77-25,982

ESIMAI, Grace Ogechukwukanma, 1944-
REGRESSION ESTIMATION FOR
MULTIVARIATE NORMAL DISTRIBUTIONS.

Iowa State University, Ph.D., 1977
Statistics

Xerox University Microfilms, Ann Arbor, Michigan 48106

Regression estimation for multivariate normal distributions

by

Grace Ogechukwukanma Esimai

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major: Statistics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa

1977

TABLE OF CONTENTS

I. INTRODUCTION

A. Introduction

Although statisticians and research workers have, for a long time, appealed to preliminary tests of significance as a technique in their investigations, it was only in the last three decades that proper evaluation of their effects on subsequent inferences is being made. These inference procedures incorporating preliminary tests usually occur in incompletely specified models. Recently Bancroft and Han (1977) has given a more appropriate designation of such inference procedures and termed them as inference based on conditional specification.

The term, conditional specification as opposed to unconditional specification, is used to describe the situation when the research worker is uncertain regarding the initial specification of a model for his investigation. He may wish to determine the final specification based on available data, usually by using preliminary tests. However, the research worker, either from experience or some knowledge about the investigation, may be able to choose a complete model for his study. In such a case, the research worker has an unconditional specification.

A bibliography on inference based on conditional specifications was recently compiled by Bancroft and Han (1977). These include estimation, prediction, hypothesis testing and others. In this dissertation, we shall consider the regression estimation of a population mean under conditional specification.

B. Literature Review

The earliest paper on the effect of preliminary tests was due to Bancroft (1944). He discussed the bias in the case of estimation of variances on the basis of a preliminary F-test. Since then, many statisticians and research workers have worked on inference procedures based on conditional specification. Most of these studies used the terminologies "inference procedures incorporating preliminary test(s)," "pooling data," or "inference for incompletely specified models." Recently, in a note by Bancroft and Han (1977), the terminology "inference based on conditional specification" was suggested as a broader representation of the phrases used in the past. In this section, we shall review briefly the estimation of the mean

In 1948, Mosteller studied the estimation of a population mean by pooling independent samples on the basis of a significance test. He investigated what he called the 'Disadvantage Coefficient' which is the efficiency of the never pool estimator relative to the preliminary test estimator. Bennett (1952) evaluated the bias and distribution of estimates of means based on one or more preliminary tests of significance. He extended the work of Mosteller to the cases where the two population variances may be known but unequal or equal but unknown. Preliminary tests were also used by Bennett (1956) to provide interval estimates for the mean and variance of a normal population.

Kitagawa (1963) continued the investigations of Bennett (1952) on the distribution of the preliminary test estimator for the mean of a normal distribution when the variance is unknown. He derived the bias and m.s.e. and expressed these as infinite sums which are very difficult to compute. However, Han and Bancroft (1968) worked on the same problem and were able to express the bias and m.s.e. as finite sums which are much easier to evaluate. They also recommended a procedure for determining a proper choice of the significance level of the preliminary test to ensure a relative efficiency to be larger than some preassigned value.

A little before this time, Kale and Bancroft (1967) had considered the problem of pooling means of two independent random samples from discrete distributions (particularly the Poisson and binomial) which can be approximated by normal distributions after appropriate transformations. They studied two samples from $N(\mu_i, \sigma^2)$ $i = 1,2$ assuming the parameter of interest was μ_1 and σ^2 was known. An estimator $\bar{x}^*$ was proposed both for the estimation of μ_1 and for the test of H_0: $\mu_1 = \mu_0$. The bias and m.s.e. of $\bar{x}^*$ and the size and power of the overall hypothesis testing procedure were studied. They recommended the preliminary test should be at the .25 level for the control of the m.s.e. and size of the test procedure based on $\bar{x}^*$.

In 1971, Brogan used a preliminary test of significance and two-stage sampling to derive an estimator for the mean of a normal distribution. He derived the bias and m.s.e. and compared the latter, for a fixed total sampling cost, to the m.s.e. of some other estimation procedures. Ahsanullah (1971) studied the problem of estimation of the mean μ_1 of one of the components of a bivariate normal distribution with equal marginal variances from a sample of size n. The result of a preliminary test of H_0: $\mu_1=\mu_2$ was used to define an estimator for μ_1 where μ_2 is the mean of the other component of the distribution. He studied the m.s.e. of the preliminary test estimator and tabulated its efficiency relative to the usual estimator. He also used the selection procedure recommended by Han and Bancroft (1968) to compute tables which can be used to determine a proper choice of the significance level of the preliminary test.

Bancroft (1972) gave a summary of some recent advances in inference procedures using preliminary tests of significance. He briefly outlined the theory behind the use of preliminary tests in estimation, tests of hypothesis and prediction. This is based primarily on the desire to make inferences for incompletely specified models. Useful applications of preliminary tests of significance based on results obtained in earlier papers were given in the text by Bancroft (1968).

In 1973, Han (1973a) introduced the use of preliminary tests into regression estimation for bivariate normal distributions. In estimating the mean μ_y of one of the components of a bivariate normal distribution and the mean μ_x of the other variable is known, the investigator can use X in a regression estimation to increase precision. When μ_x is unknown, Han proposed the use of a regression estimator which depends on the outcome of the preliminary test of H_0: $\mu_x=\mu_0$. He studied the bias and m.s.e. of the preliminary test estimator and discussed the relative efficiency. Later, the same year, Han (1973b) extended his study to the case when the mean of X is unknown and double sampling can be employed. If in addition, the investigator has partial information about μ_x, then Han proposed to perform a preliminary test and use the preliminary test estimator. He derived the bias, m.s.e. and relative efficiency of the preliminary test estimator and gave recommendations of the levels of the preliminary test and optimum allocation of sample sizes.

At the same time, many other statisticians and research workers have shown concern about estimation with high precision. Consequently, many workers in the field were also carrying out investigations and proposing new estimators based on certain criteria. One such investigation was given by

Stein (1955) who discussed the inadmissibility of the usual estimator for the mean of a multivariate normal distribution for $p \geq 3$. He proposed a spherically symmetric estimator which is essentially a shrunken estimator. James and Stein (1960) continued with the same studies and gave more precise forms and merits of the shrunken estimator for the cases when the covariance matrix is either known or unknown. In 1960, Stein investigated the improvement in m.s.e. by a transformation, on the regression coefficient $\hat{\underline{\beta}}$, of the form $C\hat{\underline{\beta}}: 0 < C < 1$ which is a shortening of the vector $\hat{\underline{\beta}}$.

In 1968, Thompson (1968a) studied various ways of shrinking the minimum variance unbiased estimator of a population mean towards some known origin, thereby reducing its m.s.e. He employed a preliminary test of significance as a shrinking procedure. Later in the same year, Thompson (1968b) extended his work to shrinkage towards an interval centered at some origin.

C. An Overview of the Present Research and Summary of Results

The present thesis is divided into three main parts. The first part is an effort to extend the studies of Han (1973a) for bivariate normal distributions to (p+1) variate normal distributions ($p+1 > 2$). The second part attempts to extend the method of double sampling with partial information on auxiliary variables first studied by Han (1973b) for one auxiliary variable to the case where the auxiliary variable is a pxl vector. The last part considers regression estimators with certain shrunken estimators for the mean of the auxiliary variable and compares them with the preliminary test estimators.

In Chapter II, Section B, we define the preliminary test estimator, $\hat{\mu}$, for the general p+l variate normal distribution and study its bias when the covariance matrix, Σ, is known. In Chapter II, Section C, we derive and discuss the m.s.e. of $\hat{\mu}$ for Σ known. In Chapter II, Section D, the relative efficiency e, of $\hat{\mu}$ is considered while Chapter II, Sections E and F, respectively, deal with the derivation and discussion of the properties of the Bias and m.s.e. of $\hat{\mu}$ when Σ is unknown. Chapter II, Section G, gives the expression for and some computed values of the relative efficiency e'.

In Chapter III, we consider double sampling with partial information on auxiliary variables and for Σ known, we define the preliminary test estimator $\hat{\mu}_{\ell r}$ and exhibit its properties in Section B. The m.s.e. and the relative efficiency of $\hat{\mu}_{\ell r}$ are given and studied in Chapter III, Sections C and D, respectively. Chapter III, Section E, furnishes a discussion of the optimal sample design and some comparisons. When Σ is unknown, the bias, m.s.e. and relative efficiency e_2 of $\hat{\mu}_{\ell r}$ are derived and investigated in Chapter III, Sections F, G and H, respectively.

In Chapter IV, we consider regression estimators with certain shrunken estimators for the mean of the auxiliary variable. A shrunken regression estimator $\hat{\mu}^*$ is given in Chapter IV, Section B, following Thompson (1968a). The relative efficiency e_3 of $\hat{\mu}^*$ is also discussed. In Chapter IV, Section C, a shrunken regression estimator $\hat{\mu}_1$ is constructed following James and Stein (1960). We also give an expression for its relative efficiency.

In general, the bias and m.s.e. of the preliminary test estimators are found to be functions of n, $\underline{\mu}_x$, Σ_{12} and α where Σ_{12} is the covariance between Y and $\underline{X}$. When Σ is known, the bias and m.s.e. are found in terms of the cumulative distribution of the noncentral Chi-squared distribution. For p = 1, Han (1973a, 1973b) expressed the bias and m.s.e. in terms of the cumulative distribution function of the standard normal distribution. Thus the computations in this dissertation afford a further empirical verification of the results of Han (1975) on some relationships between noncentral Chi-squared and normal distributions. The properties of the bias and m.s.e. for p > 1 are found to be identical with those recorded for p = 1.

When Σ is unknown, the bias and m.s.e. of the preliminary test estimators are also found to be functions of n, $\underline{\mu}_x$, Σ_{12} and α , but in terms of the cumulative distribution of the noncentral F-distribution. For p = 1, Han (1973a, 1973b) expressed these in terms of the moments of normal distribution. The properties of the bias and m.s.e. for $p > 1$ are in general found to be identical with those recorded for $p = 1$.

The m.s.e. of the shrunken regression estimator is found to be a function of n, Σ and $\underline{\mu}_x$. The efficiency of the preliminary test estimator relative to the shrunken regression estimator is generally found to be greater than unity when $\underline{\mu}_x = \underline{0}$, or when the null hypothesis of the preliminary test is true. The value of the relative efficiency then decreases to a value smaller than unity, increases to above unity and finally decreases to unity as components of $\underline{\mu}_x$ increase.

II. REGRESSION ESTIMATION FOR MULTIVARIATE NORMAL DISTRIBUTIONS

A. Introduction

Consider that we have a multivariate normal population, that is, consider the case:

$$\underline{Y} \sim N(\underline{\mu},\Sigma)$$

where

$$\underline{Y}_{(p+1)x1} = \begin{pmatrix} Y \\ X_1 \\ \cdot \\ \cdot \\ \cdot \\ X_p \end{pmatrix} = \begin{pmatrix} Y \\ \underline{X} \end{pmatrix} \begin{matrix} 1x1 \\ px1 \end{matrix} \qquad (2.1)$$

$$\underline{\mu} = \begin{pmatrix} \mu \\ \mu_1 \\ \cdot \\ \cdot \\ \cdot \\ \mu_p \end{pmatrix} = \begin{pmatrix} \mu \\ \underline{\mu}_x \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} \sigma^2 & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

Suppose we are interested in estimating the mean μ. This happens in an investigation that the investigator is interested in primarily one variable while he uses other variables as auxiliary information. Following Han (1973a), we may use the remaining p variables as ancillary variables to increase precision. If $\underline{\mu}_x$ and Σ are known and we have a random sample of size n, we can use the regression estimator defined as

$$\hat{\hat{\mu}} = \bar{y} + \Sigma_{12}\,\Sigma_{22}^{-1}\,(\underline{\mu}_x - \bar{\underline{X}})$$

where

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i ; \quad \bar{\underline{X}} = \frac{1}{n}\sum_{i=1}^{n} \underline{X}_i \, .$$

In this case we know the variance of the regression estimator is

$$\frac{1}{n}[\sigma^2 - \Sigma_{12}\,\Sigma_{22}^{-1}\,\Sigma_{21}] \text{ and if } \frac{1}{n}[\Sigma_{12}\,\Sigma_{22}^{-1}\,\Sigma_{21}]$$

is considerably large, we have an appreciable gain in precision. If $\underline{\mu}_x$ is unknown, one may use $\bar{y}$ to estimate μ. However, it may happen that from certain sources, the experimenter may expect that $\underline{\mu}_x = \underline{\mu}_0$ but not sure for certainty. In this case, a preliminary test of H_0: $\underline{\mu}_x = \underline{\mu}_0$ can be performed and the estimator is made to depend on the result of the preliminary test. In this chapter we shall consider the properties of this preliminary test estimator.

B. The Preliminary Test Estimator and its Bias when Σ is Known

Assume $(y_i, X_{1i}, \ldots X_{pi})$ $i = 1, \ldots n$ is a random sample from the (p+1) - variate normal distribution $N(\underline{\mu}, \Sigma)$. Suppose Σ is known and $\underline{\mu}$ unknown. Consider the hypotheses:

$$H_0: \ \underline{\mu}_x = \underline{\mu}_0$$
$$H_1: \ \underline{\mu}_x \neq \underline{\mu}_0$$

Wlog (without loss of generality) we take $\underline{\mu}_0$ to be the null vector $\underline{0}$. The test statistic for H_0 versus H_1 is $n(\bar{\underline{X}}' \ \Sigma_{22}^{-1} \ \bar{\underline{X}})$ which has a Chi-squared distribution, χ_p^2, with p degrees of freedom. A size α test is to reject H_0 if $n(\bar{\underline{X}}' \ \Sigma_{22}^{-1} \ \bar{\underline{X}}) > \chi_{p,\alpha}^2$ where $\chi_{p,\alpha}^2$ is the 100(1-α) percentage point of χ_p^2. Therefore if we let $\chi_{p,\alpha}^2 = c$ and denote the acceptance region $[n \ \bar{\underline{X}}' \ \Sigma_{22}^{-1} \ \bar{\underline{X}}: n \ \bar{\underline{X}}' \ \Sigma_{22}^{-1} \ \bar{\underline{X}} < c]$ by A then the preliminary test estimator can be written as

$$\hat{\mu} = \begin{cases} \bar{y} - \Sigma_{12} \ \Sigma_{22}^{-1} \ \bar{\underline{X}} & \text{given } A \\ \bar{y} & \text{given } \bar{A} \end{cases} \tag{2.2}$$

The expected value of $\hat{\mu}$ is

$$\begin{aligned} E(\hat{\mu}) &= E\{(\bar{y} - \Sigma_{12} \ \Sigma_{22}^{-1} \ \bar{\underline{X}}) | A\} \ P(A) + E(\bar{y}|\bar{A})P(\bar{A}) \\ &= E(\bar{y}) - \Sigma_{12} \ \Sigma_{22}^{-1} \ E(\bar{\underline{X}}|A)P(A) \end{aligned} \tag{2.3}$$

But $E(\bar{y}) = \mu$. Hence the bias of $\hat{\mu}$ is the second term and if we denote this by B, we can write

$$B = -\Sigma_{12} \ \Sigma_{22}^{-1} \ E(\bar{\underline{X}}|A)P(A) \tag{2.4}$$

Now we know $\bar{\underline{X}} \sim N(\underline{\mu}_x, \frac{1}{n} \Sigma_{22})$ and since Σ_{22} is positive definite, $\exists$ a nonsingular $T \ni T'T = \Sigma_{22}^{-1}$. Let $\underline{Z} = T\bar{\underline{X}}$. Therefore

$$\underline{Z} \sim N(T\underline{\mu}_x \, , \, \tfrac{1}{n}I)$$

$$\sim N(\underline{\nu}_x \, , \, \tfrac{1}{n}I) \quad \text{say}$$

and we can write

$$B = -\Sigma_{12}\,\Sigma_{22}^{-1}\,T^{-1}\,E[\underline{Z}|n(\underline{Z}'\underline{Z}) \le c] \cdot P[n(\underline{Z}'\underline{Z}) \le c]$$

where

$$\{n(\underline{Z}'\underline{Z}) : n(\underline{Z}'\underline{Z}) \le c\} = A \; .$$

Hence

$$B = -\Sigma_{12}\,\Sigma_{22}^{-1}\,T^{-1}\,E[\underline{Z}|A]P(A) \qquad (2.5)$$

In general $n(\underline{Z}'\underline{Z})$ has a noncentral Chi-squared distribution with p degrees of freedom and noncentrality parameter $\lambda = n(\underline{\mu}_x'T'T\underline{\mu}_x) = n\underline{\nu}_x'\underline{\nu}_x$. We shall denote the i-th component of the px1 vector $\underline{\nu}_x$ by $\nu_x^{(i)}$.

Now consider

$$R = P(A) = \int_0^c e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\lambda}{2}\right)^j h_{p+2j}(t)dt \qquad (2.6)$$

where $h_{p+2j}(\cdot)$ is the probability density function of χ^2_{p+2j}. Differentiating (2.6) with respect to (w.r.t.) $\nu_x^{(i)}$ following the method of justification in the Appendix, we obtain

$$\frac{\partial R}{\partial \nu_x^{(i)}} = \int_0^c e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n}{2} \nu_x^{(i)} j(\tfrac{\lambda}{2})^{j-1} h_{p+2j}(t)dt$$

$$- \int_0^c \frac{1}{2} 2n\nu_x^{(i)} e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{1}{j!} (\tfrac{\lambda}{2})^j h_{p+2_j} t(dt) \qquad (2.7)$$

or

$$\frac{\partial R}{\partial \nu_x^{(i)}} = n[H_{p+2}(c;\lambda) - P(A)]\nu_x^{(i)} \qquad (2.8)$$

where $H_{p+2}(c;\lambda)$ is the cumulative distribution function of the noncentral Chi-squared distribution with p+2 degrees of freedom and noncentrality parameter λ.

Alternatively, we can evaluate P(A) by the use of the distribution of $\underline{Z}$ and write

$$R = P(A) = \int_A \cdots \int \prod_{j=1}^{P} \frac{1}{\sqrt{2\pi}} \sqrt{n}\, e^{-\frac{n}{2}(Z^{(j)} - \nu_x^{(j)})^2} dZ^{(j)} \qquad (2.9)$$

since components of $\underline{Z}$ are independent. If we differentiate (2.9) w.r.t. $\nu_x^{(i)}$ using the method of justification in the Appendix, we obtain

$$\frac{\partial R}{\partial \nu_x^{(i)}} = \int_A \cdots \int \prod_{j=1}^{P} \frac{\sqrt{n}}{\sqrt{2\pi}} \frac{n}{2} 2(Z^{(i)} - \nu_x^{(i)}) e^{-\frac{n}{2}(Z^{(j)} - \nu_x^{(j)})^2} dZ^{(j)}$$

(2.10)

$$\frac{\partial R}{\partial \nu_x^{(i)}} = n\ E[Z^{(i)}|A]\ P(A) - n\nu_x^{(i)}\ P(A) \tag{2.11}$$

To obtain $E(Z^{(i)}|A)P(A)$ we equate (2.8) and (2.11),

$$n[H_{p+2}(c;\lambda) - P(A)]\nu_x^{(i)} = nE[Z^{(i)}|A]P(A) - n\nu_x^{(i)}P(A)$$

which gives

$$E(Z^{(i)}|A)P(A) = H_{p+2}(c;\lambda)\nu_x^{(i)} \tag{2.12}$$

Substituting (2.12) in (2.5) and noting that the conditional expectation of a vector is defined as the vector of the conditional expectation of its components, we have

$$B = -\Sigma_{12}\ \Sigma_{22}^{-1}\ T^{-1}\ H_{p+2}(c;\lambda)\underline{\nu}_x = -\Sigma_{12}\ \Sigma_{22}^{-1}\ \underline{\mu}_x H_{p+2}(c;\lambda)$$

As a partial check, when c=0, the estimator reduces to the usual estimator $\bar{y}$ which is the case when we always reject the null hypothesis. In this case, B=0. When $c=\infty$, the null hypothesis is always accepted and the regression estimator $\bar{y} - \Sigma_{12}\ \Sigma_{22}^{-1}\ \underline{\bar{X}}$ is used. The bias in this case reduces to the bias for the regression estimator, i.e.,

$$B = -\Sigma_{12}\ \Sigma_{22}^{-1}\ T^{-1}\underline{\nu}_x = -\Sigma_{12}\ \Sigma_{22}^{-1}\ \underline{\mu}_x\ .$$

We now check the result with that of Han (1973a) when $p=1$. Without loss of generality we let $\Sigma_{22} = I$ and $\sigma^2=1$. Therefore

$$\sqrt{n}\, B = -KH_{p+2}(c;\lambda) \quad \text{where} \quad K = \Sigma_{12}\mu_x\sqrt{n} \; .$$

For $p=1$, $\Sigma_{12} = \rho$ and we let $\mu_x\sqrt{n} = a$.

Hence $K = \rho a$ and we observe that $\sqrt{n}B$ changes sign with ρ or a. Therefore for $p=1$, we may only study the bias for positive values of ρ and a. It is obvious to see that $\sqrt{n}B$ is a function of ρ, a and α. The values of $-\sqrt{n}B$ for certain values of ρ, a and α were computed and examined and only very few of these are given in Table 2.1.

Table 2.1. Values of $-\sqrt{n}B$ for $p=1$.

a	α = .05			α = .50		
	ρ			ρ		
	.1	.5	.9	.1	.5	.9
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.5	0.0342	0.1712	0.3082	0.0032	0.0159	0.0286
1.0	0.0583	0.2917	0.5250	0.0045	0.0226	0.0407
1.5	0.0658	0.3288	0.5918	0.0038	0.0192	0.0345
2.0	0.0570	0.2848	0.5126	0.0023	0.0115	0.0207
2.5	0.0392	0.1959	0.3526	0.0010	0.0051	0.0092
3.0	0.0215	0.1076	0.1937	0.0003	0.0017	0.0031

From the computed values we note the following properties of the Bias.

1. The bias is zero when $\mu_x=0$. This corresponds to the case when the hypothesis is true.
2. The value of the bias generally increases with ρ and decreases as α increases.
3. For fixed n, α and ρ, the bias first increases then decreases to zero as μ_x increases.

We also note that the values of $-\sqrt{n}B$ given in Table 2.1 are identical with the values obtained by Han (1973a). The only difference is that while Han's results were given to three decimal places, the values here are computed to four decimal places. The above properties of the Bias were also recorded by Han. Furthermore, we note that Han expressed the bias in terms of functions of the distribution function and probability density function of the standard normal distribution while in this paper, the bias is expressed in terms of the cumulative distribution function of the noncentral Chi-squared distribution with an odd degree of freedom. The above results thus provide an empirical verification of the theoretical results obtained by Han (1975) on some relationships between non-central Chi-squared and normal distributions.

For p=2, the values of $-\sqrt{n}B$ for some values of Σ_{12}, $\underline{\mu}_x\sqrt{n}$ and α are given in Table 2.2. Since the bias changes sign with $\underline{\mu}_x$, the values were computed for only positive values of $\underline{\mu}_x$.

Table 2.2. Values of $-\sqrt{n}B$ for p=2.

Σ'_{12} / $(\underline{\mu}_x\sqrt{n})'$	$\alpha = .05$				
	$\begin{pmatrix}-.5\\.5\end{pmatrix}$	$\begin{pmatrix}.5\\.5\end{pmatrix}$	$\begin{pmatrix}-.5\\.7\end{pmatrix}$	$\begin{pmatrix}-.7\\-.7\end{pmatrix}$	$\begin{pmatrix}.7\\.7\end{pmatrix}$
(0, 0)	0	0	0	0	0
(.5, 0)	-.1931	.1931	-.1931	-.2704	.2704
(.5, .5)	0	.3723	.0745	-.5212	.5212
(1.0, 0)	-.3447	.3447	-.3447	-.4826	.4826
(1.0, .5)	-.1656	.4968	-.0994	-.6955	.6955
(1.0,1.0)	0	.5839	.1168	-.8175	.8175
(1.5, 0)	-.4191	.4191	-.4191	-.5868	.5868
(1.5, .5)	-.2672	.5344	-.2138	-.7482	.7482
(1.5,1.0)	-.1162	.5812	-.0232	-.8137	.8137
(1.5,1.5)	0	.5448	.1090	-.7627	.7627
(2.0, 0)	-.4018	.4018	-.4018	.5625	.5625
(2.0, .5)	-.2866	.4777	-.2484	-.6687	.6687
(2.0,1.0)	-.1637	.4912	-.0982	-.6877	.6877
(2.0,1.5)	-.0625	.4376	.0125	-.6126	.6126
(2.0,2.0)	0	.3351	.0670	-.4691	.4691
(2.5, 0)	-.3126	.3126	-.3126	-.4376	.4376
(2.5, .5)	-.2365	.3548	-.2129	-.4967	.4967
(2.5,1.0)	-.1496	.3492	-.1097	-.4888	.4888
(2.5,1.5)	-.0744	.2976	-.0298	-.4166	.4166
(2.5,2.0)	-.0242	.2179	.0145	-.3050	.3050
(2.5,2.5)	0	.1356	.0271	-.1898	.1898
(3.0, 0)	-.1978	.1978	-.1978	-.2769	.2769
(3.0, .5)	-.1551	.2171	-.1427	-.3040	.3040
(3.0,1.0)	-.1031	.2062	-.0825	-.2886	.2886
(3.0,1.5)	-.0563	.1690	-.0338	-.2366	.2366
(3.0,2.0)	-.0237	.1187	-.0047	-.1662	.1662
(3.0,2.5)	-.0064	.0708	.0064	-.0991	.0991
(3.0,3.0)	0	.0355	.0071	-.0497	.0497

Table 2.2. (continued)

Σ'_{12} / $(\underline{\mu}_x\sqrt{n})'$	α = .20				
	$\begin{pmatrix}-.5\\.5\end{pmatrix}$	$\begin{pmatrix}.5\\.5\end{pmatrix}$	$\begin{pmatrix}-.5\\.7\end{pmatrix}$	$\begin{pmatrix}-.7\\-.7\end{pmatrix}$	$\begin{pmatrix}.7\\.7\end{pmatrix}$
(0, 0)	0	0	0	0	0
(.5, 0)	-.1117	.1117	-.1117	-.1563	.1563
(.5, .5)	0	.2085	.0417	-.2919	.2919
(1.0, 0)	-.1814	.1814	-.1814	-.2540	.2540
(1.0, .5)	-.0845	.2536	-.0507	-.3550	.3550
(1.0,1.0)	0	.2728	.0546	-.3819	.3819
(1.5, 0)	-.1903	.1903	-.1903	-.2664	.2664
(1.5, .5)	-.1179	.2358	-.0943	-.3301	.3301
(1.5,1.0)	-.0472	.2361	-.0094	-.3305	.3305
(1.5,1.5)	0	.1940	.0388	-.2716	.2716
(2.0, 0)	-.1507	.1507	-.1507	-.2109	.2109
(2.0, .5)	-.1047	.1745	-.0908	-.2443	.2443
(2.0,1.0)	-.0554	.1663	-.0333	-.2328	.2328
(2.0,1.5)	-.0187	.1311	.0037	-.1836	.1836
(2.0,2.0)	0	.0855	.0171	-.1197	.1197
(2.5, 0)	-.0937	.0937	-.0937	-.1311	.1311
(2.5, .5)	-.0692	.1038	-.0623	-.1454	.1454
(2.5,1.0)	-.0408	.0953	-.0300	-.1334	.1334
(2.5,1.5)	-.0182	.0726	-.0073	-.1017	.1017
(2.5,2.0)	-.0051	.0458	.0031	-.0641	.0641
(2.5,2.5)	0	.0238	.0048	-.0333	.0333
(3.0, 0)	-.0462	.0462	-.0462	-.0647	.0647
(3.0, .5)	-.0355	.0497	-.0326	-.0695	.0695
(3.0,1.0)	-.0221	.0442	-.0177	-.0619	.0619
(3.0,1.5)	-.0109	.0327	-.0065	-.0458	.0458
(3.0,2.0)	-.0040	.0200	-.0008	-.0281	.0281
(3.0,2.5)	-.0009	.0101	.0009	-.0141	.0141
(3.0,3.0)	0	.0042	.0008	-.0058	.0058

Table 2.2. (continued)

Σ'_{12} / $(\underline{\mu}_x\sqrt{n})'$	α = .50				
	$\begin{pmatrix}-.5\\ .5\end{pmatrix}$	$\begin{pmatrix}.5\\ .5\end{pmatrix}$	$\begin{pmatrix}-.5\\ .7\end{pmatrix}$	$\begin{pmatrix}-.7\\ -.7\end{pmatrix}$	$\begin{pmatrix}.7\\ .7\end{pmatrix}$
(0, 0)	0	0	0	0	0
(.5, 0)	-.0348	.0348	-.0348	-.0487	.0487
(.5, .5)	0	.0630	.0126	-.0883	.0883
(1.0, 0)	-.0578	.0518	-.0518	-.0725	.0725
(1.0, .5)	-.0235	.0704	-.0141	-.0985	.0985
(1.0,1.0)	0	.0698	.0140	-.0977	.0977
(1.5, 0)	-.0474	.0474	-.0474	-.0663	.0663
(1.5, .5)	-.0286	.0572	-.0229	-.0801	.0801
(1.5,1.0)	-.0106	.0530	-.0021	-.0742	.0742
(1.5,1.5)	0	.0386	.0077	-.0540	.0540
(2.0, 0)	-.0314	.0314	-.0314	-.0440	.0440
(2.0, .5)	-.0213	.0355	-.0185	-.0497	.0497
(2.0,1.0)	-.0105	.0315	-.0063	-.0442	.0442
(2.0,1.5)	-.0032	.0222	.0006	-.0311	.0311
(2.0,2.0)	0	.0125	.0025	-.0174	.0174
(2.5, 0)	-.0159	.0159	-.0159	-.0222	.0222
(2.5, .5)	-.0115	.0172	-.0103	-.0241	.0241
(2.5,1.0)	-.0063	.0148	-.0047	-.0207	.0207
(2.5,1.5)	-.0025	.0102	-.0010	-.0142	.0142
(2.5,2.0)	-.0006	.0056	.0004	-.0078	.0078
(2.5,2.5)	0	.0024	.0005	-.0034	.0034
(3.0, 0)	-.0062	.0062	-.0062	-.0087	.0087
(3.0, .5)	-.0047	.0065	-.0043	-.0092	.0092
(3.0,1.0)	-.0027	.0055	-.0022	-.0077	.0077
(3.0,1.5)	-.0012	.0037	-.0007	-.0052	.0052
(3.0,2.0)	-.0004	.0020	-.0001	-.0028	.0028
(3.0,2.5)	-.0001	.0009	.0001	-.0012	.0012
(3.0,3.0)	0	.0003	.0001	-.0004	.0004

From Table 2.2 we observe the following properties of the Bias for p=2.

1. The bias is zero when $\underline{\mu}_x = \underline{0}$. Again this corresponds to the case when the hypothesis is true.

2. The value of the bias generally increases with Σ_{12} but decreases as α increases.

3. The bias is zero if $\underline{\mu}_x$ has identical components and Σ_{12} has components which differ only in sign.

4. The bias is negative if either $\underline{\mu}_x$ or Σ_{12} has non-identical but positive components and the other has components which differ only in sign.

5. If n, α, Σ_{12} and a component of $\underline{\mu}_x$ are fixed, the bias first increases then decreases to zero as the other component of $\underline{\mu}_x$ increases.

C. The M.S.E. of $\hat{\mu}$ when Σ is Known

In order to find the M.S.E. of $\hat{\mu}$, we first consider

$$V(\hat{\mu}) = E(\hat{\mu}^2) - [E(\hat{\mu})]^2 . \tag{2.13}$$

Also we can write

$$\hat{\mu} = \begin{cases} \bar{y} - \Sigma_{12}\Sigma_{22}^{-1}\, T^{-1}\underline{Z} & \text{given } A \\ \bar{y} & \text{given } \bar{A} \end{cases}$$

Therefore

$$E(\hat{\mu}^2) = E[(\bar{y} - \Sigma_{12}\Sigma_{22}^{-1} T^{-1}\underline{Z})^2|A]\ P(A)$$

$$+ E(\bar{y}^2|\bar{A})\ P(\bar{A})$$

$$= E(\bar{y}^2) - 2\Sigma_{12}\Sigma_{22}^{-1} T^{-1}E(\bar{y}\underline{Z}|A)\ P(A) \qquad (2.14)$$

$$+ \Sigma_{12}\Sigma_{22}^{-1} T^{-1}E[\underline{ZZ}'|A]\ P(A)T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

Therefore to evaluate $E(\hat{\mu}^2)$, we need to find $E(\underline{ZZ}'|A)P(A)$ and $E(\bar{y}\underline{Z}|A)P(A)$. Let us consider $E(\underline{ZZ}'|A)P(A)$ and denote the i-th component of $\underline{Z}$ by $Z^{(i)}$. We need actually consider $E[(Z^{(i)})^2|A]$ $P(A)$ and $E(Z^{(i)}Z^{(k)}|A)P(A)$ for $i \neq k$. These can be evaluated by using the second derivatives of R.

Differentiating (2.7) w.r.t. $\nu_x^{(i)}$, we have

$$\frac{\partial^2 R}{\partial \nu_x^{(i)2}} = \int_0^c e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{2n\nu_x^{(i)}}{2}\right)^2 j(j-1)\left(\frac{\lambda}{2}\right)^{j-2} h_{p+2j}(t)\ dt$$

$$+ \int_0^c e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n}{2} j\left(\frac{\lambda}{2}\right)^{j-1} h_{p+2j}(t)\ dt$$

$$- \int_0^c \frac{1}{2}\, 2n\nu_x^{(i)}\ e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n\nu_x^{(i)}}{2} j\left(\frac{\lambda}{2}\right)^{j-1} h_{p+2j}(t)\ dt$$

$$- \int_0^c \frac{1}{2}\, 2n\ e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\lambda}{2}\right)^{j} h_{p+2j}(t)\ dt$$

$$+ \int_0^c (\tfrac{1}{2}\ 2n\nu_x^{(i)})^2\ e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{1}{j!} (\tfrac{\lambda}{2})^j h_{p+2j}(t)\ dt$$

$$- \int_0^c \tfrac{1}{2}\ 2n\nu_x^{(i)}\ e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{1}{j!}\ \frac{2n\nu_x^{(i)}}{2}\ j(\tfrac{\lambda}{2})^{j-1} h_{p+2j}(t)\ dt$$

$$= n^2(\nu_x^{(i)})^2 H_{p+4}(c;\lambda) + nH_{p+2}(c;\lambda) - n^2(\nu_x^{(i)})^2 H_{p=2}(c;\lambda)$$

$$- n\ P(A) + n^2(\nu_x^{(i)})^2\ P(A) - n^2(\nu_x^{(i)})^2 H_{p+2}(c;\lambda)$$

$$= n^2(\nu_x{}^{(i)})^2 H_{p+4}(c;\lambda) + n\{1-2n(\nu_x^{(i)})^2\} H_{p+2}(c;\lambda)$$

$$+ n[n(\nu_x^{(i)})^2 - 1]\ P(A) \tag{2.15}$$

Similarly differentiating (2.9) twice w.r.t. $\nu_x^{(i)}$, we obtain

$$\frac{\partial^2 R}{\partial \nu^{(i)2}} = \int_A \cdots \int \prod_{j=1}^{P} \frac{\sqrt{n}}{\sqrt{2\pi}} (\tfrac{n}{2}\ 2)^2 Z^{(i)} (Z^{(i)}-\nu_x^{(i)}) e^{-\frac{n}{2}(Z^{(j)}-\nu_x^{(j)})^2} dZ^{(j)}$$

$$- \int_A \cdots \int \prod_{j=1}^{P} \frac{\sqrt{n}}{\sqrt{2\pi}}\ \frac{n}{2}\ 2\ e^{-\frac{n}{2}(Z^{(j)}-\nu_x^{(j)})^2} dZ^{(j)}$$

$$- \int_A \cdots \int \prod_{j=1}^{P} \frac{\sqrt{n}}{\sqrt{2\pi}} (\tfrac{n}{2}2)^2 \nu_x^{(i)} (Z^{(i)}-\nu_x^{(i)}) e^{-\frac{n}{2}(Z^{(j)}-\nu_x^{(j)})^2} dZ^{(j)}$$

$$= n^2 E[(Z^{(i)})^2|A] P(A) - n^2\nu_x^{(i)} E(Z^{(i)}|A) P(A) \qquad (2.16)$$

$$- n P(A) - n^2\nu_x^{(i)} E(Z^{(i)}|A) P(A) + n^2(\nu_x^{(i)})^2 P(A)$$

Hence from (2.12)

$$\frac{\partial^2 R}{\partial \nu_x^{(i)^2}} = n^2 E[(Z^{(i)^2})|A] P(A) - 2n^2\nu_x^{(i)^2} H_{p+2} (c;\lambda) \qquad (2.17)$$

$$+ n^2(\nu_x^{(i)})^2 P(A) - n p(A)$$

Equating (2.15) and (2.17), we have

$$n^2 E\{(Z^{(i)})^2|A\} P(A) + n^2\nu_x^{(i)2}[P(A) - 2H_{p+2}(c;\lambda)]$$

$$- n P(A) = n^2(\nu_x^{(i)})^2 H_{p+4}(c;\lambda) + n\{1-2n(\nu_x^{(i)})^2\}H_{p+2}(c;\lambda)$$

$$+ n [n(\nu_x^{(i)})^2 - 1] P(A)$$

or

$$E\{Z^{(i)^2}|A\} P(A) = (\nu_x^{(i)})^2 H_{p+4}(c;\lambda) + \frac{1}{n} H_{p+2}(c;\lambda)$$

Next we find $E(Z^{(i)}Z^{(k)}|A)P(A)$ first by differentiating (2.7) and (2.10) w.r.t. $\nu_x^{(k)}$, then equating the results.

From (2.7),

$$\frac{\partial R}{\partial \nu_x^{(k)} \partial \nu_x^{(i)}} = \int_0^c e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n\nu_x^{(i)}}{2} \frac{2n\nu_x^{(k)}}{2}\; j\;\, j-1(\frac{\lambda}{2})^{j-2} h_{p+2j}(t)dt$$

$$- \int_0^c \frac{1}{2}\, 2n\nu_x^{(k)}\; e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n\nu_x^{(i)}}{2}\; j(\frac{\lambda}{2})^{j-1} h_{p+2j}(t)dt$$

$$- \int_0^c \frac{1}{2}\, 2n\nu_x^{(i)}\; e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n\nu_x^{(k)}}{2}\; j(\frac{\lambda}{2})^{j-1} h_{p+2j}(t)dt$$

$$- \int_0^c (\frac{1}{2}\, 2n)^2 \nu_x^{(i)} \nu_x^{(k)}\; e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{1}{j!}(\frac{\lambda}{2})^j h_{p+2j}(t)dt$$

$$= n^2 \nu_x^{(i)} \nu_x^{(k)} H_{p+4}(c;\lambda) - 2n^2 \nu_x^{(i)} \nu_x^{(k)} H_{p+2}(c;\lambda)$$

$$+ n^2 {}_x^{(i)}\; {}_x^{(k)}\; P(A)\;. \qquad (2.18)$$

Similarly from (2.10),

$$\frac{\partial R}{\partial \nu_x^{(k)} \partial \nu_x^{(i)}} = \int \cdots \int_A \prod_{j=1}^{p} \frac{\sqrt{n}}{\sqrt{2\pi}} (\frac{2n}{2})^2 (Z^{(i)} - \nu_x^{(i)})(Z^{(k)} - \nu_x^{(k)})$$

$$e^{-\frac{n}{2}(Z^{(j)} - \nu_x^{(j)})^2} dZ^{(i)} = n^2 E(Z^{(i)} Z^{(k)} | A) P(A)$$

$$-n^2 \nu_x^{(i)} E(Z^{(k)} | A) P(A) - n^2 \nu_x^{(k)} E(Z^{(i)} | A) P(A)$$

$$+ n^2 \nu_x^{(i)} \nu_x^{(k)}\; P(A). \qquad (2.19)$$

Hence by (2.12), (2.19) becomes

$$\frac{\partial R}{\partial \nu_x^{(k)} \partial \nu_x^{(i)}} = n^2\ E[(Z^{(i)}Z^{(k)})|A]\ P(A) - n^2\nu_x^{(i)}H_{p+2}(c;\lambda)\nu_x^{(k)}$$

$$- n^2\nu_x^{(k)}H_{p+2}(c;\lambda)\nu_x^{(i)} + n^2\nu_x^{(i)}\nu_x^{(k)}\ P(A) \qquad (2.20)$$

Therefore, equating (2.18) and (2.20 we have

$$E[Z^{(i)}Z^{(k)}|A]\ P(A) = H_{p+4}(c;\lambda)\nu_x^{(i)}\nu_x^{(k)} \qquad (2.21)$$

We may now let $E[\underline{ZZ}'|A]P(A) = D$ where D is a pxp matrix with i-th diagonal element = $(\nu_x^{(i)})^2 H_{p+4}(c;\lambda) + \frac{1}{n} H_{p+2}(c;\lambda)$ and the (i,k)-th off diagonal element = $H_{p+4}(c;\lambda)\nu_x^{(i)}\nu_x^{(k)}$.

