned} \tag{48}$$

where the matrix T is given in Table 17. From T we note that 17 rows have dropped out because they contained only non-negative elements and six of the remaining 15 correspond to the nonbasic vectors which we already know will produce efficient adjacent extreme points when brought into the basis. Thus we are left with nine linear programs to solve.

Each of the nine programs was computed and each had a strictly positive optimal objective function value. Thus a c^+ vector does not exist for any of the nine and bringing any one of them into the basis would not produce an efficient y_F.

To see that the linear program will, in fact, give a

Table 17. Elements of matrix T for problem (48)

	YUI	YMI	YDI	YMT	YDD	YSRY
RAMC	.000	.000	.000	-1.606	.000	1.833
RADD	.000	.000	.000	-1.478	1.000	1.833
GSMC	.000	36.000	.000	-.107	.000	.000
GSDC	.000	.000	36.000	-.107	.000	.000
GSDD	.000	.000	.000	-1.478	1.000	1.833
US12	48.000	.000	.000	-.122	.000	.000
US34	48.000	.000	.000	-.143	.000	.000
XTAC	.000	18.000	.000	-.990	.000	.000
XINC	.000	.000	18.000	-1.659	.000	.000
XIND	.000	.000	.000	-2.345	.500	.917
XRAMC	.000	15.000	.000	.749	.000	-.917
XRADC	.000	.000	18.000	-.054	.000	.000
XRES1	.000	.000	.000	3.211	.000	-1.000
XRES2	.000	.000	.000	2.956	.000	-2.333
XSD	.000	.000	.000	-1.016	-.100	1.650

required c^+ if it exists we solved program (48)[18] by using t_k as the row corresponding to XRES2. We have shown above by

[18] In obtaining the numerical solution the right hand side was replaced by a vector with .2 for each element except in the row corresponding to XRES2 where a -.1 value was used. These changes were made to insure that rounding errors would not be a problem. In specific cases these changes could prevent the program from locating the required c^+ vector even when it existed.

case 2 procedures that bringing XRES2 into the basis will produce an efficient adjacent y_F. The values for the optimal c_y for program 48 are given in Table 18 along with the results of using that price vector in the original problem. Note that the efficient adjacent y_F obtained is the same as that obtained previously by introducing XRES2 into the basis.

If the basis matrix for the original efficient vector y_F were unique then we could be assured that the six different adjacent efficient vectors which we have obtained include all adjacent efficient vectors. The basis matrix for the original y_F is, however, not unique.

First of all, two nonbasic vectors (XLPG and XI2G) have zero z_j-c_j values and when either of these is brought into the basis the y_F vector remains unchanged. Furthermore XSP1 and XRES5 are in the original basis at zero level so the solution is degenerate. Thus it would be possible to bring any vector into the basis, if in the column of the tableau corresponding to that vector, the elements in the rows for either XSPI or XRES5 were nonzero. From the optimal tableau it was found that this was true for the following nonbasic vectors: INST, RAMC, XINC, XIND, SRAMC, XRADC, and XRES4. Bringing in any one of these seven nonbasic vectors would produce an alternative basis matrix for the original y_F vector. To insure that all efficient vectors adjacent to the original y_F are included in the six

Table 18. Efficient point 6 and alternative prices obtained from problem (48)

	Final Output	Price Vector	Price Range
YUI	46800.000	1.000	.032 ∞
YMI	4320.000	1.000	.591 ∞
YDI	2160.000	1.000	.990 ∞
YMT	69.834	10.728	.000 10.762
YDD	30.000	25.711	25.407 115.629
YSRY	60.737	13.637	13.593 13.593
UF		34.498	
GF		34.498	
GSMC		34.851	
GSDC		34.851	
GSMT		20.116	
GSDD		34.802	
US12		46.686	
US34		46.467	
BUDG		.000	
GSSO		.000	

that have been obtained it would be necessary to repeat for each of the alternative basis matrices the type of analysis done for the original basis. These computations were not performed because without additional computer routines they would be a much larger computational task than justified by the importance of this particular numerical example.

As indicated in Chapter IV an initial attempt was made to compute all efficient points (i.e., not only those adjacent to a particular point) for the model specified here. The efficient y_F vectors which were calculated are listed in Appendix I. Note that many of these vectors would receive consideration only under a very unusual configuration of relative prices or weights for the commodities involved. The computational expense of obtaining such vectors is likely to be wasted. The method which was substituted will eliminate much of this unnecessary computational expense.

Let us designate the final output vector of Table 14 as point 0. Then the six adjacent points given in Table 16 provide a policy-maker with information concerning the optimal trade-offs which can and must be made in order to increase any component of 0 by a small amount. For example, if a policy-maker were interested in considering relative changes in M.S. theses, Ph.D. dissertations, and standard research years while holding the levels of the other three final commodities fixed he could consider the adjacent extreme points 3, 5 and 6. These points have been plotted in Figure 1. Every point along the lines connecting these adjacent efficient extreme points is also efficient. From the figure we see that for each M.S. thesis he is willing to give up he will be able to obtain 1.25 additional standard research years; however, if he were to obtain an additional M.S. thesis he would be forced to give up 1.60 standard research

