BURSTING SPEED OF ROTATING DISCS

by

Harold Alexander Heckart

A Dissertation Submitted to the

Graduate Faculty in Partial Fulfillment of

The Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Major Subject: Applied Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State College

1957

TABLE OF CONTENTS

ABSTRACT

This thesis is an extension of a paper by Weiss and Prager[16] in which these authors have applied Tresca's yield condition and the associated flow rule to the determination of the bursting speed of a rotating annular disc, having constant initial thickness. In this thesis the results of the above paper are extended directly to the problem of the axially-symmetric annular disc with an arbitrary initial thickness. It is found that Tresca's yield condition and the associated flow rule do not appear to be applicable to the problem of the solid disc.

Von Mises' yield condition is next applied to the problem of the axially-symmetric disc with an arbitrary initial thickness. Under the assumption that the elastic strains are negligible in comparison with the plastic strains, the Von Mises' stress-strain rate law is used, rather than the more complex Prandtl-Reuss stress-strain law. A set of simultaneous equations is obtained, whose solution is the set of stresses and strains corresponding to the considered angular velocity of the disc. Bursting speed of the disc is assumed to be that value of angular speed for which the strains may increase indefinitely without further increase in rotational speed. In general, this system of equations must be solved numerically, a process which may be carried out with the aid of a desk calculator.

The bursting speed of a solid disc having a constant initial thickness is computed, using Von Mises' yield condition, and the results compared with the bursting speed of an annular disc obtained by the use of Tresca's yield condition. This comparison was made by way of some experimental results of Holms and Jenkins[6], and the agreement is found to be satisfactory.

INTRODUCTION

The problem of determining the limitations which must be placed upon the angular velocity of a rotating disc has attracted considerable attention in the past, and contemporary progress in gas-turbine and aircraft jet-engine design is accelerating research in this field. It is desirable that accurate data be available, in order that such turbine discs may be designed to operate safely. In general, there are two distinct factors which limit the angular velocity at which a disc may operate. First, in many types of machinery, the inherent tolerances may place a limit upon the amount of deformation which the disc may undergo without regard to the actual bursting speed. The second applies in situations where deformation of the disc is not in itself important, in which case the actual bursting speed of the disc may well be the only limiting factor. It is with the latter type of limitation that this paper deals.

Experimental data have been obtained by several investigators; for example, Holms and Jenkins[6], Mac Gregor and Tierney[8] and Skidmore[14]. Unfortunately, stress-strain data for the materials used in these experiments are not readily obtainable, hence it is not feasible to compare theoretical calculations with any of these results.

Under the assumption that stress and strain are linearly related - in other words, the material is elastic - the solution for stresses and strains is well-known. Stodola[15] (p. 157-169), for example, has treated discs with a thickness function of the form $h = cr^k$. By considering an arbitrary initial thickness as being formed by a number of annular rings of this type, the solution for the stresses and strains in such an arbitrary cross-section may be approximated. By means of such solutions the maximum angular velocity for which the material of the disc behaves elastically may be found.

The limitations on this type of solution are reached, however, for angular velocities considerably below that at which the disc ruptures. For somewhat greater angular velocities the material is said to become plastic; that is, the material remains deformed after the stresses have been removed. From a mathematical standpoint, there are two widely-used criteria for the transition from the elastic to the plastic state. Tresca's yield condition states that plastic yielding will occur when the maximum shear stress in the material reaches a critical value. This critical value is equal to the experimentally determined yield stress in pure shear. Von Mises' yield condition states that plastic yielding will occur when a certain invariant of the stress deviation tensor becomes equal to the square of the yield stress in simple shear. The exact forms of these criteria will be shown later.

Nadai and Donnell[10] first investigated the distribution of stresses and strains in a rotating disc for angular velocities ranging from that for which the yield point was first reached at some point in the disc to that for which the entire disc had just become plastic. A simple solution using Tresca's yield condition is also given by Hoffman and Sachs[5]. If it is assumed that the yield stress in pure shear remains constant as deformation proceeds, the material is said to be perfectly plastic. This implies that unrestricted plastic flow may take place without further increase in the stresses. This is usually a satisfactory approach when the plastic strains are fairly small, as, for example, when the angular velocity is limited by the amount of deformation to be tolerated rather than by the angular velocity at which the disc bursts.

However, the yield stress in pure shear does not remain constant in practical materials for large strains. Such materials are said to strain-harden, and this is the type of materials considered in this paper. For such materials it will be shown that the angular velocity at which the disc bursts represents a considerable increase over that at which the disc first becomes fully plastic.

GENERAL DISCUSSION

The mathematical theory of plasticity may be based upon the behavior of the material in question under a simple tensile test. When longitudinal force is applied to a cylindrical test specimen, the states of stress and strain near the ends of the specimen are quite complex. However, if the specimen is rather long, it is assumed that the states of stress and strain are homogeneous in the central portion of the specimen. It is customary to choose a 'gauge length' in this central portion, and then define the 'conventional tensile stress' S_{11} as the ratio of applied axial force to the original area of the cross-section, and the 'conventional tensile strain' ε_{11} as the ratio of elongation of the gauge length to its original length. On the other hand, 'true' stress and strain are defined with relation to the current area and current length of the specimen.

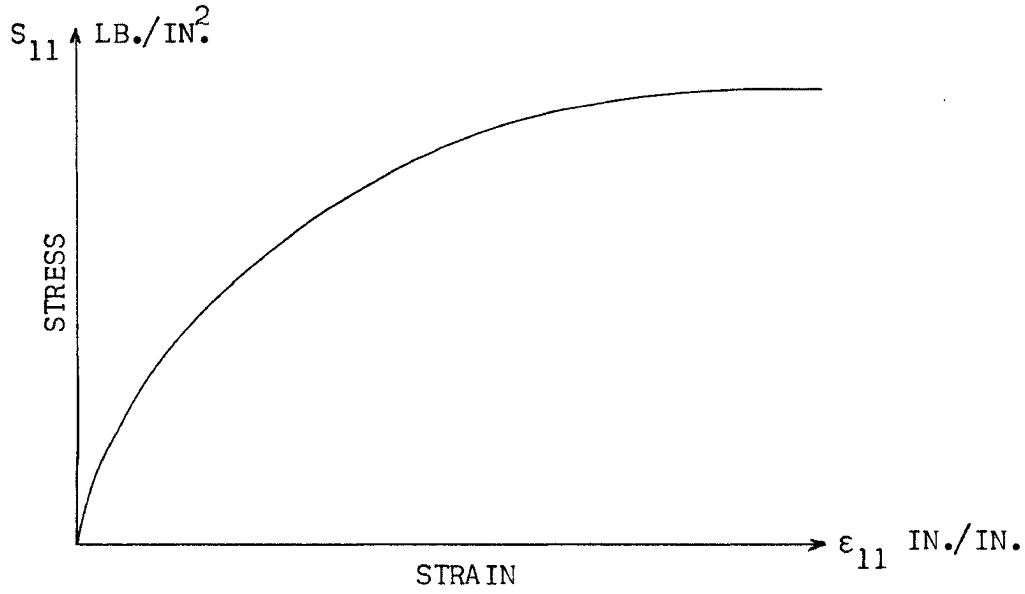

Figure 1. Stress-strain curve for a material which does not have a well-defined yield point.

If the conventional tensile stress S_{11} is plotted as a function of the conventional tensile strain ε_{11}, a curve such as that of Figure 1 results for many materials. For such materials, accurate measurements reveal the presence of small permanent strains even for very small stresses, and the transition to larger permanent strains is gradual. Such materials are said not to have a well-defined 'yield point', and will not be considered in this paper.

