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# Potentially eventually exponentially positive sign patterns

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We introduce the study of potentially eventually exponentially positive (PEEP) sign patterns and establish several results using the connections between these sign patterns and the potentially eventually positive (PEP) sign patterns. It is shown that the problem of characterizing PEEP sign patterns is not equivalent to that of characterizing PEP sign patterns. A characterization of all $2 \times 2$ and $3 \times 3$ PEEP sign patterns is given.

## 1. Introduction

A matrix $A \in \mathbb{R}^{n\times n}$ is *eventually positive* if there exists a $k_0 \in \mathbb{Z}^+$ such that for all $k \geq k_0$, $A^k > 0$ (where the inequality is interpreted entrywise). A matrix $A$ is *eventually exponentially positive* if there exists some $t_0 \geq 0$ such that for all $t \geq t_0$,

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} > 0.$$

Eventually exponentially positive matrices have applications to dynamical systems in situations where it is of interest to determine whether an initial trajectory reaches positivity at a certain time and remains positive thereafter [Noutsos and Tsatsomeros 2008]. There is a characterization of eventual exponential positivity in terms of eventual positivity:

**Theorem 1.1** [Noutsos and Tsatsomeros 2008, Theorem 3.3]. *The matrix $A \in \mathbb{R}^{n\times n}$ is eventually exponentially positive if and only if there exists $a \geq 0$ such that $A + aI$ is eventually positive* (*where $I$ is the $n \times n$ identity matrix*).

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A *sign pattern* is a matrix having entries in $\{+, -, 0\}$. For a real matrix $A$, $\operatorname{sgn}(A)$ is the sign pattern having entries that correspond to the signs of the entries in $A$. If $\mathscr{A}$ is an $n \times n$ sign pattern, the *qualitative class* of $\mathscr{A}$, denoted $\mathscr{Q}(\mathscr{A})$, is the set of all $A \in \mathbb{R}^{n\times n}$ such that $\operatorname{sgn}(A) = \mathscr{A}$; such a matrix $A$ is called a *realization* of $\mathscr{A}$. A sign pattern $\mathscr{A}$ is *potentially eventually positive* (PEP) if there exists some realization $A \in \mathscr{Q}(\mathscr{A})$ that is eventually positive. PEP sign patterns were studied in [Berman et al. 2010], and we adapt several techniques from that paper to study potentially eventually exponentially positive sign patterns.

**Definition 1.2.** A sign pattern $\mathscr{A}$ is *potentially eventually exponentially positive* (PEEP) if there exists some realization $A \in \mathscr{Q}(\mathscr{A})$ that is eventually exponentially positive.

Since an eventually positive matrix is eventually exponentially positive, a PEP sign pattern is PEEP. Theorem 1.1 leads naturally to consideration of a sign pattern with positive diagonal entries.

**Definition 1.3.** Given an $n \times n$ sign pattern $\mathscr{A} = [\alpha_{ij}]$, we denote by $\mathscr{A}_{D(+)} = [\hat{\alpha}_{ij}]$ the $n \times n$ sign pattern such that $\hat{\alpha}_{ij} = \alpha_{ij}$ for $i \neq j$ and $\hat{\alpha}_{ii} = +$ for $i, j \in \{1, \dots, n\}$. $\mathscr{A}_{D(0)}$ and $\mathscr{A}_{D(-)}$ are defined analogously, with zero and negative diagonal, respectively.

In [Berman et al. 2010] it is noted that if $\mathscr{A}$ is PEP then $\mathscr{A}_{D(+)}$ is also PEP. This observation together with Theorem 1.1 leads to the following observation.

**Observation 1.4.** If $\mathscr{A}$ is a PEEP sign pattern, then $\mathscr{A}_{D(+)}$ is a PEP sign pattern (and hence $\mathscr{A}_{D(+)}$ is also PEEP).

Given a PEEP sign pattern, we can generate a PEP sign pattern by changing every diagonal element to $+$. However, taking a PEP sign pattern and changing $+$ diagonal entries to 0 or $-$ does not always yield a PEEP sign pattern. For example,

$$\mathscr{B}_{D(+)} = \begin{bmatrix} + & - & 0 \\ + & + & - \\ - & + & + \end{bmatrix} \tag{1}$$

is PEP [Berman et al. 2010], but in Example 2.3 below it is shown that the sign pattern

$$\mathscr{B}_{D(0)} = \begin{bmatrix} 0 & - & 0 \\ + & 0 & - \\ - & + & 0 \end{bmatrix} \tag{2}$$

is not PEEP. Thus the problem of determining which sign patterns are PEEP is not equivalent to the problem of determining which sign patterns are PEP.

