Classification of small class association schemes coming from certain combinatorial objects

by

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A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Mathematics

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2005

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ABSTRACT

We explore two- or three-class association schemes. We study aspects of the structure of the relation graphs in association schemes which are not easily revealed by their parameters and spectra. The purpose is to develop some combinatorial methods to characterize the graphs and classify the association schemes, and also to delve deeply into several specific classification problems. We work with several combinatorial objects, including strongly regular graphs, distance-regular graphs, the desarguesian complete set of mutually orthogonal Latin squares, orthogonal arrays, and symmetric Bush-type Hadamard matrices, all of which give rise to many small-class association schemes. We work within the framework of the theory of association schemes.

Our focus is placed on the search for all isomorphism classes of association schemes and characterization of small-class association schemes of specific order. In particular, we examine two-class association schemes (strongly regular graphs) of order 64 and their three-class fission schemes. After we collect 'feasible' parameter sets for the putative association schemes, we make an attempt to check the realization (existence) of the parameter sets and describe the structure of the schemes chiefly by investigating the structure of their relation graphs. In the course of this thesis, we find a new way to construct orthogonal arrays and investigate their implications for strongly regular graphs, symmetric Bush-type Hadamard matrices, and three-class association schemes. We obtain several results regarding the characterization and classification of two- or three-class association schemes of order 64.

CHAPTER 1. Introduction

In this chapter we first give a brief history of the theory of association schemes. We then briefly explain our motivation and then give an overview of the thesis, outlining main results. We provide the background material in Appendix A.

The concept of (symmetric) association schemes was introduced by Bose and Shimamoto (1952) in the study of experimental designs. This concept plays a fundamental role in the analysis and classification of partially balanced incomplete block designs which were originally defined by Bose and Nair (1939). A similar concept was introduced earlier in the work of Schur (1933) and developed mostly in connection with the theory of group characters and permutation groups. It was due to the work of Delsarte (1973) that association schemes were proven to be a useful tool for the study of a wide range of combinatorics including design theory, coding theory and algebraic graph theory. The theory of association schemes has been developed rapidly since then, and has been established as a branch of mathematics over the last decade or two through the work of many algebraists, combinatorialists, geometers and group theorists (cf. [1, 72, 62]).

A d-class association scheme $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ of order $v = |X|$ may be considered a decomposition of a complete (di)graph $K_v = (X, X \times X)$ of v vertices into regular digraphs $\Gamma_i = (X, R_i)$, so that $R_1, R_2, \ldots R_d$ form a partition of $X \times X$ together with $R_0 = \{(x, x) : x \in X\}$ and satisfy certain regularity conditions. If the association scheme is symmetric, that is, all relations R_i are symmetric (binary) relations, then the (non-trivial) relation graphs $\Gamma_i = (X, R_i)$, $i = 1, 2, \ldots, d$, are undirected simple regular graphs.

An association scheme all of whose relation graphs Γ_i are connected is called a primitive association scheme. If the relation graph of any non-trivial relation is disconnected then the scheme is called imprimitive. Given an association scheme, sometimes we can obtain other association schemes either by merging two or more non-trivial relations into one or by refining a non-trivial relation into two or more. The former is called fusion process and the latter is called fission. A fusion scheme of a primitive association scheme is primitive, and a fission scheme of an imprimitive association scheme is imprimitive. Primitive association schemes are building blocks of many larger-class imprimitive association schemes.

Two association schemes $(X, \{R_i\}_{0 \le i \le d})$ and $(Y, \{S_j\}_{0 \le j \le d})$ are isomorphic if there is a bijection $\varphi : X \to Y$ such that for each $i = 1, 2, \ldots, d$, $\{(\varphi(x), \varphi(y)) : (x, y) \in R_i\} = S_j$ for some j. In this case, the bijection φ induces a permutation σ on the set $\{1, 2, \ldots, d\}$ such that the graphs (X, R_i) and (Y, S_{i^σ}) are isomorphic for every i. So, if there is a relation graph of one scheme whose isomorphic copy is not found among the relation graphs of the other scheme, then the two schemes cannot be isomorphic. Also the automorphism group of any relation graph contains the automorphisms of the association scheme.

Recently much research has been going on in the investigation and computer classification of two or three-class association schemes by many authors including Brouwer, Coolsaet, van Dam, Degraer, Fon-Der-Flaass, Haemers, and Spence (cf. [36, 17] and [18] and their references.) In [16], van Dam described several constructions of symmetric 3-class association schemes mainly through the study of eigenvalues and multiplicities, and gave a list of feasible parameter sets for symmetric 3-class association schemes on at most 100 vertices. (Here the feasibility is subject to the known necessary conditions on the parameter sets to have an association scheme satisfying the parameter sets. This will be discussed in detail later.) The list included many feasible parameter sets whose realizability has not been checked.

The structure of relation graphs often reveals crucial information for the structure of association schemes, especially for the small-class association schemes. Thus it is important to study relation graphs in the classification of association schemes. The relation graphs of a 2-class symmetric association scheme are strongly regular graphs. In this case, two relation graphs are complementary to each other. Having a strongly regular graph is equivalent to having a symmetric 2-class association scheme.

In [47], Jørgensen, Jones, Klin and Song showed several ways to construct 3-class imprimitive non-symmetric association schemes from doubly regular (m, r)-team tournaments. Inspired by their work we have started investigating the links between orientations of the completely multipartite strongly regular graph $\overline{m \circ K_r}$ and association schemes based on characterization of doubly regular (m, r)-team tournaments.

Our aim is to verify whether each of the given feasible parameter sets is realized or not. If such a scheme exists, then we check whether the scheme is unique for the given parameter set. We do this by surveying all feasible parameter sets available in the existing literature (for example, [8, 17, 16, 46]). We explore the structure of relation graphs of existing or putative association schemes. We do this on a case by case basis for the schemes of order 64 hoping to develop a general theory in the future. There are many combinatorial structures including orthogonal arrays and Bush-type Hadamard matrices that are related to the schemes of order 64.

Our results are discussed in Chapters 2, 3, and 4. In Chapter 2, we investigate the structures of two strongly regular graphs (SRGs) with the same parameters $(v, k, \lambda, \mu) = (64, 28, 12, 12)$. We do not know how many strongly regular graphs with these parameters exist. These two cospectral graphs are the halved-folded 8-cube and the graph of 'Latin square type' denoted by $L_4(8)$, which comes from the orthogonal array $\mathrm{OA}(4, 8)$. We begin the chapter with an investigation of the halved-folded Hamming 8-cube. We study some local structures of the halved-folded 8-cube, which lead us to derive necessary and sufficient conditions for any cospectral strongly regular graphs to be isomorphic to the halved-folded 8-cube. We describe a new way to construct orthogonal arrays $\mathrm{OA}(t, n)$ and strongly regular graphs $L_t(n)$ from a complete set of mutually orthogonal Latin squares. Our new construction method will be used in the subsequent chapter in which we construct Bush-type Hadamard matrices. We then investigate the local structure of $L_4(8)$ and show the structural difference between $L_4(8)$ and the halved-folded 8-cube.

In Chapter 3, we introduce a way to construct a symmetric Bush-type Hadamard matrix from a set of mutually orthogonal Latin squares. Whenever we have a symmetric Bush-type

Hadamard matrix, we can obtain imprimitive three-class association schemes. We discuss how to obtain such association schemes from given symmetric Bush-type Hadamard matrices. From a given symmetric Bush-type Hadamard matrix we can obtain other symmetric Bush-type Hadamard matrices. New Bush-type Hadamard matrices obtained from old sometimes yield cospectral but non-isomorphic strongly regular graphs. As an example we obtain another $\mathrm{SRG}(64, 28, 12, 12)$ that is not isomorphic to either of the two cospectral graphs discussed in Chapter 2.

In Chapter 4, we survey the known feasible parameter sets and collect all schemes that realize each parameter set. We then classify and characterize the relation graphs of the schemes in an attempt to discover other schemes that employ non-isomorphic cospectral graphs as their relation graphs. We discover a few new schemes that realize some feasible parameter sets. We classify 3-class association schemes of order 64 according to their fusion and fission relationship to known strongly regular graphs.

In Chapter 5, we discuss a few related problems and topics that are not resolved in this thesis.

In Appendix A, we provide all the terms and basic facts that are used throughout the thesis. Appendix B consists of many tables that are used as examples.

CHAPTER 2. Strongly regular graphs with parameters (64, 28, 12, 12)

In this chapter we characterize strongly-regular graphs (SRGs) that are obtained from Hamming 8-cube via halving and folding process and its cospectral graph obtained from orthogonal array OA$(4, 8)$. Both the halved-folded 8-cube and the strongly regular graph obtained from OA$(4, 8)$ have the same parameters but they are not isomorphic. In doing this we also introduce an easy way to construct orthogonal arrays OA(t, q) from the desarguesian complete set of mutually orthogonal Latin squares (MOLS). The basic information not covered in this chapter can be found in Appendix A or their references.

2.1 Halved and folded Hamming cubes

The structure of Hamming n-cubes is well known in connection with Hamming codes. Hamming n-cubes, denoted by $H(n, 2)$, for $n \geq 2$, are imprimitive, bipartite, antipodal distance-regular graphs (DRGs). The halved and folded graphs of Hamming n-cubes are often called *halved n-cubes* and *folded n-cubes*, respectively. We recall some facts on halved and folded n-cubes first, and then study the structure of the halved-folded 8-cube in the following section.

The halved n-cube H_n may be defined by the words of a binary code consisting of all even weight words of length n, words being adjacent if they differ in two coordinate entries; i.e., if their Hamming distance is 2. H_n has 2^{n-1} vertices with valency $k = \binom{n}{2} = n(n-1)/2$ and diameter $d = \left[\frac{n}{2}\right]$. It has the following parameters and spectrum. For $n \geq 3$,

$$b_j = \frac{1}{2}(n-2j)(n-2j-1), \qquad c_j = j(2j-1);$$

$$\theta_j = \frac{1}{2}(n-2j)^2 - \frac{1}{2}n, \qquad m_j = \binom{n}{j}, \qquad (2j < n).$$

(If n is even, then $m_d = \frac{1}{2}\binom{n}{d}$.)

Proposition 2.1.1. [58, 68] The halved n-cubes are characterized by their intersection array. That is, all graphs which are cospectral with the halved n-cubes are isomorphic.

For $n \geq 3$, the parameters, eigenvalues θ_j, and multiplicities m_j of the folded n-cube is given by $d = \left[\frac{n}{2}\right]$,

$$b_j = n - j, \qquad c_j = \begin{cases} j & \text{for} \quad 2j < n \\ n & \text{for} \quad 2j = n \end{cases}$$

$$\theta_j = n - 4j, \qquad m_j = \binom{n}{2j}.$$

Proposition 2.1.2. [6, 69] For $n \neq 6$, the folded n-cube is uniquely characterized by its intersection array, and for the folded 6-cube, there are precisely three non-isomorphic cospectral graphs.

Example 2.1.3. Folded 7-cube. This DRG with diameter 3 has the following intersection matrices.

$$B_1 = B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 7 & 0 & 2 & 0 \\ 0 & 6 & 0 & 3 \\ 0 & 0 & 5 & 4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 6 & 0 & 3 \\ 21 & 0 & 10 & 6 \\ 0 & 15 & 10 & 12 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 4 \\ 0 & 15 & 10 & 12 \\ 35 & 20 & 20 & 18 \end{bmatrix}.$$

By Proposition 2.1.2, we know that this graph is uniquely determined by its parameters (or by its spectrum). The spectrum of the graph and the character table of related P-polynomial association scheme are given by

$$\begin{pmatrix} 7 & 3 & -1 & -5 \\ 1 & 21 & 35 & 7 \end{pmatrix}, \qquad P = \begin{array}{c} \begin{bmatrix} 1 & 7 & 21 & 35 \\ 1 & 3 & 1 & -5 \\ 1 & -1 & -3 & 3 \\ 1 & -5 & 9 & -5 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 21 \\ 35 \\ 7 \end{array}$$

Example 2.1.4. Halved 7-cube. This DRG is also uniquely determined by its parameters by Proposition 2.1.1. Its intersection matrices are

$$B_1 = B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 21 & 10 & 6 & 0 \\ 0 & 10 & 12 & 15 \\ 0 & 0 & 3 & 6 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 10 & 12 & 15 \\ 35 & 20 & 18 & 20 \\ 0 & 5 & 4 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 6 \\ 0 & 5 & 4 & 0 \\ 7 & 2 & 0 & 0 \end{bmatrix}.$$

The spectrum of this graph and the character table of the associated P-polynomial scheme are given by

$$\begin{pmatrix} 21 & 1 & -3 & 9 \\ 1 & 21 & 35 & 7 \end{pmatrix}, \qquad P = \begin{bmatrix} 1 & 21 & 35 & 7 \\ 1 & 1 & -5 & 3 \\ 1 & -3 & 3 & -1 \\ 1 & 9 & -5 & -5 \end{bmatrix} \begin{array}{c} 1 \\ 21 \\ 35 \\ 7 \end{array}$$

Remark 2.1.5. The association schemes coming from the folded 7-cube and the halved 7-cube are isomorphic. Also, both of these schemes are three-class fission schemes of the two-class association scheme associated to the halved-folded 8-cube, an SRG$(64, 28, 12, 12)$, which will be discussed in the following section.

2.2 Local structure of the halved-folded 8-cube

The halved 8-cube (H_8) is an antipodal DRG with diameter 4, and the folded 8-cube (F_8) is a complete bipartite DRG with diameter 4. Their intersection arrays are

$$B(H_8) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 28 & 12 & 6 & 0 & 0 \\ 0 & 15 & 16 & 15 & 0 \\ 0 & 0 & 6 & 12 & 28 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B(F_8) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 8 & 0 & 2 & 0 & 0 \\ 0 & 7 & 0 & 3 & 0 \\ 0 & 0 & 6 & 0 & 8 \\ 0 & 0 & 0 & 5 & 0 \end{bmatrix}.$$

Although these distance-regular graphs are uniquely determined by their parameters, the folding of H_8, which is isomorphic to the halving of F_8, is not uniquely determined by its parameters. In this section we study this halved-folded 8-cube, and in the later section we describe its cospectral graph $L_4(8)$.

The *halved-folded* $2l$*-cube*, for $l \geq 3$, is a distance-regular graph with diameter $d := \lfloor \frac{l}{2} \rfloor$ and parameters

$$b_j = (l-j)(2l-2j-1), \qquad c_j = j(2j-1), \qquad (0 \leq j \leq d-1)$$

$$c_d = \begin{cases} d(2d-1), & \text{if } l \text{ is odd;} \\ 2d(2d-1), & \text{if } l \text{ is even.} \end{cases}$$

The eigenvalues and multiplicities are

$$\theta_j = 2(l-2j)^2 - l, \qquad m_j = \binom{2l}{2j}.$$

Halved-folded $2l$-cubes are characterized by their intersection array if $2l$ is sufficiently large (cf. [11]). Metsch [56] showed that the halved-folded $2l$-cubes of diameter $d \geq 5$ and the larger halved-folded $2l$-cube of diameter 4, are uniquely determined by their intersection arrays. The small diameter cases with $l = 8, 7, 6, 5, 4$ are remained to be studied more with regard to the characterization. We describe the halved-folded 8-cube Γ. The following theorem describes the structure of the induced subgraph on 14 vertices.

Theorem 2.2.1. Let Γ be the halved-folded graph of the Hamming cube $H(8,2)$. Let v and w be arbitrary vertices in Γ, and let Δ be the induced subgraph on $N = \Gamma(v) \cap \Gamma(w)$.

1. Δ is a regular graph with valency 6 if v and w are adjacent.

2. Δ is a disjoint union of two $\mathrm{SRG}(6,4,2,4)$ if v and w are non-adjacent.

3. If v and w are adjacent, then the induced subgraph on $S = N \cup \{v, w\}$ is the graph depicted in Figure 2.1.

Proof The vertices of Γ, which are equivalence classes of antipodal pairs of H_8, can be represented by a word of length eight over $\{0,1\}$. Let $v = x_1x_2\cdots x_8$. Then, the vertices adjacent to v can also be represented by 8-tuples that have either two or six coordinate places different from v. Without loss of generality, let $w = x_1x_2\cdots x_6\bar{x}_7\bar{x}_8$ be an adjacent vertex of v. With this notation, the set $N = \Gamma(v)\cap\Gamma(w)$ consists of all the vertices $x = a_1a_2\cdots a_8$ where $a_i \in \{0,1\}$ and $a_i = x_i$ for all $i = 1,2,\ldots,8$ except for one i from $\{1,2,\ldots,6\}$ and one i from $\{7,8\}$. That is,

$$
\begin{aligned}
N = \Gamma(v)\cap\Gamma(w) = \{ \quad & v_1 = \bar{x}_1x_2x_3x_4x_5x_6\bar{x}_7x_8 & w_1 = \bar{x}_1x_2x_3x_4x_5x_6x_7\bar{x}_8 \\
& v_2 = x_1\bar{x}_2x_3x_4x_5x_6\bar{x}_7x_8 & w_2 = x_1\bar{x}_2x_3x_4x_5x_6x_7\bar{x}_8 \\
& v_3 = x_1x_2\bar{x}_3x_4x_5x_6\bar{x}_7x_8 & w_3 = x_1x_2\bar{x}_3x_4x_5x_6x_7\bar{x}_8 \\
& v_4 = x_1x_2x_3\bar{x}_4x_5x_6\bar{x}_7x_8 & w_4 = x_1x_2x_3\bar{x}_4x_5x_6x_7\bar{x}_8 \\
& v_5 = x_1x_2x_3x_4\bar{x}_5x_6\bar{x}_7x_8 & w_5 = x_1x_2x_3x_4\bar{x}_5x_6x_7\bar{x}_8 \\
& v_6 = x_1x_2x_3x_4x_5\bar{x}_6\bar{x}_7x_8 & w_6 = x_1x_2x_3x_4x_5\bar{x}_6x_7\bar{x}_8 \ \}
\end{aligned}
$$

For example, $v_1 = \bar{x}_1x_2x_3x_4x_5x_6\bar{x}_7x_8$ is an element of N and v_1 has exactly 6 neighbors in N. They are the ones having the same coordinates as v_1 either except for (i) the first coordinate and one of the coordinate positions $2,3,4,5,6$, or except for (ii) the seventh and eighth coordinates. Similarly, we can see that every vertex in N has valency 6 in the induced subgraph of Γ on N.

For the second part, consider the pair of vertices that are not adjacent, say $v = x_1x_2\cdots x_8$ and $w = x_1x_2x_3x_4\bar{x}_5\bar{x}_6\bar{x}_7\bar{x}_8$. Then the vertices in $N = \Gamma(v)\cap\Gamma(w)$ have either (1) two different coordinates from v among the coordinate positions $5,6,7,8$, or (2) two different coordinates from w among the coordinate positions $1,2,3,4$. So, $N = N_1 \cup N_2$ with

$$
\begin{aligned}
N_1 = \{ \quad & x_1x_2x_3x_4\bar{x}_5\bar{x}_6x_7x_8, \quad x_1x_2x_3x_4\bar{x}_5x_6\bar{x}_7x_8, \quad x_1x_2x_3x_4\bar{x}_5x_6x_7\bar{x}_8, \\
& x_1x_2x_3x_4x_5\bar{x}_6\bar{x}_7x_8, \quad x_1x_2x_3x_4x_5\bar{x}_6x_7\bar{x}_8, \quad x_1x_2x_3x_4x_5x_6\bar{x}_7\bar{x}_8 \ \}; \\
N_2 = \{ \quad & \bar{x}_1\bar{x}_2x_3x_4\bar{x}_5\bar{x}_6\bar{x}_7\bar{x}_8, \quad \bar{x}_1x_2\bar{x}_3x_4\bar{x}_5\bar{x}_6\bar{x}_7\bar{x}_8, \quad \bar{x}_1x_2x_3\bar{x}_4\bar{x}_5\bar{x}_6\bar{x}_7\bar{x}_8, \\
& x_1\bar{x}_2\bar{x}_3x_4\bar{x}_5\bar{x}_6\bar{x}_7\bar{x}_8, \quad x_1\bar{x}_2x_3\bar{x}_4\bar{x}_5\bar{x}_6\bar{x}_7\bar{x}_8, \quad x_1x_2\bar{x}_3\bar{x}_4\bar{x}_5\bar{x}_6\bar{x}_7\bar{x}_8 \ \}.
\end{aligned}
$$

It is clear that no vertex in N_1 is adjacent to any vertex in N_2. Furthermore, every vertex $a_1a_2\cdots a_8 \in N_1$ is adjacent to others except for $a_1a_2a_3a_4\bar{a}_5\bar{a}_6\bar{a}_7\bar{a}_8$ in N_1, and a vertex $a_1a_2\cdots a_8 \in N_2$ is adjacent to others except for $\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4a_5a_6a_7a_8$ in N_2. Therefore, for each $z \in N$, $|\Gamma(z) \cap N| = 4$. Also, every pair of vertices in each component N_i has either two common neighbors or four common neighbors depending on whether they are adjacent or not. For the third part, as in the part 1, let $v = x_1x_2\cdots x_8$ and $w = x_1x_2\cdots x_6\bar{x}_7\bar{x}_8$, and consider two adjacent vertices $v_1 = \bar{x}_1x_2x_3x_4x_5x_6\bar{x}_7x_8$ and $w_1 = \bar{x}_1x_2x_3x_4x_5x_6x_7\bar{x}_8$ both in N. Then, we see that all the vertices in $\{v_1, v, w\} \cup (\Gamma(v_1) \cap N) - \{w_1\}$ are adjacent to each other in Γ. Similarly, we see that all the vertices in $\{w_1, v, w\} \cup (\Gamma(w_1) \cap N) - \{v_1\}$ are adjacent to each other in Γ, too. This tells us that the vertices v and w belong to at least two cliques of size 8. However, the induced subgraph on $((\Gamma(v_1) \cup \Gamma(w_1)) \cap N) \cup \{v, w\} = S$ has only five further edges may be unnoticed in the above two cliques; they are the edges $\{v_i, w_i\}$ for $i = 2, 3, 4, 5, 6$. In sum, we see that the induced subgraph on S has the configuration depicted as in the Figure 2.1. □

Corollary 2.2.2. In the halved-folded 8-cube Γ,

1. each adjacent pair of vertices of Γ belongs to two cliques of size 8,
2. any three mutually adjacent vertices are adjacent with 6 other vertices in common,
3. any mutually adjacent four vertices have 4 other vertices in their common neighbors.

Proof It is an immediate consequence of Part 3 of Theorem 2.2.1. □

We now have the following theorem which describes the structure of the subconstituent of the halved-folded 8-cube.

Theorem 2.2.3. Let Γ be the halved-folded 8-cube. Let v be a vertex of Γ. Then the induced subgraph of Γ on $\{v\} \cup \Gamma(v)$ possesses 8 distinct cliques of size 8.

Proof Here we use the same notations as in the proof of Part 1 and 3 of the previous theorem; namely, $v = x_1x_2\cdots x_8$, $w = x_1x_2\cdots x_6\bar{x}_7\bar{x}_8$ and $N = \{v_1, v_2, \ldots, v_6, w_1, w_2, \ldots, w_6\}$. Consider the induced subgraph on the set $\{v, w, v_1, v_2, \ldots, v_6\}$ which is a K_8 containing the pair $\{v_1, v\}$. The adjacent pair v and v_1 together with their common neighbors should form another configuration depicted by Figure 2.1, by Part 3 of the previous theorem. In fact, the five vertices

$$u_2 = \bar{x}_1\bar{x}_2x_3x_4x_5x_6x_7x_8 \quad u_3 = \bar{x}_1x_2\bar{x}_3x_4x_5x_6x_7x_8$$

$$u_4 = \bar{x}_1x_2x_3\bar{x}_4x_5x_6x_7x_8 \quad u_5 = \bar{x}_1x_2x_3x_4\bar{x}_5x_6x_7x_8$$

$$u_6 = \bar{x}_1x_2x_3x_4x_5\bar{x}_6x_7x_8$$

which are in $\Gamma(v) - N$, together with vertices v, v_1 and w_1 form another K_8 containing v and v_1.

Notice that u_i and v_i are also adjacent for each $i = 2, 3, 4, 5, 6$. Now consider the pair v and v_2. They belong to the clique formed by $\{v, v_1, v_2, v_3, v_4, v_5, v_6, w\}$. In addition to the vertices in this clique, the pair also has common neighbors w_2 and u_2 among all the vertices we have counted so far. So there will be four other vertices, that are in $\Gamma(v) \cap \Gamma(v_2)$ and that are not accounted for, say, t_3, t_4, t_5, t_6, such that $v, v_2, w_2, u_2, t_3, t_4, t_5, t_6$ form another clique containing v and v_2.

By continuing this process, we can enumerate all the vertices of $\Gamma(v)$ as well as all the cliques appeared in the induced graph on $\{v\} \cup \Gamma(v)$. The number of vertices added to the original 13 vertices in $\{w, v_1, \ldots, v_6, w_1, \ldots, w_6\}$ can be enumerated as $13+5+4+3+2+1+0 = 28 = |\Gamma(v)|$, which indicates that there are 8 cliques including the original two cliques and six new cliques coming with v_1 (or w_1), v_2 (or w_2), …, v_6 (or w_6). This completes the proof. □

Corollary 2.2.4. Let Γ be the halved-folded 8-cube.

1. For any clique K_4 of Γ, all but four vertices of Γ are adjacent to the K_4.
2. For any clique K_6 of Γ, all vertices of Γ are adjacent to the clique.

