Order Number 9003499

Qualitative studies of a convective porous medium equation with a nonlinear forcing at the boundary

Anderson, Jeffrey Ruben, Ph.D.

Iowa State University, 1989

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

Qualitative studies of a convective porous medium equation with a nonlinear forcing at the boundary

by

Jeffrey Ruben Anderson

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Department: Mathematics
Major: Applied Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa
1989

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

Looking back from this point in my academic career, I am amazed at the number of good people who have given me guidance and encouragement along the way. These people have all played a part, directly or indirectly, in the production of this thesis, and I would like to acknowledge some of them here.

Firstly, I want to thank my parents and grandparents for their constant love and support. My thanks also to the faculty of Elbow Lake - Wendell schools for instilling in me a genuine love of learning. Special thanks to Mr. Rod Melin and Mr. Al Westberg for nuturing an early love of mathematics and for teaching me not only basic techniques but also patience in my work. At the collegiate level, I had the good fortune to come under the guidance of Dr. Bill Tomhave. His introductiory calculus courses furnished me with useful mathematics as well as an example of fine teaching which I have found invaluable in forming my own teaching style. For this and also his assistance in my gaining entrance into graduate school, I am grateful. I am also grateful to Dr. Gary Krenz, Dr. Richard Smith, and Dr. Joyati Debnath for being role models during my initial years of graduate work and for being good friends.

Dr. Noemí Wolanski and Prof. Gary Lieberman kindly offered help and encouragement through various stages of this work, and Prof. Paul Sacks graciously gave of his time to guide major parts of my education and also directed me to very useful references. I want to thank them for their time and assistance. The Mathematics office secretaries, Madonna, Jan, Joanne, and Ruth, have always cheerfully helped me, especially during the typing of this thesis, and I want to thank them as well. Finally, I should like to thank the two people who played the greatest roles: my thesis advisor, Prof. Howard Levine, and my wife, Dawn.

During my first year of graduate work, I was very lucky to take a course in Analysis from Prof. Levine. Through this course he not only taught me to think effectively, but also started me to think more seriously about future research topics. In January of 1986, Prof. Levine kindly agreed to take me on as a Ph.D. student. He has patiently supported and encouraged me through some tough times, taught me a great many things, and presented me with opportunities I wouldn't have otherwise had. He has been a good friend and an inspiring mathematician. I can't begin to thank him enough for all he has done.

My years of graduate work have not passed without some trial for Dawn, but she has always remained a supporting friend, a helpful collegue, and a loving wife. As my one-time officemate, she helped to solidify and give life to my understanding of topology. As a teacher, she has given me another example of fine teaching to draw upon. She has stuck with me through the hard times and celebrated with me through some good times. I only hope that I can be as supportive of her as she now begins her Ph.D. work. It is to her that this thesis is dedicated.

This thesis was prepared in part while the author was supported by the United States Air Force Office of Scientific Research and by a fellowship through the Boeing Corporation.

1 INTRODUCTION

1.1 General Introduction

Since the time of Newton and the invention of the (fluxional) calculus, a great many differential equations have appeared in mathematical models for various physical situations. These equations are formulated by defining interesting physical quantities (called *dynamical* or *state variables*), such as velocity, density, or pressure, and then applying empirically based laws to obtain differential equations in one or more of these dynamical variables. A simple but widely used example of such a law is Newton's second law,

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt},$$

which relates the time rate of change of the velocity (acceleration) of a particle to the resultant force imparted upon it. To give an example of the formulation of a differential equation, we follow a discussion in [12]. Consider the unsaturated flow of water through a porous medium, and define the state variables

$$\begin{aligned}\theta(x, y, z, t) &\equiv \text{ volumetric moisture content,}\\ \mathbf{q}(x, y, z, t) &\equiv \text{ seepage velocity, and}\\ \Phi(x, y, z, t) &\equiv \text{ piezometric (pressure) head,}\end{aligned}$$

where x, y, and z are the usual Cartesian coordinates with z denoting the vertically upward direction. Then the law of conservation of mass

$$\frac{\partial \theta}{\partial t} + \nabla \cdot \mathbf{q} = 0,$$

and Darcy's law

$$\mathbf{q} = -K(\theta)\left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}\right),$$

together with some additional physical assumptions and appropriate rescaling in t, result in

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2(\theta^m)}{\partial x^2} + \frac{\partial^2(\theta^m)}{\partial y^2} + \frac{\partial^2(\theta^m)}{\partial z^2} + C\frac{\partial(\theta^n)}{\partial z}.$$

Here C, n, and m are positive constants. For such flow in a porous column, we have

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2(\theta^m)}{\partial z^2} + C\frac{\partial(\theta^n)}{\partial z}.$$

In addition to the differential equations, one also builds known data into the model. For instance, in the example above, suppose it is known that the ends of the porous column (of height 1) are always dry and that initially, i.e., at $t = 0$, $\theta = z(1-z)$. Such data lead to the *boundary conditions* $\theta(0,t) = \theta(1,t) = 0$ and the *initial condition* $\theta(z,0) = z(1-z)$. Thus, the complete mathematical model is the initial - boundary value problem

$$\text{(IBVP)} \qquad \begin{aligned} &\frac{\partial \theta}{\partial t} = \frac{\partial^2(\theta^m)}{\partial z^2} + C\frac{\partial(\theta^n)}{\partial z} && \text{on } (0,1)\times(0,\infty) \\ &\theta(0,t) = \theta(1,t) = 0 && \text{on } (0,\infty) \\ &\theta(z,0) = z(1-z) && \text{on } [0,1]. \end{aligned}$$

With such a model in hand, one hopes to predict the values of the state variables by working directly with the model. If anything can be done in this direction, then it is usually a vast improvement over actually taking measurements from a laboratory setup (which may be costly or impossible). The ultimate goal of this process is to actually find a function (or functions) which satisfies the model problem, i.e., find a solution of the model problem. This is done with some degree of success. However, consideration of the equation

$$\frac{d^2\psi}{dt^2} + \sin\psi = 0,$$

which models a simple pendulum [30, page 213], shows that explicit solutions (when they can even be found) can not always be immediately translated into useful information.

This is the point where a research mathematician in partial differential equations (PDEs) will often pick up the battle. A physically motivated model, such as (IBVP), which involves PDEs, is considered as a purely mathematical problem with the hope of obtaining information about its solution. However, in viewing the model (or an abstraction of it) as a mathematical problem, such a person is usually first confronted with three questions:

1. Does there exist a solution?

2. Is the solution unique?

3. Does the solution depend continuously on the data? That is, do "small" changes in the data result in "small" changes in the solution?

If these three questions are all in the affirmative, then the model is said to be well-posed in the sense of Hadamard, see [15, page 155]. But before these issues can be taken up, the question of what is meant by a solution must be addressed. Classically, a solution is simply a function possessing all the necessary derivatives which satisfies the model. However, it is known that some PDEs simply do not have such *classical solutions*, and so a weakened notion of solution is sometimes required. So-called *weak solutions* must be carefully defined. If the definition is overly weakened, then uniqueness may fail, whereas existence may fail if the definition is not weakened enough.

Once these are settled, the following additional questions can be posed for a time dependent (evolutionary) model:

4. Are there any stationary solutions, i.e., solutions which are independent of t?

5. Are any of these stationary states, w, stable in the sense that a solution, u, which starts "near" w will stay "near" w for all time? Is w asymptotically stable in the sense that if u starts "near" w, then u "converges" to w as time increases?

6. Do any of the solutions "blow-up" in finite or infinite time?

Such qualitative studies compose a large part of modern research in dynamical systems.

Let us remark here that the first three of these six questions (especially the existence question) are usually regarded by people outside the field as physically expected and, therefore, necessary only to satisfy a mathematician's need for rigor. This is unfortunate since many existence proofs are somewhat constructive and so are of use when trying to establish more "physically interesting" results. Such is the case in this work.

1.2 Statement of Problems and Motivation

Before stating the problems, let us recall the usual subscript notation for partial derivatives. If f is a function of x and t, then

$$f_x \equiv \frac{\partial f}{\partial x}, f_{xx} \equiv \frac{\partial^2 f}{\partial x^2}, f_t \equiv \frac{\partial f}{\partial t},$$

and so on. This notation can also be used similarly for functions of a higher number of independent variables and shall be used throughout the remainder of this work.

In this thesis, we will be concerned with the following model problems

$$\text{(A)} \qquad \begin{aligned} & u_t = (u^m)_{xx} + \frac{\epsilon}{n}(u^n)_x && \text{on } Q_T \\ & u(0,t) = 0 && \text{on } (0,T) \\ & (u^m)_x(1,t) = au^p(1,t) && \text{on } (0,T) \\ & u(x,0) = u_0(x) && \text{on } [0,1], \end{aligned}$$

and

$$\text{(B)} \qquad \begin{aligned} & u_t = (u^m)_{xx} + \frac{\epsilon}{n}(u^n)_x && \text{on } Q_T \\ & -(u^m)_x(0,t) = au^p(0,t) && \text{on } (0,T) \\ & u(1,t) = 0 && \text{on } (0,T) \\ & u(x,0) = u_0(x) && \text{on } [0,1], \end{aligned}$$

where $a, \epsilon, m, n, p > 0$, and $Q_T \equiv (0,1) \times (0,T)$. Our main objective is to answer the questions discussed above for nonnegative "solutions" of (A) and (B). The motivation for such a study comes from both physical and purely mathematical fronts.

From the discussion in the previous section, a physical situation giving rise to (A) or (B) can be understood once it is known how a boundary condition of the type

$$(u^m)_x(1,t) = au^p(1,t) \;\text{ or }\; -(u^m)_x(0,t) = au^p(0,t)$$

might come about. To do this, one must go all the way back to the law of conservation of mass to discover that

$$(u^m)_x + \frac{\epsilon}{n}(u^n)$$

is an expression of the seepage velocity (sometimes referred to as simply the flux). Roughly speaking, this quantity can be described as the volume of water flowing across a unit area of the porous medium per unit time. Hence, such boundary conditions are a forcing of the flux which depends nonlinearly on the volumetric moisture content. Note that in this light, specifying $(u^m)_x + \frac{\epsilon}{n}(u^n)$ on the boundary is a more natural condition. We will discuss our reasoning behind expressing the boundary conditions as in (A) or (B) later in this section.

When $\epsilon = 0$, the PDE in (A) and (B) is called the porous medium equation; see [26] for a review of recent results concerning this equation. In fluid mechanics, first order derivative terms such as $(u^n)_x$ usually come about as a result of operating in the Eulerian system. In such a coordinate system, one fixes attention at a point in space, and hence properties of the fluid, such as temperature, are carried to the observer not only by conduction but also by the flow of the fluid. The latter method of transporting properties of the fluid is called *convection.* Thus we have termed the PDE in (A) and (B) the *convective porous medium equation.*

This PDE, in various settings, has been the object of investigation by others. In a series of papers culminating in [10], Gilding developed an existence and comparison theory for various models involving the more general equation

$$u_t = a(u)_{xx} + b(u)_x,$$

which he mentioned is often called the nonlinear Fokker-Planck equation. The problems considered therein were

1. The Cauchy problem

$$\begin{aligned} &u_t = a(u)_{xx} + b(u)_x && \text{on } R \times (0,T] \\ &u(x,0) = u_0(x) \geq 0 && \text{on } R, \end{aligned}$$

2. The Cauchy-Dirichlet problem

$$\begin{aligned} &u_t = a(u)_{xx} + b(u)_x && \text{on } (0,\infty) \times (0,T] \\ &u(x,0) = u_0(x) \geq 0 && \text{on } [0,\infty) \\ &u(0,t) = \psi(t) \geq 0 && \text{on } (0,T], \end{aligned}$$

3. The first boundary value problem

$$\begin{aligned} &u_t = a(u)_{xx} + b(u)_x && \text{on } (-1,1) \times (0,T] \\ &u(-1,t) = \psi^-(t) \geq 0 && \text{on } (0,T] \\ &u(1,t) = \psi^+(t) \geq 0 && \text{on } (0,T] \\ &u(x,0) = u_0(x) \geq 0 && \text{on } [-1,1]. \end{aligned}$$

Originally, he established the existence and uniqueness results for these problems under assumptions which included

$$\frac{(b')^2}{a'} \in L^\infty(0,\epsilon)$$

for any $\epsilon > 0$ [11]. This assumption was later shown to be unnecessary by Diaz and Kersner [5]. In [31,32], Wolanski developed the existence and comparison theory for such equations with boundary conditions of the form $a(u)_x(1,t) + b(u)(1,t) = f(t)$. A similar theory for the case of nonlinear boundary conditions of the form $a(u)_x(1,t) + b(u)(1,t) = c(u)(1,t)$ was established by Xu [33]. This work actually contains problem (A) in the cases $a, \epsilon \leq 0$ and problem (B) for the cases $a, -\epsilon \leq 0$. However, the methods used do not carry over to these problems when $a, \epsilon > 0$. We shall discuss this in more detail in Chapter 2. Finally, we mention a work of DiBenedetto [6] where he has proved the continuity of solutions to more general PDEs under any of the above boundary condition under appropriate assumptions.

Concerning the long time behavior of solutions, there have been many studies conderned with the existence and behavior of the *interface curves* of solutions of the Cauchy problem for

$$u_t = (u^m)_{xx} + (u^n)_x.$$

See, for instance [12,16,13]. These are curves

$$\xi^-(t) \equiv \inf\{x \in R : u(x,t) > 0\}$$

and

$$\xi^+(t) \equiv \sup\{x \in R : u(x,t) > 0\},$$

and simply their existence for $m > 1$ is in sharp contrast to the situation when $0 < m \leq 1$. Theorems dealing with blow up of solutions of (A) or (B) in finite time can be developed using the same concavity method as was used in [20,21,22,19,23]. We shall return to this in Chapter 4.

The mathematical motivation behind these problems comes from a previous work of Levine [19]. He considered (A) and (B) for $m = 1$ and $n = 2$. Our aim is to discover how his results persist or are altered as n and m are varied. It is for purposes of comparison that we have written the boundary conditions as above. Studies of this type can be interesting from a purely mathematical point of view as dramatic changes in the character of solutions have often been observed with different values of $m > 0$.

A case in point is the porous medium equation

$$u_t = (u^m)_{xx}.$$

When $m = 1$, this equation is known as the heat equation and, due to maximum principles [27], exhibits infinite speed of propagation. This is also true for $0 < m < 1$. However, in the case $m > 1$, one has finite speed of propagation. (See [26].)

1.3 Outline of the Thesis

In Chapter 2, we make some preliminary definitions, and then introduce two more general problems which include (A) and (B), respectively, as special cases. We define a weakened notion of solutions for these problems, and state the local existence and continuation results. We also state some technical results on increased regularity for later use. The proofs of all the theorems in Chapter 2 appear in Chapter 5.

In Chapter 3, we completely parallel and generalize Levine's results on stationary solutions. This chapter concludes with the complete solution diagrams for the stationary solutions of (A) and (B). For the cases of $0 < m < n$, we observe that these diagrams are identical to those obtained by Levine. However $m = n$ is a special border case, and the case of $m > n$ gives diagrams which are roughly inversions of the first case. Another interesting aspect of these curves occurs for problem (B) in the case $0 < m < n$, where they are not completely contained in a plane.

The long time behavior is taken up in Chapter 4, and it is here that we must restrict to the case of $n, p \geq m$ and $m \geq 1$ because of the lack of a better uniqueness result. We hope to take up the issue of an improved uniqueness theorem in a later work. Comparison and continuous dependence results are presented here, and then the stability of stationary states is discussed. Finally, we show that some solutions of (A) do blow up in finite time. Results which heavily suggest that some solutions of (B) also blow up in finite time are also presented.

The thesis concludes with two appendices where we gather some modifications of known facts which are used in the development of existence and uniqueness.

2 LOCAL EXISTENCE AND CONTINUATION

2.1 Preliminary Definitions

For the purposes of defining weak solutions, we now make the following definitions. Our notation will adhere to standard usages as much as possible.

To begin, R shall always denote the real numbers,

$$R \equiv (-\infty, \infty).$$

Given an open, connected set $\Omega \subset R$ or $\Omega \subset R \times R$, we define the following function spaces:

$C(\Omega) \equiv$ continuous functions on Ω,

$C^1(\Omega) \equiv$ continuously differentiable functions on Ω,

$C^2(\Omega) \equiv$ twice continuously differentiable functions on Ω.

Furthermore, define the norms

$$\| u \|^p_{L^p(\Omega)} \equiv \int_\Omega | u(x) |^p \, dx,$$

for $1 \leq p < \infty$, and

$$\| u \|_{L^\infty(\Omega)} \equiv \text{ess sup} \mid u \mid .$$

Then, for $1 \leq p \leq \infty$,

$$L^p(\Omega) \equiv \{\text{Lebesgue measurable functions } f \text{ on } \Omega : \| f \|_{L^p(\Omega)} < \infty\}$$

are the usual Lebesgue spaces. The notation "a.e." will denote "almost everywhere". Finally, for $\Omega \subset R$,

$$\| u \|^2_{H^1(\Omega)} \equiv \int_\Omega \left\{ | u(x) |^2 + | u'(x) |^2 \right\} dx,$$

and

$$H^1(\Omega) \equiv \text{completion of } C^1(\Omega) \text{ in the norm } \| u \|_{H^1(\Omega)} .$$

2.2 Definition of Weak Solutions, Local Existence, Continuation, and Increased Regularity

Our considerations in this chapter will center around the more general problems

$$\text{(A1)} \qquad \begin{array}{ll} u_t = \phi(u)_{xx} + f(u)_x & \text{on } Q_T \\ u(0,t) = 0 & \text{on } (0,T) \\ \phi(u)_x(1,t) = g(u)(1,t) & \text{on } (0,T) \\ u(x,0) = u_0(x) & \text{on } [0,1] , \end{array}$$

and

$$\text{(B1)} \qquad \begin{array}{ll} u_t = \phi(u)_{xx} + f(u)_x & \text{on } Q_T \\ -\phi(u)_x(0,t) = g(u)(0,t) & \text{on } (0,T) \\ u(1,t) = 0 & \text{on } (0,T) \\ u(x,0) = u_0(x) & \text{on } [0,1] . \end{array}$$

Throughout this thesis we shall assume the following of $\phi, f, g,$ and u_0 :

(H1) $\phi \in C(R) \cap C^2(R \setminus \{0\})$ such that $\phi(0) = 0$ and $\phi'(u) > 0$ for all $u \neq 0$.

(H2) $f, g \in C(R) \cap C^1(R \setminus \{0\})$ such that $f(0) = g(0) = 0$.

(H3) $u_0 \in L^\infty(0,1)$ such that $u_0 \geq 0$ a.e. on $(0,1)$.

Note that these hypotheses allow problems (A1) and (B1) to include (A) and (B), respectively, as special cases. To see this one need only make the definitions $\phi(u) \equiv u \mid u \mid^{m-1}, f(u) \equiv \epsilon u \mid u \mid^{n-1}$, and $g(u) \equiv au \mid u \mid^{p-1}$. (Since we are concerned with only the case of nonnegative solutions in this work, it turns out not to be important how ϕ, f , and g are defined on $(-\infty, 0)$. However we will eventually need to consider mollifiers of these functions at which time it will be necessary that they are defined on all of R.)

The aim of the present chapter is to define a weakened notion of solution (*weak solutions*) for problems (A1) and (B1) and then state the associated local existence and continuation theorems. Finally we will state some technical results which will be needed later in our development of blow-up theorems. Although these results are new and of interest in their own right, their proofs are delegated to Chapter 5. In this way the main flow of the thesis is better preserved.

We now proceed to the definitions of subsolution, supersolution, and solution of (A1) and (B1). These definitions are just the usual weakened notions which are motivated by multiplying the equation or inequality by an appropriate "test function" and then formally integrating by parts.

Definition 2.1 **(A)** *A function $u(x,t)$ is called a subsolution (supersolution) of (A1) on Q_T if:*

(i) *$u(x,t)$ is defined everywhere on $\overline{Q_T} \setminus ([0,1] \times \{0\})$ and a.e. on $[0,1] \times \{0\}$ such that $u \in L^\infty(Q_T)$ and $\phi(u)_x \in L^2(Q_T)$.*

(ii) *$u(0,t) \leq (\geq) 0$ on $(0,T)$.*

(iii) *For every $t \in [0,T]$ and every*

$$\xi \in P_{\{0\}}(\overline{Q_T}) \equiv \{\xi \in C^1(\overline{Q_T}) \mid \xi(0,t) = 0, \xi \geq 0\},$$

$$\begin{aligned}\int_0^1 u(x,t)\xi(x,t)dx \ \leq (\geq) \ & \int_0^1 u_0(x)\xi(x,0)dx \\ & + \int_0^t \int_0^1 \{u\xi_s - (\phi(u)_x + f(u))\xi_x\}\, dxds \\ & + \int_0^t [g(u(1,s)) + f(u(1,s))]\, \xi(1,s)ds.\end{aligned}$$

A function $u(x,t)$ is called a <u>*solution of (A1) on Q_T*</u> *if it is both a subsolution and a supersolution of (A1) on Q_T .*

(B) *A function $u(x,t)$ is called a subsolution (supersolution) of (B1) on Q_T if:*

(i) *$u(x,t)$ is defined everywhere on $\overline{Q_T} \setminus ([0,1] \times \{0\})$ and a.e. on $[0,1] \times \{0\}$ such that $u \in L^\infty(Q_T)$ and $\phi(u)_x \in L^2(Q_T)$.*

(ii) *$u(1,t) \leq (\geq) 0$ on $(0,T)$.*

(iii) *For every $t \in [0,T]$ and every*

$$\xi \in P_{\{1\}}(\overline{Q_T}) \equiv \{\xi \in C^1(\overline{Q_T}) \mid \xi(1,t) = 0, \xi \geq 0\},$$

$$\begin{aligned}\int_0^1 u(x,t)\xi(x,t)dx \ \leq (\geq) \ & \int_0^1 u_0(x)\xi(x,0)dx \\ & + \int_0^t \int_0^1 \{u\xi_s - (\phi(u)_x + f(u))\xi_x\}\, dxds \\ & + \int_0^t [g(u(0,s)) - f(u(0,s))]\, \xi(0,s)ds.\end{aligned}$$

A function $u(x,t)$ is called a <u>*solution of (B1) on Q_T*</u> *if it is both a subsolution and a supersolution of (B1) on Q_T .*

The main existence and continuation theorem can now be stated as follows.

Theorem 2.1 A) *There exists a $T > 0$ ($T \leq \infty$) such that there exists a solution, $u(x,t) \geq 0$, of (A1) on Q_T and*

$$\limsup_{t \to T^-} \left[t + \| u(\cdot,t) \|_{L^\infty(0,1)} \right] = +\infty.$$

B) *There exists a $T > 0$ ($T \leq \infty$) such that there exists a solution, $u(x,t) \geq 0$, of (B1) on Q_T and*

$$\limsup_{t \to T^-} \left[t + \| u(\cdot,t) \|_{L^\infty(0,1)} \right] = +\infty.$$

For certain types of nonlinearities, e.g., the case of $f', g' \leq 0$ for problem (A1) and the case of $f' \geq 0, g' \leq 0$ for problem (B1), Theorem 2.1 can be established via the approach of semigroup theory as in [33]. See [4] for an excellent review of this topic, and see [3] for a development relating to the porous medium equation. In [31,32], Wolanski has applied this technique also. However, this approach fails for other nonlinearities. To give a brief argument as to why this is the case, we first simply state here that in a development such as [31,32,33] one has the following: If u and v are solutions of (A1), then

$$\| u(\cdot,t) - v(\cdot,t) \|_{L^1(0,1)} \leq \| u(\cdot,0) - v(\cdot,0) \|_{L^1(0,1)}$$

for all $t \geq 0$. Now suppose that $t > 0$ is such that $u \geq v$ on Q_t. Operating formally, we integrate the PDE to find

$$\begin{aligned} \int_0^1 [u(x,t) - v(x,t)]dx \;\; &\leq \;\; \int_0^1 [u(x,0) - v(x,0)]dx \\ &\quad + \int_0^t [(f+g)(u(1,s)) - (f+g)(v(1,s))]ds, \end{aligned}$$

and hence such an L^1 estimate is possible provided $f + g$ is nonincreasing. In this way, we expect that the nonlinearities for which $f' + g' \geq 0$ present a more difficult situation and will have solutions which exhibit more interesting behavior, such as blow up in finite time.