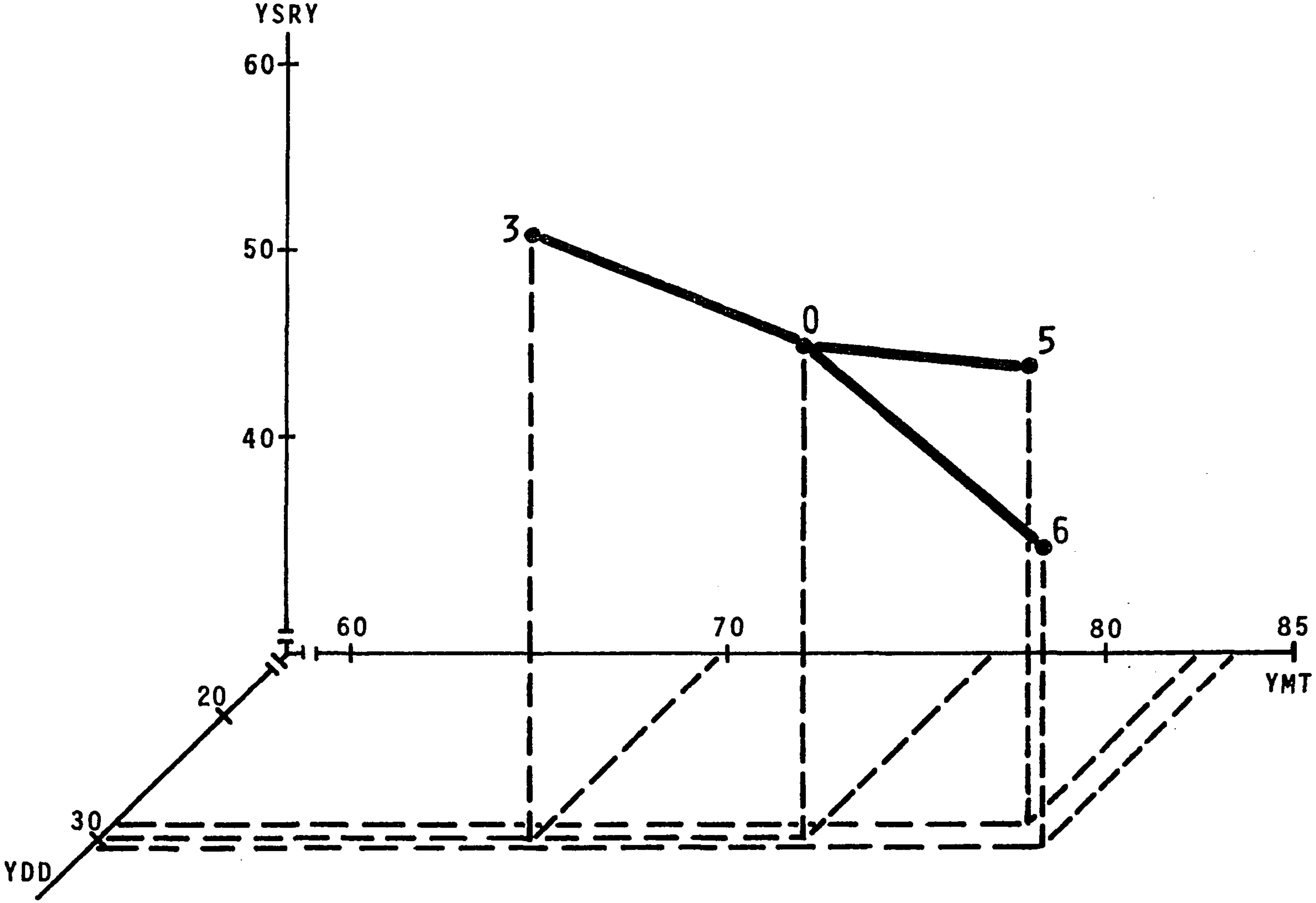

Figure 1. Efficient trade-offs among YMI, YDD, and YSRY

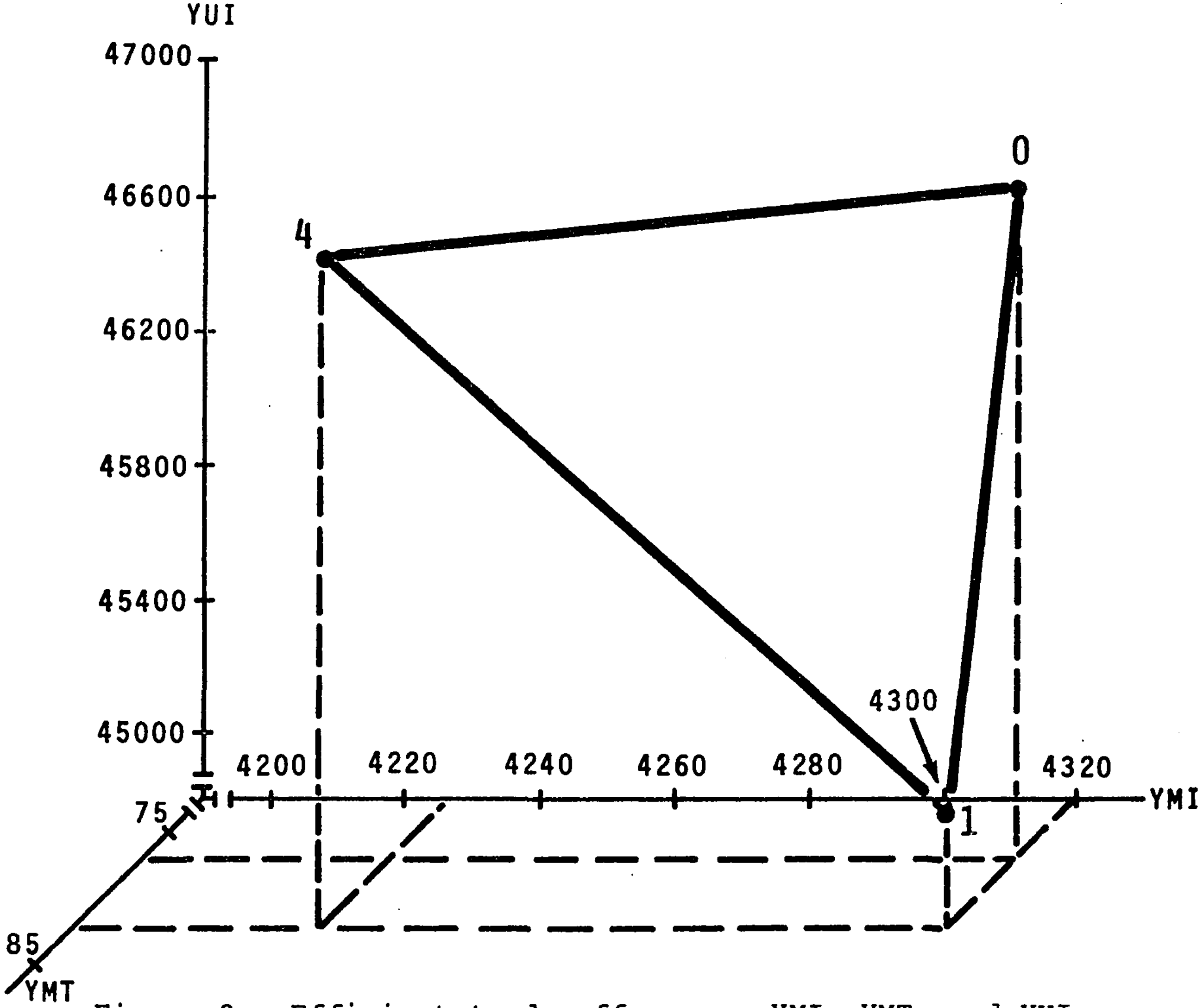

Figure 2. Efficient trade-offs among YMI, YMT, and YUI

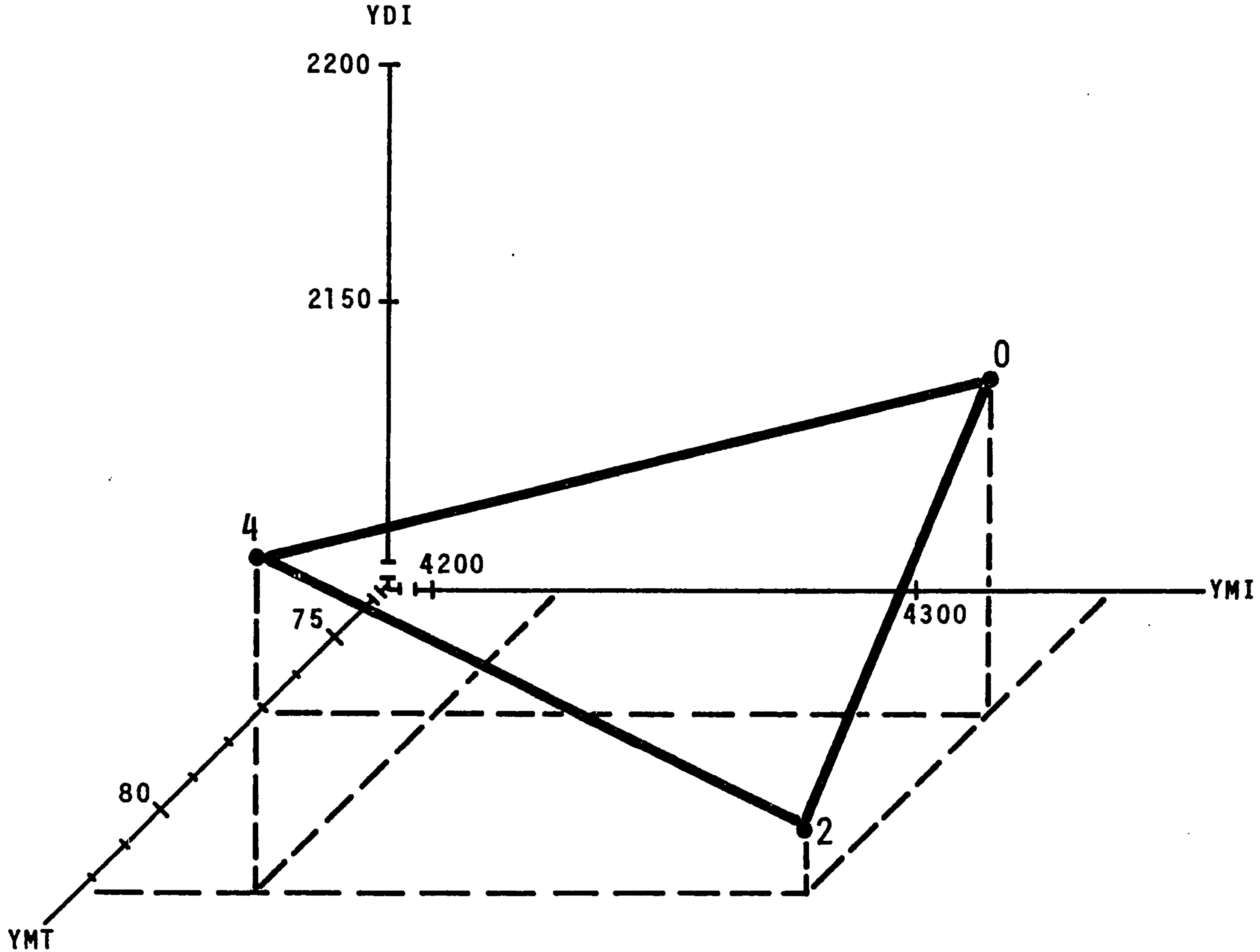

Figure 3. Efficient trade-offs among YMI, YMT, and YDI

years.

If the policy-maker were interested in relative trade-offs between undergraduate and M.S. instruction and M.S. theses he would consider points 0, 4, and 1. These points have been plotted in Figure 2. If the policy-maker were concerned with the trade-offs between graduate student instruction and M.S. theses he could consider points 0, 2, and 4, which have been plotted in Figure 3.

Other examples are, of course, possible but those given indicate the type of information concerning output possibilities which the adjacent efficient extreme points provide the policy-maker. It would seem that such information would be quite an aid in rational decision-making.

The fact that a system of dual prices is associated with each efficient output vector leads to the possibility of using this imputed price information to effect a decentralized decision process in the system. Such a proposal immediately runs into some difficulties resulting from the nonuniqueness of quantity and imputed price relationships in a linear model.

First, consider the possibility of using the vector of dual prices solely for effecting decisions within the department, completely disregarding any relationship between this vector of imputed prices and any vector of prices for similar commodities existing in the larger economic system. For this purpose the dual variables will provide a vector of imputed prices which when used to compare the value of inputs with the value of outputs of any particular activity will, in a completely decentralized manner, discriminate between unprofitable activities and activities which just break even. In general, however, for those activities which do just break even, these prices alone, without coordination concerning quantities, will not be sufficient to insure that optimal quantities will be chosen.

Alternatively, it is of interest to consider comparing the relative imputed prices of the primary commodities with their relative prices in the larger economic system. Then primary commodities for which the relative value is higher in the department than the larger economic system would be the primary commodities to be increased in later periods. Such a link is not, however, as straightforward as one would hope. The policy-maker chooses a particular efficient output vector and we can find a price vector, c_y, for final outputs which, when substituted in the original problem will give that efficient output vector as an optimal solution. In fact there is likely to be a whole set of values for c_y which will give that particular vector of final outputs. The imputed prices for primary commodities will depend upon which value for c_y is used. The arbitrariness of the choice must be removed if the imputed prices for primary commodities is to serve as a useful link to the larger economic system.

One possibility for removing the arbitrariness is to rely upon the policy-maker to choose that c_y which best represents the relative weights which he places upon the outputs. Another possibility is to choose that c_y which most nearly meets the estimated value of outputs in the larger economic system. In either case the choice of c_y is not free since it must be a c_y vector consistent with the desired efficient vector of final output. Assuming that c_y could be chosen in an acceptable way, then the dual variables for primary commodities could be used as a link to the larger economic system.

Beyond the difficulties arising from nonuniqueness, decisions concerning changes in the primary commodities over time involve important dynamic considerations, which can only be adequately dealt with by expanding the model to include sequential time periods, identifying flows through the system over time and explicitly including investment activities for primary commodities using actual or opportunity market values. Such an expansion of the model has not been attempted for this study.

It is of interest to compare the importance of prices in the different models considered. In problem (39) prices play a very central role since the relative prices are taken as given either from the policy-maker or the larger economic system and then the model is used to choose a feasible output vector which is optimal given the prices. This is opposed to the type of model just discussed in which an efficient vector of outputs is chosen from competing vectors of outputs, then a price vector is chosen which is consistent with the efficient vector chosen. If the relative prices for final outputs for problem (39) can be reliably estimated then the optimal dual variables for inputs provide an important link to the larger economic system. The goal programming models can be viewed as a case where the importance of the price vector lies in between the two extreme cases above. On the one hand the vector of outputs is chosen so as to closely approximate a particular vector of desired outputs, but the goal programming approach does require that policy-makers specify different vectors such as c_ε, c_y, and c_x each of which indicates the relative importance of different outputs and can be considered to be relative price vectors. However, these relative prices are only marginally important since c_ε and c_y apply only to commodity additions or subtractions around the particular desired level of outputs, $\hat{y}_F$. The vector, c_x, is also only marginally important since it applies only to changes in activity levels which do not violate higher level objectives. Thus the ability of the policy-maker to specify his preferences in the form of output goals, $\hat{y}_F$, as well as his ability to specify his preferences in the form of relative prices is important for the success of the goal programming approach.

The possibility of using the goal programming approach to implement a decentralized hierarchical decision process is quite evident. The high level policy-maker can set his goals with top priority in terms of a few highly aggregated outputs and leave to lower level policy-makers the decisions concerning which specific activities can best be used to meet the aggregated outputs (i.e., to leave for lower level policy-makers the responsibility to set and optimize subgoals). It is hoped that the computational examples given in this chapter help elucidate how such a decentralized process could be carried out with the aid of similar models.

The model outlined here was intended only as an example and a number of important criticisms can be directed toward it in its present form. Some of the deficiencies can be relatively easily accommodated; others pose more fundamental problems. There is little doubt that an applicable model would need to be greatly disaggregated by activity and by commodity. Such an expansion of the problem poses very little difficulty even with respect to computation expense.

The applicability of the model would be greatly enhanced if it were expanded to cover several time periods with explicit allowance for changes in the stocks of students, faculty, and physical space and with the flows of students through the system specifically included. The necessary relations linking the periods should be relatively easy to specify and an expanded model with the main matrix assuming the familiar block triangular structure should be sufficient to accomplish such a generalization without basic difficulties.

The specified model is completely deterministic and the real world is, of course, filled with many elements of uncertainty. If the important elements of uncertainty proved to be in the vector of primary commodities (e.g., the number of students available is known only as a random variable) then the use of chance constrained programming could be applied. More complicated types of uncertainty might not be accommodated in such a straightforward manner. However, Charnes, Clower, and Kortanek (1967, p. 316) have shown that "the preemptive goal method is a robust one; small errors in assignment of preemptive goals result in small errors in total profit." The stochastic approach to goal programming outlined by Contini (1968) provides a possible way of treating uncertainty, but the problem immediately becomes nonlinear and increases in the size of a nonlinear model are likely to quickly become important in terms of computational expense.

The fact that the model is linear can also be criticized. As noted above input substitution can adequately be approximated by a linear model but the assumption of constant returns to scale is implicitly accepted in the use of a linear model. To the extent that increasing or decreasing returns to scale are important, the present model is inadequate. Adequate nonlinear models could of course be specified but would fundamentally change the models we have discussed in both theoretical and computational aspects.

Finally, the assumption of complete divisibility of commodities implicit in the model used may prove too much of a simplification especially in assigning faculty and students to specific class sections. By using units of man years, which are divisible, we have evaded this difficulty. While such an approach is adequate if the model is sufficiently aggregated it is likely to be increasingly inadequate as the model is disaggregated.