Proof Both can be verified by enumerating entire neighbors of the given cliques. Recall that every pair of vertices has 12 common neighbors. Also, from Part 3 of Theorem 2.2.1 and its corollary, we know that any mutually adjacent triple has 6 common neighbors, and any mutually adjacent quadruple has 4 common neighbors. Therefore, the total number of vertices adjacent to any of the four vertices of K_4 is counted as $28 \cdot 4 - 12 \cdot 6 + 6 \cdot 4 - 4 \cdot 1 = 60$ by the inclusion and exclusion principle. For 2, the number is $28{\cdot}6 - 12{\cdot}15 + 6{\cdot}20 - 4{\cdot}15 + 3{\cdot}6 - 2{\cdot}1 = 64$, so the proof follows. □

Having the configuration on $S = (\Gamma(v) \cap \Gamma(w)) \cup \{v, w\}$, we have seen that we could build up the induced graph on $\Gamma(v) \cup \{v\}$ by adding the other 15 adjacent vertices of v to S as in Theorem 2.2.3. In this process, the way the 15 vertices are added does not depend on the label of the vertices, but it depends on the parameters of the graph and the 14-vertex configuration depicted in Figure 2.1. In the same manner, we can then add another 15 vertices that are adjacent to w to have a combined structure on 44 vertices in $\Gamma(v) \cup \Gamma(w)$. The remaining 20 vertices in $V(\Gamma) - (\Gamma(v) \cup \Gamma(w))$ will be added as neighbors of the vertices in $\Gamma(v) \cup \Gamma(w)$. This building-up process is uniquely determined up to isomorphism according to the parameters of the graph and the 14-vertex configuration depicted in Figure 2.1. So we have the following:

Corollary 2.2.5. Let Γ be an SRG$(64, 28, 12, 12)$, and let v and w be two adjacent vertices of Γ. Suppose the induced subgraph on the subset $S = (\Gamma(v) \cap \Gamma(w)) \cup \{v, w\}$ of $V(\Gamma)$ has the configuration depicted in Figure 2.1. Then Γ is isomorphic to halved-folded 8-cube.

2.3 Construction of orthogonal arrays

In this section we describe how we can obtain orthogonal arrays from a set of mutually orthogonal Latin squares. We know that there are at most $n - 1$ MOLS of order n. Having a complete set of MOLS leads to many exciting consequences. For instance, it is an important observation that the existence of $n-1$ mutually orthogonal Latin squares of order n is equivalent to the existence of a projective plane of order n. We will see that having a complete set of MOLS also implies many ways to construct orthogonal arrays in the current section and relevant combinatorial structures in the following sections.

As a generalization of the concept of the orthogonality of squares, we say that $m \times n$ matrices A and B are *orthogonal* if (a_{ij}, b_{ij}) are all distinct. Then clearly the following two $n \times n$ matrices R and C are orthogonal:

$$R = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots \\ n & n & \cdots & n \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & \cdots & n \end{bmatrix}.$$

The two $1 \times n^2$ matrices (row vectors of length n^2)

$$R^{()} = [11\ldots1\ 22\ldots2\ \cdots\ nn\ldots n] \quad \text{and} \quad C^{()} = [12\ldots n\ 12\ldots n\ \cdots\ 12\ldots n]$$

are also orthogonal. It is also clear that any Latin square of order n is orthogonal to both R and C. Conversely, if a matrix A is orthogonal to both R and C, then A is a Latin square of order n. In what follows, our alphabet is $Q = \{1, 2, \ldots, n\}$ unless otherwise specified.

Definition 2.3.1. An *orthogonal array* $\mathrm{OA}(t, n)$ is a $t \times n^2$ array with entries in Q such that any two rows are orthogonal; i.e., in any $2 \times n^2$ submatrix all possible columns occur precisely once; so, all pairs $(i, j) \in Q \times Q$ appear on the columns of any two rows.

The following well-known theorem gives the relation between a set of $t-2$ mutually orthogonal Latin squares and an orthogonal array $\mathrm{OA}(t, n)$. We sketch the proof since it shows a way to construct an orthogonal array from a set of MOLS.

Theorem 2.3.2. The existence of $t - 2$ MOLS of order n is equivalent to the existence of $\mathrm{OA}(t, n)$.

Proof Let $A_3, A_4, \ldots, A_t$ be $t - 2$ MOLS of order n, and let $A_p = \left[a_{ij}^{(p)}\right]$, $i, j \in \{1, 2, \ldots, n\}$

for $3 \le p \le t$. Then we define the following $t \times n^2$ array A.

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & \cdots & n & n & \cdots & n \\ 1 & 2 & \cdots & n & 1 & 2 & \cdots & n & \cdots & 1 & 2 & \cdots & n \\ a_{11}^{(3)} & a_{12}^{(3)} & \cdots & a_{1n}^{(3)} & a_{21}^{(3)} & a_{22}^{(3)} & \cdots & a_{2n}^{(3)} & \cdots & a_{n1}^{(3)} & a_{n2}^{(3)} & \cdots & a_{nn}^{(3)} \\ a_{11}^{(4)} & a_{12}^{(4)} & \cdots & a_{1n}^{(4)} & a_{21}^{(4)} & a_{22}^{(4)} & \cdots & a_{2n}^{(4)} & \cdots & a_{n1}^{(4)} & a_{n2}^{(4)} & \cdots & a_{nn}^{(4)} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{11}^{(t)} & a_{12}^{(t)} & \cdots & a_{1n}^{(t)} & a_{21}^{(t)} & a_{22}^{(t)} & \cdots & a_{2n}^{(t)} & \cdots & a_{n1}^{(t)} & a_{n2}^{(t)} & \cdots & a_{nn}^{(t)} \end{bmatrix}$$

Then the entries of A are clearly in Q. It is also clear that the mutual orthogonality between any two rows is inherited from that of the matrices $R, C, A_3, A_4, \ldots, A_t$.
On the other hand, let A be a $t \times n^2$ array whose entries are in Q, and let each two rows of A be orthogonal so that $(a, b) \in Q \times Q$ appears exactly once in a fixed pair of two distinct rows. Then by permuting the indices of rows and columns we can have the first two rows $R^{()}$ and $C^{()}$ as in the above. The rest of rows will be converted into Latin squares as the rows are orthogonal to both $R^{()}$ and $C^{()}$. The mutual orthogonality of the rows will guarantee that the resulting Latin squares are mutually orthogonal. This completes the proof. $\square$

The above construction is available subject to the existence of $t-2$ MOLS of order n. However, for the infinite sequence of prime powers q, orthogonal arrays $\mathrm{OA}(t, q)$ are available for every $2 \le t \le q$ because there is a complete set of MOLS for every q. For the prime power order case, we find another fairly easy way to construct $\mathrm{OA}(t, q)$ by using a desarguesian complete set of MOLS (cf. Appendix A.5). This construction of orthogonal array will be used when we construct symmetric Bush-type Hadamard matrices later.

Theorem 2.3.3. Let $L_1, L_\alpha, \ldots, L_{\alpha^{q-2}}$ be the desarguesian complete set of MOLS of order q over $Q = \mathbb{F}_q = \{0, 1, \alpha, \alpha^2, \ldots, \alpha^{q-2}\}$. Let L_0 be the $q \times q$ array all of whose rows are identically $[0\ 1\ \alpha\ \alpha^2\ \cdots\ \alpha^{q-2}]$. Let $A = [L_0|L_1|L_\alpha|\cdots|L_{\alpha^{q-2}}]$ be the $q \times q^2$ array obtained by juxtaposing the q arrays. Then (1) A is an $\mathrm{OA}(q, q)$; (2) any t rows of A for $2 \le t \le q$ form an $\mathrm{OA}(t, q)$. (This $\mathrm{OA}(t, q)$ will be denoted by $\mathrm{OA}_t(q)$ in what follows.)

Proof From the construction of the desarguesian complete set of MOLS, the set of polynomials $\{f_a : a \in \mathbb{F}_q^*\}$ represents the desarguesian complete set $\{L_a : a \in \mathbb{F}_q^*\}$ of MOLS via $f_a(x,y) = ax+y$. That is, each $q \times q$ block L_a in A is represented by f_a for $a \in \mathbb{F}_q^*$, and each x determines each row of L_a with $\{ax+y : y \in \mathbb{F}_q\}$; in particular, different rows are created as x varies over $\mathbb{F}_q$. On the other hand, if we fix x and let a and y vary over $\mathbb{F}_q$, then $ax+y$ will fill the entries of the entire row of A corresponding to x, and this row comes from a Latin square represented by the polynomial f_x via $f_x(a,y) = xa+y$. In this correspondence, L_0 may be considered to be represented by f_0 with $a=0$. That is, each row of A, except for the first being identified as L_0, is also corresponding to one of MOLS $L_1, L_\alpha, \dots L_{\alpha^{q-2}}$, and thus A is an $\mathrm{OA}(q,q)$ constructed as in the Theorem 2.3.2. It is an immediate consequence that any t rows of this $\mathrm{OA}(q,q)$ is an $\mathrm{OA}_t(q)$ for any $2 \le t \le q$. □

We note that at least $t-1$ rows (after possibly except for the first row) of $\mathrm{OA}_t(q)$ are corresponding to $t-1$ MOLS each of which is represented by $f_x(a,y)$ for $x \in \mathbb{F}_q^*$.

2.4 Multipartite SRGs $L_t(q)$ and $\mathrm{OA}_t(q)$

Given an $\mathrm{OA}_t(q)$ with $2 \le t \le q$, we can obtain a graph $\Gamma = \Gamma(\mathrm{OA}_t(q)) = L_t(q)$ as follows: The vertices of Γ are the q^2 columns of the orthogonal array (i.e., column vectors $[a_1a_2\cdots a_t]^T$ of length t), and two vertices are adjacent if and only if they have the same entry in one coordinate position (cf. [28]). This graph $L_t(q)$ is often called as a Latin square graph and is a multipartite strongly regular graph as we will see. (Also see [52, Theorem 7.29].)

Theorem 2.4.1. The graph $\Gamma = L_t(q)$ defined as above is a strongly regular graph with parameters $(q^2,\ (q-1)t,\ q-2+(t-1)(t-2),\ t(t-1))$.

Proof Let A be the $\mathrm{OA}_t(q)$. The number of columns of $\mathrm{OA}_t(q)$ is q^2 and thus $v = q^2$.
Since every symbol $i \in Q$ occurs exactly q times in each row of A, there are q columns which have the same symbol $a \in Q$ in the ith coordinate position. This is true for each $i = 1, 2, \dots, t$. Therefore, given a fixed vertex $x = [x_1x_2\cdots x_t]^T$, there are $q-1$ distinct vertices whose ith

coordinate is x_i. These vertices do not share another common coordinate with x due to the mutual orthogonality of rows. So the valency k of Γ is $t(q-1)$.

Now for the enumeration of common neighbors of two adjacent vertices, without loss of generality, let $x = [x_1 x_2 \cdots x_t]^T$ and $y = [x_1 y_2 y_3 \cdots y_t]^T$. Then there are $q-2$ other vertices that have the first coordinate x_1, and that are common neighbors of x and y. For each $j = 2, 3, \ldots, t$, among the $q-1$ neighbors of x having the jth coordinate x_j, there is exactly one vertex which has y_i in the ith coordinate and thus being a neighbor of y for each $i \in \{2, 3, \ldots, t\} - \{j\}$. This is due to the property of A that every possible combination $(x, y) \in Q \times Q$ occurs exactly once in any pair of rows. As there are $t-1$ possible j and $t-2$ choices of i for each j, we have $\lambda = (q-2) + (t-1)(t-2)$.

In the same manner, we can enumerate the common neighbors of a non-adjacent pair which is $\mu = t(t-1)$. Since all these numbers are independent from the choice of vertices, the proof follows. $\square$

Theorem 2.4.2. The SRG $\Gamma = L_t(q)$ with q a prime power, $2 \leq t \leq q$, is a q-partite graph with each part of size q. In particular, $L_q(q)$ is completely q-partite.

Proof For the technical convenience, we consider the SRG constructed in Theorem 2.3.3 by using the orthogonal array A obtained from the desarguesian complete set of MOLS of order q. By using the exact notations used in Theorem 2.3.3, we may write the $\mathrm{OA}_t(q)$ by

$$A = \left[\begin{array}{ccccc|ccccc|c} 0 & 1 & \alpha_1 & \cdots & \alpha_{q-2} & 0 & 1 & \alpha_1 & \cdots & \alpha_{q-2} & \cdots \\ 0 & 1 & \alpha_1 & \cdots & \alpha_{q-2} & 1 & 1+1 & 1+\alpha_1 & \cdots & 1+\alpha_{q-2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \alpha_1 & \cdots & \alpha_{q-2} & \alpha_{t-2} & \alpha_{t-2}+1 & \alpha_{t-2}+\alpha_1 & \cdots & \alpha_{t-2}+\alpha_{q-2} & \cdots \end{array}\right.$$

$$\left.\begin{array}{c|ccccc} \cdots & 0 & 1 & \alpha_1 & \cdots & \alpha_{q-2} \\ \cdots & \alpha_{q-2} & \alpha_{q-2}+1 & \alpha_{q-2}+\alpha_1 & \cdots & \alpha_{q-2}+\alpha_{q-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \alpha_{t-2}\alpha_{q-2} & \alpha_{t-2}\alpha_{q-2}+1 & \alpha_{t-2}\alpha_{q-2}+\alpha_1 & \cdots & \alpha_{t-2}\alpha_{q-2}+\alpha_{q-2} \end{array}\right]$$

Then the first q vertices of Γ coming from the first q columns are clearly not adjacent to each other. The next q vertices coming from the columns of the Latin square L_1 are not adjacent to each other because each symbol of alphabet occurs once in each row and column of L_1. Due to the same reason, we see that q vertices in each of the q blocks are never adjacent to each other within the blocks. So the graph is q-partite. The latter part is evident from the mutual orthogonality; i.e., one column of the ith Latin square has exactly one coordinate in common with every column of the jth Latin square for all $j \neq i$. (Thus it is imprimitive by Proposition A.2.4.) $\square$

When we choose $q = 2t$, the $\mathrm{OA}_t(2t)$ gives $L_t(2t)$ which is an $\mathrm{SRG}(4t^2, 2t^2 - t, t^2 - t, t^2 - t)$. In particular, when $q = 8$, we obtain an $\mathrm{SRG}(64, 28, 12, 12)$ whose parameters coincide with those of halved-folded 8-cube. However, it will be clear that they are non-isomorphic cospectral pairs as we will see the structural description of $L_4(8)$ in the following section.

2.5 The structure of $L_4(8)$

In the previous section we have seen that the vertex set of $L_4(8)$ can be partitioned into 8 parts, $V_1, V_2, \ldots, V_8$ such that (i) vertices in the same part are not adjacent to each other, and (ii) every vertex in V_i is adjacent to exactly four vertices in V_j for each $j \neq i$. This means that the adjacency matrix of Γ, which we denote it C, can be arranged as an 8×8 block matrix $C = [C_{ij}]$ where blocks C_{ij} are also of size 8×8 for $i, j \in \{1, 2, \ldots, 8\}$ with $C_{ii} = 0$. We know that the complement $\overline{\Gamma}$ of an $\mathrm{SRG}(64, 28, 12, 12)$ is an SRG with parameters $(64, 35, 18, 20)$ and the adjacency matrix $\overline{C} = J - C - I$. If we delete the matrix $D = (I_8 \otimes J_8) - I$ from $\overline{C}$, then the remaining matrix $J - C - (I_8 \otimes J_8)$ must be the adjacency matrix of the graph $\Delta = \overline{\Gamma} - (8 \circ K_8)$.

Theorem 2.5.1. Let Γ be $L_4(8)$ with the partition of the vertex set $V(\Gamma) = V_1 \cup V_2 \cup \cdots \cup V_8$ as above. Let Δ be the graph obtained from the complement of Γ by deleting all the edges that link between the vertices within the same parts. Then Δ is also an SRG with parameters $(64, 28, 12, 12)$.

Proof In the graph Δ, every vertex has valency $35 - 7 = 28$ as the edges between the vertices in the same part have been eliminated from the $\overline{\Gamma}$; so, Δ is a regular graph with $k = 28$. Any two vertices v and w that are adjacent in Δ belong to two different parts and they are adjacent in $\overline{\Gamma}$. Let $v \in V_i$ and $w \in V_j$ for $i \neq j$. Then, in Γ, $\Gamma(v) \cap \Gamma(w) \subset \bigcup_{k=1}^{8} V_k - (V_i \cup V_j)$. Since each element $z \in \Gamma(v) \cap \Gamma(w)$ has 4 neighbors in every part except for its own part, so in $\bigcup_{k=1}^{8} V_k - (V_i \cup V_j)$, v and w have $12 = 48 - (24 + 24 - 12)$ common non-neighbors in Γ. Hence any two vertices that are adjacent in Δ have 12 common neighbors; i.e., $\lambda(\Delta) = 12$. Any two vertices in the same part of Δ have $18 - 6 = 12$ vertices that are adjacent to both. Two vertices from two different parts that are non-adjacent in Δ (so are in $\overline{\Gamma}$) must have been adjacent in Γ. Each of these vertices had 4 adjacent vertices in each part except for the part to where it belongs. Therefore, among 48 vertices in 6 other parts to where neither belongs, they have 12 common neighbors and 12 common non-neighbors in Γ as before. Hence any two vertices belonging to two different parts that are non-adjacent in Δ have 12 common neighbors, so we have $\mu(\Delta) = 12$. $\square$

Theorem 2.5.2. Let Γ be an SRG$(64, 28, 12, 12)$ obtained from OA$_4(8)$. Then for any $v \in V(\Gamma)$, $w \in \Gamma(v)$ the induced subgraph on $(\Gamma(v) \cap \Gamma(w)) \cup \{v, w\}$ forms a configuration depicted by Figure 2.2.

Proof Consider the OA$_4(8)$ described in the proof of Theorem 2.4.2 by replacing t and q by 4 and 8, respectively, and without loss of generality, let v be the vertex corresponding to the column $v = [a_1 a_2 a_3 a_4]^T$, $a_i \in \mathbb{F}_8$ for $i = 1, 2, 3, 4$. Then there are seven other vertices each of whose first coordinate is a_1. Let $w = [a_1 b_2 b_3 b_4]^T$, $b_j \in \mathbb{F}_8$ for $j = 2, 3, 4$, and let the other six vertices be $v_1, v_2, \ldots, v_6$. Then $v, w, v_1, v_2, \ldots, v_6$ form a clique K_8. There are six other vertices which are adjacent to both v and w. These six vertices have the following properties: (i) the first coordinate of any of these is not a_1, (ii) either the second coordinate is a_2, or the third coordinate is a_3, or the fourth coordinate is a_4, and (iii) if its second coordinate is a_2, then either the third coordinate is b_3 or the fourth coordinate is b_4; if its third coordinate is a_3, then either the second coordinate is b_2 or the fourth coordinate is b_4; if the fourth coordinate is a_4, then either the second coordinate is b_2 or the third coordinate is b_3. Therefore, the six vertices are of the columns $[\Box a_2 b_3 \Box]^T$, $[\Box a_2 \Box b_4]^T$, $[\Box\Box a_3 b_4]^T$, $[\Box b_2 a_3 \Box]^T$, $[\Box b_2 \Box a_4]^T$, $[\Box\Box b_3 a_4]^T$ where each $\Box$ has a suitable element of $\mathbb{F}_8$. If we label these vertices with $w_1, w_2, \ldots, w_6$ in order, then we observe that these six vertices can not form a clique. In order to form a clique, vertices w_1, w_3, w_5 must share a common first coordinate, vertices w_2, w_4, w_6 must share a common first coordinate, w_1 and w_4 must have the same fourth coordinate, w_2 and w_5 must have the same third coordinate, and w_3 and w_6 must have the same second coordinate. However, it is impossible to fulfill all these conditions simultaneously by the mutual orthogonality of the Latin squares used. On the contrary, from the construction of the orthogonal array and the definition of Latin square, we observe that the first coordinates of the pairs w_1 and w_4, w_2 and w_5, and w_3 and w_6 coincide, which implies the fourth coordinates of w_1 and w_4 must be different, the third coordinates of w_2 and w_5 cannot be the same, and the second coordinates w_3 and w_6 cannot be the same either. In sum, the adjacency between these vertices indicates the configuration depicted in the bottom half of Figure 2.2. $\Box$

Corollary 2.5.3. In the graph $\Gamma = L_4(8)$, three mutually adjacent vertices x, v, w are adjacent with either 6 or 4 other vertices in common.

Proof It can be easily observed that the adjacent triple v_1, v, w in the Figure 2.2 has six common neighbors $v_2, v_3, v_4, v_5, v_6, w_1$, while the triple w_1, v, w has four common neighbors v_1, w_2, w_4, w_6. The neighbors of other triples can be counted in a similar manner. $\Box$

Remark 2.5.4. Figure 2.1 and Figure 2.2 show the difference between the local structure of $L_4(8)$ and that of halved-folded 8-cube. It shows that there are at least two non-isomorphic

SRGs with parameters $(64, 28, 12, 12)$. In the following chapter, we will see that there is another non-isomorphic cospectral SRG.

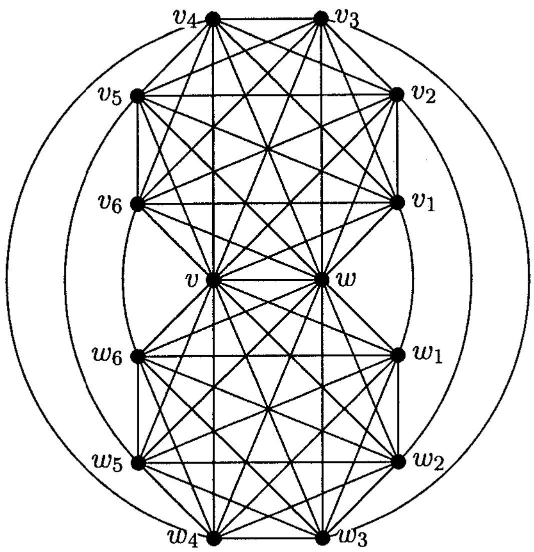

Figure 2.1 The induced subgraph on $(\Gamma(v) \cap \Gamma(w)) \cup \{v, w\}$ in halved-folded 8-cube.

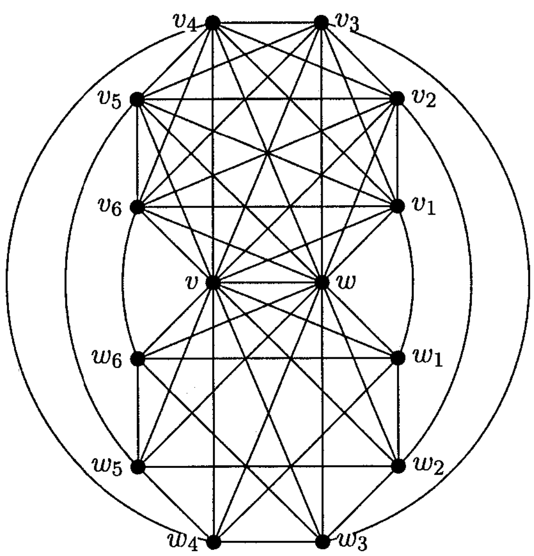

Figure 2.2 The induced subgraph on $(\Gamma(v) \cap \Gamma(w)) \cup \{v, w\}$ in $L_4(8)$.

CHAPTER 3. Symmetric Bush-type Hadamard matrices (SBHMs) and three-class imprimitive association schemes

An $n \times n$ matrix $H = [h_{ij}]$ with entries $h_{ij} = \pm 1$ is called a *Hadamard matrix of order n* if $HH^T = nI$. So, any two distinct rows of such H, row i and row j, satisfy $\sum_{k=1}^{n} h_{ik}h_{jk} = 0$. The order of Hadamard matrix must be 1 or 2 or a multiple of 4. A Hadamard matrix H is *symmetric* if $H = H^T$. A Hadamard matrix is *normalized* if all entries in its first row and first column are equal to 1. Two Hadamard matrices are *equivalent* if one can be transformed into the other by a series of row or column permutations and multiplications by -1. In this chapter, we concentrate on a special type of Hadamard matrices, called *Bush-type Hadamard matrices* which are associated with strongly regular graphs of our interest.

Definition 3.0.5. A Hadamard matrix H of order $4n^2$ is called a *Bush-type Hadamard matrix* if $H = [H_{ij}]$, where H_{ij} are blocks of order $2n$, $H_{ii} = J_{2n}$ for all i, and $H_{ij}J_{2n} = J_{2n}H_{ij} = 0$ for all $i, j \in \{1, 2, \ldots, 2n\}$, $i \neq j$.

Symmetric Bush-type Hadamard matrices of order $4n^2$ are specially interesting because if there exists a projective plane of order $2n$, then there is a symmetric Bush-type Hadamard matrix of order $4n^2$ (cf. K. A. Bush [10]); consequently the non-existence of such a matrix of order $4n^2$ would be of great significance. Wallis [70] showed that a symmetric Bush-type Hadamard matrices of order $4n^2$ is obtained from $n-1$ MOLS of order $2n$. Goldbach and Claasen [32] showed that certain 3-class association schemes can give rise to symmetric Bush-type Hadamard matrices. Recently Kharaghani and his coauthors (for example, [43, 44, 45, 50]) proved that such matrices are useful for constructions of symmetric designs and strongly regular graphs.

