OPTIMAL FIXED-TIME ORBITAL TRANSFER
WITH A RADIAL CONSTRAINT

by

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INTRODUCTION

During the last several years an increasing amount of emphasis has been placed on optimization problems in flight mechanics. In 1965 Paiewonsky (13) presented a good review of the theory and practice of optimal control as it had developed up to that time. He included 362 references of the applications of optimal control in a variety of fields. Some of the more recent advances in optimal control theory which are important for orbital transfer problems are briefly discussed below.

Bryson *et al*. (1) have extended the classical calculus of variations theory to allow inequality constraints on the state variables.

Kopp and Moyer (8) and Robbins (14) have investigated the possibility of the existence of singular subarcs in the solution of optimal control problems in which one or more of the control variables appears linearly.

Lewallen and Tapley (10) have made a comparison of several different numerical methods which are in use for the solution of optimal control problems.

Hempel (6) has found a solution of the adjoint equations which is valid for both elliptic and hyperbolic orbital subarcs.

Lion and Handlesman (11) present a method by which the optimality of a fixed-time impulsive transfer can be determined.

Edelbaum (3) presents a survey of some recent impulsive transfer results for both time-open and fixed-time problems.

The problem considered in this study is one of minimizing the propellant required to perform a fixed-time intercept or rendezvous with or without the addition of the constraint that the radius must never be less than some prescribed value.

Perhaps the most obvious application of the problem in which the radial constraint is important is the case of a transfer through a large range angle in a relatively short time where both the initial and final altitudes are low. For such a problem, an analysis without the radial constraint may result in a solution which intersects the surface of the attracting body or enters its surrounding atmosphere. Since subterranean orbits are prohibitive and atmospheric entry is usually undesirable, the radial constraint is necessary for an acceptable solution to the problem. In addition a radial constraint can be used to prohibit a heliocentric transfer from approaching too close to the sun. Also, with a slight modification of the constraint, the theory developed here may be used to provide assurance that high level radiation areas are not encountered. Various other applications may also exist.

In order to simplify the analysis, the following assumptions concerning the motion of the vehicle are made:

1. The only forces acting on the vehicle are those produced by the vehicle's propulsion unit and an inverse square law gravitational field.

2. The entire flight takes place in the plane defined by the initial position and velocity vectors of the vehicle.

3. The vehicle can be represented by a point mass.

Variational calculus is used to determine a set of necessary conditions which must be satisfied for any optimal transfer subject to the three assumptions listed above. Both bounded thrust and impulsive transfers are considered. The general theory is then applied to the problems of fixed-time, minimum propellant rendezvous and intercept with a fixed final mass and an open initial mass.

Numerical results are presented for impulsive rendezvous and intercept with initial and final radii values of 1.05 and several included angles. For the cases considered, all of the rendezvous solutions consisted of two impulses while both one and two impulse solutions were found for the intercept problem. All of the results for the rendezvous problems and some of those for the intercept problem are for the transition region, i.e., the transfers are tangent to the radial constraint at one point. In addition solutions for optimal unconstrained two impulse intercepts are compared with the corresponding nonoptimal single impulse results.

A variety of impulsive transfer maneuvers was considered in the process of determining the extent of the transition region. The behavior of the switching function for the maneuvers considered led to the conclusion that there is probably only one value of the flight time for each range angle which results in a transition solution, and for smaller values of the flight time the solution contains a singular constrained subarc. While a closed form solution of the state and adjoint equations is available for transfers which contain only coasting subarcs separated by impulses, it appears that numerical methods must be used for cases which involve singular subarcs. Due to a lack of computer time, no results are presented here for the fully constrained transfer problem.

LIST OF SYMBOLS

c	Engine exit velocity
F	Augmented function
G	Defining function for the performance index
h	Angular momentum
H	Hamiltonian
J	Performance index
K_β	Switching function
K_ϵ	Magnitude of the primer vector
m	Vehicle mass
m_p	Propellant mass
MR	Ratio of the vehicle mass before an impulse to that after an impulse
r	Radial distance measured from the center of the attracting body to the vehicle
s	Number of boundary conditions
S	Radial constraint function
T	Engine thrust
u	Horizontal velocity component
v	Radial velocity component
$\bar{V}$	Velocity vector of the vehicle
X	Adjoint transition matrix
X_{ij}	A component of X $(i, j = 1, 2, 3, 4)$
α	A real variable
β	Engine mass flow rate
β_m	Maximum value of β

γ	Lagrange multiplier associated with $\ddot{S}$
ϵ	Thrust inclination angle with respect to the local horizon
θ	Angular position of the vehicle with respect to an arbitrary reference
λ_i	Lagrange multiplier associated with φ_i $(i = 1, 2, \ldots, 6)$
μ_i	Lagrange multiplier associated with S or $\dot{S}$ $(i = 1, 2)$
τ	Normalized time
φ_i	Generalized constraint function
ω_i	Generalized boundary condition

Subscripts

a	Beginning of a constrained subarc
b	End of a constrained subarc
c	Constrained side of a junction of a constrained subarc and an unconstrained subarc
f	End of an extremal arc
I	Point of an impulse
m	Point of a midcourse impulse
u	Unconstrained side of a junction of a constrained subarc and an unconstrained subarc
o	Beginning of an extremal arc
+	Immediately following a corner point
-	Immediately preceding a corner point

Special Notation

$\dot{\theta}$	The dot denotes differentiation with respect to τ
Δv	The Δ denotes a discontinuity across an impulse

OPTIMAL ORBITAL TRANSFER

General Analysis

The method of analysis used in this study is similar to that outlined by Miele (12) with the radial inequality constraint added in the form suggested by Denham (2).

The coordinate system used for orbital transfers is shown in Figure 1. The analysis included in this section is for bounded thrust. The extension to impulsive maneuvers is given in Appendix B.

The general problem

The general optimum orbital transfer problem, subject to the assumptions listed in the introduction, may be formally stated as follows: In the class of functions $\theta(\tau)$, $r(\tau)$, $h(\tau)$, $v(\tau)$, $m(\tau)$, $\alpha(\tau)$, $\beta(\tau)$ and $\varepsilon(\tau)$, find that particular set which minimizes the difference

$$J = G_f - G_o \tag{1}$$

where G is a function of the state variables, subject to the differential constraints

$$\varphi_1 = \dot{\theta} - h/r^2 = 0 \tag{2}$$

$$\varphi_2 = \dot{r} - v = 0 \tag{3}$$

$$\varphi_3 = \dot{h} - (rc\beta/m)\cos\,\varepsilon = 0 \tag{4}$$

$$\varphi_4 = \dot{v} - h^2/r^3 + 1/r^2 - (c\beta/m)\sin\,\varepsilon = 0 \tag{5}$$

$$\varphi_5 = \dot{m} + \beta = 0 \tag{6}$$

the control variable constraint

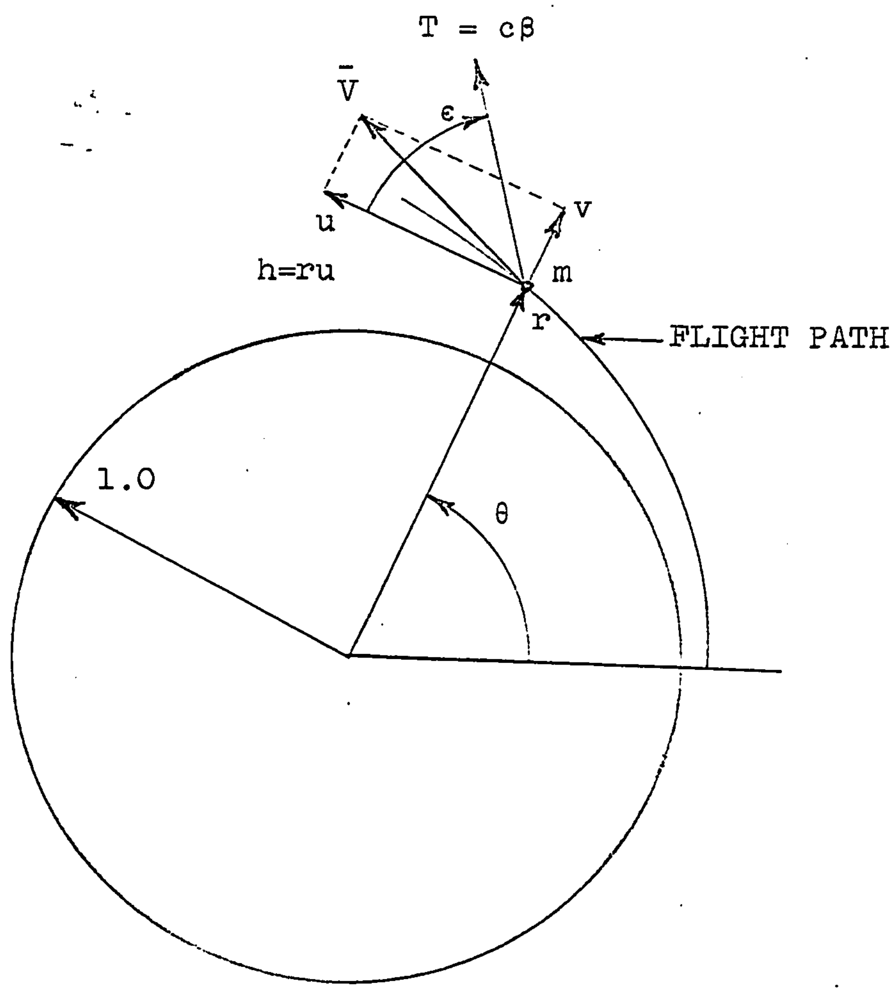


Figure 1. Coordinate system used for planar transfers.

$$\varphi_6 = \beta(\beta_m - \beta) - \alpha^2 = 0 \tag{7}$$

the radial constraint

$$S = r - 1 \geq 0 \tag{8}$$

and a set of boundary conditions

$$\omega_j(\theta, r, h, v, m, \tau)_o^f = 0 \quad j = 1, 2, \ldots, s < 12. \tag{9}$$

Necessary conditions for minimum J

Successive differentiation of Equation 8 shows that the control variables β and ϵ first appear in the expression for $\ddot{S}$. Following the definition given by Bryson et al. (1), Equation 8 is called a second order state variable constraint.

The augmented function F, given by Equation 10, forms the basis for the derivation of the necessary conditions for the solution to the problem stated above.

$$F = \sum_{i=1}^{6} \lambda_i \varphi_i + \gamma \ddot{S} = 0 \tag{10}$$

The adjoint equations are derived from Equation 10 by the application of the following two general equations:

$$\frac{d}{d\tau} \frac{\partial F}{\partial \dot{x}_i} - \frac{\partial F}{\partial x_i} = 0$$

$$\frac{\partial F}{\partial u_i} = 0 \tag{12}$$

where x_i is any state variable and u_i is any control variable. Along any subarc where $s > 0$ the value of γ is given by $\gamma = 0$. For the problem being considered the adjoint equations are given by Equations 13-20,

$$\dot{\lambda}_1 = 0 \tag{13}$$

$$\dot{\lambda}_2 = 2\lambda_1 h/r^3 + (\lambda_4 - \gamma)(3h^2/r^4 - 2/r^3) - \lambda_3 (c\beta/m)\cos\epsilon \tag{14}$$

$$\dot{\lambda}_3 = -\lambda_1/r^2 - 2(\lambda_4 - \gamma)h/r^3 \tag{15}$$

$$\dot{\lambda}_4 = -\lambda_2 \tag{16}$$

$$\dot{\lambda}_5 = (c\beta/m^2)K_\epsilon \tag{17}$$

$$\lambda_6 \alpha = 0 \tag{18}$$

$$K_\beta - \lambda_6(\beta_m - 2\beta) = 0 \tag{19}$$

$$\lambda_3 r \sin\epsilon - (\lambda_4 - \gamma)\cos\epsilon = 0 \tag{20}$$

where, by definition,

$$K_\epsilon = \lambda_3 r \cos\epsilon + (\lambda_4 - \gamma)\sin\epsilon \tag{21}$$

$$K_\beta = (c/m)K_\epsilon - \lambda_5 \, . \tag{22}$$

The Hamiltonian, H, is given by

$$H = \lambda_1 h/r^2 + \lambda_2 v + (\lambda_4 - \gamma)(h^2/r^3 - 1/r^2) + K_\beta \beta \, . \tag{23}$$

Since F, S, $\dot{S}$ and $\ddot{S}$ are not explicit functions of τ, the Hamiltonian is constant along the entire extremal arc.

The Legendre-Clebsch condition provides the following information concerning the control variables:

$$K_\beta/(2\beta - \beta_m) \geq 0 \tag{24}$$

$$K_\epsilon \geq 0 \tag{25}$$

The transversality condition, given by Equation 26, provides information concerning the Lagrange multiplier values at the initial and final points of the extremal arc.

$$[dG - H d\tau + \lambda_1 d\theta + \lambda_2 dr + \lambda_3 dh + \lambda_4 dv + \lambda_5 dm]_o^f = 0 \tag{26}$$

A corner point is a point at which the derivative of one or more of the state variables is discontinuous. At such a point Equations 27 and 28 must be satisfied.

$$(\lambda_i)_+ = (\lambda_i)_- \qquad i = 1, 2, \ldots, 5 \tag{27}$$

$$H_+ = H_- \tag{28}$$

At the beginning of any constrained subarc, i.e., any subarc along which r = 1, the following conditions must hold:

$$r = 1 \tag{29}$$

$$v = 0 \tag{30}$$

$$(\lambda_i)_+ = (\lambda_i)_- \qquad i = 1, 3, 5 \tag{31}$$

$$(\lambda_2)_+ = (\lambda_2)_- + \mu_1 \tag{32}$$

$$(\lambda_4)_+ = (\lambda_4)_- + \mu_2 \tag{33}$$

$$H_+ = H_- \ . \tag{34}$$

At the end of any constrained subarc Equations 35 and 36 must be satisfied.

$$(\lambda_i)_+ = (\lambda_i)_- \qquad i = 1, 2, \ldots, 5 \tag{35}$$

$$H_+ = H_- \tag{36}$$

Since $\ddot{S}$ must vanish along any constrained subarc, Equation 37 is valid for all constrained subarcs.

$$h^2 - 1 + (c\beta/m)\sin\varepsilon = 0 \tag{37}$$

<u>Some general results</u>

Equation 20 can be used to determine an expression for tan ε. For quadrant determination Equations 20, 21 and 25 may be combined with the result

$$\sin\varepsilon = (\lambda_4 - \gamma)/K_\varepsilon \qquad \cos\varepsilon = \lambda_3 r/K_\varepsilon \tag{38}$$

where

$$K_\epsilon = [(\lambda_3 r)^2 + (\lambda_4 - \gamma)^2]^{\frac{1}{2}}. \tag{39}$$

The primer vector is defined to be a vector with components $\lambda_3 r$ and $\lambda_4 - \gamma$. From Equation 39 it is seen that K_ϵ is simply the magnitude of the primer vector.

Equations 7, 18, 19 and 24 may be combined to give the following results:

1. $\beta = 0$ when $K_\beta < 0$
2. $\beta = \beta_m$ when $K_\beta > 0$
3. $0 < \beta < \beta_m$ when $K_\beta = 0$.