Finally we note that

$$\begin{aligned}
E(\bar{y}\underline{Z}|A)P(A) &= E\{E(\underline{Z}\bar{y}|\underline{Z},A)\}P(A) \\
&= E\{\underline{Z}\ E[\bar{y}|\underline{Z}]|A\}P(A) \\
&= E\{\underline{Z}[\mu + \Sigma_{12}\Sigma_{22}^{-1}T^{-1}(\underline{Z}-\underline{\nu}_x)]|A\}P(A) \\
&= \mu\ E(\underline{Z}|A)P(A) + E(\underline{Z}\Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{Z}|A)P(A) \\
&\quad - E(\underline{Z}\Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{\nu}_x|A)P(A)
\end{aligned}$$

But since $\Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{Z}$ is a scalar,

$$\Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{Z} = \underline{Z}'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

Similarly $\Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{\nu}_x = \underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$.

Hence

$$E(\bar{y}\underline{Z}|A)P(A) = \mu E(\underline{Z}|A)P(A) + E[\underline{Z}\underline{Z}'|A]P(A)T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

$$- E(\underline{Z}|A)P(A)\underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

or

$$E(\bar{y}\underline{Z}|A)P(A) = \mu H_{p+2}(c;\lambda)\underline{\nu}_x + DT'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

$$- H_{p+2}(c;\lambda)\underline{\nu}_x\underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21} \qquad (2.22)$$

We also note $E(\bar{y}^2) = \frac{1}{n}\sigma^2 + \mu^2$. Substituting these into (2.14) we obtain

$$E(\hat{\mu}^2) = \frac{1}{n}\sigma^2 + \mu^2 - 2\mu\Sigma_{12}\Sigma_{22}^{-1}T^{-1}H_{p+2}(c;\lambda)\underline{\nu}_x$$

$$- \Sigma_{12}\Sigma_{22}^{-1}T^{-1}DT'^{-1}\Sigma_{22}^{-1}\Sigma_{21} + 2\Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{\nu}_x\underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}H_{p+2}(c;\lambda)$$

From the last section we have

$$[E(\hat{\mu})]^2 = [\mu - \Sigma_{12}\Sigma_{22}^{-1}T^{-1}H_{p+2}(c;\lambda)\underline{\nu}_x]^2$$

$$= \mu^2 - 2\mu\Sigma_{12}\Sigma_{22}^{-1}T^{-1}H_{p+2}(c;\lambda)\underline{\nu}_x$$

$$+ \Sigma_{12}\Sigma_{22}^{-1}T^{-1}[H_{p+2}(c;\lambda)]^2\underline{\nu}_x\underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21} .$$

Substitution in (2.13) yields

$$V(\hat{\mu}) = \frac{1}{n}\,\sigma^2 + \mu^2 - 2\mu\Sigma_{12}\Sigma_{22}^{-1}T^{-1}H_{p+2}(c;\lambda)\underline{\nu}_x$$

$$- \Sigma_{12}\Sigma_{22}^{-1}T^{-1}DT^{-1}\Sigma_{22}^{-1}\Sigma_{21} + 2\Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{\nu}_x\underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}H_{p+2}(c;\lambda)$$

$$- \mu^2 + 2\mu\Sigma_{12}\Sigma_{22}^{-1}T^{-1}H_{p+2}(c;\lambda)\underline{\nu}_x$$

$$- \Sigma_{12}\Sigma_{22}^{-1}T^{-1}[H_{p+2}(c;\lambda)]^2\underline{\nu}_x\underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

$$= \frac{1}{n}\,\sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}T^{-1}DT^{-1}\Sigma_{22}^{-1}\Sigma_{21} \tag{2.23}$$

$$- \Sigma_{12}\Sigma_{22}^{-1}T^{-1}[H_{p+2}(c;\lambda)]^2\underline{\nu}_x\underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

$$+ 2\Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{\nu}_x\underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}H_{p+2}(c;\lambda)$$

Again as partial checks, when $c=0$, we always use $\hat{\mu} = \bar{y}$ and $V(\hat{\mu}) = \frac{1}{n}\,\sigma^2$. When $c=\infty$, we always use $\hat{\mu} = \bar{y} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\bar{X}}$ and $V(\hat{\mu}) = \frac{1}{n}(\sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}) = V(\hat{\hat{\mu}})$.

Now the M.S.E. is defined as M.S.E. = Variance + $(\text{Bias})^2$. Hence

$$\text{M.S.E.}(\hat{\mu}) = \frac{1}{n}\,\sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}T^{-1}DT'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

$$- \Sigma_{12}\Sigma_{22}^{-1}T^{-1}[H_{p+2}(c;\lambda)]^2\underline{\nu}_x\underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

$$+ \Sigma_{12}\Sigma_{22}^{-1}T^{-1}[H_{p+2}(c;\lambda)]^2\underline{\nu}_x\underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

$$+ 2\Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{\nu}_x\underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}H_{p+2}(c;\lambda)$$

$$\text{M.S.E.}(\hat{\mu}) = \frac{1}{n}\sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}T^{-1}DT'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

$$+ 2\Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{\nu}_x\underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}H_{p+2}(c;\lambda)$$

$$= \frac{1}{n}\sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}H_{p+4}(c;\lambda)$$

$$- \frac{1}{n}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}H_{p+2}(c;\lambda)$$

$$+ 2\,\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}H_{p+2}(c;\lambda) \qquad (2.24)$$

D. Relative Efficiency (e)

In practice, we may want to select an estimator for μ with the smallest bias and M.S.E. Since bias is a part of M.S.E., it is reasonable to consider only the M.S.E. Using (2.24) we may compare the performance of the preliminary test estimator, $\hat{\mu}$, with the usual estimator $\bar{y}$. The relative efficiency of $\hat{\mu}$ to $\bar{y}$ is defined as

$$e = \frac{1}{\text{M.S.E.}(\hat{\mu})} \Big/ \frac{1}{\text{M.S.E.}(\bar{y})} \qquad (2.25)$$

Now using (2.24) and since $\bar{y}$ is unbiased, $\text{M.S.E.}(\bar{y}) = V(\bar{y}) = \frac{1}{n}\sigma^2$. Hence

$$e = \frac{\frac{1}{n}\sigma^2}{\frac{1}{n}\sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}H_{p+4}(c;\lambda) - \frac{1}{n}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}H_{p+2}(c;\lambda) + 2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}H_{p+2}(c;\lambda)}$$

$$= \frac{1}{1+k(a)}$$

where

$$k(a) = \frac{n}{\sigma^2} \{-\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}H_{p+4}(c;\lambda)-\frac{1}{n}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}H_{p+2}(c;\lambda)$$

$$+ 2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}H_{p+2}(c;\lambda)\} \qquad (2.26)$$

Wlog we let $\Sigma_{22} = I$ and $\sigma^2 = 1$. Therefore for p=1, $\Sigma_{12} = \rho$. Table 2.3 gives the values of e for p=1 and some choices of ρ, a and α.

Table 2.3. Values of e for p=1.

a	α = .05			α = .5		
		ρ			ρ	
	.1	.5	.9	.1	.5	.9
0	1.0073	1.2199	2.4036	1.0007	1.0182	1.0613
.5	1.0044	1.1244	1.5586	1.0003	1.0084	1.0277
1.0	.9974	.9398	.8281	.9996	.9898	.9677
1.5	.9900	.7976	.5488	.9992	.9793	.9360
2.0	.9858	.7357	.4621	.9992	.9812	.9417
2.5	.9866	.7462	.4757	.9996	.9889	.9649
3.0	.9906	.8078	.5646	.9998	.9953	.9850

From Table 2.3 we observe that for fixed n, ρ and α, the relative efficiency of $\hat{\mu}$ assumes its maximum value when $\mu_x = 0$, it then decreases to a minimum and then increases as μ_x increases. For fixed n, μ_x and α, e is an increasing function of ρ while for fixed n, μ_x and ρ, e is a decreasing function of α.

The selection procedure for an estimator or the level of the preliminary test such that the relative efficiency is the largest when $\underline{\mu}_x$ equals the origin, say $\underline{0}$, suggested by the experimenter's prior knowledge, and is at least as large as some e_{min} when $\underline{\mu}_x \neq \underline{0}$ was first recommended by Han and Bancroft (1968) and was later used by Han (1973a) for the case p=1 for the present problem. The values of e_{min} and e_{max} at certain values of ρ and α are given in Table 2.4 where e_{max} is the value of e at $\mu_x = 0$.

Table 2.4. Values of e_{min} and e_{max} for p=1.

α \ ρ		.1	.5	.9
.50	e_{max}	1.0007	1.0182	1.0613
	e_{min}	0.9992	0.9793	0.9340
.05	e_{max}	1.0073	1.2199	2.4036
	e_{min}	0.9858	0.7337	0.4621

Thus for $\rho = .9$, a preliminary test at $\alpha = .05$ ensures the relative efficiency of the preliminary test estimator will be at least 0.4621 and may be as large as 2.4036 when the null hypothesis of the preliminary test is true or $\mu_x = 0$. For a more detailed table and full discussion on the properties and uses of the above table, one is referred to Han (1973a). The

Table 2.5. Values of e for p=2.

Σ_{12}' / $(\underline{\mu}_x\sqrt{n})'$	α = .05			
	$\begin{pmatrix}-.5\\.5\end{pmatrix}$	$\begin{pmatrix}.5\\.5\end{pmatrix}$	$\begin{pmatrix}-.5\\.7\end{pmatrix}$	$\begin{pmatrix}.7\\.7\end{pmatrix}$
(0, 0)	1.6673	1.6673	2.4530	4.6391
(.5, 0)	1.4792	1.4792	2.0381	2.7394
(.5, .5)	1.5930	1.1501	2.1800	1.3437
(1.0, 0)	1.3336	1.1336	1.3953	1.3004
(1.0, .5)	1.3819	.8604	1.8882	.7587
(1.0,1.0)	1.4124	.6690	1.6684	.5077
(1.5, 0)	.8723	.8723	.9879	.7771
(1.5, .5)	1.0933	.6848	1.3871	.5257
(1.5,1.0)	1.2382	.5648	1.5209	.3984
(1.5,1.5)	1.2219	.5076	1.2865	.3447
(2.0, 0)	.7348	.7348	.7908	.5857
(2.0, .5)	.9004	.6071	1.0595	.4408
(2.0,1.0)	1.0499	.5313	1.2508	.3664
(2.0,1.5)	1.1142	.5050	1.2257	.3424
(2.0,2.0)	1.0914	.5257	1.0923	.3612
(2.5, 0)	.6950	.6950	.7253	.5376
(2.5, .5)	.8169	.6053	.9072	.4390
(2.5,1.0)	.9398	.5580	1.0633	.3917
(2.5,1.5)	1.0198	.5538	1.1127	.3877
(2.5,2.0)	1.0409	.5924	1.0734	.4258
(2.5,2.5)	1.0279	.6708	1.0198	.5097
(3.0, 0)	.7273	.7273	.7444	.5764
(3.0, .5)	.8152	.6650	.8676	.5031
(3.0,1.0)	.9066	.6383	.9779	.4738
(3.0,1.5)	.9743	.6501	1.0336	.4867
(3.0,2.0)	1.0056	.6987	1.0356	.5419
(3.0,2.5)	1.0105	.7753	1.0168	.6377
(3.0,3.0)	1.0059	.8606	1.0020	.7591

Table 2.5. (continued)

Σ'_{12} / $(\underline{\mu}_x \sqrt{n})'$	$\alpha = .20$			
	$\begin{pmatrix}-.5\\.5\end{pmatrix}$	$\begin{pmatrix}.5\\.5\end{pmatrix}$	$\begin{pmatrix}-.5\\.7\end{pmatrix}$	$\begin{pmatrix}.7\\.7\end{pmatrix}$
(0, 0)	1.3141	1.3141	1.5475	1.8816
(.5, 0)	1.2197	1.2197	1.4032	1.5458
(.5, .5)	1.2634	1.0490	1.4329	1.1008
(1.0, 0)	1.0414	1.0414	1.1453	1.0845
(1.0, .5)	1.1574	.8859	1.3128	.7985
(1.0,1.0)	1.1579	.7726	1.2265	.6342
(1.5, 0)	.9099	.9099	.9633	.8375
(1.5, .5)	1.0250	.7960	1.1295	.6657
(1.5,1.0)	1.0818	.7262	1.1614	.5751
(1.5,1.5)	1.0691	.7118	1.0833	.5575
(2.0, 0)	.8570	.8570	.8844	.7536
(2.0, .5)	.9464	.7803	1.0084	.6444
(2.0,1.0)	1.0107	.7415	1.0705	.5941
(2.0,1.5)	1.0307	.7477	1.0584	.6020
(2.0,2.0)	1.0218	.7932	1.0208	.6619
(2.5, 0)	.8667	.8667	.8805	.7684
(2.5, .5)	.9271	.8196	.9611	.6986
(2.5,1.0)	.9774	.8020	1.0134	.6740
(2.5,1.5)	1.0031	.8183	1.0257	.6967
(2.5,2.0)	1.0081	.8608	1.0145	.7593
(2.5,2.5)	1.0048	.9130	1.0030	.8425
(3.0, 0)	.9095	.9095	.9156	.8367
(3.0, .5)	.9439	.8849	.9603	.7968
(3.0,1.0)	.9748	.8800	.9928	.7891
(3.0,1.5)	.9936	.8960	1.0059	.8146
(3.0,2.0)	1.0006	.9255	1.0058	.8637
(3.0,2.5)	1.0014	.9566	1.0023	.9183
(3.0,3.0)	1.0007	.9797	1.0002	.9609

Table 2.5. (continued)

$(\underline{\mu}_x \sqrt{n})'$ \ Σ'_{12}	α = .50			
	$\begin{pmatrix}-.5\\.5\end{pmatrix}$	$\begin{pmatrix}.5\\.5\end{pmatrix}$	$\begin{pmatrix}-.5\\.7\end{pmatrix}$	$\begin{pmatrix}.7\\.7\end{pmatrix}$
(0, 0)	1.0831	1.0831	1.1280	1.1769
(.5, 0)	1.0571	1.0571	1.0958	1.1184
(.5, .5)	1.0673	1.0068	1.1002	1.0134
(1.0, 0)	1.0055	1.0055	1.0313	1.0109
(1.0, .5)	1.0378	.9546	1.0703	.9148
(1.0,1.0)	1.0361	.9173	1.0489	.8498
(1.5, 0)	.9688	.9688	.9833	.9407
(1.5, .5)	1.0029	.9310	1.0266	.8732
(1.5,1.0)	1.0167	.9107	1.0322	.8389
(1.5,1.5)	1.0130	.9162	1.0151	.8480
(2.0, 0)	.9607	.9607	.9677	.9258
(2.0, .5)	.9856	.9382	.9994	.8857
(2.0,1.0)	1.0010	.9303	1.0123	.8720
(2.0,1.5)	1.0049	.9399	1.0094	.8885
(2.0,2.0)	1.0031	.9596	1.0028	.9237
(2.5, 0)	.9713	.9713	.9742	.9453
(2.5, .5)	.9852	.9606	.9917	.9256
(2.5,1.0)	.9956	.9590	1.0016	.9227
(2.5,1.5)	1.0002	.9668	1.0034	.9369
(2.5,2.0)	1.0010	.9788	1.0017	.9593
(2.5,2.5)	1.0005	.9894	1.0003	.9795
(3.0, 0)	.9853	.9853	.9863	.9717
(3.0, .5)	.9913	.9814	.9938	.9642
(3.0,1.0)	.9964	.9817	.9988	.9648
(3.0,1.5)	.9991	.9859	1.0006	.9727
(3.0,2.0)	1.0000	.9914	1.0006	.9832
(3.0,2.5)	1.0001	.9958	1.0002	.9919
(3.0,3.0)	1.0000	.9984	1.0000	.9969

few values for p=1 given here in Table 2.4 are only computed as a partial check of the general results obtained in this paper. The values agree with the results of Han (1973a).

For p=2, the values of e are given in Table 2.5 for some choices of Σ_{12}, $\underline{\mu}_x \sqrt{n}$ and α. Since e is a symmetric function of Σ_{12} and $\underline{\mu}_x$, values are computed for only positive values of Σ_{12} and $\underline{\mu}_x$ when the components are identical.

From Table 2.5 and (2.26) we note the following properties of e for p=2.

1. The relative efficiency is maximum when $\underline{\mu}_x = \underline{0}$ for fixed n, α and Σ_{12}. This corresponds to the case when the null hypothesis is true.

2. The maximum value of e increases with Σ_{12} for any given α but decreases as α increases for a given Σ_{12}.

3. For fixed n, $\underline{\mu}_x$, α and Σ_{12}, the relative efficiency is generally larger when the components of Σ_{12} have different signs than when the signs are identical.

4. The relative efficiency remains the same for values of Σ_{12} which differ only in sign.

5. For fixed α, n, Σ_{12} and some component of $\underline{\mu}_x$, the relative efficiency decreases to a minimum and then increases as the other component increases.

We also observe that since Σ is positive definite, its determinant is greater than zero. Consequently for identical components of Σ_{12}, say $\Sigma_{12} = (a, a)$, then

$$\Sigma = \begin{pmatrix} 1 & a & a \\ a & 1 & 0 \\ a & 0 & 1 \end{pmatrix}$$

$$\Longrightarrow |\Sigma| = 1 - a^2 - a^2$$
$$= 1 - 2a^2 > 0$$
$$\Longrightarrow \quad a^2 < \frac{1}{2} \quad \text{or} \quad |a| < \frac{1}{\sqrt{2}} \doteq .70$$

Thus the relative efficiency and Bias of $\hat{\mu}$ do not exist for values such as $\Sigma_{12} = (.9, .9), (.8, .8)$. Similarly for non-identical components of Σ_{12}, say $\Sigma_{12} = (a, b)$, then

$$\Sigma = \begin{pmatrix} 1 & a & b \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix}$$

$$\Longrightarrow |\Sigma| = 1 - a^2 - b^2 > 0$$
$$\Longrightarrow a^2 + b^2 < 1 .$$

Hence the relative efficiency and bias of $(\hat{\mu})$ do not exist for such values of Σ_{12} as $(.9, .7)$, etc.

Following Han (1973a), it is possible to extend the computation of e_{max} and e_{min} to any value of p so that an investigator can select an estimator or α such that e_{max} occurs when $\underline{\mu}_x = \underline{0}$ and e is at least as large as e_{min} when $\underline{\mu}_x \neq \underline{0}$. Table 2.6 gives the values of e_{max} and e_{min} for some choices of α,

Σ_{12} = (.5, .5) and p=2. It also gives $\underline{\mu}_x^*$ which is the value of $\underline{\mu}_x$ about which e_{min} occurs for a search at .05 intervals.

Table 2.6. Values of e_{min} and e_{max} for p=2.

Σ_{12} \ α		.05	.10	.20	.30	.40	.50
	e_{max}	1.6673	1.5032	1.3141	1.2040	1.1322	1.0831
(.5,.5)	e_{min}	.4976	.5904	.7098	.7941	.8587	.9085
	$\underline{\mu}_x^{*\prime}$	(1.65, 1.65)	(1.60, 1.60)	(1.40, 1.40)	(1.35, 1.35)	(1.35, 1.35)	(1.35, 1.35)

Thus for a relative efficiency of at least .75, with the above selection procedure, the investigator would use α = .30 for the preliminary test when Σ_{12} = (.5,.5). This choice guarantees a relative efficiency of at least .79. The relative efficiency in this case can be as large as 1.2040. Also from Table 2.6, we observe as before that for fixed Σ_{12},

1. e_{max} is a decreasing function of α,
2. e_{min} is an increasing function of α, and
3. $\underline{\mu}_x^*$ has identical components and is a decreasing function of α. We note that the negative values of $\underline{\mu}_x^*$ also give the same minimum values.

E. Bias of $\hat{\mu}$ when Σ is Unknown

When Σ is unknown and assume that $\underline{\mu}_0 = \underline{0}$, the preliminary test estimator is defined as

$$\hat{\mu} = \begin{cases} \bar{y} - S_{12}S_{22}^{-1}\,\underline{\bar{X}} & \text{if} \quad nm(\underline{\bar{X}}'S_{22}^{-1}\underline{\bar{X}}) \le T_0^2 \\ \bar{y} & \text{if} \quad nm(\underline{\bar{X}}'S_{22}^{-1}\underline{\bar{X}}) > T_0^2 \end{cases} \tag{2.27}$$

where $m = n-1$, T_0^2 is the $100(1-\alpha)$th percentile of a central Hotelling's T^2 distribution with m degrees of freedom, and

$$\underline{\bar{X}} = \frac{1}{n} \sum_{i=1}^{n} \underline{X}_i$$

$$S_{22} = \sum_{i=1}^{n} (\underline{X}_i - \underline{\bar{X}})(\underline{X}_i - \underline{\bar{X}})'$$

$$S_{12} = \sum_{i=1}^{n} (y_i - \bar{y})(\underline{X}_i - \underline{\bar{X}})' \tag{2.28}$$

$$S_{11} = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

In this section, we shall obtain the bias of $\hat{\mu}$. If we denote the acceptance region for the preliminary test $\{nm(\underline{\bar{X}}'S_{22}^{-1}\underline{\bar{X}}):mn(\underline{\bar{X}}'S_{22}^{-1}\underline{\bar{X}}) \le T_0^2\}$ by G, then

$$\begin{aligned} E(\hat{\mu}) &= E\{(\bar{y} - S_{12}S_{22}^{-1}\underline{\bar{X}})|G\}\ P(G) + E\{\bar{y}|\bar{G}\}\ P(\bar{G}) \\ &= E(\bar{y}) - E\{S_{12}S_{22}^{-1}\underline{\bar{X}}|G\}\ P(G) \end{aligned} \tag{2.29}$$

Since $E(\bar{y}) = \mu$, the second term is the bias and we write

$$b = -\ E\{S_{12}S_{22}^{-1}\underline{\bar{X}}|G\}\ P(G) \tag{2.30a}$$

Let $f(\underline{\bar{X}})$ be the multivariate normal density of $\underline{\bar{X}}$ and $g(S_{22},S_{12},S_{11})$ be the density of S_{22}, S_{11} and S_{12} which have a Wishart distribution. Then

$$E\{S_{12}S_{22}^{-1}\underline{\bar{X}}\,|\,G\}\ P(G)$$
(2.30b)
$$= \int\underset{G}{\cdots}\int S_{12}S_{22}^{-1}\underline{\bar{X}}\ f(\underline{\bar{X}})g(S_{22},S_{12},S_{11})d\underline{\bar{X}}dS_{22}dS_{12}dS_{11}$$

Following Han (1973a) and Rao (1965), we make the following transformations. Since S_{22} is positive definite, there exists a nonsingular matrix $B \ni B'B = S_{22}$. Also $\exists$ a nonsingular matrix $T \ni T'T = \Sigma_{22}^{-1}$. Let

$$W_1 = TB'BT'$$

$$W_2 = [S_{12} - \Sigma_{12}T'TB'B]\Sigma_{11\cdot2}^{-\frac{1}{2}}B^{-1}\ .$$

But since $\Sigma_{11\cdot2}$ is a constant scalar, we let

$$\Sigma_{11\cdot2} = K^2 \tag{2.31}$$

Therefore

$$W_2 = \frac{1}{K}[S_{12}B^{-1} - \Sigma_{12}T'TB']$$

$$W_3 = K^2(S_{11} - S_{12}B^{-1}B'^{-1}S_{21})$$

From (2.31),

$$B'B = T^{-1}W_1T'^{-1} = T^{-1}W_1^{\frac{1}{2}}W_1^{\frac{1}{2}'}T'^{-1}$$

$$S_{12} = [KW_2 + \Sigma_{12}T'TB']\,B$$

Therefore,

$$S_{12}S_{22}^{-1} = S_{12}B^{-1}B'^{-1} = KW_2W_1^{\frac{1}{2}}T + \Sigma_{12}T'T$$

Substituting in (2.30b), we have

$$E\{(S_{12}S_{22}^{-1}\underline{\bar{X}})/G\}\ P(G)$$

$$= \int_G\cdots\int (KW_2W_1^{\frac{1}{2}}T + \Sigma_{12}T'T)\underline{\bar{X}}f(\underline{\bar{X}})g(W_1,W_2,W_3)dW_3dW_2dW_1d\underline{\bar{X}} \tag{2.32}$$

We claim

$$W_1 \sim W(I,n-1)$$

$$W_2 \sim N(\underline{0},I)$$

$$W_3 \sim W(1,n-p-1)$$

and the three are mutually independent. To prove the above claim, we note

(a) $\quad S_{22} = B'B \sim W(T^{-1}T'^{-1},n-1)$

$$W_1 = TB'BT' \sim W(TT^{-1}T'^{-1}T',\ n-1)$$

$$= W(I,n-1)\ .$$

(b) Given the $\underline{X}_i$'s, the conditional distribution of W_2' is normal since $W_2'|\underline{X}_i$'s is a linear combination of the y_i's which are normally distributed. If we denote this conditional distribution by $g(W_2'|\underline{X})$, and write

$$W_2' = \frac{1}{K} B'^{-1}(S_{21} - B'BT'T\Sigma_{21})$$

then we only need find the mean and variance of $W_2'|\underline{X}$.

$$S_{21} = \sum_{i=1}^{n} (\underline{X}_i - \overline{\underline{X}})(Y_i - \overline{Y}) = \sum_{i=1}^{n} (\underline{X}_i - \overline{\underline{X}})Y_i \ .$$

Now there exists an n-th order Helmert matrix $C = (c_{ij})$ such that $\underline{U}_i' = (U_{1i} \dots U_{pi})$, U_{ji} = i-th element of $C\underline{X}_j$, ω_i = i-th element of $C\underline{Y}$ so that making an orthogonal transformation,

$$S_{21} = \sum_{i=1}^{m} \underline{U}_i \, \omega_i \quad \text{where } m = n-1.$$

$$\omega_i | \underline{U}_i \sim N(\Sigma_{12}T'T\underline{U}_i \ , \ \Sigma_{11\cdot 2})$$

$$= N(\underline{U}_i'T'T\Sigma_{21} \ , \ \Sigma_{11\cdot 2}) \ .$$

Therefore

$$E(S_{21}|\underline{U}) = \sum_{i=1}^{m} \underline{U}_i\underline{U}_i'T'T'\Sigma_{21}$$

$$= B'BT'T\Sigma_{21} \ .$$

Hence

$$E(W_2|\underline{U}) = \frac{1}{K} B'^{-1}(B'BT'T\Sigma_{21} - B'BT'T\Sigma_{21})$$

$$= \underline{0} \ .$$

$$\mathrm{Var}(W_2'|\underline{U}) = \frac{1}{K^2} B'^{-1} V(S_{21}|\underline{U})B^{-1}$$

$$= \frac{1}{K^2} B'^{-1} \left(\sum_{i=1}^{m} \underline{U}_i \ \Sigma_{11\cdot 2} \ \underline{U}_i'\right)B^{-1}$$

where recall $K^2 = \Sigma_{11\cdot 2}$ is a scalar.

$$= \frac{1}{K^2} \ \Sigma_{11\cdot 2} \ B'^{-1}B'BB^{-1} = I'$$

Therefore $(W_2|\underline{X}) \sim N(\underline{0},I)$ and this does not depend on $\underline{X}$. Hence $W_2 \sim N(\underline{0},I)$.

(c) $W_3 = \frac{1}{K^2}(S_{11} - S_{12}B^{-1}B'^{-1}S_{21})$ and from Anderson (1958), Theorems 4.3.2 and 4.3.3, we know $K^2W_3 = S_{11}-S_{12}B^{-1}B'^{-1} \sim W(\Sigma_{11\cdot 2}\ , n-p-1)$ and hence $W_3 \sim W(1, n-p-1)$. Finally, to establish the mutual independence of W_1, W_2 and W_3 we note that by fixing the $\underline{X}_i$'s, we also fix W_1 and since the conditional distribution of either W_2 or W_3 with $\underline{X}_i$ fixed does not depend on $\underline{X}$, then either conditional distribution is equivalent to the actual unconditional distribution and each is independent of W_1. Thus W_1 is independent of W_2 and W_3. To show W_2 and W_3 independent we employ Cochran's theorem as follows:

Let

$$u_i = y_i - \mu - \Sigma_{12}T'T(\underline{X}_i-\underline{\mu}_{x_i}) \sim NID(0,\Sigma_{11\cdot 2})$$

and

$$\underline{V} = \underline{X} - \underline{\mu}_x \sim N(\underline{0},T^{-1}T'^{-1})$$

be fixed. Then

$$\bar{u} = \bar{y} - \mu - \Sigma_{12}T'T(\bar{\underline{X}}_i-\underline{\mu}_x) \sim N(0,\tfrac{1}{n}\Sigma_{11\cdot 2})$$

$$\Longrightarrow \frac{n\bar{u}^2}{\Sigma_{11\cdot 2}} = \frac{n[\bar{y} - \mu - \Sigma_{12}T'T(\bar{\underline{X}}_i-\underline{\mu}_x)]^2}{\Sigma_{11\cdot 2}} \sim \chi^2(1)$$

$$S_{u\underline{V}} = \sum_{i=1}^{n} u_i(\underline{V}_i - \bar{\underline{V}})' = S_{12}-\Sigma_{12}T'TB'B \sim N(\underline{0},\Sigma_{11\cdot 2}B'B)$$

$$\Longrightarrow W_2 = \Sigma_{11\cdot 2}^{-\frac{1}{2}} (S_{12}-\Sigma_{12}T^{-1}T'^{-1}B'B)B^{-1} \sim N(\underline{0},I)$$

$$\Longrightarrow \Sigma_{11\cdot 2}^{-1}(S_{u\underline{V}}S_{\underline{VV}}^{-1}S_{\underline{V}u}) = W_2W_2' \sim \chi^2(p)$$

where

$$S_{\underline{VV}} = \sum_{i=1}^{n} (\underline{V}_i-\bar{\underline{V}})(\underline{V}_i-\bar{\underline{V}})' = B'B.$$

Similarly defining

$$S_{uu} = \sum_{i=1}^{n} (u_i-\bar{u})^2 ,$$

then finally

$$\Sigma_{11\cdot 2}^{-1}(S_{uu} - S_{u\underline{V}}S_{\underline{VV}}^{-1}S_{\underline{V}u})$$

$$= W_3 = \Sigma_{11\cdot 2}^{-1}(S_{11}-S_{12}B^{-1}B'^{-1}S_{21}) \sim \chi^2(n-p-1) \ .$$

But

$$S_{uu} = \sum_{i=1}^{n} u_i^2 - n\bar{u}^2 = S_{u\underline{V}}S_{\underline{VV}}^{-1}S_{\underline{V}u} + (S_{uu}-S_{u\underline{V}}S_{\underline{VV}}S_{\underline{V}u})$$

and hence

$$\Sigma_{11\cdot 2}^{-1} \sum_{i=1}^{n} u_i^2 = \Sigma_{11\cdot 2}^{-1} n\bar{u}^2 + \Sigma_{11\cdot 2}^{-1}(S_{u\underline{V}}S_{\underline{VV}}^{-1}S_{\underline{V}u})$$
$$+ \Sigma_{11\cdot 2}^{-1} (S_{uu}-S_{u\underline{V}}S_{\underline{VV}}^{-1}S_{\underline{V}u})$$

or

$$\Sigma_{11\cdot 2}^{-1} \sum_{i=1}^{n} (y_i-\mu)^2 - 2(y_i-\mu)\Sigma_{12}T'T(\underline{X}_i-\underline{\mu}_{x_i})$$
$$+ (\underline{X}_i-\mu_{x_i})'T'T\Sigma_{21}\Sigma_{12}T'T(\underline{X}_i-\underline{\mu}_{x_i})$$
$$= \Sigma_{11\cdot 2}^{-1} n[\bar{y}-\mu-\Sigma_{12}T'T(\bar{\underline{X}}_i-\underline{\mu}_x)]^2 + W_2W_2' + W_3$$

or

$$\chi^2(n) = \chi^2(1) + \chi^2(p) + \chi^2(n-p-1) \ .$$

Thus W_2 and W_3 are independent.

Therefore W_1, W_2 and W_3 have a joint distribution given by

$$g(W_1,W_2,W_3) = C_0 e^{-\frac{1}{2}tr(W_2'W_2+W_3+W_1)} |W_3|^{\frac{1}{2}(n-2p-2)} |W_1|^{\frac{1}{2}(n-p-2)} \qquad (2.33)$$

The region of integration is given by

$$G = \{nm(\underline{\bar{X}}'T'W_1^{-1}T\underline{\bar{X}}) \ : \ nm(\underline{\bar{X}}'T'W_1^{-1}T\underline{\bar{X}}) \leq T_0^2\}$$

The integral in (2.32) becomes

$$\int_G \cdots \int \int_{-\infty}^{\infty} \int_0^{\infty} (KW_2W_1^{-\frac{1}{2}}T+\Sigma_{12}T'T)\underline{\bar{X}}f(\underline{\bar{X}})C_0 e^{-\frac{1}{2}tr(W_2'W_2+W_3+W_1)} |W_3|^{\frac{1}{2}(n-2p-2)}$$

$$|W_1|^{-\frac{1}{2}(n-p-2)} dW_3dW_2dW_1d\underline{\bar{X}} \ .$$

$$= \int_G \cdots \int \int_{-\infty}^{\infty} \int_0^{\infty} KC_0W_1^{-\frac{1}{2}}Te^{-\frac{1}{2}tr(W_2'W_2+W_3+W_1)} |W_3|^{\frac{1}{2}(n-2p-2)}$$

$$W_2|W_1|^{\frac{1}{2}(n-p-2)} \underline{\bar{X}}f(\underline{\bar{X}})dW_3dW_2dW_1d\underline{\bar{X}}$$

$$+ \int_G \cdots \int \int_{-\infty}^{\infty} \int_0^{\infty} C_0\Sigma_{12}T'Te^{-\frac{1}{2}tr(W_2'W_2+W_3+W_1)} |W_3|^{\frac{1}{2}(n-2p-2)}$$

$$|W_1|^{\frac{1}{2}(n-p-2)} \underline{\bar{X}}f(\underline{\bar{X}})dW_3dW_2dW_1d\underline{\bar{X}}$$

But from independence and the fact $E(W_2) = \underline{0}$, we know the first term is zero. The second integral is equivalent to

$$\frac{\Sigma_{12}T'T}{2^{\frac{1}{2}p(n-1)}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma[\frac{1}{2}(n-i)]} \int_G \cdots \int e^{-\frac{1}{2}trW_1} |W_1|^{\frac{1}{2}(n-p-2)} \underline{\bar{X}}f(\underline{\bar{X}})d\underline{\bar{X}}dW_1$$

Now $\underline{\bar{X}} \sim N(\underline{\mu}_x, \frac{1}{n}T^{-1}T'^{-1})$. Let $\underline{Z} = T\underline{\bar{X}}$. Therefore

$$\underline{Z} \sim N(T\underline{\mu}_x, \frac{1}{n} I)$$

$$= N(\underline{\nu}_x, \frac{1}{n} I) \text{ say.}$$

Now $\bar{\underline{X}} = T^{-1}\underline{Z}$ and $\bar{\underline{X}}'T'W_1^{-1}T\bar{\underline{X}} = \underline{Z}'T'^{-1}T'W_1^{-1}TT^{-1}\underline{Z} = \underline{Z}'W_1^{-1}\underline{Z}$, we have $G = \{nm(\underline{Z}'W_1^{-1}\underline{Z}) : nm(\underline{Z}'W_1^{-1}\underline{Z}) \leq T_0^2\}$. Hence we wish to evaluate

$$\frac{\Sigma_{12}T'TT^{-1}}{2^{\frac{1}{2}p(n-1)}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma[\frac{1}{2}(n-i)]}\int\cdots\int_G e^{-\frac{1}{2}trW_1}|W_1|^{\frac{1}{2}(n-p-2)}\underline{Z}g(\underline{Z})d\underline{Z}dW_1 \tag{2.34}$$

where $nm(\underline{Z}'W_1^{-1}\underline{Z}) = T^2$ has the Hotelling's T^2 distribution with n-1 degrees of freedom and $\frac{T^2(n-p)}{p(n-1)} = F^*$ has the noncentral F-distribution with p and n-p degrees of freedom and non-centrality parameter $\lambda = n\underline{\nu}_x'\underline{\nu}_x$.

Following Alam and Risvi (1967), we define a random variable G given by

$$G = \frac{PF^*}{n-p} = \frac{T^2}{n-1} .$$

G has the density function

$$f(g) = f_{p,n-p}(g,\lambda) = \frac{e^{-\frac{1}{2}\lambda}}{\Gamma(n-p/2)}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\lambda}{2}\right)^j G_{(\frac{p}{2}+j,\frac{n-p}{2})}(g)dg \qquad g>0,\lambda>0 \tag{2.35}$$

where

$$G_{(\frac{p}{2}+j,\frac{n-p}{2})}(g) = \frac{g^{\frac{p}{2}+j-1}\,\Gamma(\frac{p}{2}+j+\frac{n-p}{2})}{(1+g)^{(\frac{p}{2}+j+\frac{n-p}{2})}\,\Gamma(\frac{p}{2}+j)}$$

Therefore

$$\begin{aligned} P(G) &= P(T^2 \le T_0^2) \\ &= P(T^2 \le \frac{p(n-1)}{n-p} F_{p,n-p}(\alpha)) \\ &= P(\frac{T^2}{n-1} \le \frac{p}{n-p} F_{p,n-p}(\alpha)) \\ &= P(G \le c) \end{aligned} \tag{2.36}$$

where G has the density function given in (2.35) and $c = \frac{p}{n-p} F_{p,n-p}(\alpha)$ where $F_{p,n-p}(\alpha)$ is the 100(1-α) percent point of the central F-distribution with p and n-p degrees of freedom. Therefore

$$R = P(G) = \int_0^c \frac{e^{-\frac{1}{2}\lambda}}{\Gamma(\frac{n-p}{2})} \sum_{j=0}^{\infty} \frac{1}{j!} (\frac{\lambda}{2})^j G_{(\frac{p}{2}+j,\frac{n-p}{2})}(g)dg \tag{2.37}$$

Differentiating (2.37) w.r.t. $\nu_x^{(i)}$, we have

$$\begin{aligned} \frac{\partial R}{\partial \nu_x^{(i)}} &= \int_0^c \frac{e^{-\frac{1}{2}\lambda}}{\Gamma(\frac{n-p}{2})} \sum_{j=0}^{\infty} \frac{1}{j!}\, j(\frac{\lambda}{2})^{j-1} \frac{2n\nu_x^{(i)}}{2}\, G_{(\frac{p}{2}+j,\frac{n-p}{2})}(g)dg \\ &- \int_0^c \frac{\frac{2n\nu_x^{(i)}}{2}\, e^{-\frac{1}{2}\lambda}}{\Gamma(\frac{n-p}{2})} \sum_{j=0}^{\infty} \frac{1}{j!} (\frac{\lambda}{2})^j\, G_{(\frac{p}{2}+j,\frac{n-p}{2})}(g)dg \end{aligned} \tag{2.38a}$$

For the first term if we let $j-1 = j'$, then $j = j'+1$ and consequently we may write

$$\frac{\partial R}{\partial \nu_x^{(i)}} = \int_0^c \frac{e^{-\frac{1}{2}\lambda}}{\Gamma(\frac{n-p}{2})} \sum_{j'=0}^{\infty} \frac{2n\nu_x^{(i)}}{2} \frac{1}{j'!} \left(\frac{\lambda}{2}\right)^{j'} G_{(\frac{p+2}{2}+j', \frac{n-p}{2})}(g)\,dg$$

$$- \int_0^c \frac{2n\nu_x^{(i)}}{2\Gamma(\frac{n-p}{2})} e^{-\frac{1}{2}\lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\lambda}{2}\right)^{j} G_{(\frac{p}{2}+j, \frac{n-p}{2})}(g)\,dg$$

so that

$$\frac{\partial R}{\partial \nu_x^{(i)}} = n\nu_x^{(i)}\, G^*_{p+2,n-p}(c;\lambda) - n\nu_x^{(i)}\, P(G) \tag{2.38b}$$

where $G^*_{p+2,n-p}(c;\lambda)$ is the cumulative distribution of the non-central G random variable with $p+2$, $n-p$ degrees of freedom and noncentrality parameter λ. Also making use of the separate distributions of $\underline{Z}$ and W_1 and noting that these are independent, we may write

$$R = P(G) = \int_G \cdots \int \prod_{j=1}^{p} \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{n}{2}(Z^{(j)}-\nu_x^{(j)})^2} \tag{2.39}$$

$$\cdot \frac{1}{2^{\frac{p}{2}(n-1)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma[\frac{1}{2}(n-i)]} |W_1|^{\frac{1}{2}(n-p-2)} e^{-\frac{1}{2}\mathrm{tr}W_1} dZ^{(j)} dW_1$$

Differentiating (2.39) w.r.t. $\nu_x^{(i)}$, we obtain

$$\frac{\partial R}{\partial \nu_x^{(i)}} = \int_G \cdots \int \prod_{j=1}^{p} \frac{\sqrt{n}}{\sqrt{2\pi}} \cdot \frac{n}{2} \cdot 2(Z^{(i)} - \nu_x^{(i)}) e^{-\frac{n}{2}(Z^{(j)} - \nu_x^{(j)})^2}$$

$$\cdot \frac{|W_1|^{\frac{1}{2}(n-p-2)} e^{-\frac{1}{2}\mathrm{tr}W_1}}{2^{\frac{p}{2}(n-1)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma[\frac{1}{2}(n-i)]} g(\underline{Z}) d\underline{Z}^{(j)} dW_1$$

Hence

$$\frac{\partial R}{\partial \nu_x^{(i)}} = \frac{n}{2^{\frac{p}{2}(n-1)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma[\frac{1}{2}(n-i)]} \int_G \cdots \int e^{-\frac{1}{2}\mathrm{tr}W_1} |W_1|^{\frac{1}{2}(n-p-2)} \tag{2.40}$$

$$Z^{(i)} g(\underline{Z}) d\underline{Z} dW_1 - n\nu_x^{(i)} P(G)$$

Equating (2.38b) and (2.40) we have

$$\frac{1}{2^{\frac{p}{2}(n-1)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma[\frac{1}{2}(n-i)]} \int_G \cdots \int e^{-\frac{1}{2}\mathrm{tr}W_1} |W_1|^{\frac{1}{2}(n-p-2)} \tag{2.41}$$

$$Z^{(i)} g(\underline{Z}) d\underline{Z} dW_1 = \nu_x^{(i)} G^*_{p+2,n-p}(c;\lambda)$$

Finally we let

$$\frac{1}{2^{\frac{p}{2}(n-1)}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma[\frac{1}{2}(n-i)]}\int_{G}\cdots\int e^{-\frac{1}{2}\mathrm{tr}W_1}|W_1|^{\frac{1}{2}(n-p-2)} Z^{(i)}g(\underline{Z})d\underline{Z}dW_1 = I(Z^{(i)})$$

and note (2.34) is $\Sigma_{12}T'TT^{-1}I(\underline{Z})$ where $I(\underline{Z})$ is a px1 vector with i-th component $I(Z^{(i)})$. From (2.41), $I(\underline{Z}) = \underline{\nu}_x\ G^*_{p+2,n-p}(c;\lambda)$. Hence (2.34) becomes

$$\Sigma_{12}T'TT^{-1}\underline{\nu}_x\ G^*_{p+2,n-p}(c;\lambda)$$

$$= \Sigma_{12}\Sigma_{22}^{-1}\ \underline{\mu}_x\ G^*_{p+2,n-p}(c;\lambda)$$

where

$$G^*_{p+2,n-p}(c;\lambda) = P(G \leq c)$$

and G has p+2 and n-p degrees of freedom and noncentrality parameter λ. Therefore

$$G^*_{p+2,n-p}(c;\lambda) = P(\frac{n-p}{p+2}\ G \leq \frac{n-p}{p+2}\ c)$$

$$= P(F^*_{p+2,n-p} \leq \frac{n-p}{p+2}\cdot\frac{p}{n-p}\ F_{p,n-p}(\alpha))$$

$$= P(F^*_{p+2,n-p} \leq c_2)$$

where

$$c_2 = \frac{p}{p+2}\ F_{p,n-p}(\alpha)\ .$$

Therefore Bias $= -\Sigma_{12}\Sigma_{22}^{-1}\,\underline{\mu}_x F^*_{p+2,n-p}(c_2;\lambda)$ where $F^*_{p+2,n-p}(c_2;\lambda)$ is the cumulative distribution function of the noncentral F distribution with p+2 and n-p degrees of freedom and noncentrality parameter λ.