The increased regularity theorems referred to above are results such as: "If u is a solution of (A1), then $\phi(u)_t \in L^2(Q_T)$." (Here $\phi(u)_t$ is understood to be the weak or distributional derivative of $\phi(u)$.) That is, under appropriate hypotheses, certain functions of a solution of (A1) or (B1) actually have more well behaved (in an integral sense) weak derivatives than called for in the definition. Such results (and their proofs) are useful in obtaining further a priori estimates, including the so-called energy estimate. We now state these technical results which shall be employed in Chapter 4 in the development of some blow-up theorems. At this point we have made no mention of any uniqueness of solutions to problem (A1) or (B1). Therefore, the quantifier "there exists a solution" shall appear in the following theorem. (In fact, it is the solution constructed in Chapter 5 which is used to establish it.) Once such uniqueness is proved, the quantifier may be replaced with "the solution."

Define

$$\Phi(u) \equiv \int_0^u \phi(v)dv \text{ and } \Psi(u) \equiv \int_0^u \sqrt{\phi'(v)}dv.$$

Theorem 2.2 *Assume that $f \in C^1(R)$ and that $\phi/\sqrt{\phi'}$ is bounded on compact subsets of $(0, \infty)$.*

A) *For each given u_0 such that $\phi(u_0) \in H^1(0,1)$, there exists a solution, u, of (A1) on Q_T with the following properties:*

(i) $(\Phi(u))_t, (\Psi(u))_t \in L^2(Q_T)$, *and* $(\Phi(u))_t = \dfrac{\phi(u)}{\sqrt{\phi'(u)}}(\Psi(u))_t$.

(ii)

$$\begin{aligned}\int_0^t\int_0^1(\Phi(u))_s dxds &= \int_0^1 \Phi(u(x,s))\mid_{s=0}^{s=t} dx\\ &= -\int_0^t\int_0^1 (\phi(u)_x+f(u))\,\phi(u)_x dxds\\ &\quad+\int_0^t [g(u(1,s))+f(u(1,s))]\,\phi(u(1,s))ds.\end{aligned}$$

(iii) *("Energy inequality") If $f\equiv 0$ then*

$$\int_0^t\int_0^1(\Psi(u))_s^2 dxds \le -\frac{1}{2}\int_0^1 \phi(u)_x^2(x,s)\mid_{s=0}^{s=t} dx+\int_{u_0(1)}^{u(1,t)}\phi'(s)g(s)ds.$$

B) *For each given u_0 such that $\phi(u_0)\in H^1(0,1)$, there exists a solution, u, of (B1) on Q_T with the following properties:*

(i) $(\Phi(u))_t, (\Psi(u))_t \in L^2(Q_T)$, *and* $(\Phi(u))_t = \dfrac{\phi(u)}{\sqrt{\phi'(u)}}(\Psi(u))_t$.

(ii)

$$\begin{aligned}\int_0^t\int_0^1(\Phi(u))_s dxds &= \int_0^1 \Phi(u(x,s))\mid_{s=0}^{s=t} dx\\ &= -\int_0^t\int_0^1 (\phi(u)_x+f(u))\,\phi(u)_x dxds\\ &\quad+\int_0^t [g(u(0,s))-f(u(0,s))]\,\phi(u(0,s))ds.\end{aligned}$$

(iii) *("Energy inequality") If $f\equiv 0$ then*

$$\int_0^t\int_0^1(\Psi(u))_s^2 dxds \le -\frac{1}{2}\int_0^1 \phi(u)_x^2(x,s)\mid_{s=0}^{s=t} dx+\int_{u_0(0)}^{u(0,t)}\phi'(s)g(s)ds.$$

3 STATIONARY SOLUTIONS

The goal of this chapter is to obtain solution diagrams for the stationary states of (A) and (B). This is done, as in [19], by classifying all such solutions via an integral identity and then analyzing the identity for each of the problems (A) and (B).

3.1 Classification via Integral Identity

In trying to follow the developments in [19], the first roadblock one encounters is the issue of the regularity, i.e., differentiability, of the stationary solutions. Levine's arguments depended heavily on the observation that such solutions were actually classical and so could be differentiated. However, in the present situation, we have only the weak solutions afforded by Definition 2.1. This causes no significant difficulties as we now show.

If w is a stationary solution of (A1) or (B1), then $\phi(w)_x \in L^2(Q_T)$. So $\phi(w)_x \in L^2(0,1)$, and hence, via a Sobolev embedding [18, page 61], it follows that $\phi(w) \in C([0,1])$. By the continuity of ϕ^{-1}, we thus have $w \in C([0,1])$. Such continuity allows us to prove that w satisfies

$$(\phi(w)_x + f(w))_x = 0$$

on $[0,1]$ in the classical sense, and in addition, most such solutions satisfy

$$w \in C^2(0,1).$$

This is the content of the following three lemmas.

Lemma 3.1 **(A)** *If $w(x)$ is a stationary solution of (A1), then*

$$\phi(w(x)) + \int_0^x f(w(y))dy = x[g(w(1)) + f(w(1))]$$

on $[0,1]$.

(B) *If $w(x)$ is a stationary solution of (B1), then*

$$-\phi(w(x)) + \int_x^1 f(w(y))dy = (1-x)[-g(w(0)) + f(w(0))]$$

on $[0,1]$.

Proof:

First let w be a stationary solution of (A1). Then for every $\xi \in P_{\{0\}}(\overline{Q_T})$ and every $t \in [0,T]$,

$$\begin{aligned}\int_0^1 w(x)\xi(x,t)dx &= \int_0^1 w(x)\xi(x,0)dx + \int_0^t [g(w)+f(w)](1)\xi(1,s)ds \\ &\quad + \int_0^t \int_0^1 \{w\xi_s - [\phi(w)_x + f(w)]\xi_x\}\, dxds.\end{aligned}$$

So for $\eta \equiv \eta(x)$ such that $\eta \in C^2([0,1]), \eta(0) = 0$, and $\eta \geq 0$ on $[0,1]$,

$$\int_0^1 [\phi(w)_x + f(w)]\eta_x dx = [g(w) + f(w)]\eta(1).$$

Let

$$H(x) \equiv \int_0^x f(w(y))dy - x[g(w) + f(w)](1),$$

then this last identity can be written as

$$\int_0^1 [\phi(w)_x + H']\eta_x dx = 0,$$

which upon integration by parts becomes

$$-\int_0^1 [\phi(w) + H]\eta_{xx} dx + [\phi(w) + H]\eta_x \,|_{x=0}^{x=1} = 0.$$

Given $\chi \in C([0,1])$ such that $\chi \geq 0$, define

$$\eta(x) \equiv \int_0^x \int_y^1 \chi(z) dz.$$

Then $\eta(0) = 0, \eta_x(1) = 0$, and $\eta_{xx}(x) = -\chi(x)$, so for this test function we have

$$\int_0^1 [\phi(w) + H]\chi(x) dx = 0.$$

Since χ was picked arbitrarily, it now follows that

$$\phi(w(x)) + \int_0^x f(w(y))dy = x[g(w(1)) + f(w(1))]$$

on $[0,1]$. (The above argument is actually quite standard in the Calculus of Variations.)

The statement for problem (B1) follows in a similar manner. $\square$

From Lemma 3.1 it follows that $(\phi(w)_x + f(w))_x = 0$ on $[0,1]$ for any stationary solution, $w(x)$, of (A1) or (B1) and that w is twice continuously differentiable on intervals where $w \neq 0$. Thus to prove $w \in C^2(0,1)$, it only needs to be shown that $w \neq 0$ on $(0,1)$. The following lemmas accomplish this task for most solutions. To state these we first define

$$x_0(v) \equiv \sup\{x \in [0,1] : v(x) = 0\}$$

and

$$y_0(v) \equiv \inf\{x \in [0,1] : v(x) = 0\},$$

for a given function $v : [0,1] \to R$.

Lemma 3.2 *Stationary solutions of (A1) or (B1) cannot change sign.*

Lemma 3.3 **(A)** *If $w(x)$ is a nontrivial, nonnegative stationary solution of (A1), then $w = 0$ on $[0, x_0(w)]$ and $w, w' > 0$ on $(x_0(w), 1]$.*

(B) *If $w(x)$ is a nontrivial, nonnegative stationary solution of (B1), then $w = 0$ on $[y_0(w), 1]$ and $w, -w' > 0$ on $[0, y_0(w))$.*

Once these lemmas are established it will follow from Lemma 3.1(A) and Lemma 3.3(A) that if $w(x)$ is a nontrivial, nonnegative stationary solution of (A1), then $x_0(w)[g(w(1)) + f(w(1))] = 0$. Furthermore, from Lemma 3.1(A) we have

$$\phi'(w)w_x = -f(w) + [g(w(1)) + f(w(1))] > 0$$

on $(x_0(w), 1]$. So

$$\begin{aligned} 0 &\leq \lim_{x \to x_0(w)^+} \{-f(w(x)) + [g(w(1)) + f(w(1))]\} \\ &= [g(w(1)) + f(w(1))]. \end{aligned}$$

Hence, if $g(w(1)) + f(w(1)) > 0$, then $x_0(w) = 0$ and $w > 0$ on $(0, 1]$. Similarly, if $w(x)$ is a nontrivial, nonnegative stationary solution of (B1) such that $g(w(0)) - f(w(0)) > 0$, then $y_0(w) = 1$ and $w > 0$ on $[0, 1)$. We now present the proofs of these lemmas which use the maximum principles in much the same manner as in Lemma 2.1 of [19].

Proof:(of Lemma 3.2)

Let $w(x)$ be a nontrivial stationary solution of (A1). It will be established that $w = 0$ on $[0, x_0(w)]$ which will complete the proof. To this end, suppose that

$$M \equiv \max_{0 \le x \le x_0(w)} w(x) > 0.$$

We may then find points x_1, y_1, z_1 such that $0 < y_1 < x_1 < z_1 < x_0(w), w(x_1) = M, w(y_1) = w(z_1) = \frac{M}{2}$, and $w(x) \ge \frac{M}{2}$ on $[y_1, z_1]$. By Lemma 3.1(A), w is a classical solution of

$$\phi'(w)w_{xx} + \phi''(w)w_x^2 + f'(w)w_x = 0$$

on $[y_1, z_1]$, which cannot have an interior extremum by the maximum principle. Thus we have a contradiction, and it follows that $M = 0$. Similarly it can be shown that

$$\min_{0 \le x \le x_0(w)} w(x) = 0.$$

Part (B) is proved as above. □

Proof:(of Lemma 3.3)

Let $w(x)$ be a nontrivial, nonnegative stationary solution of (A1); so $w > 0$ on $(x_0(w), 1]$. For $x_1 \in (x_0(w), 1]$ and $y_1 \in (x_0(w), x_1)$ such that $w(y_1) < w(x_1)$, we have

$$\max_{y_1 \le x \le x_1} w(x) = w(x_1)$$

by the maximum principle. Hence $w'(x_1) > 0$ by the Hopf boundary point lemma.

Part (B) follows in a similar manner. □

We now proceed to the characterizations of the nontrivial, nonnegative stationary states. This is done as in [19], except here we first state two lemmas which actually characterize positive solutions (in the classical sense) to the problems

$$(\mathrm{SA}x_0)\qquad \begin{aligned} &(\phi(w)_x + f(w))_x = 0 \quad \text{on } (x_0, 1)\\ &w(x_0) = 0\\ &\phi(w)_x(1) = g(w)(1), \end{aligned}$$

and

$$(\mathrm{SB}y_0)\qquad \begin{aligned} &(\phi(w)_x + f(w))_x = 0 \quad \text{on } (0, y_0)\\ &-\phi(w)_x(0) = g(w)(0),\\ &w(y_0) = 0, \end{aligned}$$

respectively, for $x_0 \in [0,1)$ and $y_0 \in (0,1]$. (Observe that Lemmas 3.2 and 3.3 hold for these problems.) To do this, as in [19], we must restrict to the cases where f is nondecreasing on $[0,\infty)$ for $(\mathrm{SA}x_0)$ and where $f \geq 0$ on $[0,\infty)$ for $(\mathrm{SB}y_0)$.

Lemma 3.4 (A) *Assume that f is nondecreasing on $[0,\infty)$. There exists a positive solution,*

$$w \in C([x_0,1]) \cap C^2((x_0,1]),$$

of (SAx_0) with $w_1 \equiv w(1) > 0$ iff

$$\int_0^{w_1} \frac{\phi'(\sigma)}{g(w_1) + f(w_1) - f(\sigma)} d\sigma = 1 - x_0, \tag{3.1}$$

and $g(w_1) > 0$.

Proof:

I. Suppose that $w(x)$ is a positive solution of $(\mathrm{SA}x_0)$ with $w \in C([x_0,1]) \cap C^2((x_0,1])$.

By Lemma 3.3(A), $\phi'(w)w_x > 0$ on $(x_0, 1]$, and hence

$$\phi'(w(x))w_x(x) = g(w_1) + f(w_1) - f(w(x)) > 0$$

on $(x_0, 1]$. So $g(w_1) > 0$, and letting $\hat{x} \in (x_0, 1]$, we have

$$\begin{aligned}\int_{\hat{x}}^{1} \frac{\phi'(w(x))w_x(x)}{g(w_1) + f(w_1) - f(w(x))} dx &= \int_{w(\hat{x})}^{w_1} \frac{\phi'(\sigma)}{g(w_1) + f(w_1) - f(\sigma)} d\sigma \\ &= 1 - \hat{x}.\end{aligned}$$

The integral identity (3.1) now follows by letting $\hat{x} \to x_0^+$ in the above equation.

II. Conversely, suppose $w_1 > 0$ satisfies equation (3.1) and $g(w_1) > 0$. Define

$$F(w) \equiv \int_{w}^{w_1} \frac{\phi'(\sigma)}{g(w_1) + f(w_1) - f(\sigma)} d\sigma,$$

for $w \in [0, w_1]$. By the monotonicity assumption on f, we see that

$$g(w_1) + f(w_1) - f(\sigma) \geq g(w_1) > 0$$

for all $\sigma \in [0, w_1]$, and so $F \in C([0, w_1])$. Moreover,

$$F'(w) = \frac{-\phi'(w)}{g(w_1) + f(w_1) - f(w)} < 0,$$

for $w \in (0, w_1]$, which implies that

$$F : [0, w_1] \to [0, 1 - x_0]$$

is 1-1 and onto. Hence we may define

$$w : [x_0, 1] \to [0, w_1]$$

by

$$w(x) \equiv F^{-1}(1 - x).$$

Clearly $w(x_0) = 0, w(1) = w_1$, and

$$\int_{w(x)}^{w_1} \frac{\phi'(\sigma)}{g(w_1) + f(w_1) - f(\sigma)} d\sigma = 1 - x$$

on $[x_0, 1]$. Furthermore, the differentiability of F implies that $w \in C^1((x_0, 1])$, and so differentiating the identity above gives

$$\phi'(w(x))w'(x) = g(w_1) + f(w_1) - f(w(x)).$$

From this it follows that $w \in C^2((x_0, 1])$ and is a solution of (SA_{x_0}). □

Lemma 3.5 (B) *Assume that $f \geq 0$ on $[0, \infty)$. There exists a positive solution,*

$$w \in C([0, y_0] \cap C^2([0, y_0)),$$

of (SB_{y_0}) with $w_0 \equiv w(0)$ iff

(i)

$$\int_0^{w_0} \frac{\phi'(\sigma)}{g(w_0) - f(w_0) + f(\sigma)} d\sigma = y_0, \tag{3.2}$$

and

(ii) *either $f > 0$ on $(0, w_0]$ and $g(w_0) - f(w_0) \geq 0$, or $g(w_0) - f(w_0) > 0$.*

Proof:

This result is proved much like the parallel result above. However, we will include a sketch of the proof indicating where the condition (ii) is involved.

I. Let $w \in C([0, y_0]) \cap C^2([0, y_0))$ be a solution of (SB_{y_0}). Then a quadrature yields

$$\phi'(w(x))w'(x) = -g(w_0) + f(w_0) - f(w(x))$$

on $[0, y_0)$, and this quantity is negative by Lemma 3.3(B). Hence,

$$\begin{aligned} 0 &\geq \lim_{x \to y_0^-} [-g(w_0) + f(w_0) - f(w(x))] \\ &= -g(w_0) + f(w_0). \end{aligned}$$

Furthermore, if $f(\hat{w}) = 0$ for some $\hat{w} \in (0, w_0]$, then

$$-g(w_0) + f(w_0) = \phi'(w(\hat{x}))w'(\hat{x}) + f(w(\hat{x})) < 0,$$

where $\hat{x} \in [0, y_0)$ is such that $w(\hat{x}) = \hat{w}$. Statements (i) and (ii) now follow.

II. Conversely, if (i) and (ii) hold, then the denominator, $g(w_0) - f(w_0) + f(\sigma)$, is positive for $\sigma \in [0, w_0]$. Thus, the proof concludes by defining

$$G(w) \equiv \int_w^{w_0} \frac{\phi'(\sigma)}{g(w_0) - f(w_0) + f(\sigma)} d\sigma$$

and proceeding as in the proof of Lemma 3.4(A). □

By applying these lemmas to the case of nontrivial, nonnegative stationary solutions of (A1) and of (B1), we have the following results.

Theorem 3.1 (A) *Assume that f is nondecreasing on $[0, \infty)$. There exists a nontrivial, nonnegative stationary solution, $w(x)$, of (A1) with $w_1 \equiv w(1) > 0$ iff*

$$\int_0^{w_1} \frac{\phi'(\sigma)}{g(w_1) + f(w_1) - f(\sigma)} d\sigma = 1, \tag{3.3}$$

and $g(w_1) > 0$.

Theorem 3.2 (B) *Assume that $f \geq 0$ on $[0, \infty)$. There exists a nontrivial, nonnegative stationary solution, $w(x)$, of (B1) of the form*

$$w(x) \equiv \begin{cases} 0 & \text{on } [y_0, 1] \\ v(x) & \text{on } [0, y_0), \end{cases} \tag{3.4}$$

where v is a solution of (SB_{y_0}), with $w_0 \equiv w(0)$ iff

(i)

$$\int_0^{w_0} \frac{\phi'(\sigma)}{g(w_0) - f(w_0) + f(\sigma)} d\sigma = y_0, \tag{3.5}$$

and

(ii) *either* $f > 0$ *on* $(0, w_0]$ *and* $g(w_0) - f(w_0) \geq 0$, *or* $g(w_0) - f(w_0) > 0$.

Whenever $g(w_0) > f(w_0)$, *it must be the case that* $y_0 = 1$.

Theorems 3.1 and 3.2 are actually the classification results we have been striving for, but it is interesting to pause here for a moment and examine the other apparent "solutions" of (A1) given by

$$w(x) \equiv \begin{cases} 0 & \text{on } [0, x_0] \\ v(x) & \text{on } (x_0, 1], \end{cases} \tag{3.6}$$

where v is a solution of (SA$_{x_0}$). Note that such a function has $\phi(w)_x \in L^2(Q_T)$. Moreover, for $\xi \in P_{\{0\}}(\overline{Q_T})$ and $t \in [0, T]$, an integration by parts gives

$$\begin{aligned}
&\int_0^t \int_0^1 [\phi(w)_x + f(w)]\xi_x dx ds - \int_0^t [g(w(1)) + f(w(1))]\xi(1, s) ds \\
&= -\int_0^t \int_{x_0}^1 [\phi(v)_x + f(v)]_x \xi dx ds - \left[\lim_{x \to x_0^+} \phi(v)_x(x)\right] \int_0^t \xi(x_0, s) ds \\
&= -[g(v(1)) + f(v(1))] \int_0^t \xi(x_0, s) ds,
\end{aligned}$$

since

$$\phi(v)_x + f(v) = f(v(1)) + g(v(1))$$

on $[x_0, 1]$, from which it follows that $w(x)$ is a nontrivial, nonnegative solution of (A1) on Q_T if $x_0 = 0$ or if $g(w(1)) + f(w(1)) = 0$ and $x_0 \in [0, 1)$. We also note that all the functions defined as in (3.6) are subsolutions of (A1) on Q_T.

Similar comments hold for functions defined by (3.4). If $y_0 = 1$ or if $g(w(0)) = f(w(0))$ and $y_0 \in [0,1)$, then $w(x)$ defined by (3.4) are nontrivial, nonnegative solutions of (B1) on Q_T. Moreover, all such functions, $w(x)$, are nontrivial, nonnegative subsolutions of (B1) on Q_T.

Knowing the existence of subsolutions is also useful since one usually has a comparison principle which says that a solution starting above a subsolution will remain above it (for as long as the solution exists). A similar statement holds for supersolutions.

We conclude this section with some ordering results for stationary solutions of (A1) and (B1). These are similar to the corresponding results in [19] and proved in much the same manner. However, since there is no extra work involved in establishing such statements for solutions of (SA_{x_0}) and (SB_{y_0}), we shall do this and thus have ordering results for some subsolutions of problems (A1) and (B1) as well. To help with the expression of these theorems, let us write $w_A(x, x_0)$ to denote the function defined as in (3.6) and $w_B(x, y_0)$ to denote the function defined as in (3.4).

Theorem 3.3 (A) *Suppose that* $w_1(x) \equiv w_A(x, x_1)$ *and* $w_2(x) \equiv w_A(x, x_2)$ *are given as above with* $0 \le x_2 \le x_1 < 1$. *If* $0 < w_1(1) < w_2(1)$, *then* $w_1(x) < w_2(x)$ *on* $(x_2, 1]$.

Proof:

We will borrow some techniques used to prove Theorem 2.2A in [19]. To begin, suppose that the theorem is false, and let

$$\hat{x} \equiv \sup\{x < 1 : w_2 > w_1 \text{ on } (x, 1]\}.$$

Then $x_1 < \hat{x} < 1$, and we have the following three cases to consider.

Case 1. If $f(w_1(1)) + g(w_1(1)) > f(w_2(1)) + g(w_2(1))$, then from

$$\begin{aligned}\phi(w_1)x + f(w_1) &= g(w_1(1)) + f(w_1(1)) \\ &> g(w_2(1)) + f(w_2(1)) \\ &= \phi(w_2)x + f(w_2),\end{aligned}$$

which holds on the interval $[x_1, 1]$, it follows that

$$\phi(w_1)x(\hat{x}) < \phi(w_2)x(\hat{x})$$

. Thus, for $\delta > 0$ sufficiently small, we have

$$[\phi(w_2) - \phi(w_1)]x < 0$$

on $[\hat{x}, \hat{x} + \delta] \subset (x_1, 1)$. But now

$$\begin{aligned}0 &< [\phi(w_2) - \phi(w_1)](\hat{x} + \delta) \\ &= \int_{\hat{x}}^{\hat{x}+\delta} [\phi(w_2) - \phi(w_1)]_x dx \\ &< 0,\end{aligned}$$

which is impossible.