Section 2 presents general results on PEEP sign patterns, including those obtained by perturbation analysis and connections with known results on PEP sign patterns.

At the end of Section 2 the open question of the minimum number of positive entries in an $n \times n$ PEEP sign pattern is discussed. In Section 3 small order PEEP sign patterns are characterized. The remainder of this section contains information on eventually exponentially positive matrices and terminology on digraphs and sign patterns.

The *spectrum* of $A$, denoted $\sigma(A)$, is the multiset of the eigenvalues of $A$. The *spectral radius* of $A$ is defined as $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ and an eigenvalue $\lambda \in \sigma(A)$ is a *dominant eigenvalue* if $|\lambda| = \rho(A)$. A nonzero vector $\boldsymbol{w}$ is called a *left eigenvector* of $A$ if $\boldsymbol{w}^T A = \lambda \boldsymbol{w}^T$ for some $\lambda \in \sigma(A)$ (or equivalently, $\boldsymbol{w}$ is a (right) eigenvector of $A^T$). The matrix $A$ is eventually positive if and only if $A$ has a unique dominant eigenvalue that is positive and simple, and $A$ has positive right and left eigenvectors for $\rho(A)$ [Handelman 1981] (this is called the *strong Perron–Frobenius test* for eventual positivity).

**Definition 1.5.** A real eigenvalue $\gamma \in \sigma(A)$ is called the *rightmost eigenvalue* if it is simple and for all $\lambda \in \sigma(A)$, $\lambda \neq \gamma$ implies $\mathrm{Re}(\lambda) < \gamma$, where $\mathrm{Re}(\alpha)$ denotes the real part of a complex number $\alpha$.

Not every matrix has a rightmost eigenvalue. Definition 1.5 was motivated by the following test for eventual exponential positivity, which is implicit in the proof of Theorem 3.3 in [Noutsos and Tsatsomeros 2008] (and also follows immediately from that theorem, which is Theorem 1.1 above).

**Proposition 1.6.** *Let $A \in \mathbb{R}^{n \times n}$. Then $A$ is eventually exponentially positive if and only if $A$ has a rightmost eigenvalue having positive left and right eigenvectors.*

An eventually positive matrix must have a positive entry in each row and column. This need not be the case for an eventually exponentially positive matrix (for example, an eventually exponentially positive matrix that realizes $\mathcal{B}_{D(-)}$ in (3) will not have a positive entry in each row and column). However, certain conditions on the eigenvalues require an eventually exponentially positive matrix to have a positive entry in each row and column.

**Proposition 1.7.** *Let $A$ be an eventually exponentially positive matrix.*

1. *If $A$ has an eigenvalue with nonnegative real part, then each row and column of $A$ has a positive entry.*
2. *If $A$ does not have an eigenvalue with positive real part, then each row and column of $A$ has a negative entry.*

*Proof.* If $A$ has an eigenvalue with nonnegative real part, then the rightmost eigenvalue $\gamma$ of $A$ is nonnegative. By Proposition 1.6, $A$ has positive right and left eigenvectors corresponding to $\gamma$. Suppose that row $k$ of $A$ has no positive entry. Since $A$ is an eventually exponentially positive matrix, $A$ is irreducible, so row $k$

has a negative entry. But then if $\boldsymbol{x} > 0$, $(A\boldsymbol{x})_k < 0$ and $(\gamma \boldsymbol{x})_k \geq 0$, so $\boldsymbol{x}$ is not a (right) eigenvector. Thus every row of $A$ has a positive entry. The result for column $k$ of $A$ is established with the left eigenvector. Similarly, if $A$ has no eigenvalue with positive real part, then $\gamma \leq 0$ and every row and every column of $A$ has a negative entry. □

A square sign pattern $\mathscr{A}$ (or matrix) is *reducible* if there exists a permutation matrix $P$ such that

$$P\mathscr{A}P^T = \begin{bmatrix} \mathscr{A}_{11} & 0 \\ \mathscr{A}_{21} & \mathscr{A}_{22} \end{bmatrix},$$

where $\mathscr{A}_{11}$ and $\mathscr{A}_{22}$ are nonempty square sign patterns (or matrices) and 0 is a (possibly rectangular) block consisting entirely of zero entries. If $\mathscr{A}$ is not reducible, then $\mathscr{A}$ is called *irreducible* (note any $1 \times 1$ matrix is irreducible). Since an eventually exponentially positive matrix must be irreducible, a PEEP sign pattern must be irreducible.