Symmetric Bush-type Hadamard matrices of order $4n^2$ yield strongly regular graphs with parameters $(4n^2, 2n^2-n, n^2-n, n^2-n)$. In this chapter we explore construction methods of Bush-type Hadamard matrices and associated strongly regular graphs, and then investigate the structure of strongly regular graphs and that of three-class fission schemes of the SRGs. Our aim is to construct two types of three-class fission schemes, namely symmetric three-class association schemes and non-symmetric three-class association schemes related to $(2n, 2n)$-team tournaments of type II. As a source of such association schemes, we construct symmetric Bush-type Hadamard matrices and obtain several non-isomorphic cospectral strongly regular graphs from such matrices.

3.1 Construction of SBHMs

In [70], Wallis gave a graphical interpretation of a symmetric Bush-type Hadamard matrix of order $4n^2$, and gave two construction methods as in the following two theorems:

Theorem 3.1.1. Given an integer n, having a symmetric Bush-type Hadamard matrix of order $4n^2$ is equivalent to have a strongly regular graph with parameters $(4n^2, 2n^2-n, n^2-n, n^2-n)$ whose vertex set can be partitioned into $2n$ sets of size $2n$, such that (i) no vertices in the same set are adjacent and (ii) a vertex in a given set is adjacent to exactly n of the vertices in any other set.

Remark 3.1.2. The proof is straightforward from the definition. Given a symmetric Bush-type Hadamard matrix H, we can obtain such an SRG; namely, the vertices of the graph correspond to the rows of H, the vertices i and j are adjacent if and only if H has (i, j) entry -1. On the other hand, given the adjacency matrix A of such a $2n$-partite SRG, if we set $H = J - 2A$, then H is equivalent to the desired Hadamard matrix.

Theorem 3.1.3. (1) If there exist $n-1$ MOLS of order $2n$, then there is a symmetric Bush-type Hadamard matrix of order $4n^2$.

(2) If there exists a Hadamard matrix of order $4n$, then there is a symmetric Bush-type Hadamard matrix of order $16n^2$.

Remark 3.1.4. For (1), the existence of such a Hadamard matrix is guaranteed by the construction of the corresponding strongly regular graph described in the previous theorem. Suppose $L_1, L_2, \ldots, L_{n-2}$ are $n-2$ MOLS of order $2n$. We construct a graph Γ whose vertices are the $4n^2$ ordered pairs $(1,1),(1,2),\ldots,(2n,2n)$. Two distinct vertices (a,b) and (c,d) are adjacent if and only if (i) $a=c$, (ii) $b=d$, or (iii) L_i has the same entry in positions (a,b) and (c,d) for some $i \in \{1,2,\ldots,n-2\}$. This construction is similar to the construction of $L_n(2n)$. It is easy to check that the graph is an $\mathrm{SRG}(4n^2, 2n^2-n, n^2-n, n^2-n)$. Assume that $n-1$ MOLS exist of order $2n$; denote by Γ the graph constructed using $n-2$ of them, and let L be the remaining unused square. Partition the vertex set $V(\Gamma)$ into sets $V_1, V_2, \ldots, V_{2n}$ where V_j contains all pairs (a,b) such that the (a,b) entry of L is j. Then it is clear that no two vertices in the same set are adjacent, that is Γ is a $2n$-partite graph with each part of size $2n$. It is also clear that for any vertex $(a,b) \in V_i$, (a,b) is adjacent to exactly n vertices in V_j, $i \neq j$.

For the proof of (2), we use the Kharaghani's construction (cf. [49]). Let K be a normalized Hadamard matrix of order $4n$. Let c_1, c_2, ..., c_{4n} be the columns of K. Considering the columns as $4n \times 1$ column matrices, we have $4n \times 4n$ rank-one matrices $c_i c_i^T$ for $i = 1,2,\ldots,4n$. Let C_i be $c_i c_i^T$ for each $i = 1,2,\ldots,4n$. (or if we let $C_1 = c_1 c_1^T$ and C_i be either $c_i c_i^T$ or -$c_i c_i^T$ for $i = 2,3,\ldots,4n$) Then the following are easily verified.

(1) $C_i^T = C_i$, for $i = 1,2,\ldots,4n$;

(2) $C_1 = J_{4n}$, $C_i J_{4n} = J_{4n} C_i = 0$, for $i = 2,3,\ldots,4n$ since for each C_i the number of positive ones and that of negative ones are equally $2n$;

(3) $C_i C_j^T = 0$ for $i \neq j,\ i,j \in \{1,2,\ldots,4n\}$ since the dot product of c_i and c_j equals zero for $i \neq j,\ i,j \in \{1,2,\ldots,4n\}$;

(4) $\sum_{i=1}^{4n} C_i C_i^T = 16n^2 I_{4n}$ since $\sum_{i=1}^{4n} C_i C_i^T = 4n \sum_{i=1}^{4n} C_i = 4nKK^T$ and $KK^T = 4nI_{4n}$.

Now consider a symmetric Latin square L of order $4n$ with entries $1,2,\ldots,4n$ with constant diagonal 1. (We can always find such a Latin square.) Replace each entry i of L by C_i. We then obtain a Bush-type Hadamard matrix of order $16n^2$. This completes the construction.

Kharaghani [50] conjectured that Bush-type Hadamard matrices of order $4n^2$ exist for every odd integer n. Although we can always construct a Bush-type Hadamard matrices of order $16n^2$ whenever there is a Hadamard matrix of order $4n$, it is not much obvious about the existence of Bush-type Hadamard matrices of order $4n^2$ for certain n, especially for odd n. Examples are known for $n = 3, 5, 9$ by Janko, Kharaghani and Tonchev. (See [43, 44, 45] for the details.) Muzychuk and Xiang [57] have constructed symmetric Bush-type Hadamard matrices of order $4m^4$ for all odd integers m by using reversible Hadamard difference sets. Yet there are many cases that are open. Nevertheless, for any n being of power of 2, we know fairly easy way to construct symmetric Bush-type Hadamard matrix from the $\mathrm{OA}_t(2t)$, for $t = 2^l$, $l \geq 1$ that is constructed in 2.3.3 from the desarguesian complete set of MOLS of order $2t$.

Theorem 3.1.5. Let $t = 2^l$ for some $l \geq 1$, and let A be the adjacency matrix of $2t$-partite strongly regular graph $\Gamma = L_t(2t) = \Gamma(\mathrm{OA}_t(2t))$ constructed in 2.4.1 using an $\mathrm{OA}_t(2t)$ constructed in 2.3.3. Then the matrix $H = J - 2A$ is a symmetric Bush-type Hadamard matrix of order $4t^2$.

Proof First we note that $\Gamma = \mathrm{SRG}(4t^2, t(2t-1), t(t-1), t(t-1))$. Let $H = J-2A = [H_{ij}]$, where H_{ij} are blocks of order $2t$. Then clearly all the entries of H are ± 1. $HH^T = (J-2A)(J-2A)^T = (J-2A)(J-2A) = J^2 - 4AJ + 4A^2 = 4t^2 J - 4t(2t-1)J + 4\{t(2t-1)I + t(t-1)A + t(t-1)(J - I - A)\} = 4t^2 I$ since $AJ = kJ$ and by Part (b) in Proposition A.2.5 in Appendix A. Thus H is a Hadamard matrix of order $4t^2$.

Now from the structure of the adjacency matrix of Γ, we see that H is symmetric. $H_{ii} = J_{2t}$ for $1 \leq i \leq 2t$. By the construction of Γ obtained from $\mathrm{OA}_t(2t)$ each row in H_{ij} for $i \neq j$ has exactly the same number of positive ones and negative ones. Thus $H_{ij}J_{2t} = J_{2t}H_{ij} = 0$ for all $i, j \in \{1, 2, \ldots, 2t\}$ with $i \neq j$. Therefore, H is a symmetric Bush-type Hadamard matrix of order $4t^2$. $\square$

Remark 3.1.6. Let H be the symmetric Bush-type Hadamard matrix of order $4t^2$ and let Γ denote the corresponding $L_t(2t)$. Consider the matrix $\hat{H} = -H + 2D$ where $D = I_{2t} \otimes J_{2t}$. Then

$$\hat{H}\hat{H}^T = (-H + 2D)(-H + 2D)^T = HH^T - 4DH + 4D^2 = HH^T.$$

That is, if we switch the plus and minus signs for all entries except for those in the diagonal blocks of H, then we have another symmetric Bush-type Hadamard matrix $\hat{H}$ of order $4t^2$. This Bush-type Hadamard matrix $\hat{H}$ also has a corresponding SRG $L_t(2t)$, say $\hat{\Gamma}$. It is also easy to see that $\hat{\Gamma}$ is obtained from the complement $\overline{\Gamma}$ of Γ by deleting all the edges between the vertices in each part corresponding to each diagonal block H_{ii} of H for $i = 1, 2, \ldots, 2t$; i.e.,

$$\hat{\Gamma} = \overline{\Gamma} - ((2t) \circ K_{2t}).$$

where $(2t) \circ K_{2t}$ denotes the disjoint union of $2t$ copies of the complete graph on $2t$ vertices. $\hat{\Gamma}$ is not isomorphic to Γ in general (cf. [22]). However, for the case with $t = 4$, we can easily see that $\hat{\Gamma}$ is isomorphic to Γ. This is because when the four rows of $\mathrm{OA}_8(8)$ are associated to Γ the remianing four rows of $\mathrm{OA}_8(8)$ give rise to $\hat{\Gamma}$; it can be shown that every column in the four rows associated to Γ can be found from the set of columns of the remaining four rows or that of columns of a permutation of the remaining four rows (associated to $\hat{\Gamma}$).

Whenever we obtain a symmetric Bush-type Hadamard matrix H, we have $\hat{H}$, and thus, we have a pair of strongly regular graphs with the same parameters. Such a pair, Γ and $\hat{\Gamma}$, are called 'twin' by Kharaghani. We note that Bonato, Holzmann, and Kharaghani [5] have also observed that given a Bush-type Hadamard matrix H of order $4n^2$, the matrix $M = H - I_{2n} \otimes J_{2n}$ contains two $\mathrm{SRG}(4n^2,\ 2n^2 - n,\ n^2 - n,\ n^2 - n)$. From these twin pairs, we can always obtain imprimitive symmetric three-class association schemes. We will discuss more about this later in this chapter.

Remark 3.1.7. The symmetric Bush-type Hadamard matrix H constructed in Theorem 3.1.5 also has the properties that are satisfied by the one constructed by Kharaghani in Remark 3.1.4. Let $H = [H_{ij}]$ of order $4t^2$ be the symmetric Bush-type Hadamard matrix constructed in Theorem 3.1.5. Then, for $i, j, k, l \in \{1, 2, \ldots, 2t\}$,

(i) H_{ij} are $2t \times 2t$ symmetric arrays with entries $1, -1$;

(ii) $H_{ii} = J_{2t}$, $H_{ij}J_{2t} = J_{2t}H_{ij} = 0$, for all i, j with $i \neq j$;

(iii) $H_{ij}H_{ik}^T = 0$ for all $j \neq k$;

(iv) for any i and k, $\{H_{ij} : j \in \{1, 2, \ldots, 2t\}\} = \{H_{kl} : l \in \{1, 2, \ldots, 2t\}\}$.

The first two follow easily by the way of construction of the symmetric Bush-type Hadamard matrices in Theorem 3.1.5.

For (iii), if $i = j$ or $i = k$, then it follows from (ii). Suppose $i \neq j$ and $i \neq k$. Then from the construction of $\mathrm{OA}_t(2t)$ in Theorem 2.3.3, $\mathrm{OA}_t(2t)$ consists of $2t$ parts of size $t \times 2t$. Recall that the $2t$-partite strongly regular graph $L_t(2t)$ has parameters $(4t^2, 2t^2 - t, t^2 - t, t^2 - t)$ and let $A = [A_{ij}]$ be its adjacency matrix where each A_{ij} is a $2t \times 2t$ block matrix. Then each row of A_{ij} has $\frac{t}{2}$ common positive ones with each row of A_{ik} for $j \neq k$. This implies that each pair of rows of H_{ij} and H_{ik} has t entries with the same signs and t entries with opposite signs. Hence $H_{ij}H_{ik}^T = 0$ for all $j \neq k$.

(iv) follows directly from the relationship between the Latin squares in the desarguesian complete set of MOLS.

3.2 Examples of SBHMs

Example 3.2.1. We give an example of a symmetric Bush-type Hadamard matrix of order 64 with the way of Part (2) in Remark 3.1.4. Let K and L be the following normalized Hadamard matrix of order 8 and a symmetric Latin square of order 8 repectively.

Define H_1 as follows:

$$K = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} = [c_1|c_2| \cdots |c_8]$$

$$L = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\ 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\ 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\ 6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\ 7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 \\ C_2 & C_1 & C_4 & C_3 & C_6 & C_5 & C_8 & C_7 \\ C_3 & C_4 & C_1 & C_2 & C_7 & C_8 & C_5 & C_6 \\ C_4 & C_3 & C_2 & C_1 & C_8 & C_7 & C_6 & C_5 \\ C_5 & C_6 & C_7 & C_8 & C_1 & C_2 & C_3 & C_4 \\ C_6 & C_5 & C_8 & C_7 & C_2 & C_1 & C_4 & C_3 \\ C_7 & C_8 & C_5 & C_6 & C_3 & C_4 & C_1 & C_2 \\ C_8 & C_7 & C_6 & C_5 & C_4 & C_3 & C_2 & C_1 \end{bmatrix}$$

where for each $i = 1, 2, \ldots, 8$, let $C_i = c_i c_i^T$.

Then H_1 is a symmetric Bush-type Hadamard matrix of order 64.

Example 3.2.2. Let A be the adjacency matrix of $L_4(8)$, the strongly regular graph obtained from $\mathrm{OA}_4(8)$ described in Section 2.4, and $H_2 = -(J - 2A) + 2(I_8 \otimes J_8)$ be the symmetric Bush-type Hadamard matrix of order 64 constructed as in Remark 3.1.6.

Then

$$H_2 = \begin{bmatrix} C_1 & C_5 & C_2 & C_8 & C_3 & C_6 & C_7 & C_4 \\ C_5 & C_1 & C_8 & C_2 & C_6 & C_3 & C_4 & C_7 \\ C_2 & C_8 & C_1 & C_5 & C_7 & C_4 & C_3 & C_6 \\ C_8 & C_2 & C_5 & C_1 & C_4 & C_7 & C_6 & C_3 \\ C_3 & C_6 & C_7 & C_4 & C_1 & C_5 & C_2 & C_8 \\ C_6 & C_3 & C_4 & C_7 & C_5 & C_1 & C_8 & C_2 \\ C_7 & C_4 & C_3 & C_6 & C_2 & C_8 & C_1 & C_5 \\ C_4 & C_7 & C_6 & C_3 & C_8 & C_2 & C_5 & C_1 \end{bmatrix}$$

where C_i's are the same C_i's used in Example 3.2.1.

Example 3.2.3. Here is another symmetric Bush-type Hadamard matrix we obtained from Kharaghani's matrix H_1 by permuting the positions of blocks C_2, C_3, C_5.

$$H_3 = \begin{bmatrix} C_1 & C_5 & C_2 & C_4 & C_3 & C_6 & C_7 & C_8 \\ C_5 & C_1 & C_4 & C_2 & C_6 & C_3 & C_8 & C_7 \\ C_2 & C_4 & C_1 & C_5 & C_7 & C_8 & C_3 & C_6 \\ C_4 & C_2 & C_5 & C_1 & C_8 & C_7 & C_6 & C_3 \\ C_3 & C_6 & C_7 & C_8 & C_1 & C_5 & C_2 & C_4 \\ C_6 & C_3 & C_8 & C_7 & C_5 & C_1 & C_4 & C_2 \\ C_7 & C_8 & C_3 & C_6 & C_2 & C_4 & C_1 & C_5 \\ C_8 & C_7 & C_6 & C_3 & C_4 & C_2 & C_5 & C_1 \end{bmatrix}$$

where C_i's are the same C_i's used in Example 3.2.1.

Remark 3.2.4. One interesting observation we can make is that the strongly regular graphs we obtain from the above three symmetric Bush-type Hadamard matrices are not isomorphic to each other. The twin graphs obtained from H_1 are isomorphic to halved-folded 8-cube.

(cf. Appendix B.1.) The twin graphs obtained from H_2 are isomorphic to $L_4(8)$. Recall that in Chapter 2, we have seen that every pair of adjacent vertices in halved-folded 8-cube and in $L_4(8)$ lies in a clique of size 8. However, the local structure of the third graph obtained from H_3 reveals that there are six vertices in the neighbors of a vertex v, none of which belongs to a clique of size 8 together with v. Neither of the twin graphs obtained from H_3 are isomorphic to the ones obtained from H_1 and H_2. So we have seen that there exist at least three non-isomorphic strongly regular graphs with parameters $(64, 28, 12, 12)$.

3.3 Non-symmetric 3-class schemes coming from SBHMs

Let $m \circ K_r$ denote the disjoint union of m copies of the complete graph on r vertices and let $\overline{m \circ K_r}$ denote its complement, the complete multipartite graph with m independent sets of size r. Let Γ be an orientation of $\overline{m \circ K_r}$, i.e., every edge $\{x, y\}$ in $\overline{m \circ K_r}$ is replaced by one of the arcs (x, y) or (y, x). Then we say that Γ is an *(m, r)-team tournament.*

Definition 3.3.1. An (m, r)-team tournament Γ with adjacency matrix A is said to be *doubly regular* if there exist integers k, α, β, γ such that

(i) Γ is regular with valency k,

(ii) $A^2 = \alpha A + \beta A^T + \gamma(J - I - A - A^T)$.

Note that A^T is the adjacency matrix of the graph obtained from Γ by reversing the direction of all arcs and that $J - I - A - A^T$ is the adjacency matrix of $m \circ K_r$. Since A^2 counts the number of directed paths of length 2, the equation $A^2 = \alpha A + \beta A^T + \gamma(J - I - A - A^T)$ means that the number of directed paths of length 2 from a vertex x to a vertex y is α if $(x, y) \in \Gamma$, β if $(y, x) \in \Gamma$, and γ if $\{x, y\} \in m \circ K_r$.

Lemma 3.3.2. [47] Let Γ be a doubly regular (m, r)-team tournament and let α, β, γ be as in definition 3.3.1. Let $V(\Gamma) = V_1 \cup \ldots \cup V_m$ be the partition of the vertex set into m independent sets of size r. For $x \in V_i$, let $d_j(x) = |\Gamma^+(x) \cap V_j|$ be the number of out-neighbors of x in V_j.

(1) If $x \in V_i$, $y \in V_j$ and $(x, y) \in E(\Gamma)$, then $d_j(x) - d_i(y) = \beta - \alpha$.

(2) For each pair (i, j), $i \neq j$, either (i) there exists a constant c_{ij} so that $d_j(x) = c_{ij}$ for every $x \in V_i$ or else (ii) V_i is partitioned into two non-empty sets $V_i = V_i' \cup V_i''$ so that all edges are directed from V_i' to V_j and from V_j to V_i''.

Let Δ be a finite doubly regular tournament of order m and let A be its adjacency matrix. The digraph with adjacency matrix $A \otimes J_r$ is denoted by $C_r(\Delta)$; so, it is a coclique extension of Δ. It is shown that if $m \equiv 3 (\text{mod } 4)$ and r are positive integers, then $C_r(\Delta)$ is a doubly regular (m, r)-team tournament.

Theorem 3.3.3. [47] Let Γ be a doubly regular (m, r)-team tournament. Then Γ satisfies one of the following.

(Type I.) $\beta - \alpha = r$ and Γ is isomorphic to $C_r(\Delta)$ for some doubly regular tournament Δ.

(Type II.) $\beta - \alpha = 0$, r is even and $d_i(x) = \frac{r}{2}$ for all $x \in V(\Gamma) - V_i$.

(Type III.) $\beta - \alpha = \frac{r}{2}$ and for every pair $\{i, j\}$ either V_i is partitioned into two sets V_i' and V_i'' of size $\frac{r}{2}$ so that all edges between V_i and V_j are directed from V_i' to V_j and from V_j to V_i'' or similarly with i and j interchanged.

Theorem 3.3.4. [47] Let R_1 be a doubly regular (m, r)-team tournament of Type I or Type II. Let $R_2 = R_1^T$ and $R_3 = \overline{R_1 \cup R_2}$. Then $(V(R_1), \{R_0, R_1, R_2, R_3\})$ is an imprimitive three-class association scheme.

Jørgensen proved the following relation between class of association schemes and Bush-type Hadamard matrices.

Theorem 3.3.5. [46] There exists an imprimitive 3-class association scheme of Type II (as in Theorem 3.3.3) and with $r = m$ even, if and only if there exists a Bush-type Hadamard matrix of order m^2 with the property that $H_{ij} = -H_{ji}$ for all pairs i, j with $i \neq j$.

Due to this theorem, we can obtain many imprimitive non-symmetric three-class association schemes of Type II from orthogonal arrays $\mathrm{OA}_{\frac{m}{2}}(m)$ via symmetric Bush-type Hadamard matrices of order m^2 for $m = 2^l$. We state this as follows.

Theorem 3.3.6. There exists an imprimitive non-symmetric 3-class association scheme of type II with $m = r = 2^l$ for each $l \geq 2$.

Proof By Theorem 3.1.5, we can construct symmetric Bush-type Hadamard matrices of order m^2. Let $H = [H_{ij}]$ be a symmetric Bush-type Hadamard matrix of order m^2 constructed in Theorem 3.1.5. Let $\tilde{H} = \left[\tilde{H}_{ij}\right]$ be the matrix obtained from H by multiplying every entry of H_{ij} by -1 for all i, j with $i > j$. So in $\tilde{H}$, $\tilde{H}_{ij} = -\tilde{H}_{ji}$ for all $i \neq j$. In order to show that $\tilde{H}$ is a Bush-type Hadamard matrix, it is enough to verify that $\tilde{H}_{ik}\tilde{H}_{jk} = 0$ for $i \neq j$ and $1 \leq k \leq m$. However, it follows from Part (iii) of Remark 3.1.7. Hence by Theorem 3.3.5, we have an imprimitive non-symmetric 3-class association scheme of type II with $m = r$. □

Remark 3.3.7. We construct three imprimitive non-symmetric 3-class association schemes of order 64 of type II. Their relation matrices are found in Appendix B.5.

3.4 Symmetric 3-class schemes coming from SBHMs

In Remark 3.1.6, we have seen that given a symmetric Bush-type Hadamard matrix of order $4n^2$, we can obtain twin strongly regular graphs with the same parameters $(4n^2, 2n^2 - n, n^2 - n, n^2 - n)$. The adjacency matrices A and $\hat{A}$ of these twin graphs may be expressed as $\hat{A} = J - A - (I_{2n} \otimes J_{2n})$. Thus the matrices I_{4n^2}, A, $\hat{A}$, and $I_{2n} \otimes J_{2n} - I_{4n^2}$ form a decomposition of all-1 matrix of size $4n^2$. It is easy to verify that these are the adjacency matrices of a symmetric three-class association scheme. Furthermore this association scheme is a fission scheme of the 2-class association scheme (the strongly regular graph Γ) with the adjacency matrices I, A and $J - A - I$. We now state it as follows.

Theorem 3.4.1. There exists an imprimitive symmetric 3-class association scheme of order n^2 for each $n = 2^l$, $l \geq 2$.

Proof Let A be the adjacency matrix of strongly regular graph $\Gamma = L_{\frac{n}{2}}(n)$ obtained from $\mathrm{OA}_{\frac{n}{2}}(n)$. Let H be the symmetric Bush-type Hadamard matrix obtained as in Theorem 3.1.5. Then the following are the adjacency matrices of the 3-class symmetric fission scheme of Γ that we desired.

$$
\begin{aligned}
A_0 &= I_{n^2} \\
A_1 &= \tfrac{1}{2}(J - H) \\
A_2 &= \tfrac{1}{2}(J + H - 2(I_n \otimes J_n)) \\
A_3 &= (I_n \otimes J_n) - I_{n^2}.
\end{aligned}
$$

$\square$

Remark 3.4.2. We construct three imprimitive symmetric 3-class association schemes of order 64. Their relation matrices are found in Appendix B.4.

CHAPTER 4. Three-class association schemes of order 64

In this chapter we provide the list of all feasible parameter sets of three-class association schemes of order 64 that appear in van Dam [16] and other existing literature (for example, [17, 46, 18]). We then discuss the classification of several cases of three-class association schemes of order 64. Many parameter sets are realized and some of the association schemes with such parameter sets are well-known. In order to understand their construction we survey some known association schemes.

The amorphic 3-class association schemes are precisely the 3-class association schemes in each of which all three relation graphs are strongly regular graphs, and that are not generated by one of their relations. In this case it is shown that the parameters of the graphs are either all of Latin square type, or all of negative Latin square type. Higman [41] was the first author proved this. The same results can also be found in [33], where also all such schemes on at most 25 vertices can be found.

Theorem 4.0.3. If all three relations of a 3-class association scheme are strongly regular graphs, then they either have parameters $(n^2, l_i(n-1), n-2+(l_i-1)(l_i-2), l_i(l_i-1))$ or $(n^2, l_i(n+1), -n-2+(l_i+1)(l_i+2), l_i(l_i+1))$. Here l_i, $1 \leq i \leq 3$, are positive integers such that $l_1+l_2+l_3 = n+1$ in the first case and $l_1+l_2+l_3 = n-1$ in the second case.