Case 2. If $f(w_1(1)) + g(w_1(1)) > f(w_2(1)) + g(w_2(1))$, then we may argue as above to conclude

$$[\phi(w_2) - \phi(w_1)]x > 0$$

on $[\hat{x} - \delta, \hat{x} + \delta] \subset (x_1, 1)$. So

$$\begin{aligned}[\phi(w_2) - \phi(w_1)](\hat{x} - \delta) &= -\int_{\hat{x}-\delta}^{\hat{x}} [\phi(w_2) - \phi(w_1)]_x dx \\ &< 0,\end{aligned}$$

which implies that $w_2(\hat{x}-\delta) < w_1(\hat{x}-\delta)$. Define

$$\hat{y} \equiv \sup\{y \in (x_1, \hat{x}) : w_2 < w_1 \text{ on } (y, \hat{x})\}.$$

Arguing as above once more, we find that there exists a small $\nu > 0$ such that

$$[\phi(w_2) - \phi(w_1)]_x > 0$$

on $[\hat{y}, \hat{y}+\nu] \subset [x_1, \hat{x})$. But now

$$\begin{aligned} 0 &< \int_{\hat{y}}^{\hat{y}+\nu} [\phi(w_2) - \phi(w_1)]_x dx \\ &= [\phi(w_2) - \phi(w_1)](\hat{y}+\nu) \\ &< 0, \end{aligned}$$

which again is impossible.

Case 3. If $f(w_1(1)) + g(w_1(1)) = f(w_2(1)) + g(w_2(1))$, then the conservation laws for w_1 and w_2 imply

$$\phi(w_1)_x + f(w_1) = \phi(w_2)_x + f(w_2)$$

on $[x_1, 1]$. Hence, $w \equiv w_2 - w_1$ satisfies a differential equation of the form

$$w_x + hw = 0.$$

But, since $w(\hat{x}) = 0$, we must have $w \equiv 0$ in at least some small neighborhood about $x = \hat{x}$. This contradicts the definition of $\hat{x}$. □

Let us remark here that $w_1(1) < w_2(1)$ *does not imply* that $x_1 \geq x_2$, and so the hypotheses above do not allow any further weakening. For example,

$$w_1(x) \equiv \sqrt{x},$$

and

$$w_2(x) \equiv \begin{cases} 0, & x \in [0, \frac{1}{2}] \\ \sqrt{8x-4}, & x \in (\frac{1}{2}, 1] \end{cases}$$

are such functions associated with the problem

$$\begin{aligned} &(u \mid u \mid)'' = 0 && \text{on } (0,1) \\ &u(0) = 0 \\ &(u \mid u \mid)'(1) = u \mid u \mid^2 (1) \end{aligned}$$

which clearly violate the conclusion of the theorem.

Theorem 3.4 (B) *Suppose that* $w_1(x) \equiv w_B(x, x_1)$ *and* $w_2(x) \equiv w_B(x, x_2)$ *are given with* $0 < x_1 \leq x_2 \leq 1$ *and* $g(w_i(0)) - f(w_i(0)) \geq 0$, *for* $i = 1, 2$. *If* $0 < w_1(0) < w_2(0)$, *then* $w_1(x) < w_2(x)$ *on* $[0, x_2)$.

The proof of this result is virtually identical to that of the previous theorem and, therefore, is omitted. Observe that Theorems 3.3(A) and 3.4(B) are actually stronger results than their counterparts in [19] in the sense that they do not require the additional hypotheses $f' + g' > 0$ and $f'' > 0$, respectively.

3.2 Solution Diagrams for Problem (A)

This section is devoted to applying Lemma 3.4(A) to the case of $\phi(u) \equiv u \mid u \mid^{m-1}$, $f(u) \equiv \frac{\epsilon}{n} u \mid u \mid$, and $g(u) \equiv au \mid u \mid^{p-1}$, where $a, \epsilon, m, n, p > 0$. Observe that once this analysis is done, we will have complete solution diagrams for the nontrivial, nonnegative solutions of

$$\begin{aligned}
&[(u^m)_x + \frac{\epsilon}{n}u^2]_x = 0 \quad \text{on } [x_0, 1)\\
&u(x_0) = 0\\
&(u^m)_x(1) = au^p(1).
\end{aligned}$$

The apparently restrictive choice of f above is made to simplify the following analysis and is really not a restriction at all. For note that $v(x)$ is a nontrivial, nonnegative solution of

$$\begin{aligned}
&[(v^m)_x + \frac{\epsilon}{n}v^n]_x = 0 \quad \text{on } [x_0, 1)\\
&v(x_0) = 0\\
&(v^m)_x(1) = av^p(1)
\end{aligned}$$

iff $u(x) \equiv v^{n/2}(x)$ is a solution of

$$\begin{aligned}
&[(u^{2m/n})_x + \frac{\epsilon}{n}u^2]_x = 0 \qquad \text{on } [x_0, 1)\\
&u(x_0) = 0\\
&(u^{2m/n})_x(1) = au^{2p/n}(1),
\end{aligned}$$

and so the above work will allow us to completely characterize all the nontrivial, nonnegative stationary states of (A), as well as some subsolutions as discussed in the previous section.

For the above choices of ϕ, f, and g, equation (3.1) becomes

$$\int_0^{w_1} \frac{m\sigma^{m-1}}{aw_1^p + \frac{\epsilon}{n}w_1^2 - \frac{\epsilon}{n}\sigma^2}d\sigma = 1 - x_0,$$

which, after a substitution and some manipulation, can be written as

$$H(w_1) \equiv \frac{mn}{\epsilon}w_1^{m-2}\int_0^1 \frac{\sigma^{m-1}}{\beta - \sigma^2}d\sigma = 1 - x_0,$$

where

$$\beta \equiv \frac{an}{\epsilon} w_1^{p-2} + 1.$$

The case of $p = 2$ is now easy to analyze. When $m \neq 2$, there is exactly one value of w_1 associated with each $x_0 \in [0, 1)$ such that $H(w_1) = 1 - x_0$. When $m = 2$,

$$H(w_1) = \frac{n}{\epsilon} \ln \left[1 + \frac{\epsilon}{an}\right],$$

and so for any $a > 0$ and $x_0 \in (1 - \frac{1}{a}, 1) \cap [0, 1)$, there exists exactly one $\epsilon > 0$ such that $H(w_1) = 1 - x_0$ for any $w_1 > 0$.

Now consider the case $p \neq 2$. We have $w_1 = [(\epsilon/an)(\beta - 1)]^{1/(p-2)}$. Therefore, $H(w)$ can be expressed in terms of β as

$$H(w_1) = \frac{mn}{\epsilon} \left(\frac{\epsilon}{an}\right)^{\frac{m-2}{p-2}} (\beta - 1)^{\frac{m-2}{p-2}} \int_0^1 \frac{\sigma^{m-1}}{\beta - \sigma^2} d\sigma.$$

Thus if we define

$$G(\beta) \equiv \int_0^1 \frac{\sigma^{m-1}}{\beta - \sigma^2} d\sigma,$$

then the problem of finding $w_1 > 0$ such that $H(w_1) = 1 - x_0$ is equivalent to the problem of finding $\beta > 1$ for which

$$(\beta - 1)^{\frac{m-2}{p-2}} G(\beta) = \frac{\epsilon}{mn} \left(\frac{an}{\epsilon}\right)^{\frac{m-2}{p-2}} (1 - x_0).$$

To carry out this task we shall first obtain the graph of $(\beta - 1)^q G(\beta)$ for $\beta > 1$ and $q \equiv (m-2)/(p-2)$. The key to doing this is the following lemma.

Lemma 3.6 *Define*

$$P(x) \equiv \int_0^1 \frac{\sigma^n}{x - \sigma^2} d\sigma,$$

for $x > 1$ and $n > -1$. Then

$$P'(x) = \frac{n-1}{2x} P(x) - \frac{1}{2x(x-1)}.$$

Proof:

This lemma is simply the application of some techniques from introductory calculus. From an integration by parts, it follows that

$$P(x) = \frac{1}{(n+1)(x-1)} - \frac{2}{n+1}\int_0^1 \frac{\sigma^{n+2}}{(x-\sigma^2)^2}d\sigma,$$

and, via some algebraic manipulation, we have

$$\begin{aligned}\int_0^1 \frac{\sigma^{n+2}}{(x-\sigma^2)^2}d\sigma &= \int_0^1 \frac{x\sigma^n - \left(x-\sigma^2\right)\sigma^n}{\left(x-\sigma^2\right)^2}d\sigma \\ &= x\int_0^1 \frac{\sigma^n}{(x-\sigma^2)^2}d\sigma - \int_0^1 \frac{\sigma^n}{x-\sigma^2}d\sigma.\end{aligned}$$

Thus

$$P(x) = \frac{1}{(n+1)(x-1)} - \frac{2x}{n+1}\int_0^1 \frac{\sigma^n}{(x-\sigma^2)^2}d\sigma + \frac{2}{n+1}P(x),$$

which can be simplified to

$$-\int_0^1 \frac{\sigma^n}{(x-\sigma^2)^2}d\sigma = \frac{n-1}{2x}P(x) - \frac{1}{2x(x-1)}.$$

The conclusion now follows from

$$P'(x) = -\int_0^1 \frac{\sigma^n}{(x-\sigma^2)^2}d\sigma$$

for $x > 1$. □

Using this lemma, we calculate

$$\frac{d}{d\beta}\left[(\beta-1)^q G(\beta)\right] = \frac{q(\beta-1)^{q-1}}{2\beta}\left\{[p(\beta-1)+2]G(\beta) - \frac{1}{q}\right\}, \qquad (3.7)$$

which, for $m \neq 2$, is seen to be negative in the case $q < 0$. If $q = 0$, i.e., $m = 2$, then

$$\frac{d}{d\beta}\left[(\beta-1)^q G(\beta)\right] = -\int_0^1 \frac{\sigma}{(x-\sigma^2)^2}d\sigma < 0.$$

Thus $(\beta-1)^q G(\beta)$ is strictly decreasing for $q \le 0, \beta > 1$. Furthermore,

$$\lim_{\beta\to\infty} (\beta-1)^q G(\beta) = 0$$

in this case.

Also for $q < 0$,

$$\begin{aligned}(\beta-1)^q \int_0^1 \frac{\sigma^{m-1}}{\beta-\sigma^2} d\sigma &\ge (\beta-1)^q \int_0^1 \frac{\sigma^{m-1}}{\beta} d\sigma \\ &= \frac{(\beta-1)^q}{m\beta},\end{aligned}$$

and so

$$\lim_{\beta\to 1^+} (\beta-1)^q G(\beta) = \infty.$$

When $q = 0$ we form the same conclusion from

$$(\beta-1)^q G(\beta) = \int_0^1 \frac{\sigma}{\beta-\sigma^2} d\sigma = \frac{1}{2}\ln\left(\frac{\beta}{\beta-1}\right).$$

To analyze the case of $q > 0$, define

$$R(\beta) \equiv [p(\beta-1)+2]G(\beta) - \frac{1}{q},$$

and

$$\bar{R}(\beta) \equiv \frac{1}{[p(\beta-1)+2]\beta^{(m-2)/2}} R(\beta).$$

Then note that (3.7) can be written as

$$\frac{d}{d\beta}\left[(\beta-1)^q G(\beta)\right] = \frac{q(\beta-1)^{q-1}}{2\beta} R(\beta).$$

Also, via Lemma 3.6, we calculate

$$R'(\beta) = \left[\frac{p}{p(\beta-1)+2} + \frac{m-2}{2\beta}\right] R(\beta) + \frac{p[\frac{1}{q}-1](\beta-1)-2}{[p(\beta-1)+2](\beta-1)},$$

and so

$$\bar{R}'(\beta) = \frac{1}{[p(\beta-1)+2]\beta^{(m-2)/2}}\left\{\frac{p[\frac{1}{q}-1](\beta-1)-2}{[p(\beta-1)+2](\beta-1)}\right\}.$$

Therefore, in the case $q \geq 1$, we find that $\bar{R}'(\beta) < 0$. Now if $\bar{R}(\beta)$ were ever zero, then it would be negative for all large values of β. But then $R(\beta)$ would also be negative for all large β, thus contradicting

$$\lim_{\beta\to\infty} R(\beta) = \frac{p}{m} - \frac{1}{q} = -\frac{2}{m}\left[\frac{1}{q} - 1\right] \geq 0,$$

which follows from the Lebesgue dominated convergence theorem. Hence, $\bar{R}(\beta) > 0$ for all $\beta > 1$ which implies

$$\frac{d}{d\beta}\left[(\beta-1)^q G(\beta)\right] > 0$$

for $q \geq 1$. In this case also, we have

$$\lim_{\beta\to 1^+} (\beta-1)^q G(\beta) = 0,$$

and

$$\lim_{\beta\to\infty} (\beta-1)^q G(\beta) = \begin{cases} 1/m & \text{if } q = 1 \\ \infty & \text{if } q > 1. \end{cases}$$

The first of these limits actually holds for all $q > 0$ and follows from

$$\begin{aligned}
&(\beta-1)^q G(\beta) \\
&\leq\ (\beta-1)^q \int_0^{1/2} \frac{\sigma^{m-1}}{\beta-\sigma^2} d\sigma + \max\left\{\frac{1}{2^{m-1}}\right\} (\beta-1)^q \int_{1/2}^1 \frac{1}{\beta-\sigma^2} d\sigma \\
&\leq\ (\beta-1)^q \int_0^{1/2} \frac{\sigma^{m-1}}{\beta-\sigma^2} d\sigma + \max\left\{\frac{1}{2^{m-1}}, 1\right\} (\beta-1)^q \ln\left[\frac{\sqrt{\beta}+1}{\sqrt{\beta}-1}\right],
\end{aligned}$$

and

$$\lim_{\beta\to 1^+} (\beta-1)^q \ln(\sqrt{\beta}-1) = 0,$$

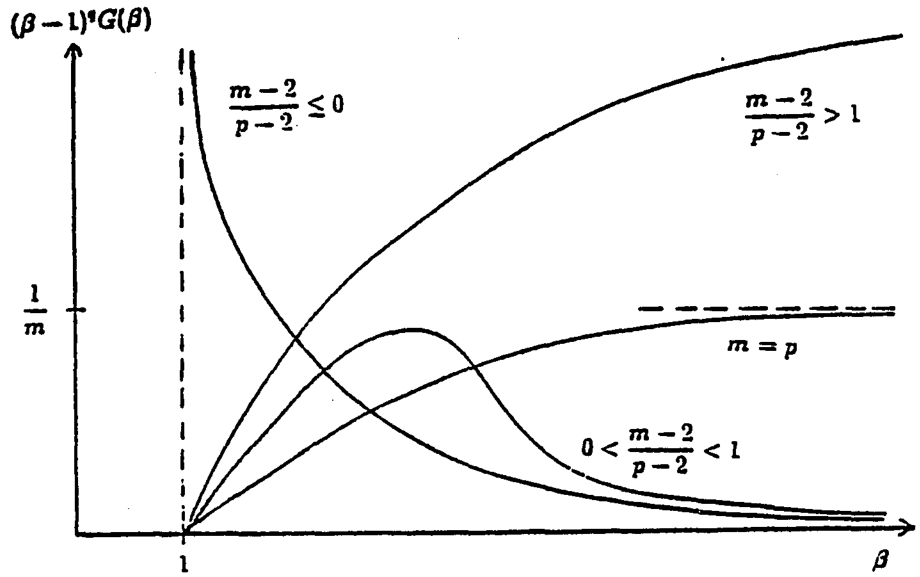

Figure 3.1: Graph of $(\beta-1)^q G(\beta)$, where $q \equiv (m-2)/(p-2)$ and $p \neq 2$.

for any positive q.

Finally in the case $0 < q < 1$, we see that $\bar{R}'$ vanishes exactly once. Hence, from

$$\lim_{\beta \to 1+} R(\beta) = \infty \text{ and } \lim_{\beta \to \infty} R(\beta) = -\frac{2}{m}\left[\frac{1}{q} - 1\right] < 0,$$

it follows that $R(\beta)$ must have exactly one zero. Therefore, there exists $\beta_0 > 1$ such that

$$\frac{d}{d\beta}\left[(\beta-1)^q G(\beta)\right] \begin{cases} > 0 & \text{on } (1, \beta_0) \\ = 0 & \text{for } \beta = \beta_0 \\ < 0 & \text{on } (\beta_0, \infty). \end{cases}$$

Also,

$$\lim_{\beta \to 1+} (\beta-1)^q G(\beta) = \lim_{\beta \to \infty} (\beta-1)^q G(\beta) = 0.$$

The graph of $(\beta-1)^q G(\beta)$ can thus be sketched as in Figure 3.1

From such sketches it is easy to determine when there exists $w_1 > 0$ and $x_0 \in$

$[0,1)$ such that

$$(\beta-1)^{(m-2)/(p-2)}G(\beta)=\frac{\epsilon}{mn}\left(\frac{an}{\epsilon}\right)^{(m-2)/(p-2)}(1-x_0),$$

for

$$\beta\equiv\frac{an}{\epsilon}w_1^{p-2}+1,$$

in the case $p\neq 2$. We now use this to determine the number of nontrivial, nonnegative solutions of $(\mathrm{SA}x_0)$ in the various cases $m,n,p,a,\epsilon>0$ and $x_0\in[0,1)$ by recalling our comments at the outset of this section and replacing m,p and $w_0^{2/n}$ above by $2m/n, 2p/n$ and w_0, respectively. (Recall that only the case $x_0=0$ represents solutions of (A).) For $a>0$, $x_0\in[0,1)$ with $a(1-x_0)<1$ define $\epsilon_0>0$ such that

$$\frac{an}{\epsilon_0}\ln\left[1+\frac{\epsilon_0}{an}\right]=a(1-x_0),$$

and for m,p,n with $0<(m-n)/(p-n)<1$ define

$$\bar{G}\equiv\max_{1<\beta<\infty}(\beta-1)^{(m-n)/(p-n)}\int_0^1\frac{\sigma^{(2m/n)-1}}{\beta-\sigma^2}d\sigma.$$

Finally, define $\epsilon_1>0$ such that

$$\frac{\epsilon_1}{mn}\left(\frac{an}{\epsilon_1}\right)^{(m-n)/(p-n)}(1-x_0)=\bar{G}.$$

Our results are summarized in the following outline.

$0<m<n$.

$0<p<m$. There are two solutions for $0<\epsilon<\epsilon_1$, exactly on solution for $\epsilon=\epsilon_1$, and no solutions for $\epsilon>\epsilon_1$.

$p=m$. There is exactly one solution when $a(1-x_0)<1$. Otherwise, there are no solutions.

$p > m$. There is exactly one solution for all $a, \epsilon > 0$ and $x_0 \in [0, 1)$.

$m = n$.

$p \neq m$. There is exactly one solution for each $a, \epsilon > 0$ and $x_0 \in [0, 1)$.

$p = m$. There are a continuum of solutions for each $a > 0$, $x_0 \in [0, 1)$ such that $a(1 - x_0) < 1$ and $\epsilon = \epsilon_0$. Otherwise, there are no solutions.

$m > n$.

$0 < p < m$. There is exactly one solution for each $a, \epsilon > 0$ and $x_0 \in [0, 1)$.

$p = m$. There is exactly one solution when $a(1 - x_0) < 1$; otherwise, there are none.

$p > m$. There are two solutions for $0 < \epsilon < \epsilon_1$, exactly one solution for $\epsilon = \epsilon_1$, and no solutions for $\epsilon > \epsilon_1$.

Let $w(x, \epsilon)$ denote the solution of (SA_{x_0}). The graphs of $w_1(\epsilon) \equiv w(1, \epsilon)$ are now sketched in Figures 3.2 - 3.7 in the various cases of $a, n, m, p > 0$ for $x_0 = 0$. The analysis which justifies the slopes and limiting behavior of these curves is presented following the sketches. We observe that for $0 < m < n$ our graphs are identical with those obtained by Levine [19] for $m = 1, n = 2$.

We now proceed to calculate the signs of $w_1'(\epsilon)$ and the limiting behavior of $w_1(\epsilon)$ along its various branches. To do this we will again consider the case $f(u) = \frac{\epsilon}{n} u \mid u \mid$ and $x_0 \in [0, 1)$. Our results can then be interpreted for the general cases $f(u) = \frac{\epsilon}{n} u \mid u \mid^{n-1}$ as done previously. Also, we can observe from this analysis that all the diagrams of the subsolutions which we have discussed look exactly like the solutions diagrams. For now we shall agree to drop the subscript on $w_1(\epsilon)$ and simply denote this by $w(\epsilon)$ or w.

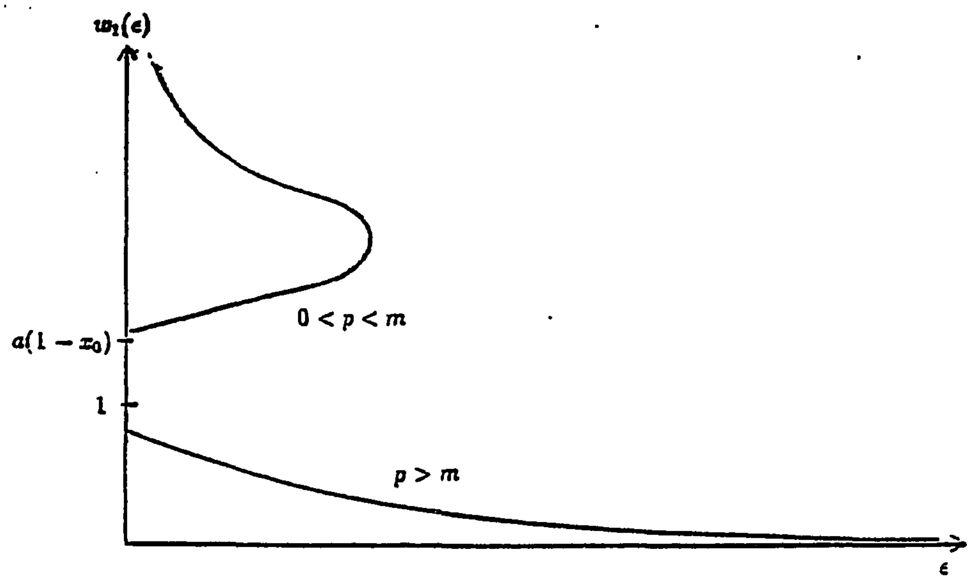

Figure 3.2: $w_1(\epsilon)$ where $(w^m)'(1) = aw^p(1), 0 < m < n, a \geq 1$, and $w_1(0) = a^{1/(m-p)}$.

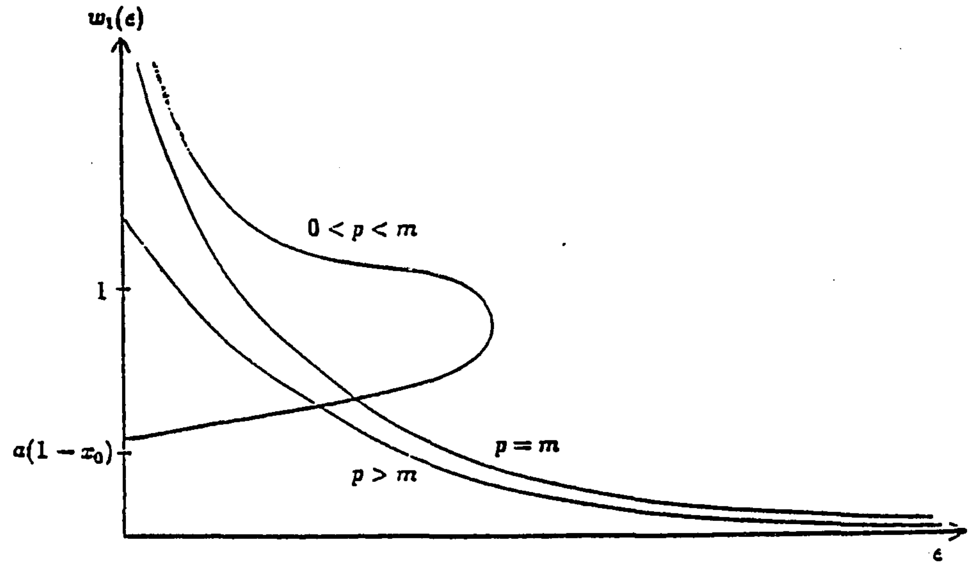

Figure 3.3: $w_1(\epsilon)$ where $(w^m)'(1) = aw^p(1), 0 < m < n, a < 1$, and $w_1(0) = a^{1/(m-p)}$.