As a partial check, when $c_2=0$, the estimator reduces to the usual estimator $\bar{y}$ which is the case when we always reject the null hypothesis. In this case Bias = 0. When $c_2=\infty$, the null hypothesis is always accepted and the regression estimator $\bar{y} - S_{12}S_{22}^{-1}\bar{\underline{X}}$ is always used. The bias in this case is the usual bias for the regression estimator since $F^*_{p+2,n-p}(c_2;\lambda) = 1$ and Bias $= -\Sigma_{12}\Sigma_{22}^{-1}\,\underline{\mu}_x$.

Now for the purpose of comparison with the results of Han (1973a), we let p = 1 and

$$\underline{\rho} = \frac{1}{\sigma}\,\Sigma_{12}T'$$

$$\Sigma_{12}T'T = \sigma\underline{\rho}T$$

so

$$-\frac{\text{Bias}}{\sigma} = \underline{\rho}T\underline{\mu}_x F^*_{p+2,n-p}(c_2;\lambda)\ . \qquad (2.42)$$

For p = 1 we have

$$-\frac{\text{Bias}}{\sigma} = \rho\,\frac{\mu_x}{\sigma_x}\,F^*_{3,n-1}\left(\frac{1}{3}F_{1,n-1}(\alpha),\ n\frac{\mu_x^2}{\sigma_x^2}\right)$$

Wlog we set $\Sigma_{22} = I$ and $\sigma^2 = 1$. Therefore for $p = 1$, $\Sigma_{12} = \rho$ and we study the bias for positive values of ρ and μ_x since Bias changes sign with either parameter. Table 2.7 gives the values of -Bias for $p = 1$, $n = 9$ and some choices of μ_x, ρ and α.

Table 2.7. Values of -Bias for $p = 1$ and $n = 9$.

	$\alpha = .05$			$\alpha = .10$		
	ρ			ρ		
μ_x	.1	.5	.9	.1	.5	.9
0	0	0	0	0	0	0
0.3	.020	.102	.184	.015	.077	.139
0.6	.028	.139	.251	.018	.090	.161
0.9	.020	.101	.180	.010	.050	.090
1.2	.008	.042	.076	.003	.015	.027
1.5	.002	.011	.019	.001	.003	.005

The above values are essentially the same as those obtained by Han (1973a) although differences are observed. The differences occur because the expression for the bias given here is in terms of noncentral F distribution, while that of Han is given in terms of moments of normal distributions. Therefore there may be rounding off errors in the computation. We can observe that Bias = 0 when $\mu_x = 0$ or when the null hypothesis of the preliminary test is true. Also it can be seen that -Bias is an increasing function of ρ for fixed n, α and μ_x but a decreasing function of α for fixed n, ρ and μ_x. However, -Bias increases and then decreases to zero as μ_x increases whenever α, n and ρ are fixed.

For p = 2 and n = 9, the values of -Bias are given in Table 2.8 for some values of Σ_{12}, $\underline{\mu}_x$ and α.

Table 2.8. Values of -Bias for p = 2 and n = 9.

Σ'_{12} / $\underline{\mu}'_x$	$\alpha = .05$			
	$\begin{pmatrix} -.5 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -.5 \\ .5 \end{pmatrix}$	$\begin{pmatrix} .5 \\ .5 \end{pmatrix}$	$\begin{pmatrix} .7 \\ .7 \end{pmatrix}$
(0, 0)	0.0	0.0	0.0	0.0
(.3, 0)	-0.1193	-0.1193	0.1193	0.1670
(.3, .3)	-0.1112	0.0	0.2225	0.3114
(.6, 0)	-0.1911	-0.1911	0.1911	0.2675
(.6, .3)	-0.1761	-0.0880	0.2641	0.3698
(.6, .6)	-0.1352	0.0	0.2704	0.3786
(.9, 0)	-0.1847	-0.1847	0.1847	0.2585
(.9, .3)	-0.1677	-0.1118	0.2236	0.3131
(.9, .6)	-0.1240	-0.0413	0.2066	0.2893
(.9, .9)	-0.0721	0.0	0.1443	0.2020
(1.2, 0)	-0.1201	-0.1201	0.1201	0.1681
(1.2, .3)	-0.1075	-0.0807	0.1344	0.1882
(1.2, .6)	-0.0766	-0.0383	0.1149	0.1609
(1.2, .9)	-0.0424	-0.0106	0.0742	0.1039
(1.2,1.2)	-0.0177	0.0	0.0354	0.0496

Table 2.8. (continued)

$\underline{\mu}_x'$ \ Σ_{12}'	$\alpha = .2$			
	$\begin{pmatrix}-.5\\0\end{pmatrix}$	$\begin{pmatrix}-.5\\.5\end{pmatrix}$	$\begin{pmatrix}.5\\.5\end{pmatrix}$	$\begin{pmatrix}.7\\.7\end{pmatrix}$
(0, 0)	0.0	0.0	0.0	0.0
(.3, 0)	-0.0691	-0.0691	0.0691	0.0968
(.3, .3)	-0.0585	0.0	0.1171	0.1639
(.6, 0)	-0.0831	-0.0831	0.0831	0.1163
(.6, .3)	-0.0696	-0.0348	0.1044	0.1462
(.6, .6)	-0.0403	0.0	0.0807	0.1130
(.9, 0)	-0.0502	0.0502	0.0502	0.0703
(.9, .3)	-0.0416	-0.0277	0.0554	0.0776
(.9, .6)	-0.0233	-0.0078	0.0389	0.0544
(.9, .9)	-0.0086	0.0	0.0173	0.0242
(1.2, 0)	-0.0172	-0.0172	0.0172	0.0241
(1.2, .3)	-0.0141	-0.0106	0.0176	0.0246
(1.2, .6)	-0.0077	-0.0038	0.0115	0.0161
(1.2, .9)	-0.0027	-0.0007	0.0047	0.0066
(1.2,1.2)	-0.0006	0.0	0.0012	0.0017

Table 2.8. (continued)

$\underline{\mu}_x'$ \ Σ_{12}'	$\alpha = .5$			
	$\begin{pmatrix}-.5\\0\end{pmatrix}$	$\begin{pmatrix}-.5\\.5\end{pmatrix}$	$\begin{pmatrix}.5\\.5\end{pmatrix}$	$\begin{pmatrix}.7\\.7\end{pmatrix}$
(0, 0)	0.0	0.0	0.0	0.0
(.3, 0)	-0.0211	-0.0211	0.0211	0.0295
(.3, .3)	-0.0159	0.0	0.0319	0.0446
(.6, 0)	-0.0181	-0.0181	0.0181	0.0254
(.6, .3)	-0.0137	-0.0068	0.0205	0.0287
(.6, .6)	-0.0058	0.0	0.0116	0.0162
(.9, 0)	-0.0065	-0.0065	0.0065	0.0091
(.9, .3)	-0.0049	-0.0033	0.0065	0.0091
(.9, .6)	-0.0020	-0.0007	0.0034	0.0048
(.9, .9)	-0.0005	0.0	0.0009	0.0013
(1.2, 0)	-0.0011	-0.0011	0.0011	0.0016
(1.2, .3)	-0.0008	-0.0006	0.0011	0.0015
(1.2, .6)	-0.0003	-0.0002	0.0005	0.0007
(1.2, .9)	-0.0001	0.0	0.0001	0.0002
(1.2,1.2)	0.0	0.0	0.0	0.0

From Table 2.8, the following properties of the Bias of $\hat{\mu}$ are apparent.

1. The bias is zero when the null hypothesis of the preliminary test of significance is true, that is, when $\underline{\mu}_x = \underline{0}$.

2. For fixed n, α and $\underline{\mu}_x$, the value of the bias generally increases with Σ_{12}.

3. For fixed n, α and Σ_{12}, the bias generally decreases as α increases.

4. The bias is zero when either $\underline{\mu}_x$ or Σ_{12} has identical components and the other has components which differ only in sign.

5. For fixed n, Σ_{12} and α and some component of $\underline{\mu}_x$, the value of the bias first increases, then decreases to zero as the other component increases.

F. The M.S.E. of $\hat{\mu}$ when Σ is Unknown

The M.S.E. of $\hat{\mu}$ is

$$\text{M.S.E.}(\hat{\mu}) = V(\hat{\mu}) + B^2 \tag{2.43}$$

where

$$B = \text{Bias}(\hat{\mu}) ,$$

and

$$V(\hat{\mu}) = E(\hat{\mu}^2) - [E(\hat{\mu})]^2 .$$

When Σ is unknown, the preliminary test estimator is given in (2.27). Hence, making use of the notations of Section E,

$$\begin{aligned} E(\hat{\mu}^2) &= E\{(\bar{y} - S_{12}S_{22}^{-1}\bar{\underline{X}})^2 | G\}P(G) \\ &\quad + E(\bar{y}^2|\bar{G})P(\bar{G}) \\ &= E(\bar{y}^2) - 2E[S_{12}S_{22}^{-1}\bar{y}\bar{\underline{X}}|G]P(G) \\ &\quad + E[(S_{12}S_{22}^{-1}\bar{\underline{X}}\bar{\underline{X}}'S_{22}^{-1}S_{21})|G]P(G) \end{aligned} \tag{2.44}$$

Thus we need to evaluate

$$E[S_{12}S_{22}^{-1}\bar{y}\underline{\bar{X}}\,|\,\mathfrak{a}]P(\mathfrak{a}) \quad \text{and} \quad E[(S_{12}S_{22}^{-1}\underline{\bar{X}\bar{X}}'S_{22}^{-1}S_{21})\,|\,\mathfrak{a}]P(\mathfrak{a})$$

Since $S_{12}S_{22}^{-1}\underline{\bar{X}\bar{X}}S_{22}^{-1}S_{21}$ is a scalar,

$$\begin{aligned} &E[(S_{12}S_{22}^{-1}\underline{\bar{X}\bar{X}}'S_{22}^{-1}S_{21})\,|\,\mathfrak{a}]P(\mathfrak{a}) \\ &= E\{tr(S_{22}^{-1}S_{21}S_{12}S_{22}^{-1}\underline{\bar{X}\bar{X}}')\,|\mathfrak{a}\}P(\mathfrak{a}) \\ &= tr\ E\{(S_{22}^{-1}S_{21}S_{12}S_{22}^{-1}\underline{\bar{X}\bar{X}};)\,|\mathfrak{a}\}P(\mathfrak{a}) \end{aligned}$$

Now using the transformation of (2.31), we have

$$tr\int\cdots\int_{\mathfrak{a}}\{(KT'W_1^{-\frac{1}{2}'}W_2' + T'T\Sigma_{21})(KW_2W_1^{-\frac{1}{2}}T + \Sigma_{12}T'T)\underline{\bar{X}\bar{X}}'f(\underline{\bar{X}})$$

$$g(W_1,W_2,W_3)dW_3dW_2dW_1d\underline{\bar{X}}$$

$$= K^2tr\int\cdots\int_{\mathfrak{a}}\int_{\infty}^{\infty}\int_{0}^{\infty}T'W_1^{-\frac{1}{2}'}W_2'W_2W_1^{-\frac{1}{2}}T\underline{\bar{X}\bar{X}}'f(\underline{\bar{X}})c\cdot e^{-\frac{1}{2}tr(W_2'W_2+W_3W_1)}|W_3|^{\frac{1}{2}(n-p-3)}$$

$$\cdot|W_1|^{\frac{1}{2}(n-p-2)}dW_3dW_2dW_1d\underline{\bar{X}}$$

$$+Ktr\int\cdots\int_{\mathfrak{a}}\int_{-\infty}^{\infty}\int_{0}^{\infty}T'T\ {}_{21}W_2W_1^{-\frac{1}{2}}T\underline{\bar{X}\bar{X}}'f(\underline{\bar{X}})c\cdot e^{-\frac{1}{2}tr(W_2'W_2+W_3+W_1)}$$

$$\cdot|W_3|^{\frac{1}{2}(n-p-3)}|W_1|^{\frac{1}{2}(n-p-2)}dW_3dW_2dW_1d\underline{\bar{X}}$$

$$+K\operatorname{tr}\int_G\cdots\int\int_{-\infty}^{\infty}\int_0^{\infty}T'W_1^{-\frac{1}{2}'}W_2'\Sigma_{12}T'T\underline{\bar{X}}\underline{\bar{X}}'f(\underline{\bar{X}})c\cdot e^{-\frac{1}{2}\operatorname{tr}(W_2'W_2+W_3+W_1)}$$

$$\cdot|W_3|^{\frac{1}{2}(n-p-3)}|W_1|^{\frac{1}{2}(n-p-2)}dW_3dW_2dW_1d\underline{\bar{X}}$$

$$+\operatorname{tr}\int_G\cdots\int\int_{-\infty}^{\infty}\int_0^{\infty}T'T\Sigma_{21}\Sigma_{12}T'T\underline{\bar{X}}\underline{\bar{X}}'f(\underline{\bar{X}})c\cdot e^{-\frac{1}{2}\operatorname{tr}(W_2'W_2+W_3+W_1)}|W_3|^{\frac{1}{2}(n-p-3)}$$

$$|W_1|^{\frac{1}{2}(n-p-2)}dW_3dW_2dW_1d\underline{\bar{X}}$$

Now from independence and the fact

$$c^*\int_{-\infty}^{\infty}W_2'W_2e^{-\frac{1}{2}\operatorname{tr}W_2'W_2}dW_2 = I\ ,$$

the first term equals

$$\frac{K^2\operatorname{tr}}{2^{\frac{1}{2}p(n-1)}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma[\frac{1}{2}(n-i)]}\int\int_G T'W_1^{-\frac{1}{2}}T\underline{\bar{X}}\underline{\bar{X}}f(\underline{\bar{X}})e^{-\frac{1}{2}\operatorname{tr}W_1}|W_1|^{\frac{1}{2}(n-p-2)}dW_1d\underline{\bar{X}}$$

$$=\frac{K^2}{2^{\frac{1}{2}(p(n-1)}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma[\frac{1}{2}(n-i)]}\int\int_G\underline{\bar{X}}'T'W_1^{-1}T\underline{\bar{X}}f(\underline{\bar{X}})e^{-\frac{1}{2}\operatorname{tr}W_1}|W_1|^{\frac{1}{2}(n-p-2)}dW_1d\underline{\bar{X}}$$

(where we recall $G = \{nm\underline{\bar{X}}'T'W_1^{-1}T\underline{\bar{X}}:nm\underline{\bar{X}}'T'W_1^{-1}T\underline{\bar{X}} \le T_0^2\}$ and $T'W_1^{-1}T = S_{22}^{-1}$)

$$= K^2\ E[\underline{\bar{X}}'S_{22}^{-1}\underline{\bar{X}} \mid nm\underline{\bar{X}}'S_{22}^{-1}\underline{\bar{X}} \leq T_0^2]$$

$$= K^2\ E[\frac{p\quad t}{n(n-p)} \mid nm\frac{p\quad t}{n(n-p)} \leq T_0^2]$$

$$= \frac{K^2 p}{n(n-p)}\ E[t \mid t \leq \frac{n(n-p)}{nmp} \cdot \frac{p(n-1)}{n-p}\ F_{p,n-p}]$$

$$= \frac{K^2 p}{n(n-p)}\ E[t \mid t \leq F_{p,n-p}(\alpha)]$$

$$= \frac{K^2 p}{n(n-p)} \int_0^d t\ f(t) = Q \tag{2.45}$$

where $d = F_{p,n-p}(\alpha)$ and t has the noncentral F distribution with p and $n-p$ degrees of freedom and noncentrality parameter $\lambda = n\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x$.

Also from independence and the fact $E(W_2) = \underline{0}$, the second and third terms are zero and the fourth term is equivalent to

$$\mathrm{tr}\iint_{G} T'T\Sigma_{21}\Sigma_{12}T'TT^{-1}\underline{ZZ}'T'^{-1}g(\underline{Z})c_1 e^{-\frac{1}{2}\mathrm{tr}W_1}|W_1|^{\frac{1}{2}(n-p-2)}\,d\underline{Z}dW_1$$

$$= \frac{\mathrm{tr}T'^{-1}T'T\Sigma_{21}\Sigma_{12}T'TT^{-1}}{2^{\frac{1}{2}p(n-1)}\ \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma[\frac{1}{2}(n-i)]} \iint_{G} \underline{ZZ}'g(2)e^{-\frac{1}{2}\mathrm{tr}W_1}|W_1|^{\frac{1}{2}(n-p-2)}\,d\underline{Z}dW_1 \tag{2.46}$$

We evaluate the diagonal elements of the above integral by differentiating each of the two representations of $P(G)$ twice w.r.t. $\nu_x^{(i)}$ and equating the results. The off-diagonal elements can similarly be evaluated by differentiating the two representations first w.r.t. $\nu_x^{(i)}$ and then w.r.t. $\nu_x^{(K)}$ and equating the results.

Differentiating (2.38a) w.r.t. $\nu_x^{(i)}$, we have

$$\frac{\partial R}{\partial \nu_x^{(i)2}} = \int_0^c \frac{e^{-\frac{1}{2}\lambda}}{\Gamma(\frac{n-p}{2})} \sum_{j=0}^{\infty} \frac{1}{j!}\, j\, (j-1)(\frac{\lambda}{2})^{j-2}(\frac{2n\nu_x^{(i)}}{2})^2 G_{(\frac{p}{2}+j,\frac{n-p}{2})}(g)dg$$

$$+ \int_0^c \frac{e^{-\frac{1}{2}\lambda}}{\Gamma(\frac{n-p}{2})} \sum_{j=0}^{\infty} \frac{1}{j!}\, j(\frac{\lambda}{2})^{j-1}(\frac{2n}{2}) G_{(\frac{p}{2}+j,\frac{n-p}{2})}(g)dg$$

$$- \int_0^c 2n\nu_x^{(i)} \frac{e^{-\frac{1}{2}\lambda}}{\Gamma(\frac{n-p}{2})} \sum_{j=0}^{\infty} \frac{1}{j!}\, j(\frac{\lambda}{2})^{j-1} \frac{2n\nu_x^{(i)}}{2} G_{(\frac{p}{2}+j,\frac{n-p}{2})}(g)dg$$

$$- \int_0^c \frac{2n}{2} \frac{e^{-\frac{1}{2}\lambda}}{\Gamma(\frac{n-p}{2})} \sum_{j=0}^{\infty} \frac{1}{j!} (\frac{\lambda}{2})^j\, G_{(\frac{p}{2}+j,\frac{n-p}{2})}(g)dg$$

$$+ \int_0^c (\frac{2n\nu_x^{(i)}}{2})^2 \frac{e^{-\frac{1}{2}\lambda}}{\Gamma(\frac{n-p}{2})} \sum_{j=0}^{\infty} \frac{1}{j!} (\frac{\lambda}{2})^j\, G_{(\frac{p}{2}+j,\frac{n-p}{2})}(g)dg$$

$$- \int_0^c \frac{2n\nu_x^{(i)}}{2} \frac{e^{-\frac{1}{2}\lambda}}{\Gamma(\frac{n-p}{2})} \sum_{j=0}^{\infty} \frac{1}{j!}\, j(\frac{\lambda}{2})^{j-1} \frac{2n\nu_x^{(i)}}{2} G_{(\frac{p}{2}+j,\frac{n-p}{2})}(g)dg$$

Therefore

$$\frac{\partial R}{\partial \nu_x^{(i)^2}} = n^2(\nu_x^{(i)})^2\; G^*_{p+4,n-p}(c;\lambda) + nG^*_{p+2,n-p}(c;\lambda)$$

$$- n^2(\nu_x^{(i)})^2\; G^*_{p+2,n-p}(c;\lambda) - nP(G) + n^2(\nu_x^{(i)})^2 P(G)$$

$$- n^2(\nu_x^{(i)})^2\; G^*_{p+2,n-p}(c;\lambda) \tag{2.47}$$

Similarly differentiating (2.40) w.r.t. $\nu_x^{(i)}$, we obtain

$$\frac{\partial^2 R}{\partial \nu_x^{(i)^2}} = \int_G \cdots \int \prod_{j-1}^{P} \frac{\sqrt{n}}{\sqrt{2\pi}} \left(\frac{n}{2}\cdot 2\right)^2 (Z^{(i)} - \nu_x^{(i)})^2\, e^{-\frac{n}{2}(Z^{(j)} - \nu_x^{(j)})^2}$$

$$\cdot \frac{|W_1|^{\frac{1}{2}(n-p-2)}\, e^{-\frac{1}{2}trW_1}}{2^{\frac{p}{2}(n-1)}\, \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{P} \Gamma[\frac{1}{2}(n-i)]}\, dZ^{(j)} dW_1$$

$$- \int_G \cdots \int \prod_{j-1}^{P} \frac{\sqrt{n}}{\sqrt{2\pi}} \left(\frac{n}{2}\cdot 2\right) e^{-\frac{n}{2}(Z^{(j)} - \nu_x^{(j)})^2}$$

$$\cdot \frac{W_1^{\frac{1}{2}(n-p-3)}\, e^{-\frac{1}{2}trW_1}}{2^{\frac{p}{2}(n-1)}\, \pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{P} \Gamma[\frac{1}{2}(n-i)]}\, dZ^{(j)} dW_1$$

Therefore

$$\frac{\partial^2 R}{\partial \nu_x^{(i)^2}} = n^2 c_1 \iint_{\mathcal{G}} z^{(i)^2} g(\underline{Z}) e^{-\frac{1}{2}\mathrm{tr}W_1} |W_1|^{\frac{1}{2}(n-p-2)} dZ^{(j)} dW_1$$

$$- 2n^2 c_1 \nu_x^{(i)} \iint_{\mathcal{G}} z^{(i)} g(\underline{Z}) e^{-\frac{1}{2}\mathrm{tr}W_1} |W_1|^{\frac{1}{2}(n-p-2)} dZ^{(j)} dW_1$$

$$+ n^2 \nu_x^{(i)^2} P(\mathcal{G}) - nP(\mathcal{G})$$

But from equating (2.38b) and (2.40), we know the middle term $= -2n^2 \nu_x^{(i)2} G^*_{p+2,n-p}(c;\lambda)$. Hence

$$\frac{\partial^2 R}{\partial \nu_x^{(i)^2}} = n^2 c_1 \iint_{\mathcal{G}} z^{(i)^2} g(\underline{Z}) e^{-\frac{1}{2}\mathrm{tr}W_1} |W_1|^{\frac{1}{2}(n-p-2)} dZ^{(j)} dW_1$$

$$- 2n^2 \nu_x^{(i)^2} G^*_{p+2,n-p}(c;\lambda) + n^2 \nu_x^{(i)^2} P(\mathcal{G}) - nP(\mathcal{G}) \tag{2.48}$$

Equating (2.47) and (2.48) yields

$$\frac{1}{2^{\frac{1}{2}p(n-1)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma[\frac{1}{2}(n-i)]} \iint_{\mathcal{G}} z^{(i)^2} g(\underline{Z}) e^{-\frac{1}{2}\mathrm{tr}W_1} |W_1|^{\frac{1}{2}(n-p-2)}$$

$$dZ^{(j)} dW_1 = (\nu_x^{(i)})^2 G^*_{p+4,n-p}(c;\lambda) + \frac{1}{n} G^*_{p+2,n-p}(c;\lambda) \tag{2.49}$$

Next we differentiate (2.38a) w.r.t. $\nu_x^{(K)}$ to get

$$\frac{\partial R}{\partial \nu_x^{(K)} \partial \nu_x^{(i)}} = \int_0^c \frac{e^{-\frac{1}{2}\lambda}}{\Gamma(\frac{n-p}{2})} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n\nu_x^{(i)}}{2} \cdot \frac{2n\nu_x^{(K)}}{2} j \cdot j-1 (\tfrac{\lambda}{2})^{j-1}$$

$$G_{(\frac{p}{2}+j, \frac{n-p}{2})}(g)dg - \int_0^c \frac{1}{2} \cdot 2n\nu_x^{(K)} \frac{e^{-\frac{1}{2}\lambda}}{\Gamma(\frac{n-p}{2})} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n\nu_x^{(i)}}{2} j (\tfrac{\lambda}{2})^{j-1}$$

$$G_{(\frac{p}{2}+j, \frac{n-p}{2})}(g)dg - \int_0^c \frac{1}{2} \cdot 2n\nu_x^{(i)} \frac{e^{-\frac{1}{2}\lambda}}{\Gamma(\frac{n-p}{2})} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n\nu_x^{(K)}}{2} \cdot j (\tfrac{\lambda}{2})^{j-1}$$

$$G_{(\frac{p}{2}+j, \frac{n-p}{2})}(g)dg + \int_0^c (\tfrac{1}{2} \cdot 2n)^2 \nu_x^{(i)} \nu_x^{(K)} \frac{e^{-\frac{1}{2}\lambda}}{\Gamma(\frac{n-p}{2})} \sum_{j=0}^{\infty} \frac{1}{j!} (\tfrac{\lambda}{2})^j$$

$$G_{(\frac{p}{2}+j, \frac{n-p}{2})}(g)dg = n^2 \nu_x^{(i)} \nu_x^{(K)} G^*_{p+4, n-p}(c;\lambda) - 2n^2 \nu_x^{(i)} \nu_x^{(K)}$$

$$G^*_{p+2, n-p}(c;\lambda) + n^2 \nu_x^{(i)} \nu_x^{(K)} P(\alpha) \qquad (2.50)$$

Similarly from (2.40),

$$\frac{\partial R}{\partial \nu_x^{(K)} \partial \nu_x^{(i)}} = \iint_G \prod_{j=1}^{p} \frac{\sqrt{n}}{\sqrt{2\pi}} \cdot \left(\frac{2n}{2}\right)^2 (Z^{(i)} - \nu_x^{(i)})(Z^{(K)} - \nu_x^{(K)})$$

$$e^{-\frac{n}{2}(Z^{(j)} - \nu_x^{(j)})^2} \frac{|W_1|^{\frac{1}{2}(n-p-2)} e^{-\frac{1}{2}\mathrm{tr}W_1}}{2^{\frac{p}{2}(n-1)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma[\frac{1}{2}(n-i)]} g(\underline{Z}) dZ^{(j)} dW_1$$

$$= \frac{n^2}{2^{\frac{p}{2}(n-1)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma[\frac{1}{2}(n-i)]} \iint_G Z^{(i)} Z^{(K)} e^{-\frac{n}{2}(Z^{(j)} - \nu_x^{(j)})^2}$$

$$\cdot |W_1|^{\frac{1}{2}(n-p-2)} e^{-\frac{1}{2}\mathrm{tr}W_1} g(\underline{Z}) dZ^{(j)} dW_1$$

$$- \frac{n^2 \nu_x^{(K)}}{2^{\frac{p}{2}(n-1)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma[\frac{1}{2}(n-i)]} \iint_G Z^{(i)} e^{-\frac{n}{2}(Z^{(j)} - \nu_x^{(j)})^2}$$

$$\cdot |W_1|^{\frac{1}{2}(n-p-2)} e^{-\frac{1}{2}\mathrm{tr}W_1} g(\underline{Z}) dZ^{(j)} dW_1$$

$$- \frac{n^2 \nu_x^{(i)}}{2^{\frac{p}{2}(n-1)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma[\frac{1}{2}(n-i)]} \iint_G Z^{(K)} e^{-\frac{n}{2}(Z^{(j)} - \nu_x^{(j)})^2}$$

$$\cdot |W_1|^{\frac{1}{2}(n-p-2)} e^{-\frac{1}{2}\mathrm{tr}W_1} g\underline{Z} dZ^{(j)} dW_1 + n^2 \nu_x^{(i)} \nu_x^{(K)} P(G)$$

Again we know each of the two middle terms equals $-n^2\nu_x^{(i)}\nu_x^{(K)}G^*_{p+2,n-p}(c;\lambda)$ from (2.38) and (2.40). Therefore

$$\frac{\partial R}{\partial\nu_x^{(K)}\partial\nu_x^{(i)}} = \frac{n^2}{2^{\frac{p}{2}(n-1)}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma[\frac{1}{2}(n-i)]}\iint_{G} Z^{(i)}Z^{(K)}$$

$$e^{-\frac{n}{2}(Z^{(j)}-\nu_x^{(j)})^2}|W_1|^{\frac{1}{2}(n-p-2)}e^{-\frac{1}{2}trW_1}g(\underline{Z})dZ^{(j)}dW_1$$

$$- 2n^2\nu_x^{(i)}\nu_x^{(K)}G^*_{p+2,n-p}(c;\lambda) + n^2\nu_x^{(i)}\nu_x^{(K)}P(G) \tag{2.51}$$

Equating (2.50) and (2.51) we have

$$\frac{n^2}{2^{\frac{p}{2}(n-1)}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma[\frac{1}{2}(n-i)]}\iint_{G} Z^{(i)}Z^{(K)}e^{-\frac{n}{2}(Z^{(j)}-\nu_x^{(j)})^2} \tag{2.52}$$

$$|W_1|^{\frac{1}{2}(n-p-2)}e^{-\frac{1}{2}trW_1}g(\underline{Z})dZ^{(j)}dW_1 = \nu_x^{(i)}\nu_x^{(K)}G^*_{p+4,n-p}(c;\lambda)$$

We may now let

$$\frac{1}{2^{\frac{p}{2}(n-1)}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma[\frac{1}{2}(n-i)]}\iint_{G}\underline{ZZ}'g(\underline{Z})e^{-\frac{1}{2}trW_1}|W_1|^{\frac{1}{2}(n-p-2)}d\underline{Z}dW_1 \tag{2.53}$$

$= M$ where M is a pxp matrix with i-th diagonal element $= (\nu_x^{(i)})^2 G^*_{p+4,n-p}(c;\lambda) + \frac{1}{n} G^*_{p+2,n-p}(c;\lambda)$ and the (i,k)th off-diagonal element $= \nu_x^{(i)} \nu_x^{(K)} G^*_{p+4,n-p}(c;\lambda)$. Hence

$$
\begin{aligned}
& E[(S_{12}S_{22}^{-1}\underline{\bar{X}}\underline{\bar{X}}'S_{22}^{-1}S_{21})|G]\ P(G) \\
&= Q + \mathrm{tr}T'^{-1}T'T\Sigma_{21}\Sigma_{12}T'TT^{-1}M \\
&= Q + \mathrm{tr}T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{12}\Sigma_{22}^{-1}T^{-1}M \qquad (2.54) \\
&= Q + \mathrm{tr}\Sigma_{12}\Sigma_{22}^{-1}T^{-1}MT'^{-1}\Sigma_{22}^{-1}\Sigma_{21} \\
&= Q + \Sigma_{12}\Sigma_{22}^{-1}T^{-1}MT'^{-1}\Sigma_{22}^{-1}\Sigma_{21}
\end{aligned}
$$

since the second term is a scalar.

Next we note that

$$
\begin{aligned}
& E[S_{12}S_{22}^{-1}\bar{y}\underline{\bar{X}}|G]\ P(G) \\
&= E\{E(S_{12}S_{22}^{-1}\underline{\bar{X}}\bar{y}|S,G)|G\}\ P(G) \\
&= E\{E(S_{12}S_{22}^{-1}\underline{\bar{X}}\bar{y}|S,\underline{\bar{X}},G)|G\}\ P(G) \\
&= E\{S_{12}S_{22}^{-1}\underline{\bar{X}}\ E[\bar{y}|\underline{\bar{X}}]|G\}\ P(G) \\
&= E\{S_{12}S_{22}^{-1}\ \underline{\bar{X}}[\mu+\Sigma_{12}\Sigma_{22}^{-1}(\underline{\bar{X}}-\underline{\mu}_x)]|G\}\ P(G) \\
&= \mu E[S_{12}S_{22}^{-1}\underline{\bar{X}}|G]\ P(G) + E[(S_{12}S_{22}^{-1}\underline{\bar{X}}\Sigma_{12}\Sigma_{22}^{-1}\underline{\bar{X}})|G]P(G) \\
&\quad - E(S_{12}S_{22}^{-1}\underline{\bar{X}}\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x|G)\ P(G)
\end{aligned}
$$

But since $\Sigma_{12}\Sigma_{22}^{-1}\bar{\underline{X}}$ and $\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x$ are scalars, they are equal to $\bar{\underline{X}}'\Sigma_{22}^{-1}\Sigma_{21}$ and $\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}$, respectively. Hence

$$E[S_{12}S_{22}^{-1}\bar{y}\bar{\underline{X}}|G]\ P(G)$$

$$= \mu E[S_{12}S_{22}^{-1}\bar{\underline{X}}|\ G]\ P(G)$$

$$+ E[(S_{12}S_{22}^{-1}\underline{\bar{X}\bar{X}}'\Sigma_{22}^{-1}\Sigma_{21}|G]\ P(G)$$

$$- E(S_{12}S_{22}^{-1}\bar{\underline{X}}\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}|G]\ P(G)$$

To evaluate the middle term,

$$E[S_{12}S_{22}^{-1}\underline{\bar{X}\bar{X}}'\Sigma_{22}^{-1}\Sigma_{21}|G]\ P(G)$$

$$= E\{tr(\Sigma_{22}^{-1}\Sigma_{21}S_{12}S_{22}^{-1}\underline{\bar{X}\bar{X}}')|G\}\ P(G)$$

$$= tr\ \Sigma_{22}^{-1}\Sigma_{21}\ E\{S_{12}S_{22}^{-1}\underline{\bar{X}\bar{X}}'|G\}\ P(G)$$

and with the transformation of (2.31)

$$= trT'T\Sigma_{21} \iiiint_G KW_2W_1^{-\frac{1}{2}}T + \Sigma_{12}T'T)\underline{\bar{X}\bar{X}}'f(\bar{\underline{X}})\ g(W_1,W_2,W_3)dW_3dW_2dW_1d\bar{\underline{X}}$$

$$= trT'T\Sigma_{21}\iint_G KW_2W_1^{-\frac{1}{2}}T\underline{\bar{X}\bar{X}}'f(\bar{\underline{X}})c_0e^{-\frac{1}{2}tr(W_2'W_2+W_3+W_1)}$$
$$|W_3|^{\frac{1}{2}(n-p-3)}\ |W_1|^{\frac{1}{2}(n-p-2)}\ dW_3dW_2dW_1d\bar{\underline{X}}$$

$$+ \operatorname{tr} T'T\Sigma_{21}\Sigma_{12}T'T \iint_{G}\int_{-\infty}^{\infty}\int_{0}^{\infty} \underline{\bar{X}}\underline{\bar{X}}' f(\underline{\bar{X}}) c_0 e^{-\frac{1}{2}\operatorname{tr}(W_2'W_2+W_3+W_1)} |W_3|^{\frac{1}{2}(n-p-3)} |W_1|^{\frac{1}{2}(n-p-2)} dW_3 dW_2 dW_1 d\underline{\bar{X}}$$

But from independence and the fact $E(W_2) = \underline{0}$, the first term $= \underline{0}$ and using (2.46) and (2.53)

$$E[S_{12}S_{22}^{-1}\underline{\bar{X}}\underline{\bar{X}}'\Sigma_{22}^{-1}\Sigma_{21}|G]\ P(G)$$

$$= \operatorname{tr} T'^{-1}T'T\Sigma_{21}\Sigma_{12}T'TT^{-1}M$$

$$= \Sigma_{12}\Sigma_{22}^{-1}T^{-1}MT'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

Therefore

$$E(S_{12}S_{22}^{-1}\bar{y}\underline{\bar{X}}|G)\ P(G)$$

$$\begin{aligned} &= \mu\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x G^*_{p+2,n-p}(c;\lambda) \\ &+ \Sigma_{12}\Sigma_{22}^{-1}T^{-1}MT'^{-1}\Sigma_{22}^{-1}\Sigma_{21} \\ &- \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}G^*_{p+2,n-p}(c;\lambda) \end{aligned} \tag{2.55}$$

Finally $E(\bar{y}^2) = \frac{1}{n}\sigma^2 + \mu^2$. Substituting into (2.44), we have

$$\begin{aligned} E(\hat{\mu}^2) &= \frac{1}{n}\sigma^2 + \mu^2 - 2\mu\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x G^*_{p+2,n-p}(c;\lambda) \\ &- \Sigma_{12}\Sigma_{22}^{-1}T^{-1}MT'^{-1}\Sigma_{22}^{-1}\Sigma_{21} \\ &+ 2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}G^*_{p+2,n-p}(c;\lambda) \\ &+ Q \end{aligned}$$

$$[E(\hat{\mu})]^2 = [\mu - \Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{\nu}_x G^*{}_{p+2,n-p}(c;\lambda)]^2$$

$$= \mu^2 - 2\mu\Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{\nu}_x G^*_{p+2,n-p}(c;\lambda)$$

$$+ \Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{\nu}_x\underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}[G^*{}_{p+2,n-p}(c;\lambda)]^2$$

Therefore

$$V(\hat{\mu}) = E(\hat{\mu}^2) - [E(\hat{\mu})]^2$$

$$= \frac{1}{n}\sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}T^{-1}MT'^{-1}\Sigma_{22}^{-1}\Sigma_{21} + Q$$

$$+ 2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}G^*_{p+2,n-p}(c;\lambda) \tag{2.56}$$

$$- \Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{\nu}_x\underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}[G^*_{p+2,n-p}(c;\lambda)]^2$$

To be able to make any partial checks, we need to compute the variance of $\hat{\mu}$ for the cases when $c = 0$ and when $c = \infty$. When $c = 0$, we always reject the null hypothesis and so the estimator reduces to $\hat{\mu} = \bar{y}$ with variance

$$V(\hat{\mu}) = V(\bar{y}) = \frac{1}{n}\sigma^2 \tag{2.57}$$

For $c = \infty$, we always accept and so use the estimator

$$\hat{\mu} = \bar{y} - S_{12}S_{22}^{-1}\bar{\underline{X}} .$$

Now

$$V(\bar{y} - S_{12}S_{22}^{-1}\bar{\underline{X}})$$

$$= E\ V(\bar{y} - S_{12}S_{22}^{-1}\bar{\underline{X}}|\underline{X}_i\text{'s}) + V\ E(\bar{y} - S_{12}S_{22}^{-1}\bar{\underline{X}}|\underline{X}_i\text{'s})$$

$$= E\ [\tfrac{1}{n}\Sigma_{11\cdot 2} - 2\mathrm{Cov}(\bar{y}, \sum_{i=1}^{n}(y_i-\bar{y})(\underline{X}_i-\bar{\underline{X}})S_{22}^{-1}\bar{\underline{X}}|\underline{X}_i\text{'s})$$

$$+ \Sigma_{11\cdot 2}\bar{\underline{X}}'S_{22}^{-1}\bar{\underline{X}}]$$

$$+ V[\mu + \Sigma_{12}\Sigma_{22}^{-1}(\bar{\underline{X}}-\underline{\mu}_x) - \Sigma_{12}\Sigma_{22}^{-1}\bar{\underline{X}}]$$

$$= \frac{1}{n}\Sigma_{11\cdot 2} + \Sigma_{11\cdot 2}\ E(\bar{\underline{X}}'S_{22}^{-1}\bar{\underline{X}})$$

But $nm(\bar{\underline{X}}'S_{22}^{-1}\bar{\underline{X}}) = T^2 \sim$ noncentral T^2 distribution with n-1 degree of freedom where recall $m = n-1$ and $\frac{T^2(n-p)}{p(n-1}$ ~ noncentral F distribution with p and n-p degrees of freedom and non-centrality parameter $n\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x$. Hence

$$V(\bar{y} - S_{12}S_{22}^{-1}\bar{\underline{X}}\ )$$

$$= \frac{1}{n}\Sigma_{11\cdot 2} + \Sigma_{11\cdot 2}E\{(\bar{\underline{X}}'S_{22}^{-1}\bar{\underline{X}})\frac{nm}{nm}\ \frac{n-p}{p(n-1)}\ \frac{p(n-1)}{n-p}\}$$

$$= \frac{1}{n}\Sigma_{11\cdot 2} + \Sigma_{11\cdot 2}\ \frac{p}{n(n-p)}\ E(t)$$

where $t \sim F_{p-n-p}(\lambda)$. But

$$E(t) = \frac{n-p}{n-p-2}\ [1 + \frac{2n\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x}{p}]$$

Hence

$$V(\bar{y} - S_{12}S_{22}^{-1}\bar{\underline{X}}) = \frac{1}{n}\Sigma_{11\cdot 2} + \frac{\Sigma_{11\cdot 2}}{n} \frac{p}{(n-p-2)} [1 + \frac{2n\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x}{p}] \tag{2.58}$$

So now for partial checks, when $c = 0$, (2.56) is $V(\hat{\mu}) = \frac{1}{n}\sigma^2$ which is the variance of the estimator when we always reject. For $c = \infty$, we note that

$$Q = K^2 \frac{p}{n(n-p)} \int_0^\infty t\ f(t)dt$$

$$= \Sigma_{11\cdot 2} \frac{p}{n(n-p)} E(t)$$

$$= \Sigma_{11\cdot 2} \frac{p}{n(n-p-2)} [1 + \frac{2n\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x}{p}]$$

and (2.56) reduces to

$$\frac{1}{n}\sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21} - \frac{1}{n}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$+ \Sigma_{11\cdot 2} \frac{p}{n(n-p-2)} [1 + \frac{2n\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x}{p}]$$

$$+ 2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}$$

$$- \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}$$

$$= \frac{1}{n}\Sigma_{11\cdot 2} + \Sigma_{11\cdot 2} \frac{p}{n(n-p-2)} [1 + \frac{2n\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x}{p}]$$

which by (2.58) is the variance of the regression estimator when we always accept H_0.