Case 3 is applicable only if K_β vanishes for a finite time interval. Such a subarc is termed singular and is discussed at length by Robbins (13). In particular he has shown that if impulsive thrusting is not allowed, then the only possible solution involving a singular subarc is that in which the entire solution is singular, and in general such solutions do not exist. Although his analysis does not allow for state variable constraints, his arguments involving the number of boundary conditions and unknowns can be used to eliminate the general existence of singular unconstrained subarcs.

The problem of singular subarcs is also discussed by Kopp and Moyer (8). They present a derivation of a necessary condition for the existence of a singular subarc without a state variable constraint. Their analysis is extended to constrained subarcs in Appendix D of this study, with the result for the problem considered here that the thrust direction must be pointed inwards along a singular radially constrained subarc.

One of the difficulties which arises in problems involving state variable constraints is the determination of when to enter and when to leave the constraint boundary. For the problem under consideration some information can be found from Equations 34 and 36. In particular, they may be reduced to Equations 40 and 41 for entrance and exit corners respectively.

$$\beta_+[(c/m)(\lambda_3^2 + \lambda_{4_-}^2)^{\frac{1}{2}} \cos(\varepsilon_+ - \varepsilon_-) - \lambda_5] = \beta_-[(c/m)(\lambda_3^2 + \lambda_{4_-}^2) - \lambda_5] \tag{40}$$

$$\beta_+[(c/m)(\lambda_3^2 + \lambda_{4_+}^2) - \lambda_5] = \beta_-[(c/m)(\lambda_3^2 + \lambda_{4_+}^2)^{\frac{1}{2}} \cos(\varepsilon_+ - \varepsilon_-) - \lambda_5] \tag{41}$$

At an entrance corner to a constrained subarc the subscript "+" refers to the constrained side of the corner while the subscript "-" refers to the unconstrained side of the corner. At an exit corner from a constrained subarc the subscripts "+" and "-" refer to the unconstrained side and constrained side of the corner respectively. Examination of Equations 40 and 41 reveals that they are identical with respect to the constrained and unconstrained sides of the entrance and exit corners. Therefore both equations are imbedded in Equation 42,

$$\beta_c[(c/m)(\lambda_3^2 + \lambda_{4u}^2)^{\frac{1}{2}} \cos(\epsilon_c - \epsilon_u) - \lambda_5] =$$

$$\beta_u[(c/m)(\lambda_3^2 + \lambda_{4u}^2)^{\frac{1}{2}} - \lambda_5] \tag{42}$$

where the subscripts "c" and "u" refer respectively to the constrained and unconstrained sides of a junction.

On the unconstrained side of a junction one of two conditions must prevail, either $\beta_u = 0$ or $\beta_u = \beta_m$. On the constrained side of a corner there are three possible conditions. Either $\beta_c = 0$, $\beta_c = \beta_m$ or $0 < \beta_c < \beta_m$. There are thus six possible combinations of constrained and unconstrained subarc junction conditions.

In the analysis which follows it is helpful to note that the coefficient of β_u in Equation 42 is identical to K_{β_u}.

Consider first the case where $\beta_c = \beta_u = \beta_m$. Then Equation 42 may be reduced to

$$\cos(\epsilon_c - \epsilon_u) = 1 \tag{43}$$

so that $\epsilon_c = \epsilon_u$ and ϵ is continuous at the corner.

Next consider the case where $\beta_u = \beta_m$ and $0 < \beta_c < \beta_m$. Since the coefficient of β_c is less than or equal to the coefficient of β_u and $\beta_c < \beta_u$, it follows that both sides of Equation 42 must vanish. But, since $\beta_c \neq 0$ and $\beta_u \neq 0$, it is concluded that the coefficients of β_c and β_u must vanish. Consequently $\epsilon_c = \epsilon_u$ and $K_{\beta_u} = 0$.

If $\beta_u = \beta_m$ and $\beta_c = 0$, Equation 42 requires that $K_{\beta_u} = 0$, however, no results for ϵ are directly available. If the lower limit on β were some very small value, $\bar{\beta}$, rather than zero, Equation 42 would require that $\epsilon_c = \epsilon_u$ and $K_{\beta_u} = 0$. By taking a limiting process in which $\bar{\beta} \to 0$, it can be concluded that $\epsilon_c = \epsilon_u$ for the case where $\beta_u = \beta_m$ and $\beta_c = 0$.

Consider now the three cases where $\beta_u = 0$. The possibility of having $\beta_c = 0$ can be eliminated immediately since if $\beta = 0$ on both the constrained and unconstrained subarcs, an unpowered transfer from a noncircular orbit to a circular orbit is required. Such transfers are physically impossible without thrust addition. Since it has been concluded that $\beta_c \neq 0$, Equation 42 requires that the coefficient of β_c must vanish. Now, if $\epsilon_c \neq \epsilon_u$, then the coefficient of β_u is greater than that of β_c, and thus $K_{\beta_u} > 0$. But it has been assumed that $\beta_u = 0$ which requires $K_{\beta_u} \leq 0$. It is therefore concluded that $\epsilon_c = \epsilon_u$ for the case where $\beta_u = 0$. Furthermore, the coefficients of β_u and β_c are then identical so that $K_{\beta_u} = 0$.

It has now been shown that ϵ must be continuous at any junction of unconstrained and constrained subarcs. From Equations 20-22, an alternate form of K_β can be found to be

$$K_\beta = (c/m)\lambda_3 r \sec \epsilon - \lambda_5 \quad (44)$$

Since all of the terms on the right hand side of Equation 44 are continuous at a junction point it follows that K_β must be continuous at a junction point.

From Equation 20,

$$\tan \epsilon = (\lambda_4 - \gamma)/\lambda_3 r \; . \tag{45}$$

At any entrance point to a constrained subarc, the required continuity of ϵ and the discontinuity in λ_4 given by Equation 33 can be used to give

$$\gamma(\tau_a) = \mu_2 \; . \tag{46}$$

Similarily at the end of any constrained subarc, the required continuity of ϵ and λ_4 can be used to give the exit junction condition

$$\gamma(\tau_b) = 0. \tag{47}$$

Minimum Propellant Transfers

Consider now the problem of performing a rendezvous at a given point in a specified time such that the propellant consumption is minimized. The performance index, G, and the boundary conditions are given by Equations 48-58.

$$G = -m \tag{48}$$

$$\theta(\tau_o) = \theta_o \tag{49}$$

$$r(\tau_o) = r_o \tag{50}$$

$$h(\tau_o) = h_o \tag{51}$$

$$v(\tau_o) = v_o \tag{52}$$

$$\theta(\tau_f) = \theta_f \tag{53}$$

$$r(\tau_f) = r_f \tag{54}$$

$$h(\tau_f) = h_f \tag{55}$$

$$v(\tau_f) = v_f \tag{56}$$

$$m(\tau_f) = 1 \tag{57}$$

$$\tau_o,\ \tau_f\text{:}\quad \text{Given} \tag{58}$$

The transversality condition provides the following additional boundary condition which must be satisfied:

$$\lambda_5(\tau_o) = 1 \tag{59}$$

For the unconstrained problem there are five unknown initial conditions, $\lambda_1(\tau_o)$, $\lambda_2(\tau_o)$, $\lambda_3(\tau_o)$, $\lambda_4(\tau_o)$ and $m(\tau_o)$, which must be properly chosen to satisfy the five final conditions for $\theta(\tau_f)$, $r(\tau_f)$, $h(\tau_f)$, $v(\tau_f)$ and $m(\tau_f)$.

For the constrained problem there are four additional unknowns, τ_a, τ_b, μ_1 and μ_2 which must be chosen to satisfy the intermediate boundary conditions for $r(\tau_a)$, $v(\tau_a)$, $\gamma(\tau_b)$ and either $K_\beta(\tau_a) = 0$ or $\ddot{S}(\tau_a) = 0$ depending on the form of the solution.

The analysis for the intercept problem is similar to that for the rendezvous problem. The only difference is that the boundary conditions given by Equations 55 and 56 are replaced with two new results from the transversality condition, given by Equations 60 and 61.

$$\lambda_3(\tau_f) = 0 \tag{60}$$

$$\lambda_4(\tau_f) = 0 \tag{61}$$

METHOD OF SOLUTION

One of the difficulties encountered in attempting to determine the solution to optimal transfer problems is the uncertainty of the number and sequence of subarcs for bounded thrust problems or the number and position of impulses for an impulsive transfer problem. In a recent paper Lion and Handlesman (11) suggest a method for determining the optimality of a two impulse reference subarc. Although their analysis is based on minimizing the total $\Delta\bar{V}$ requirement, it is equivalent to minimizing the impulsive propellant consumption. Their results concerning the primer vector can easily be translated into similar results concerning the switching function K_β for the approach used here.

It is expected that if the impulsive approximation is fairly good the results of the impulsive solution can be used as a starting point for an iterative procedure to be used to solve the two-point boundary value problem arising for the bounded thrust problem.

Bryson _et al_. (1) present an analytic example involving a second order state variable constraint. Three types of solutions are found for different values of the constraint. The first corresponds to an unconstrained problem. The second consists of an extremal arc which is tangent to the constraint boundary at only one point while the third involves a fully constrained subarc, i.e., the constraint boundary is followed for a finite time. It is suggested that other second order state variable constraint problems may have similar types of solutions. However, their example does not involve "bang-bang" control or impulses. Thus it is not known a priori whether or not three different types of solutions exist for either the bounded thrust or impulsive transfer problems.

The remainder of this chapter is concerned with the method used to determine optimal impulsive transfers for the problem under consideration.

The Unconstrained Problem

Depending on the boundary conditions for a specific problem, the radial constraint $r \geq 1$ may or may not be important. The general procedure to follow is to find a solution to the optimum transfer problem without the radial constraint and then check it to see if the radial constraint is satisfied. If it is, then the problem is completed. If it is not, then further analysis is required to determine the solution. Consider now those problems for which the radial constraint is not critical in shaping the extremal arc.

The most logical initial assumption concerning the number and placement of impulses for a rendezvous is that two impulses are required, one at each end of the transfer.

For an intercept problem it can be assumed that a single impulse applied at the beginning of the transfer is optimum. In either case the properties of the transfer orbital arc can be determined. In particular the radial and tangential components of the velocity at the beginning and end of the transfer orbital arc are given by

$$v(\tau_{o+}) = (e/h)\sin \varphi_o \tag{62}$$

$$u(\tau_{o+}) = h/r_o \tag{63}$$

$$v(\tau_{f-}) = (e/h)\sin \varphi_f \tag{64}$$

$$u(\tau_{f-}) = h/r_f \tag{65}$$

where e and φ are the eccentricity and true anomoly of the transfer orbit.

The corresponding mass ratios are then given by

$$MR_o = EXP(\Delta\bar{V}_o/c) \tag{66}$$

$$MR_f = EXP(\Delta\bar{V}_f/c) \tag{67}$$

where

$$\Delta\bar{V} = [(\Delta v)^2 + (\Delta u)^2]^{\frac{1}{2}} \tag{68}$$

$$\Delta v = v_+ - v_-, \qquad \Delta u = u_+ - u_- \ . \tag{69}$$

Since the final mass has been normalized to a value of unity, the initial mass is given by

$$m_{o-} = (MR_o)(MR_f) \ . \tag{70}$$

Since the switching function must vanish at any point where an impulse is applied,

$$K_\beta(\tau_I) = (c/m)r\lambda_3 \sec \varepsilon - \lambda_5 = 0 \tag{71}$$

or,

$$\lambda_3 = (m\lambda_5/cr) \cos \varepsilon \tag{72}$$

where τ_I is any time where an impulse occurs.

From Equation 45 it then follows that

$$\lambda_4 = (m\lambda_5/c)\sin\epsilon \; . \tag{73}$$

In Appendix B it is shown that K_β is continuous across an impulse and that the product $m\lambda_5$ is constant along the entire extremal arc.

At a point where an impulse is applied ϵ must satisfy

$$\sin\epsilon = \Delta v/\Delta\bar{V}, \qquad \cos\epsilon = \Delta u/\Delta\bar{V} \; . \tag{74}$$

Equations 59, 70 and 74 can thus be substituted into Equations 72 and 73 to give the following expressions for λ_3 and λ_4 at any impulse point.

$$\lambda_3 = (m_{o-}/cr)\cos\epsilon \tag{75}$$

$$\lambda_4 = (m_{o-}/c)\sin\epsilon \tag{76}$$

Equations 75 and 76 can be used to evaluate $\lambda_3(\tau_o)$ and $\lambda_4(\tau_o)$ for both the rendezvous and intercept problems. For the rendezvous problem Equations 75 and 76 can also be used to determine $\lambda_3(\tau_f)$ and $\lambda_4(\tau_f)$. For the intercept problem Equations 60 and 61 require the final values of λ_3 and λ_4 to vanish. Thus the initial and final values of λ_3 and λ_4 are known for both the rendezvous and intercept problems.

The results of Appendix C can now be used to determine the initial values of λ_1 and λ_2 as

$$\begin{bmatrix} \lambda_1 \\ \lambda_2(\tau_{o+}) \end{bmatrix} = \begin{bmatrix} X_{31} & X_{32} \\ X_{41} & X_{42} \end{bmatrix}^{-1}_{\tau_f} \begin{bmatrix} \lambda_3(\tau_f)-\lambda_3(\tau_o)-X_{34}(\tau_f)\lambda_4(\tau_o) \\ \lambda_4(\tau_f)-X_{44}(\tau_f)\lambda_4(\tau_o) \end{bmatrix} . \tag{77}$$

Now that the initial values of all of the Lagrange multipliers are known, the results of Appendix C can be used to determine the complete time history of the switching function K_β. The method suggested by Lion and Handlesman (11) can then be applied to determine the optimality of the assumed solution.

Consider now a problem in which the above assumed solution is not optimal, and an additional impulse is implied from the behavior of the switching function. The problem then arises as to when and where the additional impulse should be placed. Lion and Handlesman (11) suggest that it should be applied at the point where the switching function reaches its maximum value and in a direction specified by the values of λ_3 and λ_4 at that point. However, if only one additional impulse is used, the point suggested can be reached only with an initial impulse equal to that resulting from the originally assumed solution. Thus an additional impulse at the suggested point will cause an increase in the initial mass. Furthermore, the boundary conditions for θ_f and r_f cannot be satisfied with an additional impulse at that point. In spite of these objections, the suggestion of Lion and Handlesman (11) is still useful since the proper time and position to apply the midcourse impulse is probably "in the neighborhood" of that suggested. Nonetheless, the problem still exists as to where the additional impulse should be applied.

Suppose that the midcourse impulse is chosen to occur at a position and time given by the three parameters θ_m, r_m and τ_m. The elements of the two oribtal arcs which comprise the complete transfer can then be determined. Consequently the resulting required initial mass can be calculated. Thus the initial mass is a function of the three parameters θ_m, r_m and τ_m. A gradient technique can be used to determine the optimum values of θ_m, r_m and τ_m. The midcourse values of λ_3 and λ_4 can then be found from Equations 75 and 76, and the initial values of λ_1 and λ_2 can be determined from Equation 77, modified by replacing τ_f with τ_m. The time history of K_β can then be evaluated. As a check on the results of the gradient program, the final values of λ_3 and λ_4 can be compared with their required values and the value of $\dot{K}_\beta$ at the midcourse impulse point can be compared with its required value of zero.

A similar approach can be used for multiple interior impulses and initial or final coasting subarcs.