A large family of 3-class amorphic schemes comes from orthogonal arrays. Such schemes are often known as the *Latin square schemes* $L_{i,j}(n)$. Suppose that we have $l-2$ MOLS, or equivalently an orthogonal array $OA(l,n)$, an $l \times n^2$ array $A = [A_{st}]$, $s \in I_R = \{1, 2, \ldots, l\}$, $t \in I_C = \{1, 2, \ldots, n^2\}$ such that for any two rows a, b we have that $\{(A_{ai}, A_{bi}) : i \in I_C\} = \{(i,j) : i,j \in \{1, ..., n\}\}$. The relations of the scheme $L_{i,j}(n)$ are defined on the set I_C. Let S_1 and S_2 be two disjoint non-empty subsets of I_R of sizes i and j, respectively. Two distinct elements $x, y \in I_C$ are in relation R_h for $h = 1, 2$ if and only if $A_{rx} = A_{ry}$ for some $r \in S_h$, otherwise they are in third relation R_3. (For more information on Latin square schemes, we refer to [54] or [16].)

If q is a prime power, and $q \equiv 1 \pmod 3$, we can define the 3-class *cyclotomic* association scheme $Cycl(q)$ as follows. Let α be a primitive element of GF(q). As vertices we take the elements of GF(q). Two vertices will be in relation R_i if their difference equals α^{3t+i} for some t for $i = 1, 2, 3$.

The *rectangular scheme* $R(m, n)$ has as vertices the ordered pairs (i, j), with $i = 1, 2, ..., m$, and $j = 1, 2, ..., n$. For two distinct pairs we can have the following three situations. They agree in the first coordinate, or in the second coordinate, or in neither coordinate, and the relations are defined accordingly. Note that the graph of the third relation is the complement of the line graph of the complete bipartite graph $K_{m,n}$. The scheme is characterized by its parameters. van Dam [16] gave the first eigenmatrix P and intersection matrices B_1, B_2, B_3 of this 3-class association scheme.

The property that one of the relations of a d-class association scheme forms a distance-regular graph with diameter d is equivalent to the scheme being *P-polynomial* that is, the relations can be ordered such that the adjacency matrix A_i of relation R_i is a polynomial of degree i in A_1 for every i. This is also equivalent to the conditions $p^i_{1\ i+1} > 0$ and $p^i_{1j} = 0$ for $j > i + 1$, $i = 0, 1, \cdots, d - 1$. For a 3-class association scheme the conditions are equivalent to $p^1_{13} = 0$, $p^1_{12} > 0$ and $p^2_{13} > 0$ for some ordering of the relations. Dually we say that the association scheme is a *Q-polynomial scheme* if the primitive idempotents can be ordered such that the idempotent E_i is a polynomial of degree i in E_1 with respect to entry-wise multiplication for every i. Equivalent conditions are $q^i_{1\ i+1} > 0$ and $q^i_{1j} = 0$ for $j > i + 1$, $i = 0, 1, \cdots, d - 1$. In the case of a 3-class association scheme these conditions are equivalent to $q^1_{13} = 0$, $p^1_{12} > 0$ and $p^2_{13} > 0$ for some ordering of the idempotents. (In this case, we say that the association scheme has Q-polynomial of ordering 123, i.e., Q-123.) For more details refer [1].

There is a formal duality between ordinary multiplication, the numbers p^h_{ij} and the matrices

A_i and P on the one hand, and Hadamard (or Schur) product, the numbers q_{ij}^h and the matrices E_i and Q on the other hand. If two association schemes have the property that the intersection numbers of one are the Krein parameters of the other, then the converse is also true. Two such association schemes are said to be *formally dual* to each other. An association scheme is called *formally self-dual* if the eigenmatrices P and Q coincide for some ordering of the primitive idempotents; it follows that the multiplicities m_i are equal to the valencies k_i and $q_{ij}^h = p_{ij}^h$.

In the rest of the chapter we list the feasible parameter sets and current status of the existence and non-existence results on the association schemes realizing the parameter sets. In the list, each parameter set is labeled according to the following schemes: 'S' for symmetric, 'N' for non-symmetric, 'P' for primitive, and 'I' for imprimitive. Then the first three matrices are the intersection matrices $B_i = \left[p_{ij}^h\right]$, $(B_i)_{jh} = p_{ij}^h$, for $i = 1,2,3$. The last matrix is the character table P of the association scheme with the given intersection matrix, which is augmented by the column of the multiplicities of the corresponding eigenvalues. Followed by the matrices, the number of known isomorphic classes of association schemes and some relevant information of the scheme with the given intersection matrices are provided.

4.1 Symmetric primitive 3-class schemes of order 64

The feasible parameter sets for the schemes in this category are as follows.

$$SP1. \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 35 & 18 & 20 & 20 \\ 0 & 8 & 5 & 10 \\ 0 & 8 & 10 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 8 & 5 & 10 \\ 14 & 2 & 6 & 2 \\ 0 & 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 8 & 10 & 5 \\ 0 & 4 & 2 & 2 \\ 14 & 2 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 35 & 14 & 14 \\ 1 & 3 & -2 & -2 \\ 1 & -5 & 6 & -2 \\ 1 & -5 & -2 & 6 \end{bmatrix} \begin{matrix} 1 \\ 35 \\ 14 \\ 14 \end{matrix}$$

$$SP2. \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 28 & 12 & 12 & 12 \\ 0 & 9 & 8 & 12 \\ 0 & 6 & 8 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 9 & 8 & 12 \\ 21 & 6 & 8 & 6 \\ 0 & 6 & 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 6 & 8 & 4 \\ 0 & 6 & 4 & 3 \\ 14 & 2 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 28 & 21 & 14 \\ 1 & 4 & -3 & -2 \\ 1 & -4 & 5 & -2 \\ 1 & -4 & -3 & 6 \end{bmatrix} \begin{matrix} 1 \\ 28 \\ 21 \\ 14 \end{matrix}$$

$$SP3.\quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 27 & 10 & 12 & 12 \\ 0 & 8 & 9 & 6 \\ 0 & 8 & 6 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 8 & 9 & 6 \\ 18 & 6 & 2 & 6 \\ 0 & 4 & 6 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 8 & 6 & 9 \\ 0 & 4 & 6 & 6 \\ 18 & 6 & 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & 27 & 18 & 18 \\ 1 & -5 & 2 & 2 \\ 1 & 3 & -6 & 2 \\ 1 & 3 & 2 & -6 \end{bmatrix} \begin{matrix} 1 \\ 27 \\ 18 \\ 18 \end{matrix}$$

$$SP4.\quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 21 & 8 & 6 & 6 \\ 0 & 6 & 6 & 9 \\ 0 & 6 & 9 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 6 & 6 & 9 \\ 21 & 6 & 8 & 6 \\ 0 & 9 & 6 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 6 & 9 & 6 \\ 0 & 9 & 6 & 6 \\ 21 & 6 & 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 21 & 21 & 21 \\ 1 & 5 & -3 & -3 \\ 1 & -3 & 5 & -3 \\ 1 & -3 & -3 & 5 \end{bmatrix} \begin{matrix} 1 \\ 21 \\ 21 \\ 21 \end{matrix}$$

$$SP5.\quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 7 & 0 & 2 & 0 \\ 0 & 6 & 0 & 3 \\ 0 & 0 & 5 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 6 & 0 & 3 \\ 21 & 0 & 10 & 6 \\ 0 & 15 & 10 & 12 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 4 \\ 0 & 15 & 10 & 12 \\ 35 & 20 & 20 & 18 \end{bmatrix} \begin{bmatrix} 1 & 7 & 21 & 35 \\ 1 & 3 & 1 & -5 \\ 1 & -1 & -3 & 3 \\ 1 & -5 & 9 & -5 \end{bmatrix} \begin{matrix} 1 \\ 21 \\ 35 \\ 7 \end{matrix}$$

$$SP6.\quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 21 & 10 & 6 & 0 \\ 0 & 10 & 12 & 15 \\ 0 & 0 & 3 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 10 & 12 & 15 \\ 35 & 20 & 18 & 20 \\ 0 & 5 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 6 \\ 0 & 5 & 4 & 0 \\ 7 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 21 & 35 & 7 \\ 1 & 1 & -5 & 3 \\ 1 & -3 & 3 & -1 \\ 1 & 9 & -5 & -5 \end{bmatrix} \begin{matrix} 1 \\ 21 \\ 35 \\ 7 \end{matrix}$$

$$SP7.\quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 9 & 2 & 2 & 0 \\ 0 & 6 & 4 & 3 \\ 0 & 0 & 3 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 6 & 4 & 3 \\ 27 & 12 & 10 & 12 \\ 0 & 9 & 12 & 12 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 6 \\ 0 & 9 & 12 & 12 \\ 27 & 18 & 12 & 8 \end{bmatrix} \begin{bmatrix} 1 & 9 & 27 & 27 \\ 1 & 5 & 3 & -9 \\ 1 & 1 & -5 & 3 \\ 1 & -3 & 3 & -1 \end{bmatrix} \begin{matrix} 1 \\ 9 \\ 27 \\ 27 \end{matrix}$$

Remark 4.1.1. All the above parameter sets are realized. The number of non-isomorphic schemes are the following:

Schemes from	$SP1$	$SP2$	$SP3$	$SP4$	$SP5^{(1)}$	$SP6^{(2)}$	$SP7^{(3)}$
Number of isomorphism classes	≥ 1	≥ 1	≥ 1	≥ 1	1	1	2

(1) $SP5$ is realized as the P-polynomial scheme coming from folded 7-cube. This is uniquely

determined by their intersection array for by Brouwer [6] and Terwilliger [69].

[(2)] $SP6$ is realized as the P-polynomial scheme obtained from the halved 7-cubes and is characterized by their intersection array by Neumaier [58] and Terwilliger [68].

[(3)] $SP7$ is realized as the parameter sets for two DRGs, the Hamming graph $H(3,4)$ and the direct product of K_4 and Shrikhande graph. These two DRGs are the only graphs with the given parameters.

(1) The schemes realizing the parameter sets $SP1$, $SP2$, $SP3$, and $SP4$ are all amorphic schemes. In particular, the schemes for $SP1$, $SP2$ and $SP4$ are Latin square schemes and $SP4$ is also realized as a cyclotomic scheme $Cycl(64)$. Their relation graphs are all strongly regular graphs that are obtained as 'decompositions' of strongly regular graphs with higher valencies; that is, the edge set of the original SRG splits into two disjoint subsets each of which defines an SRG with a smaller valency as follows.

($SP1$) $L_{2,2}(8)$: [SRG(64,28,12,12) $\rightsquigarrow$ SRG(64,14,6,2) $\uplus$ SRG(64,14,6,2)];

$L_{2,5}(8)$: [SRG(64,49,36,42) $\rightsquigarrow$ SRG(64,35,18,20) $\uplus$ SRG(64,14,6,2)].

($SP2$) $L_{2,3}(8)$: [SRG(64,35,18,20) $\rightsquigarrow$ SRG(64,21,8,6) $\uplus$ SRG(64,14,6,2)];

$L_{2,4}(8)$: [SRG(64,42,26,30) $\rightsquigarrow$ SRG(64,28,12,12) $\uplus$ SRG(64,14,6,2)];

$L_{3,4}(8)$: [SRG(64,49,36,42) $\rightsquigarrow$ SRG(64,28,12,12) $\uplus$ SRG(64,21,8,6)].

($SP3$) SRG(64, 36, 20, 20) $\rightsquigarrow$ SRG(64, 18, 2, 6) $\uplus$ SRG(64, 18, 2, 6) ;

SRG(64, 45, 32, 30) $\rightsquigarrow$ SRG(64, 18, 2, 6) $\uplus$ SRG(64, 27, 10, 12)

($SP4$) $L_{3,3}(8)$: [SRG(64, 42, 26, 30) $\rightsquigarrow$ SRG(64, 21, 8, 6) $\uplus$ SRG(64, 21, 8, 6)].

(2) Association schemes from $SP1, SP2, SP3, SP4, SP6$, and $SP7$ are formally self-dual since $P = Q$. Thus the multiplicities are equal to the valencies and $q_{ij}^h = p_{ij}^h$.

(3) Association schemes from $SP5, SP6$, and $SP7$ are 3-class fission schemes with four integral eigenvalues.

(4) Association schemes from $SP5, SP6$, and $SP7$ are P- and Q-polynomial association schemes. $SP5$ and $SP6$ have two Q-polynomial structures [8].

Remark 4.1.2. We can also obtain $SP5$ and $SP6$ from halved-folded 8-cube Γ as follows: Let us represent the vertices (classes of antipodal pairs) of halved-folded 8-cube as opposite-pairs $\underline{x} = \underline{a} \vee \underline{\bar{a}} = a_1a_2\cdots a_8 \vee \bar{a}_1\bar{a}_2\cdots\bar{a}_8$ of even weight length 8 words $\underline{a}$ and $\underline{\bar{a}}$. Suppose we delete the first coordinate a_1 and its opposite $\bar{a}_1$ from every vertex $\underline{x}$ of Γ and have remaining pair of words of length 7. Considering the resulting pairs of length 7 words as vertices, we denote the vertices as $x = a \vee \bar{a} = a_2a_3\cdots a_8 \vee \bar{a}_2\bar{a}_3\cdots\bar{a}_8$ and $y = b \vee \bar{b} = b_2b_3\cdots b_8 \vee \bar{b}_2\bar{b}_3\cdots\bar{b}_8$ and define new relations on them by using the distance $\partial(x,y) = \min\{\partial_H(a,b), \partial_H(a,\bar{b})\}$ where $\partial_H(a,b)$ indicates the usual Hamming distance between two words.

(i) If we define relations between two vertices x, y by $(x,y) \in R_1$ iff the distance $\partial(x,y) = 1$, $(x,y) \in R_2$ iff $\partial(x,y) = 2$, and $(x,y) \in R_3$ otherwise, then we obtain the P-polynomial association scheme $SP5$ which is isomorphic to the folded 7-cube. (See Appendix B.2.)

(ii) If we define relations between two vertices x, y by $(x,y) \in R_1$ iff the distance $\partial(x,y) = 2$, $(x,y) \in R_3$ iff $\partial(x,y) = 1$, and $(x,y) \in R_2$ otherwise, then we obtain the P-polynomial association scheme $SP6$ which is isomorphic to the halved 7-cube. (See Appendix B.2.)

4.2 Symmetric imprimitive 3-class schemes of order 64

Known feasible parameter sets in this category are as follows:

$$SI1.\quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 49 & 36 & 42 & 42 \\ 0 & 6 & 0 & 7 \\ 0 & 6 & 7 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 6 & 0 & 7 \\ 7 & 0 & 6 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 1 & 0 & 0 \\ 7 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 49 & 7 & 7 \\ 1 & 1 & -1 & -1 \\ 1 & -7 & 7 & -1 \\ 1 & -7 & -1 & 7 \end{bmatrix} \begin{matrix} 1 \\ 49 \\ 7 \\ 7 \end{matrix}$$

$$SI2.\quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 42 & 26 & 30 & 30 \\ 0 & 10 & 6 & 12 \\ 0 & 5 & 6 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 10 & 6 & 12 \\ 14 & 2 & 6 & 2 \\ 0 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 5 & 6 & 0 \\ 0 & 2 & 1 & 0 \\ 7 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 42 & 14 & 7 \\ 1 & 2 & -2 & -1 \\ 1 & -6 & 6 & -1 \\ 1 & -6 & -2 & 7 \end{bmatrix} \begin{matrix} 1 \\ 42 \\ 14 \\ 7 \end{matrix}$$

$$SI3.\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 35 & 18 & 20 & 20 \\ 0 & 12 & 10 & 15 \\ 0 & 4 & 5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 12 & 10 & 15 \\ 21 & 6 & 8 & 6 \\ 0 & 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 4 & 5 & 0 \\ 0 & 3 & 2 & 0 \\ 7 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 35 & 21 & 7 \\ 1 & 3 & -3 & -1 \\ 1 & -5 & 5 & -1 \\ 1 & -5 & -3 & 7 \end{bmatrix} \begin{matrix} 1 \\ 35 \\ 21 \\ 7 \end{matrix}$$

$$SI4.\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 28 & 12 & 12 & 12 \\ 0 & 12 & 12 & 16 \\ 0 & 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 12 & 12 & 16 \\ 28 & 12 & 12 & 12 \\ 0 & 4 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 3 & 4 & 0 \\ 0 & 4 & 3 & 0 \\ 7 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 28 & 28 & 7 \\ 1 & 4 & -4 & -1 \\ 1 & -4 & 4 & -1 \\ 1 & -4 & -4 & 7 \end{bmatrix} \begin{matrix} 1 \\ 28 \\ 28 \\ 7 \end{matrix}$$

$$SI5.\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 14 & 0 & 4 & 2 \\ 0 & 12 & 8 & 12 \\ 0 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 12 & 8 & 12 \\ 42 & 24 & 28 & 30 \\ 0 & 6 & 5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 6 & 5 & 0 \\ 7 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 14 & 42 & 7 \\ 1 & 2 & -2 & -1 \\ 1 & -2 & -6 & 7 \\ 1 & -6 & 6 & -1 \end{bmatrix} \begin{matrix} 1 \\ 42 \\ 7 \\ 14 \end{matrix}$$

$$SI6.\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 15 & 2 & 4 & 0 \\ 0 & 12 & 10 & 15 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 12 & 10 & 15 \\ 45 & 30 & 32 & 30 \\ 0 & 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 15 & 45 & 3 \\ 1 & 3 & -3 & -1 \\ 1 & -1 & -3 & 3 \\ 1 & -5 & 5 & -1 \end{bmatrix} \begin{matrix} 1 \\ 30 \\ 15 \\ 18 \end{matrix}$$

$$SI7.\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 30 & 14 & 18 & 10 \\ 0 & 9 & 0 & 10 \\ 0 & 6 & 12 & 10 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 9 & 0 & 10 \\ 15 & 0 & 14 & 0 \\ 0 & 6 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 6 & 12 & 10 \\ 0 & 6 & 0 & 5 \\ 18 & 6 & 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & 30 & 15 & 18 \\ 1 & 6 & -1 & -6 \\ 1 & -2 & -1 & 2 \\ 1 & -10 & 15 & -6 \end{bmatrix} \begin{matrix} 1 \\ 15 \\ 45 \\ 3 \end{matrix}$$

$$SI8.\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 14 & 3 & 3 & 2 \\ 0 & 9 & 9 & 12 \\ 0 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 9 & 9 & 12 \\ 42 & 27 & 27 & 30 \\ 0 & 6 & 5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 6 & 5 & 0 \\ 7 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 14 & 42 & 7 \\ 1 & -2 & -6 & 7 \\ 1 & 2\sqrt{3} & -2\sqrt{3} & -1 \\ 1 & -2\sqrt{3} & 2\sqrt{3} & -1 \end{bmatrix} \begin{matrix} 1 \\ 7 \\ 28 \\ 28 \end{matrix}$$

$$SI9. \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 30 & 14 & 14 & 10 \\ 0 & 14 & 14 & 20 \\ 0 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 14 & 14 & 20 \\ 30 & 14 & 14 & 10 \\ 0 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 30 & 30 & 3 \\ 1 & -2 & -2 & 3 \\ 1 & 2\sqrt{5} & -2\sqrt{5} & -1 \\ 1 & -2\sqrt{5} & 2\sqrt{5} & -1 \end{bmatrix} \begin{matrix} 1 \\ 15 \\ 24 \\ 24 \end{matrix}$$

Remark 4.2.1. All the above feasible parameter sets are realized except for $SI8$ and $SI9$. The number of non-isomorphic association schemes with the given parameters are as follows.

Schemes from	$SI1$	$SI2$	$SI3$	$SI4$	$SI5$	$SI6$	$SI7$	$SI8$	$SI9$
Number of isomorphism classes	1	1676267*	≥ 1	≥ 3	≥ 1	94⋆	12	?	?

$*, \star$ are due to [14] and [55], respectively.

(1) Association schemes from $SI1, SI2, SI3$, and $SI4$ are amorphic, especially they are Latin square schemes.

($SI1$) $L_{1,1}(8)$: [SRG(64,14,6,2) $\rightsquigarrow$ SRG(64,7,6,0) $\uplus$ SRG(64,7,6,0)];
$L_{1,7}(8)$: [SRG(64,56,48,56) $\rightsquigarrow$ SRG(64,49,36,42) $\uplus$ SRG(64,7,6,0)].
(Note: $L_{1,1}(8) \cong R(8,8)$)

($SI2$) $L_{1,2}(8)$: [SRG(64,21,8,6) $\rightsquigarrow$ SRG(64,14,6,2) $\uplus$ SRG(64,7,6,0)];
$L_{2,6}(8)$: [SRG(64,56,48,56) $\rightsquigarrow$ SRG(64,42,26,30) $\uplus$ SRG(64,14,6,2)];
$L_{1,6}(8)$: [SRG(64,49,36,42) $\rightsquigarrow$ SRG(64,42,26,30) $\uplus$ SRG(64,7,6,0)].

($SI3$) $L_{1,3}(8)$: [SRG(64,28,12,12) $\rightsquigarrow$ SRG(64,21,8,6) $\uplus$ SRG(64,7,6,0)];
$L_{3,5}(8)$: [SRG(64,56,48,56) $\rightsquigarrow$ SRG(64,35,18,20) $\uplus$ SRG(64,21,8,6)];
$L_{1,5}(8)$: [SRG(64,42,26,30) $\rightsquigarrow$ SRG(64,35,18,20) $\uplus$ SRG(64,7,6,0)].

($SI4$) $L_{1,4}(8)$: [SRG(64,35,18,20) $\rightsquigarrow$ SRG(64,28,12,12) $\uplus$ SRG(64,7,6,0)];
$L_{4,4}(8)$: [SRG(64,56,48,56) $\rightsquigarrow$ SRG(64,28,12,12) $\uplus$ SRG(64,28,12,12)].

(2) Association schemes from $SI1, SI2, SI3$, and $SI4$ are formally self-dual since $P = Q$. Thus the multiplicities are equal to the valencies and $q_{ij}^h = p_{ij}^h$.

(3) As we have obtained three non-isomorphic strongly regular graphs of the same parameters, we can also obtain the association scheme $SI4$ from halved-folded 8-cube and the one from the strongly regular graph obtained in Remark 3.2.4. These are not isomorphic to $L_{1,4}(8)$ or $L_{4,4}(8)$. (See Appendix B.4.)

(4) Association schemes from $SI5, SI6$, and $SI7$ are 3-class fission schemes with four integral eigenvalues.

(a) The fusion scheme of the scheme from $SI5$ is a symmetric imprimitive 2-class scheme from SRG(64,56,48,56).

(b) The fusion scheme of the scheme from $SI6$ is a symmetric primitive 2-class scheme from SRG(64,18,2,6). (There are 167 non-isomorphic SRG(64,18,2,6) [36].) Association scheme from $SI6$ is a P-polynomial scheme obtained from DRG with valency 15. The second relation graph is an SRG(64,45,32,30).

There are 94 non-isomorphic cospectral three-class association schemes from $SI6$ [18], where only 5 examples were known before. The authors of [18] applied generation algorithm techniques based on backtracking with forward checking and dynamic variable ordering. Among the association schemes found in [18], 5 association schemes correspond to spreads in the unique generalized quadrangle GQ$(3,5)$. (Brouwer and Koolen have shown that there are exactly 5 non-isomorphic spreads in GQ$(3,5)$, with exactly one of them corresponding to a planar ovoid in GQ$(3,5)$ and exactly one serendipitous spread [48, p.44].) The authors of [18] have verified by computer that at least 80 of the generated schemes can be constructed by means of the method described by D. G. Fon-Der-Flaass in [24, Construction 4]. The first relation graph is a distance regular antipodal cover of K_{16}.

(c) The fusion scheme of association scheme from $SI7$ is a symmetric primitive 2-class scheme from SRG(64,45,32,30). Association scheme from $SI7$ is Q-polynomial scheme, $Q-123$. The third relation graph is an SRG(64,18,2,6).

(5) The parameter sets $SI8$ and $SI9$ are feasible for 3-class association schemes with two

integral eigenvalues. It is not known whether they are realizable or not though.

(a) If there is a scheme with $SI8$, it must have a fusion scheme which is isomorphic to a symmetric imprimitive 2-class scheme from SRG(64,56,48,56). (i.e, SRG(64, 56, 48, 56) $\cong L_8(8) \rightsquigarrow$ [two regular graphs on 64 vertices with degree 14 and 42].)

(b) If there is a scheme with $SI9$, it must have a fusion scheme which is isomorphic to a symmetric imprimitive 2-class scheme from SRG(64,60,56,60).

These schemes must have the following Krein parameters.

$$SI8.\ B_1^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 7 & 6 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix},\ B_2^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 4 \\ 28 & 12 & \alpha & \alpha^- \\ 0 & 16 & \alpha^- & \alpha \end{bmatrix},\ B_3^* = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 3 \\ 0 & 16 & \alpha^- & \alpha \\ 28 & 12 & \alpha & \alpha^- \end{bmatrix}$$

where $\alpha = 12 + \frac{4}{3}\sqrt{3}$ and $\alpha^- = 12 - \frac{4}{3}\sqrt{3}$.

$$SI9.\ B_1^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 15 & 14 & 0 & 0 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 8 & 7 \end{bmatrix},\ B_2^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 8 \\ 24 & \frac{56}{5} & 8 & 8 \\ 0 & \frac{64}{5} & 8 & 8 \end{bmatrix},\ B_3^* = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 8 & 7 \\ 0 & \frac{64}{5} & 8 & 8 \\ 24 & \frac{56}{5} & 8 & 8 \end{bmatrix}$$

4.3 Non-symmetric primitive 3-class schemes of order 64

In this section, we check the feasibility and realizability of the non-symmetric primitive 3-class fission schemes from the 2-class symmetric primitive association schemes from the existing 5 pairs of strongly regular graphs on 64 vertices listed in [7].