CHAPTER VI. THE RELATION BETWEEN A NONCOMPETITIVE PRICE SETTING RULE AND EFFICIENT PRODUCTION

In this chapter we will discuss the relation between a specific noncompetitive price setting rule and efficient production in a linear activity analysis model. The possibility of decentralizing decisions and information under the noncompetitive rule will be discussed, as well as the implications of the results with respect to the general theory of the second best. Finally a numerical example and relevant calculations will be presented.

There are, of course, many types of noncompetitive behavior, each a result of special characteristics in the economic system. Noncompetitive behavior may result, for example, if information is not complete, if increasing returns are present, or if a certain degree of monopolistic power exists. The analysis of systems having such characteristics can be approached in many different ways, but in general will require stochastic or nonlinear models and may require quite different techniques such as game theory. It should be made clear that no attempt is being made here to analyze noncompetitive behavior in general. Rather we are concerned with a very special case of noncompetitive price setting which can be analyzed within the framework of a linear activity analysis model.

The specific assumption which is made is that certain economic agents set minimum price levels for certain primary commodities. Unlike the competitive case for the linear model, these minimum prices will not be zero when there is a surplus of the primary commodity involved. The set of minimum prices are assumed to be given external to the model rather than being considered variables determined simultaneously with other variables in the model. The mechanism by which they are determined is only assumed to exist, and the exact conditions under which such minimum price levels can be maintained are not spelled out beyond assuming a sufficient degree of monopoly control or ability to form coalitions.

We will now proceed to show that for a model somewhat less general than (30) that an alternative price vector exists which satisfies conditions 1-4a given by (31) but not 4b (i.e., some primary resources may have positive prices even when not used to the limit of availability). This alternative price vector will be referred to as a noncompetitive price vector, p*, while the p satisfying condition (31) will be called the competitive price vector. Our major interest is showing the relationship between such noncompetitive price vectors and efficient commodity vectors, similar to (31) for competitive prices. Linear programming problems (32) and (33) will be used in the analysis.

The results of Nikaidô (1964) are very closely related to the relationships shown below. In fact, Nikaidô (1964, p. 298) states that the purpose of his article is "to try a challenge to the monopolistic prevalence of this linking of linear programming to competition."

We are interested in a model in which the prices of primary commodities have rigid minimums, some or all of which are strictly positive. The vector, $\lambda \geq 0$, will be used to symbolize the lowest limit which primary commodity prices are allowed to assume. This new condition is explicitly included in the former model by adding constraint rows to the linear programming problem (33) to obtain a new linear programming problem which is given as (50) below. The dual to (50) which is given as (49) corresponds to linear programming problem (32) for the competitive case:

$$\begin{array}{rl} & \min\ -e'\bar{y}_F - \lambda' s \\ \text{subject to} & A_F x - y_F = 0 \\ & A_I x = 0 \\ & A_P x - y_P = 0 \\ & y_P - s \geq \eta \\ & y_F - \bar{y}_F = \hat{y}_F \\ & x,\ \bar{y}_F,\ s \geq 0 \end{array} \qquad (49)$$

$$\begin{array}{llll} & \max \ \eta' w_P + \hat{y}_F' t_F & & \\ \text{subject to} & A_F' u_F + A_I' u_I + A_P' u_P & \leqq 0 & \\ & -u_F \qquad\qquad +t_F & = 0 & \\ & \qquad -u_P + w_P & = 0 & (50) \\ & \qquad\qquad -t_F & \leqq -e_F & \\ & \qquad\qquad -w_P & \leqq -\lambda & \\ & w_P \geqq 0. & & \end{array}$$

Using (49) and (50) and a somewhat limiting assumption that A_F be square and nonsingular we will give in (52) a result for noncompetitive prices, p*, corresponding to result (31) for competitive prices.

First we will give a preliminary result, (51), relating efficiency to an optimal solution of problem (49).

If problem (49) is such that $e' > -\lambda' A_P A_F^{-1}$, $\lambda \geq 0$, where e is a vector of ones, then $\hat{y}_F$ is efficient if and only if (49) has an optimal solution such that $\bar{y}_F = 0$. (51)

To show that efficiency implies an optimal solution such that $\bar{y}_F = 0$, note that by the definition of efficiency there must exist a feasible solution to (49) but also by the definition of efficiency of $\hat{y}_F$, it is possible to find a feasible $y_F \geqq 0$ and $\bar{y}_F \geqq 0$ such that $y_F = \bar{y}_F + \hat{y}_F$ only if $\bar{y}_F = 0$. Furthermore, the vector, s, also has a definite upper limit; therefore, (49) is bounded and an optimal solution must exist.

To show that an optimal solution such that $\bar{y}_F=0$, implies efficiency, an equivalent statement will be employed, i.e., $\hat{y}_F$ not efficient implies (49) does not have an optimal solution such that $\bar{y}_F=0$. There are two cases. Case 1. If $\hat{y}_F$ is not attainable then there does not exist a $y_F \geqq 0$ which is feasible for (49), and therefore (49) does not have an optimal solution. Case 2. If $\hat{y}_F$ is attainable but there exists a feasible y_F such that $y_F \geq \hat{y}_F$[19] then any optimal solution to (49) must have $\bar{y}_F \geq 0$. We will show that any solution with $\bar{y}_F=0$ will be strictly larger than some solution with $\bar{y}_F \geq 0$ and therefore cannot be optimal. It is at this point that the property of a square nonsingular A_F is employed.[20]

Suppose that x^* and $\hat{x}$ are both attainable, (i.e., feasible for problem (49)) where $A_F x^* = y_F^* \geq \hat{y}_F$

and $A_F \hat{x} = \hat{y}_F$

since $\bar{y}_F = y_F - \hat{y}_F$, $\bar{y}_F^* = A_F(x^* - \hat{x}) \geq 0$.

We assume $-e'\lambda' A_P A_F^{-1} < 0$ and it follows that,

$(-e' - \lambda' A_P A_F^{-1})\bar{y}_F^* < 0$, since the first vector is strictly negative and the second non-negative with at least one element positive,

thus

---

[19]Only in Chapter VI is the convention employed which distinguishes between the inequality signs $\leq$ and $\leqq$. For example, $\bar{y}_F \leq 0$ implies $\bar{y}_F \leqq 0$ and $\bar{y}_F \neq 0$.

[20]It seems quite possible that the requirement of a square nonsingular A_F matrix could be relaxed by employing the concept of a generalized inverse (Ijiri, 1965, p. 30).

$$-e'\bar{y}_F^* - \lambda' A_P A_F^{-1} A_F (x^* - \hat{x}) < 0$$

or,

$$-e'\bar{y}_F^* - \lambda' A_P (x^* - \hat{x}) < 0.$$

But this last expression is exactly the change in the value of the objective function of problem (49) when the solution corresponds to x^* rather than $\hat{x}$ since,

$$z^* - \hat{z} = (-e'\bar{y}_F^* - \lambda' s^*) - (-e'0 - \lambda'\hat{s}) = -e'\bar{y}_F^* - \lambda'(-\eta(+y_P^* + \eta - \hat{y}_P)$$

$$= -e'\bar{y}_F^* - \lambda' A_P (x^* - \hat{x}) < 0$$

or

$$z^* < \hat{z}.$$

We have shown that if there exists a $\bar{y}_F \geq 0$ which is feasible for problem (49) it must be in the optimal solution.

Next, the relation between the noncompetitive price vector, p^* and an efficient vector $\hat{y}_F$ is given in (52):

For a model with A_F square and non-singular, an attainable $\hat{y}_F$ is efficient if and only if there exists a price vector p^* such that: (52)

1. $p^{*\prime}\hat{y} = 0$
2. $p^{*\prime}A \leq 0$
3. $p_F^* \geqq e' > -\lambda' A_P A_F^{-1}, \ \lambda \geqq 0$

4a. $p_{Pi}^* \geqq \lambda_i$, all i

4b. $p_{Pi}^* = \lambda_i$, if $\hat{y}_{Pi} > \eta_i$.

Part I will show that p^* satisfying 1-4 for an attainable $\hat{y}_F$ implies that $\hat{y}_F$ is efficient. The attainability of $\hat{y}_F$ implies that (49) has a feasible solution, and p^* satisfying conditions 2-4 implies that (50) has a feasible solution with

$$p^*_F = t_F = u_F$$
$$p^*_I = u_I$$
$$p^*_P = w_P = u_P.$$

Since problem (49) and its dual problem (50) both have feasible solutions, they must both have optimal solutions,[21] the values of which must be equal. Thus there exist optimal solutions such that $-e'\bar{y}_F{}^0-\lambda's^0=\eta'w_P{}^0+\hat{y}_F't_F{}^0$ and for all feasible nonoptimal solutions we have $-e'\bar{y}_F-\lambda's>\eta'w_P+\hat{y}_F't_F$. Condition 1, $p^{*\prime}\hat{y}=0$, has not yet been used. In terms of the variables of problem (49) and (50) this condition implies that

$$[p^{*\prime}_F|p^{*\prime}_I|p^{*\prime}_P]\begin{bmatrix}\hat{y}_F\\ \hline 0\\ \hline \hat{y}_P\end{bmatrix} = 0$$

or that $0=t_F'\hat{y}_F+w_P'\hat{y}_P$, and there exists a feasible $s=\hat{y}_P-\eta$ such that $t_F'\hat{y}_F+w_P'\hat{y}_P = t_F'\hat{y}_F+w_P'(\eta+s)$ or $-w_P's = t_F'\hat{y}_F+w_P'\eta$ and by condition 4.b, $-\lambda's = -w_P's$ so $-\lambda's=t_F'\hat{y}_F+w_P'\eta$, where s,t_F,w_P are feasible for problems (49) and (50). But from above we have $-e'\bar{y}_F-\lambda's\geq t_F'\hat{y}_F+w_P'\eta$ for all feasible $\bar{y}_F$, s, t_F, and w_P,

[21]See Goldman and Tucker (1956, theorem 2, page 61).

and if equality holds, these solutions are optimal. Therefore, the s, t_F, and w_P satisfying $-\lambda's=t_F'\hat{y}_F+w_P'\eta$ must be optimal solutions such that $\bar{y}_F=0$. We have shown that the existence of a vector, p*, satisfying conditions 1-4 implies that the optimal solution of problem (49) must have $\bar{y}_F=0$. And from (51) it follows that $\hat{y}_F$ is efficient.

Part II of the proof is that an efficient $\hat{y}_F$ implies the existence of a p* satisfying conditions 1-4. From (51) the efficiency of $\hat{y}_F$ implies (49) has an optimal solution such that $\bar{y}_F=0$. For the optimal solution we have:

$$-e'\bar{y}_F{}^0-\lambda's^0 = -\lambda's^0 = \hat{y}_F't_F{}^0+\eta'w_P{}^0.$$

The first equality results from the fact that for $\hat{y}_F$ efficient $\bar{y}_F$ must equal zero. The second equality results from the equality of optimal dual objective functions. For the optimal solutions we also have:

$$-\lambda's^0 = -w_P{}^{0'}s^0 = w_P{}^{0'}(\eta-y_P{}^0).$$

The first equality results from the fact that at the optimal $w_P{}^0>\lambda_P \rightarrow s^0 = 0$ and the second since $-s^0>\eta-y_P{}^0 \rightarrow w_P{}^0=0$. Thus we have:

$$w_P{}^{0'}(\eta-y_P{}^0) = \hat{y}_F't_F{}^0+\eta'w_P{}^0$$

or

$$t_F{}^{0'}\hat{y}_F+w_P{}^{0'}y_P{}^0 = 0.$$

If we let $u_F{}^0 = t_F{}^0 = p_F^*$, $u_P{}^0 = w_P{}^0 = p_P^*$, $y_P{}^0 = \hat{y}_P$ and $u_I{}^0 = p_I^*$, and note that $y_I = 0$, then condition 1, (i.e., $p^{*\prime}\hat{y}=0$) follows from the above equation. Conditions 2, 3 and 4.a follow directly from the constraints of problem (50) and condition 4.b follows from the properties of an optimal solution to a linear programming problem.[22] This completes the proof of Part II and of (52).