Now we obtain the M.S.E. of $\hat{\mu}$.

$$\begin{aligned} \text{M.S.E.}(\hat{\mu}) &= \frac{1}{n}\sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}T^{-1}MT'^{-1}\Sigma_{22}^{-1}\Sigma_{21} + Q \\ &+ 2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}G^*_{p+2,n-p}(c;\lambda) \\ &- \Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{\nu}_x\underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}[G^*_{p+2,n-p}(c;\lambda)]^2 \\ &+ \Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{\nu}_x\underline{\nu}_x'T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}[G^*_{p+2,n-p}(c;\lambda)]^2 \end{aligned}$$

or

$$\begin{aligned} \text{M.S.E.}(\hat{\mu}) &= \frac{1}{n}\sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}T^{-1}MT'^{-1}\Sigma_{22}^{-1}\Sigma_{21} + Q \\ &+ 2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}G^*_{p+2,n-p}(c;\lambda) \\ &= \frac{1}{n}\sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}G^*{}_{p+4,n-p}(c;\lambda) \\ &- \frac{1}{n}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}G^*_{p+2,n-p}(c;\lambda) + Q \\ &+ 2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}G^*_{p+2,n-p}(c;\lambda) \end{aligned} \tag{2.59}$$

G. Relative Efficiency (e')

To evaluate the gain and loss of precision of the preliminary test estimator, we consider the relative efficiency of $\hat{\mu}$ to the usual estimator $\bar{y}$. This is defined as

$$e' = \frac{1}{\text{M.S.E.}(\hat{\mu})} \bigg/ \frac{1}{\text{M.S.E.}(\bar{y})}$$

so that using (2.59),

$$e' = \frac{1}{1 + h} \tag{2.60}$$

where

$$h = \frac{n}{\sigma^2} \{2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}G^*_{p+2,n-p}(c;\lambda)$$

$$- \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}G^*_{p+4,n-p}(c;\lambda)$$

$$- \frac{1}{n}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}G^*_{p+2,n-p}(c;\lambda) + Q\}$$

Wlog we let $\Sigma_{22} = I$, $\sigma^2 = 1$. Therefore

$$h = 2n\Sigma_{12}\underline{\mu}_x\underline{\mu}_x'\Sigma_{21}F^*_{p+2,n-p}(c_2;\lambda) - n\Sigma_{12}\underline{\mu}_x\underline{\mu}_x'\Sigma_{21}F^*_{p+4,n-p}(c_4;\lambda)$$

$$- \Sigma_{12}\Sigma_{21}F^*_{p+2,n-p}(c_2;\lambda) + (1-\Sigma_{12}\Sigma_{21})\frac{p}{n-p}\int_0^d tf(t)dt$$

If we let $\Sigma_{12}\underline{\mu}_x = K_1$, $\Sigma_{12}\Sigma_{21} = g_1$,

$$h = 2nK_1^2F^*_{p+2,n-p}(c_2;\lambda) - nK_1^2F^*_{p+4,n-p}(c_4;\lambda)$$

$$- g_1F^*_{p+2,n-p}(c_2;\lambda) + (1-g_1)\frac{p}{n-p}\int_0^d tf(t)dt$$

We note that e' is a function of n, Σ_{12}, $\underline{\mu}_x$ and α for any given p. For the computation of e' for certain choices of n, Σ_{12}, $\underline{\mu}_x$ and α, we use the incomplete Beta approximation to the noncentral F distribution. We denote the cumulative distribution function of the noncentral F random variable with ν_1 and ν_2 degrees of freedom by $F^*_{\nu_1,\nu_2}(d;\lambda)$ where λ is the noncentrality parameter. That is, we let

$$\int_0^d f(t|\nu_1,\nu_2,\lambda)dt = F^*_{\nu_1,\nu_2}(d,\lambda) \ .$$

Therefore

$$F^*_{\nu_1,\nu_2}(d,\lambda) = \int_0^d \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}(\frac{\lambda}{2})^j}{j!} (\frac{\nu_1}{\nu_2})^{\frac{1}{2}\nu_1+j} \cdot \frac{t^{\frac{1}{2}\nu_1+j-1}}{B(\frac{1}{2}\nu_1+j,\frac{\nu_2}{2})} (1+\frac{\nu_1}{\nu_2}t)^{-\frac{1}{2}(\nu_1+\nu_2+2j)} dt \tag{2.61}$$

and since $1-I_x(a,b) = I_{1-x}(b,a)$ where $I_x(a,b)$ is the incomplete β function given in Karl Pearson (1934), then from Tiku (1967),

$$F^*_{\nu_1,\nu_2}(d,\lambda) = \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}(\frac{\lambda}{2})^j}{j!} I_x(\frac{\nu_1}{2}+j,\frac{\nu_2}{2});x = \frac{\nu_1 d}{\nu_1 d+\nu_2}. \tag{2.62}$$

To obtain an analogue of (2.62) for

$$\int_0^d tf(t|\nu_1,\nu_2,\lambda)dt \ ,$$

we use (2.61) and note

$$\int_0^d t f(t|\nu_1,\nu_2,\lambda)dt$$

$$= \int_0^d \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}(\frac{\lambda}{2})^j}{j!} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{1}{2}\nu_1+j} \frac{t^{\frac{1}{2}\nu_1+j}}{B(\frac{1}{2}\nu_1+j,\frac{\nu_2}{2})} \left(1+\frac{\nu_1}{\nu_2}t\right)^{-\frac{1}{2}(\nu_1+\nu_2+2j)} dt$$

Let

$$x = \frac{\nu_1 t}{\nu_1 t+\nu_2}$$

or

$$t = \frac{x\nu_2}{\nu_1(1-x)} .$$

Therefore

$$dt = \frac{\nu_2}{\nu_1} \frac{1}{(1-x)^2} dx$$

Further let

$$d_1 = \frac{\nu_1 d}{\nu_1 d+\nu_2} ,$$

we have

$$\sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}(\frac{\lambda}{2})^j}{j!} \int_0^{d_1} \left(\frac{x}{1-x}\right)^{\frac{1}{2}\nu_1+j} \frac{\left[1 + (\frac{x}{1-x})\right]^{-\frac{1}{2}(\nu_1+\nu_2+2j)}}{B(\frac{1}{2}\nu_1+j,\frac{\nu_2}{2})} \frac{\nu_2}{\nu_1}\left(\frac{1}{(1-x)^2}\right)dx$$

$$= \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}(\frac{\lambda}{2})^j}{j!} \int_0^{d_1} \frac{\nu_2}{\nu_1} x^{\frac{1}{2}\nu_1+j} (1-x)^{-(\frac{1}{2}\nu_1+j)} \frac{(1-x)^{-2}[1-x]^{\frac{1}{2}(\nu_1+\nu_2+2j)}}{B(\frac{1}{2}\nu_1+j,\frac{\nu_2}{2})}$$

$$= \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}(\frac{\lambda}{2})^j}{j!} \int_0^{d_1} \frac{\nu_2}{\nu_1} \frac{x^{\frac{1}{2}\nu_1+j}(1-x)^{\frac{1}{2}\nu_2-2}}{B(\frac{1}{2}\nu_1+j,\frac{\nu_2}{2})} dx$$

$$= \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}(\frac{\lambda}{2})^j}{j!} \int_0^{d_1} \frac{\nu_2}{\nu_1} \frac{x^{\frac{\nu_1+2}{2}+j-1}(1-x)^{\frac{\nu_2-2}{2}-1}}{B(\frac{1}{2}\nu_1+j,\frac{\nu_2}{2})} dx$$

But

$$B(\frac{\nu_1}{2}+j,\frac{\nu_2}{2}) = \frac{\nu_2}{\nu_1+2j} B(\frac{\nu_1+2}{2}+j,\frac{\nu_2-2}{2})$$

Therefore

$$\int_0^{d} tf(t)dt = \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}(\frac{\lambda}{2})^j}{j!} \int_0^{d_1} \frac{\nu_2}{\nu_1} \frac{\nu_1+2j}{\nu_2} \frac{x^{\frac{\nu_1+2}{2}+j-1}(1-x)^{\frac{\nu_2-2}{2}-1}}{B(\frac{\nu_1+2}{2}+j,\frac{\nu_2-2}{2})} dx$$

$$= \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}(\frac{\lambda}{2})^j}{j!} \frac{\nu_1+2j}{\nu_1} \frac{1}{B(\frac{\nu_1+2}{2}+j,\frac{\nu_2-2}{2})} \int_0^{d_1} x^{\frac{\nu_1+2}{2}+j-1}(1-x)^{\frac{\nu_2-2}{2}-1} dx$$

$$= \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}(\frac{\lambda}{2})^j}{j!} \frac{\nu_1+2j}{\nu_1} I_{d_1}(\frac{\nu_1+2}{2}+j,\frac{\nu_2-2}{2}) \qquad (2.63)$$

where

$$d_1 = \frac{\nu_1 d}{\nu_1 d+\nu_2} .$$

For the purpose of comparison with the results of Han (1973a), we compute the values of e' for p = 1 and certain values of n, $\Sigma_{12} = \rho$, μ_x and α. These values are shown in Table 2.9 and reveal no significant difference from the values obtained by Han. Han's results were in terms of moments of normal densities while the present results are expressed as a function of the cumulative distribution and the expected values of the truncated noncentral F distribution. Subroutines using an incomplete Beta distribution to approximate the noncentral F distribution were used in the computation and the slight differences for small values of α are due to these approximations and rounding off errors.

Table 2.9 shows e' assumes its maximum value when $\mu_x = 0$. It then decreases to some minimum before increasing to 1.0 as μ_x increases. The value of 1.0 for large values of μ_x corresponds to the fact that when μ_x gets very large, then the difference from zero is significant and we always reject the null hypothesis, thus making the two estimators the same. For fixed n, μ_x and α, e' increases with ρ while for fixed n, μ_x and ρ, e' is a decreasing function of α.

The values of e' for p = 2 are given in Table 2.10 for some values of Σ_{12}, $\underline{\mu}_x$, n and α. From the table, we note the following properties.

1. The relative efficiency e' is maximum when $\underline{\mu}_x = \underline{0}$ for fixed n, α and Σ_{12}. This corresponds to the case when the null hypothesis is true.

2. For a fixed sample size, the maximum value of e' increases with Σ_{12} for any given α but decreases as α increases for a given Σ_{12}.

3. e' remains the same for values of Σ_{12} which differ only in sign.

4. For fixed α, n, Σ_{12} and some component of $\underline{\mu}_x$, the relative efficiency decreases to a minimum and then increases as the other component increases.

5. The value of e' equals 1.0 for large values of n or $\underline{\mu}_x$. This is because the two estimators tend to be the same as n gets large; while for large values of $\underline{\mu}_x$, we would always reject the null hypothesis and use $\hat{\mu} = \bar{y}$.

6. For a fixed α and small values of $\underline{\mu}_x$, the value of e' increases with Σ_{12}, but e' is a decreasing function of Σ_{12} for moderately large values of $\underline{\mu}_x$. For large values of $\underline{\mu}_x$, e' equals 1.0 as explained in 5 above.

Table 2.9. Values of e' for p = 1.

n = 9		α = .05		
μ_x \ ρ	.3	.5	.7	.9
.0	.9924	1.1512	1.5046	2.5468
.2	.9530	1.0290	1.1688	1.4275
.4	.8657	.8166	.7525	.6812
.6	.7979	.6753	.5488	.4391
.8	.7800	.6282	.4863	.3737
1.0	.8166	.6598	.5153	.3988
1.2	.8732	.7487	.6169	.4996
1.5	.9577	.9038	.8335	.7551
		α = .10		
.0	.9945	1.1205	1.3644	1.9434
.2	.9661	1.0201	1.1135	1.2685
.4	.9016	.8508	.7845	.7107
.6	.8611	.7520	.6318	.5208
.8	.8659	.7434	.6132	.4972
1.0	.9044	.8021	.6857	.5745
1.2	.9498	.8870	.8069	.7203
1.5	.9893	.9738	.9515	.9232
		α = .20		
.0	1.0018	1.0766	1.2122	1.4569
.2	.9803	1.0099	1.0578	1.1292
.4	.9423	0.9009	0.8452	0.7808
.6	.9266	0.8510	0.7582	0.6620
.8	.9406	0.8700	0.7819	0.6888
1.0	.9670	0.9226	0.8633	0.7951
1.2	.9871	0.9683	0.9413	0.9077
1.5	.9983	0.9958	0.9919	0.9868

Table 2.9. (continued)

n = 9	α = .30			
μ_x \ ρ	.3	.5	.7	.9
.0	1.0020	1.0477	1.1247	1.2470
.2	.9883	1.0049	1.0309	1.0677
.4	.9657	0.9363	0.8953	0.8460
.6	.9597	0.9116	0.8479	0.7756
.8	.9714	0.9326	0.8800	0.8184
1.0	.9865	0.9667	0.9383	0.9031
1.2	.9957	0.9889	0.9790	0.9661
1.5	.9996	0.9989	0.9980	0.-967
	α = .40			
.0	1.0015	1.0284	1.0715	1.1340
.2	.9932	1.0023	1.0163	1.0355
.4	.9803	.9613	.9342	.9003
.6	.9782	.9498	.9101	.8621
.8	.9860	.9656	.9366	.9005
1.0	.9942	.9852	.9720	.9549
1.2	.9984	.9958	.9919	.9867
1.5	.9999	.9997	.9994	.9990
	α = .50			
.0	1.0010	1.0157	1.0388	1.0711
.2	.9963	1.0010	1.0082	1.0179
.4	.9893	.9783	.9621	.9414
.6	.9888	.9753	.9609	.9327
.8	.9933	.9832	.9684	.9493
1.0	.9975	.9935	.9875	.9796
1.2	.9994	.9983	.9968	.9947
1.5	1.0000	.9999	.9998	.9997

Table 2.9. (continued)

n = 11		α = .05		
μ_x \ ρ	.3	.5	.7	.9
.0	1.0078	1.1632	1.5131	2.5263
.2	.9578	1.0169	1.1206	1.2968
.4	.8632	.7889	.6986	.6062
.6	.8045	.6654	.5284	.4145
.8	.8102	.6562	.5106	.3940
1.0	.8660	.7341	.5976	.4789
1.2	.9328	.8519	.7538	.6534
1.5	.9880	.9707	.9458	.9145
		α = .10		
.0	1.0083	1.1273	1.3699	1.9212
.2	.9689	1.0083	1.0739	1.1759
.4	.9002	.8304	.7438	.6530
.6	.8703	.7532	.6267	.5120
.8	.8930	.7806	.6566	.5418
1.0	.9404	.8665	.7751	.6796
1.2	.9778	.9464	.9030	.8510
1.5	.9976	.9939	.9883	.9810
		α = .20		
.0	1.0067	1.0792	1.2097	1.4424
.2	.9813	1.0012	1.0324	1.0772
.4	.9420	.8896	.8211	.7446
.6	.9344	.8591	.7664	.6700
.8	.9567	.8998	.8262	.7449
1.0	.9820	.9557	.9188	.8738
1.2	.9952	.9877	.9767	.9623
1.5	.9997	.9992	.9985	.9975

Table 2.9. (continued)

n = 11		α = .30		
μ_x \ ρ	.3	.5	.7	.9
.0	1.0048	1.0490	1.1231	1.2399
.2	.9886	.9990	1.0151	1.0374
.4	.9656	.9298	.8807	.8229
.6	.9648	.9190	.8579	.7880
.8	.9800	.9511	.9108	.8621
1.0	.9931	.9824	.9668	.9468
1.2	.9985	.9961	.9925	.9877
1.5	.9999	.9998	.9997	.9994
		α = .40		
.0	1.0031	1.0290	1.0705	1.1313
.2	.9933	.9987	1.0069	1.0181
.4	.9803	.9578	.9259	.8866
.6	.9814	.9552	.9184	.8736
.8	.9906	.9762	.9553	.9289
1.0	.9972	.9926	.9859	.9770
1.2	.9995	.9986	.9973	.9956
1.5	1.0000	1.0000	.9999	.9998
		α = .50		
.0	1.0018	1.0160	1.0381	1.0691
.2	.9963	.9990	1.0030	1.0084
.4	.9894	.9765	.9578	.9340
.6	.9906	.9767	.9565	.9309
.8	.9956	.9887	.9786	.9653
1.0	.9988	.9969	.9940	.9902
1.2	.9998	.9995	.9990	.9983
1.5	1.0000	1.0000	1.0000	1.0000

Table 2.9. (continued)

n = 19	α = .05			
μ_x \ ρ	.3	.5	.7	.9
.0	1.0330	1.1867	1.5276	2.4760
.2	.9560	.9595	.9647	.9718
.4	.8524	.7222	.5876	.4706
.6	.8487	.6987	.5522	.4316
.8	.9210	.8227	.7093	.5991
1.0	.9796	.9493	.9071	.8564
1.2	.9974	.9933	.9872	.9791
1.5	1.0000	1.0000	1.0000	1.0000
	α = .10			
.0	1.0264	1.1417	1.3730	1.8812
.2	.9653	.9600	.9520	.9417
.4	.8965	.7885	.6678	.5547
.6	.9132	.8085	.6899	.5770
.8	.9665	.9183	.8544	.7818
1.0	.9939	.9843	.9702	.9520
1.2	.9995	.9986	.9973	.9956
1.5	1.0000	1.0000	1.0000	1.0000
	α = .20			
.0	1.0170	1.0855	1.2076	1.4206
.2	.9779	.9689	.9558	.9388
.4	.9433	.8726	.7844	.6912
.6	.9633	.9110	.8424	.7655
.8	.9900	.9742	.9515	.9227
1.0	.9988	.9967	.9937	.9897
1.2	.9999	.9998	.9996	.9994
1.5	1.0000	1.0000	1.0000	1.0000

Table 2.9. (continued)

n = 19	α = .30			
μ_x \ ρ	.3	.5	.7	.9
.0	1.0107	1.0520	1.1207	1.2275
.2	.9861	.9784	.9671	.9524
.4	.9677	.9234	.8640	.7957
.6	.9825	.9555	.9178	.8719
.8	.9962	.9899	.9806	.9685
1.0	.9996	.9990	.9981	.9968
1.2	1.0000	1.0000	.9999	.9998
1.5	1.0000	1.0000	1.0000	1.0000
	α = .40			
.0	1.0065	1.0305	1.0688	1.1245
.2	.9916	.9862	.9781	.9676
.4	.9821	.9559	.9191	.8742
.6	.9915	.9778	.9579	.9327
.8	.9984	.9958	.9918	.9866
1.0	.9999	.9997	.9993	.9989
1.2	1.0000	1.0000	1.0000	1.0000
1.5	1.0000	1.0000	1.0000	1.0000
	α = .50			
.0	1.0036	1.0167	1.0371	1.0655
.2	.9953	.9920	.9870	.9804
.4	.9906	.9763	.9555	.9292
.6	.9959	.9893	.9794	.9666
.8	.9993	.9982	.9965	.9943
1.0	1.0000	.9999	.9998	.9996
1.2	1.0000	1.0000	1.0000	1.0000
1.5	1.0000	1.0000	1.0000	1.0000

Table 2.10. Values of e' for p = 2.

n = 5		α = .05		
μ_x' \ Σ_{12}'	$\begin{pmatrix} .5 \\ 0 \end{pmatrix}$	$\begin{pmatrix} .5 \\ .5 \end{pmatrix}$	$\begin{pmatrix} .7 \\ .7 \end{pmatrix}$	$\begin{pmatrix} -.5 \\ .7 \end{pmatrix}$
(0, 0)	0.9853	1.3988	7.2021	2.3427
(0, .5)	0.9107	0.9348	1.3479	0.9592
(0,1.5)	0.7600	0.3752	0.2709	0.2525
(0,3.0)	1.0	1.0	1.0	1.0
(.5, .5)	0.6977	0.5192	0.4252	1.7289
(.5,1.5)	0.6735	0.2655	0.1706	0.3955
(.5,3.0)	1.0	1.0	1.0	1.0
(1.5,1.5)	0.4660	0.2144	0.1286	0.9600
(1.5,3.0)	1.0	1.0	1.0	1.0
(3.0,3.0)	1.0	1.0	1.0	1.0
		α = .20		
(0, 0)	1.0408	1.2939	2.4263	1.6877
(0, .5)	1.0106	0.9745	1.1088	0.9423
(0,1.5)	0.9791	0.7090	0.5750	0.5605
(0,3.0)	1.0	1.0	1.0	1.0
(.5, .5)	0.8486	0.6466	0.5195	1.3323
(.5,1.5)	0.9441	0.6164	0.4621	0.7389
(.5,3.0)	1.0	1.0	1.0	1.0
(1.5,1.5)	0.9230	0.7649	0.6296	0.9993
(1.5,3.0)	1.0	1.0	1.0	1.0
(3.0,3.0)	1.0	1.0	1.0	1.0
		α = .50		
(0, 0)	1.0266	1.1007	1.2776	1.1826
(0, .5)	1.0158	0.9857	1.0013	0.9584
(0,1.5)	1.0001	0.9673	0.9400	0.9377
(0,3.0)	1.0	1.0	1.0	1.0
(.5, .5)	0.9578	0.8538	0.7617	1.0696
(.5,1.5)	0.9973	0.9588	0.9238	0.9756
(.5,3.0)	1.0	1.0	1.0	1.0
(1.5,1.5)	0.9987	0.9951	0.9905	1.0
(1.5,3.0)	1.0	1.0	1.0	1.0
(3.0,3.0)	1.0	1.0	1.0	1.0

Table 2.10. (continued)

n = 7		α = .05		
Σ'_{12} / μ'_x	$\begin{pmatrix} .5 \\ 0 \end{pmatrix}$	$\begin{pmatrix} .5 \\ .5 \end{pmatrix}$	$\begin{pmatrix} .7 \\ .7 \end{pmatrix}$	$\begin{pmatrix} -.5 \\ .7 \end{pmatrix}$
(0, 0)	1.0176	1.4274	6.2926	2.3269
(0, .5)	0.9277	0.8481	0.9950	0.7835
(0,1.5)	0.8848	0.4879	0.3545	0.3410
(0,3.0)	1.0	1.0	1.0	1.0
(.5, .5)	0.6766	0.4481	0.3308	1.5161
(.5,1.5)	0.8262	0.3918	0.2606	0.5250
(.5,3.0)	1.0	1.0	1.0	1.0
(1.5,1.5)	0.8545	0.5884	0.4217	0.9985
(1.5,3.0)	1.0	1.0	1.0	1.0
(3.0,3.0)	1.0	1.0	1.0	1.0
		α = .20		
(0, 0)	1.0376	1.2685	2.2148	1.6131
(0, .5)	1.0008	0.9023	0.9243	0.8243
(0,1.5)	0.9941	0.8856	0.8077	0.8015
(0,3.0)	1.0	1.0	1.0	1.0
(.5, .5)	0.8379	0.6148	0.4729	1.1899
(.5,1.5)	0.9858	0.8626	0.7683	0.9136
(.5,3.0)	1.0	1.0	1.0	1.0
(1.5,1.5)	1.0	1.0	1.0	1.0
(1.5,3.0)	1.0	1.0	1.0	1.0
(3.0,3.0)	1.0	1.0	1.0	1.0
		α = .50		
(0, 0)	1.0195	1.0853	1.2386	1.1569
(0, .5)	1.0080	0.9631	0.9527	0.9236
(0,1.5)	0.9999	0.9938	0.9883	0.9880
(0,3.0)	1.0	1.0	1.0	1.0
(.5, .5)	0.9592	0.8627	0.7717	1.0359
(.5,1.5)	0.9996	0.9939	0.9882	0.9963
(.5,3.0)	1.0	1.0	1.0	1.0
(1.5,1.5)	1.0	1.0	1.0	1.0
(1.5,3.0)	1.0	1.0	1.0	1.0
(3.0,3.0)	1.0	1.0	1.0	1.0

Table 2.10. (continued)

n = 9	$\alpha = .05$			
$\underline{\mu}_x'$ \ Σ_{12}'	$\begin{pmatrix} .5 \\ 0 \end{pmatrix}$	$\begin{pmatrix} .5 \\ .5 \end{pmatrix}$	$\begin{pmatrix} .7 \\ .7 \end{pmatrix}$	$\begin{pmatrix} -.5 \\ .7 \end{pmatrix}$
(0, 0)	1.0633	1.4760	5.7927	2.3526
(0, .5)	0.9662	0.7956	0.8099	0.6803
(0,1.5)	0.9664	0.6829	0.5426	0.5328
(0,3.0)	1.0	1.0	1.0	1.0
(.5, .5)	0.6838	0.4179	0.2901	0.3828
(.5,1.5)	0.9407	0.6284	0.4740	0.7384
(.5,3.0)	1.0	1.0	1.0	1.0
(1.5,1.5)	1.0	1.0	1.0	1.0
(1.5,3.0)	1.0	1.0	1.0	1.0
(3.0,3.0)	1.0	1.0	1.0	1.0
	$\alpha = .20$			
(0, 0)	1.0545	1.2741	2.1230	1.5924
(0, .5)	1.0099	0.8673	0.8312	0.7637
(0,1.5)	0.9990	0.9672	0.9401	0.9385
(0,3.0)	1.0	1.0	1.0	1.0
(.5, .5)	0.8506	0.6189	0.4688	1.1214
(.5,1.5)	0.9972	0.9670	0.9388	0.9798
(.5,3.0)	1.0	1.0	1.0	1.0
(1.5,1.5)	1.0	1.0	1.0	1.0
(1.5,3.0)	1.0	1.0	1.0	1.0
(3.0,3.0)	1.0.	1.0	1.0	1.0
	$\alpha = .50$			
(0, 0)	1.0219	1.0824	1.2214	1.1477
(0, .5)	1.0074	0.9541	0.9291	0.9081
(0,1.5)	1.0	0.9989	0.9979	0.9978
(0,3.0)	1.0	1.0	1.0	1.0
(.5, .5)	0.9663	0.8814	0.7973	1.0207
(.5,1.5)	0.9999	0.9991	0.9983	0.9995
(.5,3.0)	1.0	1.0	1.0	1.0
(1.5,1.5)	1.0	1.0	1.0	1.0
(1.5,3.0)	1.0	1.0	1.0	1.0
(3.0,3.0)	1.0	1.0	1.0	1.0

III. DOUBLE SAMPLING WITH PARTIAL INFORMATION ON AUXILIARY VARIABLES

A. Introduction

Consider a p+1 variate normal population

$$\begin{pmatrix} Y \\ \underline{X} \end{pmatrix} \sim N(\underline{\mu}, \Sigma)$$

where Y is a univariate random variable and $\underline{X}$ is a pxl random vector with $p \geq 1$,

$$\underline{\mu} = \begin{pmatrix} \mu \\ \underline{\mu}_x \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma^2 & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \qquad (3.1)$$

Suppose we are interested in estimating the population mean μ of Y. It is well known that the precision of the estimator can be increased if auxiliary information is available. For example, if the relationship is linear, a linear regression estimator may be constructed. We shall consider here the regression estimator. In the given multivariate normal distribution, the vector $\underline{X}$ is correlated with Y and so can be used as an ancillary variable to increase precision in estimating μ. To use the regression estimator we need to know the population mean $\underline{\mu}_x$ of $\underline{X}$. When $\underline{\mu}_x$ is unknown, we may take a preliminary sample to estimate it. This sampling procedure is the double sampling technique. In certain situations, an investigator may have partial information about $\underline{\mu}_x$. In order to make use of this partial information, the investigator can perform a preliminary test about the hypothesis H_0: $\underline{\mu}_x = \underline{\mu}_0$ versus H_1: $\underline{\mu}_x \neq \underline{\mu}_0$ where $\underline{\mu}_0$ is some constant vector that he believes that the population mean $\underline{\mu}_x$ should be based on the partial information.

As an example, consider estimating the average growth of some rats. It is known that the growth is highly correlated with the amount of a certain vitamin in the feed. Hence the vitamin content of the feed can be used as an auxiliary variable. The investigator usually does not know the population mean value of the vitamin content but from the growth of Neurospora mycelium (or some other fungus) on agae plates and the comparison of this with the growth on some control plates with known concentration of the vitamin, the experimenter may believe that the population mean should be $\underline{\mu}_0$. Once a preliminary sample is available, the investigator may test H_0: $\underline{\mu}_x = \underline{\mu}_0$ against H_1: $\underline{\mu}_x \neq \underline{\mu}_0$. He then will use $\underline{\mu}_0$ in the regression estimator if H_0 is accepted, otherwise he uses the sample mean based on the preliminary sample. This estimator is usually known as the preliminary test estimator. If the investigator's prior information or experience is reliable, then the true mean $\underline{\mu}_x$ of $\underline{X}$ will be expected to be very close to $\underline{\mu}_0$. In this situation, the efficiency of the preliminary test estimation is very high. Thus in practice, it is desirable to use the preliminary test estimator when some partial information is available to the investigator.

B. The Preliminary Test Estimator and its Bias when Σ is Known

Let $\binom{Y}{\underline{X}}$ have a multivariate normal distribution as given in section A. We assume $\underline{X}$ is cheaply observed while the pair $(Y,\underline{X})$ is more expensive to observe. We wish to estimate μ, the population mean of Y. Let $(y_i, X_{1i}, X_{2i}, \ldots, X_{pi})'$ $i = 1,\ldots,n_2$ be a random sample from $N(\underline{\mu},\Sigma)$. This is supplemented by m more independent observations on $\underline{X}' = (\underline{X}_1,\ldots,\underline{X}_p)'$. In practice, the sample of n_2 observations is usually a subsample from the sample of $n_1 = n_2+m$ observations. From all the observations, we define

$$\underline{\bar{X}}_1 = \left(\frac{1}{n_1}\sum_{i=1}^{n_1} X_{1i},\ldots,\frac{1}{n_1}\sum_{i=1}^{n_1} X_{pi}\right)',$$

and from the subsample in which $\underline{X}$ and Y are observed, we define

$$\underline{\bar{X}}_2 = \left(\frac{1}{n_2}\sum_{i=1}^{n_2} X_{1i},\ldots,\frac{1}{n_2}\sum_{i=1}^{n_2} X_{pi}\right)' \quad \bar{y} = \frac{1}{n_2}\sum_{i=1}^{n_2} y_i \ .$$

If the vector $\underline{\mu}_x$ and Σ are known, then the regression estimator of μ is

$$\hat{\mu} = \bar{y} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{\mu}_x - \underline{\bar{X}}_2) \ .$$

The regression estimator is unbiased with variance

$$\frac{1}{n_2}\{\sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\}.$$

If

$$\frac{1}{n_2}\{\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\}$$

is considerably large, we have an appreciable gain in precision. If $\underline{\mu}_x$ is unknown and it happens that from certain sources, the experimenter is pretty sure but not certain that $\underline{\mu}_x = \underline{\mu}_0$, then he may perform a preliminary test of H_0: $\underline{\mu}_x = \underline{\mu}_0$. In this case he can make the regression estimator depend on the result of the preliminary test. The new estimator is then the preliminary test estimator. Without loss of generality, we let $\underline{\mu}_0 = \underline{0}$. Thus the preliminary test estimator is defined as

$$\hat{\mu}_{\ell r} = \begin{cases} \bar{y} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\bar{X}}_2 & \text{if } n_1(\underline{\bar{X}}_1'\Sigma_{22}^{-1}\underline{\bar{X}}_1) \le \chi^2_{p,\alpha} \\ \bar{y} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{\bar{X}}_1-\underline{\bar{X}}_2) & \text{if } n_1(\underline{\bar{X}}_1'\Sigma_{22}^{-1}\underline{\bar{X}}_1) > \chi^2_{p,\alpha} \end{cases} \quad (3.2)$$

where the subscript ℓr denotes linear regression and $\chi^2_{p,\alpha}$ is the 100(1-α) percent point of the Chi-squared distribution with p degrees of freedom. α is the level of significance of the preliminary test.

The joint distribution of $(\underline{\bar{X}}_1, \underline{\bar{X}}_2, \bar{y})'$ is normal with mean $(\underline{\mu}_x, \underline{\mu}_x, \mu)'$ and covariance matrix

$$\begin{pmatrix} \frac{1}{n_1}\Sigma_{22} & \frac{1}{n_1}\Sigma_{22} & \frac{1}{n_1}\Sigma_{12} \\ \frac{1}{n_1}\Sigma_{22} & \frac{1}{n_2}\Sigma_{22} & \frac{1}{n_2}\Sigma_{12} \\ \frac{1}{n_1}\Sigma_{12} & \frac{1}{n_2}\Sigma_{12} & \frac{1}{n_2}\sigma^2 \end{pmatrix}$$

Denote the acceptance region for the preliminary test by A and its complement by $\bar{A}$ and let $\chi^2_{p,\alpha} = c$. The expected value of $\hat{\mu}_{\ell r}$ is

$$\begin{aligned} E(\hat{\mu}_{\ell r}) &= E\{(\bar{y} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\bar{X}}_2)|A\}P(A) \\ &+ E\{[\bar{y} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{\bar{X}}_1-\underline{\bar{X}}_2)]|\bar{A}\}P(\bar{A}) \\ &= E(\bar{y} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\bar{X}}_2) \\ &+ \Sigma_{12}\Sigma_{22}^{-1}\, E\{\underline{\bar{X}}_1|\bar{A}\}P(\bar{A}) \\ &= \mu - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x + \Sigma_{12}\Sigma_{22}^{-1}\, E\{\underline{\bar{X}}_1|\bar{A}\}P(\bar{A}) \qquad (3.3) \end{aligned}$$

Hence the bias of $\hat{\mu}_{\ell r}$ is given as

$$B_1 = \Sigma_{12}\Sigma_{22}^{-1}\, E\{\underline{\bar{X}}_1|\bar{A}\}P(\bar{A}) - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x \qquad (3.4)$$

In order to evaluate the bias, we need to find the first term.

Now $\underline{\bar{X}}_1 \sim N(\underline{\mu}_x, \frac{1}{n_1}\Sigma_{22})$ and since Σ_{22} is positive definite, a nonsingular matrix $D \ni D'D = \Sigma_{22}^{-1}$. Let

$\underline{Z} = D\underline{\bar{X}}_1$, then

$$\underline{Z} \quad N(D\underline{\mu}_x, \frac{1}{n_1}, I)$$

$$N(\underline{\gamma}_x, \frac{1}{n_1}, I) \quad \text{say.}$$

Hence $\{n_1(\underline{Z}'\underline{Z}) : n_1(\underline{Z}'\underline{Z}) > c\} \equiv \bar{A}$.

$$B_1 = \Sigma_{12}\Sigma_{22}^{-1}D^{-1} \, E[\underline{Z}|\bar{A}]P(\bar{A}) - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x \tag{3.5}$$

It is known that $n_1(\underline{Z}'\underline{Z})$ has a noncentral Chi-squared distribution with p degrees of freedom and noncentrality parameter $\delta = n_1(\underline{\mu}_x'D'D\underline{\mu}_x) = n_1(\underline{\gamma}_x'\underline{\gamma}_x)$. $\underline{\gamma}_x$ is a pxl vector and we denote the i-th component by $\gamma_x^{(i)}$. Hence

$$T = P(\bar{A}) = \int_c^\infty e^{-\frac{1}{2}\delta} \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{\delta}{2}\right)^j h_{p+2j}(t)dt \tag{3.6}$$

where $h_{p+2j}(\cdot)$ is the probability density function of χ^2_{p+2j}. Differentiating (3.6) with respect to $\gamma_x^{(i)}$, we obtain

$$\frac{\partial T}{\partial \gamma_x^{(i)}} = \frac{\partial}{\partial \gamma_x^{(i)}} \int_c^\infty e^{-\frac{1}{2}\delta} \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{\delta}{2}\right)^j h_{p+2j}(t)dt$$

and by the Lebesgue Dominated Convergence Theorem (LDCT) as justified in the Appendix, we can take the differentiation inside the integral and have

$$\frac{\partial T}{\partial \gamma_x^{(i)}} = \int_c^\infty e^{-\frac{1}{2}\delta} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n_1\gamma_x^{(i)}}{2} j(\frac{\delta}{2})^{j-1} h_{p+2j}(t)dt \qquad (3.7)$$

$$- \int_c^\infty \frac{1}{2} 2n_1\gamma_x^{(i)} e^{-\frac{1}{2}\delta} \sum_{j=0}^{\infty} \frac{1}{j!}(\frac{\delta}{2})^j h_{p+2j}(t)dt$$

$$= n_1[1-H_{p+2}(c;\delta) - P(\bar{A})]\gamma_x^{(i)}$$

$$= n_1[P(A) - H_{p+2}(c;\delta)]\gamma_x^{(i)}$$

where

$$P(A) = \int_0^c e^{-\frac{1}{2}\delta} \sum_{j=0}^{\infty} \frac{1}{j!}(\frac{\delta}{2})^j h_{p+2j}(t)dt$$

and $H_{p+2}(c;\delta)$ is the cumulative distribution function of the noncentral Chi-squared distribution with p+2 degrees of freedom and noncentrality parameter δ.