For an $n \times n$ sign pattern $\mathscr{A} = [\alpha_{ij}]$, the *digraph of* $\mathscr{A}$, denoted $\Gamma(\mathscr{A})$, has vertex set $\{1, \ldots, n\}$ and arc set $\{(i, j) : \alpha_{ij} \neq 0\}$. A nonnegative sign pattern $\mathscr{A}$ is *primitive* if $\mathscr{A}$ is irreducible and the greatest common divisor of the lengths of the cycles of $\Gamma(\mathscr{A})$ is one; for a nonnegative matrix the definition of primitive is analogous. It is well known that a primitive (necessarily nonnegative) matrix is eventually positive.

Let $\mathscr{A} = [\alpha_{ij}]$, $\hat{\mathscr{A}} = [\hat{\alpha}_{ij}]$ be sign patterns. If $\alpha_{ij} \neq 0$ implies $\alpha_{ij} = \hat{\alpha}_{ij}$, then $\mathscr{A}$ is a *subpattern* of $\hat{\mathscr{A}}$ and $\hat{\mathscr{A}}$ is a *superpattern* of $\mathscr{A}$. Define the *positive part* of $\mathscr{A}$ to be $\mathscr{A}^+ = [\alpha_{ij}^+]$, where

$$\alpha_{ij}^+ = \begin{cases} + & \text{if } \alpha_{ij} = +, \\ 0 & \text{if } \alpha_{ij} = 0 \text{ or } \alpha_{ij} = -. \end{cases}$$

Note $\mathscr{A}^+$ is a subpattern of $\mathscr{A}$.

## 2. PEEP sign patterns

In this section we establish general properties of PEEP sign patterns. Some of these results will be used in Section 3 to determine which sign patterns of order at most 3 are PEEP.

**Remark 2.1.** If $\mathscr{A}_{D(+)}$ is a PEP sign pattern, then $\mathscr{A}_{D(-)}$ is a PEEP sign pattern, because if $A \in \mathcal{Q}(\mathscr{A}_{D(+)})$ is eventually positive, there exists $t > 0$ such that $A - tI \in \mathcal{Q}(\mathscr{A}_{D(-)})$.

A PEP sign pattern must have a positive entry in each row and column. This need not be the case for an eventually exponentially positive matrix. The sign pattern

$$\mathcal{B}_{D(-)} = \begin{bmatrix} - & - & 0 \\ + & - & - \\ - & + & - \end{bmatrix} \tag{3}$$

is PEEP because the sign pattern $\mathcal{B}_{D(+)}$ in (1) is PEP. But $\mathcal{B}_{D(-)}$ does not have a $+$ entry in row 1 nor in column 3. If $A \in \mathbb{R}^{n\times n}$ is an eventually exponentially positive matrix with nonnegative trace, then $A$ has an eigenvalue with nonnegative real part. As a consequence of Proposition 1.7, we have the following observation.

**Observation 2.2.** If $\mathcal{A}$ is a PEEP sign pattern with no $-$ on the diagonal, then $\mathcal{A}$ has a $+$ in each row and column.

The next example shows that the problem of determining which sign patterns are PEEP is not equivalent to the problem of determining which sign patterns are PEP, because the fact that $\mathcal{A}_{D(+)}$ is PEP does not guarantee that $\mathcal{A}$ is PEEP.

**Example 2.3.** The sign pattern

$$\mathcal{B}_{D(0)} = \begin{bmatrix} 0 & - & 0 \\ + & 0 & - \\ - & + & 0 \end{bmatrix}$$

is not PEEP by Observation 2.2, because $\mathcal{B}_{D(0)}$ has no $-$ on the diagonal and no $+$ in row 1. Note that $(\mathcal{B}_{D(0)})_{D(+)} = \mathcal{B}_{D(+)}$ from (1) is PEP.