Since the assumption that A_F was square and nonsingular was used in the proof of (52) but is not necessary for (31), the latter applies to a more general model than the former. The assumption of a square A_F is limiting in that it implies that the model applies only to an economic system in which the number of production activities and the number of final commodities are equal. In many actual economic systems the number of possible production activities will greatly exceed the number of final commodities produced.

The importance of the square and nonsingular assumption may be given a more interesting interpretation in economic terms by noting its implications with respect to the aggregation of activities into final commodities and the disaggregation of a vector of final commodities into production activities. The matrix, A_F, can always (i.e., even when A_F is not square) be viewed as providing the weighting

[22]See Goldman and Tucker (1956; corollary 2b, p. 62).

coefficients for the aggregation of activities into a unique vector of final commodities. However, only when A_F is square and nonsingular so that A_F^{-1} exists can we, in general, disaggregate a vector of final commodities into a unique bundle of basic activities, x (i.e., $x=A_F^{-1} y_F$). In the more general case it may be possible to disaggregate a given vector of final commodities into a number of basic activity bundles and conversely a number of combinations of basic activities may be aggregated into the same vector of final commodities.

Since the problems of determining efficient vectors and corresponding price vectors may be characterized as linear programming problems, standard computational routines may be employed. However, for large economic systems the problem of obtaining all the necessary information (i.e., the values for the technology matrix, A, and the levels of primary resource availability η) at one central location may be very costly or even impossible. Even if all the necessary information can be obtained it may exceed the size of available computational equipment. The decomposition processes such as those discussed in Chapter II would seem to provide the most promising possibilities for surmounting these difficulties in situations where actual numerical solutions are desired.

Decentralization Under a Noncompetitive Pricing Rule

Certain more theoretical questions concerning the possibility of decentralizing economic decisions may be posed within the context of a linear activity analysis model such as that outlined here. It is upon certain of these questions that we will focus at this point. Specifically we will examine the possibility of developing certain decentralized pricing rules which will sustain an efficient point once it has been attained. The important questions concerning decentralized dynamic price-quantity adjustment processes and the conditions under which they are stable or converge to an efficient point when starting from a point which is not efficient will not be discussed, even though these questions have received much attention in the literature. Important examples are Arrow and Hurwicz (1960) and the survey article by Negishi (1962).

The linear activity analysis model can be viewed as decentralized with respect to technological information and decision-making. Assume that each column of the A matrix is known to only one decision-maker who is also informed of prices relative to his activity but has no information about other activities. Assuming such a decentralized economic organization, Koopmans (1951a, p. 93) has shown that the following proposal will maintain an efficient point once it has been obtained:

Let the players in our allocation game be called helmsman (or central planning board), a custodian for each commodity, and a manager for each activity. Consider the following rules of behavior:

I. For the helmsman: Choose a vector p_{fin}[23] of positive prices on all final commodities, and inform the custodian of each such commodity of its price.

II. For all custodians: Buy and sell your commodity from and to managers at one price only, which you announce to all managers. Buy all that is offered at that price. Sell all that is demanded up to the limit of availability.

III. For all custodians of final commodities: Announce to managers the price set on your commodity by the helmsman.

IV. For all custodians of intermediate commodities: Announce a tentative price on your commodity. If demand by managers falls short of supply by managers, lower your price. If demand exceeds supply, raise it.

V. For all custodians of primary commodities: Regard the available inflow from nature as a part of the supply of your commodity. Then follow the rule on custodians of intermediate commodities, with the following exception: Do not announce a price lower than zero but accept a demand below supply at a zero price if necessary.

VI. For all managers: Do not engage in activities that have negative profitability. Maintain activities of zero profitability at a constant level. Expand activities of positive profitability by increasing orders for the necessary inputs with, and offers of the outputs in question to, the custodians of these commodities.

This set of pricing rules can be considered a restatement of (31). If these rules are followed then an attainable bundle and a price, p, satisfying conditions 1-4 of (31) will be maintained, which is a necessary and sufficient condition for efficiency. Koopmans (1951a, p. 95) has discussed the relation of these rules to the competitive bidding process of

[23] p_{fin} corresponds to p_F in our notation.

competitive markets. For example note the following quote:

> The reader will have noticed that the behavior prescribed from individuals by the rules I-VI is similar to that which results from the operation of competitive markets. The rules on the custodians are only personalizations of the properties of competitive markets. The vector p_{fin}, which ultimately gives direction to the allocation of resources in production, instead of being set by a helmsman, could equally well be the result of competitive bidding by many consumers, each of which maximizes his individual utility. The behavior attributed to each manager could also come about as the result of each activity being carried out independently by many entrepreneurs bidding competitively for the input commodities of that activity and selling its output commodities competitively.

The fact that such a process will maintain an efficient output vector is proposed as support of the long-standing belief that a competitively organized economic system will produce efficiently. Such a process assumes that decision-makers are price-takers rather than price-setters; an assumption which is insured if no coalitions are formed and if the number making decisions with respect to any given activity or commodity is large.

Below we will indicate certain revisions of rules I-VI such that an efficient vector of final commodities will be maintained even though noncompetitive elements are present in the system. The term noncompetitive is used here in the limited special sense that certain primary commodities are assumed to be controlled by an individual or a coalition of individuals who may thus act as price-setters rather than price-takers. It will be assumed that a minimum positive price is set for primary commodities and any amount up to the limit of availability will be sold at this price. If the amount demanded at this minimum price exceeds the amount available, then a competitive type rule of increasing the price until the amount demanded equals the maximum amount available will be followed. We will replace rule V for custodians of primary commodities by the above noncompetitive, minimum price-setting type behavior, letting the minimum price be designated by the vector λ' and chosen such that $e' > -\lambda' A_P A_F^{-1}$. We will add to rules I and III the constraint that the price of final commodities should never be less than unity (i.e., $p_F \geq e$).[24] Then by noting the relation of (52) and this revised set of pricing rules, it is evident that the revised set will maintain an efficient final output vector. If we let $\lambda=0$, the revised rules are identical to Koopmans'.

The possibility of defining prices such that primary commodities not used to the level of their availability may continue to have a positive price would seem to be a step in the direction of reality. As simple examples, observe that theaters charge for tickets even though the theater

---

[24]Actually the unity vector, e, is used here merely because it was convenient in the proof of (51). All that is required is that $p_F > -\lambda' A_P A_F^{-1}$. No generality is lost since for any price, p^*, satisfying the conditions of (52), a new price, $p^{**} = \gamma p^*$, where γ is a positive scalar, will satisfy the same conditions with γ replaced by $\gamma\lambda$.

remains unfilled, and that laborers command positive wages even when a large amount of unemployment exists. Finally, to quote Nikaidô (1964, p. 299), "The above competitive situation is certainly unrealistic, because zero prices are imputed to factors not fully employed there, while in reality such factors seldom become free goods."

The setting of the level of the minimum price, λ, could correspond to a number of real world situations. The monopolist with exclusive control over a primary commodity is an obvious example. Probably of more importance are the numerous economic and political coalitions which are formed and which result in the setting of minimum prices, implicit or explicit. Labor union contracts, minimum wage and fair pricing laws, agricultural support prices, and understandings between businessmen about not "ruining the market" are but a few important examples. The study of how such coalitions form and the method by which final agreement is reached is quite interesting in itself; however, it would require a somewhat different analytical approach (e.g., game theory) and will not be discussed or reviewed here. We will be content to assume that agreement on a λ can be reached.

It is quite important, however, to note that the level at which λ is set is also limited by the formal conditions necessary for maintaining an efficient final commodity vector. The condition that prices of final commodities must be strictly greater than $-\lambda' A_P A_F^{-1}$ must be fulfilled to maintain efficiency. The economic meaning of this condition becomes clear once the dimensions of the expression $-\lambda' A_P A_F^{-1}$ have been ascertained. First, take $\lambda' A_P$ which is a 1xn vector and the dimensions of element j will be dollars/jth activity, since λ has dimensions of dollars/ith input and the jth column of A_P has elements with dimensions of ith input/jth activity. Since $x=A_P^{-1}y_F$, it is clear that the kth column of A_F^{-1} has elements with dimensions of jth activity/kth final commodity. So finally, the dimensions of the 1xn vector $-\lambda' A_P A_F^{-1}$ will be dollars/kth final commodity, and the vector values will simply be the value of primary commodities (priced at the minimum level λ) per unit of kth final commodity. Thus the condition $p_F'+\lambda' A_P A_F^{-1}>0$ merely states that the price of final commodities must be strictly greater than the minimum value of primary commodities necessary to produce that final commodity. If for the kth element this condition does not hold and the kth element is negative, then production of the kth commodity will involve a loss, even when all primary commodities are priced at their minimum level, and it is clear that an efficient vector of final commodities would not be maintained. A related discussion concerning the numerical example is given below.

Once λ has been set "low enough" so that the condition, $p_F'+\lambda' A_P A_F^{-1}>0$, is satisfied, the decentralized decision-making with decentralized information can be carried out. However, the level that is "low enough" for each element of λ is not independent of the values set for the other elements of λ . In fact, it is apparent that complete centralization of information (i.e., knowledge of A_P and A_F as well as λ and p_F) would be necessary to determine values of λ so that the condition would be satisfied. We conclude that a decision-making process which will sustain an efficient point when noncompetitive price-setters are present cannot be decentralized to the same degree possible for the competitive counterpart.

Implications for the Theory of Second Best

The result of (52) and the revised noncompetitive pricing rules would appear to have implications for the "General Theory of Second Best" as outlined by Lipsey and Lancaster (1957, p. 12).

> The general theorem for the second best optimum states that if there is introduced into a general equilibrium system a constraint which prevents the attainment of one of the Paretian conditions, the other Paretian conditions, although still attainable, are, in general, no longer desirable. In other words, given that one of the Paretian optimum conditions cannot be fulfilled, then an optimum situation can be achieved only by departing from all other Paretian conditions.

Koopmans (1957, p. 95) has clearly outlined the possibility of considering the linear activity analysis model as describing production in a general equilibrium model. Assuming convex preference orderings and nonsaturation of at least one consumer for each final commodity, all Pareto optimal points would be efficient points. Thus for this simplified model the "Paretian conditions" can be considered to be the price conditions in (31) and rules I-VI. The relation of these competitive pricing rules and the "Lerner-Lange" type rules has been thoroughly discussed by Koopmans (1951a, p. 95; 1951b). Lipsey and Lancaster (1957, p. 17) clearly indicate the intended application of their theorem to such situations in the following quote: "A nationalized industry conducting its price-output policy according to the Lerner-Lange 'Rule' in an imperfectly competitive economy may well diminish both the general productive efficiency of the economy and the welfare of its members."