The Transition Problem

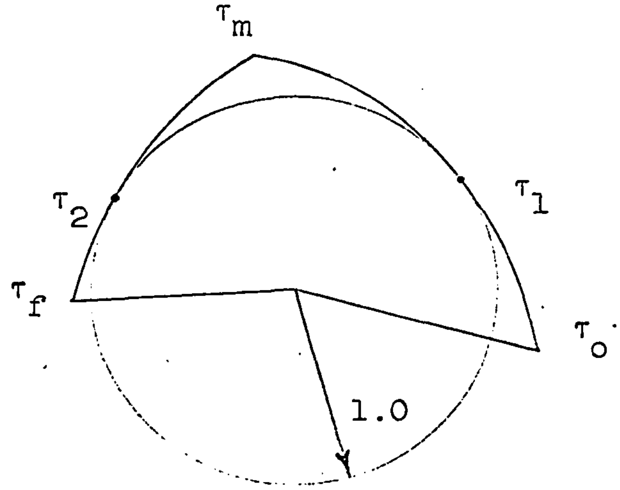
For the sets of boundary conditions in which the radial constraint is important in shaping the extremal arc additional difficulties are encountered in the determination of the solution. The most important is the possible existence of a singular constrained subarc. If such a subarc exists, it appears as though numerical methods must be used for the integration of the state and adjoint equations along the constrained subarc, and a closed form solution for the unknown initial conditions is not available.

Since it is not known a priori whether or not the extremal arc includes a singular subarc, it can be assumed that it does not and thus one can attempt to find a multiple impulse solution which is tangent to the constraint at one or more points.

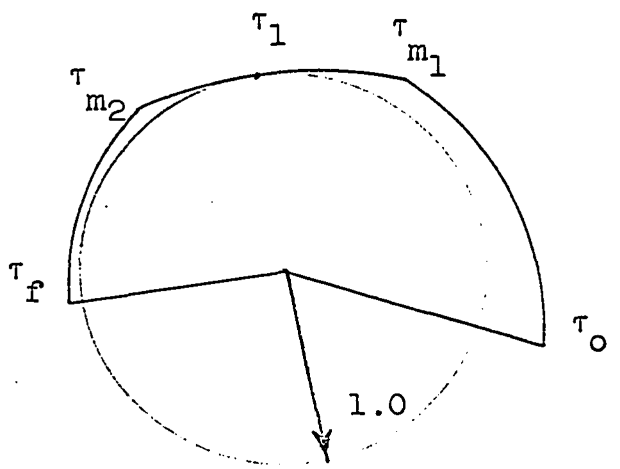
Any solution which is tangent to the constraint at one or more points but does not involve a subarc which follows the constraint is called a transition solution.

In attempting to clarify this matter, assume now that one set of boundary conditions which results in a transition solution has been found. Assume further that if one of the boundary conditions is decreased slightly, the reference solution violates the radial constraint[1]. Then consider the two proposed types of impulsive solutions to the transition problem which are shown in Figure 2. The first consists of a single additional impulse applied at τ_m. The two resulting subarcs are tangent to the radial constraint at τ_1 and τ_2. The second requires that two additional impulses

[1] The assumption that a decrease in one of the boundary conditions causes a violation of the constraint is made only for the ease in discussion. The arguments which follow are equally valid if an increase in one of the boundary conditions causes a constraint violation.



(a) One additional impulse



(b) Two additional impulses

Figure 2. Two types of proposed multiple impulse solutions for a transition problem.

be applied at τ_{m_1} and τ_{m_2} with the transfer being tangent to the constraint at τ_1. For both cases Equations 33, 46 and 47 can be combined to determine that $\mu_2 = 0$ and thus that λ_4 is continuous at a point of tangency.

Consider now the first of these types of proposed impulsive solutions. An analysis similar to that discussed for the unconstrained problem can be used to determine the initial, midcourse and final values of λ_3 and λ_4. In Appendix B it is shown that $\dot{K}_\epsilon$ must vanish at an interior impulse. Thus Equation B-5 can be used to determine an expression for λ_2 at the midcourse impulse point in terms of the unknown Lagrange multiplier λ_1. The results of Appendix C and Equations 31-33 can then be used to give

$$\lambda(\tau_m) = X(\tau_m)\ X(\tau_1)\lambda(\tau_{o+}) + X(\tau_m)\xi \tag{78}$$

where

$$\xi = [0,\ \mu_1(\tau_1),\ 0,\ 0]^T.$$

Equation 78 can be expanded to give three equations which are linear in λ_1, $\lambda_2(\tau_{o+})$ and $\mu_1(\tau_1)$ and can be solved by standard methods. A value for $\mu_1(\tau_2)$ can be determined to satisfy the final value of λ_3 or λ_4. The time history of K_β can then be determined and an analysis similar to that used for the unconstrained problem can be applied to determine the optimality of the solution.

Consider now the second type of proposed solution. A gradient technique can be used to find the optimal placement of the two impulses. The values of λ_1 and $\lambda_2(\tau_{0+})$ can be found so as to satisfy λ_3 and λ_4 at the initial and first midcourse impulse points. The value of μ_1 at the point of tangency can be determined so as to satisfy the value of either λ_3 or λ_4 at the second midcourse impulse point. The time history of K_β can then be calculated to determine the optimality of the solution.

If a study of optimal transfers for a family of boundary conditions is being made, the sets of boundary conditions which correspond to unconstrained problems should be investigated first. There will usually be a subset of boundary conditions for which the unconstrained problems result in extremal arcs which are tangent to the radial constraint at one or more points and are therefore transition cases.

RESULTS

Numerical solutions were obtained for a number of impulsive rendezvous and intercept problems. A value of $c = 0.5$ was used to represent the propulsion system for the vehicle. This is equivalent to a propellant specific impulse of about 420 seconds if the transfers are performed near the earth. The initial mass $\bar{m}_{o-}$ required to perform a transfer for some other value of $c = \bar{c}$ can be found from

$$\bar{m}_{o-} = m_{o-}^{(0.5/\bar{c})}.$$

All computations were performed on an IBM 360/50 digital computer using FORTRAN IV with double precision accuracy.

Rendezvous Problems

The following boundary conditions were used for all of the rendezvous problems investigated:

$$\theta_o = 0$$

$$r_o = r_f = 1.05$$

$$h_o = h_f = r_o^{\frac{1}{2}}$$

$$v_o = v_f = 0.$$

The initial and final velocities correspond to circular orbits at the beginning and end of the transfer. A parametric study was made for the final range angle θ_f. For each value of θ_f the elements of the transfer orbit were calculated by requiring the perigee radius to be unity.

The corresponding flight time was then calculated from the familiar Keplarian time equation. Since the perigee radius was forced to be unity, all of the rendezvous solutions obtained are for the transition between the constrained and unconstrained cases. All of the solutions involve two impulses, one at each end of the transfer. Figure 3 shows the propellant masses required to perform the various transfers. Figures 4 and 5 show typical trajectory histories of the state and adjoint variables respectively. The trajectory history of the switching function is also shown in Figure 5.

In an attempt to find other values of the flight time for a given value of θ_f which resulted in a transition solution, three and four impulse trajectories were considered for flight times slightly lower than those determined by forcing the perigee radius to a value of unity for the two impulse transfers. A four impulse transfer of the type shown in Figure 2b was considered first. The resulting switching function behavior, shown in Figure 6, implies that an impulse applied at τ_1 should replace the two impulses at τ_{m_1} and τ_{m_2}. Thus a three impulse transfer of the type shown in Figure 2a was considered. Figure 7 shows the resulting switching function history. From the results of Figure 7 one might be lead to attempt to find a five impulse solution with the impulses applied at τ_o, τ_1, τ_m, τ_2 and τ_f.

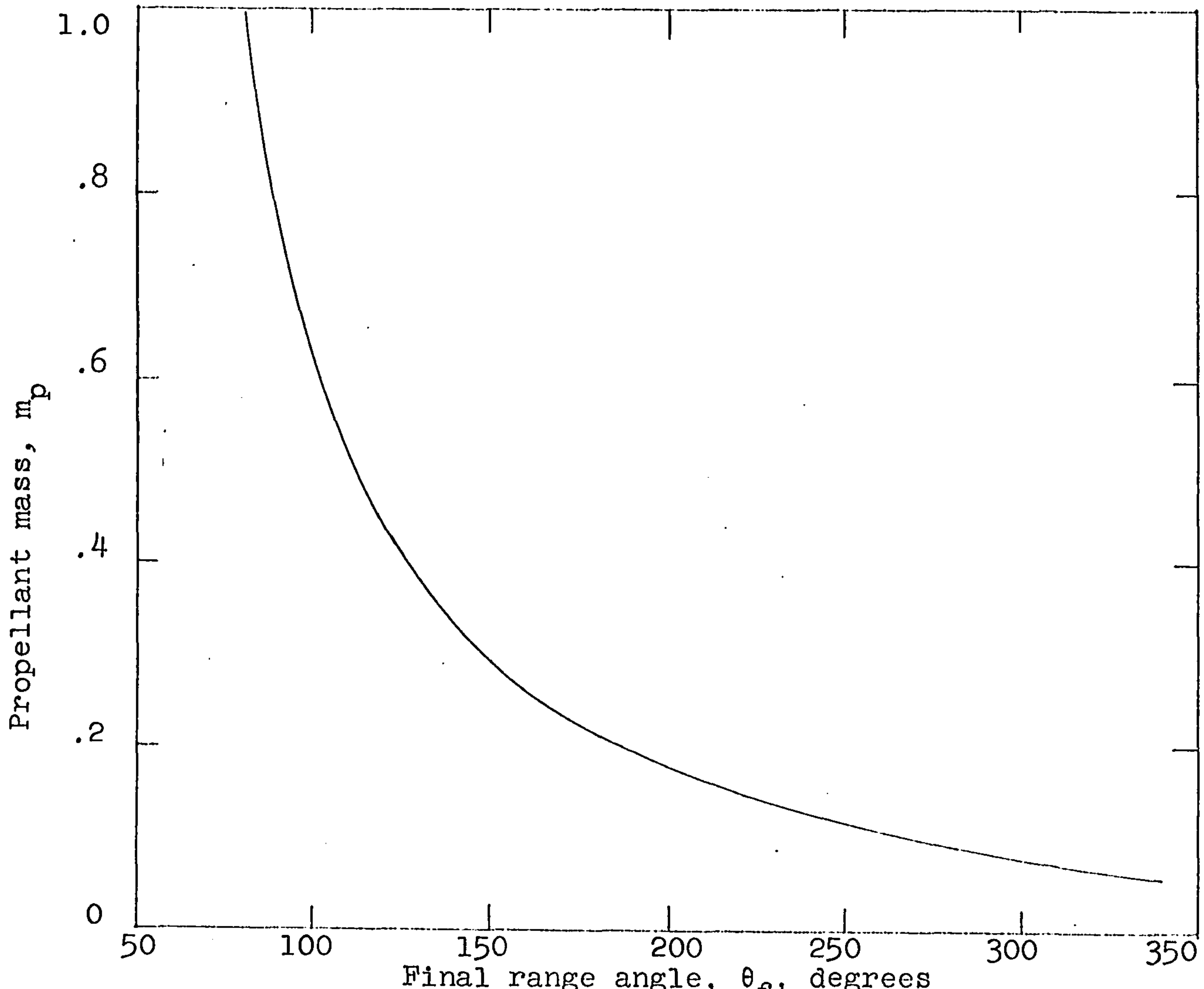


Figure 3. Propellant mass required for optimal rendezvous

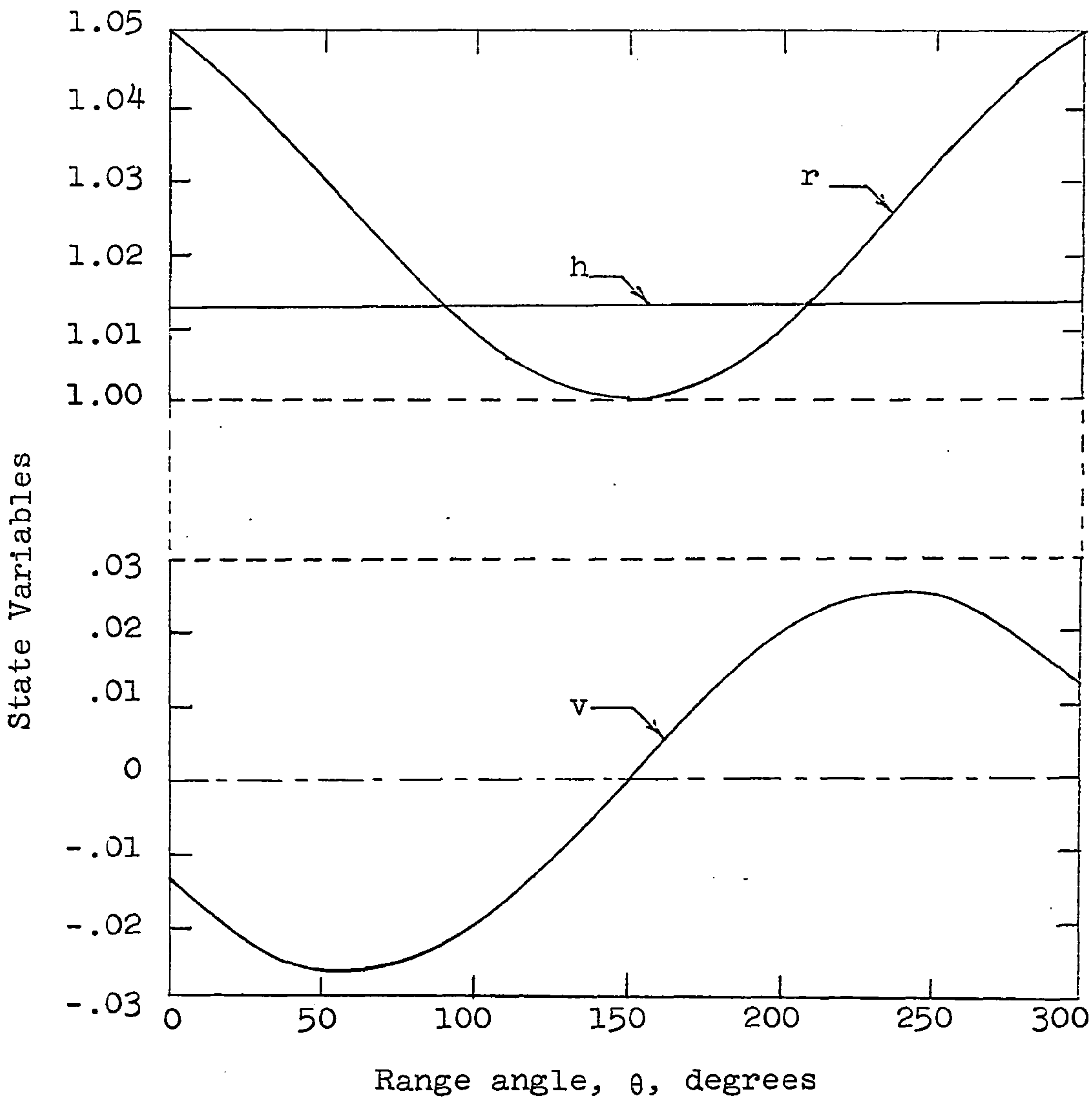


Figure 4. Trajectory histories of the state variables for an optimal rendezvous.