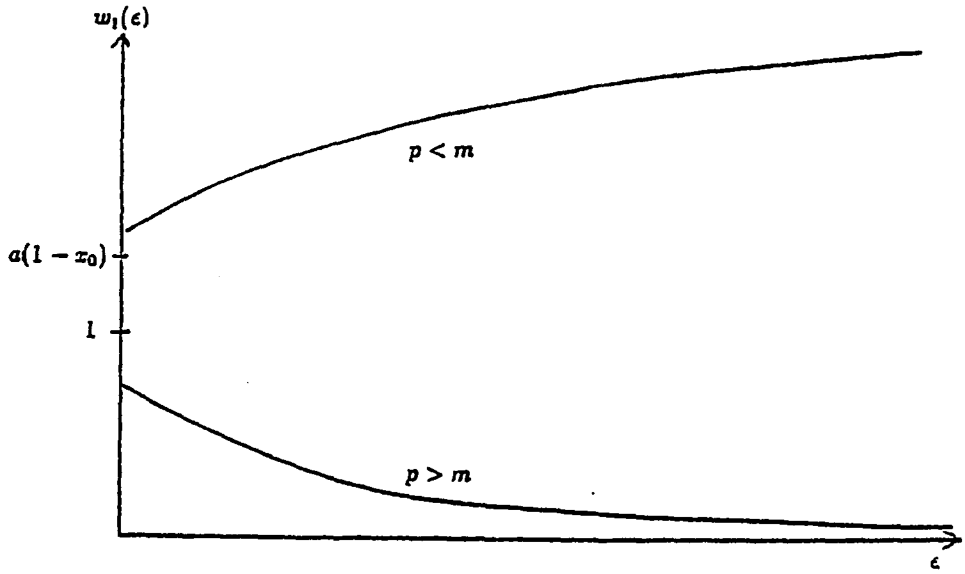

Figure 3.4: $w_1(\epsilon)$ where $(w^m)'(1) = aw^p(1), m = n, a \geq 1$, and $w_1(0) = a^{1/(m-p)}$.

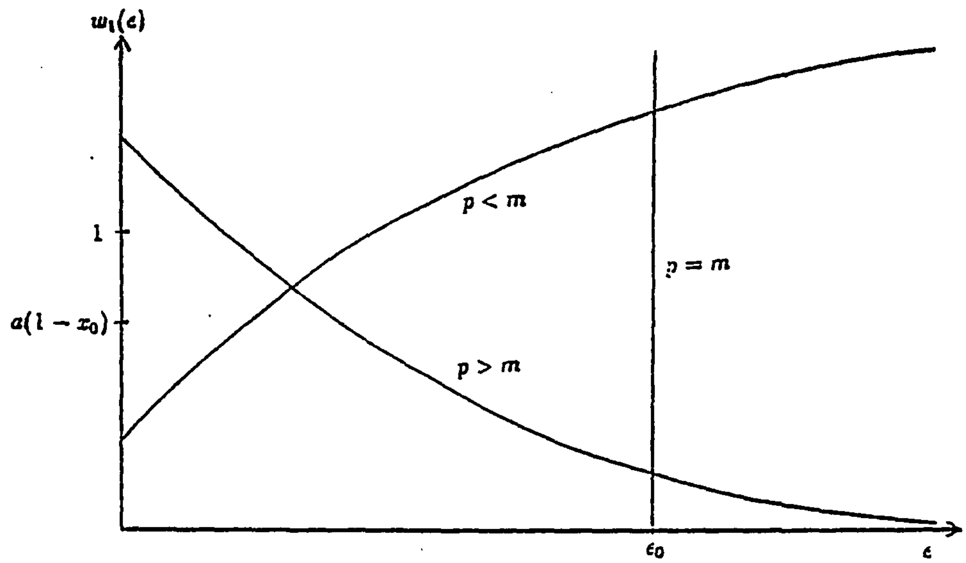

Figure 3.5: $w_1(\epsilon)$ where $(w^m)'(1) = aw^p(1), m = n, a < 1$, and $w_1(0) = a^{1/(m-p)}$.

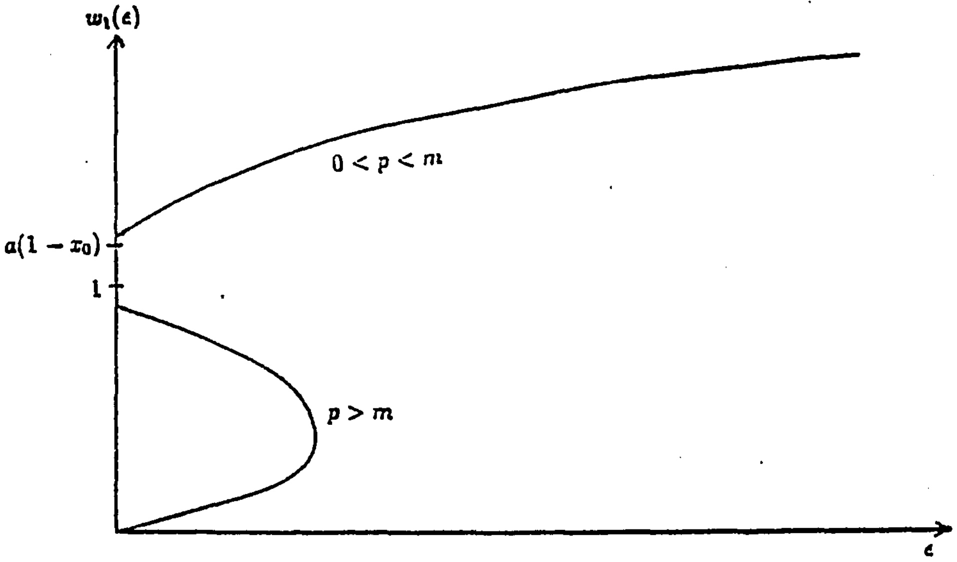

Figure 3.6: $w_1(\epsilon)$ where $(w^m)'(1) = aw^p(1), m > n, a \geq 1$, and $w_1(0) = a^{1/(m-p)}$.

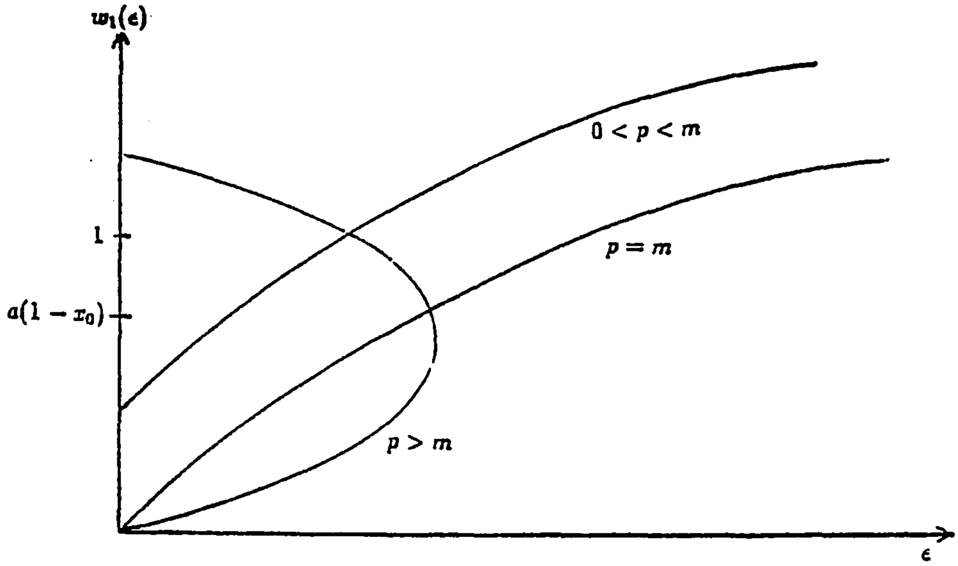

Figure 3.7: $w_1(\epsilon)$ where $(w^m)'(1) = aw^p(1), m > n, a < 1$, and $w_1(0) = a^{1/(m-p)}$.

For the case $p = 2$, recall that

$$\left[\int_0^1 \frac{\sigma^{m-1}}{2a + \epsilon(1-\sigma^2)} d\sigma\right] w^{m-2} = \frac{1 - x_0}{2m},$$

so differentiating with respect to ϵ immediately gives

$$w_\epsilon \begin{cases} < 0 & \text{if } m < 2 \\ > 0 & \text{if } m > 2, \end{cases}$$

where

$$w_\epsilon \equiv \frac{\partial w}{\partial \epsilon}.$$

Also from this identity it follows that

$$\lim_{\epsilon \to \infty} w(\epsilon) = \begin{cases} 0 & \text{if } m < 2 \\ \infty & \text{if } m > 2. \end{cases}$$

Recall, for $p \neq 2$, that

$$\frac{1}{\epsilon} w^{m-2} G(\beta) = \frac{1}{mn}(1 - x_0), \tag{3.8}$$

where $\beta \equiv (an/\epsilon) w^{p-2} + 1$. Differentiating this expression with respect to ϵ gives

$$\begin{aligned} &-\frac{1}{\epsilon^2} w^{m-2} G(\beta) + \frac{m-2}{\epsilon} w^{m-3} w_\epsilon G(\beta) \\ &= -\frac{1}{\epsilon} w^{m-2} G'(\beta) \left[\frac{an(p-2)}{\epsilon} w^{p-3} w_\epsilon - \frac{an}{\epsilon^2} w^{p-2}\right], \end{aligned}$$

which, after some manipulation, becomes

$$\frac{1}{2\beta} \{(m-2)[p(\beta - 1) + 2]G(\beta) - (p-2)\} w_\epsilon = \frac{w}{\epsilon} \frac{d}{d\beta}[(\beta - 1)G(\beta)]. \tag{3.9}$$

If $m \neq 2$, then this can be written as

$$\frac{m-2}{2\beta} R(\beta) w_\epsilon = \frac{w}{\epsilon} \frac{d}{d\beta}[(\beta - 1)G(\beta)]. \tag{3.10}$$

From a previous calculation, $\frac{d}{d\beta}[(\beta-1)G(\beta)] > 0$, and so from (3.9) we see that for $(m-2)/(p-2) < 0$,

$$w_\epsilon \begin{cases} < 0 & \text{if } m < 2 \\ > 0 & \text{if } m > 2. \end{cases}$$

Moreover, taking limits in (3.8), we have

$$\lim_{\epsilon \to \infty} \frac{1}{\epsilon} w^{m-2} G(\beta) = \lim_{\epsilon \to \infty} \frac{1}{an} w^{m-p} (\beta - 1) G(\beta) = \frac{1}{mn}(1 - x_0),$$

so it must be that $\lim_{\epsilon \to \infty} w(\epsilon) = 0$. For if this limit were positive, then $\lim_{\epsilon \to \infty} \beta = 1$ and $\lim_{\epsilon \to \infty} \frac{1}{an} w^{m-p}(\beta - 1)G(\beta) = 0 = \frac{1}{mn}(1 - x_0)$ which is impossible. Similarly, we have $\lim_{\epsilon \to \infty} w(\epsilon) = \infty$ when $m > 2$.

In the case $m = 2$, we again use equation (3.9) to conclude

$$w_\epsilon \begin{cases} > 0 & \text{if } p < 2 \\ < 0 & \text{if } p > 2. \end{cases}$$

Furthermore, from (3.8), it must be the case that

$$\lim_{\epsilon \to \infty} w(\epsilon) = \begin{cases} \infty & \text{if } p < 2 \\ 0 & \text{if } p > 2. \end{cases}$$

We previously showed $R(\beta) > 0$ when $(m-2)/(p-2) \geq 1$. So from (3.10) it follows that, for $(m-2)/(p-2) \geq 1$,

$$w_\epsilon \begin{cases} < 0 & \text{if } m < 2 \\ > 0 & \text{if } m > 2. \end{cases}$$

As done above it also follows that

$$\lim_{\epsilon \to \infty} w(\epsilon) = \begin{cases} 0 & \text{if } m < 2 \\ \infty & \text{if } m > 2. \end{cases}$$

In the case $0 < (m-2)/(p-2) < 1$, we have $R(\beta) > 0$ on $(1, \beta_0), R(\beta_0) = 0$, and $R(\beta) < 0$ on (β_0, ∞). Thus we will consider $\epsilon = \epsilon(w)$. From our previous work, $\epsilon(w)$ is bounded with some bound $\epsilon(m,p)$, and $\epsilon([a(1-x_0)]^{1/(m-p)}) = 0$, since $w(0) = [a(1-x_0)]^{1/(m-p)}$. Recalling

$$(\beta - 1)^{\frac{m-2}{p-2}} G(\beta) = \frac{\epsilon}{mn} \left(\frac{an}{\epsilon}\right)^{\frac{m-2}{p-2}} (1 - x_0),$$

we see that

$$\lim_{\epsilon \to 0+} (\beta - 1)^{\frac{m-2}{p-2}} G(\beta) = 0,$$

and so one of the solution branches of $w(\epsilon)$ must satisfy $\lim_{\epsilon \to 0+} (w^{p-2}/\epsilon) = 0$. Hence,

$$\lim_{\epsilon \to 0+} w(\epsilon) = \begin{cases} \infty & \text{if } p < 2 \\ 0 & \text{if } p > 2 \end{cases}$$

for this branch. So with $p < 2$, $m < 2$ and $\epsilon(w)$ is defined on $[[a(1-x_0)]^{1/(m-p)}, \infty)$. Equation (3.10) now implies that $w_\epsilon > 0$ on the lower branch, and $w_\epsilon < 0$ on the upper branch. Similarly, with $p > 2$, $m > 2$ and $\epsilon(w)$ is defined on $(0, [a(1-x_0)]^{1/(m-p)}]$. So $w_\epsilon > 0$ on the lower branch, and $w_\epsilon < 0$ on the upper branch.

Finally, note that $w(0) = [a(1-x_0)]^{1/(m-p)}$ provided $m \neq p$. With $m = p$, we use (3.8) to calculate

$$\lim_{\epsilon \to 0+} \frac{1}{\epsilon} w^{m-2} G(\beta) = \lim_{\epsilon \to 0+} \frac{\beta - 1}{an} G(\beta) = \frac{1}{mn}(1 - x_0).$$

Now if $\lim_{\epsilon \to 0+} w^{p-2}(\epsilon) \equiv L > 0$, then

$$\lim_{\epsilon \to 0+} \frac{\beta - 1}{an} G(\beta) = \frac{1}{an} \lim_{\beta \to \infty} (\beta - 1) G(\beta) = \frac{1}{amn},$$

and so $a(1-x_0)=1$. But we showed previously that $a(1-x_0)<1$ in order for $w(\epsilon)$ to exist at all. Hence it must be that

$$\lim_{\epsilon\to 0^+} w^{p-2}(\epsilon)=0,$$

from which it follows

$$\lim_{\epsilon\to 0^+} w(\epsilon)=\begin{cases}\infty & \text{if } m=p<2\\ 0 & \text{if } m=p>2.\end{cases}$$

3.3 Solution Diagrams for Problem (B)

We now carry out a similar analysis for positive solutions of (SB_{y_0}) and hence obtain the solution diagrams for the nontrivial, nonnegative stationary states of (B). However, in contrast to the previous section, some of the functions $w_B(x,y_0)$ are actually solutions of (B), namely, those for which $w_0\equiv w_B(0,y_0)$ satisfies $aw_0^p=(\epsilon/n)w_0^n$. Because of this, the solutions diagrams for (B) in the case $0<m<n$ will leave the plane.

In the same spirit as in the previous section, we consider the case $\phi(u)=u\mid u\mid^{m-1}, f(u)=\frac{\epsilon}{n}u\mid u\mid, g(u)=au\mid u\mid^{p-1}$. In this situation Lemma 3.5(B) says that we must seek $w_0>0$ and $y_0\in(0,1]$ such that

$$\int_0^{w_0}\frac{m\sigma^{m-1}}{aw_0^p-(\epsilon/n)[w_0^2-\sigma^2]}d\sigma=y_0,$$

and

$$aw_0^p\geq\frac{\epsilon}{n}w_0^2.$$

As before we can rewrite the integral identity as

$$F(w_0)\equiv\frac{mn}{\epsilon}w_0^{m-2}\int_0^1\frac{\sigma^{m-1}}{(\frac{an}{\epsilon}w_0^{p-2}-1)+\sigma^2}d\sigma=y_0.$$

For now we shall consider the case of positive solutions of $F(w_0) = y_0$ such that $anw_0^{p-2} > \epsilon$. The possibility of solutions which satisfy $anw_0^{p-2} = \epsilon$ will be discussed later.

If $p = 2$, then it is clear that when $m \neq 2$ and $\epsilon < an$ there will be exactly one solution of $F(w_0) = y_0$ for each $y_0 \in (0, 1]$. When $m = 2$,

$$F(w_0) = \frac{n}{\epsilon} \ln \left[\frac{an/\epsilon}{(an/\epsilon) - 1} \right],$$

so for any $a > 0$ and any $y_0 \in (1/a, 1]$ there exists an ϵ such that $\epsilon < an$ and $F(w_0) = y_0$ for all $w_0 > 0$.

To analyze the remaining cases, we define

$$\alpha \equiv \frac{an}{\epsilon} w_0^{p-2} - 1, \; G(\alpha) \equiv \int_0^1 \frac{\sigma^{m-1}}{\alpha + \sigma^2} d\sigma, \; q \equiv \frac{m-2}{p-2}$$

and seek $\alpha > 0$ such that

$$(\alpha + 1)^q G(\alpha) = \frac{\epsilon}{mn} \left(\frac{an}{\epsilon} \right)^q y_0.$$

To do this, we first sketch the graph of $(\alpha + 1)^q G(\alpha)$ for the various ranges of q.

Using the inequalities

$$\frac{(\alpha + 1)^{q-1}}{m} \leq (\alpha + 1)^q G(\alpha) \leq \frac{(\alpha + 1)^q}{m\alpha},$$

and the Lebesgue dominated convergence theorem, the following limits are found:

$$\lim_{\alpha \to \infty} (\alpha + 1)^q G(\alpha) = \begin{cases} 0 & \text{if } q < 1 \\ 1/m & \text{if } q = 1 \\ \infty & \text{if } q > 1. \end{cases}$$

Furthermore, observe that

$$G(\alpha) \geq \begin{cases} \int_0^1 \frac{1}{\alpha+\sigma^2} d\sigma & \text{if } 0 < m < 1 \\ \int_0^1 \frac{\sigma}{\alpha+\sigma^2} d\sigma & \text{if } 1 \leq m \leq 2, \end{cases}$$

and so (using this and the Lebesgue dominated convergence theorem again) we have

$$\lim_{\alpha \to 0^+} (\alpha+1)^q G(\alpha) = \begin{cases} \infty & \text{if } 0 < m \leq 2 \\ 1/(m-2) & \text{if } m > 2. \end{cases}$$

In the case $q \leq 0$,

$$\frac{d}{d\alpha}[(\alpha+1)^q G(\alpha)] = q(\alpha+1)^{q-1} G(\alpha) + (\alpha+1)^q G'(\alpha)$$

which is easily seen to be negative. To continue the analysis for positive values of q we require the following lemma, which is parallel to Lemma 3.6.

Lemma 3.7 *Define*

$$P(x) \equiv \int_0^1 \frac{\sigma^n}{x+\sigma^2} d\sigma,$$

for $x > 0, n > -1$. *Then*

$$P'(x) = \frac{n-1}{2x} P(x) - \frac{1}{2x(x+1)}.$$

Using this lemma, as before, it follows that

$$\frac{d}{d\alpha}[(\alpha+1)^q G(\alpha)] = \frac{q(\alpha+1)^{q-1}}{2\alpha} R(\alpha), \tag{3.11}$$

where

$$R(\alpha) \equiv [\alpha p + (p-2)] G(\alpha) - \frac{1}{q}$$

and $q \neq 0$. Moreover,

$$R'(\alpha) = \left[p + \frac{(m-2)[\alpha p + (p-2)]}{2\alpha}\right] G(\alpha) - \frac{\alpha p + (p-2)}{2\alpha(\alpha+1)},$$

similar to the previous section. But now the analysis becomes a little more complex because the term $[\alpha p + (p-2)]$ can change sign. (Recall that the corresponding term in the previous section, $[p(\beta-1)+2]$, did not give us this problem.) Therefore, we shall have a few more cases to consider.

In the case $q > 0, p > 2$ (so $m > 2$), we can solve for $G(\alpha)$ in terms of $R(\alpha)$ to get

$$R'(\alpha) = \left[\frac{p}{\alpha p + (p-2)} + \frac{m-2}{2\alpha}\right] R(\alpha) + \frac{p(p-m)\alpha + (p-2)(p-m+2)}{(m-2)(\alpha+1)[\alpha p + (p-2)]}.$$

Hence

$$\bar{R}(\alpha) \equiv \frac{1}{[\alpha p + (p-2)]\alpha^{(m-2)/2}} R(\alpha)$$

has

$$\bar{R}'(\alpha) = \frac{1}{[\alpha p + (p-2)]\alpha^{(m-2)/2}} \left\{\frac{p(p-m)\alpha + (p-2)(p-m+2)}{(m-2)(\alpha+1)[\alpha p + (p-2)]}\right\}. \tag{3.12}$$

From (3.12) we see that $\bar{R}'(\alpha) > 0$ for $2 < m \leq p$, and thus $\bar{R}(\alpha) < 0$ for all $\alpha > 0$ since

$$\lim_{\alpha\to\infty} \bar{R}(\alpha) = \lim_{\alpha\to\infty} \frac{\alpha}{[\alpha p + (p-2)]} \left(\frac{1}{\alpha^{m/2}}\right) R(\alpha) = 0.$$

Similarly, $\bar{R}(\alpha) > 0$ when $2 < p \leq m-2$, since $\bar{R}'(\alpha) < 0$ for all $\alpha > 0$. In the intermediate case of $m, p > 2$ such that $m - 2 < p < m$, there exists $\alpha_0 > 0$ for which $\bar{R}'(\alpha) > 0$ on $(0, \alpha_0)$, $\bar{R}'(\alpha_0) = 0$, and $\bar{R}'(\alpha) < 0$ on (α_0, ∞). Furthermore,

$$\lim_{\alpha\to 0^+} R(\alpha) = 0,$$

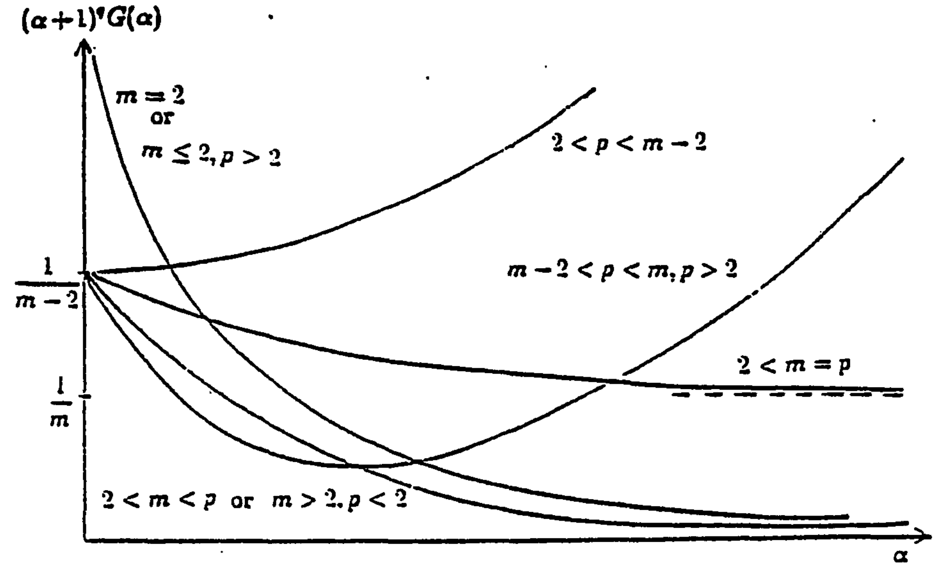

Figure 3.8: $(\alpha + 1)^{(m-2)/(p-2)}G(\alpha)$ for the cases of $(m-2)/(p-2) \leq 0$ and $(m-2)/(p-2) > 0, p > 2$.

and

$$\lim_{\alpha \to 0^+} R'(\alpha) = \left\{ \begin{array}{ll} \dfrac{p}{m-2} - \dfrac{p-2}{m-4} & \text{if } m > 4 \\ -\infty & \text{if } 2 < m \leq 4 \end{array} \right\} < 0,$$

which implies that $R(\alpha) < 0$ for small $\alpha > 0$. Therefore, there exists $\alpha_1 > 0$ such that $\bar{R}(\alpha) < 0$ on $(0, \alpha_1)$, $\bar{R}(\alpha_1) = 0$, and $\bar{R}(\alpha) > 0$ on (α_1, ∞). The graphs of $(\alpha+1)^q G(\alpha)$ can now be deduced, in the cases discussed above, from this information and appear in Figure 3.8.