On the other hand, some materials such as structural steel, exhibit a stress-strain curve of the type shown in Figure 2. This stress-strain curve is characterized by a linear portion for small values of stress and strain. For states of stress and strain corresponding to points on the portion $\overline{OP}$, stress and strain are linearly related to within a high degree of approximation. Furthermore, over a portion of the curve which nearly

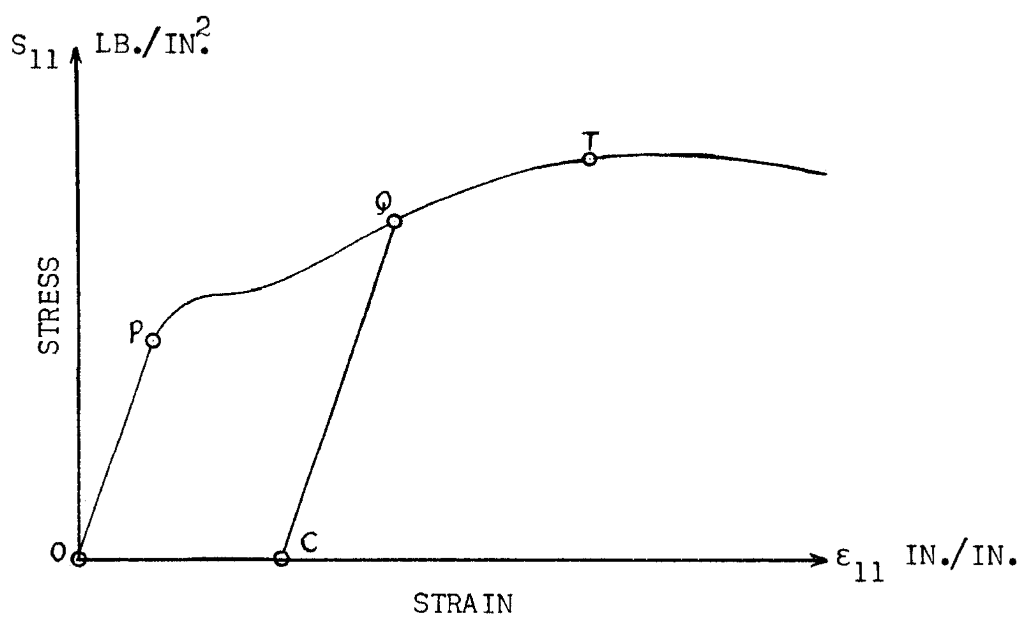

Figure 2. Stress-strain curve of a material having a well-defined yield point.

coincides with $\overline{OP}$, removal of the applied stress causes the strain to fall to zero. As long as the state of stress and strain corresponds to a point of this portion of the stress-strain curve, the material is said to be 'elastic'.

If, however, the state of stress and strain is such as to correspond to a point on the stress-strain curve which is not on this linear portion, for example point Q of Figure 2, then stress and strain are no longer linearly related. Furthermore, if the stress be reduced gradually, the state of stress and strain no longer corresponds to a point on the original stress-strain curve, but to a point on the path $\overline{QC}$, parallel to the linear portion $\overline{OP}$. The material is then said to have yielded plastically, the strain $\overline{OC}$ being the permanent or plastic strain. Upon re-applying the stress, the behavior of the material is very nearly described by the line $\overline{CQ}$ until point Q is reached. Application of further stress will then result in increased plastic yielding, and the state of stress and strain will then correspond to a point on the original stress-strain curve such as T in Figure 2.

For more general stress states, the transition from elastic to plastic behavior is more complicated. The discussion may well begin with a material which has not been strained beyond the elastic range. The stresses and strains within the material are uniquely related until plastic yielding begins. The criterion for incipient plastic yielding may then be expressed in terms of the stress components, only. The 'yield condition' may therefore be written in the form

$$Y(S_{ij}) = 0, \tag{1}$$

where S_{ij} is the collection of nine components characterizing the state of stress, called the 'stress tensor'. Since the material is usually assumed to be isotropic, the value of Y must not change when the axes to which the stresses are referred are changed. Thus, Y must be an invariant of the stress tensor. Bridgman[1] has found that hydrostatic pressure alone does not produce appreciable plastic deformation. Hence it is usually assumed that plastic deformation depends only upon the stress deviation, defined below.

If S_{11}, S_{22} and S_{33} are the normal stress components, the 'mean normal stress' is defined as

$$\bar{S} = 1/3(S_{11} + S_{22} + S_{33}). \tag{2}$$

The stress deviation then is defined by the tensor having normal components

$$\bar{S}_{11} = S_{11} - S, \quad \bar{S}_{22} = S_{22} - S, \quad \bar{S}_{33} = S_{33} - S, \tag{3}$$

and the same shear components as the stress tensor. Thus the stress deviation is related to the change in shape, whereas the mean normal stress is related to the change in volume. The yield condition may, then, be expressed as an invariant of the stress deviation tensor.

It is always possible to choose the coordinate axes in such a manner that the shear components of the stress tensor when referred to this set of axes vanish. For the remainder of this discussion it shall be assumed that axes have been so chosen, unless specifically stated otherwise. The yield condition then, is a function of three variables, the three normal stress components. Hence the yield condition may be discussed readily from a geometrical point of view.

The yield condition, of the form $Y(S_{ij}) = 0$, defines a surface in stress-space. Drucker[3,4] has shown that this yield surface is both closed and convex. For stress states corresponding to points interior to the yield surface, $Y < 0$, and elastic deformation only is possible. For stress states corresponding to points on the yield surface the material is assumed to be in a state of incipient plastic flow, since $Y = 0$.

For a disc in which the thickness is small compared to the outer radius a common assumption is that the axial stress is negligible. The following discussion will be limited to such cases of plane stress in order to take advantage of the simpler geometry associated with the yield surface. Generalization to the tri-axial stress problem offers no difficulty insofar as the basic theory goes, but the application of the theory may be extremely difficult.

Consider the trace of the yield surface - hereinafter referred to as the yield curve - in the $S_r S_\theta$ plane, as shown in Figure 3. For the sake of definiteness Von Mises' yield curve is shown, as given by Hoffman and Sachs[5] (Figure 4,3 p.42), but the argument applies to any arbitrary yield curve.

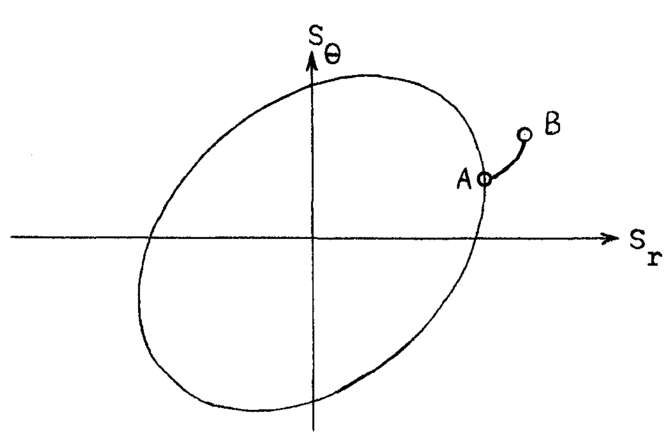

Figure 3. Yield curve with arbitrary stress path.