Related sign patterns are discussed in Corollary 3.4 and Theorem 3.5 below.

Matrix perturbations are used extensively in the study of potential eventual positivity. It is well known that for any matrix $A \in \mathbb{R}^{n\times n}$, the eigenvalues of $A$ are continuous functions of the entries of $A$. For a simple eigenvalue, the same is true of the eigenvector [Golub and Van Loan 1996, p. 323]. Because a matrix is eventually positive if and only if it passes the strong Perron–Frobenius test, eventual positivity is inherited by matrices that are small perturbations of eventually positive matrices. That is, if $A \in \mathbb{R}^{n\times n}$ is eventually positive and $C \in \mathbb{R}^{n\times n}$ is any matrix, then for $\varepsilon$ sufficiently small, $A(\varepsilon) = A + \varepsilon C$ is eventually positive (see, for example, [Ellison et al. 2010] for applications of this technique). The analogous result for eventually exponentially positive matrices follows from Proposition 1.6 and perturbation theory.

**Theorem 2.4.** *If $A \in \mathbb{R}^{n\times n}$ is eventually exponentially positive and $C \in \mathbb{R}^{n\times n}$ is any matrix, then for $\varepsilon$ sufficiently small, $A(\varepsilon) = A + \varepsilon C$ is eventually exponentially positive.*

If $\hat{\mathcal{A}}$ is a superpattern of a PEEP sign pattern $\mathcal{A}$, and $A \in \mathcal{Q}(\mathcal{A})$ is eventually exponentially positive, then a matrix $\hat{A}$ realizing $\hat{\mathcal{A}}$ can be obtained by a small perturbation of $A$.

**Corollary 2.5.** *If $\mathcal{A}$ is a PEEP sign pattern, then every superpattern of $\mathcal{A}$ is PEEP. If $\hat{\mathcal{A}}$ is a sign pattern that is not PEEP, then no subpattern of $\hat{\mathcal{A}}$ is a PEEP sign pattern.*

If a sign pattern $\mathcal{A}$ has a primitive positive part, it is PEP. There is an analogous result for PEEP sign patterns.

**Theorem 2.6.** *Let $\mathcal{A}$ be a sign pattern such that $\mathcal{A}^+$ is irreducible. Then $\mathcal{A}$ is PEEP.*

*Proof.* Let $B$ be the matrix obtained from $\mathcal{A}^+$ by replacing $+$ by $1$. Since $B+I \geq 0$, has positive entries on its diagonal, and is irreducible, $B+I$ is primitive and thus eventually positive. So $B$ is eventually exponentially positive and $\mathcal{A}^+$ is PEEP. Since $\mathcal{A}$ is a superpattern of $\mathcal{A}^+$, $\mathcal{A}$ is PEEP. □

The converse of Theorem 2.6 is false because the sign pattern $\mathcal{B}_{D(+)}$ (1) is a PEP sign pattern with reducible positive part.

Several necessary or sufficient conditions for PEP sign patterns were established in [Berman et al. 2010]. The sign patterns

$$\mathcal{B}_1 = \begin{bmatrix} - & - & + \\ + & - & - \\ - & + & - \end{bmatrix}, \quad \mathcal{B}_2 = \begin{bmatrix} - & - & - \\ + & - & - \\ - & + & - \end{bmatrix}$$

are PEEP and demonstrate that the following statements about PEP sign patterns do not necessarily hold for PEEP sign patterns:

1. For $n \geq 2$, an $n \times n$ sign pattern that has exactly one positive entry in each row and each column is not PEP.
2. If $n \geq 2$, the minimum number of $+$ entries in an $n \times n$ PEP sign pattern is $n+1$.
3. If $A$ is PEP, then $\Gamma(\mathcal{A})$ has a cycle (of length one or more) consisting entirely of $+$ entries.