For $(m-2)/(p-2) > 0$ such that $p < 2$, we have $m < 2$ and $R(\alpha) < 0$ when $\alpha \in (0, (2-p)/p]$. Observe that $[\alpha p + (p-2)]$ is positive for $\alpha \in ((2-p)/p, \infty)$, and so, for such values of α, equation (3.12) holds. In the case $p \leq m < 2$, if $\alpha > (2-p)/p$, then $\bar{R}'(\alpha) > 0$. It thus follows from $\lim_{\alpha \to \infty} \bar{R}(\alpha) = 0$ that $\bar{R}(\alpha) < 0$ for all $\alpha > (2-p)/p$. Therefore, $R(\alpha) < 0$ for all $\alpha > 0$.

When $q > 0, p < 2$ is such that $p > m$, we see from (3.12) that $\bar{R}'(\alpha)$ vanishes

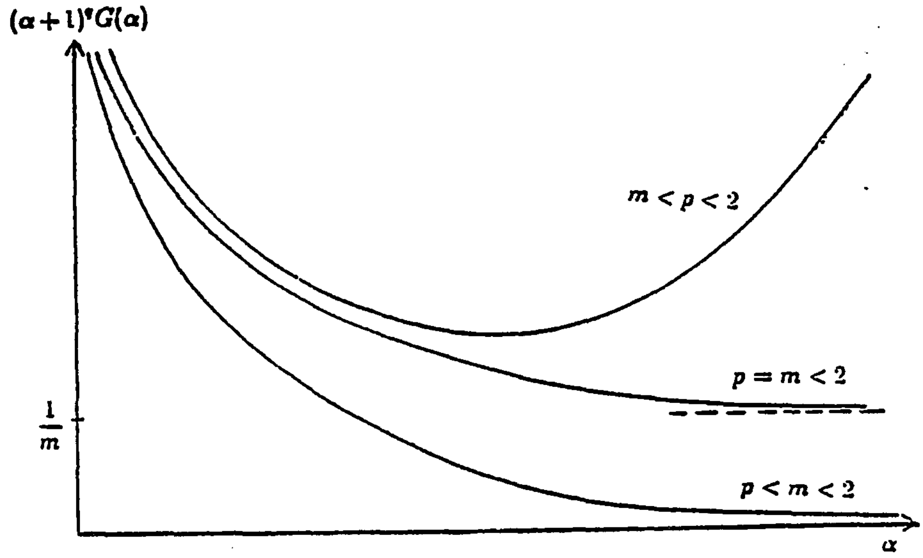

Figure 3.9: $(\alpha+1)^{(m-2)/(p-2)}G(\alpha)$ for the case of $(m-2)/(p-2)>0, p<2$.

exactly once on $((2-p)/p,\infty)$, namely, for $\hat{\alpha}\equiv\dfrac{2-p}{p}\left[1+\dfrac{2}{p-m}\right]$. In fact, $\bar{R}'(\alpha)>0$ on $((2-p)/p,\hat{\alpha})$, $\bar{R}'(\hat{\alpha})=0$, and $\bar{R}'(\alpha)<0$ on $(\hat{\alpha},\infty)$. Hence from $R((2-p)/p)<0$ and $\lim_{\alpha\to\infty}\bar{R}(\alpha)=0$, it follows that there must exist $\alpha_1>0$ such that $\bar{R}(\alpha)<0$ on $((2-p)/p,\alpha_1)$, $\bar{R}(\alpha_1)=0$, and $\bar{R}(\alpha)>0$ on (α_1,∞). From the above information we now have the graph of $(\alpha+1)^qG(\alpha)$ for the remaining cases as given in Figure 3.9. Using these diagrams it is now easy to determine when there exist $w_0>0$ such that $anw_0^{p-2}>\epsilon$ and $F(w_0)=y_0$.

It only remains to consider solutions of $F(w_0)=y_0$ for which $anw_0^{p-2}=\epsilon$. (Note that such solutions give rise to stationary solutions of (B). This did not happen in the previous section.) If $anw_0^{p-2}=\epsilon$, then

$$F(w_0)=\frac{mn}{\epsilon}w_0^{m-2}\int_0^1\sigma^{m-3}d\sigma.$$

Hence, we must have $m > 2$, $w_0^{p-2} = \epsilon/an$, and

$$w_0 = \left[\frac{y_0\epsilon(m-2)}{mn}\right]^{1/(m-2)}.$$

So in the case $p = 2$, if $\epsilon = an$, then for any $y_0 \in (0,1]$ there exists a solution

$$w_0 = \left[\frac{y_0 a(m-2)}{m}\right]^{1/(m-2)}.$$

When $p \neq 2$, then for ϵ such that

$$\frac{mn}{\epsilon(m-2)}\left(\frac{\epsilon}{an}\right)^{(m-2)/(p-2)} \in (0,1]$$

there exists a solution,

$$w_0 = \left[\frac{y_0\epsilon(m-2)}{mn}\right]^{1/(m-2)} = \left(\frac{\epsilon}{an}\right)^{1/(p-2)},$$

with

$$y_0 = \frac{mn}{\epsilon(m-2)}\left(\frac{\epsilon}{an}\right)^{(m-2)/(p-2)}.$$

We now use the above work to determine the number of nontrivial, nonnegative solutions of (SB_{y_0}) in the various cases $m, n, p, a, \epsilon > 0$ and $y_0 \in (0,1]$. (This is done, as in the previous section, by replacing m, p, and $w_0^{2/n}$ by $2m/n, 2p/n$, and w_0, respectively.) To do this we make the following definitions. For the cases $p > n, m-n < p < m$ and $m < p < n$, define

$$\underline{G} \equiv \min_{\alpha>0}(\alpha+1)^{(m-n)/(p-n)}\int_0^1 \frac{\sigma^{(2m/n)-1}}{\alpha+\sigma^2}d\sigma,$$

and $\epsilon_1 > 0$ is defined such that

$$\frac{\epsilon_1}{nm}\left(\frac{an}{\epsilon_1}\right)^{(m-n)/(p-n)} y_0 = \underline{G}.$$

Finally,

$$\epsilon_0(p) \equiv an \left[\frac{a(m-n)y_0}{m}\right]^{(p-n)/(m-p)}$$

for $m \neq n$, $m \neq p$, and define $\epsilon_0 > 0$ such that

$$\frac{an}{\epsilon_0} \ln\left[\frac{(an/\epsilon_0)}{(an/\epsilon_0)-1}\right] = ay_0,$$

for $a > 0$, $y_0 \in (0,1]$ with $ay_0 > 1$ and $m = n = p$. Our results are now summarized in the following outline.

$o < m < n$.

- **$o < p < m$.** There is exactly one solution for each $a, \epsilon > 0$ and $y_0 \in (0,1]$.
- **$p = m$.** There is exactly one solution when $ay_0 > 1$. Otherwise, there are none.
- **$m < p < n$.** There are two solutions for $0 < \epsilon < \epsilon_1$, exactly one solution for $\epsilon = \epsilon_1$, and no solutions for $\epsilon > \epsilon_1$.
- **$p = n$.** There is exactly one solution when $0 < \epsilon < an$. For $\epsilon \geq an$, there are no solutions.
- **$p > n$.** There is exactly one solution for each $a, \epsilon > 0$ and $y_0 \in (0,1]$.

$m = n$.

- **$p \neq m$.** There is exactly one solution for each $a, \epsilon > 0$ and $y_0 \in (0,1]$.
- **$p = m$.** There are a continuum of solutions when $ay_0 > 1$ and $\epsilon = \epsilon_0$. Otherwise, there are no solutions.

$m > n$.

- **$o < p \leq n$, $n < p \leq m - n$, or $p > m$.** There is exactly one solution for $0 < \epsilon \leq \epsilon_0(p)$ and no solutions for $\epsilon > \epsilon_0(p)$.

$m - n < p < m$ and $p > n$. There are two solutions for $\epsilon_0(p) \leq \epsilon < \epsilon_1$, exactly one solution for $0 < \epsilon < \epsilon_0(p)$ and for $\epsilon = \epsilon_1$, and no solutions for $\epsilon > \epsilon_1$.

$p = m$. There is exactly one solution for each $\epsilon > 0$ whenever $1 < ay_0 \leq m/(m-n)$. Otherwise, there are no solutions.

Observe that this settles the question of stationary solution diagrams for (B) in the case $0 < m \leq n$ upon setting $y_0 = 1$. However, when $0 < n < m$, we must consider the bounded surface $w_0(\epsilon, y_0) \equiv w_B(0, y_0)$. The outline above gives the cross-sections of this surface for fixed y_0. The actual solution curve is made up of the curve $w_0(\epsilon, 1)$ and the curve made up of the endpoints of $w_0(\epsilon, y_0)$, given by $w_0(\epsilon(p), y_0)$ for $y_0 \in (0, 1]$. Note that we have an explicit representation for this last curve, since these are solutions which satisfy

$$aw_0^p = \frac{\epsilon_0(p)}{n} w_0^n.$$

We now give the stationary solution curves for problem (B) in Figures 3.10 - 3.22. In the case $0 < m \leq n$, these are the graphs of $w_0(\epsilon) \equiv w(0, \epsilon)$, where $w(x, \epsilon)$ is the solution of (SB_{y_0}) and $y_0 = 1$. In the case $0 < n < m$, we are actually drawing the edge of the surface $w_0(\epsilon, y_0)$ which corresponds to solutions of (B). For this case the notation $\epsilon_0 \equiv \epsilon_0(p)$ is used, and the vertical axis is labeled $w_0(\epsilon, y_0)$ eventhough we are only graphing a curve which lies on this surface.

To calculate the signs of $w_0'(\epsilon)$ and the limiting behavior along the various branches of w_0, we drop the subscript on w_0 for now and recall that

$$w^{m-2} \int_0^1 \frac{\sigma^{m-1}}{anw^{p-2} - \epsilon(1 - \sigma^2)} d\sigma = \frac{y_0}{mn},$$

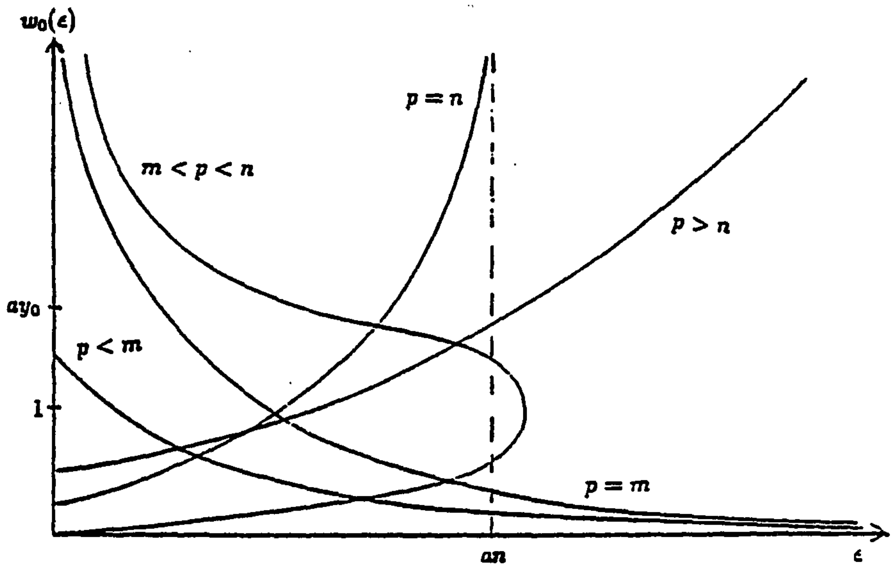

Figure 3.10: $w_0(\epsilon)$ where $-(w^m)'(0) = aw^p(0), a > 1, 0 < m < n$, and $w_0 = a^{1/(m-p)}$

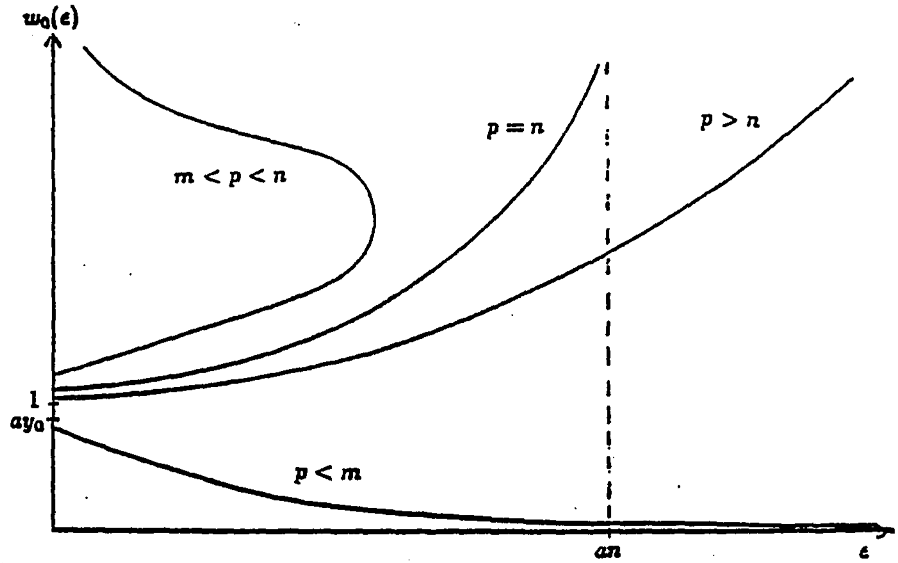

Figure 3.11: $w_0(\epsilon)$ where $-(w^m)'(0) = aw^p(0), 0 < a \leq 1, 0 < m < n$, and $w_0(0) = a^{1/(m-p)}$

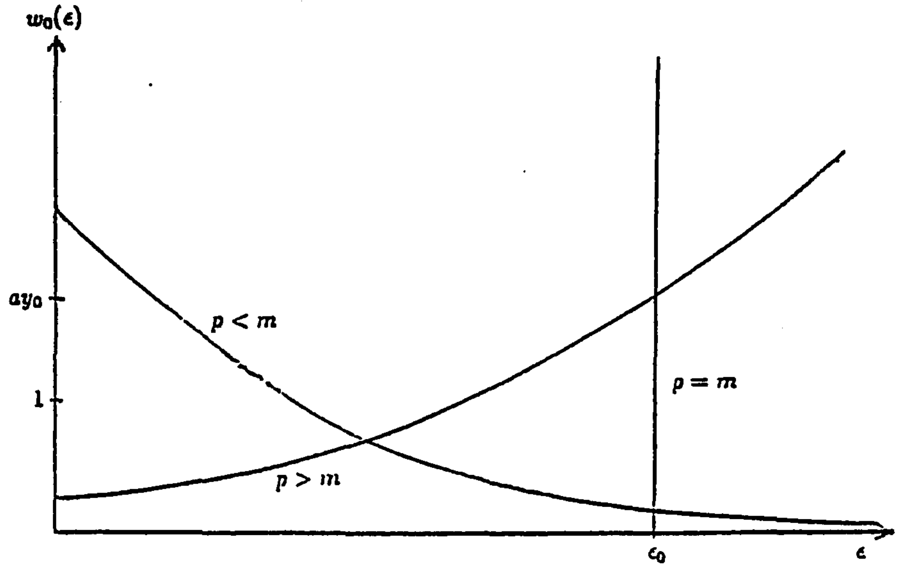

Figure 3.12: $w_0(\epsilon)$ where $-(w^m)'(0) = aw^p(0), a > 1$, $0 < m = n$, $w_0(0) = a^{1/(m-p)}$.

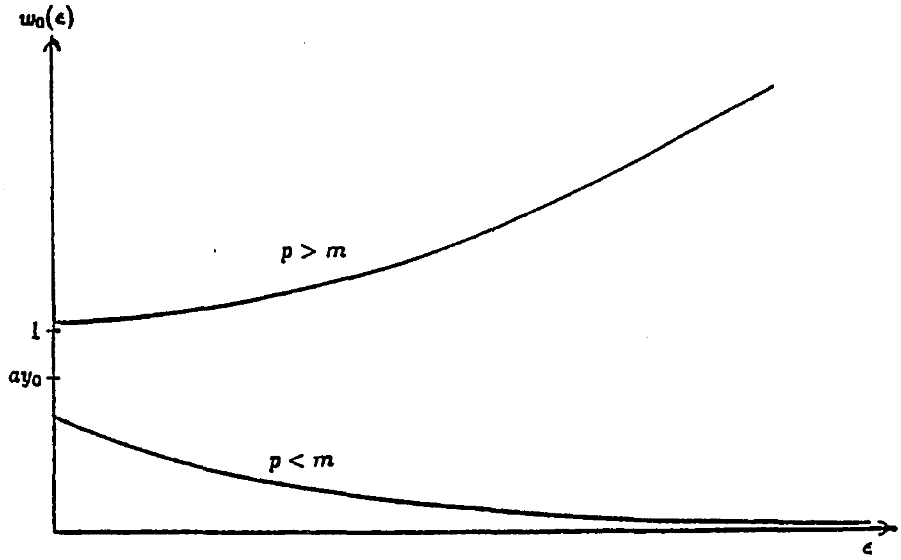

Figure 3.13: $w_0(\epsilon)$ where $-(w^m)'(0) = aw^p(0), 0 < a \leq 1$, $0 < m = n$, and $w_0(0) = a^{1/(m-p)}$

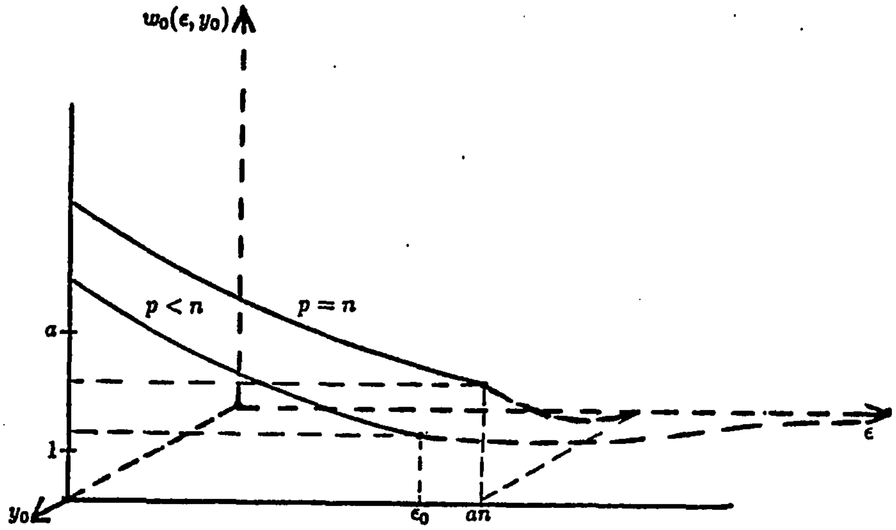

Figure 3.14: $w_0(\epsilon)$ where $-(w^m)'(0) = aw^p(0), a \geq m/(m-n)$, $0 < n < m$, $0 < p \leq n$, and $w_0(0) = a^{1/(m-p)}$.