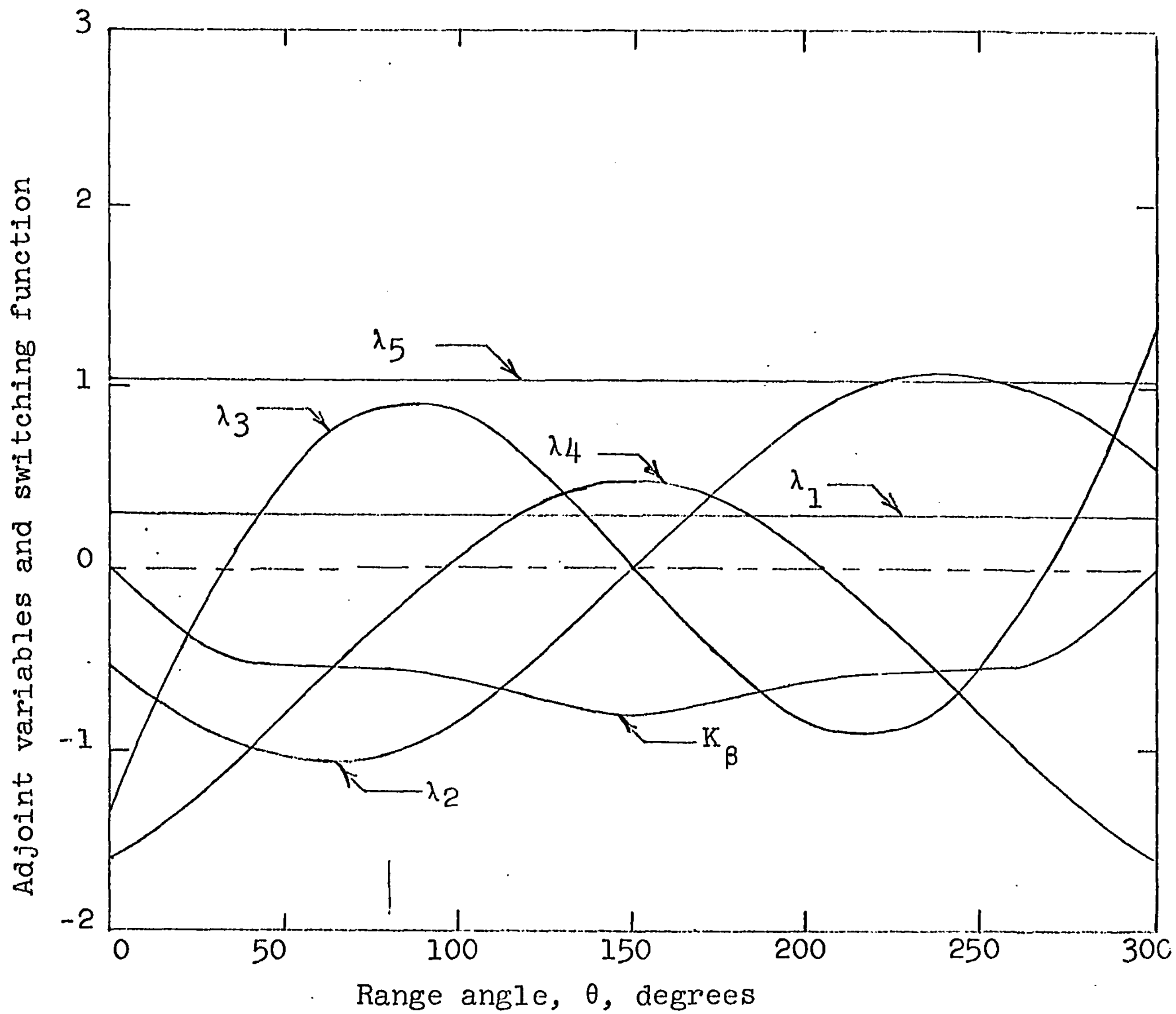


Figure 5. Trajectory histories of the adjoint variables and switching function for an optimal rendezvous.

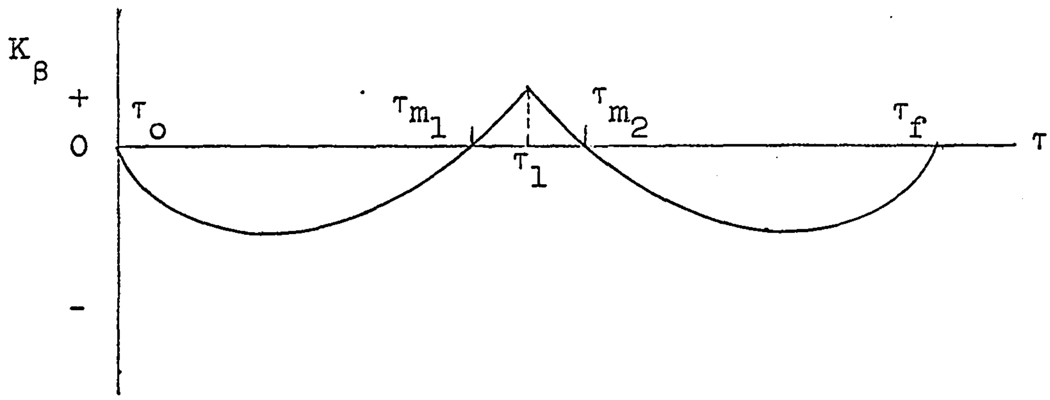


Figure 6. Switching function behavior for a nonoptimal four impulse rendezvous.

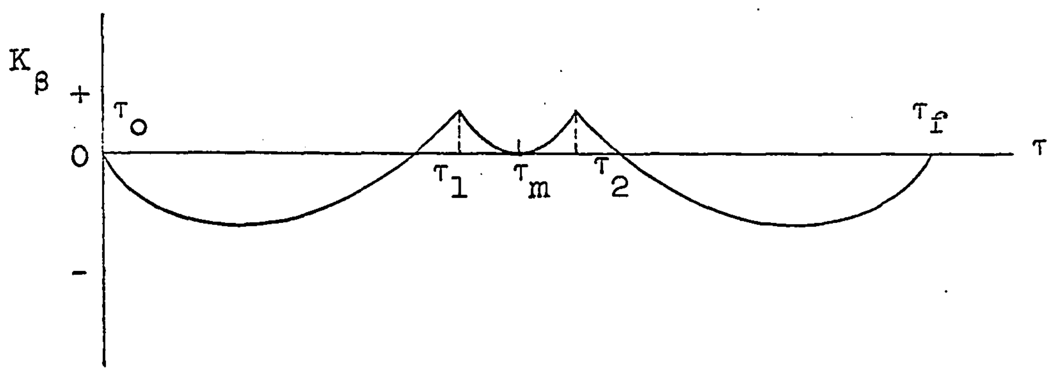


Figure 7. Switching function behavior for a nonoptimal three impulse rendezvous.

However, the discussion which follows indicates that only one value of τ_f results in a transition solution for each value of θ_f, and for lower values of τ_f the solution is fully constrained with a singular constrained subarc.

For the two impulse transition solutions previously discussed, Equations C-41 - C-44 are satisfied, if τ_o corresponds to the perigee point of the transfer arc. Because of the symmetry of the state and adjoint variables for the unconstrained problem, it is to be expectected that any other transition solutions or a constrained solution will show similar symmetry. If the two impulse transition transfers are analyzed as constrained problems, τ_1, τ_2 and τ_m in Figure 7 converge to the same value so that the switching function is described by Figure 8. Figure 8 thus implies that an impulse should be applied at τ_m.

However, for the three impulse cases considered, the thrust direction at τ_m is directed inward along the radius. Since $r(\tau_m) = 1$ and $v(\tau_m) = 0$, an impulse at τ_m results in a constraint violation. On the other hand, if a singular constrained subarc is considered, the switching function could take the form shown in Figure 9. The discontinuities of $\dot{K}_\beta$ are possible through the discontinuity in λ_2 due to μ_1 at τ_a and in $\dot{\gamma}$ at τ_b.

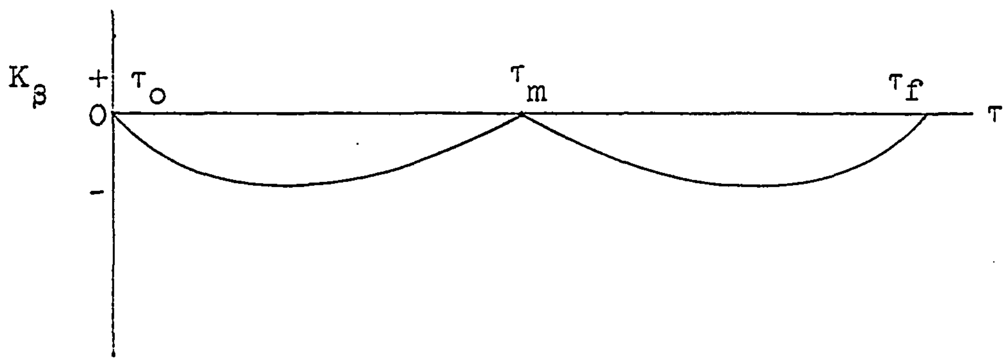


Figure 8. Switching function behavior for an optimal two impulse transition rendezvous analyzed as a constrained problem.

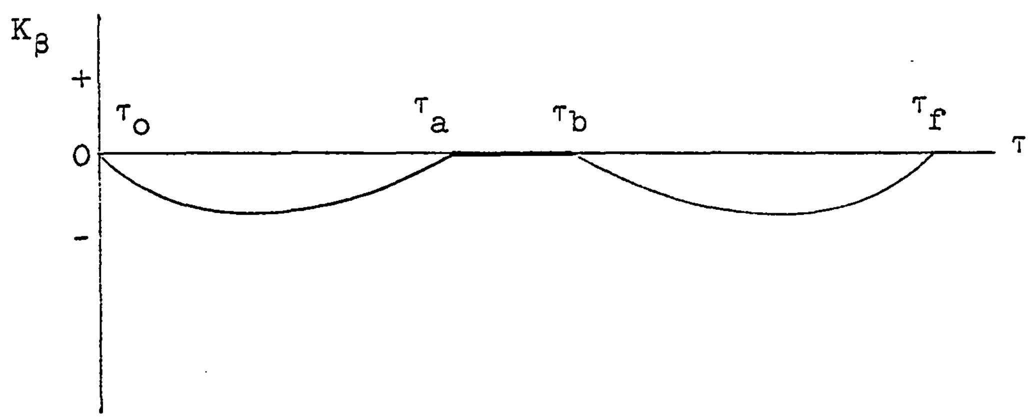


Figure 9. Possible switching function behavior for an optimal constrained rendezvous with a singular constrained subarc.

Intercept Problems

The boundary conditions for the intercept problems considered were identical to those for the rendezvous problems except that the final values of the angular momentum and radial velocity were not specified.

The first type of intercept transfers considered was a single impulse transfer similar to the two impulse transfers considered for the rendezvous problem. From Figure 10, which describes the resulting behavior of the switching function, it is seen that for range angles less than 169° the single impulse transfers are optimal, but for greater range angles two impulses are required.

In order to determine the amount of propellant that can be saved by using two impulses in place of one, a series of two impulse transfers was considered for range angles greater than 169°, with the flight times determined from the corresponding single impulse transfers.

Because of the greater complexity involved in attempting to obtain numerical solutions to problems involving finite burning periods and a limited amount of available computer time, no attempt was made to solve the bounded thrust or fully constrained impulsive-singular subarc problems. A considerable amount of time was spent in the development of a reasonably efficient and reliable gradient process which was used for determining the point where the second impulse

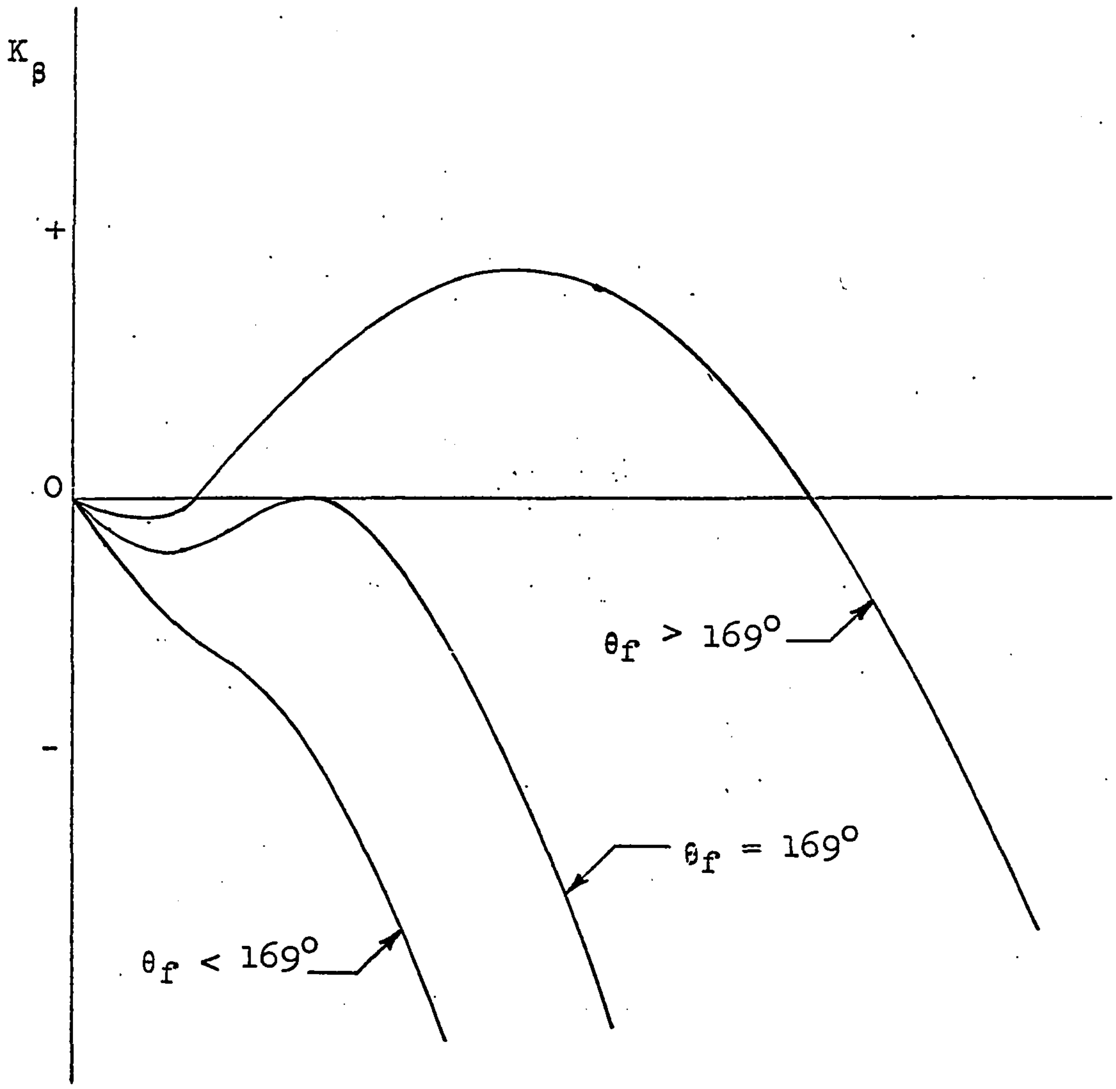


Figure 10. Typical switching function behavior for one impulse intercept problems.

should be applied. The resulting program is based on the ideas presented by Hauge (5) with the three parameters θ_m, r_m and τ_m taking the role of control variables. It was found that the solutions for the two impulse transfers violate the radial constraint so that they are actually unconstrained problems. Figure 11 shows the propellant mass required for the optimal one and two impulse intercept problems. From Figure 12 it is seen that a nonoptimal single impulse transfer can require as much 13% more propellant than the corresponding optimal two impulse transfer. Figures 13-16 describe the state and adjoint variable histories for typical optimal one and two impulse intercepts.