For the simple linear activity analysis model, which admittedly leaves much to be desired in terms of representing a realistic general equilibrium system, the general theorem for the second best apparently does hold. In (52) the condition 4b for $\lambda_i > 0$ clearly "prevents the attainment of one of the Paretian conditions" (i.e., condition 4b of (31)); nevertheless, the other conditions (i.e., 1-3) remain necessary for a Paretian optimal which in this case reduces to being an efficient point. We have indicated how the strictly positive λ_i values can be attributed to monopolistic power or imperfectly competitive situations. The fact that the theorem does not hold for this model is apparent in that the pricing rules I-VI, when revised for the presence of monopolistic power, retained all the rules for the competitive case, except for rule V involving custodians who set minimum price levels. Not only are the other rules still "desirable" they are necessary for an efficient output vector and if $p_F^* + \lambda' A_P A_F^{-1} > 0$ is also fulfilled they are sufficient conditions.

Numerical Examples

In order to illustrate the relationship between efficient points and the alternative pricing systems, a number of computations have been made using two very simple linear activity analysis models.[25] Written in a format corresponding to (30) the first model is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -3 & -2 & 0 & 0 \\ -5 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{matrix} = \\ = \\ = \\ = \\ = \\ = \\ = \end{matrix} \begin{bmatrix} y_{F1} \\ y_{F2} \\ y_{I1} \\ y_{I2} \\ y_{P1} \\ y_{P2} \\ y_{P3} \end{bmatrix} \begin{matrix} \geqq \\ \geqq \\ = \\ = \\ \geqq \\ \geqq \\ \geqq \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -12 \\ -10 \\ -9 \end{bmatrix} \qquad (53)$$

$x,\ y_F \geqq 0,\ y_P \leqq 0.$

---

[25]The first model used was taken from Charnes and Cooper (1961, p. 292) in order to help facilitate comparisons. It is somewhat unfortunate that the intermediate commodities do not play a significant role in the model.

The linear programming problem (54) which corresponds to (50) was used to perform the actual computations:

$$
\left[\begin{array}{cc|cc|ccc|ccc|cc}
1 & 0 & -1 & 0 & -3 & -5 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{array}\right]
\begin{bmatrix} u_{F1} \\ u_{F2} \\ u_{I1} \\ u_{I2} \\ u_{P1} \\ u_{P2} \\ u_{P3} \\ w_{P1} \\ w_{P2} \\ w_{P3} \\ t_{F1} \\ t_{F2} \end{bmatrix}
\begin{array}{c} \leq \\ \\ \\ \\ = \\ \\ = \\ \\ \\ \leq \\ \\ \leq \\ \\ \\ \end{array}
\left[\begin{array}{c}
0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline -1 \\ -1 \\ \hline -\lambda_1^i \\ \lambda_2^i \\ -\lambda_3^i
\end{array}\right]
\tag{54}
$$

$$u_F \qquad u_I \qquad u_F \qquad w_P \qquad t_F$$

$w_P \geq 0$

where

$$
\hat{y}_F^{\,1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \hat{y}_F^{\,2} = \begin{bmatrix} 1.50 \\ 3.75 \end{bmatrix}, \quad \hat{y}_F^{\,3} = \begin{bmatrix} .0 \\ 4.5 \end{bmatrix}, \quad \hat{y}_F^{\,4} = \begin{bmatrix} .00 \\ 4.25 \end{bmatrix}, \quad \hat{y}_F^{\,5} = \begin{bmatrix} .00 \\ 4.75 \end{bmatrix}
$$

and

$$
\lambda^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \lambda^2 = \begin{bmatrix} .13 \\ .00 \\ .00 \end{bmatrix}, \quad \lambda^3 = \begin{bmatrix} .13 \\ .02 \\ .02 \end{bmatrix}
$$

Note that when $\lambda=0$ the problem corresponds to (33). The first model, corresponding to (53) and (54), involves no joint production but the matrix, A_F, is altered in the second model, (55), so that the first two activities jointly produce both final commodities:

Problem (55) is identical to problem (54) except for the upper left submatrix which is as follows:

$$\begin{bmatrix} 1 & .8 \\ .3 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{55}$$

where

$$\hat{y}_F^{\,a} = \begin{bmatrix} 2.9 \\ 4.6 \end{bmatrix}, \quad \hat{y}_F^{\,b} = \begin{bmatrix} 2.625 \\ 4.950 \end{bmatrix}, \quad \hat{y}_F^{\,c} = \begin{bmatrix} .0 \\ 4.5 \end{bmatrix}$$

and

$$\lambda^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \lambda^3 = \begin{bmatrix} .13 \\ .02 \\ .02 \end{bmatrix}.$$

The efficient final outputs for the first model consist of the points $\hat{y}_F^{\,1}$, $\hat{y}_F^{\,2}$, $\hat{y}_F^{\,3}$; those points on the straight line joining $\hat{y}_F^{\,1}$ to $\hat{y}_F^{\,2}$; as well as the points on the line joining $\hat{y}_F^{\,2}$ to $\hat{y}_F^{\,3}$. These efficient final outputs are plotted in Figure 4. The efficient final outputs for the joint production model are plotted in Figure 5 and consist of $\hat{y}_F^{\,a}$, $\hat{y}_F^{\,b}$ and the points on the straight line joining these points. The results of the

Table 19. Solutions to problem (54)

$\hat{y}_F$:		Primary Commodities 1	2	3	Final Commodities 1	2
$\hat{y}_F^{\ 1}$	λ^1	.00	.00	.00		
	p	.50	.00	.00	1.50	1.00
	y	-12.00	-10.00	-8.00	2.00	3.00
	$\eta - y_P$	.00	.00	-1.00		
$\hat{y}_F^{\ 1}$	λ^3	.13	.02	.02		
	p*	.48	.02	.02	1.56	1.00
	y	-12.00	-10.00	-8.00	2.00	3.00
	$\eta - y_P$	.00	.00	1.00		
$\hat{y}_F^{\ 2}$	λ^1	.00	.00	.00		
	p	.50	.00	.00	1.50	1.00
	y	-12.00	-7.50	-9.00	1.50	3.75
	$\eta - y_P$	.00	-2.50	.00		
$\hat{y}_F^{\ 2}$	λ^3	.13	.02	.02		
	p*	.13	.02	.51	1.00	1.28
	y	-12.00	-7.50	-9.00	1.50	3.75
	$\eta - y_P$	.00	-2.50	.00		
$\hat{y}_F^{\ 3}$	λ^1	.00	.00	.00		
	p	.00	.00	1.00	1.00	2.00
	y	-9.00	.00	-9.00	.00	4.50
	$\eta - y_P$	-3.00	-10.00	.00		

Table 19 (Continued)

$\hat{y}_F$:		Primary Commodities 1	2	3	Final Commodities 1	2
$\hat{y}_F{}^3$	λ^2	.13	.00	.00		
	p*	.13	.00	.61	1.00	1.48
	y	-9.00	.00	-9.00	.00	4.50
	$-y_P$	-3.00	-10.00	.00		
$\hat{y}_F{}^3$	λ^3	.13	.02	.02		
	p*	.13	.02	.51	1.00	1.28
	y	-9.00	.00	-9.00	.00	4.50
	$\eta-y_P$	-3.00	-10.00	.00		
$\hat{y}_F{}^4$	λ^1	.00	.00	.00		
	p	.00	.00	1.00	1.00	2.00
	y	-10.00	-2.50	-9.00	.50	4.25
	$\eta-y_P$	-2.00	-7.50	.00		
$\hat{y}_F{}^4$	λ^2	.13	.00	.00		
	p*	.13	.00	.61	1.00	1.48
	y	-10.00	-2.50	-9.00	.50	4.25
	$\eta-y_P$	-2.00	-7.50	.00		
$\hat{y}_F{}^4$	λ^3	.13	.02	.02		
	p*	.13	.02	.51	1.00	1.28
	y	-10.00	-2.50	-9.00	.50	4.25
	$\eta-y_P$	-2.00	-7.50	.00		

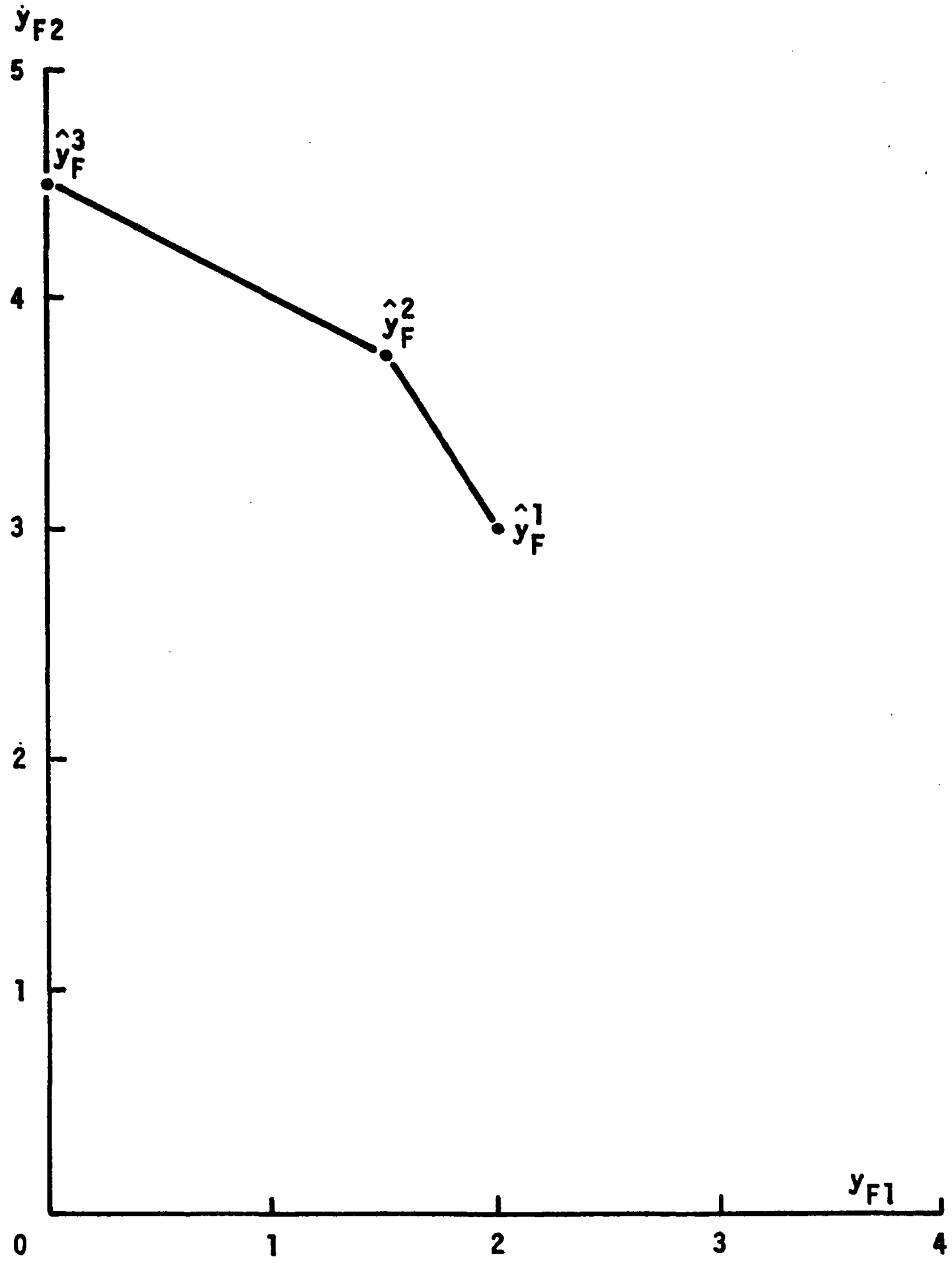

Figure 4. Efficient points for problem (54)