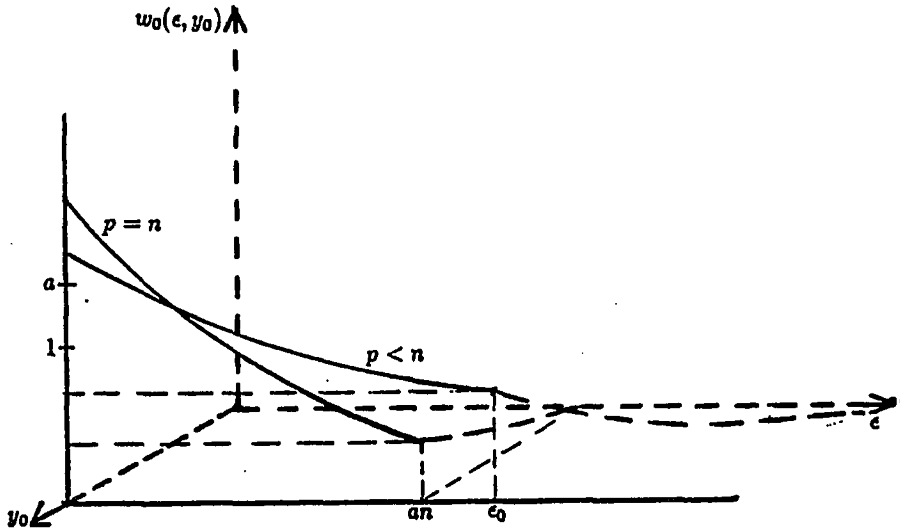

Figure 3.15: $w_0(\epsilon)$ where $-(w^m)'(0) = aw^p(0), 1 < a < m/(m-n)$, $0 < n < m$, $0 < p \leq n$, and $w_0(0) = a^{1/(m-p)}$

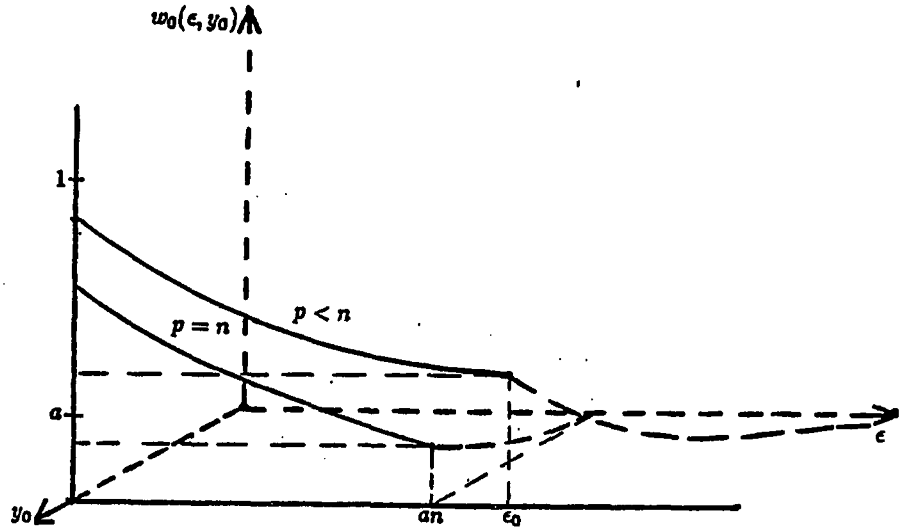

Figure 3.16: $w_0(\epsilon)$ where $-(w^m)'(0) = aw^p(0), 0 < a \leq 1,\ 0 < n < m,\ 0 < p \leq n$, and $w_0(0) = a^{1/(m-p)}$

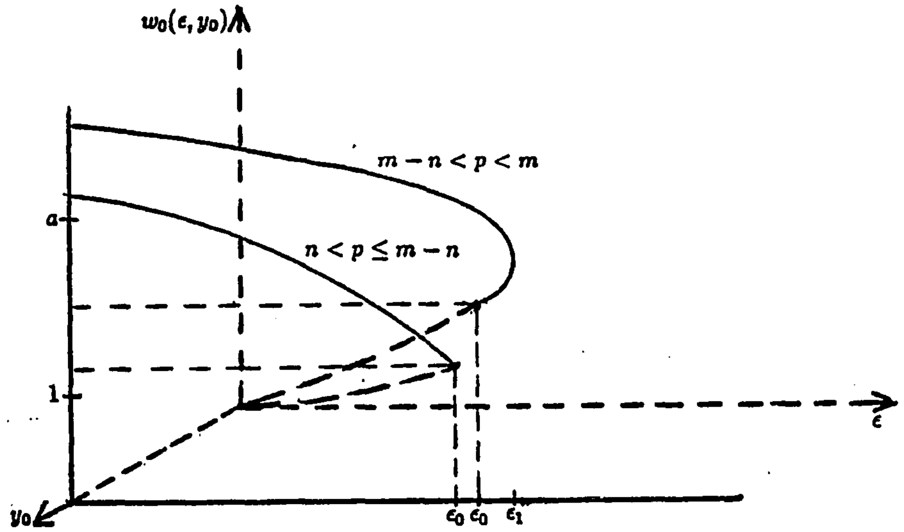

Figure 3.17: $w_0(\epsilon)$ where $-(w^m)'(0) = aw^p(0), a \geq m/(m-n),\ 0 < n < m$, $n < p < m$, and $w_0(0) = a^{1/(m-p)}$.

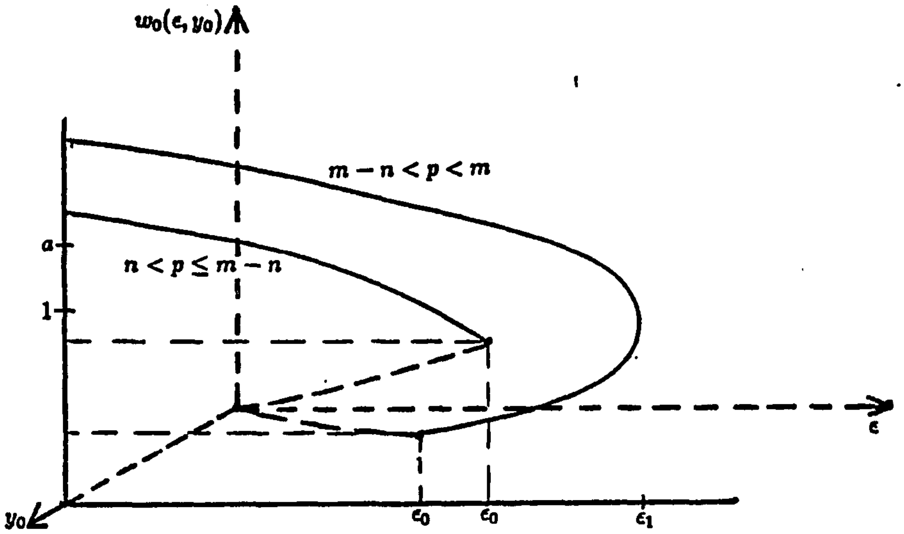

Figure 3.18: $w_0(\epsilon)$ where $-(w^m)'(0) = aw^p(0), 1 < a < m/(m-n)$, $0 < n < m$, $n < p < m$, and $w_0(0) = a^{1/(m-p)}$

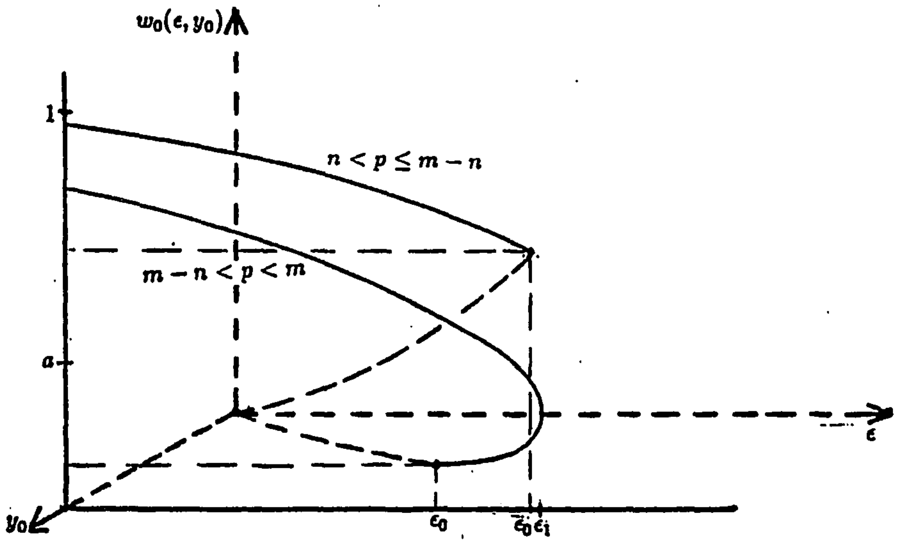

Figure 3.19: $w_0(\epsilon)$ where $-(w^m)'(0) = aw^p(0), 0 < a \leq 1$, $0 < n < m$, $n < p < m$, and $w_0(0) = a^{1/(m-p)}$

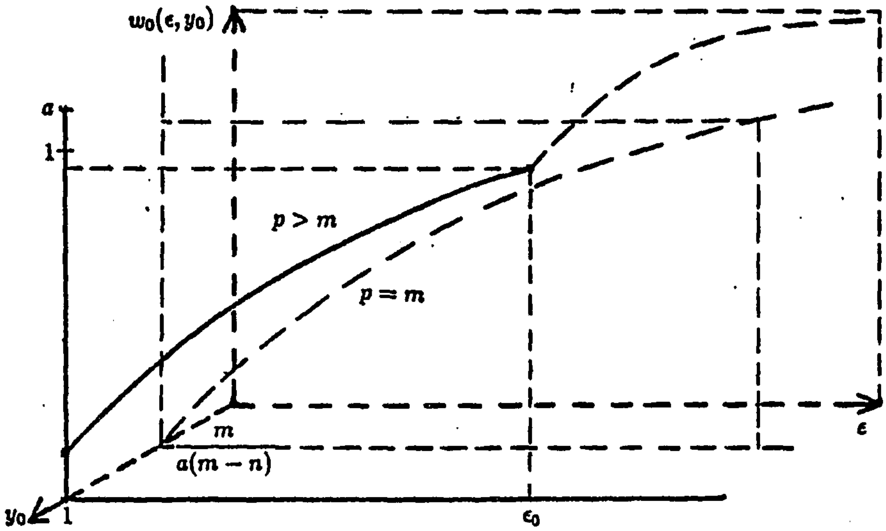

Figure 3.20: $w_0(\epsilon)$ where $-(w^m)'(0) = aw^p(0), a \geq m/(m-n)$, $0 < n < m$, $p \geq m$, and $w_0(0) = a^{1/(m-p)}$.

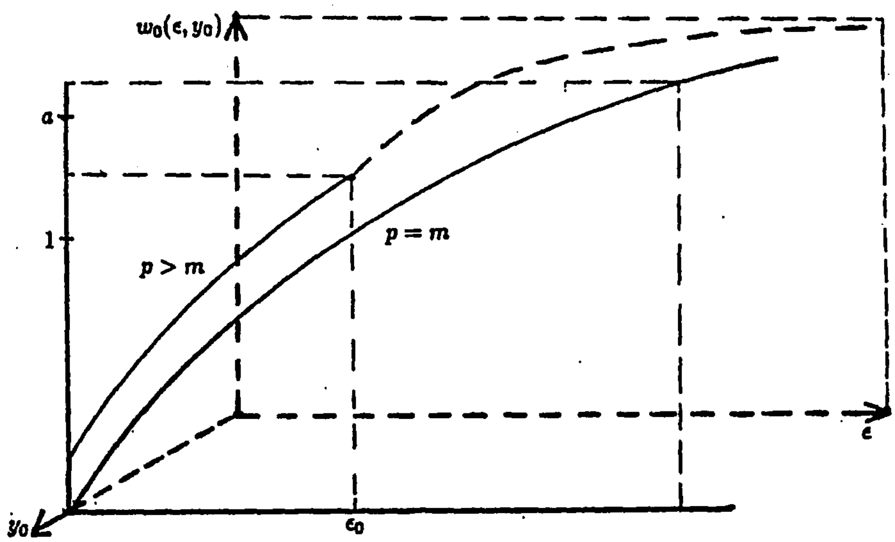

Figure 3.21: $w_0(\epsilon)$ where $-(w^m)'(0) = aw^p(0), 1 < a < m/(m-n)$, $0 < n < m$, $p \geq m$, and $w_0(0) = a^{1/(m-p)}$

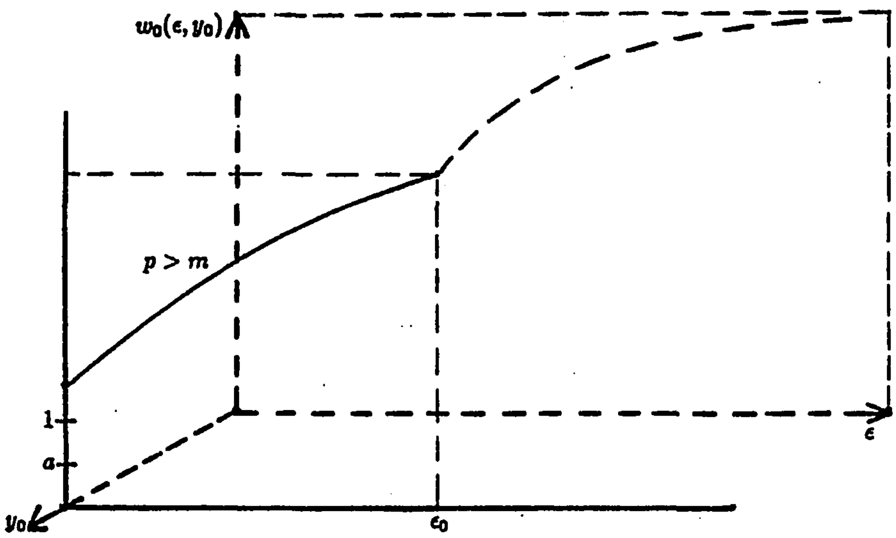

Figure 3.22: $w_0(\epsilon)$ where $-(w^m)'(0) = aw^p(0), 0 < a \leq 1$, $0 < n < m$, $p \geq m$, and $w_0(0) = a^{1/(m-p)}$

for $0 < \epsilon < anw^{p-2}$. Differentiating this expression with respect to ϵ in the case $p = 2$ gives

$$(m-2)w_\epsilon \int_0^1 \frac{\sigma^{m-1}}{an - \epsilon(1-\sigma^2)} d\sigma = -\int_0^1 \frac{(1-\sigma^2)\sigma^{m-1}}{[an - \epsilon(1-\sigma^2)]} d\sigma.$$

From this it follows that

$$w_\epsilon \begin{cases} > 0 & \text{if } m < 2 \\ < 0 & \text{if } m > 2. \end{cases}$$

Furthermore,

$$\begin{aligned} \lim_{\epsilon \to an^-} w^{m-2}(\epsilon) &= \frac{y_0}{mn} \lim_{\epsilon \to an^-} \left[\int_0^1 \frac{\sigma^{m-1}}{an - \epsilon(1-\sigma^2)} d\sigma \right]^{-1} \\ &= \begin{cases} 0 & \text{if } m < 2 \\ a(m-2)y_0/m & \text{if } m > 2, \end{cases} \end{aligned}$$

and so

$$\lim_{\epsilon\to an^-} w(\epsilon) = \begin{cases} \infty & \text{if } m < 2 \\ \left[\dfrac{a(m-2)y_0}{m}\right]^{1/(m-2)} & \text{if } m > 2. \end{cases}$$

In the case of $p \neq 2$, we have

$$\frac{1}{\epsilon} w^{m-2} G(\alpha) = \frac{y_0}{mn},$$

which upon differentiation yields

$$[(m-2)G(\alpha) + (p-2)(\alpha+1)G'(\alpha)]w_\epsilon = \frac{w}{\epsilon}\frac{d}{d\alpha}[(\alpha+1)G(\alpha)] < 0.$$

Therefore, we immediately see that

$$w_\epsilon \begin{cases} < 0 & \text{if } m \geq 2, p < 2 \\ > 0 & \text{if } m \leq 2, p > 2. \end{cases}$$

Moreover,

$$\lim_{\epsilon\to\infty} w(\epsilon) = \begin{cases} 0 & \text{if } m = 2, p < 2 \\ \infty & \text{if } m \leq 2, p > 2 \end{cases}$$

follows from $\epsilon < anw^{p-2}(\epsilon)$. In the case $m > 2, p < 2$, there exists an $\epsilon_0 > 0$ such that $w(\epsilon)$ exists on $(0, \epsilon_0]$ but not on (ϵ_0, ∞). Explicitly, ϵ_0 is the number for which

$$\frac{\epsilon_0}{nm}\left(\frac{an}{\epsilon_0}\right)^{(m-2)/(p-2)} y_0 = \frac{1}{m-2},$$

and thus we have

$$\lim_{\epsilon\to\epsilon_0^-} (\alpha+1)^{(m-2)/(p-2)} G(\alpha) = \frac{1}{m-2}.$$

So it must be the case that

$$\lim_{\epsilon\to\epsilon_0^-} \alpha = 0,$$

which implies

$$\lim_{\epsilon \to \epsilon_0^-} w(\epsilon) = \left(\frac{\epsilon_0}{an}\right)^{1/(p-2)}.$$

To continue, we use Lemma 3.7 to rewrite (3.13) as

$$\frac{m-2}{2\alpha} R(\alpha) w_\epsilon = \frac{w}{\epsilon}\frac{d}{d\alpha}[(\alpha+1)G(\alpha)] < 0 \tag{3.14}$$

for $m \neq 2, p \neq 2$. Recalling the properties of $R(\alpha)$, the following can be seen:

$$w_\epsilon \begin{cases} > 0 & \text{if } p \geq m > 2 \\ < 0 & \text{if } 2 < p \leq m-2 \\ < 0 & \text{if } p \leq m < 2. \end{cases}$$

Also, as done above, it follows that

$$\lim_{\epsilon \to \epsilon_0^-} = \begin{cases} (\epsilon_0/an)^{1/(p-2)} & \text{if } p > m > 2 \text{ or if } 2 < p \leq m-2 \\ \infty & \text{if } p < m < 2, \end{cases}$$

and in the case $p = m \neq 2$,

$$\lim_{\epsilon \to \infty} w^{p-2}(\epsilon) = \infty$$

implies that

$$\lim_{\epsilon \to \infty} w(\epsilon) = \begin{cases} 0 & \text{if } p = m < 2 \\ \infty & \text{if } p = m > 2. \end{cases}$$

When $m < p < 2$, we define $\epsilon_1 > 0$ by the equation

$$\frac{\epsilon_1}{mn}\left(\frac{an}{\epsilon_1}\right)^{(m-2)/(p-2)} y_0 = \min_{\alpha>0}(\alpha+1)^q G(\alpha).$$

Now $(\alpha+1)^q G(\alpha) = (\epsilon/mn)(an/\epsilon)^q y_0$ has two solutions if $0 < \epsilon < \epsilon_1$, one solution if $\epsilon = \epsilon_1$, and none if $\epsilon > \epsilon_1$. So $\epsilon = \epsilon(w)$ is bounded by ϵ_1. Also

$$\lim_{\epsilon \to 0+} (\alpha+1)^q G(\alpha) = \infty,$$

and

$$w(0) = (ay_0)^{1/(m-p)},$$

which implies that one of the solution branches must satisfy $\lim_{\epsilon \to 0^+} w(\epsilon) = \infty$. Hence $\epsilon(w)$ is defined on $[(ay_0)^{1/(m-p)}, \infty)$ with $\epsilon'(w) > 0$ on $[(ay_0)^{1/(m-p)}, w(\epsilon_1))$ and $\epsilon'(w) < 0$ on $(w(\epsilon_1, \infty)$.

Similarly, when $m, p > 2$ are such that $m - 2 < p < m$, then there are two solutions of $(\alpha + 1)^q G(\alpha) = (\epsilon/mn)(an/\epsilon)^q y_0$ if $\epsilon_0 \le \epsilon < \epsilon_1$, one solution if $\epsilon = \epsilon_1$ or $\epsilon < \epsilon_0$, and no solutions if $\epsilon > \epsilon_1$. (Here ϵ_0 and ϵ_1 are defined as above.) Also one of the branches of $w(\epsilon)$ must satisfy

$$\lim_{\epsilon \to \epsilon_0^+} w(\epsilon) = \left(\frac{\epsilon_0}{an}\right)^{1/(p-2)} < (ay_0)^{1/(m-p)} = w(0).$$

Furthermore, $w(\epsilon) > (\epsilon/an)^{1/(p-2)}$ for all ϵ on which it is defined. So $\epsilon(w)$ is defined on $((\epsilon_0/an)^{1/(p-2)}, (ay_0)^{1/(m-p)}]$ with $\epsilon' < 0$ on $(w(\epsilon_1), (ay_0)^{1/(m-p)}]$ and $\epsilon' > 0$ on $((\epsilon_0/an)^{1/(m-p)}, w(\epsilon_1))$.

Finally, as we have used above, $w(0) = (ay_0)^{1/(m-p)}$ provided $m \neq p$. If $m = p \neq 2$, then from

$$\lim_{\epsilon \to 0^+} (\alpha + 1)G(\alpha) = \frac{ay_0}{m} > \frac{1}{m}$$

it follows that $\lim_{\epsilon \to 0^+} w^{p-2}(\epsilon) = 0$. Hence

$$\lim_{\epsilon \to 0^+} w(\epsilon) = \begin{cases} 0 & \text{if } p = m > 2 \\ \infty & \text{if } p = m < 2. \end{cases}$$

4 UNIQUENESS, COMPARISON, AND STABILITY

In this chapter we restrict attention to nonlinearities ϕ, f, and g for which we have obtained uniqueness and comparison results for problems (A1) and (B1). The following additional hypotheses will be needed:

(H4) $\phi, f \in C^1(R)$.

(H5) For each $M > 0$, there exists a constant C (which may depend on M) such that

$$| f'(u) + g'(u) | \leq C\phi'(u)$$

for all $u \in (0, M]$.

(H6) For each $M > 0$, there exists a constant C (which may depend on M) such that

$$| -f'(u) + g'(u) | \leq C\phi'(u)$$

for all $u \in (0, M]$.

Under the hypotheses (H1) - (H5) we shall be able to obtain such results for problem (A1); the hypotheses (H1) - (H4), (H6) will turn out to be sufficient for problem (B1).

With these further restricted types of nonlinearities, we proceed to explore the stability/instability of the stationary states found in Chapter 3. It is also established that some solutions of problems (A1) or (B1) blow up in finite time.

In section 1, we develop the above mentioned comparison results. Theorems which relate to the stability of instability of stationary solutions are gathered in section 2. Also, in section 2 the stability/instability of stationary solutions of (A) and (B) is completely analyzed for $p, n \geq m \geq 1$ and solution diagrams displaying these results are given. Finally in section 3 we develop blow up results which are parallel to those in [19, section 3].

4.1 Uniqueness and Comparison

For the developments of this section we will need to use the solution of (A1) (and eventually, the solution of (B1)) which was constructed in the existence proof. Let us call this solution, so constructed, the *limit solution.* We shall prove comparison principles between nonnegative sub- and supersolutions of the problems (A1) and (B1) and the associated limit solution. It will follow from this that the limit solution and the *weak solution* (introduced in Chapter 2) must be the same. Thus the desired uniqueness and comparison principles will appear as corollaries to these above results. Our technique closely parallels some work of Diaz and Kersner [5].

Let us mention here that, for the remainder of this chapter, when discussing problem (A1), we will always assume (H1) - (H5). Likewise, when we are discussing problem (B1), (H1) - (H4) and (H6) will be in force. Also, when we are discussing any solutions, subsolutions, or supersolutions, all times (t or T) under consideration will be understood to lie strictly inside the maximum time interval of existence for those solutions. That is, our results are not necessarily global.

Theorem 4.1 (A) *Let $u(x,t)$ denote the limit solution of (A1) on Q_T.*

(i) *If $\bar{u}(x,t)$ is a nonnegative supersolution of (A1) on Q_T, then*

$$\int_0^1 [u(x,t) - \bar{u}(x,t)]^+ dx \le C \int_0^1 [u(x,0) - \bar{u}(x,0)]^+ dx$$

for all $t \in [0,T]$.

(ii) *If $\underline{u}(x,t)$ is a nonnegative subsolution of (A1) on Q_T, then*

$$\int_0^1 [\underline{u}(x,t) - u(x,t)]^+ dx \le C \int_0^1 [\underline{u}(x,0) - u(x,0)]^+ dx$$

for all $t \in [0,T]$.

Here C is a constant which depends on T.

A similar result holds for problem (B1).

Theorem 4.2 (B) *Let $u(x,t)$ denote the limit solution of (B1) on Q_T.*

(i) *If $\bar{u}(x,t)$ is a nonnegative supersolution of (B1) on Q_T, then*

$$\int_0^1 [u(x,t) - \bar{u}(x,t)]^+ dx \le C \int_0^1 [u(x,0) - \bar{u}(x,0)]^+ dx$$

for all $t \in [0,T]$.

(ii) *If $\underline{u}(x,t)$ is a nonnegative subsolution of (B1) on Q_T, then*

$$\int_0^1 [\underline{u}(x,t) - u(x,t)]^+ dx \le C \int_0^1 [\underline{u}(x,0) - u(x,0)]^+ dx$$

for all $t \in [0,T]$.

Here C is a constant which depends on T.

Before presenting the proof of these theorems let us first state some important corollaries of them. We have the following comparison results.

Corollary 4.1 (A) *Let $u(x,t)$ and $v(x,t)$ be a nonnegative subsolution and a nonnegative supersolution, respectively, of (A1) on Q_T. If $u(x,0) \leq v(x.0)$ a.e on $(0,1)$, then for each $t \in (0,T], u(x,t) \leq v(x,t)$ on $[0,1]$.*

Corollary 4.2 (B) *Let $u(x,t)$ and $v(x,t)$ be a nonnegative subsolution and a nonnegative supersolution, respectively, of (B1) on Q_T. If $u(x,0) \leq v(x.0)$ a.e on $(0,1)$, then for each $t \in (0,T], u(x,t) \leq v(x,t)$ on $[0,1]$.*

We also obtain the following continuous dependence results.