In order to determine the flight times which correspond to transition solutions for the two impulse transfers, several problems with flight times slightly greater than those for the unconstrained cases were solved. A plot of the minimum radius value along the trajectory versus the flight time was then made for each value of θ_f and interpolated to give a new value of τ_f. The procedure was repeated until solutions were found for which the minimum radius was approximately unity. The results are shown in Table 1.

As for the rendezvous problems, an attempt was made to find more than one transition solution for each value of θ_f.

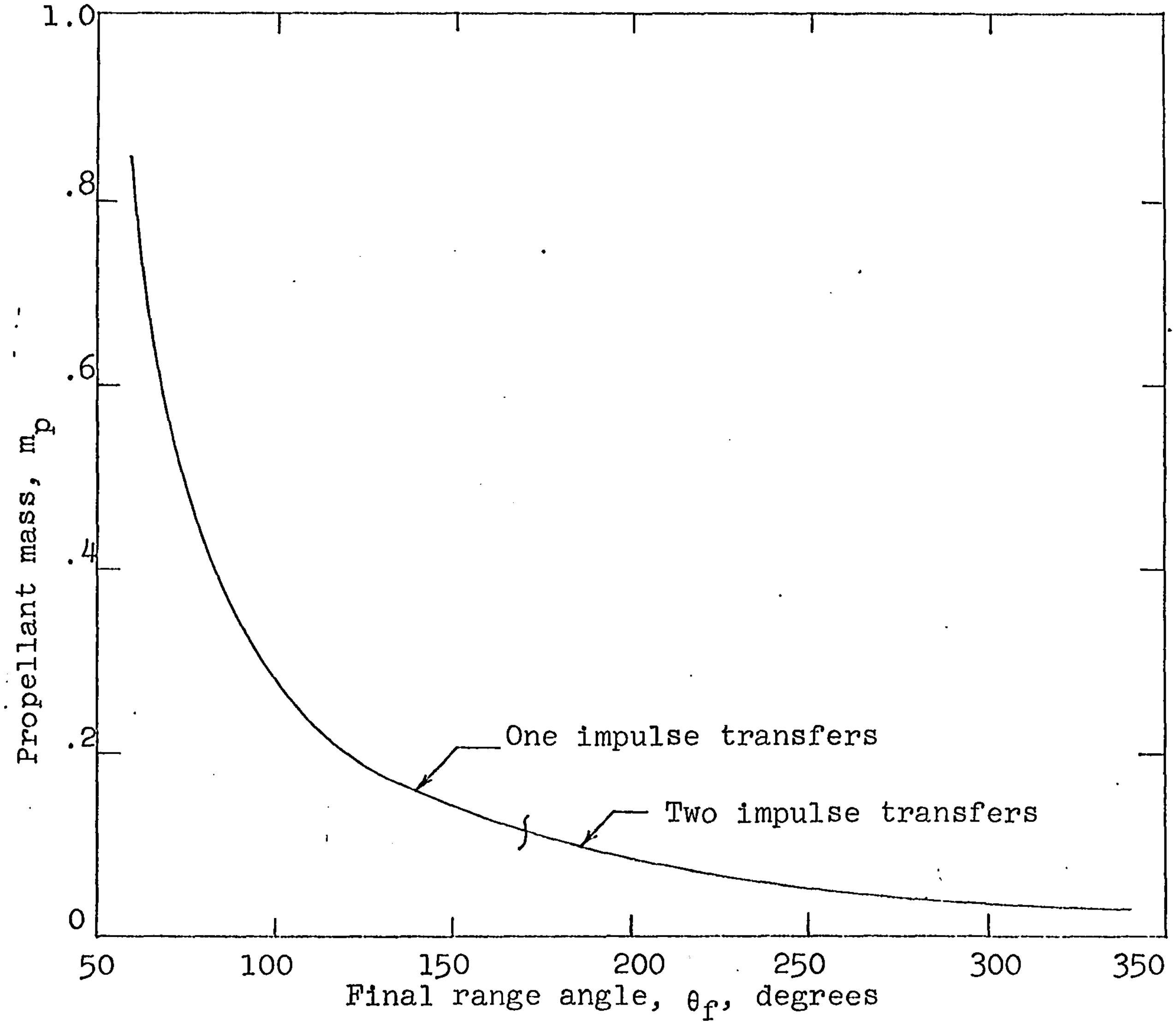


Figure 11. Propellant mass required for optimal intercept.

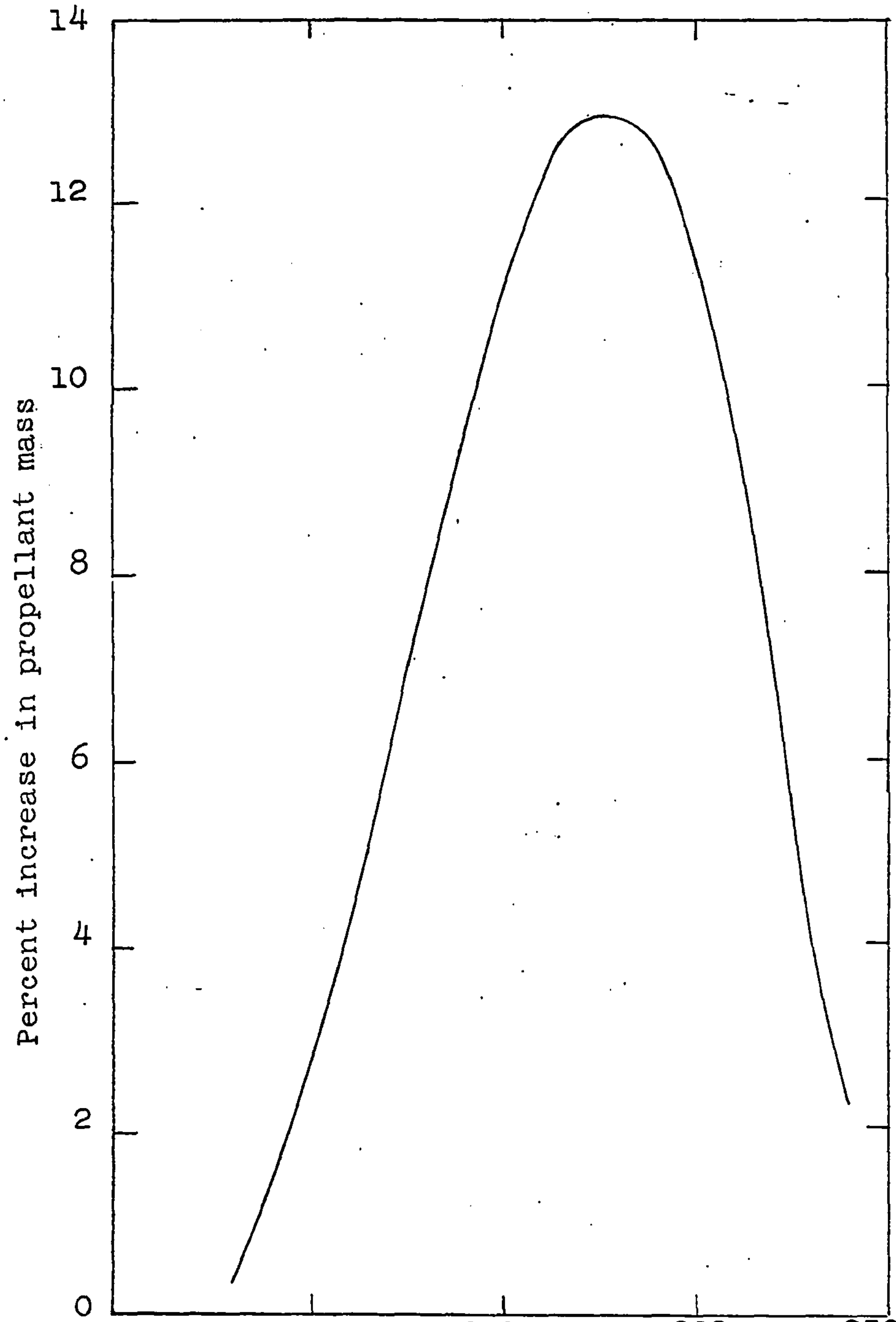


Figure 12. Percent increase in propellant mass required for nonoptimal single impulse intercept based on optimal two impulse intercept.

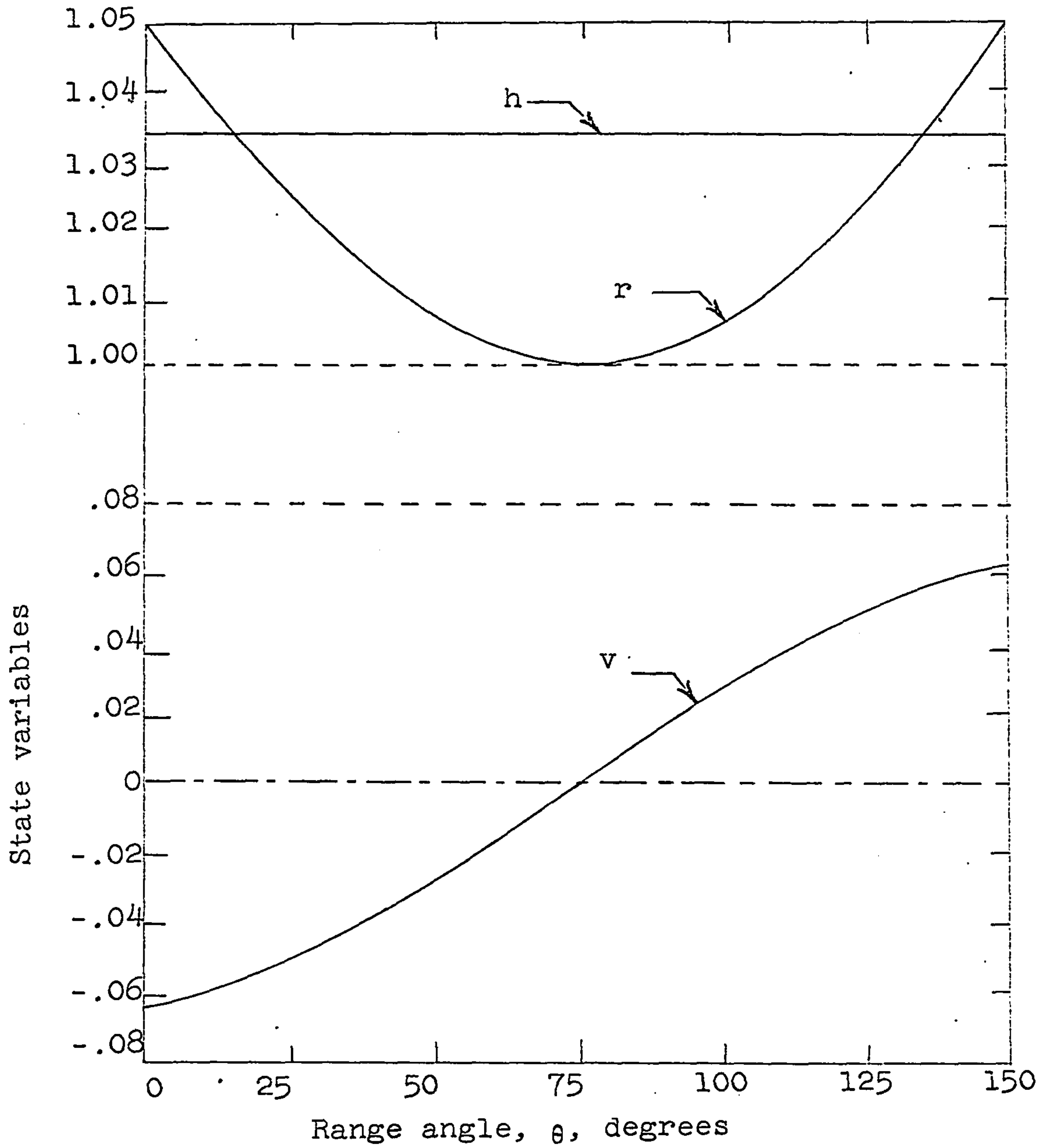


Figure 13. Trajectory histories of the state variables for an optimal one impulse intercept.

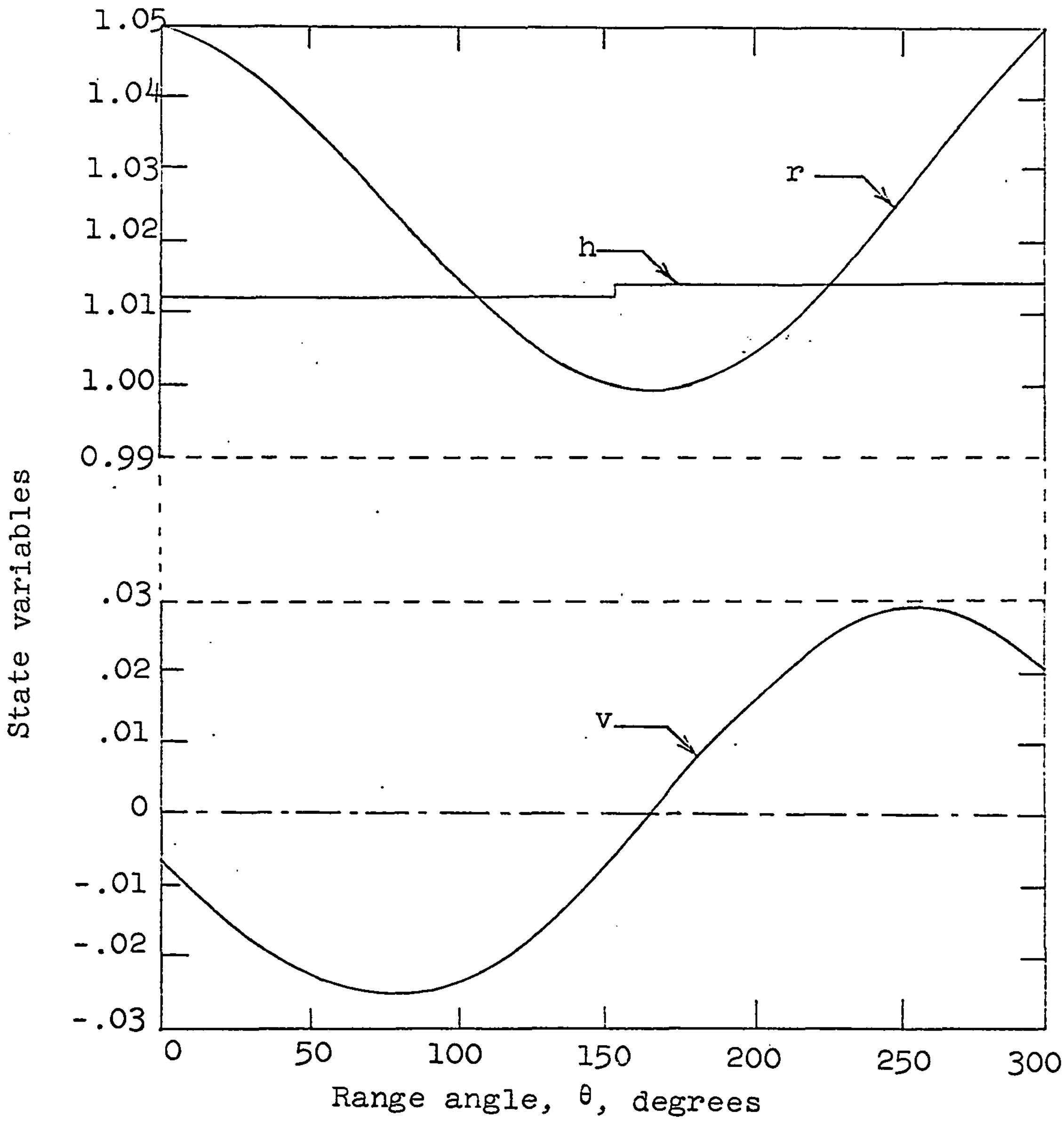


Figure 14. Trajectory histories of the state variables for an optimal two impulse intercept.