Certain conditions that prevent a sign pattern from being PEP also prevent a sign pattern from being PEEP:

**Theorem 2.7** [Berman et al. 2010]. *Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern with $n \geq 2$ such that for every $k = 1, \ldots, n$,*

1. $\alpha_{kk} = +$, *and*
2. (a) *no off-diagonal entry in row $k$ is $+$, or*
   (b) *no off-diagonal entry in column $k$ is $+$.*

*Then $\mathcal{A}$ is not PEP.*

**Corollary 2.8.** *Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern with $n \geq 2$ such that for every $k = 1, \ldots, n$,*

(a) *no off-diagonal entry in row $k$ is $+$, or*

(b) *no off-diagonal entry in column $k$ is $+$.*

*Then $\mathcal{A}$ is not PEEP.*

*Proof.* By Theorem 2.7, $\mathcal{A}_{D(+)}$ is not PEP, so $\mathcal{A}$ is not PEEP. □

**Corollary 2.9.** *If $\mathcal{A}$ is a PEEP sign pattern, then there exists $k$ such that both row and column $k$ have an off-diagonal $+$. Hence, a PEEP sign pattern must have at least 2 positive off-diagonal entries.*

A square sign pattern $\mathcal{A}$ is a *Z sign pattern* if $\alpha_{ij} \neq +$ for all $i \neq j$.

**Corollary 2.10.** *If $\mathcal{A}$ is an $n \times n$ Z sign pattern with $n \geq 2$, then $\mathcal{A}$ is not PEEP.*

**Proposition 2.11** [Berman et al. 2010]. *Let*

$$\mathcal{K} = \begin{bmatrix} [+] & [-] & [+] & \ldots \\ [-] & [+] & [-] & \ldots \\ [+] & [-] & [+] & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

*be a square checkerboard block sign pattern where the block $[+]$ (respectively, $[-]$) consists of entirely positive (respectively, entirely negative) entries, and the diagonal blocks are square. Then $-\mathcal{K}$ is not PEP, and if $\mathcal{K}$ has a negative entry, then $\mathcal{K}$ is not PEP.*

**Corollary 2.12.** *No subpattern of a checkerboard pattern $\mathcal{K}$ that contains a negative entry is PEEP.*

**Remark 2.13.** Provided the sign pattern $\mathcal{K}$ contains a negative entry,

$$-\mathcal{K} = \begin{bmatrix} [-] & [+] & [-] & \ldots \\ [+] & [-] & [+] & \ldots \\ [-] & [+] & [-] & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is PEEP because the positive part of $(-\mathcal{K})_{D(+)}$ is primitive.

For a PEP sign pattern $\mathcal{A}$, Lemma 4.3 in [Berman et al. 2010] establishes the existence of a standard form of a matrix $C \in \mathcal{Q}(\mathcal{A})$ with $\rho(C) = 1$ and $C\mathbb{1} = \mathbb{1}$. We have a related result for PEEP sign patterns.

**Proposition 2.14.** *Let $\mathcal{A}$ be a PEEP sign pattern. There is an eventually exponentially positive matrix $C \in \mathcal{Q}(\mathcal{A})$ such that the rightmost eigenvalue $\gamma(C)$ lies in $\{-1, 0, 1\}$ and $C\mathbb{1} = \gamma(C)\mathbb{1}$.*

*Proof.* There exists $A \in \mathcal{Q}(\mathscr{A})$ that is eventually exponentially positive. Let $\gamma(A)$ be the rightmost eigenvalue of $A$ and $\boldsymbol{v} = [v_1, \ldots, v_n]^T$ be the corresponding positive eigenvector. If $\gamma(A) \neq 0$, let $B = (1/|\gamma(A)|)A$; otherwise, $B = A$. Then $B \in \mathcal{Q}(\mathscr{A})$, $B$ is eventually exponentially positive, $\gamma(B) \in \{-1, 0, 1\}$, and $B\boldsymbol{v} = \gamma(B)\boldsymbol{v}$. Let $C = D^{-1}BD$ for $D = \operatorname{diag}(v_1, \ldots, v_n)$. Then $C \in \mathcal{Q}(\mathscr{A})$ is eventually exponentially positive and $\gamma(C) \in \{-1, 0, 1\}$ with $C\mathbb{1} = \gamma(C)\mathbb{1}$. □

We have only started the study of PEEP sign patterns and there are many open questions. Here we highlight one particular question.

**Question 2.15.** What is the minimum number of positive entries in an $n \times n$ PEEP sign pattern, or equivalently, what is the minimum number of positive entries in an eventually exponentially positive $n \times n$ matrix?