$(NP1)$ Fission of $\Gamma = SRG(64,\ 28,\ 12,\ 12)$;

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 28 & 12 & 12 \\ 0 & 15 & 16 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 15 & 16 \\ 35 & 20 & 18 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 28 & 35 \\ 1 & 4 & -5 \\ 1 & -4 & 3 \end{bmatrix}$$

These yield the following feasible parameter set for a non-symmetric 3-class fission scheme.

$$NP1. \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 4 \\ 14 & 4 & 4 & 2 \\ 0 & 5 & 10 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 14 & 4 & 4 & 2 \\ 0 & 0 & 4 & 4 \\ 0 & 10 & 5 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 5 & 10 & 8 \\ 0 & 10 & 5 & 8 \\ 35 & 20 & 20 & 18 \end{bmatrix} \begin{bmatrix} 1 & 14 & 14 & 35 \\ 1 & \bar{\rho} & \rho & -5 \\ 1 & \rho & \bar{\rho} & -5 \\ 1 & -2 & -2 & 3 \end{bmatrix} \begin{matrix} 1 \\ 14 \\ 14 \\ 35 \end{matrix}$$

where $\rho = 2 + 4i$.

$(NP2)$ Fission of $\Gamma = SRG(64,\ 42,\ 26,\ 30)$;

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 42 & 26 & 30 \\ 0 & 15 & 12 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 15 & 12 \\ 21 & 6 & 8 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 42 & 21 \\ 1 & 2 & -3 \\ 1 & -6 & 5 \end{bmatrix}$$

These yield the following feasible parameter set.

$$NP2. \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 7 & 5 & 9 \\ 21 & 7 & 7 & 6 \\ 0 & 6 & 9 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 21 & 7 & 7 & 6 \\ 0 & 5 & 7 & 9 \\ 0 & 9 & 6 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 6 & 9 & 6 \\ 0 & 9 & 6 & 6 \\ 21 & 6 & 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 21 & 21 & 21 \\ 1 & \bar{\rho} & \rho & -3 \\ 1 & \rho & \bar{\rho} & -3 \\ 1 & -3 & -3 & 5 \end{bmatrix} \begin{matrix} 1 \\ 21 \\ 21 \\ 21 \end{matrix}$$

where $\rho = 1 + 4i$.

$(NP3)$ Fission of $\Gamma = SRG(64,\ 36,\ 20,\ 20)$;

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 36 & 20 & 20 \\ 0 & 15 & 16 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 15 & 16 \\ 27 & 12 & 10 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 36 & 27 \\ 1 & -4 & 3 \\ 1 & 4 & -5 \end{bmatrix}$$

These yield the following feasible set.

$$NP3. \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 4 & 8 & 4 \\ 18 & 4 & 4 & 6 \\ 0 & 9 & 6 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 18 & 4 & 4 & 6 \\ 0 & 8 & 4 & 4 \\ 0 & 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 9 & 6 & 8 \\ 0 & 6 & 9 & 8 \\ 27 & 12 & 12 & 10 \end{bmatrix} \begin{bmatrix} 1 & 18 & 18 & 27 \\ 1 & \bar{\rho} & \rho & 3 \\ 1 & \rho & \bar{\rho} & 3 \\ 1 & 2 & 2 & -5 \end{bmatrix} \begin{matrix} 1 \\ 18 \\ 18 \\ 27 \end{matrix}$$

where $\rho = -2 + 4i$.

$(NP4)$ Fission of $\Gamma = SRG(64,\ 14,\ 6,\ 2)$;

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 14 & 6 & 2 \\ 0 & 7 & 12 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 7 & 12 \\ 49 & 42 & 36 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 14 & 49 \\ 1 & 6 & -7 \\ 1 & -2 & 1 \end{bmatrix}$$

These yield the following fission character table, but it is not feasible since some putative intersection numbers calculated from this character table are negative. (For example, $\hat{p}_{11}^2 = -3$.)

$$\hat{P}_\rho = \begin{bmatrix} 1 & 7 & 7 & 49 \\ 1 & \rho & \bar{\rho} & -7 \\ 1 & \bar{\rho} & \rho & -7 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad \text{where } \rho = 3 + 4i$$

$(NP5)$ Fission of $\Gamma = SRG(64,\ 18,\ 2,\ 6)$;

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 18 & 2 & 6 \\ 0 & 15 & 12 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 15 & 12 \\ 45 & 30 & 32 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 18 & 45 \\ 1 & -6 & 5 \\ 1 & 2 & -3 \end{bmatrix}.$$

These yield the following fission table, but it is not feasible since the intersection number $\hat{p}_{11}^1$ appears to be -1.

$$\hat{P}_\rho = \begin{bmatrix} 1 & 9 & 9 & 45 \\ 1 & \rho & \bar{\rho} & 5 \\ 1 & \bar{\rho} & \rho & 5 \\ 1 & 1 & 1 & -3 \end{bmatrix} \quad \text{where } \rho = -3 + 4i$$

Remark 4.3.1. (1) $NP1, NP2, NP3$ are all formally self-dual. (This can be checked by calculating Krein parameters using Theorem II.3.5 in [1].)

(2) Enomoto and Mena in [21] constructed a scheme with this parameter set $NP1$. It is not known if there are others.

(3) The parameter sets $NP2$ and $NP3$ are feasible, but it is not known whether they are realizable.

4.4 Non-symmetric imprimitive 3-class schemes of order 64

$(NI1)$ Fission of $\Gamma = SRG(64,\ 56,\ 48,\ 56)$;

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 56 & 48 & 56 \\ 0 & 7 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 7 & 0 \\ 7 & 0 & 6 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 56 & 7 \\ 1 & 0 & -1 \\ 1 & -8 & 7 \end{bmatrix}$$

These yield one feasible parameter set for a 3-class fission. This is realized and there are at least three non-isomorphic schemes.

$$NI1. \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 12 & 12 & 16 \\ 28 & 12 & 12 & 12 \\ 0 & 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 28 & 12 & 12 & 12 \\ 0 & 12 & 12 & 16 \\ 0 & 4 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 3 & 4 & 0 \\ 0 & 4 & 3 & 0 \\ 7 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 28 & 28 & 7 \\ 1 & \bar{\rho} & \rho & -1 \\ 1 & \rho & \bar{\rho} & -1 \\ 1 & -4 & -4 & 7 \end{bmatrix} \begin{matrix} 1 \\ 28 \\ 28 \\ 7 \end{matrix}$$

where $\rho = 4i$.

Remark 4.4.1. As we have seen earlier, there are three cospectral non-isomorphic strongly regular graphs with parameters (64, 28, 12, 12). They are halved-folded 8-cube, $L_4(8)$, and the strongly regular graph discussed in Remark 3.2.4. Thus the number of non-isomorphic symmetric imprimitive 3-class association schemes with the parameter set $SI4$ in Section 4.2 is greater than or equal to 3. We also note that these three cospectral non-isomorphic strongly regular graphs are corresponding to three symmetric Bush-type Hadamard matrices as we have discussed in Section 3.2. By Theorem 3.3.6, we can obtain three non-isomorphic non-symmetric imprimitive 3-class fission schemes from these three non-isomorphic strongly regular graphs as well; so, there are at least 3 schemes with the parameter set $NI1$. (See Appendix B.5 for the relation matrices of the three-class association schemes.)

CHAPTER 5. Future research problems

We provide a few remarks in this final chapter. This includes a few problems and related topics.

Remark 5.1. We have constructed the third non-isomorphic cospectral strongly regular graph with parameters $(64, 28, 12, 12)$ from the symmetric Bush-type Hadamard matrix H_3 given in Example 3.2.3. We have seen that the matrix H_3 has been obtained by permuting the positions of some blocks of other symmetric Bush-type Hadamard matrices which are associated with other cospectral strongly regular graphs. (While I am writing up this dissertation, we have discovered many more cospectral strongly regular graphs from distinct SBHMs. We have not yet completely examined how many other non-isomorphic cospectral strongly regular graphs coming from other symmetric Bush-type Hadamard matrices of order 64 would be. It will be discussed fully in [22].) We hope to find all symmetric Bush-type Hadamard matrices and their associated strongly regular graphs. Once we settle the case on the order 64, then the obvious next step would be to look for the construction of all symmetric Bush-type Hadamard matrices of different order and their associated graphs.

Remark 5.2. With regard to the construction of symmetric Bush-type Hadamard matrices, we have discussed the following in Chapter 3: Kharaghani [50] conjectured that Bush-type Hadamard matrices of order $4n^2$ exist for every odd integer n. While the construction of Bush-type Hadamard matrices of order $16n^2$ is provided for which a Hadamard matrix of order $4n$ exists, it is not much obvious about the existence or non-existence of such matrices of order $4n^2$ for certain n, especially for odd n. Examples are known for $n = 3, 5, 9$ by Janko, Kharaghani and Tonchev. (See [43, 44, 45].) In [57], Muzychuk and Xiang constructed symmetric Bush-type Hadamard matrices of order $4m^4$ for all odd integers m by using reversible Hadamard difference sets. Yet there are many cases that are open. Nevertheless, for any n being of power of 2, we know a fairly easy way to construct a symmetric Bush-type Hadamard matrix from the $\mathrm{OA}(t, 2t)$, for $t = 2^l$, $l \geq 1$ that is constructed in 2.3.3 from the desarguesian complete set of MOLS of order $2t$. We see a possibility of implementation of these constructions to find some non-symmetric imprimitive 3-class association schemes of type II due to [46].

In fact, in [46], Jørgensen obtained two non-symmetric imprimitive 3-class association schemes of order 36 of type II with $m = r = 6$ by using non-symmetric Bush-type Hadamard matrices satisfying conditions in Theorem 3.3.5. Non-symmetric imprimitive 3-class association schemes of order 100 of type II with $m = r = 10$ have not been constructed yet although a non-symmetric Bush-type Hadamard matrix of order 100 was constructed in [44]. From the symmetric Bush-type Hadamard matrices of order $4m^4$ constructed by Muzychuk and Xiang, we can easily construct non-symmetric imprimitive 3-class association schemes of type II if the matrices satisfy conditions in Theorem 3.3.5. In order to construct non-symmetric imprimitive 3-class association schemes of type II for other open cases it would be useful to find all Bush-type Hadamard matrices satisfying the conditions in Theorem 3.3.5.

Remark 5.3. In Section 4.3, we have found two feasible parameter sets $NP2$ and $NP3$ for non-symmetric primitive 3-class association schemes, but we have not been able to determine whether they are realizable. It will be interesting to know whether there exist such association schemes as the non-symmetric case is linked to many combinatorial objects. It is not easy to find primitive association schemes in general. So, we need to look at more examples of primitive non-symmetric association schemes of small class. For class 3 case, the smallest example of such a scheme is of order 36 which comes from the action of $\mathrm{PSU}(3, 3)$ on 36 points and have valencies 7, 7, 21. It was first discovered by Ivanov, Faradžev and Klin in 1982 whose description appeared in [23, p.115]. This was rediscovered by Goldbach and Classen [31]. It is interesting to know whether there is a way to construct such schemes from certain digraphs.

APPENDIX A. Preliminaries

In this appendix we recall some basic terminologies and facts about regular graphs, strongly regular graphs, distance-regular graphs, association schemes, and Latin squares, which are not our results and which are included for reference purposes. For more information we refer to, for example, [1, 4, 8].

A.1 Graphs, digraphs and tournaments

A *graph* $\Gamma = (V, E)$ is a set V of vertices and a subset E of the unordered pairs of vertices, called edges. In this dissertation, a graph is finite, undirected, and simple, i.e., without loops and multiple edges, unless indicated otherwise. We say that two vertices x and y are *adjacent* if the pair $\{x, y\}$ is an edge; i.e., $\{x, y\} \in E$. Given a graph Γ, sometimes the vertex set and edge set of Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively.

The vertices adjacent to a given vertex x are called *neighbors* of x. Let $k(x) := |\Gamma(x)| = |\{y \in V(\Gamma) : \{x, y\} \in E(\Gamma)\}|$; the number of neighbors of x, which is called the *degree* or *valency* of x. A graph is *regular* with degree k if $k(x) = k$ for all $x \in V(\Gamma)$. This is the most obvious kind of combinatorial regularity, and it has interesting consequences for the eigenvalues of the graph as it will be seen shortly.

A graph is *complete* if any two vertices are adjacent. So a complete graph on n vertices has $n(n-1)/2$ edges. In a complete graph on n vertices, every vertex has degree $n - 1$. The complement $\bar{\Gamma}$ of a graph Γ is the graph on the same vertices, but with complementary edge set; i.e., two vertices are adjacent in $\bar{\Gamma}$ if and only if they are not adjacent in Γ.

Given a graph $\Gamma = (X, R)$, and a subset Y of X, the *induced subgraph* of Γ on Y is the graph with its vertex set Y and edge set $S \subset R$ consisting of all edges between the vertices of

Y. A *clique* is an induced subgraph that is a complete graph.

A *walk* of length l between two vertices x, y is a sequence of vertices $x = x_0, x_1, \cdots, x_l = y$ such that for any i the vertices x_i and x_{i+1} are adjacent. If all vertices in the sequence are distinct then the walk is called a *path.* The length of the path $x_0, x_1, \ldots, x_l$ is l, the number of edges connecting the initial vertex x_0 and the terminal vertex x_l. If there is a path between any two vertices of the graph, then the graph is called *connected.* Otherwise the graph is called *disconnected.*

The *adjacency matrix* $A = A(\Gamma)$ of a graph $\Gamma = (X, E)$ is the $|X| \times |X|$ matrix with rows and columns are indexed by the vertices, with its (x, y)-entry $A_{xy} = 1$ if $\{x, y\} \in E$ and $= 0$ otherwise. It is a symmetric matrix with entries 0 or 1. The eigenvalues, eigenvectors and spectrum of this matrix are also called those of the graph Γ, respectively.

Lemma A.1.1. Let Γ be a regular graph of degree k. Then: (a) k is an eigenvalue of Γ, (b) the multiplicity of k is 1 if Γ is connected, and (c) for any eigenvalue λ of Γ, we have $|\lambda| \leq k$.

A *directed graph* or *digraph* D consists of a vertex set $V(D)$ and an arc set $E(D)$, where each arc is an ordered pair of vertices. We write (x, y) for the arc. The adjacency matrix $A = A(D)$ is defined as $A_{xy} = 1$ if $(x, y) \in E(D)$ and $= 0$ otherwise. Let $D^+(x) := \{y \in V(D) : (x, y) \in E(D)\}$ denote the set of *out-neighbors* of x and $D^-(x)$ denote the set of *in-neighbors* of x. Let $d^+(x)$ denote the *outdegree*, the number of out-neighbors of x, and let $d^-(x)$ denote the *indegree* of x.

A *tournament* is a directed graph T with the property that for any two vertices x and y, exactly one of the arcs (x, y) or (y, x) belongs to $E(T)$. In terms of the adjacency matrix A, a tournament is a graph with the property that $A + A^t + I = J$, where I and J denote the identity and all-one matrices, respectively. If every vertex in a tournament T with n vertices has outdegree k then every vertex has indegree $n - k - 1$ and so $k = n - k - 1$, i.e., $n = 2k + 1$, and thus such a tournament is a regular directed graph of degree k.

An *orientation* of a graph Γ is a (simple) digraph D obtained from Γ by choosing an arc either (x, y) or (y, x) (but not both) for each edge $\{x, y\} \in E(\Gamma)$. An oriented graph is an orientation of a simple graph and it is the same thing as a loopless simple digraph. A tournament is an orientation of a complete graph. A tournament T is called *doubly regular* if it is regular and for every vertex x in T the out-neighbors of x span a regular tournament.

Definition A.1.2. Let Γ and Γ' be (di)graphs. An *isomorphism* $\phi : \Gamma \to \Gamma'$ is a bijection from $V(\Gamma)$ to $V(\Gamma')$ which induces a bijection from $E(\Gamma)$ to $E(\Gamma')$; that is, $(x, y) \in E(\Gamma)$ if and only if $(\phi(x), \phi(y)) \in E(\Gamma')$. An automorphism of a (di)graph Γ is an isomorphism $\phi : \Gamma \to \Gamma$.

A.2 Strongly regular and distance-regular graphs

A regular graph in which the number of vertices adjacent to two vertices x and y depends only on whether x and y are adjacent or not, is called a *strongly regular graph.* In terms of parameters we have the following definition.

Definition A.2.1. A *strongly regular graph* with parameters (v, k, λ, μ) is a graph, Γ, with v vertices, which is neither complete nor the complement of a complete graph, in which

$$|\Gamma(x) \cap \Gamma(y)| = \begin{cases} k & \text{if } x = y \\ \lambda & \text{if } \{x, y\} \in E(\Gamma) \\ \mu & \text{if } \{x, y\} \notin E(\Gamma) \end{cases}$$

Such a graph will be denoted by $\mathrm{SRG}(v, k, \lambda, \mu)$.

Lemma A.2.2. If Γ is an $\mathrm{SRG}(v, k, \lambda, \mu)$, then $k(k - \lambda - 1) = (v - k - 1)\mu$.

Lemma A.2.3. The complement $\overline{\Gamma}$ of an SRG Γ with parameters (v, k, λ, μ) is also an SRG with parameters

$$(\overline{v}, \overline{k}, \overline{\lambda}, \overline{\mu}) = (v,\ v - 1 - k,\ v - 2 - 2k + \mu,\ v - 2k + \lambda).$$

We call a strongly regular graph Γ *primitive* if both Γ and its complement $\overline{\Gamma}$ are connected, and if it is not primitive we say it is *imprimitive.* We will recall these concepts as a little bit general manner when we deal with distance-regular graph. It follows the next result that an imprimitive strongly regular graph is either a complete multipartite graph or the disjoint union of a number of copies of a complete graph.

Proposition A.2.4. A strongly regular graph Γ with parameters (v, k, λ, μ) is imprimitive if and only if $\mu = k$ or $\mu = 0$.

Proof See [27]. $\square$

By Lemma A.1.1, the adjacency matrix of a regular graph with degree k has an eigenvalue k with eigenvector $\mathbf{1}$ (the all-one vector). For convenience we call an eigenvalue *restricted* if it has an eigenvector perpendicular to $\mathbf{1}$.

Proposition A.2.5. For a graph Γ of order n, not complete or empty, with adjacency matrix A, the following are equivalent:

(a) Γ is an $\mathrm{SRG}(v, k, \lambda, \mu)$ for certain non-negative integers k, λ, and μ.

(b) $A^2 = kI + \lambda A + \mu(J - I - A)$ for certain reals k, λ, and μ.

(c) A has precisely two distinct restricted eigenvalues.

Proof See [34, Theorem 1.1]. $\square$

Next result is known as the *integrality condition*, and depends on an analysis of the eigenvalues of the graph.

Proposition A.2.6. If there is an $\mathrm{SRG}(v, k, \lambda, \mu)$ with adjacency matrix A and let f and g be respective multiplicities of restricted eigenvalues of A, then the numbers

$$f, g = \frac{1}{2}\left\{(v-1) \pm \frac{(v-1)(\mu-\lambda) - 2k}{\sqrt{(\mu-\lambda)^2 + 4(k-\mu)}}\right\}.$$

are non-negative integers.

Proof See [12, Theorem 3.1] or [34, Theorem 1.2]. $\square$

The *distance* $\partial(x, y)$ between two vertices x and y in a graph Γ is the length of the shortest path from x to y. The *diameter* of a graph Γ is the maximum distance between two distinct vertices.

Definition A.2.7. Let $\Gamma = (V, E)$ be a connected regular graph. Γ is called a *distance-regular graph* (DRG) with diameter d if it satisfies the following distance-regularity condition:
For all $h, i, j \in \{0, 1, ..., d\}$, and x, y with $\partial(x, y) = h$, the number

$$p_{ij}^h = |\{z \in V : \partial(x, z) = i, \ \partial(z, y) = j\}|$$

is constant in that it depends only on h, i, j but does not depend on the choice of x and y. If $\Gamma_i(x) = \{y \in V : \partial(x, y) = i\}$, then $p_{ij}^h := |\Gamma_i(x) \cap \Gamma_j(y)|$ for all $x, y \in V(\Gamma)$ with $\partial(x, y) = h$. This number is called the *intersection number*.

Let $\partial(x, y) = i$ and define

$$a_i := p_{1\ i}^i = |\Gamma_1(x) \cap \Gamma_i(y)|$$

$$b_i := p_{1\ i+1}^i = |\Gamma_1(x) \cap \Gamma_{i+1}(y)|$$

$$c_i := p_{1\ i-1}^i = |\Gamma_1(x) \cap \Gamma_{i-1}(y)|$$

Then $b_0 = p_{11}^0 = |\Gamma(x)| = k$; the degree of Γ, and we will define $c_0 := 0$, $b_d := 0$, and $a_0 := 0$. The following matrix is called the *intersection array* (matrix) for the DRG of diameter d.

$$B\ (= B_1) = \begin{pmatrix} a_0 & c_1 & & & \\ b_0 & a_1 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & c_d \\ & & & b_{d-1} & a_d \end{pmatrix}.$$

The $d+1$ intersection matrices $B_0, B_1, B_2, \ldots, B_d$ of Γ are defined as follows:

$$B_i = [(B_i)_{jh}] \quad \text{where} \quad (B_i)_{jh} = p_{ij}^h.$$

For a graph Γ of diameter d we define the i-th *distance graph* Γ_i to be the graph with the same vertex set as Γ, and with two vertices adjacent if and only if they are at distance i in the graph Γ. We call Γ *imprimitive* if for some i, $(1 \leq i \leq d)$, the graph Γ_i is disconnected. A graph which is not imprimitive is called *primitive*. Denote by k_i $(0 \leq i \leq d)$ the number of vertices in $\Gamma_i(x)$ for any vertex x; in particular $k_0 = 1$ and $k_1 = k$. The following result gives us the basic properties of the parameters a_i, b_i, c_i, and k_i of a distance-regular graph, see [8] for the details.

Proposition A.2.8. Let Γ be a distance-regular graph with degree k and diameter d. Then the following hold:

(a) $k_{i-1}b_{i-1} = k_i c_i$ for $i = 1, 2, ..., d$.

(b) $1 = c_1 \leq c_2 \leq \cdots \leq c_d$.

(c) $k = b_0 \geq b_1 \geq \cdots \geq b_{d-1} > 0$.

(d) $c_i + a_i + b_i = k$ for $0 \leq i \leq d$.

All DRGs with diameter 2 are SRGs. Conversely, all connected SRGs are DRGs with diameter 2. The DRGs form a class of P-polynomial association schemes, and SRGs form a special class of symmetric association schemes of class two.

Definition A.2.9. Let F be a set of q-elements $(q \geq 2)$, and let $X = F^n$; the set of ordered n-tuples of elements of F. A *Hamming graph*, $\Gamma = H(n, q)$, is a graph whose vertex set is given by $V(\Gamma) = X = \{x = (x_1, x_2, \ldots, x_n) \in F^n\}$, and whose edge set, $E(\Gamma)$, is defined by

$$\{x, y\} \in E(\Gamma) \text{ if and only if } x_i = y_i \text{ for all but one } i$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in V(\Gamma)$. The Hamming graph is a DRG with diameter $d = n$.

In $\Gamma = H(n, q)$, the degree of Γ is $k = |\Gamma(x)| = b_0 = n(q-1)$, and the intersection numbers of $H(n, q)$ are given by

$$c_i = i, \qquad b_i = (n-i)(q-1),$$

$$a_i = k - b_i - c_i = n(q-1) - i - (n-i)(q-1) = i(q-2),$$

and,

$$p_{ij}^h = \sum_{l=0}^{n} (q-2)^{j+2l-i-h}(q-1)^{i-l} \binom{l}{j+2l-i-h} \binom{h}{l} \binom{n-h}{i-l}.$$

When $q = 2$ and $F = \mathbb{F}_2 = \{0, 1\}$, the Hamming graphs are known as *Hamming cubes.*

The *direct product* of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is the graph defined on the vertex set $V_1 \times V_2$ such that two vertices $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$ are adjacent if and only if either $x_1 = y_1$ and $\{x_2, y_2\} \in E_2$, or $\{x_1, y_1\} \in E_1$ and $x_2 = y_2$. Hamming graphs $H(d, q)$ is the direct power K_q^d of d-copies of K_q.

Cospectral graphs are the graphs with the same spectrum. There are many non-isomorphic cospectral distance-regular graphs. Among the small distance-regular graphs, the Hamming $H(2,4)$ graph, which is also known as the 'lattice graph, $L_2(4)$', and the line graph $L(K_{4,4})$ of the complete bipartite graph $K_{4,4}$, has a cospectral graph known as the 'Shrikhande graph'.(cf. [8, p.105].) Doob graphs which are obtained by taking the direct product of copies of $H(2,4)$ and the Shrikhande graph are all cospectral as long as the number of factors is the same.

A.3 Halved and folded distance-regular graphs

A graph $\Gamma = (V, E)$ is called *bipartite* if the vertex set V is partitioned into two parts X and Y such that $E \subseteq \{\{x, y\} : x \in X, y \in Y\}$. In particular, if $E = \{\{x, y\} : x \in X, y \in Y\}$, then Γ is called a *complete bipartite* graph. The adjacency matrix of a bipartite graph is congruent to a matrix of the form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A and D are $|X| \times |X|$ and $|Y| \times |Y|$ zero matrices, respectively. Similarly, we call a graph $\Gamma = (V, E)$ a *multipartite* or *m-partite graph* if V is partitioned into m non-empty subsets, say, $V = V_1 \cup V_2 \cup \cdots \cup V_m$ (with $m > 2$) and no two vertices in the same part is adjacent.

Definition A.3.1. Let Γ be a bipartite distance-regular graph with bipartition $V(\Gamma) = X \cup Y$. Let $V(\Gamma') = X$, and for $x, y \in X$, let x and y be adjacent in Γ' if and only if $\partial(x, y) = 2$ in Γ. Γ' is distance-regular and is called the *halved graph* of Γ.