Assume an existing stress state corresponding to point A on the yield curve, and allow the stresses to change along the indicated path to point B. For such a stress change, the yield curve may change in either of three ways, or some combination thereof. The yield curve may distort locally, as in Figure 4; it may translate as a whole as in Figure 5; or

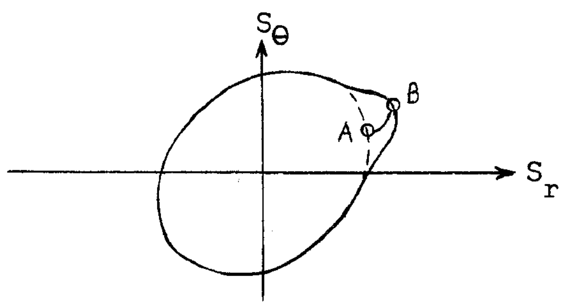

Figure 4. Local distortion of yield curve.

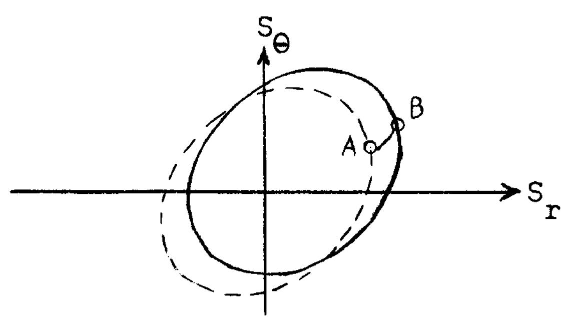

Figure 5. Translation of yield curve.

it may dilate as in Figure 6.

It seems likely that some combination of these three effects would be found in a practical material. However, it is desirable to effect some simplification of the problem, with the understanding that only experimental evidence can determine how well the simplified approach approximates actual conditions. Due to the complications presented by anisotropy, it is desirable to assume that the material remains isotropic under any program of stressing. This assumption requires that the yield curve change by dilatation only. The remaining problem in connection with the yield curve is the manner in which the dilatation is related to the strain program applied to the material. This will be covered later, in connection with the application of the different yield criteria.

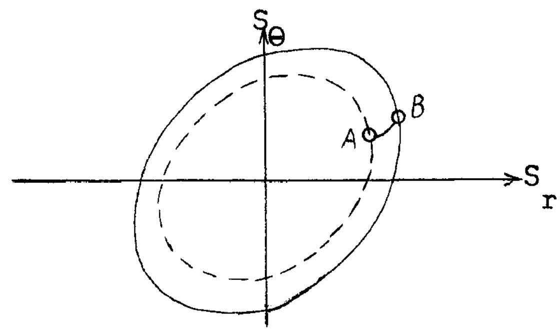

Figure 6. Dilatation of yield curve.

There are two widely used theories of the behavior of strain-hardening materials. The deformation theory assumes a unique relation between stress and strain, providing the prior history of the material does not include removal of stresses after plastic deformation has taken place; i.e., unloading has not occurred. On the other hand, the flow theory assumes that the rate at which the strains are changing is uniquely determined by the stresses. Zaid[17,18] and Manson[9], among others, have obtained solutions for the rotating disc problem making use of the deformation theory as a means of simplifying the problem, although it was known to these authors that there are theoretical objections to the deformation theory. These objections have been pointed out by Prager[11], who showed that the deformation theory does not yield unique results for certain changes in stress state, as does the flow theory.

Weiss and Prager[16] have obtained a solution for the bursting speed of a rotating annular disc with uniform initial thickness, using Tresca's yield condition and the associated flow rule. An extension of this work to the solid disc with an arbitrary initial thickness is desired.

There are several basic assumptions which are made in this paper. As noted above, it is commonly assumed in working with this problem that the axial stress is zero, and that the remaining stresses do not vary in the axial direction. This assumption is justified by the agreement of theoretical calculations with experimental evidence, where such is available, and will be adopted in this paper. The problem is further simplified by the assumption that the material is incompressible while undergoing plastic deformation. Empirical evidence in favor of this assumption was presented by Bridgman[1], who found that the compressibility of most metals is negligible under conditions of plastic deformation.

In the case of uniaxial tension the plastic strain obtaining when the material is near the breaking point is many times the elastic strain present in the material. In comparison with the plastic strains, the elastic strains are therefore negligible, and are usually ignored. The theory of plasticity arising from this assumption is known as the rigid-plastic theory, and is the theory proposed by Von Mises. It should be noted that this theory would not be tenable if the deformation were limited to values not greatly different from the maximum elastic deformation. Since this paper deals with large plastic strains, the rigid-plastic theory is adopted.

Finally, the assumption is made that the material remains isotropic under plastic deformation. This is probably not strictly true of practical materials, but is justified in previous work by the agreement with experimental evidence.

Given a number of assumptions such as those which are made in this paper, it is then the task of mathematics to determine the logical consequences thereof. From a strictly mathematical point of view, any set of assumptions is admissible which leads to a consistent system of results. From an engineering standpoint, however, it is necessary that the assumptions represent the behavior of the material in question to a sufficient degree so that the theory yields usable results. It would appear that tests of rotating discs should furnish additional information as to the justification of certain of these assumptions.

INVESTIGATION

A. Notation

Throughout this paper the following notation will be employed:

r = radial distance to a generic point on the disc, before deformation.

θ = angular position of a generic point on the disc, with relation to any convenient reference line.

z = axial distance to a generic point of the disc, measured from the central plane of the disc.

R = radial distance to a generic point on the disc, after deformation.

h = thickness of the disc at a generic point, before deformation.

H = thickness of the disc at a generic point, after deformation.

a, b = initial inner and outer radii, respectively, of the disc.

S_r, S_θ, S_z = conventional normal stresses in the radial, tangential and axial directions, respectively.

$\sigma_r, \sigma_\theta, \sigma_z$ = true normal stresses in the radial, tangential and axial directions, respectively.

$\overline{S}_r, \overline{S}_\theta, \overline{S}_z$ = normal stress deviation components in the radial, tangential and axial directions, respectively.

$\overline{S}$ = $1/3\ (S_r + S_\theta + S_z)$, mean normal stress.

u = radial displacement of a generic point on the disc.

α = b/a.

η = u/a.

ν = r/a.

ρ = density of the material.

e_r, e_θ, e_z = true rates of normal strain in the radial, tangential and axial directions respectively.

$\varepsilon_r, \varepsilon_\theta, \varepsilon_z$ = conventional normal strains in the radial, tangential and axial directions, respectively.

σ = twice the value of the true critical shearing stress.

c, k = constants in the thickness formula.

ω = angular velocity of the rotating disc.

g = function appearing in the Von Mises' yield condition.

f = functional relation between stress and strain, as given by a uniaxial tensile test.

λ, λ' = parameters appearing in Von Mises' stress-strain relations.

Y = yield function.

B. Basic Assumptions

For convenience, the basic assumptions discussed under the heading General Discussion are collected below. These assumptions are to hold throughout the paper, unless stated otherwise.

(a) The disc is composed of an isotropic, homogeneous material, which remains isotropic during plastic deformation.

(b) The thickness of the disk is small compared to its radius.

(c) Elastic strains are neglected, as being negligible compared with the plastic strains.

(d) The material is incompressible.

C. Tresca's Yield Condition and Flow Rule.

Due to axial symmetry, the principal stresses are the tangential stress σ_θ, the radial stress σ_r and the axial stress σ_z, as shown in Figure 7. The principal shearing stresses are, therefore, $\frac{1}{2}|\sigma_\theta - \sigma_z|$, $\frac{1}{2}|\sigma_\theta - \sigma_r|$ and $\frac{1}{2}|\sigma_r - \sigma_z|$. Tresca's yield condition states that plastic flow can occur if at least one of these principal shearing stresses is equal to a critical value, which depends upon the considered state of strain-hardening. Furthermore, none of the principal shearing stresses may exceed this critical value.