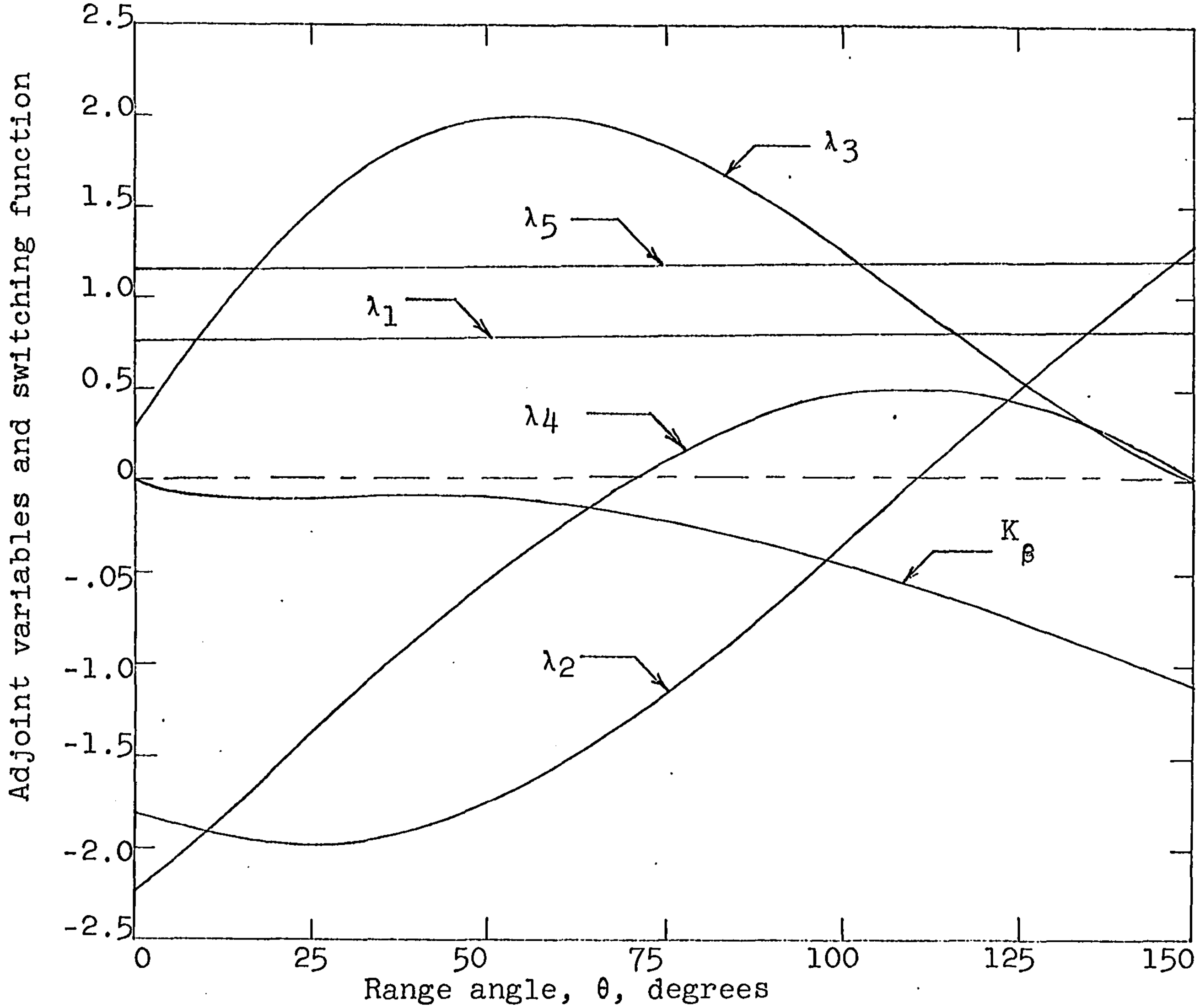


Figure 15. Trajectory histories of the adjoint variables and switching function for an optimal one impulse intercept.

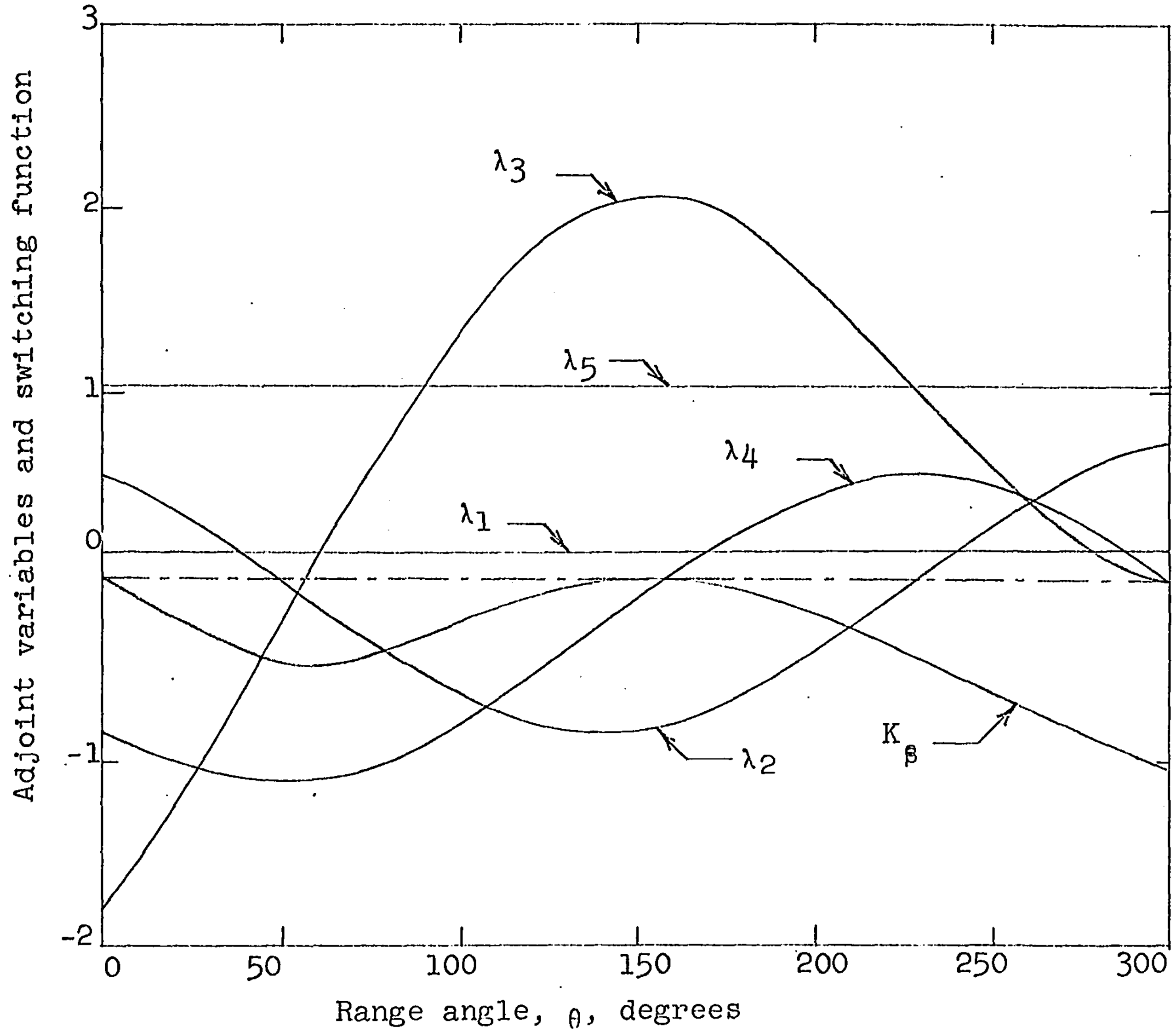


Figure 16. Trajectory histories of the adjoint variables and switching function for an optimal two impulse intercept.

Table 1. Two impulse transition intercept data

Total Range Angle	Total Flight Time	Minimum Radius	Midcourse Impulse Range Angle	Midcourse Impulse Radius	Midcourse Impulse Time
180	3.1800	0.999916	63	1.00721	1.1368
210	3.7391	1.000007	83	1.00712	1.5133
240	4.2934	0.999995	105	1.00527	1.9122
270	4.8456	1.000026	128	1.00295	2.3263
300	5.3983	1.000063	150	1.00108	2.7223
320	5.7685	1.000065	169	1.00010	3.0577

The resulting switching function behavior led to the same conclusions that are discussed in connection with the rendezvous problems.

SUMMARY

Variational calculus is used to determine a set of necessary conditions for optimal planar transfers between two terminals for both bounded and impulsive thrusting. A constraint is included to restrict the trajectory to remain above a radius of unity if so desired.

The necessary condition for singular subarcs presented by Kopp and Moyer (8) is extended to the case of a singular constrained subarc. The result for the problem under consideration is that the thrust direction must be directed inward along any singular constrained subarc.

Numerical results are presented for impulsive fixed-time minimum fuel rendezvous and intercept problems. For the boundary conditions considered, all of the rendezvous solutions involved two impulses, one at each end of the transfer, while both one and two impulse solutions were found for the intercept problem. All of the rendezvous solutions and some of the intercept solutions are for the transition region. The remainder of the intercept solutions are for the unconstrained problem. In an attempt to find multiple values of the flight time which would result in transition solutions for any given range angle, it was found that there is probably only one transition solution for each range angle. All other solutions are then either constrained with a singular constrained subarc or unconstrained.

For the cases considered it was found that a non-optimal single impulse transfer can require up to 13% more propellant than the corresponding two impulse transfer.

RECOMMENDATIONS FOR FURTHER STUDY

Several extensions of the analysis and results presented here are apparent. In particular, a solution to the fully constrained problem is desired. Perhaps it would be helpful to first consider a problem in which the thrust level is fixed at a constant level throughout the flight. It might be expected that the solution to such a problem would involve the three types of solutions discussed by Bryson *et al*. (1). A solution to both the constrained and unconstrained problems with bounded thrust is another area of interest.

Other problems which might be considered include minimum time, interplanetary and three-dimensional transfers.

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APPENDIX A

The Normalized Equations of Motion

The basic dimensional equations of motion for powered exoatmospheric flight in an inverse square law gravitational field are[1]

$$\tilde{\theta}' = \tilde{h}/\tilde{r}^2 \qquad \text{(A-1)}$$

$$\tilde{r}' = \tilde{v} \qquad \text{(A-2)}$$

$$\tilde{h}' = (\tilde{r}\tilde{c}\tilde{\beta}/\tilde{m})\cos\,\epsilon \qquad \text{(A-3)}$$

$$\tilde{v}' = \tilde{h}^2/\tilde{r}^3 - k/\tilde{r}^2 + (\tilde{c}\tilde{\beta}/\tilde{m})\sin\,\epsilon \qquad \text{(A-4)}$$

$$\tilde{m}' = -\,\tilde{\beta} \qquad \text{(A-5)}$$

where k is the gravitational constant of the attracting body.

In this study it is assumed that the final value of the vehicle mass is specified to be M and that an inequality constraint of the form $\tilde{r} \geq R$ is desired. In order to make the results as general as possible the following normalized variables are introduced:

$$\theta = \tilde{\theta} \qquad \text{(A-6)}$$

$$r = \tilde{r}/R \qquad \text{(A-7)}$$

$$h = \tilde{h}/Rq \qquad \text{(A-8)}$$

$$v = \tilde{v}/q \qquad \text{(A-9)}$$

$$m = \tilde{m}/M \qquad \text{(A-10)}$$

---

[1]In this appendix the symbol "'" denotes differentiation with respect to t and the symbol "~" denotes a dimensional variable.

$$\beta = \tilde{\beta}/(Mq/R) \tag{A-11}$$

$$c = \tilde{c}/q \tag{A-12}$$

$$\tau = t/(R/q) \tag{A-13}$$

where

$$q = (k/R)^{\frac{1}{2}}. \tag{A-14}$$

The normalized differential equations now become

$$\dot{\theta} = h/r^2 \tag{A-15}$$

$$\dot{r} = v \tag{A-16}$$

$$\dot{h} = (rc\beta/m)\cos\,\epsilon \tag{A-17}$$

$$\dot{v} = h^2/r^3 - 1/r^2 + (c\beta/m)\sin\,\epsilon \tag{A-18}$$

$$\dot{m} = -\beta \tag{A-19}$$

where the symbol "$\cdot$" denotes differentiation with respect to τ.

In addition the radial inequality constraint becomes

$$r \geq 1. \tag{A-20}$$

APPENDIX B

Impulsive Analysis

The differential state and adjoint equations are given by Equations A-15 to A-19 and 13-17 respectively. For a powered subarc the independent variable can be changed from τ to m with the transformation

$$\frac{d}{dm} = -\frac{1}{\beta}\frac{d}{d\tau} \ . \tag{B-1}$$

The resulting equations for an unconstrained subarc ($\gamma = 0$) become[1]

$$\begin{aligned}
\theta' &= -h/\beta r^2 \\
r' &= -v/\beta \\
h' &= -(rc/m)\cos\epsilon \\
v' &= -(h^2/r^3 - 1/r^2)/\beta - (c/m)\sin\epsilon \\
m' &= 1 \\
\lambda_1' &= 0 \\
\lambda_2' &= -2\lambda_1 h/\beta r^3 - \lambda_4(3h^2/r^4 - 2/r^3)/\beta + \lambda_3(c/m)\cos\epsilon \\
\lambda_3' &= \lambda_1/\beta r^2 + 2\lambda_4 h/\beta r^3 \\
\lambda_4' &= \lambda_2/\beta \\
\lambda_5' &= -(c/m^2)(r^2\lambda_3^2 + \lambda_4^2)^{\frac{1}{2}} \ .
\end{aligned} \tag{B-2}$$

In arriving at the impulsive approximation from the above powered case it is assumed that $\beta \to \infty$ and $\Delta\tau \to 0$ in such a way that the product $\beta\Delta\tau$ remains constant at the value given

[1] In this appendix the symbol "'" refers to differentiation with respect to m.

by $-\Delta m$. Integration of Equations B-2 then gives

$$
\begin{aligned}
\Delta\theta &= 0 \\
\Delta r &= 0 \\
\Delta h &= rc \cos \epsilon \ln MR \\
\Delta v &= c \sin \epsilon \ln MR \\
\Delta m &= m_+ - m_- \\
\Delta\lambda_1 &= 0 \\
\Delta\lambda_2 &= -c\,\lambda_3 \cos \epsilon \ln MR \\
\Delta\lambda_3 &= 0 \\
\Delta\lambda_4 &= 0 \\
\Delta\lambda_5 &= c\,[r^2\lambda_3^2 + \lambda_4^2]^{\frac{1}{2}}(1/m_+ - 1/m_-)
\end{aligned}
\tag{B-3}
$$

where, by definition,

$$
\begin{aligned}
MR &= m_-/m_+ \\
\Delta(\ ) &= (\ )_+ - (\ )_- \ .
\end{aligned}
\tag{B-4}
$$

Lawden (9) has shown that the following conditions must be satisfied for all optimal impulsive transfers:

a) K_β, K_ϵ, $\dot{K}_\beta$ and $\dot{K}_\epsilon$ must be continuous everywhere;

b) $K_\beta \leq 0$ everywhere;

c) $K_\beta = 0$ at every impulse point;

d) $\dot{K}_\beta = \dot{K}_\epsilon = 0$ at every interior impulse point;

e) The product $m\lambda_5$ is constant along the entire extremal arc.