Alternatively, we can evaluate $P(\bar{A})$ by the use of the distribution of $\underline{Z}$ and write

$$T = P(\bar{A}) = \int_{\bar{A}}\cdots\int \prod_{j=1}^{P} \frac{1}{\sqrt{2\pi}} \sqrt{n_1}\, e^{-\frac{n_1}{2}(Z^{(j)}-\gamma_x^{(j)})^2} dZ^{(j)} \qquad (3.8)$$

If we now differentiate (3.8) w.r.t. $\gamma_x^{(i)}$ by the LDCT as shown in the Appendix, we have

$$\frac{\partial T}{\partial \gamma_x^{(i)}} = \int_{\bar{A}}\cdots\int \prod_{j=1}^{P} \frac{\sqrt{n_1}}{\sqrt{2\pi}} \frac{n_1}{2} 2(Z^{(i)}-\gamma_x^{(i)}) e^{-\frac{n_1}{2}(Z^{(j)}-\gamma_x^{(j)})^2} dZ^{(j)}$$

$$= n_1\, E[Z^{(i)}|\bar{A}]\, P(\bar{A}) - n_1\gamma_x^{(i)} P(\bar{A}) \qquad (3.9)$$

Hence we may obtain $E(Z^{(i)}|\bar{A})P(\bar{A})$ by equating (3.7) and (3.9). That is

$$n_1[1-H_{p+2}(c;\delta) - P(\bar{A})]\gamma_x^{(i)} = n_1[E(\underline{Z}^{(i)}|\bar{A})P(\bar{A}) - \gamma_x^{(i)}P(\bar{A})]$$

$$E(Z^{(i)}|\bar{A})P(\bar{A}) = [1-H_{p+2}(c;\delta)]\gamma_x^{(i)} \qquad (3.10)$$

Substituting (3.10) in (3.5), then

$$B_1 = \Sigma_{12}\Sigma_{22}^{-1}D^{-1}[1-H_{p+2}(c;\delta)]\underline{\gamma}_x - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x$$

$$= -\Sigma_{12}\Sigma_{22}^{-1}D^{-1}\underline{\gamma}_x H_{p+2}(c;\delta) = -\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x H_{p+2}(c;\delta) \qquad (3.11)$$

As a partial check, when $c = 0$, the estimator reduces to $\bar{y}$ + $\Sigma_{12}\Sigma_{22}^{-1}(\bar{\underline{X}}_1-\bar{\bar{\underline{X}}}_2)$ with zero bias which is the case when we always reject the null hypothesis. In this case $H_{p+2}(c;\delta) = 0$ and $B_1 = 0$. When $c = \infty$, the null hypothesis is always accepted and the estimator reduces to $\hat{\mu}_{\ell r} = \bar{y} - \Sigma_{12}\Sigma_{22}^{-1}\bar{\underline{X}}_2$. Here $H_{p+2}(c;\delta) = 1$ and $B_1 = -\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x$ which is the bias for the regression estimator, $\bar{y} - \Sigma_{12}\Sigma_{22}^{-1}\bar{\underline{X}}_2$.

Wlog we let $\Sigma_{22} = I$ and $\sigma^2 = 1$. Again for $p = 1$, we observe that B_1 changes sign with $\Sigma_{12} = \rho$ or μ_x so we need only study the bias for $\mu_x > 0$ and $\rho > 0$. The values of $-B_1$ for $p = 1$ and $n_1 = 30$ and certain values of ρ, μ_x and α are given in Table 3.1 and are independent of n_2.

Table 3.1. Values of $-B_1$ for $n_1 = 30$ and $p = 1$.

	$\alpha = .05$		$\alpha = .10$		$\alpha = .25$	
			ρ			
μ_x	.7	.9	.7	.9	.7	.9
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.0474	0.0610	0.0362	0.0465	0.0173	0.0222
0.2	0.0781	0.1005	0.0561	0.0722	0.0245	0.0315
0.3	0.0827	0.1063	0.0543	0.0698	0.0207	0.0266
0.4	0.0648	0.0833	0.0380	0.0489	0.0121	0.0156
0.5	0.0387	0.0497	0.0199	0.0256	0.0052	0.0067
0.6	0.0176	0.0227	0.0079	0.0102	0.0017	0.0021
0.7	0.0061	0.0079	0.0024	0.0030	0.0004	0.0005
0.8	0.0016	0.0021	0.0005	0.0007	0.0001	0.0001
0.9	0.0003	0.0004	0.0001	0.0001	0.0000	0.0000
1.0	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000

From Table 3.1 the following properties of the bias are obvious.

1. The bias is zero when $\mu_x = 0$ which is when the null hypothesis is true.

2. The bias is an increasing function of ρ, but a decreasing function of α.

3. For fixed n, α and ρ, the bias first increases from zero and then decreases to zero as μ_x increases from zero to one.

We observe that the values obtained here correspond with those of Han (1973b) which are used as a further check of the expression for the bias.

For p = 2, the values of $-B_1$ for n_1 = 30 and certain values of Σ_{12}, $\underline{\mu}_x$ and α are given in Table 3.2.

Table 3.2. Values of $-B_1$ for p = 2 and n_1 = 30.

$\underline{\mu}_x'$ \ Σ_{12}'	$\begin{pmatrix} .7 \\ 0 \end{pmatrix}$	$\begin{pmatrix} .5 \\ .5 \end{pmatrix}$	$\begin{pmatrix} .7 \\ .7 \end{pmatrix}$	$\begin{pmatrix} -.5 \\ .7 \end{pmatrix}$
		α = .05		
(0, 0)	0.0	0.0	0.0	0.0
(.5, 0)	0.0659	0.0471	0.0659	-0.0471
(.5, .5)	.0622	0.0888	0.1244	0.0178
(1.0, 0)	.0002	0.0001	0.0002	0.0001
(1.0, .5)	.0002	0.0002	0.0003	0.0000
(1.0,1.0)	.0002	0.0002	0.0003	0.0000
		α = .10		
(0, 0)	0.0	0.0	0.0	0.0
(.5, 0)	0.0383	0.0274	0.0383	-0.0274
(.5, .5)	0.0358	0.0511	0.0715	0.0102
(1.0, 0)	0.0001	0.0000	0.0001	0.0000
(1.0, .5)	0.0001	0.0001	0.0001	0.0000
(1.0,1.0)	0.0000	0.0001	0.0001	0.0000
		α = .25		
(0, 0)	0.0	0.0	0.0	0.0
(.5, 0)	0.0126	0.0090	0.0126	-0.0090
(.5, .5)	0.0116	0.0165	0.0232	0.0033
(1.0, 0)	0.0	0.0	0.0	0.0
(1.0, .5)	0.0	0.0	0.0	0.0
(1.0,1.0)	0.0	0.0	0.0	0.0

From Table 3.2 we note that for $p = 2$ and $n_1 = 30$

1. The bias is zero when $\underline{\mu}_x = \underline{0}$. This once more corresponds to the case when the null hypothesis is true.

2. The bias is generally an increasing function of Σ_{12}, but a decreasing function of α.

3. For fixed n, α and Σ_{12}, the bias first increases from zero and then decreases to zero as $\underline{\mu}_x'$ increases from (0,0) to (1.0,1.0).

C. The M.S.E. of $\hat{\mu}_{\ell r}$ when Σ is Known

By definition, the M.S.E. of $\hat{\mu}_{\ell r}$ is given by M.S.E.$(\hat{\mu}_{\ell r}) = V(\hat{\mu}_{\ell r}) + (\text{Bias})^2$. Therefore to find M.S.E.$(\hat{\mu}_{\ell r})$, we may first find

$$V(\hat{\mu}_{\ell r}) = E(\hat{\mu}_{\ell r})^2 - [E(\hat{\mu}_{\ell r})]^2 \tag{3.12}$$

From (3.2), we have

$$\begin{aligned} E(\hat{\mu}^2_{\ell r}) &= E[(\bar{y} - \Sigma_{12}\Sigma_{22}^{-1}\bar{\underline{X}}_2)^2 | A]P(A) \\ &+ E[(\bar{y} - \Sigma_{12}\Sigma_{22}^{-1}\bar{\underline{X}}_2)^2 | \bar{A}]P(\bar{A}) + E[(\Sigma_{12}\Sigma_{22}^{-1}D^{-1}\underline{Z})^2 | \bar{A}]P(\bar{A}) \\ &+ 2\Sigma_{12}\Sigma_{22}^{-1}D^{-1}E[(\bar{y}\underline{Z}) | \bar{A}]P(\bar{A}) - 2\Sigma_{12}\Sigma_{22}^{-1}E[(\bar{\underline{X}}_1\bar{\underline{X}}_2') | \bar{A}]P(\bar{A})\Sigma_{22}^{-1}\Sigma_{21} \end{aligned}$$

Therefore

$$\begin{aligned} E(\hat{\mu}^2_{\ell r}) &= E(\bar{y} - \Sigma_{12}\Sigma_{22}^{-1}\bar{\underline{X}}_2)^2 + 2\Sigma_{12}\Sigma_{22}^{-1}D^{-1}E[(\bar{y}\underline{Z}) | \bar{A}]P(\bar{A}) \\ &- 2\Sigma_{12}\Sigma_{22}^{-1}E[(\bar{\underline{X}}_1\bar{\underline{X}}_2') | \bar{A}]P(\bar{A})\Sigma_{22}^{-1}\Sigma_{21} \\ &+ \Sigma_{12}\Sigma_{22}^{-1}D^{-1}E[\underline{Z}\underline{Z}' | \bar{A}]P(\bar{A})D'^{-1}\Sigma_{22}^{-1}\Sigma_{21} \end{aligned} \tag{3.13}$$

Therefore to evaluate $E(\hat{\mu}^2_{\ell r})$, we need to find $E[\underline{Z}\underline{Z}'|\bar{A}]P(\bar{A})$, $E[(\bar{y}\underline{Z})|\bar{A}]P(\bar{A})$ and $E[\bar{\underline{X}}_1\bar{\underline{X}}_2'|\bar{A}]P(\bar{A})$. As before, we denote the i-th component of $\underline{Z}$ by $Z^{(i)}$ and note that it is sufficient to consider only $E[(Z^{(i)})^2|\bar{A}]P(\bar{A})$ and $E[Z^{(i)}Z^{(K)}|\bar{A}]P(\bar{A})$ for $i \neq K$. To evaluate these, we use the second derivatives of T where T is given in (3.6). Thus differentiating (3.7) w.r.t. $\gamma_x^{(i)}$, we have

$$\frac{\partial^2 T}{\partial \gamma_x^{(i)2}} = \int_c^\infty e^{-\frac{1}{2}\delta} \sum_{j=0}^\infty \frac{1}{j!}\left(\frac{2n_1\gamma_x^{(i)}}{2}\right)^2 j(j-1)\left(\frac{\delta}{2}\right)^{j-2} h_{p+2j}(t)dt$$

$$+ \int_c^\infty e^{-\frac{1}{2}\delta} \sum_{j=0}^\infty \frac{1}{j!} \frac{2n_1}{2} j\left(\frac{\delta}{2}\right)^{j-1} h_{p+2j}(t)dt$$

$$- \int_c^\infty \frac{1}{2}\, 2n_1\gamma_x^{(i)} e^{-\frac{1}{2}\delta} \sum_{j=0}^\infty \frac{1}{j!} \frac{2n_1\gamma_x^{(i)}}{2} j\left(\frac{\delta}{2}\right)^{j-1} h_{p+2j}(t)dt$$

$$- \int_c^\infty \frac{1}{2}\, 2n_1\, e^{-\frac{1}{2}\delta} \sum_{j=0}^\infty \frac{1}{j!}\left(\frac{\delta}{2}\right)^j h_{p+2j}(t)dt$$

$$+ \int_c^\infty \left(\frac{1}{2}\, 2n_1\gamma_x^{(i)}\right)^2 e^{-\frac{1}{2}\delta} \sum_{j=0}^\infty \frac{1}{j!}\left(\frac{\delta}{2}\right)^j h_{p+2j}(t)dt$$

$$- \int_c^\infty \frac{1}{2}\, 2n_1\gamma_x^{(i)}\, e^{-\frac{1}{2}\delta} \sum_{j=0}^\infty \frac{1}{j!} \frac{2n_1\gamma_x^{(i)}}{2} j\left(\frac{\delta}{2}\right)^{j-1} h_{p+2j}(t)dt$$

Therefore

$$\frac{\partial^2 T}{\partial \gamma_x^{(i)2}} = n_1{}^2(\gamma_x^{(i)})^2[1-H_{p+4}(c;\delta)] + n_1[1-H_{p+2}(c;\delta)]$$

$$- n_1{}^2(\gamma_x^{(i)})^2[1-H_{p+2}(c;\delta)] - n_1 P(\bar{A}) + n_1{}^2(\gamma_x^{(i)})^2 P(\bar{A})$$

$$- n_1{}^2(\gamma_x^{(i)})^2[1-H_{p+2}(c;\delta)]$$

Hence

$$\frac{\partial^2 T}{\partial \gamma_x^{(i)2}} = n_1{}^2(\gamma_x^{(i)})^2[1-H_{p+4}(c;\delta)] + n_1\{1-2n_1(\gamma_x^{(i)})^2\}[1-H_{p+2}(c;\delta)]$$

$$+ n_1[n_1(\gamma_x^{(i)})^2 - 1]P(\bar{A}) \qquad (3.14)$$

Similarly, differentiating T twice w.r.t. $\gamma_x^{(i)}$ where T is given in (3.8), we obtain

$$\frac{\partial^2 T}{\partial \gamma_x^{(i)2}} = \int_{\bar{A}}\cdots\int \prod_{j=1}^{P} \frac{\sqrt{n_1}}{\sqrt{2\pi}} (\frac{n_1}{2}\,2)^2 Z^{(i)}(Z^{(i)}-\gamma_x^{(i)})e^{-\frac{n_1}{2}(Z^{(j)}-\gamma_x^{(j)})^2} dZ^{(j)}$$

$$- \int_{\bar{A}}\cdots\int \prod_{j=1}^{P} \frac{\sqrt{n_1}}{\sqrt{2\pi}} \frac{n_1}{2}\,2 e^{-\frac{n_1}{2}(Z^{(j)}-\gamma_x^{(i)})^2} dZ^{(j)}$$

$$- \int_{\bar{A}}\cdots\int \prod_{j=1}^{P} \frac{\sqrt{n_1}}{\sqrt{2\pi}} (\frac{n_1}{2}\,2)^2 \gamma_x^{(i)}(Z^{(i)}-\gamma_x^{(i)})e^{-\frac{n_1}{2}(Z^{(j)}-\gamma_x^{(j)})^2} dZ^{(j)}$$

Therefore

$$\frac{\partial^2 T}{\partial \gamma_x^{(i)^2}} = n_1^{\,2} E[Z^{(i)^2}|\bar{A}]P(\bar{A}) - n_1^{\,2}\gamma_x^{(i)} E(Z^{(i)}|\bar{A})P(\bar{A}) \tag{3.15}$$

$$- n_1 P(\bar{A}) - n_1^{\,2}\gamma_x^{(i)} E(Z^{(i)}|\bar{A})P(\bar{A}) + n_1^{\,2}(\gamma_x^{(i)})^2 P(\bar{A})$$

Hence from (3.10),

$$\frac{\partial^2 T}{\partial \gamma_x^{(i)^2}} = n_1^{\,2} E[Z^{(i)^2}|\bar{A}]P(\bar{A}) - 2n_1^{\,2}(\gamma_x^{(i)})^2[1-H_{p+2}(c;\delta)]$$

$$+ n_1^{\,2}(\gamma_x^{(i)})^2 P(\bar{A}) - n_1 P(\bar{A}) \tag{3.16}$$

Equating (3.16) and (3.14), we have

$$n_1^{\,2} E[Z^{(i)^2}|\bar{A}]P(\bar{A}) - 2n_1^{\,2}(\gamma_x^{(i)})^2[1-H_{p+2}(c;\delta)] + n_1^{\,2}(\gamma_x^{(i)})^2 P(\bar{A})$$

$$- n_1 P(\bar{A}) = n_1^{\,2}(\gamma_x^{(i)})^2[1-H_{p+4}(c;\delta)]$$

$$+ n_1\{1-2n_1(\gamma_x^{(i)})^2\}[1-H_{p+2}(c;\delta)] + n_1[n_1(\gamma_x^{(i)})^2 - 1]P(\bar{A})$$

or

$$E(Z^{(i)^2}|\bar{A})P(\bar{A}) = (\gamma_x^{(i)})^2[1-H_{p+4}(c;\delta)] + \frac{1}{n_1}[1-H_{p+2}(c;\delta)]$$

Next we find $E[Z^{(i)}Z^{(K)}|\bar{A}]P(\bar{A})$. Differentiating T w.r.t. $\gamma_x^{(i)}$ and then w.r.t. $\gamma_x^{(K)}$ where T is given in (3.6), we have

$$\frac{\partial T}{\partial\gamma_x^{(K)}\partial\gamma_x^{(i)}} = \int_c^\infty e^{-\frac{1}{2}\delta} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n_1\gamma_x^{(i)}}{2} \frac{2n_1\gamma_x^{(K)}}{2} j\; j-1 (\frac{\delta}{2})^{j-2} h_{p+2j}(t)dt$$

$$- \int_c^\infty \frac{1}{2}\, 2n_1\gamma_x^{(K)} e^{-\frac{1}{2}\delta} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n_1\gamma_x^{(i)}}{2}\, j (\frac{\delta}{2})^{j-1} h_{p+2j}(t)dt$$

$$- \int_c^\infty \frac{2n_1\gamma_x^{(i)}}{2}\, e^{-\frac{1}{2}\delta} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n_1\gamma_x^{(K)}}{2}\, j\, (\frac{\delta}{2})^{j-1} h_{p+2j}(t)dt$$

$$+ \int_c^\infty (\frac{2n_1}{2})^2 \gamma_x^{(i)}\gamma_x^{(K)} e^{-\frac{1}{2}\delta} \sum_{j=0}^{\infty} \frac{1}{j!} (\frac{\delta}{2})^j h_{p+2j}(t)dt$$

$$\frac{\partial T}{\partial\gamma_x^{(K)}\partial\gamma_x^{(i)}} = n_1{}^2\gamma_x^{(i)}\gamma_x^{(K)}[1-H_{p+4}(c;\delta)] - 2n_1{}^2\gamma_x^{(i)}\gamma_x^{(K)}[1-H_{p+2}(c;\delta)]$$

$$+ n_1{}^2\gamma_x^{(i)}\gamma_x^{(K)} P(\bar{A}) \qquad (3.17)$$

Similarly using (3.8) we have

$$\frac{\partial T}{\partial\gamma_x^{(K)}\partial\gamma_x^{(i)}} = \int_{\bar{A}}\cdots\int \prod_{j=1}^{P} \frac{\sqrt{n_1}}{\sqrt{2\pi}} (\frac{2n_1}{2})^2 (Z^{(i)}-\gamma_x^{(i)})(Z^{(K)}-\gamma_x^{(K)})$$

$$e^{-\frac{n_1}{2}(Z^{(j)}-\gamma_x^{(j)})^2} dZ^{(j)}$$

Therefore

$$\frac{\partial T}{\partial \gamma_x^{(K)} \partial \gamma_x^{(i)}} = n_1{}^2 E(Z^{(i)} Z^{(K)} | \bar{A}) P(\bar{A}) - n_1{}^2 \gamma_x^{(i)} E(Z^{(K)} | \bar{A}) P(\bar{A})$$

$$- n_1{}^2 \gamma_x^{(K)} E(Z^{(i)} | \bar{A}) P(\bar{A}) + n_1{}^2 \gamma_x^{(i)} \gamma_x^{(K)} P(\bar{A}) \qquad (3.18)$$

Therefore using (3.10)

$$\frac{\partial T}{\partial \gamma_x^{(K)} \partial \gamma_x^{(i)}} = n_1{}^2 E[Z^{(i)} Z^{(K)} | \bar{A}] P(\bar{A}) - n_1{}^2 \gamma_x^{(i)} [1-H_{p+2}(c;\delta)] \gamma_x^{(K)}$$

$$- n_1{}^2 \gamma_x^{(K)} [1-H_{p+2}(c;\delta)] \gamma_x^{(i)} + n_1{}^2 \gamma_x^{(i)} \gamma_x^{(K)} P(\bar{A}) \qquad (3.19)$$

Equating (3.17) and (3.19), we have

$$E[Z^{(i)} Z^{(K)} | \bar{A}] P(\bar{A}) = [1-H_{p+4}(c;\delta)] \gamma_x^{(i)} \gamma_x^{(K)} \qquad (3.20)$$

and so we have evaluated $E(\underline{Z}\underline{Z}' | \bar{A}) P(\bar{A})$ completely. For convenience, we let $E[\underline{Z}\underline{Z}' | \bar{A}] P(\bar{A}) = W$ where W is a pxp matrix with the i-th diagonal element

$$(\gamma_x^{(i)})^2 [1-H_{p+4}(c;\delta)] + \frac{1}{n_1} [1-H_{p+2}(c;\delta)]$$

and the (i,K)th off diagonal element $[1-H_{p+4}(c;\delta)] \gamma_x^{(i)} \gamma_x^{(K)}$.

Next we evaluate other terms in (3.13).

$$E(\bar{y} - \Sigma_{12} \Sigma_{22}^{-1} \underline{\bar{X}}_2)^2 = E(\bar{y}^2) - 2\Sigma_{12} \Sigma_{22}^{-1} E(\bar{y} \underline{\bar{X}}_2) + \Sigma_{12} \Sigma_{22}^{-1} E(\underline{\bar{X}}_2 \underline{\bar{X}}_2') \Sigma_{22}^{-1} \Sigma_{21}$$

$$= \frac{1}{n_2} \sigma^2 + \mu^2 - 2\Sigma_{12} \Sigma_{22}^{-1} \{ Cov(\bar{y} \underline{\bar{X}}_2) + E(\bar{y}) E(\underline{\bar{X}}_2) \}$$

$$+ \Sigma_{12} \Sigma_{22}^{-1} \{ \Sigma_{\underline{\bar{X}}_2} + [E(\underline{\bar{X}}_2)][E(\underline{\bar{X}}_2)]' \} \Sigma_{22}^{-1} \Sigma_{21}$$

$$= \frac{1}{n_2}\sigma^2 + \mu^2 - \frac{2}{n_2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} - 2\Sigma_{12}\Sigma_{22}^{-1}\mu\underline{\mu}_x$$

$$+ \frac{1}{n_2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} + \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}$$

$$= \frac{1}{n_2}\sigma^2 + \mu^2 - \frac{1}{n_2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} - 2\Sigma_{12}\Sigma_{22}^{-1}\mu\underline{\mu}_x + \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}$$

Now

$$E[(\bar{y}\underline{Z})|\bar{A}]P(\bar{A}) = E\{E(\underline{Z}\bar{y}|\underline{Z},\bar{A})\}P(\bar{A})$$

$$= E\{\underline{Z}E[\bar{y}|\underline{Z}]|\bar{A}\}P(\bar{A})$$

$$= E\{\underline{Z}[\mu+\Sigma_{12}\Sigma_{22}^{-1}D^{-1}(\underline{Z}-\underline{\gamma}_x)]|\bar{A}\}P(\bar{A})$$

$$= \mu E[\underline{Z}|\bar{A}]P(\bar{A}) + E(\underline{Z}\Sigma_{12}\Sigma_{22}^{-1}D^{-1}\underline{Z}|\bar{A})P(\bar{A})$$

$$- E(\underline{Z}\Sigma_{12}\Sigma_{22}^{-1}D^{-1}\underline{\gamma}_x|\bar{A})P(\bar{A})$$

But since $\Sigma_{12}\Sigma_{22}^{-1}D^{-1}\underline{Z}$ is a scalar and $\Sigma_{12}\Sigma_{22}^{-1}D^{-1}\underline{\gamma}_x$ is a constant,

$$\Sigma_{12}\Sigma_{22}^{-1}D^{-1}\underline{Z} = \underline{Z}'D'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

and

$$\Sigma_{12}\Sigma_{22}^{-1}D^{-1}\underline{\gamma}_x = \underline{\gamma}_x'D'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

Hence

$$E(\bar{y}\underline{Z}|\bar{A})P(\bar{A}) = \mu E(\underline{Z}|\bar{A})P(\bar{A}) + E[\underline{ZZ'}|\bar{A}]P(\bar{A})D'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

$$- E(\underline{Z}|\bar{A})P(\bar{A})\underline{\gamma}_x'D'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

or

$$E(\bar{y}\underline{Z}|\bar{A})P(\bar{A}) = \mu[1-H_{p+2}(c;\delta)]\underline{\gamma}_x + WD'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

$$[1-H_{p+2}(c;\delta)]\underline{\gamma}_x\underline{\gamma}_x' D'^{-1}\Sigma_{22}^{-1}\Sigma_{21} \tag{3.21}$$

Finally,

$$E[\underline{\bar{X}}_1\underline{\bar{X}}_2'|\bar{A}]P(\bar{A}) = E\{E(\underline{\bar{X}}_1\underline{\bar{X}}_2'|\underline{\bar{X}}_1,\bar{A})\}P(\bar{A})$$

$$= E\{\underline{\bar{X}}_1 E[\underline{\bar{X}}_2' \; \underline{\bar{X}}_1]|\bar{A}\}P(\bar{A})$$

$$= E\{\underline{\bar{X}}_1[\underline{\mu}_x' + (\underline{\bar{X}}_1'-\underline{\mu}_x')]|\bar{A}\}P(\bar{A})$$

$$= E[(\underline{\bar{X}}_1\underline{\bar{X}}_1')|\bar{A}]P(\bar{A}) \; .$$

Therefore

$$E[\underline{\bar{X}}_1\underline{\bar{X}}_2'|\bar{A}]P(\bar{A}) = D^{-1}WD'^{-1} \tag{3.22}$$

and substituting these into (3.13) and then into (3.12), we have

$$V(\hat{\mu}_{\ell r}) = \frac{1}{n_2}\sigma^2 - \frac{1}{n_2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} - 2\Sigma_{12}\Sigma_{22}^{-1}\mu\underline{\mu}_x + \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}$$

$$+ 2\Sigma_{12}\Sigma_{22}^{-1}\mu\underline{\mu}_x[1-H_{p+2}(c;\delta)] - 2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}[1-H_{p+2}(c;\delta)]$$

$$+ \Sigma_{12}\Sigma_{22}^{-1}D^{-1}WD'^{-1}\Sigma_{22}^{-1}\Sigma_{21} + 2\mu\underline{\mu}_x\Sigma_{12}\Sigma_{22}^{-1}H_{p+2}(c;\delta)$$

$$- \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}[H_{p+2}(c;\delta)]^2$$

Noting that

$$\Sigma_{12}\Sigma_{22}^{-1}D^{-1}WD'^{-1}\Sigma_{22}^{-1}\Sigma_{21} = \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}[1-H_{p+4}(c;\delta)]$$

$$+ \frac{1}{n_1}\Sigma_{12}\Sigma_{22}^{-1}D^{-1}D'^{-1}\Sigma_{22}^{-1}\Sigma_{21}[1-H_{p+2}(c;\delta)]$$

and that $D^{-1}D'^{-1} = \Sigma_{22}$, then

$$V(\hat{\mu}_{\ell r}) = \frac{1}{n_2}\sigma^2 - \frac{1}{n_2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} + \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}$$

$$- 2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}[1-H_{p+2}(c;\delta)]$$

$$+ \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}[1-H_{p+4}(c;\delta)]$$

$$+ \frac{1}{n_1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}[1-H_{p+2}(c;\delta)]$$

$$- \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}[H_{p+2}(c;\delta)]^2 \qquad (3.23)$$

As a partial check, when c = 0 and we always reject the null hypothesis, then $\hat{\mu}_{\ell r}$ reduces to $\bar{y} + \Sigma_{12}\Sigma_{22}^{-1}(\bar{\underline{X}}_1-\bar{\underline{X}}_2)$. Now

$$V[\bar{y} + \Sigma_{12}\Sigma_{22}^{-1}(\bar{\underline{X}}_1-\bar{\underline{X}}_2)] = V(\bar{y}) + V(\Sigma_{12}\Sigma_{22}^{-1}\bar{\underline{X}}_1) + V(\Sigma_{12}\Sigma_{22}^{-1}\bar{\underline{X}}_2)$$

$$+ 2\Sigma_{12}\Sigma_{22}^{-1}\mathrm{Cov}(\bar{y},\bar{\underline{X}}_1) - 2\Sigma_{12}\Sigma_{22}^{-1}\mathrm{Cov}(\bar{\underline{X}}_1,\bar{\underline{X}}_2)\Sigma_{22}^{-1}\Sigma_{21}$$

$$- 2\Sigma_{12}\Sigma_{22}^{-1}\mathrm{Cov}(\bar{y},\bar{\underline{X}}_2) = \frac{1}{n_2}\sigma^2 + \frac{1}{n_1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} + \frac{1}{n_2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$+ \frac{2}{n_1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} - \frac{2}{n_1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} - \frac{2}{n_2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

Therefore

$$V(\bar{y} + \Sigma_{12}\Sigma_{22}^{-1}(\bar{\underline{X}}_1-\bar{\underline{X}}_2)) = \frac{1}{n_2}\sigma^2 + \frac{1}{n_1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} - \frac{1}{n_2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \quad (3.24)$$

Therefore putting $c = 0$ in (3.23), i.e. $H_{p+2}(c;\delta) = H_{p+4}(c;\delta) = 0$

$$V(\hat{\mu}_{\ell r}) = \frac{1}{n_2}\sigma^2 - \frac{1}{n_2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} + \frac{1}{n_1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

which is the same as (3.24).

When $c = \infty$, we always accept H_0 and the preliminary test estimator reduces to $\bar{y} - \Sigma_{12}\Sigma_{22}^{-1}\bar{\underline{X}}_2$. Now

$$V(\bar{y} - \Sigma_{12}\Sigma_{22}^{-1}\bar{\underline{X}}_2) = V(\bar{y}) + V(\Sigma_{12}\Sigma_{22}^{-1}\bar{\underline{X}}_2) - 2\Sigma_{12}\Sigma_{22}^{-1}\mathrm{Cov}(\bar{y},\bar{\underline{X}}_2)$$

$$= \frac{1}{n_2}\sigma^2 + \frac{1}{n_2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} - \frac{2}{n_2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = \frac{1}{n_2}\sigma^2 - \frac{1}{n_2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \quad (3.25)$$

putting $c = \infty$ in (3.23), then $H_{p+2}(c;\delta) = H_{p+4}(c;\delta) = 1$ and

$$V(\hat{\mu}_{\ell r}) = \frac{1}{n_2}\sigma^2 - \frac{1}{n_2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

which is the same as (3.25). We now give the M.S.E. of $\hat{\mu}_{\ell r}$.

$$\text{M.S.E.}(\hat{\mu}_{\ell r}) = V(\hat{\mu}_{\ell r}) + \text{Bias}^2 = \frac{1}{n_2}\sigma^2 + \frac{1}{n_1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}[1-H_{p+2}(c;\delta)]$$

$$- \frac{1}{n_2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} + \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21} - 2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}[1-H_{p+2}(c;\delta)]$$

$$+ \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}[1-H_{p+4}(c;\delta)]$$

or

$$\text{M.S.E.}(\hat{\mu}_{\ell r}) = g_1 + h_1 \tag{3.26}$$

where

$$g_1 = \frac{1}{n_2}\sigma^2 + \frac{1}{n_1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} - \frac{1}{n_2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

and

$$h_1 = \text{M.S.E.}(\hat{\mu}_{\ell r}) - g_1 .$$

We note that g_1 is the variance of $\bar{y} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{\bar{X}}_1 - \underline{\bar{X}}_2)$ which is the linear regression estimator ignoring the information of $\underline{\mu}_x$.

D. Relative Efficiency (e_1)

In practice, we would want to select an estimator for μ with the smallest bias and M.S.E. Again we consider only the M.S.E. of the preliminary test estimator since bias is a part of M.S.E. Using (3.26), we compare the performance of the preliminary test estimator $\hat{\mu}_{\ell r}$ with the usual linear regression estimator, $\bar{y} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{\bar{X}}_1 - \underline{\bar{X}}_2)$, when the information of $\underline{\mu}_x$ is ignored. The relative efficiency of $\hat{\mu}_{\ell r}$ to $\bar{y} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{\bar{X}}_1 - \underline{\bar{X}}_2)$ is defined as

$$e_1 = \frac{\text{M.S.E.}(\bar{y} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{\bar{X}}_1 - \underline{\bar{X}}_2))}{\text{M.S.E.}(\hat{\mu}_{\ell r})} \tag{3.27}$$

and since $\bar{y} + \Sigma_{12}\Sigma_{22}^{-1}(\bar{\underline{X}}_1-\bar{\underline{X}}_2)$ is unbiased, its M.S.E. is equal to its variance. Therefore, using (3.24) and (3.26), we have

$$e_1 = \frac{g_1}{g_1 + h_1} \tag{3.28}$$

Wlog we let $\Sigma_{22} = I$ and $\sigma^2 = 1$. Hence for $p = 1$, $\Sigma_{12} = \rho$. Table 3.3 gives the values of e_1 for $n_1 = 30$, $n_2 = 10$, $p = 1$ and certain values of ρ, μ_x and α.

Table 3.3. Values of e_1 for $p = 1$, $n_1 = 30$, $n_2 = 10$.

	$\alpha = .05$		$\alpha = .10$		$\alpha = .25$	
	ρ					
μ_x	.7	.9	.7	.9	.7	.9
0.0	1.2119	1.7336	1.1574	1.4907	1.0719	1.1936
0.1	1.1044	1.2964	1.0717	1.1932	1.0291	1.0735
0.2	.9096	.8061	.9207	.8275	.9547	.8970
0.3	.7769	.5901	.8279	.6653	.9171	.8205
0.4	.7380	.5379	.8168	.6482	.9263	.8385
0.5	.7762	.5891	.8636	.7235	.9569	.9017
0.6	.8574	.7130	.9273	.8405	.9826	.9588
0.7	.9347	.8553	.9726	.9361	.9950	.9880
0.8	.9788	.9503	.9926	.9823	.9990	.9975
0.9	.9951	.9882	.9986	.9965	.9998	.9996
1.0	.9992	.9980	.9998	.9995	1.0000	1.0000

From Table 3.3 we can easily observe the following properties.

1. The relative efficiency of $\hat{\mu}_{\ell r}$ assumes its maximum value when $\mu_x = 0$.

2. For $\mu_x \leq .1$ and fixed α, n_1 and n_2, e_1 is larger for $\rho = .9$ than for $\rho = .7$ and is a decreasing function of α within this range.

3. For $.2 \leq \mu_x \leq .7$, e_1 is larger for $\rho = .7$ than for $\rho = .9$ and is an increasing function of α within this range.

4. For $\mu_x \geq .8$, there is no appreciable difference in the values of e_1 for either values of ρ or different values of α.

5. For fixed n_1, n_2, ρ and α, e_1 first decreases from a value above unity to some minimum and then increases again to unity as μ_x increases.

Table 3.4 gives the values of e_1 for $p = 2$, $n_1 = 30$, $n_2 = 10$ and certain values of Σ_{12}, α and $\underline{\mu}_x$. From the table, the following properties of e_1 are apparent.

1. e_1 has its maximum when $\underline{\mu}_x = \underline{0}$.

2. The maximum at $\underline{\mu}_x = \underline{0}$ is an increasing function of ρ for fixed α, n_1 and n_2.

3. For fixed α, n_1, n_2 and Σ_{12}, e_1 decreases from the maximum value to a minimum and then increases to unity as $\underline{\mu}_x'$ increases from (0,0) to (1.0,1.0).

E. The Optimal Sample Design and Comparisons

The problem here is to find the optimum allocation of the sample sizes n_1 and n_2 for some given cost function. Usually the cost function is of the form

Table 3.4. Values of e_1 for $p = 2$, $n_1 = 30$ and $n_2 = 10$.

μ_x' \ Σ_{12}'	$\begin{pmatrix} .7 \\ 0 \end{pmatrix}$	$\begin{pmatrix} .5 \\ .5 \end{pmatrix}$	$\begin{pmatrix} .7 \\ .7 \end{pmatrix}$	$\begin{pmatrix} -.5 \\ .7 \end{pmatrix}$
		$\alpha = .05$		
(0, 0)	1.2410	1.2502	4.0695	1.6385
(.5, 0)	.6869	.8255	.5566	.8010
(.5, .5)	.6987	.5172	.2213	1.0363
(1.0, .0)	.9965	.9982	.9934	.9977
(1.0, .5)	.9968	.9963	.9861	.9999
(1.0,1.0)	.9975	.9948	.9805	.9998
		$\alpha = .10$		
(0, 0)	1.1939	1.2010	2.7094	1.4836
(.5, 0)	.7803	.8839	.6688	.8654
(.5, .5)	.7915	.6375	.3182	1.0188
(1.0, 0)	.9990	.9995	.9981	.9993
(1.0, .5)	.9991	.9989	.9959	1.0000
(1.0,1.0)	.9993	.9985	.9944	.9999
		$\alpha = .25$		
(0, 0)	1.1085	1.1122	1.6131	1.2444
(.5, 0)	.9079	.9544	.8474	.9459
(.5, .5)	.9147	.8329	.5695	1.0051
(1.0, 0)	.9999	.9999	.9998	.9999
(1.0, .5)	.9999	.9999	.9996	1.0000
(1.0,1.0)	.9999	.9998	.9994	1.0000

$$\text{cost} = C = C(n_1,n_2) = n_1c_1 + n_2c_2 \tag{3.29}$$

where c_1 is the cost of observing the vector $\underline{X}$ and c_2 is the cost of observing Y. The optimum values of n_1 and n_2 are obtained by minimizing the m.s.e.$(\hat{\mu}_{\ell r})$ given in (3.26) subject to the constraint (3.29). We recall that in practice, under the supposition of a conditional specification, the experimenter has only partial information based on which he believes that $\underline{\mu}_x$ is close to $\underline{0}$. The relative efficiency of $\hat{\mu}_{\ell r}$ is largest at $\underline{\mu}_x = \underline{0}$ and so it would be best to consider the problem of optimum allocation under the optimum situation by letting $\underline{\mu}_x = \underline{0}$ in m.s.e.$(\hat{\mu}_{\ell r})$.

When $\underline{\mu}_x = \underline{0}$, from (3.26)

$$\text{m.s.e.}(\hat{\mu}_{\ell r}) = \frac{K_1}{n_1} + \frac{K_2}{n_2} \tag{3.30}$$

where

$$K_1 = \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}[1-H_{p+2}(c;0)]$$

and

$$K_2 = \sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

Thus we wish to minimize (3.30) subject to (3.29). From (3.29),

$$n_1 = \frac{C-n_2c_2}{c_1} .$$

Thus

$$\text{m.s.e.}(\hat{\mu}_{\ell r}) = \frac{K_1c_1}{C-n_2c_2} + \frac{K_2}{n_2}$$

and

$$\frac{\partial \text{m.s.e.}(\hat{\mu}_{\ell r})}{\partial n_2} = \frac{K_1c_1c_2}{(C-n_2c_2)^2} - \frac{K_2}{n_2{}^2} = 0$$

or

$$n_2 = \frac{C\sqrt{K_2}}{\sqrt{K_1c_1c_2} + c_2\sqrt{K_2}}$$

Substituting in (3.29 we have

$$n_1 = \frac{C\sqrt{K_1}}{\sqrt{K_2c_1c_2} + c_1\sqrt{K_1}}$$

Substituting for n_1 and n_2 in (3.30), the optimum value of m.s.e.$(\hat{\mu}_{\ell r})$ is

$$\begin{aligned} \text{m.s.e.}(\hat{\mu}_{\ell r})_{\text{opt}} &= \frac{K_1\{\sqrt{K_2c_1c_2}+c_1\sqrt{K_1}\}}{C\sqrt{K_1}} + \frac{K_2\{\sqrt{K_1c_1c_2}+c_2\sqrt{K_2}\}}{C\sqrt{K_2}} \\ &= \frac{K_1c_1 + 2\sqrt{K_1K_2c_1c_2}+c_2K_2}{C} \\ &= \frac{(\sqrt{K_1c_1} + \sqrt{K_2c_2})^2}{C} \qquad (3.31) \end{aligned}$$

The regression estimator under double sampling without using the preliminary test is $\bar{y} + \Sigma_{12}\Sigma_{22}^{-1}(\bar{\underline{X}}_1-\bar{\underline{X}}_2)$ with variance

$$\frac{\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}{n_1} + \frac{\sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}{n_2}$$

$$= \frac{K_1'}{n_1} + \frac{K_2'}{n_2}$$

where

$$K_1' = \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

and

$$K_2' = \sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

Next we note that since $\bar{y} + \Sigma_{12}\Sigma_{22}^{-1}(\bar{\underline{X}}_1-\bar{\underline{X}}_2)$ is unbiased, its variance equals its m.s.e. and so denoting this m.s.e. by M and following the above method of minimizing m.s.e.$(\hat{\mu}_{\ell r})$ we find that

$$M_{opt} = \frac{(\sqrt{K_1'c_1} + \sqrt{K_2'c_2})^2}{C} \tag{3.32}$$

Now to compare (3.31) and (3.32) we note from (3.31) that $(1-H_{p+2}(c;0)$ is a decreasing function of c with a maximum equal to unity at $c = 0$. Hence the numerator of m.s.e.$(\hat{\mu}_{\ell r})$ is at most as large as that of M_{opt} and so we are led to conclude that m.s.e.$(\hat{\mu}_{\ell r})_{opt} \leq M_{opt}$ with equality holding for $c = 0$ which is the point at which the two estimators coincide.