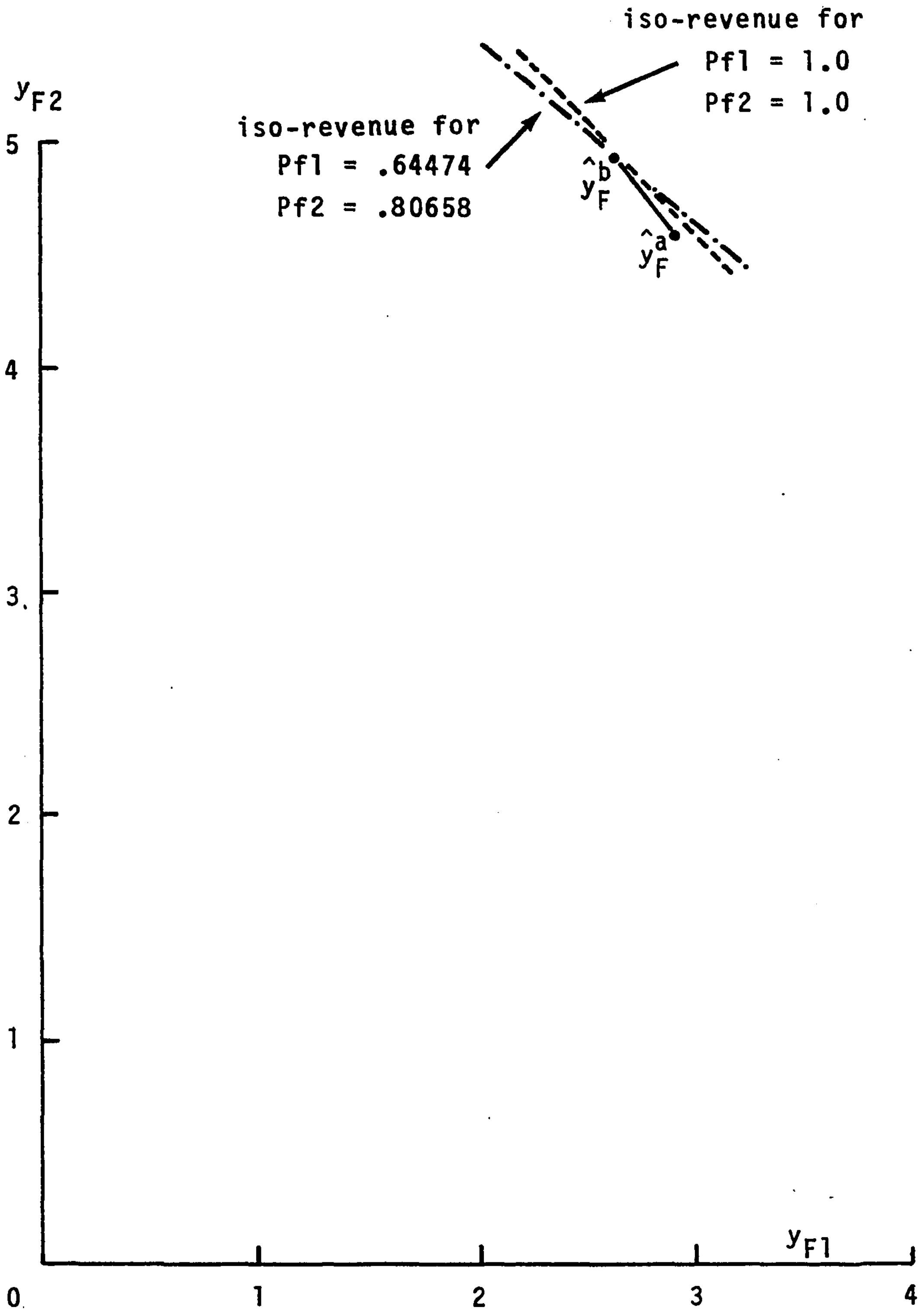

Figure 5. Efficient points for problem (55)

Table 20. Solutions to problem (55)

$\hat{y}_F$		Primary Commodities			Final Commodities	
		1	2	3	1	2
$\hat{y}_F{}^a$	λ^1	.000	.000	.000		
	p	.691	.000	.000	1.273	1.000
	y	-12.000	-10.000	-8.000	2.900	4.600
	$\eta - y_P$	0	0	-1		
$\hat{y}_F{}^a$	λ^3	.130	.020	.020		
	p*	.687	.020	.020	1.382	1.000
	y	-12.000	-10.000	-8.000	2.900	4.600
	$\eta - y_P$	.000	.000	-1.000		
$\hat{y}_F{}^b$	λ^1	.000	.000	.000		
	p	.691	.000	.000	1.273	1.000
	y	-12.000	-7.500	-9.000	2.625	4.950
	$\eta - y_P$	.000	-2.500	.000		
$\hat{y}_F{}^b$	λ^3	.130	.020	.020		
	p*	.525	.020	.125	1.000	1.000
	y	-12.000	-7.500	-9.000	2.625	4.950
	$\eta - y_P$	.000	2.500	.000		
$\hat{y}_F{}^c$	λ^1	.000	.000	.000		
	p	.575	.000	.075	1.000	1.000
	y	-12.000	-7.500	-9.000	2.625	4.950
	$\eta - y_P$	.000	-2.500	.000		

Table 20 (Continued)

$\hat{y}_F$:		Primary Commodities 1	2	3	Final Commodities 1	2
$\hat{y}_F{}^c$	λ^2	.130	.020	.020		
	p*	.525	.020	.125	1.000	1.000
	y	-12.000	-7.500	-9.000	2.625	4.950
	$\eta - y_P$	.000	-2.500	.000		

solutions are given in Tables 19 and 20 respectively. The t vector of (54) and (55) gives the prices for final commodities, p_F; w_p gives the primary commodity prices, p_P, and u_I the intermediate commodity prices p_I. However, for this particular model $p_I=0$ for all solutions and thus is not given in the tables. The dual variables for (54) and (55) correspond to activity levels and commodity values. With reference to the partitioned right hand side vector of (54) dual values corresponding to the next three partitions given values for y_F, y_P, $\bar{y}_F$ respectively, and when $\lambda_j>0$ those corresponding to the last partition give values for $s_j=y_{Pj}+\eta_j$. Thus from the optimal simplex tableau, all of the information present in Tables 19 and 20 may be obtained.

All of the numerical results conform to the theoretical results of (31) and (51) as of course they must unless there is a mistake in the theory. Those problems with $\lambda^i=0$ and $\hat{y}_F{}^i$ efficient result in prices satisfying the conditions in (31).

Those with $\lambda^i \geq 0$ and $\hat{y}_F{}^i$ efficient result in prices satisfying the conditions in (52). Those with $\hat{y}_F{}^i$ which are not attainable result in unbounded solutions to problems (54) and (55). This is as expected since an unattainable $\hat{y}_F{}^i$ vector results in the dual linear programming problems of (54) and (55) being infeasible. The results of these unbounded solutions are not recorded in the tables. Finally, those problems which were run with $\hat{y}_F{}^i$ which are attainable but not efficient (i.e., $\hat{y}_F{}^4$ for the first example and $\hat{y}_F{}^c$ for the second example) have positive $\bar{y}_F$ elements such that $y_F=\hat{y}_F+\bar{y}_F$ is efficient, and it is this vector of values which is given in the tables along with corresponding price vectors.

The pricing solutions given in the tables are not necessarily unique. Any positive scalar multiple of the competitive price vectors in Table 19 will also be competitive price vectors for the corresponding efficient vector of final commodities. Any positive scalar multiple of the noncompetitive prices given in Table 20 will satisfy our conditions for a noncompetitive price vector if the corresponding minimum price vector level is multiplied by the same positive scalar. There often exist other price vectors which differ by more than a scalar constant. For example, alternative price vectors for the problems with $\hat{y}_F{}^2$, λ^1; $\hat{y}_F{}^2$, λ^2; and $\hat{y}_F{}^b$, λ^1 are respectively:

$$p_P=\begin{bmatrix}0\\0\\0\end{bmatrix},\ p_F=\begin{bmatrix}1\\2\end{bmatrix},\ p_P^*=\begin{bmatrix}.13\\.00\\.61\end{bmatrix},\ p_F=\begin{bmatrix}1.00\\1.48\end{bmatrix},\ \text{and } p_P=\begin{bmatrix}.395\\.000\\.105\end{bmatrix},\ p_F=\begin{bmatrix}.64474\\.80658\end{bmatrix}.$$

To gain a better understanding of the economic meaning of the vector quantity $m'=\lambda' A_P A_F^{-1}$ and its relationship to the noncompetitive and competitive price vectors for a given commodity vector we will compare the price vectors for $\hat{y}_F{}^b$. The noncompetitive vector for $\lambda^{3'}=[.13, .02, .02]$ is given in Table 20 as $[p_{Pi}^{*'}\ p_F^{*'}] = [.525, .02, .125, 1.0, 1.0]$. The corresponding competitive price for $\hat{y}_F{}^b$ is $[p_P'|p_F'] = [.395, 0, .105, .64474, .30658]$. We can see that these two price vectors are related through the vector $[\lambda'|m']$ in the following manner:

$$[p_P^{*'}|p_F^{*'}] + [-\lambda'|m'] = [p_P'|p_F'] .$$

For our specific example we have:

$$[.525,.02,.125,1,1] + [-.13,-.02,-.02,-.35526,-.19342]$$

$$= [.395,0,.105,.64474,.80658]$$

since

$$m'=\lambda' A_P A_F^{-1}= [.13,.02,.02]\begin{bmatrix}-3 & -2\\-5 & 0\\-1 & -2\end{bmatrix}\begin{bmatrix}1 & -.3\\-.8 & 1\end{bmatrix}\frac{1}{.76}$$

$$= [-.35526,-.19342] .$$

Thus m represents a kind of "lump sum" amount which measures the difference between the noncompetitive and competitive pricesof final commodities. It is that portion of the value of final commodities which is imputed to primary commodities as a result of the ability of primary commodity custodians to set a minimum price level λ even before any type of maximization process takes place. But the levels of the various elements of λ are not unimportant. First of all, as was discussed above, λ must be small enough so that $p_F+\lambda' A_P A_F^{-1}>0$, if we desire that an efficient output vector will result. And the particular values for λ affect not only absolute but also relative prices of final commodities. Note the different slopes of the iso-revenue lines in Figure 5. These different lines correspond to the noncompetitive and competitive prices given above for efficient point $\hat{y}_F^b$.

Up to this point in the discussion we have implicitly assumed that the economic system corresponding to the model was a closed system and that no possibility existed for exchange of commodities with some other economic system. Koopmans (1951a, p. 91) has allowed for the possibility of exchange by augmenting the existing model with an exchange matrix, Π, of the following form:

$$\Pi = \begin{bmatrix} -\Pi_2 & -\Pi_3 & -\Pi_4 & \cdots & -\Pi_n \\ \Pi_1 & 0 & \cdots & & 0 \\ 0 & \Pi_1 & & & \vdots \\ \vdots & & \Pi_1 & & 0 \\ 0 & \cdots & & 0 & \Pi_1 \end{bmatrix}.$$

This matrix defines a set of relative exchange rates or prices, $\Pi=(\Pi_1 \ldots \Pi_n)$, between commodity one and all other commodities. Commodity one can be considered the <u>numéraire</u> and Π_1 must be strictly positive for the matrix to have any meaning.