Differentiation of Equation 39 and condition d) above gives Equation B-5 which must be satisfied at any interior impulse point.

$$\lambda_3^2 rv - \lambda_1\lambda_3 - \lambda_2\lambda_4 - 2\lambda_3\lambda_4 h/r = 0 \tag{B-5}$$

APPENDIX C

Solution of the Adjoint Equations for Coasting Subarcs

Hempel (6) has given a solution to the adjoint equations which is valid for elliptic and hyperbolic coasting subarcs. However, he uses parameters in the normalization of the equations of motion which are different from those used in this study. In particular he introduces a normalized angular momentum parameter of unity and uses the length of the semilatus rectum of the orbit to normalize the radius. Two difficulties are encountered in attempting to apply his results to the present study. The first arises because of the difference in the normalization processes. The second, and more important, arises when problems of multiple subarcs occur. In such cases the angular momentum and semilatus rectum are in general different for each subarc. As a result, the use of Hempel's procedure requires a different normalization for each subarc. In spite of this difficulty in attempting to use his results, his article is very helpful since he outlines a method by which the adjoint equations may be solved.

During any coasting subarc $\dot{\lambda}_5 = \dot{h} = 0$ so that $\bar{\lambda}_5$ and h are constant on all coasting subarcs.

<u>Elliptic and hyperbolic orbits</u>

The adjoint equations may be written in matrix form as

$$\dot{\lambda}(\tau) = B(\tau)\lambda(\tau) \tag{C-1}$$

where $B(\tau)$ is a four by four matrix given by

$$B(\tau) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2h/r^3 & 0 & 0 & 3h^2/r^4 - 2/r^3 \\ -1/r^2 & 0 & 0 & -2h/r^3 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{C-2}$$

Equation C-1 forms a set of four ordinary linear first order differential equations with time varying coefficients. Gibson (4) states that the solution to such a system can be written in the general form

$$\lambda(\tau) = X(\tau)\ \lambda(\tau_o) \tag{C-3}$$

where $X(\tau)$ is a four by four matrix which satisfies the differential equation

$$\dot{X}(\tau) = B(\tau)X(\tau) \tag{C-4}$$

with initial conditions given by

$$X(\tau_o) = I \tag{C-5}$$

where I is the identity matrix.

Expansion of Equation C-4 gives

$$\dot{X}_{1j} = 0 \tag{C-6}$$

$$\dot{X}_{2j} = (2h/r^3)X_{1j} + (3h^2/r^4 - 2/r^3)X_{4j} \qquad j = 1,2,3,4 \tag{C-7}$$

$$\dot{X}_{3j} = -(1/r^2)\ X_{1j} - (2h/r^3)\ X_{4j} \tag{C-8}$$

$$\dot{X}_{4j} = -X_{2j} \tag{C-9}$$

Integration of Equation C-6 and application of the initial conditions gives

$$X_{11} = 1, \quad X_{12} = X_{13} = X_{14} = 0 \tag{C-10}$$

Differentiation of Equation C-9 and substitution of Equations C-7 and C-10 into the result gives

$$\ddot{X}_{41} + (3h^2/r^4 - 2/r^3)\ X_{41} = -2h/r^3 \qquad \text{(C-11)}$$

$$\ddot{X}_{4j} + (3h^2/r^4 - 2/r^3)\ X_{4j} = 0 \qquad j = 2,3,4 \qquad \text{(C-12)}$$

Equations C-11 and C-12 are linear, second order ordinary differential equations whose solution may be written as

$$X_{41} = a_1 u_1 + b_1 u_2 + u_3 \qquad \text{(C-13)}$$

$$X_{4j} = a_j u_1 + b_j u_2 \qquad j = 2,3,4 \qquad \text{(C-14)}$$

where u_1 and u_2 are linearly independent solutions of the homogeneous Equation C-12 and u_3 is a solution of the non-homogeneous Equation C-11. The coefficients a_j and b_j, $j = 1,2,3,4$ must be determined so as to satisfy the given initial conditions.

It can be verified that the functions u_1, u_2 and u_3 are given by

$$\begin{aligned} u_1 &= v \\ u_2 &= h^2(3\tau u_1 - 2r - hu_3)/(1-e^2) \\ u_3 &= h(1-h^2/r)/e^2 \end{aligned} \qquad \text{(C-15)}$$

Differentiation of Equations C-13 and C-14 and use of Equation C-9 gives

$$X_{21} = -(a_1\dot{u}_1 + b_1\dot{u}_2 + \dot{u}_3) \qquad \text{(C-16)}$$

$$X_{2j} = -(a_j\dot{u}_1 + b_j\dot{u}_2) \quad j = 2,3,4 \qquad \text{(C-17)}$$

Equations C-13, C-14, C-16 and C-17 may be evaluated at τ_o to determine the coefficients a_j and b_j, $j = 1,2,3,4$ with the following results:

$$
\begin{aligned}
a_1 &= \dot{u}_2(\tau_o)u_3(\tau_o) - u_2(\tau_o)\dot{u}_3(\tau_o) \\
a_2 &= -u_2(\tau_o) \\
a_3 &= 0 \\
a_4 &= -\dot{u}_2(\tau_o) \\
b_1 &= -\dot{u}_1(\tau_o)u_3(\tau_o) + u_1(\tau_o)\dot{u}_3(\tau_o) \\
b_2 &= u_1(\tau_o) \\
b_3 &= 0 \\
b_4 &= \dot{u}_1(\tau_o)
\end{aligned}
\tag{C-18}
$$

Substitution of Equations C-10, C-13 and C-14 into Equation C-8 gives

$$\dot{X}_{31} = -1/r^2 - (2h/r^3)(a_1u_1 + b_1u_2 + u_3) \tag{C-19}$$

$$\dot{X}_{32} = -(2h/r^3)(a_2u_1 + b_2u_2) \tag{C-20}$$

$$\dot{X}_{33} = 0 \tag{C-21}$$

$$\dot{X}_{34} = -(2h/r^3)(a_4u_1 + b_4u_4) \tag{C-22}$$

Equation C-21 may be integrated directly and with the given initial conditions the solution is

$$X_{33} = 1 \tag{C-23}$$

Now let v_1, v_2 and v_3 be any functions which satisfy the following equations:

$$
\begin{aligned}
\dot{v}_1 &= -(2h/r^3)u_1 \\
\dot{v}_2 &= -(2h/r^3)u_2 \\
\dot{v}_3 &= -(1/r^2) - (2h/r^3)u_3
\end{aligned}
\tag{C-24}
$$

The solution of Equations C-19, C-20 and C-22 is then given by

$$X_{31} = a_1v_1 + b_1v_2 + v_3 + c_1$$

$$X_{32} = a_2 v_1 + b_2 v_2 + c_2 \qquad \text{(C-25)}$$
$$X_{34} = a_4 v_1 + b_4 v_2 + c_4 .$$

where the constants c_1, c_2 and c_4 may be evaluated from the initial conditions with the result

$$c_1 = -[a_1 v_1(\tau_o) + b_1 v_2(\tau_o) + v_3(\tau_o)]$$
$$c_2 = -[a_2 v_1(\tau_o) + b_2 v_2(\tau_o)] \qquad \text{(C-26)}$$
$$c_4 = -[a_4 v_1(\tau_o) + b_4 v_2(\tau o)]$$

It can be verified that the functions v_1, v_2 and v_3 are given by

$$v_1 = h/r^2$$
$$v_2 = h^2(3_\tau\ v_1 - h\ v_3)/(1-e^2) \qquad \text{(C-27)}$$
$$v_3 = v(1+h^2/r)/e^2$$

The solution to the adjoint equations for elliptic and hyperbolic orbits is now complete. The results are summarized below.

$$\lambda(\tau) = X(\tau)\ \lambda(\tau_o) \qquad \text{(C-28)}$$

$$X_{11} = X_{33} = 1$$
$$X_{12} = X_{13} = X_{14} = X_{23} = X_{43} = 0$$
$$X_{21} = -[-v_2(\tau_o)\dot{u}_1 + v_1(\tau_o)\dot{u}_2 + \dot{u}_3]$$
$$X_{22} = -[-u_2(\tau_o)\dot{u}_1 + u_1(\tau_o)\dot{u}_2]$$
$$X_{24} = -[-\dot{u}_2(\tau_o)\dot{u}_1 + \dot{u}_1(\tau_o)\dot{u}_2]$$
$$X_{31} = -v_2(\tau_o)v_1 + v_1(\tau_o)v_2 + v_3 - v_3(\tau_o) \qquad \text{(C-29)}$$
$$X_{32} = -u_2(\tau_o)v_1 + u_1(\tau_o)v_2 - u_3(\tau_o)$$
$$X_{34} = -\dot{u}_2(\tau_o)v_1 + \dot{u}_1(\tau_o)v_2 - \dot{u}_3(\tau_o)$$
$$X_{41} = -v_2(\tau_o)u_1 + v_1(\tau_o)u_2 + u_3$$

$$X_{42} = -u_2(\tau_o)u_1 + u_1(\tau_o)u_2$$
$$X_{44} = -\dot{u}_2(\tau_o)u_1 + \dot{u}_1(\tau_o)u_2$$

$$\begin{aligned}
u_1 &= v \\
u_2 &= h^2(3\tau u_1 - 2r - hu_3)/(1-e^2) \\
u_3 &= h(1-h^2/r)/e^2 \\
v_1 &= h/r^2 \\
v_2 &= h^2(3\tau v_1 - hv_3)/(1-e^2) \\
v_3 &= v(1+h^2/r)/e^2 \\
\dot{u}_1 &= (h^2/r-1)/r^2 \\
\dot{u}_2 &= h^2(3\tau\dot{u}_1 + u_1 - h\dot{u}_3)/(1-e^2) \\
\dot{u}_3 &= h^3v/(r^2e^2)
\end{aligned} \tag{C-30}$$

For computational purposes it may be desirable to write Equation C-28 in a slightly different form. The matrix $X(\tau)$ may be written as the product of two matrices as shown below.

$$X(\tau) = F(\tau)\, G(\tau_o) \tag{C-31}$$

where $F(\tau)$ and $G(\tau_o)$ are

$$F(\tau) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -\dot{u}_1 & -\dot{u}_2 & -\dot{u}_3 \\ 1 & v_1 & v_2 & v_3 \\ 0 & u_1 & u_2 & u_3 \end{bmatrix} \tag{C-32}$$

$$G(\tau_o) = \begin{bmatrix} -v_3 & -u_3 & 1 & -\dot{u}_3 \\ -v_2 & -u_2 & 0 & -\dot{u}_2 \\ v_1 & u_1 & 0 & \dot{u}_1 \\ 1 & 0 & 0 & 0 \end{bmatrix}_{\tau = \tau_o} \tag{C-33}$$

Equation C-28 may now be written as

$$\lambda(\tau) = F(\tau)\, g(\tau_o) \tag{C-34}$$

where $g(\tau_o)$ is given by

$$g(\tau_o) = G(\tau_o)\, \lambda(\tau_o) \tag{C-35}$$

A comparison of Equations C-28 and C-34 shows that both forms require the multiplication of a matrix and a vector. However, a comparison of Equations C-29 and C-32 indicates that the amount of computation required to determine $F(\tau)$ is much less than required for $X(\tau)$.

From Equations C-30 it is seen that several of the terms on the right side require a division by e^2 or $1-e^2$. Along a circular orbit $e=0$ and along a parabolic orbit $e = 1$. Thus different solutions must be found for circular and parabolic orbits.

Circular orbits

For a circular orbit the radial velocity v vanishes and the radius r is constant. In addition h and r are related by

$$h^2 = r \tag{C-36}$$

For an unconstrained subarc the adjoint equations then become

$$\begin{aligned} \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= (2h/r^3)\lambda_1 + (1/r^3)\lambda_4 \\ \dot{\lambda}_3 &= -(1/r^2)\lambda_1 - (2h/r^3)\lambda_4 \\ \dot{\lambda}_4 &= -\lambda_2 \end{aligned} \tag{C-37}$$

Equations C-37 form a set of four linear ordinary first order differential equations with constant coefficients and may be solved by standard methods. As in the preceding section the solution in matrix form is

$$\lambda(\tau) = X(\tau)\, \lambda(\tau_o) \tag{C-38}$$

where $X(\tau)$ is a four by four matrix with elements

$$\begin{aligned}
X_{11} &= X_{33} = 1 \\
X_{12} &= X_{13} = X_{14} = X_{23} = X_{43} = 0 \\
X_{21} &= 2h\omega \sin \omega\tau \\
X_{22} &= \cos \omega\tau \\
X_{24} &= \omega \sin \omega\tau \\
X_{31} &= 4h^2\omega(\omega\tau - \sin \omega\tau) - \tau/r^2 \\
X_{32} &= 2h(1 - \cos \omega\tau) \\
X_{34} &= -2h\omega \sin \omega\tau \\
X_{41} &= -2h(1 - \cos \omega\tau) \\
X_{42} &= -(1/\omega)\sin \omega\tau \\
X_{44} &= \cos \omega\tau
\end{aligned} \tag{C-39}$$

where

$$\omega^2 = 1/r^3. \tag{C-40}$$

Parabolic orbits

The solution to the adjoint equations for parabolic orbits was not required for the present study. Consequently no effort was made to find such a solution.