As shown by Prager[12], if only one of the principal shearing stresses has the critical value, the flow rule associated with Tresca's yield con-

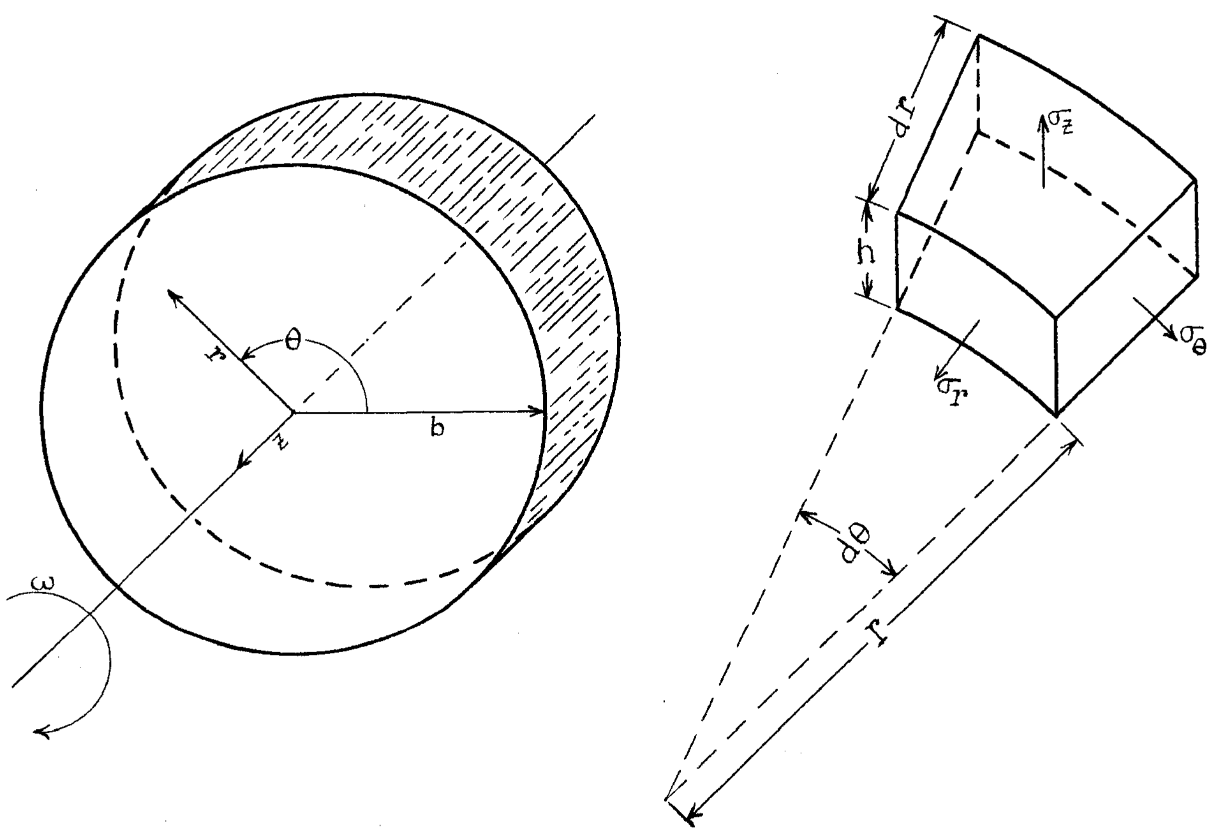

Figure 7. Rotating disc, and elementary portion showing stresses.

dition states that the instantaneous strain rate corresponds to pure shear in the plane of maximum shearing stress. The sense of this strain rate must, of course, agree with that of the maximum shearing stress.

If two principal shearing stresses have the critical value, then the resulting strain rate may be that resulting from the superposition of two states of pure shear in the two planes of maximum shearing stress.

Assumption (b) leads to the assumption that $\sigma_z = 0$. Lee[7] has investigated discs having an initial thickness function given as an exponential function of r; i.e., $h = c \exp(kr)$. For such discs he has shown that the stresses are always tensile, and that $\sigma_\theta \geqslant \sigma_r$. Since an arbitrary thickness function may be approximated to as great a degree of accuracy as desired by means of a number of annular rings of this form, it may reasonably be assumed that $\sigma_\theta \geqslant \sigma_r \geqslant 0$.

Weiss and Prager[16] have shown that, under these conditions, the plastic flow satisfies the equations:

$$\sigma_\theta = \sigma > \sigma_r > 0, \quad \sigma_z = 0, \tag{4}$$

$$e_r = 0, \quad e_\theta = - e_z \geqslant 0. \tag{5}$$

In addition, the relation between conventional tangential stress S_θ and conventional tangential strain ε_θ may be taken as

$$S_\theta = f(\varepsilon_\theta) \tag{6}$$

where $S_{11} = f(\varepsilon_{11})$ is the relation between stress and strain as obtained from a simple tensile test of the material.

Much of the following discussion parallels that of Weiss and Prager[16].

It is desirable, however, to consider this material in the light of an arbitrary initial thickness, and to examine the possibility of extension of the method to the problem of the solid disc.

Consider a rotating disc of initial thickness $h = h(r)$ and initial radius b. From the familiar elastic stress analysis, it is known that, as the angular velocity ω of the disc is gradually increased, the yield limit is first reached at the center of the disc. As ω is further increased, there will be a plastic region at the center of the disc surrounded by an elastic region. If plastic flow occurs within the plastic region, Eqs. (5) must be satisfied. Letting the distribution of radial velocity be $v = v(r)$, it follows that $e_r = \partial v/\partial r$. This must vanish according to Eqs. (5), hence the radial velocity is independent of r. Furthermore, since in accordance with assumption (c) the elastic strains are neglected, the radial displacement u at the elastic-plastic interface is zero. Thus v and also the tangential strain rate $e_\theta = v/r$, must vanish throughout the plastic region. Hence this region remains rigid until the elastic-plastic interface reaches the exterior of the disc. The angular velocity for which this occurs is given by Hoffman and Sachs[5] (Eq. 9.38, p. 101).

For larger values of ω, plastic flow will take place. However, in order that $e_\theta = v/r$ may be finite at the center of the disc, it is necessary that $v = 0$ for $r = 0$. But, according to Eqs. (5), v is independent of r. Thus $v \equiv 0$ and no flow can occur. Therefore, the use of Tresca's yield condition and the associated flow rule must apparently be restricted to annular discs. The remainder of this section deals with such an annular disc, having inner radius a and outer radius b.

A particle, initially at radius r, will be found at radius R when the angular velocity is ω. The relation between initial and instantaneous radii is, then;

$$R = r + u. \tag{7}$$

Here, R and u are functions of the independent variables r and ω^2.