This exchange matrix Π is added to the former model in such a way that the dual linear programs (49) and (50) would be enlarged as follows:

$$\begin{array}{lllllll} & \min & -e'\bar{y}_F - \lambda' s & & & = 0 & \\ \text{subject to} & \Pi_F\xi + A_F x - y_F & & & & = 0 & \qquad (56) \\ & \Pi_I\xi + A_I x & & & & = 0 & \\ & \Pi_P\xi + A_P x & -y_P & & & = 0 & \\ & & y_P & & -s & \geq \eta & \\ & & y_F & -\bar{y}_F & & = \hat{y}_F & \\ & x,\ \bar{y}_F,\ s \geq 0 & & & & & \end{array}$$

$$
\begin{array}{llll}
 & \max\ \eta' w_P + \hat{y}_F t_F & & \\
\text{subject to} & \Pi_F' u_F + \Pi_I' u_I + \Pi_P' u_P & = 0 & (57) \\
 & A_F' u_F + A_I' u_I + A_P' u_P & \leq 0 & \\
 & -u_F \qquad\qquad\qquad + t_F & = 0 & \\
 & \qquad\quad -u_P + w_P & = 0 & \\
 & \qquad\qquad\qquad\quad -t_F & \leq -e_F & \\
 & \qquad\qquad\quad w_P & \leq -\lambda & \\
 & w_P \geqq 0 \ . & &
\end{array}
$$

Koopmans (1951a, p. 93) has proven that a necessary and sufficient condition that an efficient vector remain efficient after such an exchange matrix has been added is that the exchange prices, Π, be competitive prices. Charnes and Cooper (1961, p. 318) have shown that for $u_1 > 0$ the following relation must hold between the vectors Π and u:

$$\frac{\Pi_j}{\Pi_1} = \frac{u_j}{u_1} \ .$$

This follows from noting that the initial constraints of problem (57) give $-\Pi_j u_1 + \Pi_1 u_j = 0$ for j=2,n.

We are interested in what happens to a system with the noncompetitive imputed prices once exchange opportunities with an outside economic system are introduced. It is immediately obvious that Koopman's results for competitive prices will not carry over. We merely note that with noncompetitive prices it is possible for primary commodities to have positive prices even when not used to the extent of the availability. Thus with exchange opportunities corresponding to such a price it would be possible to continue producing the same final output as before and exchange the unused quantity of primary commodities for a positive amount of some final commodity, so the old commodity vector would no longer be efficient.

Four different numerical examples using $\hat{y}_F{}^b$ were run using two exchange matrices corresponding to the competitive and noncompetitive price vectors for λ^1 and λ^3 (see Table 20). The results of these solutions are given in Table 21. The first example corresponds to Koopmans' case and gives the expected result (i.e., that $\hat{y}_F{}^b$ remains efficient). The second example using exchange prices, Π, equal to the competitive price vector, p, but a nonzero λ results in an infeasible solution for the maximization problem (57) and will be unbounded for (56). The fact that (56) will be unbounded is immediately obvious when we note that primary commodities 2 and 3 may be obtained through the exchange matrix in unlimited quantities to increase the values of s_2 and s_3 which both have values of -.02 in the objective function. The last two examples are of more interest. They show that using exchange prices, Π, equal to noncompetitive prices it is possible to obtain no less of y_{F2} and a larger amount of y_{F1}

Table 21. Solution to problem (55) with exchange matrix added

$\hat{y}_F$:		Primary Commodities 1	2	3	Final Commodities 1	2
$\hat{y}_F{}^b$	Π^1	.691	.000	.000	1.000	1.273
	λ^1	.000	.000	.000		
	p	.691	.000	.000	1.000	1.273
	y	-12.000	-7.500	-9.000	2.625	4.950
	$\eta - y_P$	.000	-2.500	.000		
$\hat{y}_F{}^b$	Π^1	.691	.000	.000	1.000	1.273
	λ^3	.130	.020	.020		
	p					
	y	Solution was infeasible				
	$\eta - y_P$					
$\hat{y}_F{}^b$	Π^3	.525	.020	.020	1.000	1.000
	λ^1	.000	.000	.000		
	p	.525	.020	.125	1.000	1.000
	y	-12.000	-10.000	-9.000	2.675	4.950
	$\eta - y_P$	.000	.000	.000		
$\hat{y}_F{}^b$	Π^3	.525	.020	.020	1.000	1.000
	λ^3	.130	.020	.020		
	p	.525	.020	.125	1.000	1.000
	y	-12.000	-10.000	-9.000	2.675	4.950
	$\eta - y_P$	.000	.000	.000		

than was possible without the exchange opportunities. Thus an efficient output will not necessarily remain efficient when exchange opportunities at noncompetitively imputed prices are added.

The above discussion is closely related to the fact that when noncompetitive pricing is present the well known correspondence between imputed prices and productivity of resources (Samuelson, 1958) is no longer valid. Nikaidô (1964, p. 300) has discussed this fact for a model in which the net value added due to each activity is given as data for the problem rather than obtained as a result of the maximization process.

To focus the discussion we will take as an example the efficient commodity vector $\hat{y}_F{}^b$ and noncompetitive price vector corresponding to λ^3 (see Table 20). If we increase the availability of the second primary commodity from -10 to -11 units there will be no increase in output since even -10 units results in an excess over the amount used. So for the second primary commodity the value of the final commodities resulting from one more unit is zero but the imputed price is strictly positive, (i.e., .02).

If, on the other hand, we increase the available amount of the first primary commodity from -12 to -12.1 there will be an increase in the final commodity vector. The amount of final commodity one will increase from 2.625 to 2.6675 and y_{F2} will increase from 4.95 to 4.965. The value of this increase is .9575 since the corresponding noncompetitive price vector does not change. The noncompetitive price of the first primary commodity is .525 or .0525 for .1 units. Thus the noncompetitive price of y_{P1} is clearly less than the value of its productivity in this case. The difference between the two values, .0525 and .0575, is imputed to the additional -.25 units of y_{P2} used, the noncompetitive price of which is .02 even though in neither case is y_{P2} used to the level of its availability. Since the introduction of noncompetitive price setting behavior to such a simple model results in clear deviations between prices and the value of marginal product, we concur with the statement of Nikaidô (1964, p. 301) that "marginal productivity can hardly account for factor price imputation in a more realistic situation."

CHAPTER VII. TOPICS FOR FURTHER RESEARCH

A number of possibilities for research beyond that which has been done in this study can be suggested. First of all, to actually determine the computational efficiency of the solution procedures proposed in Chapter II, these procedures need to be applied to a number of large scale decomposable programming models. By applying the original Dantzig-Wolfe algorithm to the same models it would be possible to determine whether the proposed procedures are successful in decreasing the number of major iterations.

Beyond the purely technical consideration of computational efficiency much more study is needed of the possibility of using such a decision process in an actual planning situation. Use of the decomposition principle to simulate a market adjustment process is an interesting possibility. While it is not entirely clear how one should proceed, it seems quite possible that by obtaining solutions for relevant parameterizations of the coefficients and applying regression analysis to the results, one might be able to obtain an interesting picture of the behavior of the modeled economic system.

There is of course the need to move beyond the restrictive static and deterministic linear models used. The possibility of relaxing these restrictions has received attention in the literature, but much more research in this direction seems to be needed. For example, the possibility of decomposing a large system and treating it as a group of linked subsystems has also been analyzed theoretically using the methods of control theory. Such work seems to hold promising possibilities for analyzing economic models which are essentially dynamic (Mesarović, Macko, and Takahara, 1970).

The next step in determining the usefulness of goal programming and efficiency criteria for university decision-making might be to obtain the opinion of relevant university decision-makers. It is important to know whether they feel that their objectives could be characterized in the form of goals or their decisions aided by knowledge of efficient alternatives open to them. Given that it is felt that such analysis could aid their decision-making, a much more comprehensive and disaggregated model would need to be constructed.

The results of Chapter VI apply to a very specific type of noncompetitive price setting embodied in a quite restrictive model. At the very least the possibility of relaxing the assumption of a square A_F matrix should be explored. A more interesting generalization would be an investigation of how the specific noncompetitive price setting rule would affect efficiency in nonlinear models satisfying the usual convexity assumptions.

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ACKNOWLEDGMENTS

I would like to express my gratitude to Professor Jati K. Sengupta, my adviser and co-major professor for the Ph.D. degree. He has contributed greatly to my knowledge of economics in general and has provided invaluable stimulation, guidance, and encouragement with respect to this disseration. I am also especially grateful for the helpful advice and ideas which I received from Professor Karl A. Fox who served as my other co-major professor.

The research was supported through NSF funds from projects directed by Professors Karl A. Fox and Jati K. Sengupta. Additional computational services were provided by the University of Pittsburgh. I am indeed grateful for both of these sources of support.

I wish to acknowledge my debt to the other members of my committee, Dr. Arnold M. Faden, Dr. Wayne A. Fuller, Dr. James R. Prescott, and Dr. George Zyskind. I deeply appreciate their time and patience.

I also wish to gratefully acknowledge the help and encouragement received from fellow graduate students at Iowa State University, especially Dr. Amitabha Sen and Dr. Edmond Seay, and from colleagues at the University of Pittsburgh.

Finally, I must thank my wife, Marcia, for her understanding help and encouragement throughout this project.

APPENDIX I

Efficient Output Vectors, y_F, for Departmental Model Given in Table 9

YUI	46800.000	46800.000	46800.000	46800.000	46800.000
YMI	4320.000	4320.000	4320.000	4320.000	4320.000
YDI	2160.000	2160.000	2160.000	2160.000	2160.000
YMT	82.477	77.012	83.478	84.984	88.889
YDD	28.854	30.000	30.628	33.333	33.333
YSRY	52.899	55.001	44.643	20.499	6.872
YUI	46800.000	46800.000	46800.000	46800.000	46800.000
YMI	4320.000	4320.000	4320.000	4320.000	4320.000
YDI	2160.000	2160.000	2160.000	2160.000	2160.000
YMT	69.834	80.000	80.000	86.557	88.889
YDD	30.000	24.494	10.492	10.492	11.658
YSRY	60.737	62.996	67.663	44.779	28.504
YUI	46800.000	46800.000	46800.000	46800.000	46800.000
YMI	4320.000	4320.000	3628.679	3469.615	3680.903
YDI	1457.712	1457.712	1457.712	1169.398	1288.366
YMT	88.889	82.222	82.222	82.222	80.000
YDD	33.333	30.000	30.000	30.000	30.000
YSRY	27.030	73.564	87.007	95.706	97.039
YUI	46800.000	28800.000	25210.000	25210.000	-2.000
YMI	-2.000	-2.000	-2.000	-2.000	-2.000
YDI	-2.000	-2.000	-2.000	-2.000	-2.000
YMT	80.000	80.000	80.000	80.000	80.000
YDD	30.000	30.000	30.000	22.058	23.697
YSRY	102.011	114.194	116.277	116.764	110.495
YUI	-2.000	25210.080	25210.080	28800.000	46800.000
YMI	-2.000	-2.000	-2.000	-2.000	-2.000
YDI	1292.325	1350.655	1543.954	1621.237	2073.375
YMT	80.000	80.000	80.000	80.000	80.000
YDD	24.102	22.482	30.000	30.000	30.000
YSRY	118.177	115.387	114.732	112.466	99.209
YUI	46800.000	46800.000	28800.000	25210.080	25210.080
YMI	-2.000	2552.620	3616.645	3798.539	4320.000
YDI	2160.000	2160.000	2160.000	2160.000	2160.000
YMT	80.000	80.000	80.000	80.000	80.000
YDD	30.000	30.000	30.000	30.000	30.000
YSRY	96.669	92.731	91.090	90.809	80.669