Corollary 4.3 (A) *Let $u(x,t)$ and $v(x,t)$ be solutions of (A1) on Q_T. Then, for each $t \in [0,T]$,*

$$\int_0^1 \mid u(x,t) - v(x,t) \mid dx \leq C \int_0^1 \mid u(x,0) - v(x,0) \mid dx.$$

Corollary 4.4 (B) *Let $u(x,t)$ and $v(x,t)$ be solutions of (B1) on Q_T. Then, for each $t \in [0,T]$,*

$$\int_0^1 \mid u(x,t) - v(x,t) \mid dx \leq C \int_0^1 \mid u(x,0) - v(x,0) \mid dx.$$

We will prove one of the four statements in Theorems 4.1(A) and 4.2(B). The remaining assertions follow in a similar manner. To this end let $\bar{u}(x,t)$ be a nonnegative supersolution of (A1) on Q_T, and let $u(x,t)$ be the limit solution of (A1) on Q_T, i.e.,

$$u(x,t) = \lim_{\kappa \to 0^+} u_\kappa(x,t)$$

as in Appendix A. For each function, u_κ, in this limit we have

$$\begin{aligned}\int_0^1 u_\kappa(x,t)\xi(x,t)dx &= \int_0^1 [u(x,0)+\kappa]\xi(x,0)dx \\ &\quad + \int_0^t \int_0^1 \{u_\kappa \xi_s - [\phi(u_\kappa)_x + f(u_\kappa)]\xi_x\}\, dxds \\ &\quad + \int_0^t [g(u_\kappa(1,s)) - g(\kappa) + f(u_\kappa(1,s))]\xi(1,s)ds,\end{aligned}$$

for every $t \in [0,T]$ and every $\xi \in P_{\{0\}}(\overline{Q_T})$. Now if we take such a ξ with the additional property that $\xi \in C^{2,1}(\overline{Q_T})$, integrate by parts in both the above equality and also in the inequality satisfied by $\bar{u}(x,t)$, and subtract the two resultant expressions, then

$$\int_0^1 [u_\kappa(x,t) - \bar{u}(x,t)]\xi(x,t)dx$$

$$\begin{aligned} \leq\ & \int_0^1 [u(x,0) + \kappa - \bar{u}(x,0)]\xi(x,0)dx \\ &+ \int_0^t \int_0^1 (u_\kappa - \bar{u}) \{\xi_s + \Phi_\kappa \xi_{xx} - F_\kappa \xi_x\} \, dxds \\ &+ \int_0^t (u_\kappa - \bar{u})(1,s) \, [H_\kappa(s)\xi(1,s) - \Phi_\kappa(1,s)\xi_x(1,s)] \, ds \\ &\int_0^t \{-g(\kappa)\xi(1,s) + [\phi(\kappa) - \phi(\bar{u}(0,s))]\xi_x(0,s)\} \, ds, \end{aligned}$$

where

$$\Phi_\kappa(x,t) \equiv \int_0^1 \phi'[\theta u_\kappa(x,t) + (1-\theta)\bar{u}(x,t)]d\theta,$$
$$F_\kappa(x,t) \equiv \int_0^1 f'[\theta u_\kappa(x,t) + (1-\theta)\bar{u}(x,t)]d\theta,$$

and

$$H_\kappa(t) \equiv \int_0^1 g'[\theta u_\kappa(1,t) + (1-\theta)\bar{u}(1,t)]d\theta + F_\kappa(1,t).$$

Let

$$M \equiv \max\left\{ \| \bar{u} \|_{L^\infty(Q_T)}, \max_{0<\kappa<1} \| u_\kappa \|_{L^\infty(Q_T)} \right\}.$$

Observe that we have

$$\frac{\kappa}{2M} \min_{\kappa/2 \leq v \leq M} \phi'(v) \leq \Phi_\kappa \leq \max_{0 \leq v \leq M} \phi'(v)$$

and

$$| F_\kappa | \leq \max_{0 \leq v \leq M} | f'(v) |$$

on Q_T. The lower bound for Φ_K is found from the following analysis.

(i) If $\bar{u}(x,t) \geq \kappa/2$, then

$$\Phi_K(x,t) \geq \min_{\kappa/2 \leq v \leq M} \phi'(v).$$

(ii) If $0 \leq \bar{u}(x,t) \leq \kappa/2$, then

$$\begin{aligned}\Phi_K(x,t) &= \frac{\phi(u_K(x,t)) - \phi(\bar{u}(x,t))}{u_K(x,t) - \bar{u}(x,t)} \\ &\geq \frac{1}{M}\left[\phi(\kappa) - \phi\left(\frac{\kappa}{2}\right)\right] \\ &= \frac{\kappa}{2M}\phi'(\theta)\end{aligned}$$

for some $\theta \in (\kappa/2, \kappa)$.

The upper bounds follow from (H4).

Thus, for fixed $\kappa \in (0,1)$, we may define sequences $\{\Phi_{K,n}\}$ and $\{F_{K,n}\}$ and constants μ, which is independent of n and κ, and ν, which is independent of n, in such a way that

1. $\Phi_{K,n}, F_{K,n} \in C^\infty(\overline{Q_T})$,

2. $\Phi_{K,n} \to \Phi_K$ and $F_{K,n} \to F_K$ as $n \to \infty$ a.e. in Q_T,

3. $0 < \nu \leq \Phi_{K,n} \leq \mu$ in Q_T,

4. $| F_{K,n} | \leq \mu$ in Q_T, and $F_{K,n}(0,t) = 0$.

Let us choose $\chi \in C^\infty([0,1])$ such that supp$\chi \subset (0,1)$ and $0 \leq \chi \leq 1$, and $\psi \in C^\infty([0,T])$.

Now let ξ_n denote the solution to the "adjoint problem"

$$
\text{(AP)} \qquad
\begin{array}{ll}
\xi_s + \Phi_{\kappa,n}\xi_{xx} - F_{\kappa,n}\xi_x = 0 & \text{on } Q_t \\
\xi(0,s) = 0 & \text{on } (0,t) \\
\xi_x(1,s) = \psi(s)\xi(1,s) & \text{on } (0,t) \\
\xi(x,t) = \chi(x) & \text{on } [0,1].
\end{array}
$$

With the test function $\xi = \xi_n$, we thus have

$$
\int_0^1 [u_\kappa(x,t) - \bar{u}(x,t)]\chi(x)dx
$$

$$
\begin{aligned}
\leq\; & \int_0^1 [u(x,0) + \kappa - \bar{u}(x,0)]\xi(x,0)dx \\
& + \int_0^t \int_0^1 (u_\kappa - \bar{u}) \left\{ \left[\Phi_\kappa - \Phi_{\kappa,n}\right] \xi_{xx} - \left[F_\kappa - F_{\kappa,n}\right] \xi_x \right\} dxds \\
& + \int_0^t (u_\kappa - \bar{u})(1,s)\xi(1,s) \left[H_\kappa(s) - \psi(s)\Phi_\kappa(1,s)\right] ds \\
& \int_0^t \{-g(\kappa)\xi(1,s) + [\phi(\kappa) - \phi(\bar{u}(0,s))]\xi_x(0,s)\} \, ds.
\end{aligned}
$$

(Recall that $\xi(0,t) = 0$ and $\xi \geq 0$, so we have $\xi_x(0,t) \geq 0$.) By the analysis in Appendix B, there exist constants C_1, C_2, C_3 such that

$$
\| \xi_n \|_{L^\infty(Q_T)} \leq C_1, \; \| \xi_{nx} \|_{L^\infty(Q_T)} \leq C_2, \; \| \xi_{nxx} \|_{L^2(Q_T)} \leq C_3.
$$

Here C_1 depends only on

$$
\mu, \; \| \psi \|_{L^\infty(0,T)}, \; t;
$$

C_2 depends on

$$
C_1, \; \| \chi' \|_{L^\infty(0,1)};
$$

C_3 depends on

$$
C_2, \; \nu, \; \| \psi' \|_{L^\infty(0,T)} \, .
$$

Therefore, applying appropriate bounds on ξ_n and its derivatives and then letting $n \to \infty$ in the above inequality, we have

$$\int_0^1 [u_\kappa(x,t) - \bar{u}(x,t)]\chi(x)dx$$

$$\begin{aligned}
\leq\ & C_1 \int_0^1 [u(x,0) + \kappa - \bar{u}(x,0)]^+ dx \\
& + C_1 \int_0^t \mid u_\kappa(1,s) - \bar{u}(1,s) \mid\mid H_\kappa(s) - \psi(s)\Phi_\kappa(1,s) \mid ds \\
& C_2 \int_0^t \{\mid g(\kappa) \mid + \phi(\kappa)\}\, ds,
\end{aligned}$$

We now take a sequence of such functions ψ, say $\{\psi_q\}$, which converge to $H_\kappa/\Phi_\kappa(1,\cdot)$ a.e. on $(0,t)$ and are bounded independent of q and κ. The independence of κ in this bound is possible by (H5). Thus, letting $q \to \infty$ and then $\kappa \to 0^+$, it follows that

$$\int_0^1 [u(x,t) - \bar{u}(x,t)]\chi(x)dx \leq C_1 \int_0^1 [u(x,0) - \bar{u}(x,0)]^+ dx.$$

Since this inequality holds for every $\chi \in C^\infty([0,1])$ with compact support in $(0,1)$ and $0 \leq \chi \leq 1$ and the constant C_1 does not depend on χ, the theorem now follows upon consideration of a sequence of such functions χ which converge a.e. on $(0,1)$ to the function given by

$$\begin{cases} 1 & \text{if } u(x,t) > \bar{u}(x,t) \\ 0 & \text{if } u(x,t) \leq \bar{u}(x,t). \end{cases} \quad \square$$

4.2 Stability/Instability of Stationary Solutions

We now present some lemmas which are special to the case of $\phi(u) = u^m, m \geq 1$. The essential ingredient in these results is that (under appropriate hypotheses on f and g) if $w(x)$ is a stationary solution of (A1), then, for $\sigma \in (0,1)$, $(1-\sigma)w(x)$ is a supersolution and $(1+\sigma)w(x)$ is a subsolution of (A1). There are corresponding statements for stationary solutions of (B1). Such facts together with the comparison theorems of the previous section, allow us to discover the stability or instability of the stationary states. We now make the notion of stability more precise in the following definition. For this purpose, define the class of functions

$$S \equiv \{v \in C([0,1]) : \phi(v) \in C^1([0,1]),\ v(0)=0,\ v \geq 0 \text{ on } [0,1]\}.$$

Definition 4.1 (Stability for (A1)) *A stationary solution, $w(x)$, of (A1) is stable from above if for any given $\epsilon > 0$ there exists function, $z \in S$, with $\phi(z)_x > \phi(w)_x$ on $[0,1]$ such that the following is true: If $u(x,t)$ is a nonnegative solution of (A1) with $w(x) \leq u(x,0) \leq z(x)$ a.e. on $(0,1)$, then*

$$\| u(\cdot,t) - w(\cdot) \|_{L^\infty(0,1)} < \epsilon$$

for all $t \geq 0$ and

$$\lim_{t\to\infty} u(x,t) = w(x)$$

for each $x \in [0,1]$. If the statement above holds with all the inequalities reversed, then $w(x)$ is stable from below.

The stationary solution $w(x)$ is stable if it is both stable from above and below.

The analogous notions for problem (B1) are defined as above with the obvious modifications. This type of stability is often (and more accurately) referred to as asymptotic stability. The notion of stability above is the same as used in [19] and is in the spirit of some works on strongly order preserving systems [14,25].

Lemma 4.1 (A) *Assume that $\phi(u) = u^m$, $m \geq 1$, and that $f'(u)/\phi'(u)$ and $g(u)/u\phi'(u)$ are nondecreasing on $(0,\infty)$. Let $w(x)$ be a nontrivial, nonnegative stationary solution of (A1), and let $u(x,t)$ be a solution of (A1) on Q_T.*

(i) *If* $0 \le u(x,0) \le (1-\sigma)w(x)$ *a.e. on* $(0,1)$ *for some* $\sigma \in [0,1]$*, then* $0 \le u(x,t) \le (1-\sigma)w(x)$ *on* $[0,1] \times [0,\infty)$*, i.e., the solution is* global.

(ii) *If* $u(x,0) \ge (1+\sigma)w(x)$ *a.e. on* $(0,1)$ *for some* $\sigma > 0$*, then* $u(x,t) \ge (1+\sigma)w(x)$ *on* $\overline{Q_T}$.

Proof:

(i) Set $v(x) = (1-\sigma)w(x)$. Then $\phi(v) = (1-\sigma)^m\phi(w)$, so

$$\begin{aligned} \phi(v)_{xx} + f(v)_x &= (1-\sigma)^m\phi(w)_{xx} + (1-\sigma)w_x f'((1-\sigma)w) \\ &= (1-\sigma)^m\phi'(w)w_x\left[\frac{f'((1-\sigma)w)}{\phi'((1-\sigma)w)} - \frac{f'(w)}{\phi'(w)}\right] \\ &\le 0, \end{aligned}$$

on $(0,1]$. Also,

$$\begin{aligned} &\phi(v)_x(1,t) - g(v)(1,t) \\ &= (1-\sigma)^m\phi'(w)w\left[\frac{g(w)}{w\phi'(w)} - \frac{g((1-\sigma)w)}{(1-\sigma)w\phi'((1-\sigma)w)}\right](1) \\ &\ge 0. \end{aligned}$$

Thus, multiplying by $\xi \in P_{\{0\}}(\overline{Q_T})$ and integrating by parts over Q_t for $t \in [0,T]$, we have

$$-\int_0^t\int_0^1 [\phi(v)_x + f(v)]\xi_x dx ds + \int_0^t [\phi(v)_x + f(v)]\xi(1,s)ds \le 0,$$

from which it follows that v is a supersolution of (A1) on Q_T. The conclusion now follows by Corollary 4.3(A) and Theorem 2.2(A).

(ii) Let $v(x) = (1+\sigma)w(x)$. It follows from calculations similar to those in (i) that v is a subsolution of (A1) on Q_T. □

There are parallel results for (B1), and, although they only apply to (B) in the cases for which we have a uniqueness theorem when $p = m$, we present them now. Their proofs are essentially the same arguments as in the previous lemma.

Lemma 4.2 (B) *Assume that $\phi(u) = u^m$, $m \geq 1$ and that $f'(u)/\phi'(u)$ and $-g(u)/u\phi'(u)$ are nondecreasing on $(0, \infty)$. Let $w(x)$ be a nontrivial, nonnegative stationary solution of (B1), and let $u(x,t)$ by a solution of (B1) on Q_T.*

(i) *If $0 \leq u(x,0) \leq (1+\sigma)w(x)$ a.e. on $(0,1)$ for some $\sigma > 0$, then $0 \leq u(x,t) \leq (1+\sigma)w(x)$ on $[0,1] \times [0,\infty)$, i.e., the solution is global.*

(ii) *If $u(x,0) \geq (1-\sigma)w(x)$ a.e. on $(0,1)$ for some $\sigma \in (0,1)$, then $u(x,t) \geq (1-\sigma)w(x)$ on $\overline{Q_T}$.*

To apply these lemmas and get statements about stability or instability of solutions, we first establish that if $u(x,t)$ is a solution of (A1) such that

$$\lim_{t \to \infty} u(x,t) = \hat{u}(x)$$

for all $x \in [0,1]$ and some $\hat{u} \in L^\infty(0,1)$, then $\hat{u}$ is a stationary solution of (A1). (A similar statement will also be established for such solutions of (B1).) The existence of this limit for $u(x,0) = (1-\sigma)w(x)$ and $u(x,0) = (1+\sigma)w(x)$ is a consequence of the comparison theorems. From this, the desired results on stability/instability will follow. In this direction, we first show that solutions of (A1) or (B1) satisfy a certain integral equation.

Lemma 4.3 (A) *Let $u(x,t)$ be a solution of (A1) on Q_T. Then*

$$\int_0^t \left\{ \phi(u(x,s)) + \int_0^x f(u(y,s))dy \right\} ds$$

$$
\begin{aligned}
= & \int_0^t x[g(u(1,s)) + f(u(1,s))]ds \\
& - \int_0^x \int_y^1 [u(z,t) - u_0(z)]dzdy
\end{aligned}
$$

for $(x,t) \in \overline{Q_T}$.

Proof:

This is proved much in the same spirit as the similar results for stationary solutions. By the definition of a solution of (A1),

$$
\begin{aligned}
\int_0^1 u\xi(x,s) \mid_{s=0}^{s=t} dx &= \int_0^t \int_0^1 \{u\xi_s + \phi(u)\xi_{xx} - f(u)\xi_x\} \, dxds \\
&+ \int_0^t \{[g(u) + f(u)]\xi(1,s) - \phi(u)\xi_x(1,s)\} \, ds
\end{aligned}
$$

for every $\xi \in P_{\{0\}}(\overline{Q_T}) \cap C^{2,1}(\overline{Q_T})$. Define

$$
\begin{aligned}
H(x,t) &\equiv \int_0^x f(u(y,t))dy - x[g(u(1,t)) + f(u(1,t))], \\
\zeta(x) &\equiv \int_0^x \int_y^1 \chi(z)dz, \\
\xi(x,t) &\equiv \theta(t)\zeta(x), \text{and} \\
K(x,t) &\equiv -\int_0^x \int_y^1 \left\{ \int_0^t u(z,s)\theta'(s)ds - [u(z,t)\theta(t) - u_0(z)\theta(0)] \right\} dzdy,
\end{aligned}
$$

where $\chi \in C^\infty([0,1])$ and $\theta \in C^\infty([0,T])$ are both nonnegative. Then the above identity can be written as

$$
\int_0^1 K_{xx}\zeta dx + \int_0^t \int_0^1 \left[\phi(u)\theta\zeta'' - H_x\theta\zeta'\right] dxds = 0
$$

which, upon integration by parts, is equivalent to

$$
\int_0^1 \left\{ K(x,t) + \int_0^t [\phi(u(x,s)) + H(x,s)] \, \theta(s)ds \right\} \chi(x)dx = 0.
$$

Since χ was arbitrarily chosen, it follows that

$$K(x,t) + \int_0^t [\phi(u(x,s)) + H(x,s)]\,\theta(s)ds = 0$$

for $x \in [0,1]$. Manipulating terms in $K(x,t)$, we see that if $\theta(t) = 0$, then

$$K(x,t) = -\int_0^t \left\{ \int_0^x \int_y^1 [u(z,s) - u_0(z)]dz \right\} \theta'(s)ds.$$

Hence,

$$\int_0^t \left\{ [\phi(u(x,s)) + H(x,s)]\theta(s) - \theta'(s) \int_0^x \int_y^1 [u(z,s) - u_0(z)]dzdy \right\} ds = 0,$$

from which another integration by parts yields

$$\int_0^t \left\{ \int_0^s [\phi(u(x,r)) + H(x,r)]dr + \int_0^x \int_y^1 [u(z,s) - u_0(z)]dzdy \right\} \theta'(s)ds = 0.$$

Since $\theta' \geq 0$ can be picked arbitrarily, the lemma now follows. □

Now suppose f is monotone and $\hat{u}(x) = \lim_{t\to\infty} u(x,t)$, where $\hat{u} \in L^\infty(0,1)$ and $u(x,t)$ is monotone in t. (Alternatively, we could simply assume that $u(x,t)$ is uniformly bounded and $\hat{u}(x) = \lim_{t\to\infty} u(x,t)$.) Then, for fixed $x \in [0,1]$,

$$\lim_{t\to\infty} \int_0^x \int_y^1 u(z,t)dz = \int_0^x \int_y^1 \hat{u}(z)dz,$$

and so

$$\frac{\partial}{\partial t} \int_0^x \int_y^1 u(z,t_n)dz \to 0$$

for some sequence $t_n \to \infty$. The lemma above implies

$$\phi(\hat{u}(x)) + \int_0^x f(\hat{u}(y))dy - x[g(\hat{u}(1)) + f(\hat{u}(1))] = 0$$

from which it follows that $\hat{u}$ is a stationary solution of (A1). Similar statements hold for problem (B1). We now collect all these results in the following lemmas.

Lemma 4.4 (B) *Let $u(x,t)$ be a solution of (B1) on Q_T. Then*

$$\begin{aligned}
&\int_0^t \left\{\phi(u(x,s)) - \int_x^1 f(u(y,s)dy\right\} ds \\
&= \int_0^t (1-x)[g(u(0,s)+f(u(0,s))]ds \\
&\quad - \int_x^1 \int_0^y [u(z,t)-u_0(z)]dzdy
\end{aligned}$$

for $(x,t) \in \overline{Q_T}$.

Lemma 4.5 *Suppose that f is monotone on $[0,\infty)$, and let $u(x,t)$ be a solution of (A1), respectively of (B1), such that $u(x,t) \to \hat{u}(x)$ monotonically as $t \to \infty$, where $\hat{u} \in L^{\infty}(0,1)$. Then $\hat{u}$ is a stationary solution of (A1), respectively of (B1).*

As a consequence of these results we have the following stability/instability theorems, the statements of which are very similar to those in [19].

Theorem 4.3 (A) *Let $\phi(u) = u^m, m \geq 1$. Assume that f'/ϕ' is strictly increasing and that g/ϕ and f are nondecreasing on $(0,\infty)$. If the roots of (3.3) are isolated, then there is at most one nontrivial, nonnegative stationary solution of (A1), $w(x)$. Furthermore, the trivial solution is stable from above, and w is unstable.*

Proof:

Observe that once we have established the first assertion, the stability result is a consequence of Lemma 4.1(A), Corollary 4.1(A), and Lemma 4.5. To see this, let $u(x,0) = (1-\sigma)w(x)$ as in Lemma 4.1(A). Then $u(x,t) \leq u(x,0)$, and hence $u(x,t)$ is monotone decreasing by Corollary 4.1(A). (This is because $v(x,s) \equiv u(x,t+s)$ is a solution of (A1) such that $v(x,0) \leq u(x,0)$ for $s \in [0,\infty)$ and any fixed $t > 0$.) Now $\lim_{t\to\infty} u(x,t)$ exists and must equal a stationary solution of (A1). Similarly, consider $u(x,0) = (1+\sigma)w(x)$ to see that w is unstable from above.