We note that Γ has another halved graph with vertex set Y which need not be isomorphic to Γ' unless Γ is a bipartite 'distance-transitive graph', however, this fact has no bearing on what follows. Let the parameters of a bipartite distance-regular graph Γ be designated as a_i, b_i, c_i. Those of its halved graph Γ' will be designated a'_i, b'_i, c'_i. Since Γ is a bipartite graph, it has no odd cycles, which forces $a_i = 0$, for all i. As a result, $b_i + c_i = k$, the degree of Γ. Due to Gardiner [26], we get: If $c_3 = c_2$, then $c_3 = c_2 = 1$, and by Hemmeter [38], we have:

Lemma A.3.2. $b'_i = \frac{b_{2i}b_{2i+1}}{c_2}$ and $c'_j = \frac{c_{2j}c_{2j-1}}{c_2}$.

Proof Let x and y be vertices of Γ' such that $x \in \Gamma_{2i}(y)$. Count the number of pairs (w, z) with $w \in \Gamma_1(y) \cap \Gamma_{2i+1}(x)$ and $z \in \Gamma_1(w) \cap \Gamma_{2i+2}(x)$. There are b_{2i} such w, each of which has b_{2i+1} such z. Counting z's first gives $b'_i c_2$. This implies the former. The latter can be proved in the same manner, with x, y being as above, by counting the pairs (w, z) with $w \in \Gamma_1(y) \cap \Gamma_{2j-1}(x)$, $z \in \Gamma_1(w) \cap \Gamma_{2j-2}(x)$. $\square$

Lemma A.3.3. [39] Suppose Γ' is not a complete graph. Then for every $y \in Y$, $\Gamma(y)$ is a maximal clique in Γ'. Furthermore, if $y_1 \neq y_2$, then $\Gamma(y_1) \neq \Gamma(y_2)$.

We now introduce another class of distance-regular graphs that are obtained from antipodal distance-regular graphs.

Definition A.3.4. Let Γ be a distance-regular graph with diameter d. For x and y in $V(\Gamma)$, we say that x is *opposite* to y if $\partial(x, y) = d$, or $x = y$. The graph Γ is called *antipodal* if being opposite is an equivalence relation on Γ; i.e., $x \sim y$ in Γ if and only if x and y are opposite. For an antipodal graph Γ, we can define a graph $\check{\Gamma}$ whose vertices are the equivalence classes of Γ. If $\bar{x}$ and $\bar{y}$ are in $V(\check{\Gamma})$, we say that $\{\bar{x}, \bar{y}\} \in E(\check{\Gamma})$ if for some $x \in \bar{x}$ and $y \in \bar{y}$, $\{x, y\} \in E(\Gamma)$. Then $\check{\Gamma}$ is distance-regular, and is called the *folded graph*, or *antipodal quotient* of Γ. In this case, Γ is called the *antipodal cover* of $\check{\Gamma}$.

Let Γ be an antipodal distance-regular graph, and let $\check{\Gamma}$ be its folded graph. Then we have the canonical surjective map ϕ defined by $x \mapsto \bar{x}$. With this, we have:

Lemma A.3.5. [39] Let $x \in V(\Gamma)$ and Υ be the subgraph induced by $\bigcup_{j=0}^{[\frac{d}{2}]-1} \Gamma_j(x)$. Then ϕ restricted to Υ is an isomorphism onto $\phi(\Upsilon)$. If $d \geq 4$, $\phi(\Upsilon)$ contains all of the neighbors of $\bar{x}$.

Corollary A.3.6. [39] If $d \geq 4$, then the structure of the cliques which contain a vertex $\bar{x} \in V(\check{\Gamma})$ is isomorphic to the structure of the cliques containing the vertex $x \in V(\Gamma)$.

It is easy to see that bipartite and antipodal distance-regular graphs of diameter $d \geq 2$ are imprimitive graphs. In fact, if Γ is a bipartite or an antipodal graph, then Γ_2 or Γ_d are disconnected, respectively. It is clear that when Γ is connected bipartite graph of diameter $d \geq 2$, then Γ_2 has two components, and the graphs induced on these components by Γ_2 are halved graphs of Γ. Also recall that if Γ is an antipodal DRG, then we define the folded graph of Γ by taking connected components of Γ_d as vertices where two components (equivalence classes) are adjacent whenever they contain adjacent vertices in Γ. The following theorem is due to Smith, Martinov, and Gardiner. (cf.[8, p.140])

Theorem A.3.7. An imprimitive distance-regular graph of valency $k > 2$ is bipartite or antipodal (or both).

Proof Let Γ be a imprimitive distance-regular graph of valency $k > 2$ and diameter d. Since if $d = 2$ then it is clear, so we may assume that $d \geq 3$. Let i be the smallest positive integer such that Γ_i is disconnected. If $x, y, z \in V(\Gamma)$ form a triple of type (j, i, i); i.e., $\partial(x, y) = j, \partial(x, z) = \partial(y, z) = i$, then $p^j_{ii} > 0$ whence any path in Γ_j gives rise to a path in Γ_i. Hence the minimality of i implies that $j \geq i$; so Γ contains no triples of type (j, i, i) with $j < i$.

Suppose $i = 2 < d$. Then pick z_0 and z_3 at distance 3 and consider the path z_0, z_1, z_2, z_3. If $y \in \Gamma(z_0) \cap \Gamma(z_1)$ then either the triple y, z_0, z_2 or y, z_1, z_3 is a triple of forbidden type $(1, 2, 2)$; so $a_1 = 0$ and there are no triangles. If there is a closed cycle C of odd length greater than 3, then all vertices of C lie in the same connected component Δ of Γ_2; in particular, Δ contains edges of Γ. Now for any $y, z \in \Delta$ which are adjacent in Γ, all points adjacent in Γ to y and z are in Δ. This implies $\Delta = \Gamma$ as Γ is connected. This is a contradiction. Therefore, Γ has no closed odd cycles if $i = 2$, and thus Γ is bipartite.

Suppose $2 < i < d$. Then pick z_0, z_d at distance d, and let $z_0, z_1, \ldots, z_d$ be a path of length d. Since $k \geq 3$, there is $y \in \Gamma$ such that $y \in \Gamma(z_{i+1}) - \{z_i, z_{i+2}\}$. But then $\partial(y, z_0) = i + l$ with $l \in \{0, 1, 2\}$ and y, z_{i+l}, z_l is a triple of forbidden type (j, i, i), $j \leq 2$, which is a contradiction. Finally, if $i = d$, then all triples (j, d, d) with $j < d$ are forbidden whence Γ_d is a disjoint union of cliques, hence Γ is antipodal. □

Corollary A.3.8. Let Γ be an imprimitive distance-regular graph of valency $k > 2$ and diameter d. Let I be a subset of $\{0, 1, \ldots, d\}$ such that having distance in I is an equivalence relation on $V(\Gamma)$. Then I is one of the sets $\{0\}$, $\{0, d\}$, $\{0, 2, 4, \ldots\}$, or $\{0, 1, \ldots, d\}$.

Proof Straightforward from the proof of A.3.7. □

The following easy observations are useful.

Lemma A.3.9. [8, p.141]

(1) Let Γ be an antipodal graph with folded graph $\breve{\Gamma}$. Then $\breve{\Gamma}$ is not antipodal except when the diameter $d(\Gamma) \leq 3$, in which case $\breve{\Gamma}$ is complete, or when Γ is bipartite of $d(\Gamma) = 4$, in which case $\breve{\Gamma}$ is complete bipartite.

(2) If Γ is a bipartite distance-regular graph with $k > 2$ and its halved graph Γ', then Γ' is not bipartite.

A.4 Commutative association schemes

In this section, we will recall the basic facts about association schemes. (cf. [1], [8], [19])

Definition A.4.1. Let X be a finite set of cardinality v and $R_0, R_1, \ldots, R_d$ be a (non-empty) binary relation of X; i.e., $R_i \subseteq X \times X = \{(x, y) : x, y \in X\}$. The configuration $(X, \{R_i\}_{0 \leq i \leq d})$ is called a d-class (*commutative*) *association scheme* if it satisfies the following axioms:

(1) $R_0 = \{(x, x) : x \in X\}$, the diagonal relation

(2) $R_0 \dot{\cup} R_1 \dot{\cup} \ldots \dot{\cup} R_d = X \times X$

(3) $R_i^T = \{(y, x) : (x, y) \in R_i\}$ must be a member of $\{R_0, R_1, \ldots, R_d\}$. Denote $R_i^T = R_{i'}$ for some $i' \in \{1, 2, \ldots, d\}$

(4) For any $h, i, j \in \{1, 2, \ldots, d\}$, and for any $(x, y) \in R_h$, the number

$$p_{ij}^h = |\{z \in X : (x, z) \in R_i, (z, y) \in R_j\}|$$

does not depend on the choice of x and y, but depends only on i, j, h.

(5) $p_{ij}^h = p_{ji}^h$ for all $h, i, j = 0, 1, \ldots, d$,

Furthermore, an association scheme $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is called *symmetric* if $R_i^T = R_i$ for all i. A symmetric association scheme is necessarily commutative but not vice versa. The numbers p_{ij}^h are called the *intersection numbers* or *parameters* of $\mathcal{X}$. The number of elements that are in ith relation from a fixed element x is called the ith *valency* and denoted by k_i; so $k_i = |\{z \in X : (x, z) \in R_i\}|$ for any $x \in X$.

Lemma A.4.2. In a d-class association scheme $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$,

1. (a) $k_i = p_{ii'}^0$, $0 \leq i \leq d$, (b) $k_0 = 1$, $k_i > 0$, (c) $k_i = k_{i'}$, and (d) $v = \sum_{i=0}^{d} k_i$.

2. The intersection numbers have the following identities:

   (a) $p_{0j}^h = \delta_{jh}$, $p_{i0}^h = \delta_{ih}$

   (b) $p_{ij}^0 = \delta_{ij'} k_i$

   (c) $p_{ij}^h = p_{i'j'}^{h'}$

   (d) $\sum_{j=0}^{d} p_{ij}^h = k_i$

   (e) $k_h p_{ij}^h = k_j p_{i'h}^j = k_i p_{hj'}^i$

   (f) $\sum_{\alpha=0}^{d} p_{\alpha r}^h p_{ij}^\alpha = \sum_{\beta=0}^{d} p_{i\beta}^h p_{jr}^\beta$

Given a d-class association scheme $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$, let $A_0, A_1, \ldots, A_d$ be adjacency matrices of $\mathcal{X}$. Then by the definition, we have

(1) $A_0 = I$

(2) $A_0 + A_1 + \cdots + A_d = J$

(3) $A_i^T = A_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$

(4) $A_iA_j = \sum\limits_{h=0}^{d} p_{ij}^h A_h$

(5) $A_iA_j = A_jA_i$

(6) $\mathcal{X}$ is symmetric if and only if $A_i^T = A_i$ for all i.

So $A_0, A_1, \ldots, A_d$ generate a $(d+1)$-dimensional commutative $\mathbb{C}$-algebra $\mathcal{U}$ of $v \times v$ matrices called the *Bose-Mesner* algebra of $\mathcal{X}$. Since $A_0, A_1, \ldots, A_d$ are pairwise commutative normal matrices, they can be diagonalized simultaneously, and thus we have the decomposition $\mathbb{C}^v = V_0 \oplus V_1 \oplus \cdots \oplus V_d$, where each V_j is a common eigenspace for the matrices A_i. We may suppose that V_0 is the 1-dimensional eigenspace corresponding to the eigenvalue v. Let E_i be the orthogonal projection $\mathbb{C}^v \to V_i$ expressed in a matrix form with respect to the standard basis $\{e_x : x \in X\}$. Then $E_0 + E_1 + \cdots + E_d = I$, $E_iE_j = \delta_{ij}E_i$, and $E_0 = \frac{1}{|X|}J$. By the theorem II.3.1[1], $E_0, E_1, \cdots, E_d$ form a basis of the primitive idempotents of $\mathcal{U}$. Let $p_i(j)$ be the eigenvalue of A_i on V_j. Then $A_i = \sum\limits_{j=0}^{d} p_i(j)E_j$. Let $E_i = \frac{1}{|X|}\sum\limits_{j=0}^{d} q_i(j)A_j$ and set $P = (p_i(j))$ and $Q = (q_i(j))$ where the (j,i) entry of P is $p_i(j)$ and the (j,i) entry of Q is $q_i(j)$. The matrices P and Q of degree $d+1$ are called the *first* and *second eigenmatrices* of $\mathcal{X}$ respectively. We also call P the *character table* of $\mathcal{X}$. Note that $PQ = QP = vI$. Let $m_i = dimV_i = rankE_i$. The positive integer m_i are called the *multiplicities* of $\mathcal{X}$.

Theorem A.4.3. Let $P = (p_i(j))$ and $Q = (q_i(j))$ be the first and second eigenmatrices of a commutative association scheme $\mathcal{X}$ respectively. Then the following hold:

(1) $p_0(i) = 1$, $p_i(0) = k_i$,

(2) $q_0(i) = 1$, $q_i(0) = m_i$,

(3) $p_{i'}(j) = \overline{p_i(j)}$, where $\overline{p_i(j)}$ is the complex conjugate of $p_i(j)$,

(4) $q_j(i)/m_j = \overline{p_i(j)}/k_i$,

(5) (The First Orthogonality Relation) $\sum\limits_{r=0}^{d} \frac{1}{k_r} p_r(i)\overline{p_r(j)} = \frac{|X|}{m_i}\delta_{ij}$,

(6) (The Second Orthogonality Relation) $\sum\limits_{r=0}^{d} m_r p_i(r)\overline{p_j(r)} = |X|k_i\delta_{ij}$,

(7) $p_i(r)p_j(r) = \sum\limits_{h=0}^{d} p_{ij}^h p_h(r)$ $(i, j, r \in \{0, 1, \ldots, d\})$,

(8) $\sum\limits_{i=0}^{d} p_i(j) = 0$ for all $1 \leq j \leq d$.

The multiplicities, the Krein parameters, and the intersection numbers are calculated from the character table as in the following theorem:

Theorem A.4.4. The following hold:

(1) $m_i = |X| \left(\sum\limits_{j=0}^{d} \frac{|p_j(i)|^2}{k_j} \right)^{-1}$,

(2) $p_{ij}^h = \frac{1}{|X|k_h} \sum\limits_{r=0}^{d} p_i(r)p_j(r)\overline{p_h(r)}m_r = \frac{k_i k_j}{|X|} \sum\limits_{r=0}^{d} q_r(i)q_r(j)\overline{q_r(h)}/m_r^2$,

(3) $q_{ij}^h = \frac{m_i m_j}{|X|} \sum\limits_{r=0}^{d} p_r(i)p_r(j)\overline{p_r(h)}/k_r^2$.

For matrices A, B of the same size, $A \circ B$ is, by definition, the matrix obtained by the entry-wise product of A and B, i.e., $(A \circ B)_{ij} = A_{ij}B_{ij}$. We call $A \circ B$ the *Hadamard product* of A and B. Note that $A_0 + A_1 + \cdots + A_d = J$, $A_i \circ A_j = \delta_{ij}A_i$. It followes that $\mathcal{U}$ is closed under the Hadamard product. This commutative algebra with respect to the Hadamard product is denoted by $\hat{\mathcal{U}}$ and $A_0, A_1, \cdots, A_d$ are the primitive idempotents. The numbers q_{ij}^h which satisfy $E_i \circ E_j = \frac{1}{|X|} \sum\limits_{h=0}^{d} q_{ij}^h E_h$ are called the *Krein parameters.* $\hat{\mathcal{U}}$ together with the basis $E_0, E_1, \cdots E_d$ is a dual of $\mathcal{U}$ together with the basis $A_0, A_1, \cdots, A_d$ in the sense that

(1) $|X|E_0 = J$, the identity of $\hat{\mathcal{U}}$,

(2) $E_0 + E_1 + \cdots + E_d = I$, the identity of $\mathcal{U}$,

(3) $E_i^T = E_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$,

(4) $E_i \circ E_j = \frac{1}{|X|}\sum_{h=0}^{d} q_{ij}^h E_h$,

(5) $E_i \circ E_j = E_j \circ E_i$,

(6) $E_i^T = E_i$ for all i if it is symmetric.

Proposition A.4.5. The following hold:

(1) $q_{0j}^h = \delta_{jh}, \quad q_{i0}^h = \delta_{ih}$,

(2) $q_{ij}^0 = \delta_{ij'} m_i$,

(3) $q_{ij}^h = q_{i'j'}^{h'}$,

(4) $\sum_{j=0}^{d} q_{ij}^h = m_i$,

(5) $m_h q_{ij}^h = m_j q_{i'h}^j = m_i q_{hj'}^i$,

(6) $\sum_{\alpha=0}^{d} q_{ij}^\alpha q_{h\alpha}^l = \sum_{\beta=0}^{d} q_{ki}^\beta q_{\beta j}^l$,

(7) $q_i(r)q_j(r) = \sum_{h=0}^{d} q_{ij}^h q_h(r)$ $(i, j, r \in \{0, 1, \ldots, d\})$.

Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be a commutative association scheme of class d. Let $\Gamma_i = (X, R_i)$ be the graph whose vertex and edge sets are X and R_i respectively. These graphs are called the *relation graphs* of the association scheme. $\mathcal{X}$ is said to be *primitive* if all the Γ_i are connected $(i = 1, 2, \cdots, d)$. $\mathcal{X}$ is said to be *imprimitive* if it is not primitive. Let $\mathcal{X}$ and $\mathcal{Y}$ be two association schemes. Then they are *isomorphic* if there is a permutation matrix Π such that the matrices $\Pi A_i \Pi^T$ are the adjacency matrices of $\mathcal{Y}$ where A_i are adjacency matrices of $\mathcal{X}$.

Now we will review briefly about fusion and fission of a commutative association scheme. Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ and $\tilde{\mathcal{X}} = (X, \{\tilde{R}_\alpha\}_{0 \le \alpha \le e})$ be commutative association schemes defined on X. If for every $i \in \{0, 1, 2, \cdots, d\}$, $R_i \subseteq \tilde{R}_\alpha$ for some $\alpha \in \{0, 1, 2, \cdots, e\}$, then we say that $\tilde{\mathcal{X}}$ is a *fusion* scheme of $\mathcal{X}$, and $\mathcal{X}$ is a *fission* scheme of $\tilde{\mathcal{X}}$. For several equivalent criteria for fusion refer [2], [3], and [64]. The character table of a commutative fission scheme of class 3 can be calculated as follows:

Proposition A.4.6. [3, Theorem 2.3] Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq 2})$ be a symmetric association scheme of class 2 with character table

$$P = \begin{bmatrix} 1 & k_1 & k_2 \\ 1 & r & t \\ 1 & s & u \end{bmatrix} \begin{array}{c} 1 \\ m_1 \\ m_2 \end{array}$$

Suppose $\hat{\mathcal{X}} = (X, \{\hat{R}_j\}_{0 \leq j \leq 3})$ is a non-symmetric fission scheme of $\mathcal{X}$ with three classes such that $\hat{R}_2 = \hat{R}_1'$, $\hat{R}_1 \cup \hat{R}_2 = R_1$, $\hat{R}_3 = R_2$, $\hat{E}_1 \cup \hat{E}_2 = E_1$, $\hat{E}_3 = E_2$. Then the character table $\hat{P}$ of $\hat{\mathcal{X}}$ is given by

$$\hat{P} = \begin{bmatrix} 1 & \frac{1}{2}k_1 & \frac{1}{2}k_1 & k_2 \\ 1 & \rho & \bar{\rho} & t \\ 1 & \bar{\rho} & \rho & t \\ 1 & \frac{1}{2}s & \frac{1}{2}s & u \end{bmatrix} \begin{array}{c} 1 \\ \frac{1}{2}m_1 = \hat{m}_1 \\ \frac{1}{2}m_1 = \hat{m}_2 \\ m_2 = \hat{m}_3 \end{array}$$

where $\rho = \frac{1}{2}(r + \sqrt{-vk_1}/\sqrt{m_1})$.

Remark A.4.7. For a given symmetric association scheme $\mathcal{X}$ of class 2, we can calculate several putative fission tables $\hat{P}$ using the above proposition A.4.6 without knowing the existence of such fission schemes of class 3 in advance. We shall call a character table $\hat{P}$ *feasible* if it satisfies the following necessary conditions:

(1) All the multiplicities $\hat{m}_j$ are integers.

(2) All the intersection numbers $\hat{p}_{ij}^h$ (including the $\hat{k}_i = \hat{p}_{ii'}^0$) are non-negative integers.

(3) All the Krein parameters $\hat{q}_{ij}^h$ are non-negative real numbers.

The fission scheme $\hat{\mathcal{X}}$ of $\mathcal{X}$ is *realizable* if $\hat{\mathcal{X}}$ exists.

A.5 Mutually orthogonal Latin squares

Definition A.5.1. A *Latin square of order n* over an alphabet $Q = \{1, 2, \ldots, n\}$ is an $n \times n$ array of elements from Q, that satisfies each element of the alphabet appears exactly once in any column and row.

It is easily seen that a Latin square of order n exists for every positive integer n. For example, one may label the rows and columns by $1, 2, \ldots, n$, and take the entry in row i and column j to be $i + j$, where the addition is modulo n.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two Latin squares of order n over the alphabet Q. We say that A and B are *orthogonal* if n^2 ordered pairs (a_{ij}, b_{ij}), for $i, j \in \{1, 2, \ldots, n\}$ are distinct in $Q \times Q$; so, as a set, $\{(a_{ij}, b_{ij}) : i, j \in \{1, 2, \ldots, n\}\} = Q \times Q$. A set $\mathcal{L}$ of Latin squares is called *mutually orthogonal* if every pair of Latin squares in $\mathcal{L}$ is orthogonal. The maximum number $N(n)$ of mutually orthogonal Latin squares (MOLS) of order n is not known in general. However, it is easy to see that $N(n) \leq n - 1$.

A set of $n - 1$ MOLS of order n is called a *complete set* of MOLS. It is not known whether there exists a complete set of MOLS of order n for any n that is not a power of a prime number. For any prime power q, one can easily construct a complete set of $q - 1$ MOLS of order q.

The following lemma gives a useful tool when we play with MOLS.

Lemma A.5.2. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two orthogonal Latin squares over the alphabet $Q = \{1, 2, \ldots, n\}$. For any permutation σ, τ on Q, A^σ and B^τ are orthogonal where the action of the permutations is entrywise; i.e., $A^\sigma = \left[a_{ij}^\sigma\right]$.

Proof Suppose we deny the conclusion. Then there exist two distinct pairs (i, j) and (k, l) such that $(a_{ij}^\sigma, b_{ij}^\tau) = (a_{kl}^\sigma, b_{kl}^\tau)$. This implies that $a_{ij}^\sigma = a_{kl}^\sigma$ and $b_{ij}^\tau = b_{kl}^\tau$ which in turn implies that $a_{ij} = a_{kl}$ and $b_{ij} = b_{kl}$ which contradicts to the orthogonality of A and B. $\square$

We now sketch the proof of the following.

Theorem A.5.3. For every $n > 1$, $1 \leq N(n) \leq n - 1$.

Proof Let $L_1, L_2, \ldots, L_t$ be MOLS of order n. Choose $\sigma_1, \sigma_2, \ldots, \sigma_t$ permutations on $Q = \{1, 2, \ldots, n\}$ such that $L_i^{\sigma_i}$ has $[1, 2, \ldots, n]$ in its first row for each $i = 1, 2, \ldots, t$. Then by the above lemma $L_1^{\sigma_1}, L_2^{\sigma_2}, \ldots, L_t^{\sigma_t}$ are MOLS. Now consider the symbols that can be $(2, 1)$-entries of these MOLS. The $(2, 1)$-entries must be all distinct by the mutual orthogonality, and they must come from the set $\{2, 3, \ldots, n\}$ being Latin squares. Therefore, t cannot exceed $n - 1$.

$\square$

Suppose we take the finite field $\mathbb{F}_q$ of order q as an alphabet Q. Construct $q - 1$ MOLS $\{L_a : a \in \mathbb{F}_q^*\}$ of order q as follows. Consider the $q-1$ polynomials of the form $f_a(x, y) = ax+y$ for $a \in \mathbb{F}_q^*$, and define (x, y)-entry of L_a by $ax + y$ for $x, y \in \mathbb{F}_q$. Then it is straightforward to verify that each L_a is a Latin square and the set $\{L_a : a \in \mathbb{F}_q^*\}$ is a complete set of $q-1$ MOLS. (cf. for example, [37, Theorem 8.3].) This set of MOLS is called the *desarguesian* complete set of MOLS of order q. So we have:

Theorem A.5.4. If $q = p^e$ is a prime power, then $N(q) = q - 1$.

One might ask whether for a given prime power q there exist other complete sets of $q - 1$ MOLS of order q that are different from the desarguesian complete set of MOLS constructed above. The previous lemma A.5.2 indicates that there are indeed other complete sets, if by different one only means that, as matrices, the squares are different. For example if we simply permute the symbols of any of the $q-1$ squares, the resulting set of squares is still orthogonal. Such squares are of course very similar to the original ones, and so we ask is there a set which is "structurally different" in some fundamental sense? This requires precise meaning to the notion of different complete sets of MOLS, where we assume that all are based on the same set of symbols. Two complete sets of MOLS of the same order are said to be *isomorphic* if the squares of one set can be obtained from the squares of the other set by applying a fixed permutation to the rows of all of the squares of the first set, then by similarly applying a fixed permutation to the columns of the resulting squares, and finally by applying a third permutation to the symbols. The key point is that the three permutations must be applied to each square of the first set. It is known that for all prime powers $q = p^e > 8$ with $e > 1$, there are at least two non-isomorphic sets of MOLS of order q. Thus for such q there is always at least one non-desarguesian complete set of MOLS of order q. However, for each of the prime powers $q = 2, 3, 4, 5, 7, 8$, any complete set of MOLS of order q is isomorphic to those obtained from Theorem A.5.4; see [20]. Lam, Kolesova, and Thiel [51] have recently shown that there are exactly 4 non-isomorphic complete sets of MOLS of order 9. (In this thesis, we have used only the desarguesian complete set when we need a complete set of MOLS.)