We shall now compare $\hat{\mu}_{\ell r}$ with the preliminary test estimator, $\hat{\mu}$, of Chapter II for a fixed total budget. For the double sampling scheme, the cost function in (3.29) remains unchanged and when $\underline{\mu}_x = \underline{0}$, we are led to the optimum value of m.s.e. $(\hat{\mu}_{\ell r})$ in (3.31). Under the optimum situation, we shall find the optimum value of m.s.e.$(\hat{\mu})$ which we denote by m.s.e.$(\hat{\mu})_{opt}$. When $\underline{\mu}_x = \underline{0}$, from (2.24), the m.s.e. of the preliminary test estimator is $\frac{V}{n}$ where

$$V = \sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}H_{p+2}(c;0) .$$

If the total budget is devoted to a single sample, this sample has size

$$n = \frac{C}{c_2}$$

and

$$\text{m.s.e.}(\hat{\mu})_{opt} = \frac{c_2 V}{C} .$$

Hence under the optimum situation, i.e. $\underline{\mu}_x = \underline{0}$, double sampling gives a smaller m.s.e. if

$$c_2 V > (\sqrt{K_1 c_1} + \sqrt{K_2 c_2})^2 .$$

When $\underline{\mu}_x \neq \underline{0}$, from (3.26)

$$\text{m.s.e.}(\hat{\mu}_{\ell r}) = \frac{K_1^*}{n_1} + \frac{K_2}{n_2} + \theta_1$$

where K_2 is as defined in (3.30) and

$$K_1^* = \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}[1-H_{p+2}(c;\delta)]$$

$$\theta_1 = 2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}H_{p+2}(c;\delta)$$

$$- \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}H_{p+4}(c;\delta)$$

Similarly, when $\underline{\mu}_x \neq \underline{0}$, from (2.24), m.s.e.$(\hat{\mu}) = \frac{V^*}{n} + \theta_2$ where

$$V^* = \sigma^2 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}H_{p+2}(c;\lambda)$$

$$\theta_2 = 2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}H_{p+2}(c;\lambda)$$

$$- \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}H_{p+4}(c;\lambda)$$

We may now compare the two mean square errors by substituting for n_1, n_2 and n in the expression,

$$\text{m.s.e.}(\hat{\mu}) - \text{m.s.e.}(\hat{\mu}_{\ell r}) = \left(\frac{V^*}{n} + \theta_2\right) - \left(\frac{K_1^*}{n_1} + \frac{K_2}{n_2} + \theta_1\right) .$$

Double sampling gives a smaller m.s.e. if the expression is positive.

The detailed expression is complicated and would not be given here.

F. Bias of $\hat{\mu}_{\ell r}$ when Σ is Unknown

When Σ is unknown, the linear regression preliminary test estimator becomes

$$\hat{\mu}_{\ell r} = \begin{cases} \bar{y} - S_{12}S_{22}^{-1}\bar{\underline{X}}_2 & \text{if } m_1 n_1(\bar{\underline{X}}_1' S_{22}^{-1}\bar{\underline{X}}_1) \leq T_0^2 \\ \\ \bar{y} + S_{12}S_{22}^{-1}(\bar{\underline{X}}_1 - \bar{\underline{X}}_2) & \text{if } m_1 n_1(\bar{\underline{X}}_1' S_{22}^{-1}\bar{\underline{X}}_1) > T_0^2 \end{cases} \tag{3.33}$$

where $m_1 = n_1 - 1$, T_0^2 is the $100(1-\alpha)$th percentile of the Hotelling's T^2 distribution with m_1 degrees of freedom and we define

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

where

$$\bar{y} = \frac{1}{n_2} \sum_{i=1}^{n_2} y_i$$

$$\bar{\underline{X}}_1 = \frac{1}{n_1} \sum_{i=1}^{n_1} \underline{X}_i$$

$$\bar{\underline{X}}_2 = \frac{1}{n_2} \sum_{i=1}^{n_2} \underline{X}_i$$

$$S_{11} = \sum_{i=1}^{n_2} (y_i - \bar{y})^2$$

$$S_{12} = \sum_{i=1}^{n_2} (y_i-\bar{y})(\underline{X}_i-\bar{\underline{X}}_2)'$$

$$S_{22} = \sum_{i=1}^{n_1} (\underline{X}_i-\bar{\underline{X}}_1)(\underline{X}_i-\bar{\underline{X}}_1)'$$

In this section, we shall obtain the bias of $\hat{\mu}_{\ell r}$ when Σ is unknown. If we denote the rejection region for the preliminary test

$$\{m_1n_1(\bar{\underline{X}}_1'S_{22}^{-1}\bar{\underline{X}}_1):m_1n_1(\bar{\underline{X}}_1'S_{22}^{-1}\bar{\underline{X}}_1) > T_0^2\} \quad \text{by} \quad \bar{\mathfrak{a}} ,$$

then

$$E(\hat{\mu}_{\ell r}) = E\{\bar{y}-S_{12}S_{22}^{-1}\bar{\underline{X}}_2|\mathfrak{a}\}P(\mathfrak{a}) + E\{\bar{y}+S_{12}S_{22}^{-1}(\bar{\underline{X}}_1-\bar{\underline{X}}_2)|\bar{\mathfrak{a}}\}P(\bar{\mathfrak{a}})$$

$$= E(\bar{y}-S_{12}S_{22}^{-1}\bar{\underline{X}}_2) + E\{S_{12}S_{22}^{-1}\bar{\underline{X}}_1|\bar{\mathfrak{a}}\}P(\bar{\mathfrak{a}}) \qquad (3.34)$$

Now $(\bar{\underline{X}}_1, \bar{\underline{X}}_2, \bar{y})$ has a normal distribution as in section B and is independent of (S_{22}, S_{11}, S_{12}) which has a Wishart distribution.

$E(\bar{y}) = \mu$ and so if we write $E(\hat{\mu}_{\ell r}) = \mu + B_2$, we see that the bias is

$$B_2 = E\{S_{12}S_{22}^{-1}\bar{\underline{X}}_1|\bar{\mathfrak{a}}\}P(\bar{\mathfrak{a}}) - E(S_{12}S_{22}^{-1}\bar{\underline{X}}_2) .$$

Since $S_{12}S_{22}^{-1}$ and $\bar{\underline{X}}_2$ are independent, we know

$$E(S_{12}S_{22}^{-1}\bar{\underline{X}}_2) = E(S_{12}S_{22}^{-1})\cdot E(\bar{\underline{X}}_2)$$

where

$$E(S_{12}S_{22}^{-1}) = E\{E(S_{12}S_{22}^{-1}|\underline{X})\}$$

$$= E\{E[S_{12}|\underline{X}]S_{22}^{-1}\}$$

$$= E\{E[\sum_{i=1}^{n_2} (y_i-\bar{y})(\underline{X}_i-\bar{\underline{X}}_2)'|\underline{X}]S_{22}^{-1}\}$$

$$E(y_i|\underline{X}) = \mu + \Sigma_{12}\Sigma_{22}^{-1}(\underline{X}_i-\underline{\mu}_x)$$

$$\Longrightarrow \quad E(S_{12}S_{22}^{-1}) = E\{\Sigma_{12}\Sigma_{22}^{-1}S_{22}S_{22}^{-1}\} = \Sigma_{12}\Sigma_{22}^{-1}$$

and

$$E(S_{12}S_{22}^{-1}\bar{\underline{X}}_2) = \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x \quad .$$

Hence

$$B_2 = E\{S_{12}S_{22}^{-1}\bar{\underline{X}}_1|\bar{\mathfrak{a}}\}P(\bar{\mathfrak{a}}) - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x \tag{3.35}$$

It remains to evaluate the first term.

Let $f(\bar{\underline{X}}_1)$ be the multivariate normal density of $\bar{\underline{X}}_1$ and $g(S_{22},S_{12},S_{11})$ the joint density of S_{22}, S_{12} and S_{11}, then

$$E\{S_{12}S_{22}^{-1}\bar{\underline{X}}_1|\bar{\mathfrak{a}}\}P(\bar{\mathfrak{a}})$$

$$= \int_{\bar{\mathfrak{a}}}\cdots\int S_{12}S_{22}^{-1}\bar{\underline{X}}_1 f(\bar{\underline{X}}_1)g(S_{22},S_{12},S_{11})d\bar{\underline{X}}_1 dS_{22}dS_{12}dS_{11} \tag{3.36}$$

We make the following transformations as we did in (2.31).

$$W_1 = TB'BT'$$

$$W_2 = [S_{12} - \Sigma_{12}T'TB'B]\Sigma_{11\cdot 2}^{-\frac{1}{2}}B^{-1} \tag{3.37}$$

$$W_3 = (S_{11}^2 - S_{12}B^{-1}B'^{-1}S_{21})\Sigma_{11\cdot 2}$$

Substituting in (3.36) we have

$$E\{S_{12}S_{22}^{-1}\underline{\bar{X}}_1|\bar{\mathcal{C}}\}P(\bar{\mathcal{C}}) \tag{3.38}$$

$$= \int\cdots\int_{\bar{\mathcal{C}}} (KW_2W_1^{-\frac{1}{2}}T + \Sigma_{12}T'T)\underline{\bar{X}}_1 f(\underline{\bar{X}}_1)g(W_1,W_2,W_3)dW_3dW_2dW_1d\underline{\bar{X}}_1$$

where as before $W_1 \sim W(I,n_1-1)$, $W_2 \sim N(\underline{0},I)$, $W_3 \sim W(1,n_1-K)$ and they are independent. The joint density is

$$g(W_1,W_2,W_3) = c_0 e^{-\frac{1}{2}tr(W_2'W_2+W_3+W_1)}|W_3|^{\frac{1}{2}(n_1-p-3}|W_1|^{\frac{1}{2}(n_1-p-2)} \tag{3.39}$$

The region of integration is given by

$$\bar{\mathcal{C}} = \{n_1m_1(\underline{\bar{X}}_1'T'W_1^{-1}T\underline{\bar{X}}_1):(\underline{\bar{X}}_1'T'W_1^{-1}T\underline{\bar{X}}_1)n_1m_1 > T_0^2\} \quad .$$

Hence (3.38) becomes

$$\int_{\bar{G}}\cdots\int\int_{-\infty}^{\infty}\int_{0}^{\infty}(KW_2W_1^{-\frac{1}{2}}T+\Sigma_{12}T'T)\underline{\bar{X}}_1 f(\underline{\bar{X}}_1)c_0 e^{-\frac{1}{2}tr(W_2'W_2+W_3+W_1)}$$

$$|W_3|^{\frac{1}{2}(n_1-p-3)}|W_1|^{\frac{1}{2}(n_1-p-2)}dW_3dW_2dW_1d\underline{\bar{X}}_1$$

$$= \int_{\bar{G}}\cdots\int\int_{-\infty}^{\infty}\int_{0}^{\infty}Kc_0\Sigma_{22}^{-\frac{1}{2}}e^{-\frac{1}{2}tr(W_2'W_2+W_3+W_1)}|W_3|^{\frac{1}{2}(n_1-p-3)}|W_1|^{\frac{1}{2}(n_1-p-2)}W_2$$

$$W_1^{-\frac{1}{2}}\underline{\bar{X}}_1 f(\underline{\bar{X}}_1)dW_3dW_2dW_1d\underline{\bar{X}}_1$$

$$+ \int_{\bar{G}}\cdots\int\int_{-\infty}^{\infty}\int_{0}^{\infty}c_0\Sigma_{12}\Sigma_{22}^{-\frac{1}{2}}e^{-\frac{1}{2}tr(W_2'W_2+W_3+W_1)}|W_3|^{\frac{1}{2}(n_1-p-3)}$$

$$|W_1|^{\frac{1}{2}(n_1-p-2)}\underline{\bar{X}}_1 f(\underline{\bar{X}}_1)dW_3dW_2dW_1d\underline{\bar{X}}_1$$

But $E(W_2) = \underline{0}$ so that from the independence of the W_i's, the first term is zero. Hence (3.38) is equal to

$$\frac{\Sigma_{12}T'T}{2^{\frac{1}{2}p(n_1-1)}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma[\frac{1}{2}(n_1-i)]}\int_{\bar{G}}\cdots\int e^{-\frac{1}{2}trW_1}|W_1|^{\frac{1}{2}(n_1-p-2)}$$

$$\underline{\bar{X}}_1 f(\underline{\bar{X}}_1)dW_1d\underline{\bar{X}}_1$$

$$\underline{\bar{X}}_1 \sim N(\underline{\mu}_x,\frac{1}{n_1}T^{-1}T'^{-1})$$

We let $\underline{Z} = T\underline{\bar{X}}_1$ and $\underline{\nu}_x = T\underline{\mu}_x$. Therefore $\underline{Z} \sim N(\underline{\nu}_x, \frac{1}{n_1}I)$. Therefore $\bar{G} = \{n_1 m_1(\underline{Z}'W_1^{-1}\underline{Z}) : n_1 m_1(\underline{Z}'W_1^{-1}\underline{Z}) > T_0^2\}$. Hence we wish to evaluate

$$\frac{\Sigma_{12}T'T\ T^{-1}}{2^{\frac{1}{2}p(n-1)}\pi^{\frac{1}{4}p(n-1)}\prod_{i=1}^{p}\Gamma[\frac{1}{2}(n_1-i)]}\int_{\bar{G}}\cdots\int e^{-\frac{1}{2}trW_1}|W_1|^{\frac{1}{2}(n_1-p-2)} \underline{Z}g(\underline{Z})d\underline{Z}dW_1 \tag{3.40}$$

and

$$F' = n_1(\underline{Z}'W^{-1}\underline{Z})\ \frac{(n_1-p)}{p}$$

has the noncentral F distribution with p and n_1-p degrees of freedom and noncentrality parameter $\lambda = n_1\underline{\nu}_x'\underline{\nu}_x$. Therefore

$$P(\bar{G}) = P(F' > F_{p,n_1-p}(\alpha))$$

$$= P(G > \frac{p}{n_1-p}\ F_{p,n_1-p}(\alpha)$$

where

$$G = \frac{p}{n_1-p}\ F'$$

Let

$$c = \frac{p}{n_1-p}\ F_{p,n_1-p}(\alpha)\ ,$$

then

$$\bar{R} = P(\bar{G}) = \int_c^\infty e^{-\frac{1}{2}\lambda} \sum_{j=0}^\infty \frac{1}{j!} (\tfrac{\lambda}{2})^j G_{(\frac{p}{2}+j,\frac{n_1-p}{2})}(g)dg \tag{3.41}$$

where

$$G_{(\frac{p}{2}+j,\frac{n_1-p}{2})}(g)$$

is as defined in (2.35). Differentiating (3.41) w.r.t. $\nu_x^{(i)}$, we have

$$\frac{\partial\bar{R}}{\partial\nu_x^{(i)}} = \int_c^\infty e^{-\frac{1}{2}\lambda} \sum_{j'=0}^\infty \frac{2n_1\nu_x^{(i)}}{2} \frac{1}{j'!}(\tfrac{\lambda}{2})^{j'} G_{(\frac{p+2}{2}+j',\frac{n_1-p}{2})}(g)dg$$

$$- \int_c^\infty \frac{2n_1\nu_x^{(i)}}{2} e^{-\frac{1}{2}\lambda} \sum_{j=0}^\infty \frac{1}{j!}(\tfrac{\lambda}{2})^j G_{(\frac{p}{2}+j,\frac{n_1-p}{2})}(g)dg$$

or

$$\frac{\partial\bar{R}}{\partial\nu_x^{(i)}} = n_1\nu_x^{(i)}[1-G^*_{p+2,n_1-p}(c;\lambda)] - n_1\nu_x^{(i)}P(\bar{G}) \tag{3.42}$$

where

$$G^*_{p+2,n_1-p}(c;\lambda)$$

is the cumulative distribution of the noncentral G distribution with p+2, n_1-p degrees of freedom and noncentrality parameter λ.

Next we make use of the distributions of $\underline{Z}$ and W_1. From the independence, we may write

$$\bar{R} = P(\bar{\alpha}) = \int_{\bar{\alpha}}\cdots\int \prod_{j=1}^{p} \frac{\sqrt{n_1}}{\sqrt{2\pi}} e^{-\frac{n_1}{2}(Z^{(j)}-\nu_x^{(j)})^2} \tag{3.43}$$

$$\frac{|W_1|^{\frac{1}{2}(n_1-p-2)} e^{-\frac{1}{2}\mathrm{tr}W_1}}{2^{\frac{p}{2}(n_1-1)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma[\frac{1}{2}(n_1-i)]} dZ^{(j)} dW_1$$

Differentiating (3.43) w.r.t. $\nu_x^{(i)}$, we obtain

$$\frac{\partial \bar{R}}{\partial \nu_x^{(i)}} = \int_{\bar{\alpha}}\cdots\int \prod_{j=1}^{p} \frac{\sqrt{n_1}}{\sqrt{2\pi}} \frac{n_1}{2}\, 2(Z^{(i)}-\nu_x^{(i)}) e^{-\frac{n_1}{2}(Z^{(j)}-\nu_x^{(j)})^2}$$

$$\frac{|W_1|^{\frac{1}{2}(n_1-p-2)} e^{-\frac{1}{2}\mathrm{tr}W_1}}{2^{\frac{p}{2}(n_1-1)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma[\frac{1}{2}(n_1-i)]} dZ^{(j)} dW_1$$

$$= \frac{n_1}{2^{\frac{p}{2}(n_1-1)} \pi^{\frac{p}{4}(p-1)} \prod_{i=1}^{p} \Gamma[\frac{1}{2}(n_1-i)]} \int_{\bar{\alpha}}\cdots\int e^{-\frac{1}{2}\mathrm{tr}W_1} |W_1|^{\frac{1}{2}(n_1-p-2)} \tag{3.44}$$

$$Z^{(i)} g(\underline{Z}) d\underline{Z} dW_1 - n_1 \nu_x^{(i)} P(\bar{\alpha})$$

Therefore equating (3.42) and (3.44), we have

$$\frac{1}{2^{\frac{p}{2}(n_1-1)}\,\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma[\frac{1}{2}(n_1-i)]}\int\cdots\int_{\bar{G}} e^{-\frac{1}{2}\mathrm{tr}W_1}|W_1|^{\frac{1}{2}(n_1-p-2)} Z^{(i)}g(\underline{Z})d\underline{Z}dW_1$$

$$= \nu_x^{(i)}[1-G^*_{p+2,n_1-p}(c;\lambda)] = I(Z^{(i)}) \text{ say.} \tag{3.45}$$

This gives (3.40) to be $\Sigma_{12}\Sigma_{22}^{-1}T^{-1}I(\underline{Z})$ where $I(\underline{Z})$ is a px1 vector with i-th component = $I(Z^{(i)})$. From (3.45)

$$I(\underline{Z}) = \underline{\nu}_x[1-G^*_{p+2,n_1-p}(c;\lambda)]$$

Hence (3.40) is

$$\Sigma_{12}\Sigma_{22}^{-1}T^{-1}\underline{\nu}_x[1-G^*_{p+2,n_1-p}(c;\lambda)]$$

We now obtain the bias of $\hat{\mu}_{\ell r}$ to be

$$\begin{aligned} B_2 &= \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x[1-G^*_{p+2,n_1-p}(c;\lambda)] - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x \\ &= -\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x G^*_{p+2,n_1-p}(c;\lambda) \\ &= -\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x F^*_{p+2,n_1-p}(c_2;\lambda) \end{aligned}$$

where

$$c_2 = \frac{p}{p+2}F_{p,n_1-p}(\alpha)$$

As a partial check, when $c = 0$, we always reject H_0 and the estimator reduces to $\bar{y} + S_{12}S_{22}^{-1}(\bar{\underline{X}}_1-\bar{\underline{X}}_2)$ which has zero bias. In this case, $F^*_{p+2,n_1-p}(c_2;\lambda) = 0$ and $B_2 = 0$.

When $c = \infty$, we always accept H_0 and the estimator is $\bar{y} - S_{12}S_{22}^{-1}\bar{\underline{X}}_2$ with Bias $= -\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x$. Here $F^*_{p+2,n_1-p}(c_2;\lambda) = 1$ and $B_2 = -\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x$.

In order to evaluate B_2, we let $\Sigma_{22} = I$ and $\sigma^2 = 1$ wlog. The values of $-B_2$ for $p = 2$ and $n_1 = 15$ are given in Table 3.5 for a few values of α, $\underline{\mu}_x$ and Σ_{12}. From Table 3.5, the following properties of the bias can easily be observed.

1. The bias is zero when the null hypothesis H_0: $\underline{\mu}_x = \underline{0}$ is true.

2. For fixed n_1, $\underline{\mu}_x$ and Σ_{12}, the bias generally decreases as α increases.

3. The bias is zero when either $\underline{\mu}_x$ or Σ_{12} has identical components and the other has components which differ only in sign.

4. For fixed n, Σ_{12} and α and some component of $\underline{\mu}_x$, the value of B_2 first increases and then decreases to zero as the other component of $\underline{\mu}_x$ increases from 0.0 to 1.0.

5. For fixed n, α and $\underline{\mu}_x$, the value of the bias is an increasing function of Σ_{12}.

Table 3.5. Values of $-B_2$ for p = 2 and n_1 = 15.

$\underline{\mu}_x'$ \ Σ_{12}'	α = .05			
	$\begin{pmatrix}-.5\\0\end{pmatrix}$	$\begin{pmatrix}-.5\\.5\end{pmatrix}$	$\begin{pmatrix}.5\\.5\end{pmatrix}$	$\begin{pmatrix}.7\\.7\end{pmatrix}$
(0, 0)	0.0	0.0	0.0	0.0
(0, .2)	0.0	0.0780	0.0780	0.1092
(0, .4)	0.0	0.1264	0.1264	0.1769
(0, .6)	0.0	0.1253	0.1253	0.1754
(0, .8)	0.0	0.0847	0.0847	0.1185
(0,1.0)	0.0	0.0392	0.0392	0.0548
(.2, .2)	-0.0730	0.0	0.1459	0.2043
(.2, .4)	-0.0585	0.0585	0.1755	0.2457
(.2, .6)	-0.0381	0.0763	0.9525	0.2135
(.2, .8)	-0.0191	0.0572	0.0953	0.1334
(.2,1.0)	-0.0070	0.0279	0.0418	0.0585
(.4, .4)	-0.0912	0.0	0.1824	0.2554
(.4, .6)	-0.0573	0.0287	0.1433	0.2006
(.4, .8)	-0.0276	0.0276	0.0828	0.1159
(.4,1.0)	-0.0097	0.0146	0.0341	0.0477
(.6, .6)	-0.0514	0.0	0.1029	0.1440
(.6, .8)	-0.0235	0.0078	0.0548	0.0767
(.6,1.0)	-0.0079	0.0053	0.0210	0.0295
(.8, .8)	-0.0135	0.0	0.0270	0.0378
(.8,1.0)	-0.0043	0.0011	0.0096	0.0135
(1.0,1.0)	-0.0000	0.0	0.0000	0.0000

Table 3.5. (continued)

$\underline{\mu}_x'$ \ Σ_{12}'	$\alpha = 0.2$			
	$\begin{pmatrix} -.5 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -.5 \\ .5 \end{pmatrix}$	$\begin{pmatrix} .5 \\ .5 \end{pmatrix}$	$\begin{pmatrix} .7 \\ .7 \end{pmatrix}$
(0, 0)	0.0	0.0	0.0	0.0
(0, .2)	0.0	0.0448	0.0448	0.0627
(0, .4)	0.0	0.582	0.0582	0.0815
(0, .6)	0.0	0.0407	0.0407	0.0570
(0, .8)	0.0	0.0174	0.0174	0.0243
(0,1.0)	0.0	0.0046	0.0046	0.0064
(.2, .2)	-0.0389	0.0	0.0778	0.1090
(.2, .4)	-0.0251	0.0251	0.0753	0.1054
(.2, .6)	-0.0116	0.0232	0.0463	0.0648
(.2, .8)	-0.0037	0.0110	0.0183	0.2570
(.2,1.0)	-0.0008	0.0031	0.0046	0.0065
(.4, .4)	-0.0317	0.0	0.0635	0.0888
(.4, .6)	-0.0143	0.0071	0.0357	0.0499
(.4, .8)	-0.0044	0.0044	0.0132	0.0185
(.4,1.0)	-0.0009	0.0014	0.0032	0.0044
(.6, .6)	-0.0093	0.0	0.0186	0.0260
(.6, .8)	-0.0028	0.0009	0.0064	0.0090
(.6,1.0)	-0.0005	0.0004	0.0015	0.0020
(.8, .8)	-0.0010	0.0	0.0021	0.0029
(.8,1.0)	-0.0002	0.0000	0.0004	0.0006
(1.0,1.0)	-.0000	0.0	0.0000	0.0000

Table 3.5. (continued)

μ_x' \ Σ_{12}'	$\alpha = 0.5$			
	$\begin{pmatrix}-.5\\0\end{pmatrix}$	$\begin{pmatrix}-.5\\.5\end{pmatrix}$	$\begin{pmatrix}.5\\.5\end{pmatrix}$	$\begin{pmatrix}.7\\.7\end{pmatrix}$
(0, 0)	0.0	0.0	0.0	0.0
(0, .2)	0.0	0.0137	0.0137	0.0192
(0, .4)	0.0	0.0142	0.0142	0.0198
(0, .6)	0.0	0.0069	0.0069	0.0097
(0, .8)	0.0	0.0019	0.0019	0.0026
(0,1.0)	0.0	0.0003	0.0003	0.0004
(.2, .2)	-0.0110	0.0	0.0221	0.0309
(.2, .4)	-0.0057	0.0057	0.0170	0.0238
(.2, .6)	-0.0018	0.0037	0.0074	0.0103
(.2, .8)	-0.0004	0.0011	0.0019	0.0026
(.2,1.0)	-0.0000	0.0002	0.0003	0.0004
(.4, .4)	-0.0058	0.0	0.0116	0.0162
(.4, .6)	-0.0019	0.0009	0.0047	0.0065
(.4, .8)	-0.0004	0.0004	0.0011	0.0016
(.4,1.0)	-0.0000	0.0001	0.0002	0.0002
(.6, .6)	-0.0009	0.0	0.0018	0.0025
(.6, .8)	-0.0002	0.0001	0.0004	0.0006
(.6,1.0)	0.0000	0.0000	0.0001	0.0001
(.8, .8)	0.0000	0.0	0.0001	0.0001
(.8,1.0)	0.0000	0.0000	0.0000	0.0000
(1.0,1.0)	0.0000	0.0	0.0000	0.0000

G. The M.S.E. of $\hat{\mu}_{\ell r}$ when Σ is Unknown

In this section, we find the mean squared error of $\hat{\mu}_{\ell r}$.

$$\text{M.S.E.}(\hat{\mu}_{\ell r}) = V(\hat{\mu}_{\ell r}) + (\text{Bias } \hat{\mu}_{\ell r})^2$$

For Σ unknown, $\hat{\mu}_{\ell r}$ is given in (3.33) and hence

$$E(\hat{\mu}_{\ell r}{}^2) = E[(\bar{y} - S_{12}S_{22}^{-1}\underline{\bar{X}}_2)^2|\alpha]P(\alpha) + E[(\bar{y} - S_{12}S_{22}^{-1}\underline{\bar{X}}_2)^2|\bar{\alpha}]P(\bar{\alpha})$$

$$+ E[(S_{12}S_{22}^{-1}\underline{\bar{X}}_1)^2|\bar{\alpha}]P(\bar{\alpha}) + 2E(S_{12}S_{22}^{-1}\bar{y}\underline{\bar{X}}_1|\bar{\alpha})P(\bar{\alpha})$$

$$- 2E[S_{12}S_{22}^{-1}\underline{\bar{X}}_1\underline{\bar{X}}_2'S_{22}^{-1}S_{21}|\bar{\alpha}]P(\bar{\alpha}) = E[S_{12}S_{22}^{-1}\underline{\bar{X}}_1\underline{\bar{X}}_1'S_{22}^{-1}S_{21}|\bar{\alpha}]P(\bar{\alpha})$$

$$+ 2E(S_{12}S_{22}^{-1}\bar{y}\underline{\bar{X}}_1|\bar{\alpha})P(\bar{\alpha}) - 2E[S_{12}S_{22}^{-1}\underline{\bar{X}}_1\underline{\bar{X}}_2'S_{22}^{-1}S_{21}|\bar{\alpha}]P(\bar{\alpha})$$

$$+ E(\bar{y} - S_{12}S_{22}^{-1}\underline{\bar{X}}_2)^2 . \tag{3.46}$$

Recall that

$$\bar{\alpha} = \{n_1m_1(\underline{\bar{X}}_1'S_{22}^{-1}\underline{\bar{X}}_1):n_1m_1(\underline{\bar{X}}_1'S_{22}^{-1}\underline{\bar{X}}_1) > T_0^2\}$$

$$= \{n_1m_1(\underline{Z}'W_1^{-1}\underline{Z}):n_1m_1(\underline{Z}'W_1^{-1}\underline{Z}) > T_0^2\}$$

Then following arguments similar to those used to obtain (2.54), we obtain the first term of (3.46).

$$E[S_{12}S_{22}^{-1}\underline{\bar{X}}_1\underline{\bar{X}}_1'S_{22}^{-1}S_{21}|\bar{\alpha}]P(\bar{\alpha}) = Q^* + \Sigma_{12}\Sigma_{22}^{-1}T^{-1}M^*T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

where

$$Q^* = \Sigma_{11\cdot 2} \frac{p}{n_1(n_1-p)} \int_d^\infty tf(t)dt \ ,$$

t has a noncentral F distribution with p and n_1-p degrees of freedom and noncentrality parameter $n_1\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x$ and M* is a pxp matrix with i-th diagonal element

$$= (\nu_x^{(i)})^2\,[1-G^*_{p+4,n_1-p}(c;\lambda)] + \frac{1}{n_1}\,[1-G^*_{p+2,n_1-p}(c;\lambda)]$$

and the (i,K)th off-diagonal element

$$= \nu_x^{(i)}\nu_x^{(K)}[1-G^*_{p+4,n_1-p}(c;\lambda)]$$

Similarly, by arguments analogous to those used to obtain (2.55), the second term equals

$$2E(S_{12}S_{22}^{-1}\bar{\underline{y}}\bar{\underline{X}}_1\,|\bar{G})P(\bar{G}) = 2\underline{\mu}\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x(1-G^*_{p+2,n_1-p}(c;\lambda))$$

$$+\ 2\Sigma_{12}\Sigma_{22}^{-1}T^{-1}M^*T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

$$-\ 2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}[1-G^*_{p+2,n_1-p}(c;\lambda)]$$

For the third term,

$$E[S_{12}S^{-1}\bar{\underline{X}}_1\bar{\underline{X}}_2'S_{22}^{-1}S_{21}\,|\bar{G}]P(\bar{G})$$

$$= trE\{S_{22}^{-1}S_{21}S_{12}S_{22}\bar{\underline{X}}_1\bar{\underline{X}}_2'\,|\bar{G}\}P(\bar{G})$$

$$= \operatorname{tr}\bar{E}\{ES_{22}^{-1}S_{21}S_{12}S_{22}^{-1}\underline{\bar{X}}_1\underline{\bar{X}}_2'|S,\underline{\bar{X}}_1,\bar{a}\}P(\bar{a})$$

$$= \operatorname{tr}\bar{E}\{S_{22}^{-1}S_{21}S_{12}S_{22}^{-1}\underline{\bar{X}}_1E(\underline{\bar{X}}_2'|\underline{\bar{X}}_1)|\bar{a}\}P(\bar{a})$$

$$= \operatorname{tr}\bar{E}\{S_{22}^{-1}S_{21}S_{12}S_{22}^{-1}\underline{\bar{X}}_1[\underline{\mu}_x' + (\underline{\bar{X}}_1'-\underline{\mu}_x')]|\bar{a}\}P(\bar{a})$$

$$= \operatorname{tr}E[S_{22}^{-1}S_{21}S_{12}S_{22}^{-1}\underline{\bar{X}}_1\underline{\bar{X}}_1'|\bar{a}]P(\bar{a})$$

$$= Q^* + \Sigma_{12}\Sigma_{22}^{-1}T^{-1}M^*T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

and hence third term of (3.46)

$$= -2Q^* - 2\Sigma_{12}\Sigma_{22}^{-1}T^{-1}M^*T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

Finally the fourth term of (3.46) is

$$E(\bar{y} - S_{12}S_{22}^{-1}\underline{\bar{X}}_2)^2 = V(\bar{y} - S_{12}S_{22}^{-1}\underline{\bar{X}}_2) + [E(\bar{y} - S_{12}S_{22}^{-1}\underline{\bar{X}}_2)]^2$$

$$= \frac{1}{n_2}\Sigma_{11\cdot 2} + \Sigma_{11\cdot 2}\frac{p}{n_2(n_2-p-2)}[1 + \frac{2n_2\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x}{p}]$$

$$+ (\mu - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x)^2$$

by using (2.58). Substituting into (3.46) we have

$$E(\hat{\mu}_{\ell r}^2) = \Sigma_{12}\Sigma_{22}^{-1}T^{-1}M^*T'^{-1}\Sigma_{22}^{-1}\Sigma_{21} - 2\Sigma_{12}\Sigma_{22}^{-1}\mu\underline{\mu}_x G^*_{p+2,n_1-p}(c;\lambda)$$

$$- 2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}[1-G^*_{p+2,n_1-p}(c;\lambda)] - Q^*$$

$$+ \frac{1}{n_2}\Sigma_{11\cdot 2} + \Sigma_{11\cdot 2}\frac{p}{n_2(n_2-p-2)}[1 + \frac{2n_2\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x}{p}]$$

$$+ \mu^2 + \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}$$

$$V(\hat{\mu}_{\ell r}) = E(\hat{\mu}_{\ell r}^2) - [E(\hat{\mu}_{\ell r})]^2 = \Sigma_{12}\Sigma_{22}^{-1}T^{-1}M^*T'^{-1}\Sigma_{22}^{-1}\Sigma_{21}$$

$$- 2\Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}[1-G^*_{p+2,n_1-p}(c;\lambda)] - Q^* \tag{3.47}$$

$$+ \frac{1}{n_2}\Sigma_{11\cdot 2} + \Sigma_{11\cdot 2}\frac{p}{n_2(n_2-p-2)}\left[1 + \frac{2n_2\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x}{p}\right]$$

$$+ \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_x\underline{\mu}_x'\Sigma_{22}^{-1}\Sigma_{21}[G^*_{p+2,n_1-p}(c;\lambda)]^2$$

In order to check the above results, we consider (3.33). When $c = 0$, we always reject H_0 and

$$\hat{\mu}_{\ell r} = \bar{y} - S_{12}S_{22}^{-1}\underline{\bar{X}}_2 + S_{12}S_{22}^{-1}\underline{\bar{X}}_1 .$$

Therefore

$$V(\hat{\mu}_{\ell r}|c=0) = V(\bar{y}-S_{12}\,S_{22}^{-1}\underline{\bar{X}}_2) + V(S_{12}S_{22}^{-1}\underline{\bar{X}}_1)$$

$$- 2\mathrm{Cov}(S_{12}S_{22}^{-1}\underline{\bar{X}}_2, S_{12}S_{22}^{-1}\underline{\bar{X}}_1)$$

$$+ 2\mathrm{Cov}(\bar{y}, S_{12}S_{22}^{-1}\underline{\bar{X}}_1)$$

$$= V(\bar{y}-S_{12}S_{22}^{-1}\underline{\bar{X}}_2) - V(S_{12}S_{22}^{-1}\underline{\bar{X}}_1)$$

$$+ 2\mathrm{Cov}(\bar{y}, S_{12}S_{22}^{-1}\underline{\bar{X}}_1)$$

where

$$\begin{aligned}\mathrm{Cov}(\bar{y}, S_{12}S_{22}^{-1}\underline{\bar{X}}_1) &= \mathrm{Cov}[E(\bar{y}|\underline{X}_i\text{'s}), E(S_{12}S_{22}^{-1}\underline{\bar{X}}_1|\underline{X}_i\text{'s})] \\ &+ E[\mathrm{Cov}(\bar{y}, S_{12}S_{22}^{-1}\underline{\bar{X}}_1)|\underline{X}_i\text{'s}] \\ &= \mathrm{Cov}[\mu + \Sigma_{12}\Sigma_{22}^{-1}(\underline{\bar{X}}_1 - \underline{\mu}_x), \Sigma_{12}\Sigma_{22}^{-1}\underline{\bar{X}}_1] \\ &= \frac{1}{n_1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\end{aligned}$$

Now

$$\begin{aligned}V(S_{12}S_{22}^{-1}\underline{\bar{X}}_1) &= EV(S_{12}S_{22}^{-1}\underline{\bar{X}}_1|\underline{X}_i\text{'s}) + VE(S_{12}S_{22}^{-1}\underline{\bar{X}}_1|\underline{X}_i\text{'s}) \\ &= \Sigma_{11\cdot 2}E(\underline{\bar{X}}_1'S_{22}^{-1}\underline{\bar{X}}_1) + V(\Sigma_{12}\Sigma_{22}^{-1}\underline{\bar{X}}_1) \\ &= \Sigma_{11\cdot 2}\frac{p}{n_1(n_1-p-2)}[1 + \frac{2n_1\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x}{p}] + \frac{1}{n_1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\end{aligned}$$

Hence

$$\begin{aligned}V(\hat{\mu}_{\ell r}|c=0) &= \frac{1}{n_2}\Sigma_{11\cdot 2} + \frac{\Sigma_{11\cdot 2}}{n_2}\frac{p}{n_2-p-2}[1 + \frac{2n_2\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x}{p}] \\ &- \Sigma_{11\cdot 2}\frac{p}{n_1(n_1-p-2)}[1 + \frac{2n_1\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x}{p}] \\ &+ \frac{1}{n_1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \qquad (3.48)\end{aligned}$$

Similarly

$$\begin{aligned}V(\hat{\mu}_{\ell r}|c=\infty) &= V(\bar{y} - S_{12}S_{22}^{-1}\underline{\bar{X}}_2) \qquad (3.49) \\ &= \frac{1}{n_2}\Sigma_{11\cdot 2} + \Sigma_{11\cdot 2}\frac{p}{n_2(n_2-p-2)}[1 + \frac{2n_2\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x}{p}]\end{aligned}$$

As a partial check, when $c = 0$, (3.47) becomes

$$V(\hat{\mu}_{\ell r}) = \frac{1}{n_2}\Sigma_{11\cdot 2} + \Sigma_{11\cdot 2}\frac{p}{n_2(n_2-p-2)}[1 + \frac{2n_2\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x}{p}]$$

$$- \Sigma_{11\cdot 2}\frac{p}{n_1(n_1-p-2)}[1 + \frac{2n_1\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x}{p}] + \frac{1}{n_1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

which is identical with (3.48), the variance of $\hat{\mu}_{\ell r}$ when we always reject H_0. When $c = \infty$, (3.47) reduces to

$$\frac{1}{n_2}\Sigma_{11\cdot 2} + \Sigma_{11\cdot 2}\frac{p}{n_2(_2-p-2)}[1 + \frac{2n_2\underline{\mu}_x'\Sigma_{22}^{-1}\underline{\mu}_x}{p}]$$

which is identical with (3.49), the variance of the preliminary test estimator when we always accept H_0.