Since $\partial v/\partial r = 0$, all particles have the same radial velocity at any particular instant. Thus the radial displacement is independent of r. The material initially bounded between surfaces r and r + dr undergoes radial displacement u and is then bounded by the surfaces $R = r + u$ and $R + dR = r + dr + u$. Also, the thickness decreases from h to H. In accordance with assumption (d), which states that the material is incompressible, $hr\ dr = HR\ dR$. But $dr = dR$, from above, and therefore

$$H = hr/R. \tag{8}$$

The true radial stress σ_r is transmitted across an area element which is proportional to HR, while the conventional radial stress S_r is transmitted across an area element which is proportional to hr. It follows that $\sigma_r = S_r$, since $hr = HR$ by Eq. (8). The true tangential stress σ_θ is transmitted across an area element which is proportional to H dR, while the conventional tangential stress S_θ is transmitted across an area element which is proportional to $h\ dr = h\ dR$. Thus the conventional tangential stress is given by

$$S_\theta = H\ \sigma_\theta/h. \tag{9}$$

The equation of equilibrium in the deformed state is

$$d(RH\sigma_r)/dR = H\sigma_\theta - \rho H\,\omega^2 R^2. \tag{10}$$

In this equation, a total derivative is used, since the quantities involved are assumed to be independent of the axial coordinate, and are independent of the tangential coordinate from symmetry. When Eqs. (6), (8) and (9) are substituted into Eq. (10), the latter may be written in terms of the undeformed state as follows:

$$d(rhS_r)/dr = hS_\theta - h\rho\omega^2 r(r + u) = hf(u/r) - h\rho\omega^2 r(r + u). \tag{11}$$

Integration of this equation yields

$$\int_a^b d(rhS_r)/dr\ dr = \int_a^b hf(u/r)\ dr - \int_a^b h\rho\omega^2 r(r + u)\ dr. \tag{12}$$

The left hand integral must vanish, since $S_r = 0$ at both $r = a$ and $r = b$. Upon setting $\alpha = b/a$, $\eta = u/a$ and $v = r/a$, this reduces to

$$\rho a^2\omega^2 = \frac{\int_1^\alpha h(av) f(\frac{\eta}{v})\ dv}{\int_1^\alpha h(av)\left[v^2 + \eta v\right]\ dv} . \tag{13}$$

For a given strain-hardening function f and thickness function h, this equation may be evaluated, analytically or numerically, for a set of values of η. A plot may then be made of ω^2 vs. η. Bursting speed would correspond to the maximum of this curve, since this indicates that η, and consequently u, can increase indefinitely without further increase in ω^2.

As an example, the thickness function was taken as

$$h = cr^k \tag{14}$$

with c and k constant. This substitution reduces Eq. (13) to

$$\rho a^2 \omega^2 = \frac{\int_1^{\alpha} v^k f\left(\frac{\eta}{v}\right) dv}{1/(3+k)\left[\alpha^{3+k} - 1\right] + \eta/(2+k)\left[\alpha^{2+k} - 1\right]}. \qquad (15)$$

In order to compare results directly with those of Weiss and Prager[16], the stress-strain function was taken as that of AL 24S-T4 aluminum, as shown in Weiss and Prager[16] (Figure 3, p. 199), and reproduced in Figure 8. This curve is extrapolated backwards, neglecting the elastic range of behavior. Furthermore, α was taken as 2.

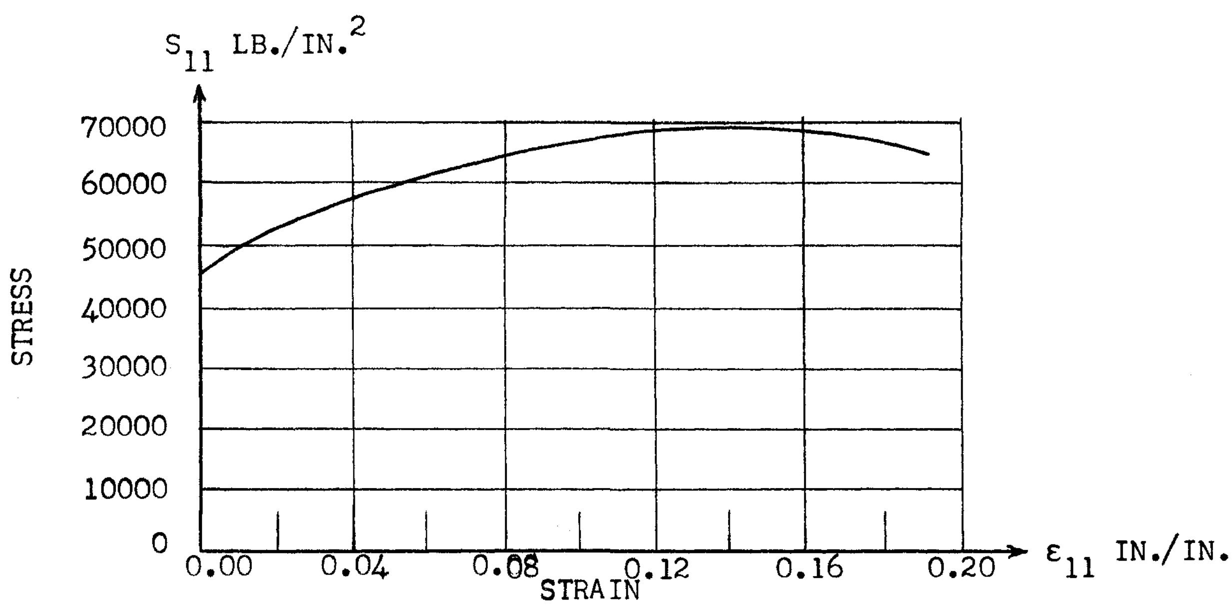

Figure 8. Stress-strain curve for AL 24S-T4 aluminum.

To show the dependence of bursting speed upon the thickness function, calculations were made for values of k ranging from -2 to 2; the results of these calculations are given in Table 1 and plotted in Figure 9. The tabulated value for k = 0 is, of course, that obtained by Weiss and Prager[16].

Table 1. Calculated values of $(\rho a^2\omega^2/6)_{Max}$ for selected values of k.

k	-2.00	-1.75	-1.50	-1.25	-1.00	-0.75	-0.50
$(\rho a^2\omega^2/6)_{Max}$	4956	4853	4759	4659	4563	4473	4386
k	-0.25	0.00	0.25	0.50	0.75	1.00	1.25
$(\rho a^2\omega^2/6)_{Max}$	4302	4222	4146	4072	4003	3936	3875
k	1.50	1.75	2.00				
$(\rho a^2\omega^2/6)_{Max}$	3814	3758	3705				

As was to be expected, removing material from the outer portion of the disc resulted in an increase in bursting speed. It is noteworthy that changing from the condition in which the outer rim is 1/4 the thickness of the inner rim to that in which the outer rim is 4 times the thickness of the inner rim resulted in a bursting speed reduction of only about 31 per cent. Hence the bursting speed appears to be not too sensitive to changes in the thickness function, within the limits for which the disc could be considered thin.