APPENDIX B. Adjacency and relation matrices of relevant graphs and association schemes

B.1 Adjacency matrices of halved-folded 8-cube, $L_4(8)$ and some induced subgraphs

Vertex labeling and adjacency matrix of halved-folded 8-cube

Vertices of halved-folded 8-cube

1 00000000 11111111	2 00000011 11111100
3 00000101 11111010	4 00000110 11111001
5 00001001 11110110	6 00001010 11110101
7 00001100 11110011	8 00010001 11101110
9 00010010 11101101	10 00010100 11101011
11 00011000 11100111	12 00100001 11011110
13 00100010 11011101	14 00100100 11011011
15 00101000 11010111	16 00110000 11001111
17 01000001 10111110	18 01000010 10111101
19 01000100 10111011	20 01001000 10110111
21 01010000 10101111	22 01100000 10011111
23 11000000 00111111	24 10100000 01011111
25 10010000 01101111	26 10001000 01110111
27 10000100 01111011	28 10000010 01111101
29 10000001 01111110	30 00001111 11110000
31 00010111 11101000	32 00011011 11100100
33 00011101 11100010	34 00011110 11100001

35 00100111 11011000	36 00101011 11010100
37 00101101 11010010	38 00101110 11010001
39 00110011 11001100	40 00110101 11001010
41 00110110 11001001	42 00111001 11000110
43 00111010 11000101	44 00111100 11000011
45 01000111 10111000	46 01001011 10110100
47 01001101 10110010	48 01001110 10110001
49 01010011 10101100	50 01010101 10101010
51 01010110 10101001	52 01011001 10100110
53 01011010 10100101	54 01011100 10100011
55 01100011 10011100	56 01100101 10011010
57 01100110 10011001	58 01101001 10010110
59 01101010 10010101	60 01101100 10010011
61 01110001 10001110	62 01110010 10001101
63 01110100 10001011	64 01111000 10000111

Adjacency matrix of halved-folded 8-cube

0111111111111111111111111111100000000000000000000000000000000000

1011110110011000110000000001111100110010000111001000011000010011

1101101101010100101000000010111010101001001010100100100100100101

1110011011001100011000000011011001100100110010010011000011001001

1110011100110010100100000100110110011000110001100011000011000110

1101101010101010010100000101010101010100100101010010010010101010

1011110001100110001100000110010011001110000100111000011000011100

1110100011110001100010001000101110000111010000011101000010111000

1101010101101001010010001001001101001010101000101010100101010100

1011001110100101001010001010001011010001100101000110011001100010

10001111110000110001100011000001111000000111100000001111110000001

1110100100001111100001010000100001111011010000010010111101001000

1101010010010111010001010001000010110110101000100101011010100100

1011001001011011001001010010000100101101100101001001100110010010

1000111000111101000101010100001000011100011110001110000001110001

1000000111111110000011011000010000000011111111110000000000001111

111010010001000001111100000100001000100101111101101001101001000

1101010010001000101111100001000010001001010111011010101010100100

1011001001000100110111100010000100010010011010110110010110010010

1000111000100010111011100100001000100011100001110001110001110001

1000000111100001111101101000010000111000000000011111100000011111

1000000000011111111110110000011111000000000000000000001111111111

1000000000000000111111011111111111111111111100000000000000000000

1000000000011111000001101111111111000000000011111111110000000000

10000001111000010000101101111100001111000000111100000001111110000

1000111000100010000100111011101000100011100010001110001110001110

1011001001000100001000111101100100010010011001001001101001101101

1101010010001000010000111110100010001001010100100101010101011011

1110100100010000100000111111000001000100101100010010110010110111

0111111000000001000011111000001111111100000011110000000000001111

0111000111000010000101110100010111100011100010001110000001110001

0100110110100100001001110010011011010010011001001001100110010010

0010101101101000010001110001011101001001010100100101011010100100

0001011011110000100001110000111110000100101100010010111101001000

0111000000111110000011010110001100001111110001000000111111000000 1

0100110001011010001010101010010100101110011001000110011001100010

0010101010010110010010101001010010110101010100101010100101010100

0001011100001110100010101000110001111000101100011101000010111000

0100001110011001001100100110001100110001111000111000011000011100

0010010101010101010100100101001010101010110101010100100100101010

0001100011001101100100100100101001100111001101100011000011000110

0001100100110011011000100011000110011011001110010011000011001001

0010010010101011101000100010100101010110110110100100100100100101

0100001001100111110000100001100011001101111011001000011000010011

0111000000100011111000011100011000100000011101111110001110000001

0100110001000101110100011010010100010001100110111001101001100010

0010101010001001101100011001010010001010101011010101010101010100

0001011100010001011100011000110001000111010011100010110010111000

0100001110000110110010010110001100001110000111000111101000011100

0010010101001010101010010101001010010101001010101011010100101010

0001100011010010011010010100101001011000110010011100110011000110

0001100100101100100110010011000110100100110001101100110011001001

0010010010110100010110010010100101101001001001011011010100100101

0100001001111000001110010001100011110010000100110111101000010011

0100001001111000110001001110000011110010000111001000010111101100

0010010010110100101001001101000101101001001010100100101011011010

0001100100101100011001001100100110100100110010010011001100110110

0001100011010010100101001011001001011000110001100011001100111001

0010010101001010010101001010101010010101001001010100101011010101

0100000111000001100011010010011011000011100001001110000101111100011

0001011100010001100011000111010001000111010000011101001101000111

0010101010001001010011000110110010001010101000101010101010101011

0100110001000101001011000101110100010001100101000110010110011101

0111000000010001100011100001111100010000001111000000111000111111110

Vertex labeling and the adjacency matrix of $OA_4(8)$

Vertices of $OA_4(8)$

1 0000 2 1111 3 2222 4 3333 5 4444 6 5555 7 6666 8 7777

9 0123 10 1032 11 2301 12 3210 13 4567 14 5476 15 6745 16 7654

17 0246 18 1357 19 2064 20 3175 21 4602 22 5713 23 6420 24 7531

25 0365 26 1274 27 2147 28 3056 29 4721 30 5630 31 6503 32 7412

33 0451 34 1540 35 2673 36 3762 37 4015 38 5104 39 6237 40 7326

41 0572 42 1463 43 2750 44 3641 45 4136 46 5027 47 6314 48 7205

49 0617 50 1706 51 2435 52 3524 53 4253 54 5342 55 6071 56 7160

57 0734 58 1625 59 2516 60 3407 61 4370 62 5261 63 6152 64 7043

Adjacency matrix of $OA_4(8)$

00000000 11110000 10101010 10010110 11001100 10100101 11000011 10011001

00000000 11110000 01010101 01101001 11001100 01011010 11000011 01100110

00000000 11110000 10101010 01101001 00110011 10100101 00111100 01100110

00000000 11110000 01010101 10010110 00110011 01011010 00111100 10011001

00000000 00001111 10101010 01101001 11001100 01011010 00111100 10011001

00000000 00001111 01010101 10010110 11001100 10100101 00111100 01100110

00000000 00001111 10101010 10010110 00110011 01011010 11000011 01100110

00000000 00001111 01010101 01101001 00110011 10100101 11000011 10011001

11110000 00000000 10010110 10101010 10100101 11001100 10011001 11000011

11110000 00000000 01101001 01010101 01011010 11001100 01100110 11000011

11110000 00000000 01101001 10101010 10100101 00110011 01100110 00111100

11110000 00000000 10010110 01010101 01011010 00110011 10011001 00111100

00001111 00000000 01101001 10101010 01011010 11001100 10011001 00111100

00001111 00000000 10010110 01010101 10100101 11001100 01100110 00111100

00001111 00000000 10010110 10101010 01011010 00110011 01100110 11000011

00001111 00000000 01101001 01010101 10100101 00110011 10011001 11000011

10101010 10010110 00000000 11110000 11000011 10011001 11001100 10100101

01010101 01101001 00000000 11110000 11000011 01100110 11001100 01011010

10101010 01101001 00000000 11110000 00111100 01100110 00110011 10100101

01010101 10010110 00000000 11110000 00111100 10011001 00110011 01011010

10101010 01101001 00000000 00001111 00111100 10011001 11001100 01011010

01010101 10010110 00000000 00001111 00111100 01100110 11001100 10100101

10101010 10010110 00000000 00001111 11000011 01100110 00110011 01011010

01010101 01101001 00000000 00001111 11000011 10011001 00110011 10100101

10010110 10101010 11110000 00000000 10011001 11000011 10100101 11001100

01101001 01010101 11110000 00000000 01100110 11000011 01011010 11001100

01101001 10101010 11110000 00000000 01100110 00111100 10100101 00110011

10010110 01010101 11110000 00000000 10011001 00111100 01011010 00110011

01101001 10101010 00001111 00000000 10011001 00111100 01011010 11001100

10010110 01010101 00001111 00000000 01100110 00111100 10100101 11001100

10010110 10101010 00001111 00000000 01100110 11000011 01011010 00110011

01101001 01010101 00001111 00000000 10011001 11000011 10100101 00110011

11001100 10100101 11000011 10011001 00000000 11110000 10101010 10010110

11001100 01011010 11000011 01100110 00000000 11110000 01010101 01101001

00110011 10100101 00111100 01100110 00000000 11110000 10101010 01101001

00110011 01011010 00111100 10011001 00000000 11110000 01010101 10010110

11001100 01011010 00111100 10011001 00000000 00001111 10101010 01101001

11001100 10100101 00111100 01100110 00000000 00001111 01010101 10010110

00110011 01011010 11000011 01100110 00000000 00001111 10101010 10010110

00110011 10100101 11000011 10011001 00000000 00001111 01010101 01101001

10100101 11001100 10011001 11000011 11110000 00000000 10010110 10101010

01011010 11001100 01100110 11000011 11110000 00000000 01101001 01010101

10100101 00110011 01100110 00111100 11110000 00000000 01101001 10101010

01011010 00110011 10011001 00111100 11110000 00000000 10010110 01010101

01011010 11001100 10011001 00111100 00001111 00000000 01101001 10101010

10100101 11001100 01100110 00111100 00001111 00000000 10010110 01010101

01011010 00110011 01100110 11000011 00001111 00000000 10010110 10101010

10100101 00110011 10011001 11000011 00001111 00000000 01101001 01010101

11000011 10011001 11001100 10100101 10101010 10010110 00000000 11110000

11000011 01100110 11001100 01011010 01010101 01101001 00000000 11110000

00111100 01100110 00110011 10100101 10101010 01101001 00000000 11110000

00111100 10011001 00110011 01011010 01010101 10010110 00000000 11110000

00111100 10011001 11001100 01011010 10101010 01101001 00000000 00001111

00111100 01100110 11001100 10100101 01010101 10010110 00000000 00001111

11000011 01100110 00110011 01011010 10101010 10010110 00000000 00001111

11000011 10011001 00110011 10100101 01010101 01101001 00000000 00001111

10011001 11000011 10100101 11001100 10010110 10101010 11110000 00000000

01100110 11000011 01011010 11001100 01101001 01010101 11110000 00000000

01100110 00111100 10100101 00110011 01101001 10101010 11110000 00000000

10011001 00111100 01011010 00110011 10010110 01010101 11110000 00000000

10011001 00111100 01011010 11001100 01101001 10101010 00001111 00000000

01100110 00111100 10100101 11001100 10010110 01010101 00001111 00000000

01100110 11000011 01011010 00110011 10010110 10101010 00001111 00000000

10011001 11000011 10100101 00110011 01101001 01010101 00001111 00000000

Adjacency matrix of induced subgraph on $\Gamma(v) \cup \{v\}$ in halved-folded 8-cube

0 1 1 1 1 0 1 1 0 0 0 1 0 0 1 1 1 0 0 0 0 0 1 0 0 0 1 1 0

1 0 1 1 0 1 1 0 1 0 0 0 1 0 1 1 0 1 0 0 0 0 1 0 0 1 0 1 0

1 1 0 0 1 1 0 1 1 0 0 1 1 0 0 0 1 1 0 0 0 0 1 0 0 1 1 0 0

1 1 0 0 1 1 1 0 0 1 0 0 0 1 1 1 0 0 1 0 0 0 1 0 1 0 0 1 0

1 0 1 1 0 1 0 1 0 1 0 1 0 1 0 0 1 0 1 0 0 0 1 0 1 0 1 0 0

0 1 1 1 1 0 0 0 1 1 0 0 1 1 0 0 0 1 1 0 0 0 1 0 1 1 0 0 0

1 1 0 1 0 0 0 1 1 1 1 0 0 0 1 1 0 0 0 1 0 0 1 1 0 0 0 1 0

1 0 1 0 1 0 1 0 1 1 1 1 0 0 0 0 1 0 0 1 0 0 1 1 0 0 1 0 0

0 1 1 0 0 1 1 1 0 1 1 0 1 0 0 0 0 1 0 1 0 0 1 1 0 1 0 0 0

0 0 0 1 1 1 1 1 1 0 1 0 0 1 0 0 0 0 1 1 0 0 1 1 1 0 0 0 0

0 0 0 0 0 0 1 1 1 1 0 1 1 1 1 0 0 0 0 1 1 0 1 1 0 0 0 0 1

1 0 1 0 1 0 0 1 0 0 1 0 1 1 1 0 1 0 0 0 1 0 1 0 0 0 1 0 1

0 1 1 0 0 1 0 0 1 0 1 1 0 1 1 0 0 1 0 0 1 0 1 0 0 1 0 0 1

0 0 0 1 1 1 0 0 0 1 1 1 1 0 1 0 0 0 1 0 1 0 1 0 1 0 0 0 1

1 1 0 1 0 0 1 0 0 0 1 1 1 1 0 1 0 0 0 0 1 0 1 0 0 0 0 1 1

1 1 0 1 0 0 1 0 0 0 0 0 0 0 1 0 1 1 1 1 1 1 1 0 0 0 0 1 0

1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 1 0 1 1 1 1 1 1 0 0 0 1 0 0

0 1 1 0 0 1 0 0 1 0 0 0 1 0 0 1 1 0 1 1 1 1 1 0 0 1 0 0 0

0 0 0 1 1 1 0 0 0 1 0 0 0 1 0 1 1 1 0 1 1 1 1 0 1 0 0 0 0

0 0 0 0 0 0 1 1 1 1 1 0 0 0 0 1 1 1 1 0 1 1 1 1 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 0 1 1 0 0 0 0 0 1

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 0 1 1 1 1 1 1 1

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1

0 0 0 0 0 0 1 1 1 1 1 0 0 0 0 0 0 0 0 1 0 1 1 0 1 1 1 1 1

0 0 0 1 1 1 0 0 0 1 0 0 0 1 0 0 0 0 1 0 0 1 1 1 0 1 1 1 1

0 1 1 0 0 1 0 0 1 0 0 0 1 0 0 0 0 1 0 0 0 1 1 1 1 0 1 1 1

1 0 1 0 1 0 0 1 0 0 0 1 0 0 0 0 1 0 0 0 0 1 1 1 1 1 0 1 1

1 1 0 1 0 0 1 0 0 0 0 0 0 0 1 1 0 0 0 0 0 1 1 1 1 1 1 0 1

0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 0 0 0 0 0 1 1 1 1 1 1 1 1 0

Adjacency matrix of induced subgraph on $\Gamma(v) \cup \{v\}$ in Kharaghani's one of twin SRGs with parameters (64, 28, 12, 12)

0 1 1 0 1 1 0 0 1 1 1 0 0 1 0 1 0 0 0 0 1 0 1 0 1 0 0 1 0

1 0 1 1 0 1 0 1 1 0 0 0 1 1 0 0 1 0 0 0 1 0 1 0 0 0 1 1 0

1 1 0 1 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 0 0 0 1 0 1 0 1 0 0

0 1 1 0 1 1 1 1 0 0 0 1 1 0 0 0 1 0 1 0 0 0 1 0 0 1 1 0 0

1 0 1 1 0 1 1 0 0 1 1 1 0 0 0 1 0 0 1 0 0 0 1 0 1 1 0 0 0

1 1 0 1 1 0 1 0 1 0 0 1 0 1 0 0 0 0 1 0 1 0 1 0 0 1 0 1 0

0 0 0 1 1 1 0 1 1 1 0 1 0 0 1 0 0 0 1 1 0 0 1 1 0 1 0 0 0

0 1 1 1 0 0 1 0 1 1 0 0 1 0 1 0 1 0 0 1 0 0 1 1 0 0 1 0 0

1 1 0 0 0 1 1 1 0 1 0 0 0 1 1 0 0 0 0 1 1 0 1 1 0 0 0 1 0

1 0 1 0 1 0 1 1 1 0 1 0 0 0 1 1 0 0 0 1 0 0 1 1 1 0 0 0 0

1 0 1 0 1 0 0 0 0 1 0 1 1 1 1 1 0 1 0 0 0 0 1 0 1 0 0 0 1

0 0 0 1 1 1 1 0 0 0 1 0 1 1 1 0 0 1 1 0 0 0 1 0 0 1 0 0 1

0 1 1 1 0 0 0 1 0 0 1 1 0 1 1 0 1 1 0 0 0 0 1 0 0 0 1 0 1

1 1 0 0 0 1 0 0 1 0 1 1 1 0 1 0 0 1 0 0 1 0 1 0 0 0 0 1 1

0 0 0 0 0 0 1 1 1 1 1 1 1 1 0 0 0 1 0 1 0 0 1 1 0 0 0 0 1

1 0 1 0 1 0 0 0 0 1 1 0 0 0 0 0 1 1 1 1 1 1 1 0 1 0 0 0 0

0 1 1 1 0 0 0 1 0 0 0 0 1 0 0 1 0 1 1 1 1 1 1 0 0 0 1 0 0

0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 0 1 1 1 1 1 0 0 0 0 0 1

0 0 0 1 1 1 1 0 0 0 0 1 0 0 0 1 1 1 0 1 1 1 1 0 0 1 0 0 0

0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 1 0 1 1 1 1 0 0 0 0 0

1 1 0 0 0 1 0 0 1 0 0 0 0 1 0 1 1 1 1 1 0 1 1 0 0 0 0 1 0

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 0 1 1 1 1 1 1 1

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1

0 0 0 0 0 0 1 1 1 1 0 0 0 0 1 0 0 0 0 1 0 1 1 0 1 1 1 1 1

1 0 1 0 1 0 0 0 0 1 1 0 0 0 0 1 0 0 0 0 0 1 1 1 0 1 1 1 1

0 0 0 1 1 1 1 0 0 0 0 1 0 0 0 0 0 0 1 0 0 1 1 1 1 0 1 1 1

0 1 1 1 0 0 0 1 0 0 0 0 1 0 0 0 1 0 0 0 0 1 1 1 1 1 0 1 1

1 1 0 0 0 1 0 0 1 0 0 0 0 1 0 0 0 0 0 0 1 1 1 1 1 1 1 0 1

0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 0 0 1 0 0 0 1 1 1 1 1 1 1 0

Adjacency matrix of induced subgraph on $\Gamma(v) \cup \{v\}$ in Kharaghani's the other twin SRG(64, 28, 12, 12)

0 1 1 1 1 0 1 1 0 0 0 0 0 1 1 0 1 1 0 0 0 0 1 0 1 1 0 0 0
1 0 1 1 0 1 0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 1 0 1 0 0 0 1
1 1 0 0 1 1 0 1 0 1 1 0 0 0 1 1 0 1 0 0 0 0 1 0 1 0 1 0 0
1 1 0 0 1 1 1 0 1 0 0 1 0 1 0 0 1 0 0 0 1 0 1 0 0 1 0 0 1
1 0 1 1 0 1 1 0 0 1 1 0 0 1 0 1 1 0 0 0 0 0 1 0 0 1 1 0 0
0 1 1 1 1 0 0 0 1 1 1 1 0 0 0 1 0 0 0 0 1 0 1 0 0 0 1 0 1
1 0 0 1 1 0 0 1 1 1 0 0 1 1 0 0 1 0 0 1 0 0 1 0 0 1 0 1 0
1 1 1 0 0 0 1 0 1 1 0 0 1 0 1 0 0 1 0 1 0 0 1 0 1 0 0 1 0
0 1 0 1 0 1 1 1 0 1 0 1 1 0 0 0 0 0 1 1 0 1 0 0 0 0 0 1 1
0 0 1 0 1 1 1 1 1 0 1 0 1 0 0 1 0 0 0 1 0 0 1 0 0 0 1 1 0
0 0 1 0 1 1 0 0 0 1 0 1 1 1 1 1 0 0 1 0 0 0 1 1 0 0 1 0 0
0 1 0 1 0 1 0 0 1 0 1 0 1 1 1 0 0 0 1 0 1 0 1 1 0 0 0 0 1
0 0 0 0 0 0 1 1 1 1 1 1 0 1 1 0 0 0 1 1 0 0 1 1 0 0 0 1 0
1 0 0 1 1 0 1 0 0 0 1 1 1 0 1 0 1 0 1 0 0 0 1 1 0 1 0 0 0
1 1 1 0 0 0 0 1 0 0 1 1 1 1 0 0 0 1 1 0 0 0 1 1 1 0 0 0 0
0 0 1 0 1 1 0 0 0 1 1 0 0 0 0 0 1 1 1 1 1 1 1 0 0 0 1 0 0
1 0 0 1 1 0 1 0 0 0 0 0 1 0 1 0 1 1 1 1 1 1 0 0 1 0 0 0
1 1 1 0 0 0 0 1 0 0 0 0 0 1 1 1 0 1 1 1 1 1 0 1 0 0 0 0
0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 0 1 1 1 1 1 0 0 0 0 0
0 0 0 0 0 0 1 1 1 1 0 0 1 0 0 1 1 1 1 0 1 1 1 0 0 0 0 1 0
0 1 0 1 0 1 0 0 1 0 0 1 0 0 0 1 1 1 1 1 0 1 1 0 0 0 0 0 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 0 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 1
0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 0 0 0 1 0 0 1 1 0 1 1 1 1 1
1 1 1 0 0 0 0 1 0 0 0 0 0 0 1 0 0 1 0 0 0 1 1 1 0 1 1 1 1
1 0 0 1 1 0 1 0 0 0 0 0 0 1 0 0 1 0 0 0 0 1 1 1 1 0 1 1 1

0 0 1 0 1 1 0 0 0 1 1 0 0 0 0 1 0 0 0 0 0 1 1 1 1 1 0 1 1

0 0 0 0 0 0 1 1 1 1 0 0 1 0 0 0 0 0 0 1 0 1 1 1 1 1 1 0 1

0 1 0 1 0 1 0 0 1 0 0 1 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 0

B.2 Relation matrices of 3-class fission schemes of halved-folded 8-cube

The one isomorphic to Folded 7-cube

0222222222222222222222111111333333333333333333333333333333333333
2022223223322333223333333311222332233233331223323333123333133311
2202232232323233323233333313122323232332331323233233133233133131
2220322322332233332233333311322332233233213323323321333321331331
22230222332233232332333331331232233223331233332233312333312333113
22322023232323233323233333131323232323231332332323133233133231313
2322220332233223333223333311332332233322133332333221333321333321133
22232330222233332233323331333132223333122323333312232333313112333
2232323202232333232332333133133223233313232323333132323233131313233
2322332220233323233323233313133323223133322332313333223321331133323
233322222203332233322333311333332221333333322213333332221113333332
22232332333022222333332313333133331222322323333313313112232332333
22323233233202223233332313331333313223223232333133131312323233233
2322332332322022332333231331333313323222322333231331331133223323323
23332223333222202333323231313333133332223333222133311133333332223332
233333322222222033333223113333313333333322222211133333333333332222
2223233233323333022222133333133331333133131122232232332232332333
2232323323332333202222133331333313331331313122322323232323233233
23223323323332333220222133313333133313313311323223223323223323323
233322233323333232220221331333313331333311133332223332223332223332
23333333222233333222220213133331333311113333333333322222223333332222
233333333332222222222011333333111113333333333333333333332222222222
1333333333333333311111102222222222222222222223333333333333333333
133333333331111133333312022222222222333333333332222222222333333333
13333331111333331333313220222223333222223333333222233333332222223333
1333111333133313333133222022223233323333222333323333222333322233332223

1311331331333133331333222202233233323323332233233233223233223232232
113131331333133331333322220233323333233232323323323232323232322322
1113133133313333133333222222033332333233232233332332322332322322222
322222233333333133331122233330222222223333333222233333333333331111
3222333222333313333131223233320222233322233332333222333333331113331
32332232232331333313312233233220223233233223323323322331133133133
33232322322231333313331223333232220233233232323323323232131313133133
3332322322213333133331223333222220333233232233332332322211313311333
32223333331222333333113232233322333022222233323333333111222233333331
3233223331322323331313232323323233202223322332333113312332233313
333232323133232223313313232332323323220232323233231313133232323133
3332322133332223133313232333223332222033232233332113133332322213333
3233333122332233233113323322333223322330222233311233331233331222333
3323313232323232313133233232332323232320223231313233133233132323
3332133322332232133133233233232332233222032231133321333321333223
3331233233223322311333233322333223322322302213313312333312332332
33133233232323222131333233323233232323223220213133133233133233232
313333233223332221133332333322333223322322220113313333213333323322
322233333313333112223333222333223332333333111022222233322233333331
3233223331333131223233322323323233323331133120222332232332233313
33232323133313312322333223323233233323131313220232323232323232313 3
3332322133313331322233332233322333233332113133222033232233232221333
323333312233333113223323323223333223333112333312233022223233333122 33
33233132323313132323233232323332323313132331323232022323233132323
3332133322313313322323323233323233231133321332333222032233213332 23
3331233233231133233223323322333223133133123333223223022331233 2332
331332333232131333323223323323233232131331332332322322202313323323 2
313333233322113333332223323333223333221133133332333223222220133332 3322