H. Relative Efficiency (e_2)

As in section D, we compare the performance of the preliminary test estimator $\hat{\mu}_{\ell r}$ with the usual linear regression estimator, $\bar{y} + S_{12}S_{22}^{-1}(\underline{\bar{X}}_1-\underline{\bar{X}}_2)$, when the information of $\underline{\mu}_x$ is ignored. We denote the relative efficiency of $\hat{\mu}_{\ell r}$ to $\bar{y} + S_{12}S_{22}^{-1}(\underline{\bar{X}}_1-\underline{\bar{X}}_2)$ by e_2 and define

$$e_2 = \frac{\text{M.S.E.}(\bar{y} + S_{12}S_{22}^{-1}(\underline{\bar{X}}_1-\underline{\bar{X}}_2))}{\text{M.S.E.}(\mu_{\ell r})} \tag{3.50}$$

Since $\bar{y} + S_{12}S_{22}^{-1}(\bar{X}_1-\bar{X}_2)$ is unbiased, its M.S.E. equals its variance which is given in (3.48), we denote it by g_2. Using (3.47), we obtain

$$
\begin{aligned}
\text{M.S.E.}(\hat{\mu}_{\ell r}) &= \Sigma_{12}\Sigma_{22}^{-1}T^{-1}M^*T'^{-1}\Sigma_{22}^{-1}\Sigma_{21} \\
&\quad - 2\Sigma_{12}\Sigma_{22}^{-1}\mu_x\mu_x'\Sigma_{22}^{-1}\Sigma_{21}\,[1-F^*_{p+2,n_1-p}(c_2;\lambda)] \\
&\quad - Q^* + \frac{1}{n_2}\Sigma_{11\cdot 2} + \Sigma_{11\cdot 2}\,\frac{p}{n_2(n_2-p-2)}[1+\frac{2n_2\mu_x'\Sigma_{22}^{-1}\mu_x}{p}] \\
&\quad + \Sigma_{12}\Sigma_{22}^{-1}\mu_x\mu_x'\Sigma_{22}^{-1}\Sigma_{21} \\
&= \frac{1}{n_2}\,\Sigma_{11\cdot 2} + \frac{1}{n_1}\,\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}[1-F^*_{p+2,n_1-p}(c_2;\lambda)] \\
&\quad - \Sigma_{12}\Sigma_{22}^{-1}\mu_x\mu_x'\Sigma_{22}^{-1}\Sigma_{21}F^*_{p+4,n_1-p}(c_4;\lambda) \\
&\quad + 2\Sigma_{12}\Sigma_{22}^{-1}\mu_x\mu_x'\Sigma_{22}^{-1}\Sigma_{21}F^*_{p+2,n_1-p}(c_2;\lambda) \\
&\quad - \Sigma_{11\cdot 2}\,\frac{p}{n_1(n_1-p)}\int_d^\infty tf(t)dt \\
&\quad + \Sigma_{11\cdot 2}\,\frac{p}{n_2(n_2-p-2)}[1+\frac{2n_2\mu_x'\Sigma_{22}^{-1}\mu_x}{p}] = h_2 \text{ say.}
\end{aligned}
$$

Therefore

$$e_2 = \frac{g_2}{h_2} \tag{3.51}$$

Wlog we let $\Sigma_{22} = I$, $\sigma^2 = 1$ and write

$$e_2 = \frac{g_2}{h_2}$$

where

$$g_2 = \frac{1}{n_2}(1-g) + \frac{1}{n_1}g + (1-g)\frac{p}{n_2(n_2-p-2)}[1 + \frac{2n_2\mu_x'\mu_x}{p}]$$

$$- (1-g)\frac{p}{n_1(n_1-p-2)}[1 + \frac{2n_1\mu_x'\mu_x}{p}]$$

$$h_2 = \frac{1}{n_2}(1-g) + \frac{1}{n_1}g[1-F^*_{p+2,n_1-p}(c_2;\lambda)]$$

$$- K_1^2 F^*_{p+4,n_1-p}(c_4;\lambda) + 2K_1^2 F^*_{p+2,n_1-p}(c_2;\lambda)$$

$$- (1-g)\frac{p}{n_1(n_1-p)}\int_d^\infty tf(t)dt$$

$$+ (1-g)\frac{p}{n_2(n_2-p-2)}[1 + \frac{2n_2\mu_x'\mu_x}{p}]$$

and

$$g = \Sigma_{12}\Sigma_{21} \ ; \ K_1 = \Sigma_{12}\mu_x \ ; \ c_2 = \frac{p}{p+2} F_{p,n_1-p}(\alpha)$$

$$c_4 = \frac{p}{p+4} F_{p,n_1-p}(\alpha) \ ; \ d = F_{p,n_1-p}(\alpha) \ ; \ \lambda = n_1\mu_x'\mu_x \ .$$

In the computation of the values of e_2, we again use the incomplete Beta distribution to approximate the noncentral F distribution. For $\int_d^\infty tf(t)dt$, we use the fact that

$$\int_0^\infty t\ f(t)dt - \int_0^d t\ f(t) = \int_d^\infty t\ f(t)dt \tag{3.52}$$

since $\int_0^\infty t\ f(t)dt$ exists and

$$\int_0^\infty t\ f(t)dt = E(t) = \frac{n_1-p}{n_1-p-2}\left[1 + \frac{2n_1\mu_x'\mu_x}{p}\right]$$

Using (3.52), we write

$$h_2 = \frac{1}{n_2}(1-g) + \frac{1}{n_1}g[1-F^*_{p+2,n_1-p}(c_2;\lambda)]$$

$$- K_1^2\ F^*_{p+4,n_1-p}(c_4;\lambda) + 2K_1^2\ F^*_{p+2,n_1-p}(c_2;\lambda)$$

$$-(1-g)\frac{p}{n_1(n_1-p-2}\left[1 + \frac{2n_1\mu_x'\mu_x}{p}\right] + (1-g)\frac{p}{n_1(n_1-p)}\int_0^d t\ f(t)dt$$

$$+ (1-g)\frac{p}{n_2(n_2-p-2)}\left[1 + \frac{2n_2\mu_x'\mu_x}{p}\right]$$

For the purpose of comparison with the results of Han (1973b), we compute the values of e_2 for $p = 1$ and certain values of n_1, n_2, ρ, μ_x and α. These values are shown in Table 3.6 and again reveal no significant difference from the values obtained by Han. Any differences are due to the approximations and rounding off errors in the computations.

From Table 3.6 we observe that e_2 assumes its maximum value when $\mu_x = 0$. It then decreases to a minimum and then increases to 1.0 as μ_x increases. This is because for large values of μ_x, we would always reject H_0 and use the usual linear regression estimator. For fixed n_1, n_2, α and small values of μ_x, e_2 increases with ρ while for moderately large values of μ_x, it decreases as ρ increases. The values of e_2 is a decreasing function of α for fixed n_1, n_2, ρ and small values of μ_x, while for moderately large values of μ_x, it is an increasing function of α.

The values of e_2 for p = 2 are given in Table 3.7 for some values of Σ_{12}, $\underline{\mu}_x$, n_1, n_2 and α. From this table we observe the following.

1. For fixed values of n_1, n_2, Σ_{12} and α, the relative efficiency e_2 is maximum when the null hypothesis is true, i.e. when $\underline{\mu}_x = \underline{0}$.

2. For fixed n_1 and n_2, the maximum value of e_2 is an increasing function of Σ_{12}, but a decreasing function of α.

3. For fixed α, n_1, n_2, Σ_{12} and some component of $\underline{\mu}_x$, the relative efficiency decreases to a minimum and then increases to 1.0 as the other component increases.

4. For moderately large values of $\underline{\mu}_x$, e_2 is a decreasing function of Σ_{12} and increasing function of α.

Table 3.6. Values of e_2 for p = 1

$n_1 = 30$, $n_2 = 15$	$\alpha = 0.05$		$\alpha = 0.10$	
μ_X	.5	.7	.5	.7
0.0	1.0985	1.2833	1.0755	1.2088
0.1	1.0458	1.1326	1.0320	1.0907
0.2	0.9398	0.8853	0.9489	0.8988
0.3	0.8599	0.7338	0.8953	0.7916
0.4	0.8380	0.6928	0.8924	0.7821
0.5	0.8672	0.7358	0.9243	0.8381
0.6	0.9195	0.8270	0.9619	0.9132
0.7	0.9643	0.9174	0.9862	0.9670
0.8	0.9884	0.9718	0.9964	0.9910
0.9	0.9972	0.9930	0.9993	0.9982
1.0	1.0000	1.0000	1.0000	1.0000
	$\alpha = 0.25$		$\alpha = 0.50$	
0.0	1.0365	1.0946	1.0093	1.0231
0.1	1.0132	1.0366	1.0029	1.0081
0.2	0.9718	0.9410	0.9922	0.9830
0.3	0.9520	0.8969	0.9885	0.9740
0.4	0.9597	0.9109	0.9919	0.9813
0.5	0.9780	0.9493	0.9964	0.9915
0.6	0.9917	0.9801	0.9989	0.9974
0.7	0.9978	0.9945	0.9998	0.9994
0.8	0.9996	0.9989	1.0000	0.9999
0.9	0.9999	0.9998	1.0000	1.0000
1.0	1.0000	1.0000	1.0000	1.0000

Table 3.6. (continued)

$n_1 = 50$, $n_2 = 10$	$\alpha = 0.05$		$\alpha = 0.10$	
μ_x	.5	.7	.5	.7
0.0	1.0395	1.1151	1.0305	1.0876
0.1	1.0081	1.0264	1.0044	1.0149
0.2	0.9528	0.8889	0.9636	0.9122
0.3	0.9308	0.8382	0.9547	0.8899
0.4	0.9498	0.8779	0.9730	0.9317
0.5	0.9789	0.9458	0.9909	0.9761
0.6	0.9947	0.9860	0.9982	0.9952
0.7	0.9992	0.9978	0.9998	0.9994
0.8	1.0000	1.0000	1.0000	1.0000
0.9	1.0000	1.0000	1.0000	1.0000
1.0	1.0000	1.0000	1.0000	1.0000
	$\alpha = 0.25$		$\alpha = 0.50$	
0.0	1.0149	1.0417	1.0038	1.0105
0.1	1.0008	1.0035	1.0000	1.0002
0.2	0.9825	0.9563	0.9956	0.9888
0.3	0.9832	0.9571	0.9966	0.9911
0.4	0.9926	0.9807	0.9988	0.9969
0.5	0.9982	0.9953	0.9998	0.9994
0.6	0.9998	0.9993	1.0000	0.9999
0.7	1.0000	0.9999	1.0000	1.0000
0.8	1.0000	1.0000	1.0000	1.0000
0.9	1.0000	1.0000	1.0000	1.0000
1.0	1.0000	1.0000	1.0000	1.0000

Table 3.7. Values of e_2 for p = 2.

$n_1 = 30$, $n_2 = 10$		$\alpha = 0.05$		
μ_x' \ Σ_{12}'	$\begin{pmatrix}0.5\\0.0\end{pmatrix}$	$\begin{pmatrix}0.5\\0.5\end{pmatrix}$	$\begin{pmatrix}0.7\\0.7\end{pmatrix}$	$\begin{pmatrix}-0.5\\0.7\end{pmatrix}$
(0.0,0.0)	1.0534	1.1822	4.0500	1.5020
(0.0,0.2)	1.0377	1.0190	1.1762	0.9871
(0.0,0.3)	1.0241	0.9180	0.7166	0.7713
(0.0,0.4)	1.0126	0.8662	0.5544	0.6796
(0.0,0.5)	1.0052	0.8665	0.5261	0.6814
(0.2,0.2)	0.9642	0.8076	0.4662	1.2459
(0.2,0.3)	0.9704	0.7555	0.3729	1.0521
(0.2,0.4)	0.9788	0.7553	0.3530	0.9139
(0.2,0.5)	0.9873	0.7997	0.3911	0.8646
(0.3,0.3)	0.9373	0.7315	0.3294	1.1010
(0.3,0.4)	0.9586	0.7526	0.3369	0.9980
(0.3,0.5)	0.9770	0.8101	0.3950	0.9453
(0.4,0.4)	0.9524	0.7872	0.3656	1.0271
(0.4,0.5)	0.9746	0.8475	0.4443	0.9899
(0.5,0.5)	0.9797	0.8987	0.5429	1.0038
		$\alpha = 0.10$		
(0.0,0.0)	1.0448	1.1495	2.7384	1.3924
(0.0,0.2)	1.0303	1.0085	1.0934	0.9719
(0.0,0.3)	1.0184	0.9307	0.7462	0.8037
(0.0,0.4)	1.0090	0.9005	0.6321	0.7478
(0.0,0.5)	1.0035	0.9129	0.6402	0.7752
(0.2,0.2)	0.9706	0.8367	0.5136	1.1704
(0.2,0.3)	0.9781	0.8057	0.4424	1.0297
(0.2,0.4)	0.9862	0.8227	0.4494	0.9379
(0.2,0.5)	0.9929	0.8726	0.5233	0.9152
(0.3,0.3)	0.9566	0.7988	0.4161	1.0614
(0.3,0.4)	0.9744	0.8314	0.4506	0.9964
(0.3,0.5)	0.9876	0.8870	0.5449	0.9686
(0.4,0.4)	0.9731	0.8688	0.5069	1.0137
(0.4,0.5)	0.9874	0.9182	0.6168	0.9945
(0.5,0.5)	0.9910	0.9526	0.7284	1.0015

Table 3.7. (continued)

$n_1 = 30,\ n_2 = 10$		$\alpha = 0.25$		
μ_x' \ Σ_{12}'	$\begin{pmatrix}0.5\\0.0\end{pmatrix}$	$\begin{pmatrix}0.5\\0.5\end{pmatrix}$	$\begin{pmatrix}0.7\\0.7\end{pmatrix}$	$\begin{pmatrix}-0.5\\0.7\end{pmatrix}$
(0.0,0.0)	1.0271	1.0866	1.6345	1.2084
(0.0,0.2)	1.0169	0.9986	1.0152	0.9676
(0.0,0.3)	1.0093	0.9589	0.8310	0.8783
(0.0,0.4)	1.0040	0.9526	0.7895	0.8680
(0.0,0.5)	1.0014	0.9672	0.8320	0.9063
(0.2,0.2)	0.9831	0.9003	0.6477	1.0756
(0.2,0.3)	0.9894	0.8954	0.6190	1.0087
(0.2,0.3)	0.9894	0.8954	0.6693	0.9723
(0.2,0.5)	0.9978	0.9551	0.7722	0.9704
(0.3,0.3)	0.9815	0.9040	0.6269	1.0219
(0.3,0.4)	0.9910	0.9331	0.6976	0.9975
(0.3,0.5)	0.9965	0.9646	0.8057	0.9904
(0.4,0.4)	0.9919	0.9566	0.7730	1.0036
(0.4,0.5)	0.9969	0.9784	0.8664	0.9985
(0.5,0.5)	0.9982	0.9899	0.9285	1.0002
		$\alpha = 0.50$		
(0.0,0.0)	1.0104	1.0318	1.1763	1.0714
(0.0,0.2)	1.0059	0.9975	0.9944	0.9827
(0.0,0.3)	1.0030	0.9851	0.9315	0.9539
(0.0,0.4)	1.0011	0.9862	0.9295	0.9590
(0.0,0.5)	1.0003	0.9925	0.9566	0.9774
(0.2,0.2)	0.9940	0.9625	0.8384	1.0219
(0.2,0.3)	0.9968	0.9659	0.8425	1.0013
(0.2,0.4)	0.9987	0.9788	0.8892	0.9925
(0.2,0.5)	0.9996	0.9906	0.9435	0.9938
(0.3,0.3)	0.9951	0.9727	0.8637	1.0050
(0.3,0.4)	0.9980	0.9843	0.9919	0.9992
(0.3,0.5)	0.9994	0.9934	0.9581	0.9982
(0.4,0.4)	0.9985	0.9915	0.9473	1.0006
(0.4,0.5)	0.9995	0.9966	0.9765	0.9997
(0.5,0.5)	0.9998	0.9987	0.9901	1.0000

Table 3.7. (continued)

$n_1 = 30,\ n_2 = 15$		$\alpha = 0.05$		
$\underline{\mu}'_x$ \ Σ'_{12}	$\begin{pmatrix}0.5\\0.0\end{pmatrix}$	$\begin{pmatrix}0.5\\0.5\end{pmatrix}$	$\begin{pmatrix}0.7\\0.7\end{pmatrix}$	$\begin{pmatrix}-0.5\\0.7\end{pmatrix}$
(0.0,0.0)	1.0897	1.3-01	4.4866	1.7881
(0.0,0.2)	1.0643	1.0294	1.1836	0.9827
(0.0,0.3)	1.0418	0.8768	0.7086	0.7100
(0.0,0.4)	1.0223	0.7995	0.5434	0.5991
(0.0,0.5)	1.0094	0.7943	0.5132	0.5930
(0.2,0.2)	0.9408	0.7283	0.4568	1.3712
(0.2,0.3)	0.9497	0.6585	0.3631	1.0746
(0.2,0.4)	0.9629	0.6517	0.3421	0.8804
(0.2,0.5)	0.9770	0.7010	0.3781	0.8109
(0.3,0.3)	0.8942	0.6246	0.3192	1.1511
(0.3,0.4)	0.9273	0.6442	0.3254	0.9971
(0.3,0.5)	0.9580	0.7115	0.3810	0.9195
(0.4,0.4)	0.9152	0.6827	0.3524	1.0413
(0.4,0.5)	0.9530	0.7590	0.4284	0.9845
(0.5,0.5)	0.9615	0.8310	0.5252	1.0060
		$\alpha = 0.10$		
(0.0,0.0)	1.0749	1.2419	2.8992	1.5916
(0.0,0.2)	1.0513	1.0131	1.0970	0.9626
(0.0,0.3)	1.0317	0.8952	0.7387	0.7482
(0.0,0.4)	1.0159	0.8479	0.6217	0.6763
(0.0,0.5)	1.0064	0.8619	0.6281	0.7013
(0.2,0.2)	0.9511	0.7659	0.5041	1.2494
(0.2,0.3)	0.9625	0.7213	0.4320	1.0422
(0.2,0.4)	0.9756	0.7377	0.4375	0.9128
(0.2,0.5)	0.9870	0.8009	0.5096	0.8788
(0.3,0.3)	0.9257	0.7081	0.4050	1.0903
(0.3,0.4)	0.9545	0.7458	0.4378	0.9947
(0.3,0.5)	0.9772	0.8193	0.5302	0.9533
(0.4,0.4)	0.9513	0.7938	0.4926	1.0208
(0.4,0.5)	0.9764	0.8641	0.6015	0.9916
(0.5,0.5)	0.9828	0.9176	0.7140	1.0023

Table 3.7. (continued)

$n_1 = 30,\ n_2 = 15$	$\alpha = 0.25$			
$\underline{\mu}'_x$ \ Σ'_{12}	$\begin{pmatrix}0.5\\0.0\end{pmatrix}$	$\begin{pmatrix}0.5\\0.5\end{pmatrix}$	$\begin{pmatrix}0.7\\0.7\end{pmatrix}$	$\begin{pmatrix}-0.5\\0.7\end{pmatrix}$
(0.0,0.0)	1.0448	1.1355	1.6682	1.2944
(0.0,0.2)	1.0283	0.9979	1.0157	0.9568
(0.0,0.3)	1.0160	0.9368	0.8255	0.8397
(0.0,0.4)	1.0071	0.9252	0.7819	0.8225
(0.0,0.5)	1.0025	0.9461	0.8246	0.8682
(0.2,0.2)	0.9717	0.8522	0.6390	1.1066
(0.2,0.3)	0.9817	0.8424	0.6090	1.0122
(0.2,0.4)	0.9904	0.8753	0.6586	0.9605
(0.2,0.5)	0.9960	0.9258	0.7624	0.9566
(0.3,0.3)	0.9677	0.8519	0.6160	1.0316
(0.3,0.4)	0.9837	0.8924	0.6866	0.9964
(0.3,0.5)	0.9935	0.9404	0.7963	0.9855
(0.4,0.4)	0.9850	0.9276	0.7628	1.0054
(0.4,0.5)	0.9942	0.9626	0.8588	0.9976
(0.5,0.5)	0.9964	0.9818	0.9236	1.0004
	$\alpha = 0.50$			
(0.0,0.0)	1.0170	1.0484	1.1829	1.0963
(0.0,0.2)	1.0099	0.9961	0.9942	0.9769
(0.0,0.3)	1.0050	0.9767	0.9290	0.9375
(0.0,0.4)	1.0020	0.9778	0.9265	0.9428
(0.0,0.5)	1.0006	0.9875	0.9544	0.9672
(0.2,0.2)	0.9899	0.9426	0.8332	1.0303
(0.2,0.3)	0.9945	0.9465	0.8368	1.0018
(0.2,0.4)	0.9976	0.9655	0.8844	0.9892
(0.2,0.5)	0.9992	0.9841	0.9405	0.9907
(0.3,0.3)	0.9914	0.9561	0.8582	1.0072
(0.3,0.4)	0.9964	0.9739	0.9077	0.9989
(0.3,0.5)	0.9988	0.9887	0.9556	0.9973
(0.4,0.4)	0.9972	0.9855	0.9443	1.0009
(0.4,0.5)	0.9991	0.9940	0.9749	0.9996
(0.5,0.5)	0.9995	0.9976	0.9893	1.0000

Table 3.7. (continued)

$n_1 = 50,\ n_2 = 10$		$\alpha = 0.05$		
$\underline{\mu}_x'$ \\ Σ_{12}'	$\begin{pmatrix}0.5\\0.0\end{pmatrix}$	$\begin{pmatrix}0.5\\0.5\end{pmatrix}$	$\begin{pmatrix}0.7\\0.7\end{pmatrix}$	$\begin{pmatrix}-0.5\\0.7\end{pmatrix}$
(0.0,0.0)	1.0352	1.1133	3.4684	1.3150
(0.0,0.2)	1.0219	0.9725	0.8563	0.8837
(0.0,0.3)	1.0116	0.9122	0.5854	0.7548
(0.0,0.4)	1.0044	0.9134	0.5627	0.7628
(0.0,0.5)	1.0012	0.9505	0.6778	0.8539
(0.2,0.2)	0.9659	0.8176	0.3762	1.1049
(0.2,0.3)	0.9789	0.8185	0.3609	0.9875
(0.2,0.4)	0.9903	0.8731	0.4412	0.9396
(0.2,0.5)	0.9968	0.9407	0.6229	0.9544
(0.3,0.3)	0.9689	0.8436	0.3878	1.0227
(0.3,0.4)	0.9867	0.9033	0.5033	0.9888
(0.3,0.5)	0.9960	0.9597	0.7024	0.9861
(0.4,0.4)	0.9904	0.9475	0.6458	1.0017
(0.4,0.5)	0.9973	0.9806	0.8246	0.9977
(0.5,0.5)	0.9989	0.9935	0.9302	0.9999
		$\alpha = 0.10$		
(0.0,0.0)	1.0295	1.0935	2.4876	1.2520
(0.0,0.2)	1.0167	0.9738	0.8560	0.8953
(0.0,0.3)	1.0079	0.9346	0.6582	0.8095
(0.0,0.4)	1.0027	0.9457	0.6792	0.8417
(0.0,0.5)	1.0006	0.9748	0.8087	0.9217
(0.2,0.2)	0.9746	0.8586	0.4489	1.0684
(0.2,0.3)	0.9862	0.8745	0.4653	0.9896
(0.2,0.4)	0.9947	0.9259	0.5887	0.9652
(0.2,0.5)	0.9986	0.9720	0.7829	0.9785
(0.3,0.3)	0.9819	0.9036	0.5235	1.0119
(0.3,0.4)	0.9934	0.9497	0.6714	0.9941
(0.3,0.5)	0.9983	0.9828	0.8502	0.9941
(0.4,0.4)	0.9958	0.9766	0.8085	1.0006
(0.4,0.5)	0.9990	0.9928	0.9277	0.9991
(0.5,0.5)	0.9996	0.9980	0.9770	1.0000

Table 3.7. (continued)

$n_1 = 50,\ n_2 = 10$		$\alpha = 0.25$		
$\underline{\mu}_x'$ \ Σ_{12}'	$\begin{pmatrix}0.5\\0.0\end{pmatrix}$	$\begin{pmatrix}0.5\\0.5\end{pmatrix}$	$\begin{pmatrix}0.7\\0.7\end{pmatrix}$	$\begin{pmatrix}-0.5\\0.7\end{pmatrix}$
(0.0,0.0)	1.0178	1.0550	1.5722	1.1397
(0.0,0.2)	1.0086	0.9820	0.8923	0.9309
(0.0,0.2)	1.0034	0.9684	0.8035	0.9016
(0.0,0.4)	1.0009	0.9806	0.8589	0.9389
(0.0,0.5)	1.0002	0.9936	0.9437	0.9791
(0.2,0.2)	0.9876	0.9269	0.6285	1.0274
(0.2,0.3)	0.9947	0.9474	0.6918	0.9947
(0.2,0.4)	0.9984	0.9768	0.8277	0.9891
(0.2,0.5)	0.9997	0.9937	0.9423	0.9951
(0.3,0.3)	0.9942	0.9671	0.7744	1.0033
(0.3,0.4)	0.9984	0.9869	0.8905	0.9984
(0.3,0.5)	0.9997	0.9967	0.9676	0.9989
(0.4,0.4)	0.9992	0.9952	0.9545	1.0001
(0.4,0.5)	0.9998	0.9989	0.9881	0.9999
(0.5,0.5)	1.0000	0.9997	0.9967	1.0000
		$\alpha = 0.50$		
(0.0,0.0)	1.0067	1.0203	1.1617	1.0492
(0.0,0.2)	1.0028	0.9927	0.9528	0.9724
(0.0,0.3)	1.0009	0.9906	0.9333	0.9695
(0.0,0.4)	1.0002	0.9957	0.9654	0.9861
(0.0,0.5)	1.0000	0.9990	0.9906	0.9966
(0.2,0.2)	0.9963	0.9770	0.8495	1.0071
(0.2,0.3)	0.9987	0.9867	0.9024	0.9985
(0.2,0.4)	0.9997	0.9956	0.9624	0.9979
(0.2,0.5)	1.0000	0.9991	0.9915	0.9993
(0.3,0.3)	0.9988	0.9932	0.9447	1.0006
(0.3,0.4)	0.9997	0.9979	0.9809	0.9997
(0.3,0.5)	1.0000	0.9996	0.9960	0.9999
(0.4,0.4)	0.9999	0.9994	0.9940	1.0000
(0.4,0.5)	1.0000	0.9999	0.9988	1.0000
(0.5,0.5)	1.0000	1.0000	1.0000	1.0000

IV. THE REGRESSION ESTIMATOR WITH A CERTAIN SHRUNKEN ESTIMATOR FOR THE MEAN OF THE AUXILIARY VARIABLE

A. Introduction

Let μ be the mean of Y and $\underline{\mu}_x$ be the mean of the px1 vector of auxiliary variables $\underline{X}$. We consider in this chapter a regression estimator of μ by using a shrunken estimator of the form $c\underline{\bar{X}}: 0 < c \leq 1$ for $\underline{\mu}_x$, when prior information about $\underline{\mu}_x$ is available, i.e. $\underline{\mu}_x$ is close to $\underline{\mu}_0$, instead of the usual minimum variance unbiased linear estimator $\underline{\bar{X}}$. We first consider the case p = 1 and following Thompson (1968a), we find the optimal value of c which minimizes the m.s.e. of $\hat{\mu}^*$, the regression estimator of μ which is defined below. The m.s.e. of $\hat{\mu}^*$ will be derived and the efficiency of the preliminary test estimator of Chapter II relative to $\hat{\mu}^*$ will be discussed. Since μ_0 is known, without loss of generality, we let $\mu_0 = 0$.

Let $\hat{\mu}_x = c\bar{X}$ and assume σ_x^2, σ_y^2, ρ known, then $\hat{\mu}^*$ is defined as $\hat{\mu}^* = \bar{y} - \beta c\bar{X}$ where

$$\beta = \frac{\sigma_{xy}}{\sigma_x^2}$$

$$\text{m.s.e.}(\hat{\mu}^*) = E(\bar{y} - \beta c\bar{X} - \mu)^2 \tag{4.1}$$

In order to find c to minimize (4.1), we differentiate w.r.t. c and equate to zero. Thus

$$\frac{\partial}{\partial c} E(\bar{y}-\beta c\bar{X}-\mu)^2 = 0$$

and since integrand is absolutely integrable,

$$E \frac{\partial}{\partial c} (\bar{y}-\beta c\bar{X}-\mu)^2 = 0$$

$$E\ 2(\bar{y}-\beta c\bar{X}-\mu)(-\beta\bar{X}) = 0$$

$$E\ \bar{y}\bar{X} - \beta cE\bar{X}^2 - \mu E\bar{X} = 0$$

$$\beta c(\mu_x^2 + \frac{\sigma_x^2}{n}) = \frac{1}{n}\beta\sigma_x^2$$

Therefore

$$c = \frac{\frac{1}{n}\sigma_x^2}{\mu_x^2 + \frac{\sigma_x^2}{n}} \tag{4.2}$$

Since μ_x is unknown, we may estimate it by $\bar{X}$ as in Thompson (1968a). Therefore

$$\hat{c} = \frac{\frac{1}{n}\sigma_x^2}{\bar{X}^2 + \frac{\sigma_x^2}{n}} \tag{4.3}$$

Hence the regression estimator of μ using a shrunken estimator for μ_x is

$$\hat{\mu}^* = \bar{y} - \frac{\frac{\beta}{n}\sigma_x^2\bar{X}}{\bar{X}^2 + \frac{\sigma_x^2}{n}}$$

or

$$\hat{\mu}^* = \bar{y} - \frac{\beta\, \sigma_x^2 \bar{X}}{n\bar{X}^2 + \sigma_x^2} \tag{4.4}$$

The case p = 2 can be treated similarly even though the derivations are more difficult. This case will not be treated here. The case $p \geq 3$ will be treated in section C of the present chapter.

B. The M.S.E. of $\hat{\mu}^*$ and Relative Efficiency (e_3)

$$\text{m.s.e.}(\hat{\mu}^*) = E[\bar{y} - \frac{\beta\sigma_x^2\bar{X}}{n\bar{X}^2 + \sigma_x^2} - \mu]^2 \tag{4.5}$$

$$= E(\bar{y}-\mu)^2 - 2\beta E\,\frac{\sigma_x^2\bar{X}\bar{y}}{n\bar{X}^2 + \sigma_x^2} + 2\beta\mu E\,\frac{\sigma_x^2\,\bar{X}}{n\bar{X}^2 + \sigma_x^2}$$

$$+ \beta^2 E\left(\frac{\sigma_x^2\bar{X}}{n\bar{X}^2 + \sigma_x^2}\right)^2$$

The second term can be evaluated as

$$-2\beta\sigma_x^2 E\,\frac{\bar{X}\bar{y}}{n\bar{X}^2 + \sigma_x^2} = -2\beta\sigma_x^2 EE[\frac{\bar{X}\bar{y}}{n\bar{X}^2 + \sigma_x^2}|\bar{X}]$$

$$= -2\beta\sigma_x^2 E\frac{\bar{X}}{n\bar{X}^2 + \sigma_x^2}E[\bar{y}|\bar{X}]$$

$$= -2\beta\sigma_x^2 E\frac{\bar{X}}{n\bar{X}^2 + \sigma_x^2}[\mu + \beta(\bar{X}-\mu_x)]$$

$$= -2\beta\sigma_x^2(\mu-\beta\mu_x)E\frac{\bar{X}}{n\bar{X}^2 + \sigma_x^2} - 2\beta^2\sigma_x^2 E\frac{\bar{X}^2}{n\bar{X}^2 + \sigma_x^2} \tag{4.6}$$

Therefore

$$\text{m.s.e.}(\hat{\mu}^*) = E(\bar{y}-\mu)^2 + 2\beta^2\mu_x\sigma_x^2 E\frac{\bar{X}}{n\bar{X}^2 + \sigma_x^2} - 2\beta^2\sigma_x^2 E\frac{\bar{X}^2}{n\bar{X}^2 + \sigma_x^2}$$

$$+ \beta^2\sigma_x^4 E\frac{\bar{X}^2}{(n\bar{X}^2 + \sigma_x^2)^2}$$

$$= \frac{1}{n}\sigma^2 + 2\beta^2\mu_x\sigma_x^2 E\frac{\bar{X}}{n\bar{X}^2 + \sigma_x^2} - 2\beta^2\sigma_x^2 E\frac{\bar{X}^2}{n\bar{X}^2 + \sigma_x^2}$$

$$+ \beta^2\sigma_x^4 E\frac{\bar{X}^2}{(n\bar{X}^2 + \sigma_x^2)^2} \tag{4.7}$$

We may now use the Gauss-Hermite quadrature to evaluate the above expected values. The relevant approximation given in equation 25.4.46 and Table 25.10 of Davis and Polonsky (1964) is

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx = \sum_{i=1}^{k} W_i f(x_i) + R_k \tag{4.8}$$

where x_i are the i-th zeros of Hermite polynomials $H_k(x)$, which are the related orthogonal polynomials. The weights

$$W_i = \frac{2^{k-1}k!\,\sqrt{\pi}}{k^2[H_{k-1}(x_i)]^2}$$

The remainder

$$R_k = \frac{k!\sqrt{\pi}}{2^k(2k)} f^{2k}(\xi) \quad (-\infty<\xi<\infty)$$

To use (4.8) we make the following transformation.

$$\bar{X} = y \sim N(\mu_x, \frac{\sigma_x^2}{n}) \ .$$

Therefore

$$E\frac{\bar{X}}{n\bar{X}^2 + \sigma_x^2} = \int_{-\infty}^{\infty} \frac{y}{ny^2 + \sigma_x^2} \frac{\sqrt{n}}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{n}{2}(\frac{y-\mu_x}{\sigma_x})^2} dy$$

Let

$$(\frac{y-\mu_x}{\sigma_x})\sqrt{\frac{n}{2}} = x$$

$$\Longrightarrow \quad y = \sqrt{\frac{2}{n}}(x\sigma_x) + \mu_x$$

$$\Longrightarrow \quad dy = \sigma_x\sqrt{\frac{2}{n}}\, dx \ .$$

Therefore

$$E\frac{\bar{X}}{n\bar{X}^2 + \sigma_x^2} = \int_{-\infty}^{\infty} \frac{\sqrt{\frac{2}{n}}(x\sigma_x) + \mu_x}{n[\sqrt{\frac{2}{n}}(x\sigma_x) + \mu_x]^2+\sigma_x^2} \frac{\sqrt{n}}{\sqrt{2\pi\sigma_x^2}} e^{-x^2}\sqrt{\frac{2}{n}}\, \sigma_x dx$$

Therefore

$$E\frac{\bar{X}}{n\bar{X}^2 + \sigma_x^2} = \int_{-\infty}^{\infty} \frac{\sqrt{\frac{2}{n}}(x\sigma_x) + \mu_x}{n[\sqrt{\frac{2}{n}}(x\sigma_x) + \mu_x]^2 + \sigma_x^2} \frac{1}{\sqrt{\pi}} e^{-x^2} dx \tag{4.9}$$

Similarly

$$E\frac{\bar{X}^2}{n\bar{X}^2 + \sigma_x^2} = \int_{-\infty}^{\infty} \frac{(\sqrt{\frac{2}{n}}(x\sigma_x) + \mu_x)^2}{n[\sqrt{\frac{2}{n}}(x\sigma_x) + \mu_x]^2 + \sigma_x^2} \frac{1}{\sqrt{\pi}} e^{-x^2} dx \tag{4.10}$$

and

$$E\frac{\bar{X}^2}{(n\bar{X}^2 + \sigma_x^2)^2} = \int_{-\infty}^{\infty} \frac{(\sqrt{\frac{2}{n}}(x\sigma_x) + \mu_x)^2}{[n(\sqrt{\frac{2}{n}}(x\sigma_x) + \mu_x)^2 + \sigma_x^2]^2} \frac{1}{\sqrt{\pi}} e^{-x^2} dx \tag{4.11}$$

Efficiency of the preliminary test estimator ($\hat{\mu}$) relative to $\hat{\mu}^*$ is

$$e_3 = \frac{1}{\text{M.S.E.}(\hat{\mu})} \Big/ \frac{1}{\text{M.S.E.}(\hat{\mu}^*)}$$

$$= \frac{\text{M.S.E.}(\hat{\mu}^*)}{\text{M.S.E.}(\hat{\mu})} \tag{4.12}$$

Therefore, using (2.24) and noting that $\beta = \Sigma_{12}\Sigma_{22}^{-1}$, we have

$$e_3 = \frac{\frac{1}{n}\sigma^2 + 2\beta^2\mu_x\sigma_x^2 E\frac{\bar{X}}{n\bar{X}^2+\sigma_x^2} - 2\beta^2\sigma_x^2 E\frac{\bar{X}^2}{n\bar{X}^2+\sigma_x^2} + \beta^2\sigma_x^4 E\frac{\bar{X}^2}{(n\bar{X}^2+\sigma_x^2)^2}}{\frac{1}{n}\sigma^2 - \beta^2\mu_x^2 H_{p+4}(c;\lambda) - \frac{1}{n}\beta\sigma_{12}H_{p+2}(c;\lambda) + 2\beta^2\mu_x^2 H_{p+2}(c;\lambda)}$$

and wlog we let $\sigma_x^2 = \sigma^2 = 1$. Hence

$$e_3 = \frac{\frac{1}{n} + 2\rho^2\mu_x E\frac{\bar{X}}{n\bar{X}^2+1} - 2\rho^2 E\frac{\bar{X}^2}{n\bar{X}^2+1} + \rho^2 E\frac{\bar{X}^2}{(n\bar{X}^2+1)^2}}{\frac{1}{n} - \rho^2\mu_x^2 H_{p+4}(c;\lambda) - \frac{1}{n}\rho^2 H_{p+2}(c;\lambda) + 2\rho^2\mu_x^2 H_{p+2}(c;\lambda)} \quad (4.13)$$

The values of e_3 for n = 9, and k = 20 are given in Table 4.1 for certain choices of μ_x, ρ and α. From the table we observe that

1. e_3 has a maximum greater than unity at $\mu_x = 0$. Again this corresponds to the case when the null hypothesis of the preliminary test is true.

2. For fixed n, μ_x and α, e_3 is in general a decreasing function of ρ.

3. For fixed n, μ_x and ρ, e_3 is also generally a decreasing function of α.

4. For fixed n, ρ and α, e_3 first decreases to a minimum, then increases to above unity and then finally drops back to unity as μ_x increases.

Table 4.1. Values of e_3 for n = 9 and k = 20.

	α = .05		α = .10		α = .25	
			ρ			
μ_x	.7	.9	.7	.9	.7	.9
0.0	1.1430	1.3674	1.0193	1.0424	0.8550	0.7330
0.1	1.0836	1.1930	0.9821	0.9632	0.8491	0.7286
0.2	0.9568	0.9184	0.9023	0.8243	0.8398	0.7270
0.3	0.8299	0.7269	0.8217	0.7155	0.8342	0.7331
0.4	0.7352	0.6174	0.7650	0.6541	0.8414	0.7551
0.5	0.6751	0.5586	0.7359	0.6293	0.8630	0.7933
0.6	0.6444	0.5325	0.7320	0.6318	0.8966	0.8449
0.7	0.6407	0.5332	0.7515	0.6596	0.9395	0.9087
0.8	0.6608	0.5574	0.7904	0.7091	0.9844	0.9761
0.9	0.7030	0.6055	0.8440	0.7782	1.0250	1.0391
1.0	0.7638	0.6767	0.9047	0.8600	1.0558	1.0888
1.3	0.9694	0.9528	1.0390	1.0625	1.0810	1.1332
1.6	1.0478	1.0782	1.0583	1.0961	1.0624	1.1032
1.9	1.0444	1.0734	1.0450	1.0744	1.0452	1.0747
2.2	1.0338	1.0559	1.0338	1.0559	1.0338	1.0559
2.5	1.0262	1.0433	1.0262	1.0433	1.0262	1.0433
2.8	1.0209	1.0345	1.0209	1.0345	1.0209	1.0345
3.1	1.0170	1.0282	1.0170	1.0282	1.0170	1.0282

C. The Shrunken Regression Estimator for $p \geq 3$

In this section we consider the shrunken regression estimator for $p \geq 3$. Suppose Σ is known. Wlog we let $\Sigma_{22} = I$ and $\sigma^2 = 1$ and consider the regression estimator

$$\hat{\mu}_1 = \bar{y} - \Sigma_{12}\underline{\bar{X}}\left(1 - \frac{p-2}{\underline{\bar{X}}'\underline{\bar{X}}}\right) \tag{4.14}$$

where following James and Stein (1960), we use

$$\underline{\bar{X}}\left(1 - \frac{p-2}{\underline{\bar{X}}'\underline{\bar{X}}}\right)$$

as an estimator of $\underline{\mu}_x$.