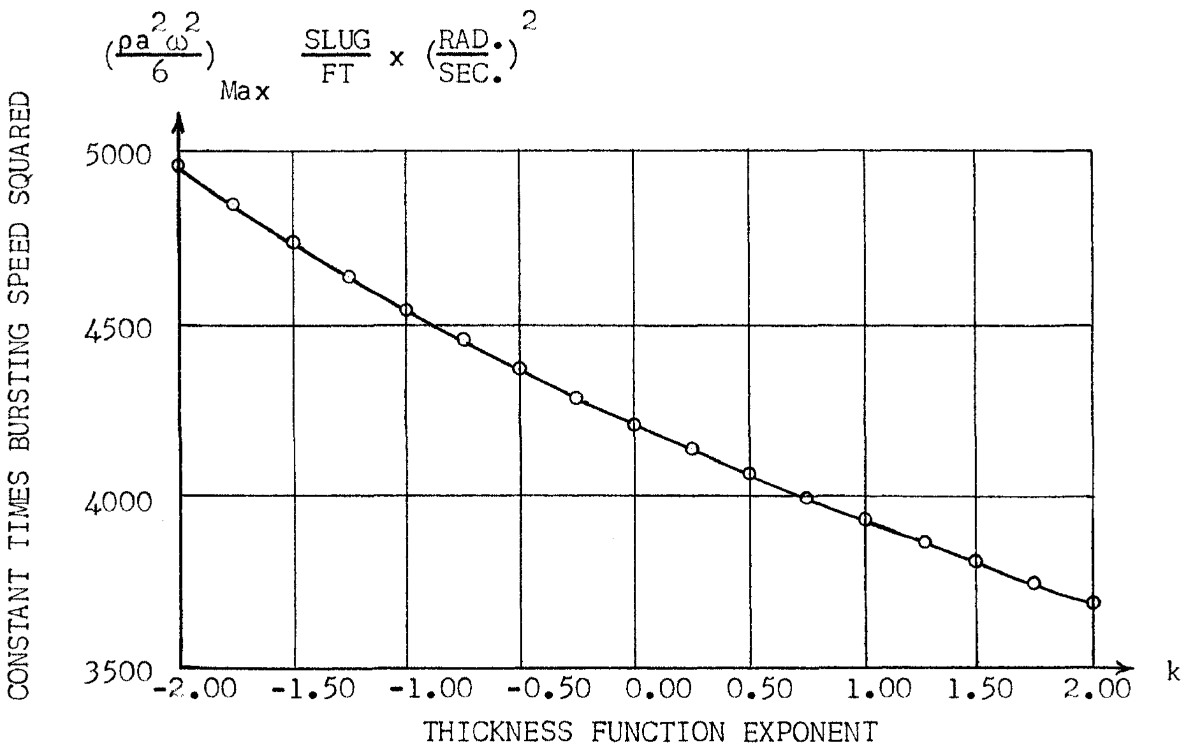

Figure 9. Relation between bursting speed and thickness function exponent.

D. Von Mises' Yield Condition

Inasmuch as Tresca's yield condition and the associated flow rule do not appear to be applicable to the problem of the solid disc, Von Mises' yield condition and the associated stress-strain-rate relationship will be applied to this case. The stress-strain-rate relations of Von Mises are chosen in preference to the more general Prandtl-Reuss relations in accordance with assumption (c); i.e., that the elastic strains are negligible in comparison with the plastic strains. This point is discussed in Prager and Hodge[13] (p.27-32). As was pointed out in the General Discussion, the assumption is made that the yield curve is subject only to dilatation and not to translation or distortion.

When the state of stress is referred to a set of principal axes, Von

Mises' yield condition is

$$\frac{1}{2}(\bar{S}_{11}^2 + \bar{S}_{22}^2 + \bar{S}_{33}^2) = g(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}), \tag{16}$$

where $\bar{S}_{11}$, $\bar{S}_{22}$ and $\bar{S}_{33}$ are the principal stress deviation components and ε_{11}, ε_{22}, and ε_{33} are the principal strains, and where the function g expresses the considered state of strain-hardening. In the case of plane stress, referred to cylindrical coordinates, $S_z = 0$ and $\bar{S} = 1/3(S_r + S_\theta)$. It follows that the stress deviation components are:

$$\bar{S}_r = 1/3(2S_r - S_\theta),\ \bar{S}_\theta = 1/3(2S_\theta - S_r) \text{ and } \bar{S}_z = -\ 1/3(S_r + S_\theta). \tag{17}$$

When Eqs.(17) are substituted in Eq. (16) the following equation results :

$$S_r^2 - S_r S_\theta + S_\theta^2 = g(\varepsilon_r, \varepsilon_\theta, \varepsilon_z). \tag{18}$$

From a purely mathematical viewpoint, any arbitrary function g might be selected, provided only that it would lead to a consistent theory of strain hardening. For the solution of engineering problems, however, it would be necessary to choose a function which would describe the behavior of the particular material under consideration. At present, to the author's knowledge, sufficient experimental evidence for the determination of such a function does not exist. Drucker[2] has suggested:

$$\frac{1}{2}(\bar{S}_{11}^2 + \bar{S}_{22}^2 + \bar{S}_{33}^2) = g\left(\int \sqrt{d\varepsilon_{ij} d\varepsilon_{ij}}\right), \tag{19}$$

where $\bar{S}_{11}$, $\bar{S}_{22}$ and $\bar{S}_{33}$ are the principle stress deviations, $d\varepsilon_{ij}$ the plastic strain increments and the integration is carried out over the strain path followed in arriving at the considered state.

For uniaxial tension

$$S_{22} = S_{33} = 0, \quad S_{11} = f(\varepsilon_{11}), \tag{20}$$

where f is the experimental stress-strain curve, and where S_{11}, S_{22} and S_{33} are the principle stresses. In accordance with assumption (d), the material is considered as being incompressible; hence Poisson's ratio is equal to 1/2,

$$\varepsilon_{22} = \varepsilon_{33} = -\tfrac{1}{2}\varepsilon_{11} \text{ and } \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0. \tag{21}$$

Thus,

$$\int\sqrt{d\varepsilon_{ij}d\varepsilon_{ij}} = \int\sqrt{(d\varepsilon_{11})^2 + \tfrac{1}{4}(d\varepsilon_{11})^2 + \tfrac{1}{4}(d\varepsilon_{11})^2} = (\sqrt{6}/2)\varepsilon_{11} \tag{22}$$

for uniaxial stress. But on the tensile stress-strain curve, $S_{11} = f(\varepsilon_{11})$ and $\frac{1}{2}(\bar{S}_{11}^2 + \bar{S}_{22}^2 + \bar{S}_{33}^2) = S_{11}^2/3$ for uniaxial stress. Hence,

$$\tfrac{1}{2}(\bar{S}_{11}^2 + \bar{S}_{22}^2 + \bar{S}_{33}^2) = 1/3\ f^2(\sqrt{6}/3\int\sqrt{d\varepsilon_{ij}d\varepsilon_{ij}}) \tag{23}$$

is a yield condition which reduces to $S_{11} = f(\varepsilon_{11})$ in the case of uniaxial stress.

In the application of Eq. (18) to general stress states, the assumption must be made that the functional form of g remains the same for the different stress states. The extent to which this assumption is justified can only be determined by experiment.

Associated with Von Mises' yield condition is the set of stress-strain-rate relations:

$$e_r = \lambda'\ \bar{S}_r;\ e_\theta = \lambda'\ \bar{S}_\theta;\ e_z = \lambda'\ \bar{S}_z, \tag{24}$$

where $\overline{S}_r$, $\overline{S}_\theta$ and $\overline{S}_z$ are the stress deviations in the radial, tangential and axial directions, respectively. If angular velocity is considered as a monotone increasing function of time, we may write these relations as:

$$\partial\varepsilon_r/\partial\omega = \lambda\,\overline{S}_r; \quad \partial\varepsilon_\theta/\partial\omega = \lambda\,\overline{S}_\theta; \quad \partial\varepsilon_z/\partial\omega = \lambda\,\overline{S}_z. \tag{25}$$

From these relations, λ may be eliminated to obtain:

$$\overline{S}_\theta\partial\varepsilon_r/\partial\omega = \overline{S}_r\partial\varepsilon_\theta/\partial\omega \quad \text{and} \quad \overline{S}_\theta\partial\varepsilon_z/\partial\omega = \overline{S}_z\partial\varepsilon_\theta/\partial\omega. \tag{26}$$

When Eqs. (17) are substituted in Eqs. (26) and ε_z is eliminated by means of the condition of incompressibility, the following result:

$$(2S_\theta - S_r)\partial\varepsilon_r/\partial\omega = (2S_r - S_\theta)\partial\varepsilon_\theta/\partial\omega, \tag{27}$$

and

$$(2S_\theta - S_r)\partial\varepsilon_r/\partial\omega = (2S_r - S_\theta)\partial\varepsilon_\theta/\partial\omega. \tag{28}$$

But Eq. (27) and Eq. (28) are identical. Hence the stress-strain rate relations of Von Mises are not independent for plane stress and an incompressible material.