YUI	28800.000	28800.000	28799.990	46800.000	46800.000
YMI	4320.000	3798.538	3611.403	3368.745	3215.926
YDI	2160.000	2160.000	2160.000	2160.000	2160.000
YMT	80.000	80.000	82.855	86.557	88.889
YDD	27.152	27.152	27.152	10.492	0.000
YSRY	78.771	88.910	82.585	63.276	61.116
YUI	46800.000	28800.000	25210.080	25210.080	25210.080
YMI	4320.000	4320.000	4320.000	3652.885	3211.227
YDI	2160.000	2160.000	2160.000	2160.000	1359.464
YMT	88.889	82.855	82.222	82.222	82.222
YDD	-2.000	27.152	30.000	30.000	30.000
YSRY	29.647	68.807	72.914	85.886	110.039
YUI	25210.080	-2.000	-2.000	-2.000	-2.000
YMI	3422.515	3367.929	3299.512	3070.083	3156.641
YDI	1478.432	1379.490	1255.480	1168.615	1260.522
YMT	80.000	80.000	80.000	82.769	82.222
YDD	30.000	30.000	25.126	27.538	30.000
YSRY	111.373	114.358	114.850	112.945	113.025
YUI	-2.000	0.000	25210.080	28800.000	39511.530
YMI	3315.800	-2.000	-2.000	-2.000	-2.000
YDI	1250.243	1286.202	1342.850	1386.640	1539.078
YMT	80.000	80.000	80.000	80.000	80.000
YDD	30.525	30.636	30.811	30.946	31.417
YSRY	114.217	117.431	114.435	112.119	104.058
YUI	46800.000	46800.000	46800.000	46800.000	46800.000
YMI	0.000	0.000	0.000	2674.274	3895.985
YDI	1330.879	1712.279	2160.000	2160.000	1457.712
YMT	80.000	80.000	80.000	80.000	80.000
YDD	31.768	31.951	33.333	33.333	33.333
YSRY	99.211	94.899	71.223	67.098	66.287
YUI	46800.000	28800.000	25210.000	25210.000	28800.000
YMI	4320.000	4320.000	4320.000	3920.192	3916.658
YDI	1457.712	2057.472	2160.000	2160.000	2057.472
YMT	80.000	80.000	80.000	80.000	80.000
YDD	33.333	33.333	33.333	33.333	33.333
YSRY	58.052	57.497	57.402	65.176	65.340
YUI	28800.000	46800.000	46800.000	28800.000	25210.080
YMI	3334.045	3313.373	3522.394	3377.870	3353.164
YDI	2057.472	1457.712	895.367	1246.456	1306.486
YMT	88.889	88.889	80.000	80.000	80.000
YDD	33.333	33.333	31.598	30.830	30.699
YSRY	45.647	46.604	96.610	109.058	111.185

YUI	25.210	28800.000	46800.000	25210.080	25210.080
YMI	3091.785	3097.546	3131.249	3337.579	4320.000
YDI	1213.735	1146.994	756.571	2160.000	2160.000
YMT	83.048	83.269	84.561	88.889	88.889
YDD	30.413	30.523	31.169	33.333	33.333
YSRY	109.433	107.178	93.988	45.483	26.380

YUI	28800.000	46800.000	46800.000	46800.000	46800.000
YMI	4320.000	4320.000	3774.332	3680.903	3487.711
YDI	2057.472	1457.712	1457.712	1288.366	1432.756
YMT	88.889	80.000	80.000	80.000	80.000
YDD	33.333	30.000	30.000	30.000	15.347
YSRY	26.475	81.319	91.930	97.039	97.988

APPENDIX II

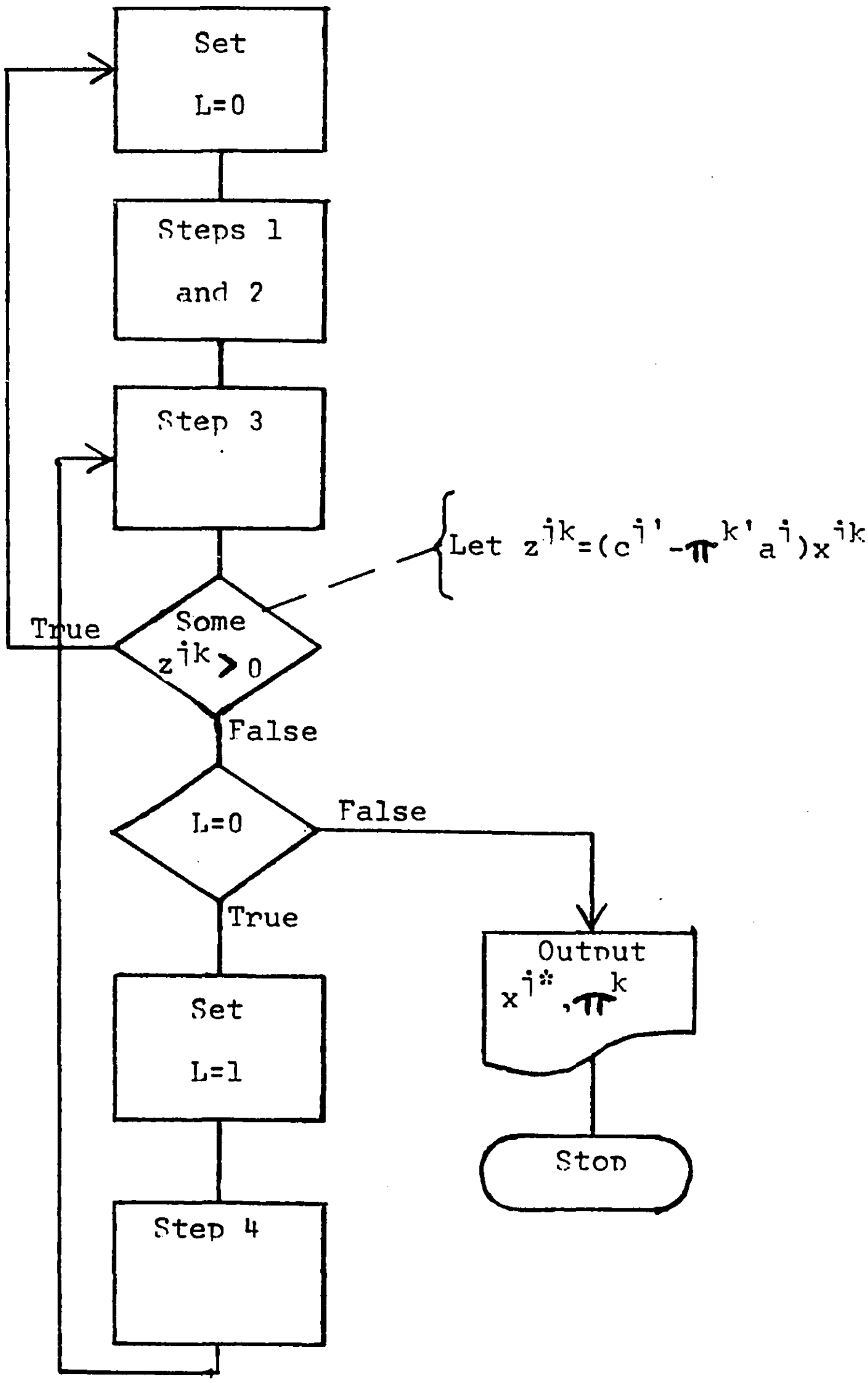

Figure 6. Flow chart for steps on pages 26 and 27

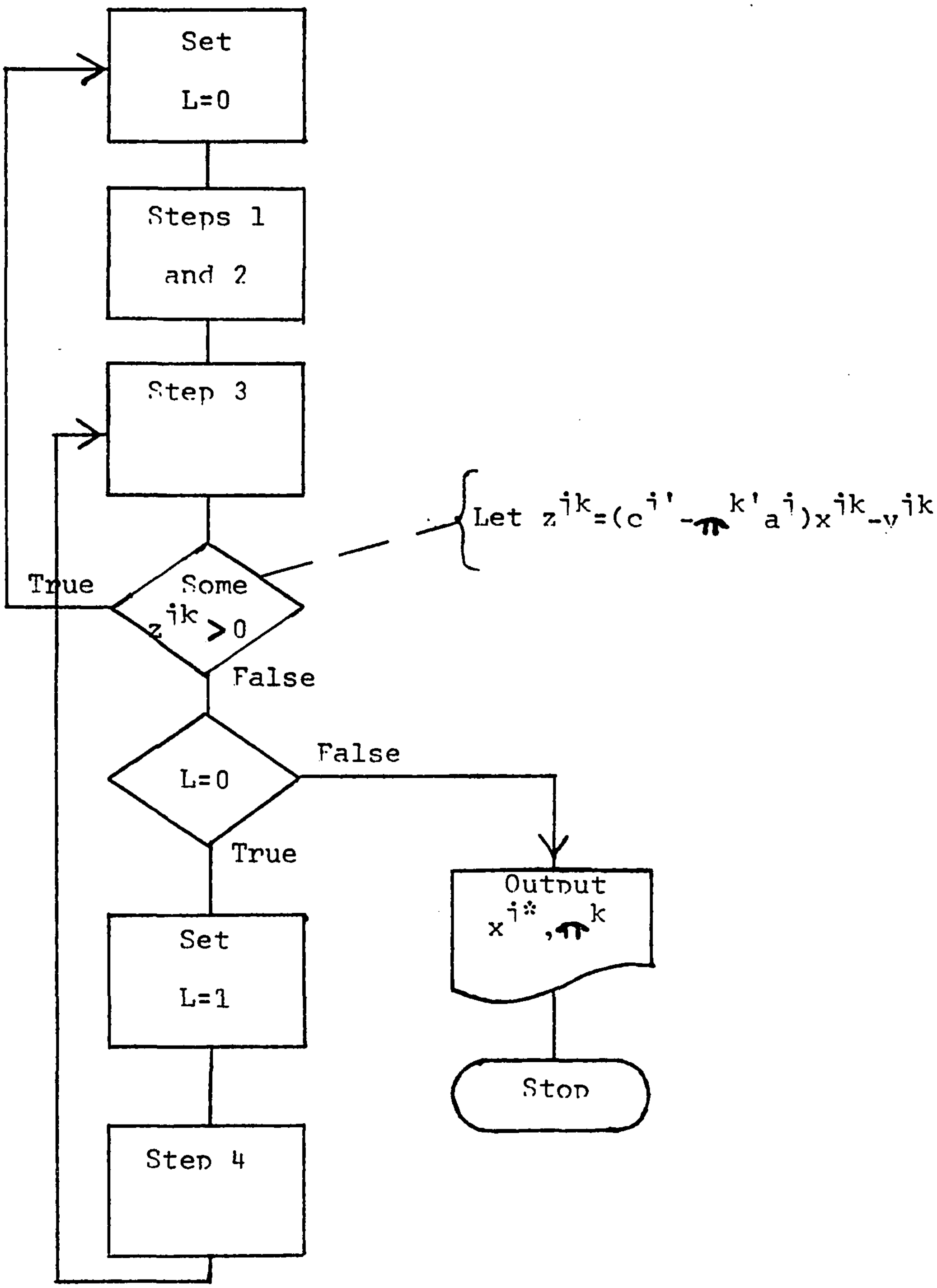

Figure 7. Flow chart for steps on pages 41 and 42