We shall now derive the m.s.e. of $\hat{\mu}_1$ and the efficiency of the preliminary test estimator, $\hat{\mu}$, relative to $\hat{\mu}_1$. We shall denote this relative efficiency by e_4.

$$\text{m.s.e.}(\hat{\mu}_1) = E\left[\bar{y} - \Sigma_{12}\underline{\bar{X}}\left(1 - \frac{p-2}{\underline{\bar{X}}'\underline{\bar{X}}}\right) - \mu\right]^2 \tag{4.15}$$

$$= E(\bar{y}-\mu)^2 - 2\Sigma_{12}E\bar{y}\underline{\bar{X}}\left(1 - \frac{p-2}{\underline{\bar{X}}'\underline{\bar{X}}}\right) + 2\mu\Sigma_{12}E\underline{\bar{X}}\left(1 - \frac{p-2}{\underline{\bar{X}}'\underline{\bar{X}}}\right)$$

$$+ \Sigma_{12}E\left[\left(1 - \frac{p-2}{\underline{\bar{X}}'\underline{\bar{X}}}\right)\underline{\bar{X}\bar{X}}'\left(1 - \frac{p-2}{\underline{\bar{X}}'\underline{\bar{X}}}\right)\right]\Sigma_{21}$$

Now

$$E\bar{y}\underline{\bar{X}} = EE\bar{y}\underline{\bar{X}}\,|\,\underline{\bar{X}}$$

$$= E\underline{\bar{X}}[\mu + \Sigma_{12}(\underline{\bar{X}}-\underline{\mu}_x)]$$

$$= \mu\underline{\mu}_x - \underline{\mu}_x\underline{\mu}_x'\Sigma_{21} + E(\underline{\bar{X}\bar{X}}')\Sigma_{21}$$

Similarly

$$E\frac{\bar{y}\bar{\underline{X}}}{\bar{\underline{X}}'\bar{\underline{X}}} = E\frac{\bar{\underline{X}}}{\bar{\underline{X}}'\bar{\underline{X}}}[\mu + \Sigma_{12}(\bar{\underline{X}}-\underline{\mu}_x)]$$

$$= \mu E\frac{\bar{\underline{X}}}{\bar{\underline{X}}'\bar{\underline{X}}} - E(\frac{\bar{\underline{X}}}{\bar{\underline{X}}'\bar{\underline{X}}})\underline{\mu}_x'\Sigma_{21} + E(\frac{\bar{\underline{X}}\bar{\underline{X}}'}{\bar{\underline{X}}'\bar{\underline{X}}})\Sigma_{21}$$

Therefore

$$\text{m.s.e.}(\hat{\mu}_1) = \frac{1}{n} + 2\Sigma_{12}\underline{\mu}_x\underline{\mu}_x'\Sigma_{21} - 2(p-2)\Sigma_{12}E(\frac{\bar{\underline{X}}}{\bar{\underline{X}}'\bar{\underline{X}}})\underline{\mu}_x'\Sigma_{21}$$

$$- \Sigma_{12}E(\bar{\underline{X}}\bar{\underline{X}}')\Sigma_{21} + (p-2)^2\Sigma_{12}E\frac{(\bar{\underline{X}}\bar{\underline{X}}')}{(\bar{\underline{X}}'\bar{\underline{X}})^2}\Sigma_{21}$$

Using (2.24) and wlog letting $\Sigma_{22} = I$, $\sigma^2 = 1$, the efficiency of the preliminary test estimator relative to $\hat{\mu}_1$

$$e_4 = \frac{\frac{1}{n} + 2\Sigma_{12}\underline{\mu}_x\underline{\mu}_x'\Sigma_{21} - 2(p-2)\Sigma_{12}E(\frac{\bar{\underline{X}}}{\bar{\underline{X}}'\bar{\underline{X}}})\underline{\mu}_x'\Sigma_{21}}{\frac{1}{n} - \Sigma_{12}\underline{\mu}_x\underline{\mu}_x'\Sigma_{21}H_{p+4}(c;\lambda) - \frac{1}{n}\Sigma_{12}\Sigma_{21}H_{p+2}(c;\lambda)}$$

$$\frac{+ (p-2)^2\Sigma_{12}E\frac{(\bar{\underline{X}}\bar{\underline{X}}')}{(\bar{\underline{X}}'\bar{\underline{X}})^2}\Sigma_{21} - \Sigma_{12}E(\bar{\underline{X}}\bar{\underline{X}}')\Sigma_{21}}{+ 2\Sigma_{12}\underline{\mu}_x\underline{\mu}_x'\Sigma_{21}H_{p+2}(c;\lambda)} \qquad (4.16)$$

To evaluate e_4, we need to evaluate

(1) $$E\frac{\bar{\underline{X}}}{\bar{\underline{X}}'\bar{\underline{X}}},$$

(2) $$E\underline{\bar{X}}\underline{\bar{X}}' \quad , \text{ and}$$

(3) $$E \frac{\underline{\bar{X}}\underline{\bar{X}}'}{(\underline{\bar{X}}'\underline{\bar{X}})^2}$$

where $\underline{\bar{X}} \sim N(\underline{\mu}_x, \frac{1}{n}I)$. Let $\underline{\bar{X}} = \underline{U}$ so that $\bar{X}_i = U_i \sim N(\mu_i, \frac{1}{n})$ where

$$\underline{\mu}_x = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix}$$

then

$$E \frac{\underline{\bar{X}}}{\underline{\bar{X}}'\underline{\bar{X}}} = \begin{pmatrix} E \dfrac{U_1}{\sum_{i=1}^{p} U_i^2} \\ E \dfrac{U_2}{\sum_{i=1}^{p} U_i^2} \\ \vdots \\ E \dfrac{U_p}{\sum_{i=1}^{p} U_i^2} \end{pmatrix}$$

We now consider the i-th component of (1) and note that

$$\underline{U}'\underline{U} = \Sigma U_i^2 = ||\underline{U}||^2 \sim \chi_p^2(n\,||\underline{\mu}_x||^2) \qquad (4.17)$$
$$\sim \chi^2_{p+2K}$$

where

$$K \sim \text{Poisson}(\tfrac{n}{2}\,||\underline{\mu}_x||^2)$$

i.e.

$$p(K=k) = \frac{e^{-\frac{n}{2}||\underline{\mu}_x||^2}\,(\frac{n}{2}||\underline{\mu}_x||^2)^k}{k!}$$

Therefore

$$E\left(\frac{1}{||\underline{U}||^2}\right) = E\left(\frac{1}{\chi^2_{p+2K}}\right)$$
$$= EE\left(\frac{1}{\chi^2_{p+2K}}\Big|K\right)$$
$$= E\frac{1}{p-2+2K}$$

Therefore

$$E\left(\frac{1}{||\underline{U}||^2}\right) = E\left(\frac{1}{p-2+2K}\right) = \sum_{k=0}^{\infty} \frac{1}{p-2+2k}\,\frac{e^{-\frac{n}{2}\sum_{i=1}^{p}\mu_i^2}\,(\frac{n}{2}\sum_{i=1}^{p}\mu_i^2)^k}{k!} \qquad (4.18)$$

Alternatively using the independence and marginal density of each component, we may also write

$$E\left(\frac{1}{\|\underline{U}\|^2}\right) = \int_{-\infty}^{\infty}\cdots\int \frac{n^{\frac{p}{2}}}{\sum_{j=1}^{p} U_j^2} \frac{e^{-\frac{n}{2}\sum_{j=1}^{p}(U_j-\mu_j)^2}}{(2\pi)^{\frac{p}{2}}}\, dU_1 \ldots dU_p \qquad (4.19)$$

Differentiating (4.18) w.r.t. μ_i we have

$$\frac{\partial}{\partial\mu_i} E\left(\frac{1}{\|\underline{U}\|^2}\right) = \sum_{k=0}^{\infty} \frac{1}{p-2+2k} (-n\mu_i) \frac{e^{-\frac{n}{2}\sum_1^p \mu_j^2} \left(\frac{n}{2}\sum_{j=1}^{p}\mu_j^2\right)^k}{k!}$$

$$+ \sum_{k=1}^{p} \frac{1}{p-2+2k} \frac{e^{-\frac{n}{2}\sum_1^p \mu_j^2}\, k\left(\frac{n}{2}\sum_{j=1}^{p}\mu_j^2\right)^{k-1}}{k!} n\mu_i$$

$$= n\mu_i \sum_{k=0}^{\infty} \frac{1}{p-2+2(k+1)} \frac{e^{-\frac{n}{2}\sum_{j=1}^{p}\mu_j^2}\left(\frac{n}{2}\sum_1^p \mu_j^2\right)^k}{k!}$$

$$- n\mu_i \sum_{k=0}^{\infty} \frac{1}{p-2+2k} \frac{e^{-\frac{n}{2}\sum_{j=1}^{p}\mu_j^2}\left(\frac{n}{2}\sum_1^p \mu_j^2\right)^k}{k!}$$

Therefore

$$\frac{\mu}{\partial\mu_i} E\left(\frac{1}{\|\underline{U}\|^2}\right) = n\mu_i\, E\frac{1}{p+2K} - n\mu_i\, E\left(\frac{1}{\|\underline{U}\|^2}\right) \qquad (4.20$$

Similarly differentiating (4.19) w.r.t. μ_i, we have

$$\frac{\partial}{\partial\mu_i} E\left(\frac{1}{||\underline{U}||^2}\right) = \int_{-\infty}^{\infty}\cdots\int \frac{n^{\frac{p}{2}}}{\sum_1^p U_i^2}\, n(U_i-\mu_i)\, \frac{e^{-\frac{n}{2}\sum_1^p (U_j-\mu_j)^2}}{(2\pi)^{\frac{p}{2}}}\, dU_1\ldots dU_p$$

$$= n\, E\left(\frac{U_i-\mu_i}{||U||^2}\right) \qquad (4.21)$$

Equating (4.20) and (4.21), we obtain

$$E\left(\frac{U_i}{||\underline{U}||^2}\right) = \mu_i\, E\left(\frac{1}{p+2K}\right) \qquad (4.22)$$

Hence we may denote

$$E\left(\frac{\underline{\bar{X}}}{\underline{\bar{X}}'\underline{\bar{X}}}\right)=$$

by L where L is a pxl vector whose i-th component

$$= \mu_i\, E\left(\frac{1}{p+2K}\right)$$

$$= \mu_i \sum_{k=0}^{\infty} \frac{1}{p+2k}\, \frac{e^{-\frac{n}{2}\sum_{j=1}^{p}\mu_j^2}\left(\frac{n}{2}\sum_1^p \mu_j^2\right)^k}{k!}$$

Now the second term is

$$E(\underline{\bar{X}}\underline{\bar{X}}') = V(\underline{\bar{X}}) + \underline{\mu}_x\underline{\mu}_x'$$

$$= \frac{1}{n} I + \underline{\mu}_x\underline{\mu}_x' \tag{4.23}$$

The third term,

$$E\frac{\underline{\bar{X}}\underline{\bar{X}}'}{(\underline{\bar{X}}'\underline{\bar{X}})^2} ,$$

is a pxp matrix with i-th diagonal element

$$E \frac{U_i^2}{(\sum_{j=1}^{p} U_j^2)^2}$$

and i-ℓth off-diagonal element

$$E\frac{U_i U_\ell}{(\sum_{j=1}^{p} U_j^2)^2}$$

to evaluate

$$E\frac{U_i^2}{(\sum_{j=1}^{p} U_j^2)^2} ,$$

we may twice differentiate alternative expressions for

$$E \left(\frac{1}{||\underline{U}||^2}\right)^2$$

w.r.t. μ_i and then equate the results. Thus using an expression similar to (4.18), we have

$$\frac{\partial^2}{\partial\mu_i^2} E\left(\frac{1}{\|\underline{U}\|^2}\right)^2 = nE\left(\frac{1}{p+2K}\right)^2$$

$$+ n\mu_i \sum_{k=1}^{\infty} \left(\frac{1}{p+2k}\right)^2 \frac{e^{-\frac{n}{2}\Sigma\mu_j^2} k\left(\frac{n}{2}\sum_{j=1}^{p}\mu_j^2\right)^{k-1}}{k!} n\mu_i$$

$$+ n\mu_i \sum_{k=0}^{\infty} \left(\frac{1}{p+2k}\right)^2 (-n\mu_i) \frac{e^{-\frac{n}{2}\sum_{j=1}^{p}\mu_j^2}\left(\frac{n}{2}\sum_{1}^{p}\mu_j^2\right)^k}{k!}$$

$$- nE\left(\frac{1}{\|\underline{U}\|^2}\right)^2 - n\mu_i\left\{n\mu_i E\left(\frac{1}{p+2K}\right)^2 - n\mu_i E\left(\frac{1}{\|\underline{U}\|^2}\right)^2\right\}$$

$$= nE\left(\frac{1}{p+2K}\right)^2 + (n\mu_i)^2 E\left(\frac{1}{p+2+2K}\right)^2 - (n\mu_i)^2 E\left(\frac{1}{p+2K}\right)^2$$

$$- nE\left(\frac{1}{\|\underline{U}\|^2}\right)^2 - (n\mu_i)^2 E\left(\frac{1}{p+2K}\right)^2 + (n\mu_i)^2 E\left(\frac{1}{\|\underline{U}\|^2}\right)^2 \tag{4.24}$$

Next, using an expression similar to (4.19), we have

$$\frac{\partial^2}{\partial\mu_i^2} E\left(\frac{1}{\|\underline{U}\|^2}\right)^2 = n\frac{\partial}{\partial\mu_i} E\frac{U_i}{(\|\underline{U}\|^2)^2} - n\frac{\partial}{\partial\mu_i}\left\{\mu_i E\left(\frac{1}{\|\underline{U}\|^2}\right)^2\right\}$$

$$= nEnU_i \frac{(U_i-\mu_i)}{(\|\underline{U}\|^2)^2} - nE\left(\frac{1}{\|\underline{U}\|^2}\right)^2 - n\mu_i nE\frac{(U_i-\mu_i)}{(\|\underline{U}\|^2)^2}$$

$$= n^2 E\frac{U_i^2}{(\,\|\underline{U}\|^2)^2} - n^2\mu_i E\frac{U_i}{(\,\|\underline{U}\|^2)^2} - nE\left(\frac{1}{(\,\|\underline{U}\|^2}\right)^2$$

$$- n^2\mu_i E\frac{U_i}{(\,\|\underline{U}\|^2)^2} + n^2\mu_i^2 E\left(\frac{1}{\|\underline{U}\|^2}\right)^2 \qquad (4.25)$$

Therefore equating (4.24) and (4.25), we have

$$nE\left(\frac{1}{p+2K}\right)^2 + n^2\mu_i^2 E\left(\frac{1}{p+2+2K}\right)^2 - 2n^2\mu_i^2 E\left(\frac{1}{p+2K}\right)^2 - nE\left(\frac{1}{\|\underline{U}\|^2}\right)^2$$

$$+ n^2\mu_i^2 E\left(\frac{1}{\|\underline{U}\|^2}\right)^2$$

$$= n^2 E\frac{U_i^2}{(\,\|\underline{U}\|^2)^2} - 2n^2\mu_i^2 E\left(\frac{1}{p+2K}\right)^2 - nE\left(\frac{1}{\|U\|^2}\right)^2 + n^2\mu_i^2 E\left(\frac{1}{\|\underline{U}\|^2}\right)^2$$

$$n^2 E\frac{U_i^2}{(\,\|\underline{U}\|^2)^2} = nE\left(\frac{1}{p+2K}\right)^2 + n^2\mu_i^2 E\left(\frac{1}{p+2+2K}\right)^2$$

$$E\frac{U_i^2}{(\,\|\underline{U}\|^2)^2} = \frac{1}{n}E\left(\frac{1}{p+2K}\right)^2 + \mu_i^2 E\left(\frac{1}{p+2+2K}\right)^2 \qquad (4.26)$$

Similarly to evaluate

$$E\frac{U_i U_\ell}{(\sum_{j=1}^{p} U_j^2)^2} \quad ,$$

we differentiate each of the alternative expressions for

$$E\left(\frac{1}{\|\underline{U}\|^2}\right)^2$$

first w.r.t. μ_i and then w.r.t. μ_ℓ and equate the results. Thus using

$$E\left(\frac{1}{\|\underline{U}\|}\right)^2 = E\left(\frac{1}{p-2+2K}\right)^2$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{p-2+2k}\right)^2 \frac{e^{-\frac{n}{2}\sum_{j=1}^{p}\mu_j^2}\left(\frac{n}{2}\sum_{j=1}^{p}\mu_j^2\right)^k}{k!} ,$$

we have

$$\frac{\partial^2}{\partial\mu_\ell\partial\mu_i} E\left(\frac{1}{\|\underline{U}\|}\right)^2 = n\mu_i \sum_{k=0}^{\infty} \left(\frac{1}{p+2k}\right)^2(-n\mu_\ell) \frac{e^{-\frac{n}{2}\sum_{j=1}^{p}\mu_j^2}\left(\frac{n}{2}\Sigma\mu_j^2\right)^k}{k!}$$

$$+ n\mu_i \sum_{k=1}^{\infty} \left(\frac{1}{p+2k}\right)^2 \frac{e^{-\frac{n}{2}\sum_{j=1}^{p}\mu_j^2} k\left(\frac{n}{2}\sum_{j}^{p}\mu_j^2\right)^{k-1}}{k!} n\mu_\ell$$

$$- n\mu_i \sum_{k=0}^{\infty} \left(\frac{1}{p-2+2k}\right)^2(-n\mu_\ell) \frac{e^{-\frac{n}{2}\sum_{j=1}^{p}\mu_j^2}\left(\frac{n}{2}\Sigma\mu_j^2\right)^k}{k!}$$

$$- n\mu_i \sum_{k=1}^{\infty}\left(\frac{1}{p-2+2k}\right)^2 \frac{e^{-\frac{n}{2}\sum_{j=1}^{p}\mu_j^2} k\left(\frac{n}{2}\sum_{j=1}^{p}\mu_j^2\right)^{k-1}}{k!} n\mu_\ell$$

$$= n^2\mu_i\mu_\ell\, E\left(\frac{1}{p+2+2K}\right)^2 - 2n^2\mu_i\mu_\ell\, E\left(\frac{1}{p+2K}\right)^2$$

$$+ n^2\mu_i\mu_\ell\, E\left(\frac{1}{\|\underline{U}\|^2}\right)^2 \qquad (4.27)$$

Next using

$$E(\frac{1}{\|\underline{U}\|})^2 = \int_{\infty}^{\infty}\cdots\int \frac{n^{\frac{p}{2}}}{(\sum_{j=1}^{p} U_j^2)^2} \frac{e^{-\frac{n}{2}\sum_{j=1}^{p}(U_j-\mu_j)^2}}{(2\pi)^{\frac{p}{2}}} dU_1\ldots dU_p ,$$

we have

$$\frac{\partial^2}{\partial\mu_\ell\partial\mu_i} E(\frac{1}{\|\underline{U}\|^2})^2 = nE\frac{nU_i(U_\ell-\mu_\ell)}{(\|\underline{U}\|^2)^2} - n\mu_i nE\frac{(U_\ell-\mu_\ell)}{(\|\underline{U}\|^2)^2}$$

$$= n^2E\frac{U_iU_\ell}{(\|\underline{U}\|^2)^2} - n^2\mu_\ell E\frac{U_i}{(\|\underline{U}\|^2)^2}$$

$$- n^2\mu_i E\frac{U_\ell}{(\|\underline{U}\|^2)^2} + n^2\mu_i\mu_\ell E\frac{1}{(\|\underline{U}\|^2)^2} \qquad (4.28)$$

Further we note that from (4.22) we can deduce that

$$E\frac{U_i}{(\|\underline{U}\|^2)^2} = \mu_i E(\frac{1}{p+2K})^2$$

and

$$E\frac{U_\ell}{(\|\underline{U}\|^2)^2} = \mu_\ell E(\frac{1}{p+2K})^2$$

Thus (4.28) is equivalent to

$$\frac{\partial^2}{\partial\mu_\ell\partial\mu_i} E(\frac{1}{\|\underline{U}\|^2})^2 = n^2E\frac{U_iU_\ell}{(\|\underline{U}\|^2)^2} - 2n^2\mu_i\mu_\ell E(\frac{1}{p+2K})^2$$

$$+ n^2\mu_i\mu_\ell E(\frac{1}{\|\underline{U}\|^2})^2 \qquad (4.29)$$

Therefore equating (4.27) and (4.29), we have

$$n^2\mu_i\mu_\ell E\left(\frac{1}{p+2+2K}\right)^2 - 2n^2\mu_i\mu_\ell E\left(\frac{1}{p+2K}\right)^2 + n^2\mu_i\mu_\ell E\left(\frac{1}{\|\underline{U}\|^2}\right)^2$$

$$= n^2 E\frac{U_iU_\ell}{(\|\underline{U}\|^2)^2} - 2n^2\mu_i\mu_\ell E\left(\frac{1}{p+2K}\right)^2 + n^2\mu_i\mu_\ell E\left(\frac{1}{\|\underline{U}\|^2}\right)^2$$

$$\Longrightarrow \quad E\frac{U_iU_\ell}{(\|\underline{U}\|^2)^2} = \mu_i\mu_\ell E\left(\frac{1}{p+2+2K}\right)^2 \qquad (4.30)$$

Denote

$$E\frac{\underline{\bar{X}}\underline{\bar{X}}'}{(\underline{\bar{X}}'\underline{\bar{X}})^2}$$

by M, then M is a pxp matrix whose i-th diagonal element

$$E\frac{U_i^2}{(\|\underline{U}\|^2)^2} = \frac{1}{n}E\left(\frac{1}{p+2K}\right)^2 + \mu_i^2 E\left(\frac{1}{p+2+2K}\right)^2$$

$$= \frac{1}{n}\sum_{k=0}^{\infty}\left(\frac{1}{p+2k}\right)^2 \frac{e^{-\frac{n}{2}\sum_{j=1}^{p}\mu_j^2}\left(\frac{n}{2}\sum_{j=1}^{p}\mu_j^2\right)^k}{k!}$$

$$+ \mu_i^2\sum_{k=0}^{\infty}\left(\frac{1}{p+2+2k}\right)^2 \frac{e^{-\frac{n}{2}\sum_{j=1}^{p}\mu_j^2}\left(\frac{n}{2}\sum_{j=1}^{p}\mu_j^2\right)^k}{k!}$$

and the i-ℓth off-diagonal element

$$E\frac{U_i U_\ell}{(\|\underline{U}\|^2)^2} = \mu_i \mu_\ell E\left(\frac{1}{p+2+2K}\right)^2$$

$$= \mu_i \mu_\ell \sum_{k=0}^{\infty} \left(\frac{1}{p+2+2k}\right)^2 \frac{e^{-\frac{n}{2}\sum_{j=1}^{p}\mu_j^2}\left(\frac{n}{2}\sum_{j=1}^{p}\mu_j^2\right)^k}{k!}$$

Thus substituting

$$E\left(\frac{\overline{\underline{X}}}{\overline{\underline{X}}'\overline{\underline{X}}}\right) ,$$

$$E\frac{\overline{\underline{X}}\overline{\underline{X}}'}{(\overline{\underline{X}}'\overline{\underline{X}})^2}$$

and

$$E(\overline{\underline{X}}\overline{\underline{X}}')$$

into (4.16), we may write

$$e_4 = \frac{h(a)}{k(a)} \tag{4.31}$$

where

$$h(a) = \frac{1}{n} + \Sigma_{12}\underline{\mu}_x\underline{\mu}_x'\Sigma_{21} - 2(p-2)\Sigma_{12}L\underline{\mu}_x'\Sigma_{21}$$

$$+ (p-2)^2\Sigma_{12}M\Sigma_{21} - \frac{1}{n}\Sigma_{12}\Sigma_{21}$$

and

$$k(a) = \frac{1}{n} - \Sigma_{12}\underline{\mu}_x\underline{\mu}_x'\Sigma_{21}H_{p+4}(c;\lambda) - \frac{1}{n}\Sigma_{12}\Sigma_{21}H_{p+2}(c;\lambda)$$

$$+ 2\Sigma_{12}\underline{\mu}_x\underline{\mu}_x'\Sigma_{21}H_{p+2}(c;\lambda)$$

V. BIBLIOGRAPHY

Ahsanullah, M. 1971. On the estimation of means in a bivariate normal distribution with equal marginal variances. Biometrika, 58, 230-233.

Alam, K. and Risvi, M. H. 1967. On Non-Central Chi-squared and Non-Central F Distributions. The Am. Statist. 21 (4): 21-22.

Anderson, T. W. 1958. An Introduction to Multivariate Statistical Analysis. John Wiley, New York.

Bancroft, T. A. 1944. On biases in estimation due to the use of preliminary tests of significance. Ann. Math. Stat., 15, 190-204.

Bancroft, T. A. 1968. Topics in Intermediate Statistical Methods Vol. 1. The Iowa State Univ. Press, Ames.

Bancroft, T. A. 1972. Some recent advances in inference procedures using preliminary tests of significance. Chapter 2, pages 19-30 in T. A. Bancroft, ed. Statistical papers in honour of George W. Snedecor. The Iowa State Univ. Press, Ames.

Bancroft, T. A. and Han, C. P. ca. 1977. Inference Based on Conditional Specification. A Note and a Bibliography. International Statistical Review. (To be published.)

Bennett, B. M. 1952. Estimation of means on the basis of preliminary tests of significance. Ann. Inst. Statist. Math., 4, 31-43.

Bennett, B. M. 1956. On the use of preliminary tests in certain statistical procedures. Ann. Inst. Statist. Math., 8, 45-52.

Brogan, D. R. 1971. Estimating the mean of a normal distribution using a preliminary test of significance and two-stage sampling. Biometrics, 27. 245-246. (Abstract)

Cochran, W. G. 1963. Sampling Techniques. 2nd ed. John Wiley, New York.

Davis, P. J. and Polonsky, I. 1964. Numerical Interpolation, Differentiation and Integration. Pages 875-924 in M. Abramowitz and J. A. Stegun, eds. Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables. National Bureau of Standards Applied Mathematics Series 55.

Han, C. P. 1973a. Regression estimation for bivariate normal distributions. Ann. Inst. Stat. Math., 25, 335-344.

Han, C. P. 1973b. Double sampling with partial information on auxiliary variables. J. Amer. Statist. Assoc., 68, 914-918.

Han, C. P. 1975. Some relationships between noncentral Chi-squared and normal distributions. Biometrika, 62, 213.

Han, C. P. and Bancroft, T. A. 1968. On pooling means when variance is unknown. J. Amer. Statist. Assoc., 63, 1333-1342.

James, W. and Stein, C. 1960. Estimation with Quadratic Loss. Fourth Berkeley Symposium on Mathematical Statistics and Probability, I, 361-379.

Kale, B. K. and Bancroft, T. A. 1967. Inference for some incompletely specified models involving normal approximations to discrete data. Biometrics, 23, 335-348.

Kitagawa, T. 1963. Estimation after preliminary tests of significance. Univ. Cal. Pub. In Statist., 3, 147-186.

Mosteller, F. 1948. On pooling data. J. Amer. Statist. Assoc., 43, 231-242.

Pearson, Karl. 1934. Tables of The Incomplete β-function. University Press, London.

Rao, C. R. 1965. Linear Statistical Inference and Its Applications. John Wiley, New York.

Royden, H. L. 1968. Real Analysis. Macmillan, London.

Snedecor, G. W. and Cochran, W. G. 1967. Statistical Methods. 6th ed. The Iowa State Univ. Press, Ames.

Stein, C. 1955. Inadmissibility of the usual estimator for the mean of a multivariate normal distribution. Third Berkeley Symposium on Mathematical Statistics and Probability, Vol. I, 197-206.

Stein, J. 1960. Multiple regression. Essays in Honor of Harold Hotelling. Stanford Univ. Press, Stanford, Calif.

Sukhatme, P. V. and Sukhatme, B. V. 1954. Sampling Theory of Surveys with Applications. 2nd ed. The Iowa State Univ. Press, Ames.

Thompson, J. R. 1968a. Some shrinkage techniques for estimating the mean. J. Amer. Statist. Assoc., 63, 113-122.

Thompson, J. R. 1968b. Accuracy borrowing in the estimation of the mean by shrinkage to an interval. J. Amer. Statist. Assoc., 63, 953-963.

Tiku, M. L. 1967. Tables of the power of the F-test. J. Amer. Statist. Assoc., 62, 525-539.

VI. ACKNOWLEDGMENTS

I wish to express my deepest appreciation to Professor Chien-Pai Han for his continued interest and encouragement in my research and studies. Without his deep understanding and patience I certainly would not have completed this work.

I also wish to express my appreciation to Dr. H. T. David for his helpful suggestions, Dr. M. Ghosh for many helpful discussions and Dr. H. A. David for his support at various times.

Finally, I am greatly indebted to my husband, Charles N. Esimai, who has been a constant source of encouragement during the past year. He turned the potentially difficult and frustrating final year of graduate study into a time which I shall always fondly remember.

VII. APPENDIX

To justify the result in (3.7) we consider, wlog, the differentiation of T w.r.t. δ.

$$\frac{\partial T}{\partial \delta} = \frac{\partial}{\partial \delta} \int_c^{\infty} \sum_{j=0}^{\infty} e^{-\frac{1}{2}\delta} \frac{1}{j!}\left(\frac{\delta}{2}\right)^j h_{p+2j}(t)dt \tag{A.1}$$

Let

$$g(t,\delta) = \sum_{j=0}^{\infty} e^{-\frac{1}{2}\delta} \frac{1}{j!}\left(\frac{\delta}{2}\right)^j h_{p+2j}(t) \tag{A.2}$$

We can differentiate under the integral sign in (A.1) by the Lebesgue Dominated Convergence Theorem if $\forall$ t,

$$\left| \frac{g(t,\delta+s) - g(t,\delta)}{s} \right| \leq G(t)$$

for every $|s| \leq s_0$ where G(t) is integrable over (c,∞). Using triangular inequality,

$$|g(t,\delta) - g(t,\delta+s)| \leq |g(t,\delta) - g_0(t,\delta+s)| + |g_0(t,\delta+s) - g(t,\delta+s)|$$

where

$$g_0(t,\delta+s) = \sum_{j=0}^{\infty} e^{-\frac{1}{2}(\delta+s)} \frac{1}{j!}\left(\frac{\delta}{2}\right)^j h_{p+2j}(t)$$

then

$$\left| \frac{g(t,\delta) - g(t,\delta+s)}{s} \right| \le \sum_{j=0}^{\infty} e^{-\frac{1}{2}\delta} \left| \frac{(e^{-\frac{1}{2}s} - 1)}{s} \right| \frac{1}{j!}\left(\frac{\delta}{2}\right)^j h_{p+2j}(t)$$

$$+ \sum_{j=0}^{\infty} e^{-\frac{1}{2}(\delta+s)} \left| \frac{\{(\delta+s)^j - \delta^j\}}{s} \right| \frac{1}{2^j j!} h_{p+2j}(t) \tag{A.3}$$

$$\left| \frac{e^{-\frac{1}{2}s} - 1}{s} \right| \le \frac{e^{\frac{1}{2}s_0}}{s_0} \quad \text{for} \quad |s| \le s_0$$

so that the integral of the first term of right side of (A.3)

$$\le \frac{e^{\frac{1}{2}s_0}}{s_0} \int_c^{\infty} \sum_{j=0}^{\infty} e^{-\frac{1}{2}\delta} \frac{1}{j!}\left(\frac{\delta}{2}\right)^j h_{p+2j}(t)dt = \frac{e^{\frac{1}{2}s_0}}{s_0} [1 - H_p(c;\delta)] < \frac{e^{\frac{1}{2}s_0}}{s_0} < \infty$$

Next $e^{-\frac{1}{2}s} \le e^{\frac{1}{2}|s|} \le e^{\frac{1}{2}s_0}$

Also, if we let $f(x) = x^j$, $f'(x) = jx^{j-1}$.

$$(\delta+s)^j = f(\delta+s) = f(\delta) + sf'(\delta+\theta s)$$

by the Mean Value Theorem $= \delta^j + sj(\delta+\theta s)^{j-1}$.

$$\left| \frac{(\delta+s)^j - \delta^j}{s} \right| \le j(\delta+\theta|s|)^{j-1} \le j(\delta+\theta s_0)^{j-1}$$

$$\le j(\delta+s_0)^{j-1} \text{ for } j \ge 1; \ (0 < \theta = \theta(s) < 1)$$

This implies that the integral of the second term of (A.3)

$$\leq e^{\frac{1}{2}s_0} \int_c^\infty \sum_{j=1}^\infty e^{-\frac{1}{2}\delta} \frac{j(\delta+s_0)^{j-1}}{2^j\ j!} h_{p+2j}(t)dt$$

$$= \frac{1}{2} e^{\frac{1}{2}s_0} \int_c^\infty \sum_{j=1}^\infty e^{-\frac{1}{2}\delta} \frac{(\delta+s_0)^{j-1}}{2^{j-1}(j-1)!} h_{p+2j}(t)dt$$

$$= e^{s_0} \int_c^\infty \sum_{j=0}^\infty \frac{e^{-\frac{1}{2}(\delta+s_0)}}{j!} \left(\frac{\delta+s_0}{2}\right)^j h_{p+2j+2}(t)dt$$

$$= e^{s_0} [1-H_{p+2}(c;\delta+s_0)] < e^{s_0} < \infty$$

Therefore

$$\frac{\partial T}{\partial \delta} = \int_c^\infty \frac{\partial}{\partial \delta} \sum_{j=0}^\infty e^{-\frac{1}{2}\delta} \frac{1}{j!}\left(\frac{\delta}{2}\right)^j h_{p+2j}(t)dt$$

Let

$$g_j(t,\delta) = e^{-\frac{1}{2}\delta} \frac{(\frac{\delta}{2})^j}{j!} h_{p+2j}(t) \tag{A.4}$$

In order to differentiate under the summation, we must show that for every fixed t,

$$\left| \frac{g_j(t,\delta) - g_j(t,\delta+s)}{s} \right| \leq U_j(t), \quad |s| \leq s_1$$

where

$$\sum_{j=0}^{\infty} U_j(t) < \infty .$$

$$h_{p+2j}(t) = \frac{e^{-\frac{t}{2}}(\frac{t}{2})^{\frac{p+2j}{2} - 1}}{2\,\Gamma(\frac{p+2j}{2})}$$

$$\Gamma(\frac{p+2j}{2}) = \int_0^{\infty} e^{-x}\, x^{\frac{p+2j}{2} - 1}\, dx$$

$$\geq \int_0^{\frac{t}{2}} e^{-x}\, x^{\frac{p+2j}{2} - 1}\, dx$$

$$\geq e^{-\frac{t}{2}} \int_0^{\frac{t}{2}} x^{\frac{p+2j}{2} - 1}\, dx$$

$$= e^{-\frac{t}{2}} \frac{(\frac{t}{2})^{\frac{p+2j}{2}}}{\frac{p+2j}{2}}$$

Therefore

$$h_{p+2j}(t) \leq \frac{e^{-\frac{t}{2}}\,(\frac{t}{2})^{\frac{p+2j}{2} - 1}}{e^{-\frac{t}{2}}\,(\frac{t}{2})^{\frac{p+2j}{2}}}\,(\frac{p+2j}{2}) = \frac{1}{t}\,\frac{p+2j}{2} \qquad (A.5)$$

By triangular inequality,

$$|g_j(t,\delta) - g_j(t,\delta+s)| \le |g_j(t,\delta) - g_{j0}(t,\delta+s)|$$

$$+ |g_{j0}(t,\delta+s) - g_j(t,\delta+s)|$$

where

$$g_{j0}(t,\delta+s) = e^{-\frac{1}{2}(\delta+s)} \frac{1}{j!}(\frac{\delta}{2})^j h_{p+2j}(t)$$

By similar arguments as above and using (A.5),

$$\left| \frac{g_j(t,\delta) - g_j(t,\delta+s)}{s} \right|$$

$$\le k_1 \sum_{j=0}^{\infty} (p+2j)e^{-\frac{1}{2}\delta} \frac{1}{j!}(\frac{\delta}{2})^j$$

$$+ k_2 \sum_{j=0}^{\infty} (p+2j) \frac{e^{-\frac{1}{2}(\delta+s_1)}}{j!} (\frac{\delta+s_1}{2})^j = \sum_{j=0}^{\infty} U_j(t) < \infty$$

since each term is some moment of a Poisson distribution and k_1, k_2 are fixed constants.

To justify the result in (3.9), we let

$$(\frac{\sqrt{n_1}}{\sqrt{2\pi}})^p \prod_{\substack{j=1 \\ j\neq i}}^{p} e^{-\frac{n_1}{2}(Z^{(j)}-\gamma_x{}^{(j)})^2} dZ^{(j)} = TdT \qquad (A.6)$$

Therefore (3.8)

$$= \int \underset{\bar{A}}{\cdots} \int e^{-\frac{n_1}{2}(Z^{(i)}-\gamma_x{}^{(i)})^2} \, T dZ^{(i)} dT$$

Let

$$e^{-\frac{n_1}{2}(Z^{(i)}-\gamma_x{}^{(i)})^2} T = g(T, \gamma_x{}^{(i)}) \ .$$

We must show that for every fixed T

$$\left| \frac{g(T, \gamma_x^{(i)}+h) - g(T, \gamma_x{}^{(i)})}{h} \right| \leq \Phi(T), |h| \leq h_0 \qquad (A.7)$$

where

$$\int \underset{\bar{A}}{\cdots} \int \Phi(T) dZ^{(i)} dT < \infty$$

Now

$$\left| \frac{e^{-\frac{n_1}{2}[Z^{(i)}-(\gamma_x{}^{(i)}+h)]^2} - e^{-\frac{n_1}{2}(Z^{(i)}-\gamma_x{}^{(i)})}}{h} \right|$$

$$= e^{-\frac{n_1}{2}[Z^{(i)}-\gamma_x{}^{(i)}]^2} \left| \frac{e^{\frac{n_1}{2} h(Z^{(i)}-\gamma_x{}^{(i)})} e^{-\frac{n_1}{2} h^2} - 1}{h} \right| \qquad (A.8)$$

If $f_1(h) = e^{kh}$, $f_1'(h) = ke^{kh}$. Let

$$e^{\frac{n_1}{2}h(Z^{(i)}-\gamma_x{}^{(i)})} = f_1(h) = f_1(0) + hf_1'(\theta_1 h) \qquad \text{(A.9)}$$

$$= 1 + h\frac{n_1}{2}(Z^{(i)}-\gamma_x{}^{(i)})e^{\frac{n_1\theta_1 h}{2}(Z^{(i)}-\gamma_x{}^{(i)})}$$

Let

$$e^{-\frac{n_1}{2}h^2} = f_2(h) = f_2(0) + hf_2'(\theta_2 h)$$

$$= 1 - n_1\theta_2 h^2 e^{-\frac{n_1\theta_2^2 h^2}{2}} \qquad \text{(A.10)}$$

Multiplying (A.9) and (A.10) we have

$$e^{\frac{n_1}{2}h(Z^{(i)}-\gamma_x{}^{(i)})}\, e^{-\frac{n_1}{2}h^2}$$

$$= 1 + h\frac{n_1}{2}(Z^{(i)}-\gamma_x{}^{(i)})e^{\frac{n_1\theta_1 h}{2}(Z^{(i)}-\gamma_x{}^{(i)})} - n_1\theta_2 h^2 e^{-\frac{n_1\theta_2^2 h^2}{2}}$$

$$- \frac{n_1^2}{2}\theta_2 h^3(Z^{(i)}-\gamma_x{}^{(i)})e^{-\frac{n_1\theta_2^2 h^2}{2} + \frac{n_1\theta_1 h}{2}(Z^{(i)}-\gamma_x{}^{(i)})}$$

Therefore for $|h| \le h_1$, the expression in (A.8)

$$\leq e^{-\frac{n_1}{2}[Z^{(i)}-\gamma_x^{(i)}]^2}\{1 + \frac{h_1 n_1}{2}\left|Z^{(i)}-\gamma_x^{(i)}\right| e^{\frac{n_1 h_1}{2}|Z^{(i)}-\gamma_x^{(i)}|}$$

$$+ n_1 h_1^2 + \frac{n_1^2 h_1^3}{2}|Z^{(i)}-\gamma_x^{(i)}| e^{\frac{n_1 h_1}{2}|Z^{(i)}-\gamma_x^{(i)}|}\}$$

$$= e^{-\frac{n_1}{2}[Z^{(i)}-\gamma_x^{(i)}]^2} + c_1|Z^{(i)}-\gamma_x^{(i)}| e^{-\frac{n_1}{2}[Z^{(i)}-\Psi(\gamma_x^i)]^2}$$

$$+ c_2 e^{-\frac{n_1}{2}[Z^{(i)}-\gamma_x^{(i)}]^2}$$

where

$$c_1 = \frac{n_1 h_1}{2}(1 + n_1 h_1^2)$$

and $\Psi(\gamma_x^{(i)}$ is some function of $\gamma_x^{(i)}$. Finally we note

$$(1+c_2)\int \cdots \int_{\bar{A}} \prod_{j=1}^{P} \frac{\sqrt{n_1}}{\sqrt{2\pi}} e^{-\frac{n_1}{2}(Z^{(j)}-\gamma_x^{(j)})^2} dZ^{(j)}$$

$$+ c_1 \int \cdots \int_{\bar{A}} |Z^{(i)}-\gamma_x^{(i)}| \frac{\sqrt{n_1}}{\sqrt{2\pi}} e^{-\frac{n_1}{2}[Z^{(i)}-\Psi(\gamma_x^{(i)})]^2} \prod_{\substack{j=1 \\ j\neq i}}^{P} \frac{\sqrt{n_1}}{\sqrt{2\pi}}$$

$$e^{-\frac{n_1}{2}(Z^{(j)}-\gamma_x^{(j)})^2} dZ^{(i)} dZ^{(j)}$$

$$= (1+c_2)\int\underset{\bar{A}}{\cdots}\int\frac{\sqrt{n_1}}{\sqrt{2\pi}}\, e^{-\frac{n_1}{2}(Z^{(i)}-\gamma_x{}^{(i)})^2}\, T\, dZ^{(i)}dT$$

$$+ c_1\int\underset{\bar{A}}{\cdots}\int|Z^{(i)}-\gamma_x{}^{(i)}\ \frac{\sqrt{n_1}}{\sqrt{2\pi}}\, e^{-\frac{n_1}{2}(Z^{(i)}-\Psi(\gamma_x^{(i)}))^2}\, T\, dZ^{(i)}dT$$

$$= \int\underset{\bar{A}}{\cdots}\int\Phi(T)\,dZ^{(i)}dT < \infty$$