The equilibrium equation, in terms of true stress and the deformed state is

$$d(HR\sigma_r)/dR = H\sigma_\theta - \rho R^2 H\omega^2, \tag{29}$$

as given in part C above.

The true radial stress σ_r is transmitted across an area which is proportional to HR, and the conventional radial stress S_r is transmitted across an area which is proportional to hr. Thus,

$$HR\sigma_r = hrS_r. \qquad (30)$$

The true tangential stress σ_θ is transmitted across an area which is proportional to H dR and the conventional tangential stress S_θ is transmitted across an area which is proportional to h dr. Hence,

$$H\sigma_\theta \, dR = hS_\theta \, dr. \qquad (31)$$

From the condition of incompressibility,

$$HR \, dR = hr \, dr. \qquad (32)$$

When Eqs. (30), (31) and (32) are substituted into Eq. (29), the equilibrium equation in terms of the undeformed state may be written as follows:

$$d(hrS_r)/dr = hS_\theta - \rho hr\omega^2 R = hS_\theta - \rho h\omega^2 r^2 (1 + \varepsilon_\theta). \qquad (33)$$

A compatibility equation may also be obtained, since $\varepsilon_\theta = u/r$ and $\varepsilon_r = du/dr$. When these relations are combined, the following equation results:

$$\varepsilon_r = rd\,\varepsilon_\theta/dr + \varepsilon_\theta = d(r\,\varepsilon_\theta)/dr. \qquad (34)$$

Eqs. (23), (27), (33) and (34) provide four equations for the four unknowns: S_r, S_θ, ε_θ and ε_r. In addition, boundary conditions are specified, depending upon the particular problem. For convenience, these equations are now collected, reading:

$$\varepsilon_r = d(r\,\varepsilon_\theta)/dr, \qquad (35)$$

$$(2S_\theta - S_r)\partial\varepsilon_r/\partial\omega = (2S_r - S_\theta)\partial\varepsilon_\theta/\partial\omega, \tag{36}$$

$$S_r^2 - S_r S_\theta + S_\theta^2 = f^2 \quad (\sqrt{6}/3 \int \sqrt{d\varepsilon_{ij} d\varepsilon_{ij}}), \tag{37}$$

$$d(hrS_r)/dr = hS_\theta - \rho h\omega^2 r^2 (1 + \varepsilon_\theta). \tag{38}$$

In general, it is not possible to obtain an analytic solution for this system of equations. In dealing with any engineering problem, the stress-strain relation $S_{11} = f(\varepsilon_{11})$ is known only as an empirical curve from experimental data. Even if an analytic expression were prescribed as an approximation to a stress-strain relation, the integrand appearing in Eq. (37) is a function of the particular program of strains which was followed in reaching the considered state. Without a priori knowledge of the manner in which the considered state of strains is reached, the author does not know of any means by which this system of equations may be solved analytically.

For a particular material, an approximate solution may be obtained by numerical methods. One such method involves replacing the differential equations by finite-difference equations, and the integral in Eq. (37) by a finite sum. In order to do this, suppose the radius of the disc to be divided into N equal parts, so that $r_n = nb/N$. Also, consider a finite set of values ω_j, where ω_o is the angular velocity at which the disc first becomes fully plastic. The finite difference equations obtained in the case of a disc with constant initial thickness are:

$$\varepsilon_{r,n,j} = 2\left[n\varepsilon_{\theta,n,j} - (n-1)\varepsilon_{\theta,n-1,j}\right] - \varepsilon_{r,n-1,j}, \tag{39}$$

$$S_{\theta,n,j} = \frac{2(\varepsilon_{\theta,n,j} - \varepsilon_{\theta,n,j-1}) + (\varepsilon_{r,n,j} - \varepsilon_{r,n,j-1})}{(\varepsilon_{\theta,n,j} - \varepsilon_{\theta,n,j-1}) + 2(\varepsilon_{r,n,j} - \varepsilon_{r,n,j-1})} S_{r,n,j} , \tag{40}$$

$$S^2_{r,n,j} - S_{r,n,j}S_{\theta,n,j} + S^2_{\theta,n,j} = f^2\left[2\sqrt{3}/3 \sum_{i=1}^{j} \left\{(\varepsilon_{r,n,i} - \varepsilon_{r,n,i-1})^2 + (\varepsilon_{r,n,i} - \varepsilon_{r,n,i-1})(\varepsilon_{\theta,n,i} - \varepsilon_{\theta,n,i-1}) + (\varepsilon_{\theta,n,i} - \varepsilon_{\theta,n,i-1})^2 \right\}^{1/2}\right], \tag{41}$$

$$nS_{r,n,j} - (n - 1)S_{r,n-1,j} - \frac{1}{2}(S_{\theta,n,j} + S_{\theta,n-1,j}) + \rho b^2\omega_j^2/2N^2 \left[n^2(1 + \varepsilon_{\theta,n,j}) + (n - 1)^2(1 + \varepsilon_{\theta,n-1,j}) \right] = 0. \tag{42}$$

This system of equations may be treated as follows. Select a value of ε_θ at the center of the disc, and a tentative value for ω_1. Eq. (34) states that $\varepsilon_r = \varepsilon_\theta$ at the center of the disc. Eq. (36) then shows that $S_r = S_\theta$ at the center of the disc, and S_r can then be computed from Eq. (41). Eq. (42) does not apply at the center of the disc, as the index $(n - 1)$ is meaningless if $n = 0$.

Next an estimate of $\varepsilon_{\theta,1,1}$, the value of the tangential strain at $r_1 = b/N$ is made, which defines $\varepsilon_{r,1,1}$ through Eq. (39). Eq. (40) then furnishes the relation between $S_{\theta,1,1}$ and $S_{r,1,1}$. By means of this relation and Eq. (41) the values of $S_{\theta,1,1}$ and $S_{r,1,1}$ are found. These values should now satisfy Eq. (42). In case Eq. (42) is not satisfied, the estimate of $\varepsilon_{\theta,1,1}$ must be revised, and the process repeated.

In this manner, one proceeds from station to station until the outer edge of the disc is reached. The value of $S_{r,N,1}$ which has been obtained should match the boundary condition prescribed. In case the boundary condition is not satisfied, the entire process must be repeated with a revised value of ω_1.

By this means, values of ω corresponding to discrete values of $\varepsilon_{\theta,0}$ may be obtained. The value of ω for which plastic instability occurs is that value for which the curve of ω^2 vs. $\varepsilon_{\theta,0}$ has a maximum. This should correspond to the bursting speed of the disc.

As an example, the calculations were carried through for a solid disc with a constant initial thickness, making use of the stress-strain data of Figure 8. In performing these calculations, N was taken as 10, and computations were made at increments of $\varepsilon_{\theta,0}$ of 0.01, except near the maximum of the curve, where smaller increments were taken in order to define the shape of the curve better. The results of these calculations are shown in Table 2 and plotted in Figure 10. Although the calculations were not carried far enough to show a true maximum, the curve appears to level off. It does not appear feasible to perform calculations for larger values of the strain, since the argument of f in Eq. (41) is twice the tangential strain, for the station at the center of the disc. Thus, the use of larger values of tangential strain would require the use of a portion of the curve, Figure 8, beyond that which is usually considered valid. The bursting speed of this disc is therefore indicated as $\rho b^2\omega^2/200 = 964$, approximately. It is interesting to compare this result with that obtained for the annular disc of the same material, using Tresca's yield condition. Holms and Jenkins[6] have found experimentally that the bursting speed of an annular disc has been decreased from that of a solid disc by approximately the percentage of the material of the disc which has been removed.