An important property of $X(\tau)$

Let $\tau_o = 0$ be a point on a single or multiple impulse transfer such that the following relationships hold:

$$
\begin{aligned}
r(\tau) &= r(-\tau) \\
h(\tau) &= h(-\tau) \\
v(\tau) &= -v(-\tau) \\
e(\tau) &= e(-\tau).
\end{aligned}
\tag{C-41}
$$

A straightforward substitution of Equations C-41 into Equations C-29 and C-39 gives the following results:

$$
\begin{aligned}
X_{21}(\tau) &= -X_{21}(-\tau) \\
X_{22}(\tau) &= X_{22}(-\tau) \\
X_{24}(\tau) &= -X_{24}(-\tau) \\
X_{31}(\tau) &= -X_{31}(-\tau) \\
X_{32}(\tau) &= X_{32}(-\tau) \\
X_{34}(\tau) &= -X_{34}(-\tau) \\
X_{41}(\tau) &= X_{41}(-\tau) \\
X_{42}(\tau) &= -X_{42}(-\tau) \\
X_{44}(\tau) &= X_{44}(-\tau) .
\end{aligned}
\tag{C-42}
$$

Now let

$$
\begin{aligned}
\lambda_2(\tau_{o+}) &= -\lambda_2(\tau_{o-}) \\
\lambda_3(\tau_o) &= 0.
\end{aligned}
\tag{C-43}
$$

Substitution of Equations C-42 and C-43 into Equation C-28 then gives

$$
\begin{aligned}
\lambda_2(\tau) &= -\lambda_2(-\tau) \\
\lambda_3(\tau) &= -\lambda_3(-\tau) \\
\lambda_4(\tau) &= \lambda_4(-\tau) .
\end{aligned}
\tag{C-44}
$$

APPENDIX D

A Necessary Condition for Singular Constrained Subarcs

Kopp and Moyer (8) present a derivation of a set of necessary conditions for singular subarcs without a state variable constraint. Two difficulties are encountered in attempting to apply their results directly to the problem considered in this study. The first arises because of the additional constraint that $\ddot{S}$ must vanish on the constrained subarc. The second arises from their assumption that the payoff function depends on the open final state variables. For minimum propellant consumption with the final mass fixed and the initial mass open, the payoff function depends on the open initial state variable $m(\tau_o)$. It is the purpose of this appendix to extend their analysis to include the constraint on $\ddot{S}$ and the modified expression for the payoff function. In so doing it is convenient to follow the approach used by Kopp (7). The results for the problem under consideration follow the general analysis.

General analysis

Consider a system of differential equations of the form

$$\dot{x}_i = f_i(\bar{x}, \bar{u}, t) \quad i = 1, \ldots, n \tag{D-1}$$

where $\bar{x}$ is an n-dimensional state vector and $\bar{u}$ is a p-dimensional control vector, subject to a set of initial and final boundary conditions and the constraint

$$Q(\bar{x}, \bar{u}, t) = 0$$

along a singular constrained subarc.

For simplicity of analysis it is now assumed that it is desired to minimize one of the open initial state variables, $x_k(t_o)$.

The payoff function P then becomes

$$P = x_k(t_o) \quad . \tag{D-2}$$

The Hamiltonian H and the adjoint equations are given by

$$H = \sum_{i=1}^{n} \lambda_i f_i - \gamma Q \tag{D-3}$$

$$\dot{\lambda}_i = - \sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial x_i} + \gamma \frac{\partial Q}{\partial x_i} \quad i = 1, \ldots, n \tag{D-4}$$

Let I_o be the set of all i's such that $x_i(t_o)$ is open except for i=k, and let I_f be the set of all i's such that $x_i(t_f)$ is open. The initial and final values of the Lagrange multipliers are then given by

$$\lambda_k(t_o) = 1 \tag{D-5}$$

$$\lambda_i(t_o) = 0 \qquad i \in I_o \tag{D-6}$$

$$\lambda_i(t_f) = 0 \qquad i \in I_f \tag{D-7}$$

Kopp (7) gives the following result which is valid for the present problem:

$$\sum_{i=1}^{n} \lambda_i \Delta x_i \Big|_{t_o}^{t_f} = \int_{t_o}^{t_f} \sum_{i=1}^{n} \dot{\lambda}_i \Delta x_i \, dt + \int_{t_o}^{t_f} \sum_{i=1}^{n} \lambda_i \Delta f_i \, dt \tag{D-8}$$

where

$$\Delta f i = f_i(\bar{x}^* + \Delta\bar{x}, \bar{u}^* + \Delta\bar{u}, t) - f_i(\bar{x}^*, \bar{u}^*, t). \tag{D-9}$$

In the above equations $\bar{x}^*$ and $\bar{u}^*$ correspond to the optimal trajectory and $\Delta\bar{x}$ and $\Delta\bar{u}$ are perturbations from $\bar{x}^*$ and $\bar{u}^*$.

From the given boundary conditions and Equations D-5 to D-7, the left hand side of Equation D-8 may be written as

$$\sum_{i=1}^{n} \lambda_i \Delta x_i \Big|_{t_o}^{t_f} = -\lambda_k(t_o)\Delta x_k(t_o) = -\Delta x_k(t_o) = -\Delta P. \tag{D-10}$$

Thus Equation D-8 becomes

$$\Delta P = -\int_{t_o}^{t_f} \sum_{i=1}^{n} \dot{\lambda}_i \Delta x_i \, dt - \int_{t_o}^{t_f} \sum_{i=1}^{n} \lambda_i \Delta f_i \, dt. \tag{D-11}$$

Equations D-1 and D-4 can be put in the canonical form

$$\dot{x}_i = \partial H/\partial \lambda_i \tag{D-12}$$

$$\dot{\lambda}_i = -\partial H/\partial x_i . \tag{D-13}$$

Expansion of Equation D-9 in a Taylor series, substitution of the result and Equations D-12 and D-13 into Equation D-11 and retention of only second order terms gives the following expression for the second variation in P which is accurate to second order terms:

$$\Delta P_2 \cong -\int_{t_o}^{t_f} \sum_{i=1}^{n} \sum_{j=1}^{p} \frac{\partial^2 H}{\partial u_j \partial x_i} \delta x_i \delta u_j \, dt -$$

$$\frac{1}{2} \int_{t_o}^{t_f} \sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial x_i \partial x_j} \delta x_i \delta x_j \, dt . \tag{D-14}$$

On the constrained subarc the varied path must satisfy

$$\delta Q = \sum_{j=1}^{n} \frac{\partial Q}{\partial x_j} \delta x_j + \sum_{j=1}^{p} \frac{\partial Q}{\partial u_j} \delta u_j = 0. \tag{D-15}$$

Since it is assumed that x_j, $j = 1,\ldots,n$ is continuous, Equation D-15 can be satisfied for a control variable pertubation of the form suggested by Kopp and Moyer (8) only if $p \geq 2$ and Q is a function of at least two control variables. Therefore it is assumed that Q is a function of at least u_1 and u_2, where u_1 appears linearly in H. Furthermore, for the remainder of this analysis, let $\delta u_j = 0$, $j = 3,\ldots, p$.

Equation D-15 can now be used to determine an expression for δu_2 as[1]

$$\delta u_2 = -\left(\frac{\partial Q}{\partial u_2}\right)^{-1}\left(\sum_{j=1}^{n} \frac{\partial Q}{\partial x_j}\, \delta x_j + \frac{\partial Q}{\partial u_1}\, \delta u_1\right). \tag{D-16}$$

Substitution of Equation D-16 into Equation D-14 then gives

$$\Delta P_2 = -\int_{t_o}^{t_f} \sum_{i=1}^{n} \left[\frac{\partial^2 H}{\partial u_1 \partial x_i} - \frac{\partial^2 H}{\partial u_2 \partial x_i} \left(\frac{\partial Q}{\partial u_2}\right)^{-1} \frac{\partial Q}{\partial u_1}\right]\delta x_i\, \delta u_1 dt - $$
$$\frac{1}{2} \int_{t_o}^{t_f} \sum_{i,j=1}^{n} \left[\frac{\partial^2 H}{\partial x_i \partial x_j} - 2\, \frac{\partial^2 H}{\partial u_2 \partial x_i}\left(\frac{\partial Q}{\partial u_2}\right)^{-1} \frac{\partial Q}{\partial x_j}\right]\delta x_i\, \delta x_j dt \tag{D-17}$$

where

$$\delta \dot{x}_i = \sum_{j=1}^{n} \left[\frac{\partial^2 H}{\partial \lambda_i \partial x_j} - \frac{\partial^2 H}{\partial \lambda_i \partial u_2}\left(\frac{\partial Q}{\partial u_2}\right)^{-1} \frac{\partial Q}{\partial x_j}\right]\delta x_j + $$
$$\left[\frac{\partial^2 H}{\partial \lambda_i \partial u_i} - \frac{\partial^2 H}{\partial \lambda_i \partial u_2} \left(\frac{\partial Q}{\partial u_2}\right)^{-1} \frac{\partial Q}{\partial u_1}\right] \delta u_1 \tag{D-18}$$

$$\delta x_i(t_o) = 0 \qquad i \in I_o \tag{D-19}$$

$$\delta x_i(t_f) = 0 \qquad i \in I_f\ . \tag{D-20}$$

---

[1]It is tacitly assumed that $\partial Q/\partial u_2$ is not singular.

A solution of Equation D-18 for the case of a control variation $\delta u_1 = \varphi_o^q(t,\tau)$ as defined by Kopp and Moyer (8) is given by

$$\delta x_i = \sum_{v=1}^{q+1} A_{i,v}\varphi_v^q + \xi_i^q \qquad \text{(D-21)}$$

where

$$A_{i,1} = \frac{\partial^2 H}{\partial\lambda_i \partial u_1} - \frac{\partial^2 H}{\partial\lambda_i \partial u_2}\left(\frac{\partial Q}{\partial u_2}\right)^{-1} \frac{\partial Q}{\partial u_1} \qquad \text{(D-22)}$$

$$A_{i,v} = \sum_{j=1}^{n} \left[\frac{\partial^2 H}{\partial\lambda_i \partial x_j} - \frac{\partial^2 H}{\partial\lambda_i \partial u_2}\left(\frac{\partial Q}{\partial u_2}\right)^{-1} \frac{\partial Q}{\partial x_j}\right]A_{j,v-1} - \dot{A}_{i,v-1}$$

$$v = 2,\ldots,q+1 \qquad \text{(D-23)}$$

$$\dot{\xi}_i^q = \sum_{j=1}^{n} \left[\frac{\partial^2 H}{\partial\lambda_i \partial x_j} - \frac{\partial^2 H}{\partial\lambda_i \partial u_2}\left(\frac{\partial Q}{\partial u_2}\right)^{-1} \frac{\partial Q}{\partial x_j}\right] \xi_j^q +$$

$$\left\{\sum_{j=1}^{n} \left[\frac{\partial^2 H}{\partial\lambda_i \partial x_j} - \frac{\partial^2 H}{\partial\lambda_i \partial u_2}\left(\frac{\partial Q}{\partial u_2}\right)^{-1} \frac{\partial Q}{\partial x_j}\right] A_{j,q+1} - A_{i,q+1}\right\}\varphi_{q+1}^q$$

$$\text{(D-24)}$$

$$\xi_i^q(t_o) = 0 \qquad \text{(D-25)}$$

Equations D-17 to D-25 are analogous to Equations 8-11 given by Kopp and Moyer (8). Thus the procedure from this point is identical to theirs. The result for a control variation $\delta u_1 = \varphi_o^1$ is

$$\frac{1}{2}\frac{d}{dt} \sum_{i=1}^{n} \left[\frac{\partial^2 H}{\partial u_1 \partial x_i} - \frac{\partial^2 H}{\partial u_2 \partial x_i}\left(\frac{\partial Q}{\partial u_2}\right)^{-1} \frac{\partial Q}{\partial u_1}\right]A_{i,1} +$$

$$\sum_{i=1}^{n} \left[\frac{\partial^2 H}{\partial u_1 \partial x_i} - \frac{\partial^2 H}{\partial u_2 \partial x_i}\left(\frac{\partial Q}{\partial u_2}\right)^{-1} \frac{\partial Q}{\partial u_1}\right]A_{i,2} - \frac{1}{2}\sum_{i,j=1}^{n} \left[\frac{\partial^2 H}{\partial x_i \partial x_j} -\right.$$

$$\left. 2\,\frac{\partial^2 H}{\partial u_2 \partial x_i}\left(\frac{\partial Q}{\partial u_2}\right)^{-1} \frac{\partial Q}{\partial x_j}\right] A_{i,1}\, A_{j,1} \geq 0 \qquad \text{(D-26)}$$

Application to the radial constraint problem

For the problem under consideration Equation D-1 represents the following five differential equations:

$$\begin{array}{llll} f_1 = \dot{\theta} = h/r^2 & (x_1 = \theta) & & \\ f_2 = \dot{r} = v & (x_2 = r) & & \\ f_3 = \dot{h} = (rc\beta/m)\cos\epsilon & (x_3 = h) & (u_1 = \beta) & \\ f_4 = \dot{v} = h^2/r^3 - 1/r^2 + (c\beta/m)\sin\epsilon & (x_4 = v) & (u_2 = \epsilon) & \text{(D-27)} \\ f_5 = \dot{m} = -\beta & (x_5 = m) & & \end{array}$$

The Hamiltonian and constraint functions are given by

$$H = \lambda_1 h/r^2 + \lambda_2 v + \beta\{(c/m)[\lambda_3 r\cos\epsilon + (\lambda_4 - \gamma)\sin\epsilon] - \lambda_5\} \qquad \text{(D-28)}$$

$$Q = \ddot{S} = h^2/r^3 - 1/r^2 + (c\beta/m)\sin\epsilon = 0 \qquad \text{(D-29)}$$

Straightforward application of Equations D-22 and D-23 gives the following results:

$$\begin{aligned} A_{1,1} &= A_{2,1} = A_{4,1} = A_{2,2} = A_{4,2} = A_{5,2} = 0 \\ A_{3,1} &= (rc/m)\sec\epsilon \\ A_{5,1} &= -1 \\ A_{1,2} &= (c/rm)\sec\epsilon \\ A_{3,2} &= (c/m)[(2h/r)\tan\epsilon - v - r\dot{\epsilon}\tan\epsilon]\sec\epsilon. \end{aligned} \qquad \text{(D-30)}$$

Similarly Equation D-26 becomes

$$\frac{1}{2}\frac{d}{d\tau}\{(c/m^2)[\lambda_3 r\cos\epsilon + (\lambda_4 - \gamma)\sin\epsilon]\} -$$

$$\lambda_3 \sec\epsilon[(c/m)^2\tan\epsilon/\cos\epsilon + rc\beta/m^3] \geq 0 . \qquad \text{(D-31)}$$

Completion of the differentiation indicated in the above equation gives

$$(c/m^2)[\dot{\lambda}_3 \cos\ \epsilon + (\dot{\lambda}_4 - \dot{\gamma})\sin\ \epsilon] - \lambda_3 (c/m)^2 \sec^2\epsilon\ \tan\ \epsilon \geq 0 \qquad \text{(D-32)}$$

Differentiation of Equation 20 provides the following expression which is valid along a constrained subarc $(r = 1,\ v = 0)$:

$$\dot{\lambda}_4 - \dot{\gamma} = \dot{\lambda}_3 \tan\ \epsilon + \lambda_3 \sec^2\epsilon\ \dot{\epsilon}\ . \qquad \text{(D-33)}$$

Equation 32 now becomes

$$(c/m^2)\sec\ \epsilon\ [\dot{\lambda}_3 + \lambda_3\dot{\epsilon}\ \tan\ \epsilon] - \lambda_3 (c/m)^2 \sec^2\epsilon \tan\ \epsilon \geq 0. \qquad \text{(D-34)}$$

Differentiation of Equation 71 gives the following result for a singular constrained subarc $(\dot{K}_\beta = 0)$:

$$\sec\ \epsilon[\dot{\lambda}_3 + \lambda_3\dot{\epsilon}\ \tan\ \epsilon] = 0. \qquad \text{(D-35)}$$

Thus Equation D-34 reduces to

$$\lambda_3 \tan\ \epsilon \leq 0. \qquad \text{(D-36)}$$

Substitution of Equation 20 into Equation D-36 gives

$$\lambda_4 - \gamma \leq 0. \qquad \text{(D-37)}$$

From Equations 38 and D-37 it is finally concluded that $\sin\ \epsilon \leq 0$ so that the thrust direction must point inward on any singular constrained subarc.