This question is motivated by Corollary 4.5 in [Berman et al. 2010], which states that the minimum number of positive entries in an $n \times n$ PEP sign pattern is $n+1$ (for $n \geq 2$). An upper bound for the minimum number of $+$ entries in a PEEP sign pattern is given by the following example.

**Example 2.16.** Let $\mathscr{C}_n$ be the $n \times n$ sign pattern

$$\mathscr{C}_n = \begin{bmatrix} 0 & + & 0 & \cdots & 0 \\ 0 & 0 & + & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & + \\ + & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Since $\mathscr{C}_n$ is nonnegative and irreducible, it is PEEP; note that $\mathscr{C}_n$ has $n$ positive entries.

**Corollary 2.17.** *The minimum number of positive entries in an $n \times n$ PEEP sign pattern is at most $n$.*

The sign pattern $\mathscr{B}_{D(-)}$ in (3) is a $3 \times 3$ pattern that has only 2 positive entries, and from Theorem 3.5 in the next section it follows that the minimum number of positive entries in a $3 \times 3$ PEEP sign pattern is exactly 2. But we do not have examples of PEEP sign patterns having fewer than $n$ positive entries for $n > 3$.

## 3. Classification of small order PEEP sign patterns

In this section we classify all $2 \times 2$ and $3 \times 3$ sign patterns as to whether the pattern is PEEP.

Two $n \times n$ sign patterns $\mathscr{A}$ and $\mathscr{A}'$ are *equivalent* if $\mathscr{A}' = P^T \mathscr{A} P$ or $\mathscr{A}' = P^T \mathscr{A}^T P$ (where $P$ is a permutation matrix). Throughout this section: ? is one of $0, +, -$; $\oplus$ is one of $0, +$; $\ominus$ is one of $0, -$.

It is clear that every $1\times 1$ sign pattern is PEEP. The classification of $2\times 2$ sign patterns as to whether they are PEEP is immediate from the classification as to whether they are PEP.

**Proposition 3.1.** *A $2\times 2$ sign pattern is PEEP if and only if it is of the form*

$$\begin{bmatrix} ? & + \\ + & ? \end{bmatrix}. \tag{4}$$

*Proof.* Sign patterns of the form (4) have $\mathcal{A}^+$ irreducible and so by Theorem 2.6, they are PEEP. For the converse, let $\mathcal{A}$ be a $2\times 2$ PEEP sign pattern. Then $\mathcal{A}_{D(+)}$ is PEP. In [Berman et al. 2010] it was shown that any $2\times 2$ PEP sign pattern has both off-diagonal entries equal to $+$, so $\mathcal{A}$ must also have both off-diagonal entries equal to $+$. □

The classification of $3\times 3$ sign patterns as to whether they are PEEP makes use of the following classification as to whether they are PEP.

**Theorem 3.2** [Berman et al. 2010]. *A $3\times 3$ sign pattern $\mathcal{A}$ is PEP if and only if $\mathcal{A}^+$ is primitive or $\mathcal{A}$ is equivalent to a sign pattern of the form*

$$\mathcal{B} = \begin{bmatrix} + & - & \ominus \\ + & ? & - \\ - & + & + \end{bmatrix}. \tag{5}$$

**Theorem 3.3.** *Let*

$$B = \begin{bmatrix} x_1 & -b_{12} & -b_{13} \\ b_{21} & x_2 & -b_{23} \\ -b_{31} & b_{32} & x_3 \end{bmatrix}, \qquad \text{with } b_{ij} > 0 \text{ for all } i, j = 1, 2, 3,$$

*be an eventually exponentially positive matrix (note there is no restriction on the signs of $x_i$, $i = 1, 2, 3$). Then $x_2 < \min\{x_1, x_3\}$.*

*Proof.* Let $\gamma$ be the rightmost eigenvalue of $B$. Observe that $B-\gamma I$ is eventually exponentially positive with rightmost eigenvalue 0. By Proposition 1.7, $B-\gamma I$ must have a positive entry in each row and column, so $x_1, x_3 > \gamma$. Since the rightmost eigenvalue of $B-\gamma I$ is simple, $0 > \operatorname{tr}(B-\gamma I) = (x_1-\gamma)+(x_2-\gamma)+(x_3-\gamma)$. The first and third term in this sum are positive, so $\operatorname{tr}(B-\gamma I) < 0$ implies that $x_2 < \gamma$. □