For the annular disc treated previously, $\alpha = 2$; therefore the relation

$$\omega_a = 3/4\omega_s \tag{43}$$

should hold, where ω_a is the bursting speed of the annular disc, and ω_s is the bursting speed of the solid disc. For the annular disc,

$$\rho a^2 \omega_a^2/6 = 4222. \tag{44}$$

When Eq. (43) is substituted in Eq. (44) and the relation $b = 2a$ used, it is found that $\rho b^2 \omega_s^2/200 = 901$. Thus it is seen that the two methods differ by some 7 per cent. As pointed out in Prager and Hodge[13] (p. 24) the results obtained by use of the two yield conditions may differ by as much as 15 per cent. Hence the author feels that these results are in good agreement.

Table 2. Calculated values of $\rho b^2 \omega^2/200$ for selected values of $\varepsilon_{\theta,0}$.

$\varepsilon_{\theta,0}$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.075	0.08	0.085
$\rho b^2 \omega^2/200$	702	781	839	877	905	926	942	955	960	964	964

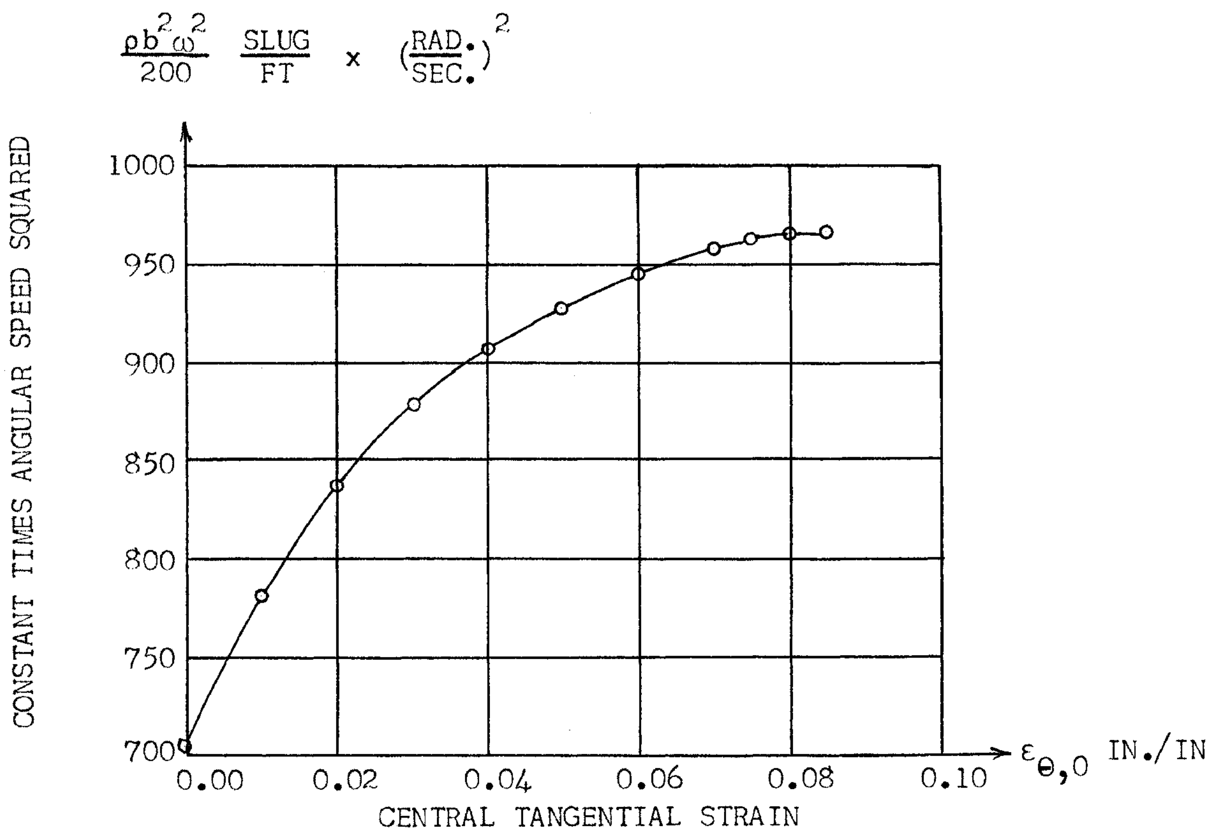

Figure 10. Angular velocity as a function of tangential strain.

LITERATURE CITED

1. Bridgman, P. W. The Compressibility of Thirty Metals as a Function of Pressure and Temperature. Proc. Amer. Acad. Arts Sci. 58: 165-242. 1922-1923.

2. Drucker, D. C. The Significance of the Criterion for Additional Plastic Deformation of Metals. Jour. of Colloid Sci. 4: 299-311. 1949.

3. _______ Some Implications of Work-Hardening and Ideal Plasticity. Quarterly of App. Math. Vol. 7, No. 4: 411-418. January, 1950.

4. _______ A More Fundamental Approach to Plastic Stress-Strain Relations. Proc. 1'st U. S. National Cong. of App. Mech.: 487-491. ASME 1951.

5. Hoffman, Oscar and Sachs, George. Introduction to the Theory of Plasticity for Engineers. N.Y., McGraw-Hill Book Co., Inc. 1953.

6. Holms, Arthur G. and Jenkins, Joseph E. Effect of Strength and Ductility on Burst Characteristics of Rotating Discs. NACA TN 1667. 1948.

7. Lee, Ti-Chiang. On the Stress in a Rotating Disk of Variable Thickness. Jour. of App. Mech., Trans. of ASME, Vol. 74: 263-266. 1952.

8. Mac Gregor, C. W. and Tierney, W. D. Developments in High Speed Rotating Disk Research at M.I.T. The Welding Journal Research Supplement. Vol. 8, No. 6: 303-309. June, 1948.

9. Manson, S. S. Analysis of Rotating Disks of Arbitrary Contour and Radial Temperature Distribution in the Region of Plastic Deformation. Proc. 1'st U. S. National Cong. of App. Mech.: 569-577. ASME 1951.

10. Nadai, A. and Donnell, L. H. Stress Distribution in Rotating Disks of Ductile Material after the Yield Point has been Reached. Trans. ASME APM-51-61 Vol. 51: 173-181. 1929.

11. Prager, William. Theory of Plastic Flow Versus Theory of Plastic Deformation. Jour. of App. Physics. Vol. 19, No. 6: 540-543. June, 1948.

12. _______ On the Use of Singular Yield Conditions and the Associated Flow Rules. Jour. of App. Mech. Vol. 20, No. 3: 317-320. September, 1953.

13. ________, and Hodge, Philip G. Theory of Perfectly Plastic Solids. N.Y., John Wiley and Sons, Inc. 1951.

14. Skidmore, Wallace E. Bursting Tests of Rotating Disks Typical of Small Gas Turbine Design. Proc. of Soc. for Exp. Stress Analysis. Vol. 8, No. 2: 29-49. 1951.

15. Stodola, A. Steam Turbines. English Translation by Loewenstein, Louis, C. N.Y., D. Van Nostrand Co. 1905.

16. Weiss, H. J. and Prager, W. The Bursting Speed of a Rotating Plastic Disc. Jour. of the Aero. Sciences. Vol. 21, No. 3: 196-200. March, 1954.

17. Zaid, M. Partially Plastic Discs. Jour. of the Aero. Sciences. Vol. 19, No. 10: 697-704. October, 1952.

18. ________ Fully Plastic Rotating Disc with Large Strains. Jour. of the Aero. Sciences. Vol. 20, No. 6: 369-377. June, 1953.

ACKNOWLEDGEMENT

The author wishes to express his appreciation to Dr. Harry J. Weiss for his kind criticism and helpful suggestions during the preparation of this manuscript.