3233331331122333223332332223333311223323333122332333310222232233

3323313313123233232332332232333131232332331323233233132022322323

3332133133132233322332332233233113233233213323323321332203223223

3331233311323323233232332322331331322333123332233312332230222332

3313323131332323323232332323231313323231332332323133232322023232

3133332113333223332232332332231133332213333233221333323222203322

333131123332333322333223332223133313331223233333122323322323330222

3313131323332332323322333223213313331323232333132323232323232022

3133113332333232332322333232213133313332233231333223323223322202

311133333323333223332223333222113331333333222133333322233332222220

The one isomorphic to Halved 7-cube

0111111111111111111111133333332222222222222222222222222222222222

1011112112211222112222222222331112211221222231122122223122223223 3

1101121121212122212122222223231121212122122321212212232212232232 3

1110211211221122211222222223321122112212213221221221322221322322 3

1112011122112212122122222322312112211222312221122231222231222332

1121101212121212212122222323212121212123221221212322122322123232

1211110221122112221122222332212211221132222122113222213222213322

1112122011112221122212223222321112222311212222231121222232331222

1121212101121222121221222322322112122321212122232121212232323212 2

1211221110122121221212223232222121123222112212322211221322332221 2

1222111111022211222112223322222111322222221113222222211133322222 21

1112122122201111122222123222232222311121121222222322323311212212 22

1121212212210111212222123222322223211211212122232232323121212212 2

1211221221211011221221232232222322212112112212322322332211221221 2

1222111222111101222121232322223222211122211132223332222222111222 1

1222222111111110222211233222232222222221111113333222222222222221111

1112122122212222011111322222322223222322323311121121221121221222

1121212212221222101111322223222232222322232323112112121212122122

1211221221222122110111322232222322232232233212112112212112212212

1222111222122212111011322322222322232222333222211122211122211122 21

1222222111122222111110132322222322223333222222222221111112222222 1111

1222222222221111111111033222222333332222222222222222222222111111111

3222222222222222223333330111111111111111111112222222222222222222222

3222222222223333322222310111111111122222222222111111111122222222222

3222222333322223222232110111111222211112222222111122222221111112222

3222333222322232222322111011112122212221112222122211122221112222111 2

3233322322322232222322211110112212221221221122122122112122112112 1

332323223222322222322222111110122212222122121212212221212121212211211

333232232222322222322222211111022221222122121122212221211221211211 1

2111111222222222322222331112222201111111222222211112222222222222223333

2111222111222232222232311212222101111222111222122211122222223332223

2122211211212223222223223112212221101121222122112212221221122332232232

2212121121123222223222311222121110122122121212212212121323232322322

2221211211132222232222311222221111102221221211222122212113323223222

21111222222231112222223321211222211222011111122212222222333111222222223

2122211222321121222323212121221212221011122112212222332223122111 22232

2212121232212112232232121221212212110121212122212323232212121212322

2221211322221112322232121222112221111022121122221332322222121132222

2122222311221122122233221221122211221122011112223312222312222311 22

22122321212121212123232212212122121212121011212323212232212232121 2

2221322211221121322322122122121221122111021123322213222221322211 2

22231221221122112332221222112221122112112011322322312222312212 21

2232212212121211323222122212122121212112110132322322122322122121

2322222122112211133222212222112221122112111103322232222132222122 11

211122222223222233111222221112222112221222222233301111112222111222222223

2122211222322232311212222112122121222122233222310111221121221122232

2212121232222322231211222211221212212222123232321101212121212121212322

2221211322232222321112222112221122221222213323221110221211221212113222

2122222311222223321122122121122221122222331222231122011112122222231122

2212232121222323212121221212122121222323212232121210112121222321212

2221322211232232221121221212212122123322213221221110211221322211 2

22231221221233221221122122112221123223223122211211201122312212 21

2232212212132322212112212212122121323223221221211211012322122121

2322222122113322222111221222112221133223222212211211110322221221 1

2122222322331122211222122111222222331122122222311221222223011112 1122

221223223231212212122122112122232312122122321212212232101121 1212

2221322322321122211221221122122332122122132212212213221102112112

2223122233212212122121221211223223211222312221122231221120111221

2232212323221212212121221212123232212123221221212322121211012121

2322221332222112221121221221123322221132222122113222212111102211

2223233122212221122211222111232223222311212222231121221121220111

2232323212221221212211222112132232222321212122232121212121211011

2322332221222121221211222121132322232221122123222112212112211101

2333222222122211222111222211133222322222211132222222111222211111 10

B.3 Symmetric Bush-type Hadamard matrices (SBHMs) of order 64

The symbol '5' in the table indicates the entry '−1' throughout this section.

The SBHMs H_1, H_2, H_3 are the ones discussed in Section 3.2.

SBHM H_1

11111111 15151515 11551155 15511551 11115555 15155151 11555511 15515115

11111111 51515151 11551155 51155115 11115555 51511515 11555511 51151551

11111111 15151515 55115511 51155115 11115555 15155151 55111155 51151551

11111111 51515151 55115511 15511551 11115555 51511515 55111155 15515115

11111111 15151515 11551155 15511551 55551111 51511515 55111155 51151551

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11111111 15151515 55115511 51155115 55551111 51511515 11555511 15515115

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51515151 11111111 51155115 11551155 51511515 11115555 51151551 11555511

15151515 11111111 51155115 55115511 15155151 11115555 51151551 55111155

51515151 11111111 15511551 55115511 51511515 11115555 15515115 55111155

15151515 11111111 15511551 11551155 51511515 55551111 51151551 55111155

51515151 11111111 51155115 11551155 15155151 55551111 15515115 55111155

15151515 11111111 51155115 55115511 51511515 55551111 15515115 11555511

51515151 11111111 15511551 55115511 15155151 55551111 51151551 11555511

11551155 15511551 11111111 15151515 11555511 15515115 11115555 15155151

11551155 51155115 11111111 51515151 11555511 51151551 11115555 51511515

55115511 51155115 11111111 15151515 55111155 51151551 11115555 15155151

55115511 15511551 11111111 51515151 55111155 15515115 11115555 51511515

11551155 15511551 11111111 15151515 55111155 51151551 55551111 51511515

11551155 51155115 11111111 51515151 55111155 15515115 55551111 15155151

55115511 51155115 11111111 15151515 11555511 15515115 55551111 51511515

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15511551 11551155 15151515 11111111 15515115 11555511 15155151 11115555

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55551111 51511515 55111155 51151551 11111111 15151515 11551155 15511551

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51511515 11115555 15515115 55111155 51515151 11111111 15511551 55115511

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11555511 51151551 11115555 51511515 11551155 51155115 11111111 51515151

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11555511 15515115 55551111 51511515 55115511 51155115 11111111 15151515

11555511 51151551 55551111 15155151 55115511 15511551 11111111 51515151

15515115 11555511 15155151 11115555 15511551 11551155 15151515 11111111

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51151551 55111155 15155151 11115555 51155115 55115511 15151515 11111111

15515115 55111155 51511515 11115555 15511551 55115511 51515151 11111111

51151551 55111155 51511515 55551111 15511551 11551155 15151515 11111111

15515115 55111155 15155151 55551111 51155115 11551155 51515151 11111111

15515115 11555511 51511515 55551111 51155115 55115511 15151515 11111111

51151551 11555511 15155151 55551111 15511551 55115511 51515151 11111111

SBHM H_2

11111111 11115555 15151515 15515115 11551155 15155151 11555511 15511551

11111111 11115555 51515151 51151551 11551155 51511515 11555511 51155115

11111111 11115555 15151515 51151551 55115511 15155151 55111155 51155115

11111111 11115555 51515151 15515115 55115511 51511515 55111155 15511551

11111111 55551111 15151515 51151551 11551155 51511515 55111155 15511551

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51515151 15515115 11111111 11115555 55111155 15511551 55115511 51511515

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51155115 11555511 51511515 55115511 15515115 15151515 55551111 11111111

15511551 11555511 15155151 55115511 51151551 51515151 55551111 11111111

SBHM H_3

11111111 11115555 15151515 15511551 11551155 15155151 11555511 15515115

11111111 11115555 51515151 51155115 11551155 51511515 11555511 51151551

11111111 11115555 15151515 51155115 55115511 15155151 55111155 51151551

11111111 11115555 51515151 15511551 55115511 51511515 55111155 15515115

11111111 55551111 15151515 15511551 11551155 51511515 55111155 51151551

11111111 55551111 51515151 51155115 11551155 15155151 55111155 15515115

11111111 55551111 15151515 51155115 55115511 51511515 11555511 15515115

11111111 55551111 51515151 15511551 55115511 15155151 11555511 51151551

11115555 11111111 15511551 15151515 15155151 11551155 15515115 11555511

11115555 11111111 51155115 51515151 51511515 11551155 51151551 11555511

11115555 11111111 51155115 15151515 15155151 55115511 51151551 55111155

11115555 11111111 15511551 51515151 51511515 55115511 15515115 55111155

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55551111 11111111 15511551 51515151 15155151 55115511 51151551 11555511

15151515 15511551 11111111 11115555 11555511 15515115 11551155 15155151

51515151 51155115 11111111 11115555 11555511 51151551 11551155 51511515

15151515 51155115 11111111 11115555 55111155 51151551 55115511 15155151

51515151 15511551 11111111 11115555 55111155 15515115 55115511 51511515

15151515 15511551 11111111 55551111 55111155 51151551 11551155 51511515

51515151 51155115 11111111 55551111 55111155 15515115 11551155 15155151

15151515 51155115 11111111 55551111 11555511 15515115 55115511 51511515

51515151 15511551 11111111 55551111 11555511 51151551 55115511 15155151

15511551 15151515 11115555 11111111 15515115 11555511 15155151 11551155

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51155115 15151515 11115555 11111111 51151551 55111155 15155151 55115511

15511551 51515151 11115555 11111111 15515115 55111155 51511515 55115511

15511551 15151515 55551111 11111111 51151551 55111155 51511515 11551155

51155115 51515151 55551111 11111111 15515115 55111155 15155151 11551155

51155115 15151515 55551111 11111111 15515115 11555511 51511515 55115511

15511551 51515151 55551111 11111111 51151551 11555511 15155151 55115511

11551155 15155151 11555511 15515115 11111111 11115555 15151515 15511551

11551155 51511515 11555511 51151551 11111111 11115555 51515151 51155115

55115511 15155151 55111155 51151551 11111111 11115555 15151515 51155115

55115511 51511515 55111155 15515115 11111111 11115555 51515151 15511551

11551155 51511515 55111155 51151551 11111111 55551111 15151515 15511551

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55115511 51511515 11555511 15515115 11111111 55551111 15151515 51155115

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15155151 11551155 15515115 11555511 11115555 11111111 15511551 15151515

51511515 11551155 51151551 11555511 11115555 11111111 51155115 51515151

15155151 55115511 51151551 55111155 11115555 11111111 51155115 15151515

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51511515 11551155 51151551 55111155 55551111 11111111 15511551 15151515

15155151 11551155 15515115 55111155 55551111 11111111 51155115 51515151

51511515 55115511 15515115 11555511 55551111 11111111 51155115 15151515

15155151 55115511 51151551 11555511 55551111 11111111 15511551 51515151

11555511 15515115 11551155 15155151 15151515 15511551 11111111 11115555

11555511 51151551 11551155 51511515 51515151 51155115 11111111 11115555

55111155 51151551 55115511 15155151 15151515 51155115 11111111 11115555

55111155 15515115 55115511 51511515 51515151 15511551 11111111 11115555

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11555511 15515115 55115511 51511515 15151515 51155115 11111111 55551111

11555511 51151551 55115511 15155151 51515151 15511551 11111111 55551111

15515115 11555511 15155151 11551155 15511551 15151515 11115555 11111111

51151551 11555511 51511515 11551155 51155115 51515151 11115555 11111111

51151551 55111155 15155151 55115511 51155115 15151515 11115555 11111111

15515115 55111155 51511515 55115511 15511551 51515151 11115555 11111111

51151551 55111155 51511515 11551155 15511551 15151515 55551111 11111111

15515115 55111155 15155151 11551155 51155115 51515151 55551111 11111111

15515115 11555511 51511515 55115511 51155115 15151515 55551111 11111111

51151551 11555511 15155151 55115511 15511551 51515151 55551111 11111111

B.4 Relation Matrices of 3-class imprimitive symmetric schemes obtained from SBHMs of order 64

The SBHMs H_1, H_2, H_3 are the ones discussed in Section 3.2.

Scheme from SBHM H_1

03333333 12121212 11221122 12211221 11112222 12122121 11222211 12212112
30333333 21212121 11221122 21122112 11112222 21211212 11222211 21121221
33033333 12121212 22112211 21122112 11112222 12122121 22111122 21121221
33303333 21212121 22112211 12211221 11112222 21211212 22111122 12212112
33330333 12121212 11221122 12211221 22221111 21211212 22111122 21121221
33333033 21212121 11221122 21122112 22221111 12122121 22111122 12212112
33333303 12121212 22112211 21122112 22221111 21211212 11222211 12212112
33333330 21212121 22112211 12211221 22221111 12122121 11222211 21121221
12121212 03333333 12211221 11221122 12122121 11112222 12212112 11222211
21212121 30333333 21122112 11221122 21211212 11112222 21121221 11222211
12121212 33033333 21122112 22112211 12122121 11112222 21121221 22111122
21212121 33303333 12211221 22112211 21211212 11112222 12212112 22111122
12121212 33330333 12211221 11221122 21211212 22221111 21121221 22111122
21212121 33333033 21122112 11221122 12122121 22221111 12212112 22111122
12121212 33333303 21122112 22112211 21211212 22221111 12212112 11222211
21212121 33333330 12211221 22112211 12122121 22221111 21121221 11222211
11221122 12211221 03333333 12121212 11222211 12212112 11112222 12122121
11221122 21122112 30333333 21212121 11222211 21121221 11112222 21211212
22112211 21122112 33033333 12121212 22111122 21121221 11112222 12122121
22112211 12211221 33303333 21212121 22111122 12212112 11112222 21211212
11221122 12211221 33330333 12121212 22111122 21121221 22221111 21211212
11221122 21122112 33333033 21212121 22111122 12212112 22221111 12122121

22112211 21122112 33333303 12121212 11222211 12212112 22221111 21211212

22112211 12211221 33333330 21212121 11222211 21121221 22221111 12122121

12211221 11221122 12121212 03333333 12212112 11222211 12122121 11112222

21122112 11221122 21212121 30333333 21121221 11222211 21211212 11112222

21122112 22112211 12121212 33033333 21121221 22111122 12122121 11112222

12211221 22112211 21212121 33303333 12212112 22111122 21211212 11112222

12211221 11221122 12121212 33330333 21121221 22111122 21211212 22221111

21122112 11221122 21212121 33333033 12212112 22111122 12122121 22221111

21122112 22112211 12121212 33333303 12212112 11222211 21211212 22221111

12211221 22112211 21212121 33333330 21121221 11222211 12122121 22221111

11112222 12122121 11222211 12212112 03333333 12121212 11221122 12211221

11112222 21211212 11222211 21121221 30333333 21212121 11221122 21122112

11112222 12122121 22111122 21121221 33033333 12121212 22112211 21122112

11112222 21211212 22111122 12212112 33303333 21212121 22112211 12211221

22221111 21211212 22111122 21121221 33330333 12121212 11221122 12211221

22221111 12122121 22111122 12212112 33333033 21212121 11221122 21122112

22221111 21211212 11222211 12212112 33333303 12121212 22112211 21122112

22221111 12122121 11222211 21121221 33333330 21212121 22112211 12211221

12122121 11112222 12212112 11222211 12121212 03333333 12211221 11221122

21211212 11112222 21121221 11222211 21212121 30333333 21122112 11221122

12122121 11112222 21121221 22111122 12121212 33033333 21122112 22112211

21211212 11112222 12212112 22111122 21212121 33303333 12211221 22112211

21211212 22221111 21121221 22111122 12121212 33330333 12211221 11221122

12122121 22221111 12212112 22111122 21212121 33333033 21122112 11221122

21211212 22221111 12212112 11222211 12121212 33333303 21122112 22112211

12122121 22221111 21121221 11222211 21212121 33333330 12211221 22112211

11222211 12212112 11112222 12122121 11221122 12211221 03333333 12121212

11222211 21121221 11112222 21211212 11221122 21122112 30333333 21212121

22111122	21121221	11112222	12122121	22112211	21122112	33033333	12121212
22111122	12212112	11112222	21211212	22112211	12211221	33303333	21212121
22111122	21121221	22221111	21211212	11221122	12211221	33330333	12121212
22111122	12212112	22221111	12122121	11221122	21122112	33333033	21212121
11222211	12212112	22221111	21211212	22112211	21122112	33333303	12121212
11222211	21121221	22221111	12122121	22112211	12211221	33333330	21212121
12212112	11222211	12122121	11112222	12211221	11221122	12121212	03333333
21121221	11222211	21211212	11112222	21122112	11221122	21212121	30333333
21121221	22111122	12122121	11112222	21122112	22112211	12121212	33033333
12212112	22111122	21211212	11112222	12211221	22112211	21212121	33303333
21121221	22111122	21211212	22221111	12211221	11221122	12121212	33330333
12212112	22111122	12122121	22221111	21122112	11221122	21212121	33333033
12212112	11222211	21211212	22221111	21122112	22112211	12121212	33333303
21121221	11222211	12122121	22221111	12211221	22112211	21212121	33333330

Scheme from SBHM H_2

03333333 11112222 12121212 12212112 11221122 12122121 11222211 12211221

30333333 11112222 21212121 21121221 11221122 21211212 11222211 21122112

33033333 11112222 12121212 21121221 22112211 12122121 22111122 21122112

33303333 11112222 21212121 12212112 22112211 21211212 22111122 12211221

33330333 22221111 12121212 21121221 11221122 21211212 22111122 12211221

33333033 22221111 21212121 12212112 11221122 12122121 22111122 21122112

33333303 22221111 12121212 12212112 22112211 21211212 11222211 21122112

33333330 22221111 21212121 21121221 22112211 12122121 11222211 12211221

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11112222 33033333 21121221 12121212 12122121 22112211 21122112 22111122

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21212121 21121221 30333333 11112222 11222211 21122112 11221122 21211212

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21121221 21212121 11112222 30333333 21122112 11222211 21211212 11221122

21121221 12121212 11112222 33033333 21122112 22111122 12122121 22112211

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12211221	22111122	21211212	11221122	21121221	12121212	22221111	33330333
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Scheme from SBHM H_3

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12212112	11222211	21211212	22112211	21122112	12121212	22221111	33333303
21121221	11222211	12122121	22112211	12211221	21212121	22221111	33333330

B.5 Relation matrices of 3-class imprimitive non-symmetric schemes obtained from SBHMs of order 64

The SBHMs H_1, H_2, H_3 are the ones discussed in Section 3.2.

Scheme from SBHM H_1

03333333 12121212 11221122 12211221 11112222 12122121 11222211 12212112

30333333 21212121 11221122 21122112 11112222 21211212 11222211 21121221

33033333 12121212 22112211 21122112 11112222 12122121 22111122 21121221

33303333 21212121 22112211 12211221 11112222 21211212 22111122 12212112

33330333 12121212 11221122 12211221 22221111 21211212 22111122 21121221

33333033 21212121 11221122 21122112 22221111 12122121 22111122 12212112

33333303 12121212 22112211 21122112 22221111 21211212 11222211 12212112

33333330 21212121 22112211 12211221 22221111 12122121 11222211 21121221

21212121 03333333 12211221 11221122 12122121 11112222 12212112 11222211

12121212 30333333 21122112 11221122 21211212 11112222 21121221 11222211

21212121 33033333 21122112 22112211 12122121 11112222 21121221 22111122

12121212 33303333 12211221 22112211 21211212 11112222 12212112 22111122

21212121 33330333 12211221 11221122 21211212 22221111 21121221 22111122

12121212 33333033 21122112 11221122 12122121 22221111 12212112 22111122

21212121 33333303 21122112 22112211 21211212 22221111 12212112 11222211

12121212 33333330 12211221 22112211 12122121 22221111 21121221 11222211

22112211 21122112 03333333 12121212 11222211 12212112 11112222 12122121

22112211 12211221 30333333 21212121 11222211 21121221 11112222 21211212

11221122 12211221 33033333 12121212 22111122 21121221 11112222 12122121

11221122 21122112 33303333 21212121 22111122 12212112 11112222 21211212

22112211 21122112 33330333 12121212 22111122 21121221 22221111 21211212

22112211 12211221 33333033 21212121 22111122 12212112 22221111 12122121

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12211221	22112211	12121212	33333033	12212112	22111122	12122121	22221111
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22221111	21211212	22111122	21121221	03333333	12121212	11221122	12211221
22221111	12122121	22111122	12212112	30333333	21212121	11221122	21122112
22221111	21211212	11222211	12212112	33033333	12121212	22112211	21122112
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Scheme from SBHM H_2

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33033333	11112222	12121212	21121221	22112211	12122121	22111122	21122112
33303333	11112222	21212121	12212112	22112211	21211212	22111122	12211221
33330333	22221111	12121212	21121221	11221122	21211212	22111122	12211221
33333033	22221111	21212121	12212112	11221122	12122121	22111122	21122112
33333303	22221111	12121212	12212112	22112211	21211212	11222211	21122112
33333330	22221111	21212121	21121221	22112211	12122121	11222211	12211221
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22221111	30333333	21121221	21212121	21211212	11221122	21122112	11222211
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22221111	33303333	12212112	21212121	21211212	22112211	12211221	22111122
11112222	33330333	21121221	12121212	21211212	11221122	12211221	22111122
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22111122 12211221 11221122 12122121 21212121 21121221 33333303 22221111

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21122112	11222211	12122121	22112211	12212112	21212121	11112222	33330333
12211221	11222211	21211212	22112211	21121221	12121212	11112222	33333033
12211221	22111122	12122121	11221122	21121221	21212121	11112222	33333303
21122112	22111122	21211212	11221122	12212112	12121212	11112222	33333330

Scheme from SBHM H_3

03333333 11112222 12121212 12211221 11221122 12122121 11222211 12212112

30333333 11112222 21212121 21122112 11221122 21211212 11222211 21121221

33033333 11112222 12121212 21122112 22112211 12122121 22111122 21121221

33303333 11112222 21212121 12211221 22112211 21211212 22111122 12212112

33330333 22221111 12121212 12211221 11221122 21211212 22111122 21121221

33333033 22221111 21212121 21122112 11221122 12122121 22111122 12212112

33333303 22221111 12121212 21122112 22112211 21211212 11222211 12212112

33333330 22221111 21212121 12211221 22112211 12122121 11222211 21121221

22221111 03333333 12211221 12121212 12122121 11221122 12212112 11222211

22221111 30333333 21122112 21212121 21211212 11221122 21121221 11222211

22221111 33033333 21122112 12121212 12122121 22112211 21121221 22111122

22221111 33303333 12211221 21212121 21211212 22112211 12212112 22111122

11112222 33330333 12211221 12121212 21211212 11221122 21121221 22111122

11112222 33333033 21122112 21212121 12122121 11221122 12212112 22111122

11112222 33333303 21122112 12121212 21211212 22112211 12212112 11222211

11112222 33333330 12211221 21212121 12122121 22112211 21121221 11222211

21212121 21122112 03333333 11112222 11222211 12212112 11221122 12122121

12121212 12211221 30333333 11112222 11222211 21121221 11221122 21211212

21212121 12211221 33033333 11112222 22111122 21121221 22112211 12122121

12121212 21122112 33303333 11112222 22111122 12212112 22112211 21211212

21212121 21122112 33330333 22221111 22111122 21121221 11221122 21211212

12121212 12211221 33333033 22221111 22111122 12212112 11221122 12122121

21212121 12211221 33333303 22221111 11222211 12212112 22112211 21211212

12121212 21122112 33333330 22221111 11222211 21121221 22112211 12122121

21122112 21212121 22221111 03333333 12212112 11222211 12122121 11221122

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12211221	21212121	11112222	33333303	12212112	11222211	21211212	22112211
21122112	12121212	11112222	33333330	21121221	11222211	12122121	22112211
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12212112	22111122	21211212	11221122	21122112	12121212	11112222	33333330

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ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my advisor, Prof. Sung-Yell Song for his guidance, patience and support throughout this research and its writing. I would also like to thank my committee members for their help in writing this thesis: Dr. Clifford Bergman, Dr. Fritz Keinert, Dr. Ling Long, and Dr. Jonathan D. H. Smith. I would additionally like to thank Prof. Hyoung June Ko for his moral support and encouragement throughout the initial stages of my graduate career.

I would like to acknowledge my mother, brother and sisters, wife's family members, especially parents-in-law, and Prof. Song's wife, Tae-Hee Song for their encouragement and love to me. Special thanks are due to my lovely wife, Minjung Lee. I am indebted to her and my daughter and son, Sehyun and Moohyun who are the hope and bliss of my life, for their endless love, patience and encouragement throughout my graduate career.