**Corollary 3.4.** *A sign pattern equivalent to one of the forms*

$$\mathcal{M}_1 = \begin{bmatrix} - & - & - \\ + & + & - \\ - & + & - \end{bmatrix} \quad \text{or} \quad \mathcal{M}_2 = \begin{bmatrix} - & - & - \\ + & + & - \\ - & + & + \end{bmatrix}$$

*is not PEEP.*

**Theorem 3.5.** *A* $3\times 3$ *sign pattern is PEEP if and only if it is equivalent to one of the following four forms*:

$$\mathcal{A}_1 = \begin{bmatrix} ? & + & ? \\ ? & ? & + \\ + & ? & ? \end{bmatrix}, \quad \mathcal{A}_2 = \begin{bmatrix} ? & + & + \\ + & ? & \ominus \\ + & \ominus & ? \end{bmatrix},$$

$$\mathcal{A}_3 = \begin{bmatrix} ? & - & \ominus \\ + & - & - \\ - & + & ? \end{bmatrix}, \quad \mathcal{A}_4 = \begin{bmatrix} + & - & \ominus \\ + & \oplus & - \\ - & + & + \end{bmatrix}.$$

*Proof.* The sign patterns $\mathcal{A}_1$ and $\mathcal{A}_2$ are PEEP by Theorem 2.6. Note that $\mathcal{A}_4$ is of the form $\mathcal{B}$ from Theorem 3.2; therefore $\mathcal{A}_4$ is PEP and hence is PEEP. Let

$$A = \begin{bmatrix} 0 & -10 & 0 \\ 22 & -33 & -8 \\ -16 & 22 & 0 \end{bmatrix}.$$

Since the spectrum of $A$ is $\{-5, -14+2i\sqrt{15}, -14-2i\sqrt{15}\}$, $\gamma = -5$ is the rightmost eigenvalue of $A$, and $\gamma$ has the right and left eigenvectors $[2, 1, 2]^T$ and $[18, 25, 40]^T$ respectively. Thus $A$ is eventually exponentially positive by Proposition 1.6. Note that $A \in Q(\mathcal{A}_3(0))$ where $\mathcal{A}_3(0)$ is the form of $\mathcal{A}_3$ with all flexible entries set to zero. Therefore $\mathcal{A}_3(0)$ is PEEP, and by Corollary 2.5 every superpattern of $\mathcal{A}_3(0)$ is PEEP. Hence every sign pattern of the form $\mathcal{A}_3$ is PEEP.

Let $\mathcal{A}$ be a $3\times 3$ PEEP sign pattern. Then by Observation 1.4, $\mathcal{A}_{D(+)}$ is PEP. By Theorem 3.2 either $(\mathcal{A}_{D(+)})^+$ is primitive or $\mathcal{A}_{D(+)}$ is of the form $\mathcal{B}$ in (5). If $(\mathcal{A}_{D(+)})^+$ is primitive, then $\mathcal{A}$ is of the form $\mathcal{A}_1$ or $\mathcal{A}_2$. Now suppose that $(\mathcal{A}_{D(+)})^+$ is not primitive. Then we must consider all possible sign patterns $\mathcal{A}$ such that

$$\mathcal{A}_{D(+)} = \begin{bmatrix} + & - & \ominus \\ + & + & - \\ - & + & + \end{bmatrix}.$$

The sign patterns $\mathcal{M}_1$ and $\mathcal{M}_2$ in Corollary 3.4 and their subpatterns rule out all of the sign patterns that could possibly have this $\mathcal{A}_{D(+)}$ except for those of the form $\mathcal{A}_3$ and $\mathcal{A}_4$. Therefore if $\mathcal{A}$ is a $3\times 3$ PEEP sign pattern, it must be of one of the forms $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ or $\mathcal{A}_4$. □

The symbols $\ominus$ and $\oplus$ are used in Theorem 3.5 so that the listed patterns are disjoint classes. For example, if the (2, 2)-entry of $\mathcal{A}_4$ were changed to ?, then one sign pattern of that form would be equivalent to one sign pattern of the form of $\mathcal{A}_3$.

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