To prove that there is at most one nontrivial, nonnegative stationary solution of (A1), suppose that w_1 and w_2 are two such solutions, i.e., $0 < w_1(1) < w_2(1)$ with no solutions of (3.3) in $(w_1(1), w_2(1))$, Then $0 < w_1(x) < w_2(x)$ and $w_1'(x), w_2'(x) > 0$ on $(0,1]$ by Theorem 3.3(A) and Lemma 3.3(A). Proceeding as in [19, pages 319–320], set

$$q(x) \equiv \frac{\phi(w_1)_x}{\phi(w_2)_x}.$$

We have

$$q(1) = \frac{g(w_1(1))/\phi(w_1(1))}{g(w_2(1))/\phi(w_2(1))} \frac{\phi(w_1(1))}{\phi(w_2(1))} < 1,$$

and

$$q'(x) = \left[\frac{f'(w_2)}{\phi'(w_2)} - \frac{f'(w_1)}{\phi'(w_1)}\right] q(x) > 0$$

on $(0,1]$. Hence for $\gamma_1 \equiv 1 - q(1)$, we have $\gamma_1 \in (0,1)$, and from $q(x) < q(1)$ on $(0,1)$ there follows $\phi(w_1(x)) < (1-\gamma_1)\phi(w_2(x))$ on $(0,1]$. (If $g(u)/\phi(u)$ is strictly increasing, we now have a contradiction from $\phi(w_1(1))/\phi(w_2(1)) \le (1-\gamma_1) < \phi(w_1(1))/\phi(w_2(1))$.) Thus,

$$w_1(x) < (1-\gamma_1)^{1/m} w_2(x).$$

Let $u(x,t)$ be the solution of (A1) with

$$u(x,0) = (1-\gamma_1)^{1/2m} w_2(x).$$

Then

$$(1-\gamma_1)^{-1/2m} w_1(x) < u(x,0) \le (1-\gamma_1)^{1/2m} w_2(x)$$

on $(0,1]$, so

$$(1-\gamma_1)^{-1/2m} w_1(x) \le u(x,t) \le (1-\gamma_1)^{1/2m} w_2(x)$$

for all $(x,t) \in [0,1] \times [0,\infty)$ by Lemma 4.1(A). But now, by Lemma 4.5,

$$\hat{u}(x) \equiv \lim_{t\to\infty} u(x,t)$$

is a stationary solution of (A1) with

$$w_1(x) < (1-\gamma_1)^{-1/2m} w_1(x) \leq \hat{u}(x) \leq (1-\gamma_1)^{1/2m} w_2(x) < w_2(x)$$

on $(0,1]$. This is impossible, since $\hat{u}(1) \in (w_1(1), w_2(1))$. □

Theorem 4.4 (B) *Let $\phi(u) = u^m, m \geq 1$. Assume that on $(0,\infty)$, f and $-g(u)/u\phi'(u)$ are nondecreasing and that g and f'/ϕ' are strictly increasing. Suppose the roots of (3.2) are isolated for $y_0 = 1$. Then there is at most one nontrivial, nonnegative stationary solution of (B1), $w(x)$. Furthermore, w is stable, and the trivial solution is unstable from above.*

Proof:

We first show that, under the above hypotheses, if $w_B(x,y_0)$ is a solution of (B1), then $y_0 = 1$. This follows by recalling that such a function with $y_0 \in (0,1)$ is a solution of (B1) iff $w_0 \equiv w_B(0,y_0)$ satisfies

$$\int_0^{w_0} \frac{\phi'(\sigma)}{f(\sigma)} d\sigma = y_0 < \infty.$$

But, since f'/ϕ' is strictly increasing, we have

$$\begin{aligned} f(\sigma) &= \int_0^\sigma f'(\tau)d\tau \\ &\leq \frac{f'(w_0)}{\phi'(w_0)} \int_0^\sigma \phi'(\tau)d\tau \\ &\equiv C\phi(\sigma), \end{aligned}$$

which implies that

$$\begin{aligned}\int_0^{w_0} \frac{\phi'(\sigma)}{f(\sigma)} d\sigma &\geq \frac{1}{C}\int_0^{w_0} \frac{\phi'(\sigma)}{\phi(\sigma)} d\sigma \\ &= \frac{1}{C}\int_0^{\phi(w_0)} \frac{1}{\tau} d\tau \\ &= \infty.\end{aligned}$$

Suppose w_1 and w_2 are two solutions of (B1) such that $0 < w_1(0) < w_2(0)$ and (3.2) has no solutions in $(w_1(0), w_2(0))$ for $y_0 = 1$. Recall that we then have $w_1 < w_2$ and $w_1', w_2' < 0$ on $[0, 1)$ by Lemma 3.3(B) and Theorem 3.4(B). As done previously, observe that $q(x)$ again has $q'(x) > 0$ on $(0, 1)$, so there follows

$$\phi(w_1(x)) < q(0)\phi(w_2(x)) + \phi(w_1(0)) - q(0)\phi(w_2(0)).$$

Now

$$q(0) = \frac{-g(w_1(0))}{-g(w_2(0))} < 1,$$

and

$$q(0) = \frac{-g(w_1(0))/\phi(w_1(0))}{-g(w_2(0))/\phi(w_2(0))}\frac{\phi(w_1(0))}{\phi(w_2(0))} \geq \frac{\phi(w_1(0))}{\phi(w_2(0))}.$$

Hence, for $\gamma_0 \equiv 1 - q(0)$, we have

$$w_1(x) < (1 - \gamma_0)^{1/m} w_2(x)$$

on $[0, 1)$, and $\gamma_0 \in (0, 1)$. Following [19], let $\delta > 0$ such that

$$(1 - \gamma_0)^{1/m} w_2(0) < (1 + \delta) w_1(0) < w_2(0),$$

and set

$$v(x) \equiv \frac{1 + \delta}{(1 - \gamma_0)^{1/m}} w_1(x) - w_2(x).$$

Then $v(0) > 0$, $v(1) = 0$, and $\phi(v)_{xx} < -f'(w_2)v'$ on $(0,1)$. It follows that $v(x) > 0$, and hence

$$w_1(x) < (1-\gamma_0)^{1/m} w_2(x) < (1+\delta)w_1(x)$$

on $[0,1)$. Now let $u(x,t)$ solve (B1) with $u(x,0) = (1+\delta)w_1(x)$. Then

$$(1-\gamma_0)^{1/m} w_2(x) \leq u(x,t) \leq (1+\delta)w_1(x)$$

on $[0,1] \times [0,\infty)$ by Lemma 4.2(B). Therefore, by Lemma 4.5,

$$\hat{u}(x) \equiv \lim_{t\to\infty} u(x,t)$$

is a stationary solution of (B1), and $\hat{u}(0) \in (w_1(0), w_2(0))$ which is a contradiction. The rest of the theorem now follows from our previous work. □

The section is now concluded with some results which are more useful than those above for determining stability or instability when more than one nontrivial, nonnegative stationary solution is present. Again, these theorems (and their proofs) are virtually identical to the corresponding results in [19]. After this is done for each problem, (A1) and (B1), we apply the results and characterize the stability or instability of the stationary states of (A) and (B). Diagrams which display these results are given.

Theorem 4.5 (A) *In (A1) replace f by ϵf where $\epsilon \geq 0$ is a parameter. Suppose that $f'(u) > 0$ for $u > 0$. Let $w(x,\epsilon)$ be a branch of nontrivial, nonnegative stationary solutions such that $w(x,\cdot)$ is C^1 (on its interval of definition). For $w_1(\epsilon) \equiv w(1,\epsilon)$, if $w_1'(\epsilon) > 0$ on this branch, then the solutions are stable. If $w_1'(\epsilon) < 0$ on this branch, then the solutions are unstable.*

Proof:

From the proof of Theorem 3.1(A),

$$\int_{w(x,\epsilon)}^{w_1(\epsilon)} \frac{\phi'(\sigma)}{g(w_1(\epsilon)) + \epsilon[f(w_1(\epsilon)) - f(\sigma)]} d\sigma = 1 - x$$

on $[0,1]$. So

$$\int_0^{w(x,\epsilon)} \frac{\phi'(\sigma)}{g(w_1(\epsilon)) + \epsilon[f(w_1(\epsilon)) - f(\sigma)]} d\sigma = x,$$

which upon differentiation with respect to ϵ yields

$$\frac{\phi'(w(x,\epsilon))}{g(w_1(\epsilon)) + \epsilon[f(w_1(\epsilon)) - f(w(x,\epsilon))]} \frac{\partial}{\partial \epsilon} w(x,\epsilon)$$

$$= \int_0^{w(x,\epsilon)} \frac{\{[g'(w_1) + \epsilon f'(w_1)]w_1'(\epsilon) + (f(w_1) - f(\sigma))\}\phi'(\sigma)}{[g(w_1) + \epsilon(f(w_1) - f(\sigma)]^2} d\sigma.$$

Hence, if $w_1'(\epsilon) > 0$, then $\partial w(x,\epsilon)/\partial\epsilon > 0$ on $(0,1]$ from which it follows that $w(x,\epsilon_1) < w(x,\epsilon_2)$ on $(0,1]$ for $[\epsilon_1, \epsilon_2]$ contained in the domain of this branch. Observe that

$$\begin{aligned} &\phi(w)_{xx}(x,\epsilon_2) + \epsilon_1 f(w)_x(x,\epsilon_2) \\ &< \phi(w)_{xx}(x,\epsilon_2) + \epsilon_2 f'(w(x,\epsilon_2))w_x(x,\epsilon_2) \\ &= 0, \end{aligned}$$

so $w(x,\epsilon_2)$ is a supersolution of (A1) with $\epsilon = \epsilon_1$. Therefore, if $u(x,t,\epsilon_1)$ is the solution of (A1) with $u(x,0,\epsilon_1) = w(x,\epsilon_2)$, then

$$w(x,\epsilon_1) \le u(x,t,\epsilon_1) \le w(x,\epsilon_2)$$

for all $(x,t) \in [0,1] \times [0,\infty)$ by the comparison theorem. Now we must have

$$\lim_{t\to\infty} u(x,t,\epsilon_1) = w(x,\epsilon_1)$$

by Lemma 4.5. It can be shown similarly that $w(x, \epsilon_1)$ is stable from below.

If $w_1'(\epsilon) < 0$ on $[\epsilon_1, \epsilon_2]$, then $w(1, \epsilon_2) < w(1, \epsilon_1)$ and consequently $w(x, \epsilon_2) < w(x, \epsilon_1)$ in some interval $(1 - \delta, 1]$. A simple calculation shows that $w(x, \epsilon_1)$ is a subsolution of (A1) with $\epsilon = \epsilon_2$. Hence, if $u(x, t, \epsilon_2)$ is the solution of (A1) which satisfies $u(x, 0, \epsilon_2) = w(x, \epsilon_1)$, then

$$w(x, \epsilon_2) < w(x, \epsilon_1) \leq u(x, t, \epsilon_2)$$

on $(1 - \delta, 1)$ for as long as u exists. Therefore, $w(x, \epsilon_2)$ is unstable from above. Instability from below follows from an identical argument. □

The next corollary is proved using Theorem 4.5(A) and an argument exactly as printed in [19, page 323]. Its proof is therefore omitted.

Corollary 4.5 (A) *Let f, g be as in Theorem 4.5(A). If $g'(w_1(0)) < 1$, then the branch of stationary solutions emanating from $\epsilon = 0$ is stable; if $g'(w_1(0)) > 1$, then it is unstable.*

To apply this corollary to (A), we recall that $w_1(0) = a^{1/(m-p)}$ for $m \neq p$. So

$$1 - g'(w_1(0)) = 1 - pa^{(1-m)/(p-m)},$$

and thus nontrivial, nonnegative solutions of (A) are unstable when $p > p_0$, where $p_0 \geq m \geq 1$ is such that

$$p_0 = a^{(m-1)/(p_0-m)}.$$

However, in the cases of $p, n \geq m \geq 1$, we always have $w_1'(\epsilon) < 0$ (whenever $w_1(\epsilon)$ exists). Hence the solution branch $w_1(\epsilon)$ is unstable by Theorem 4.5(A), and the trivial solution is stable by Theorem 4.3(A). In case $p = m < n$ and $a \geq 1$, the instability of the trivial solution can be seen by considering solutions, $u_b(x,t)$, of (A) satisfying

$$(u^m)_x(1,t) = bu^p(1,t)$$

with $b \in (0,1)$. Observe that

$$(u_b^m)_x(1,t) \le au_b^p(1,t),$$

i.e., $u_b(x,t)$ is a subsolution of (A). Hence the instability of the trivial solution in this case follows from the case $p = m < n$, $0 < a < 1$. It only remains to determine the stability or instability of the trivial solution in the cases $m = n = p$. We note that, from our work in Chapter 3, when $a \ge 1$ and $m = n = p$, there exists $x_0 \in (0,1)$ such that

$$\frac{an}{\epsilon} \ln \left[1 + \frac{\epsilon}{an}\right] = a(1 - x_0).$$

So there are subsolutions, $w_A(x,x_0)$, of (A) for which $w_A(1,x_0)$ can be any positive number. Thus the trivial solution is unstable from above. When $a < 1$, let $\epsilon_0 > 0$ be such that

$$\frac{an}{\epsilon_0} \ln \left[1 + \frac{\epsilon_0}{an}\right] = a.$$

Then for $\epsilon > \epsilon_0$ the instability of the trivial solution follows as above. The case $\epsilon = \epsilon_0$ has a continuum of stationary states, including the trivial solution, which are all unstable. Finally, for $0 < \epsilon < \epsilon_0$, the argument above does not apply (because

$$\frac{an}{\epsilon} \ln \left[1 + \frac{\epsilon}{an}\right] > a(1 - x_0)$$

for all $x_0 \in [0,1)$). However, if we let $u(x) = w(x,\epsilon_0)$, then $u_x \ge 0$, and

$$(u^m)_{xx} + \frac{\epsilon}{n}(u^n)_x \le (u^m)_{xx} + \frac{\epsilon_0}{n}(u^n)_x = 0,$$

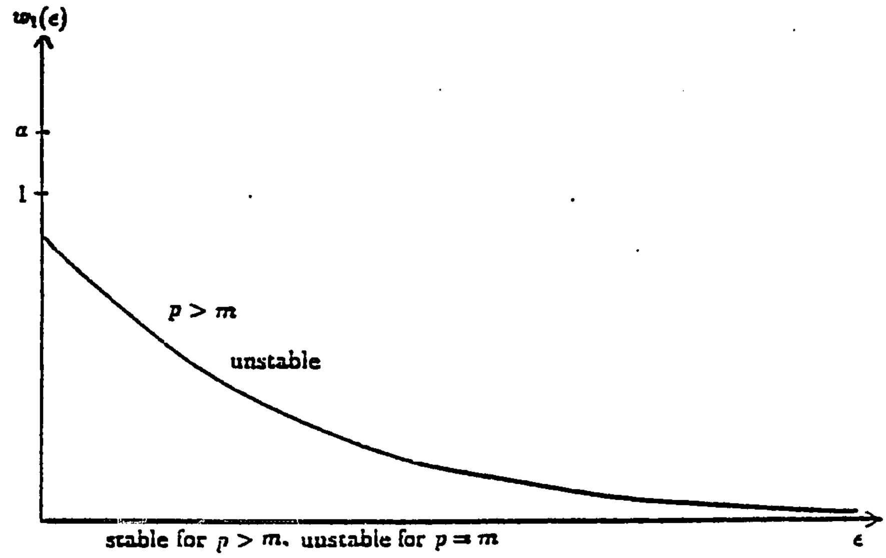

Figure 4.1: $w_1(\epsilon)$ where $(w^m)'(1) = aw^p(1)$, $1 \leq m < n$, $a \geq 1$, and $w_1(0) = a^{1/(m-p)}$.

i.e., u is a supersolution of (A). It follows that the trivial solution is stable. These results are now displayed in the following graphs. Before leaving this example let us remark that if a comparison result for (A1) can be proved without the need for (H5), then one can continue in the above manner to determine the stability of instability of more branches of the solution diagrams for (A). We hope to pursue this matter in a future work.

Replacing f by ϵf, $\epsilon \geq 0$, in (B1), we have the following parallel results.

Theorem 4.6 (B) *Let $f' > 0$ on $(0, \infty)$ and suppose that $w(x, \epsilon)$ is a branch of nontrivial, nonnegative stationary solutions of (B1) with $w(x, \cdot)$ being C^1 for each x and satisfying $g(w_0) - f(w_0) \geq 0$. If $w_0'(\epsilon) < 0$, then this is a branch of unstable stationary solutions; if $w_1'(\epsilon) > 0$, then the branch is stable.*

Corollary 4.6 *With f and g as in Theorem 4.6(B), if $g'(w_0(0)) > 1$, then the branch of stationary solutions emanating from $\epsilon = 0$ is unstable. If $g'(w_0(0)) < 1$, then this*

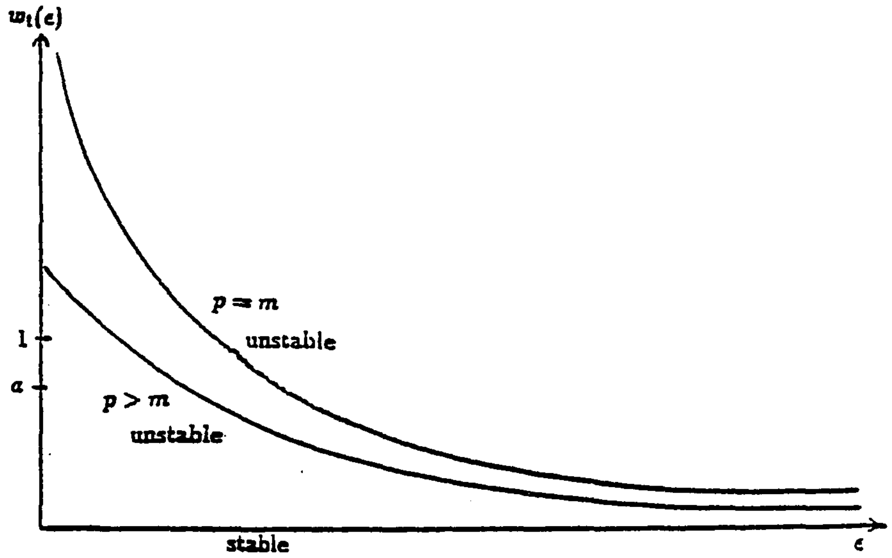

Figure 4.2: Same as above except with $0 < a < 1$.

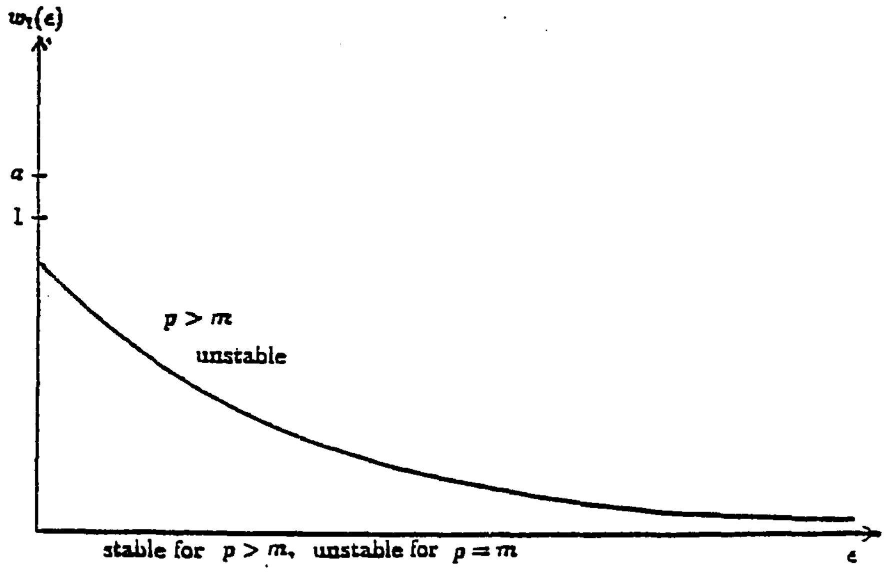

Figure 4.3: $w_1(\epsilon)$ where $(w^m)'(1) = aw^p(1)$, $1 \leq m = n$, $a \geq 1$, and $w_1(0) = a^{1/(m-p)}$.

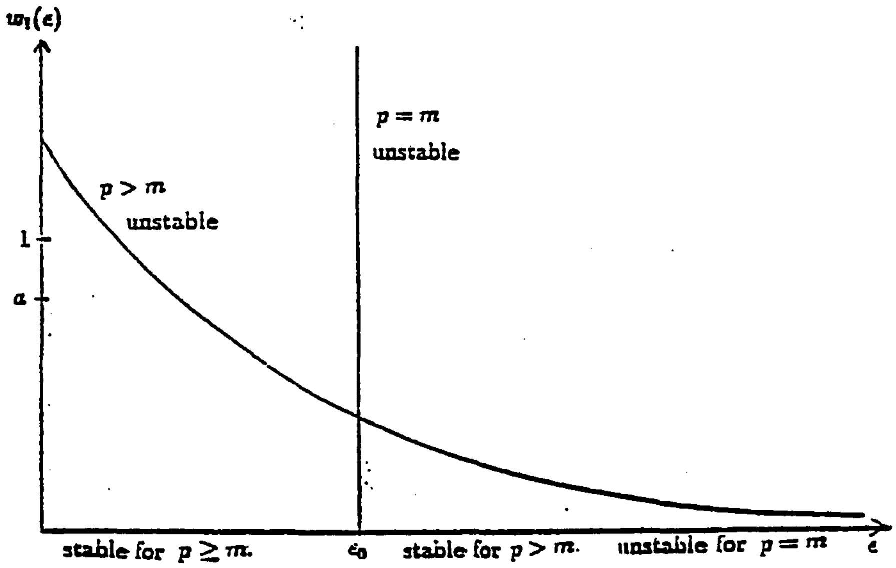

Figure 4.4: Same as above except with $0 < a < 1$.

branch is stable.

To apply these results and conclude the stability or instability of the stationary solutions of (B), we first note that this question is easily settled by Theorem 4.6(B) for all the branches of nontrivial, nonnegative solutions. Hence, we only need to discuss the trivial solution. From Lemma 4.2(B) this solution is unstable in the case $p = m < n$, $a > 1$. Since a solution of (B) for some $a > 1$ is a supersolution of (B) with $0 < a \leq 1$, we may conclude the stability of the trivial solution in the cases of $p = m < n$, $p > n > m$, and $p > n = m$, when $0 < a \leq 1$. This is achieved by considering a solution of these problems which is initially equal to a stationary state of the corresponding problem with $a > 1$ and then applying Lemma 4.5. Arguing similarly, we arrive at the same conclusion for $m < p \leq n$ when $0 < a \leq 1$ and $0 < \epsilon \leq \epsilon_0$. That the trivial solution is stable for $m < p \leq n$, $0 < a \leq 1$, $\epsilon > \epsilon_0$

follows by observing that $w(x, \epsilon_0)$ satisfies

$$(w^m)_{xx} + \frac{\epsilon}{n}(w^n)_x \leq (w^m)_{xx} + \frac{\epsilon_0}{n}(w^n)_x = 0$$

for $\epsilon > \epsilon_0$, i.e., $w(x, \epsilon_0)$ is a supersolution of (B) with $\epsilon > \epsilon_0$. Now apply Lemma 4.5 for the solution of (B), $u(x,t)$, such that $u(x,0) = w(x, \epsilon_0)$. In the case of $p = n = m$, $0 < a \leq 1$, we see that the trivial solution is stable by considering the solution of (B), $u(x,t)$, such that $u(x,0) = u_0(x) = (1-x)^{1/m}$. Since

$$\begin{aligned}
&(u_0^m)_{xx} + (\epsilon/n)(u_0^m)_x = -\epsilon/n \quad \text{on } (0,1) \\
&u_0(1) = 0 \\
&-(u_0^m)_x(0) = 1 \geq a = a u_0^m(0),
\end{aligned}$$

i.e., u_0 is a supersolution of (B), it follows that $\lim_{t\to\infty} u(x,t) = 0$ by the comparison theorem and Lemma 4.5. It only remains to consider the cases where $a > 1$. First, arguing as above, we use the continuum of stationary states for $m = n = p$, $\epsilon = \epsilon_0$ to establish that the trivial solution is stable for $m = n = p$, $\epsilon > \epsilon_0$ and unstable for $0 < \epsilon \leq \epsilon_0$. Furthermore,

$$v(x) \equiv e^{\lambda(1-x)} - 1,$$

where $0 < \lambda < \epsilon/n$ is chosen so that

$$\lambda e^{\lambda} \geq a[e^{\lambda} - 1]^{p/m},$$

is a supersolution of (B) when $m = n < p$, and hence the trivial solution must be stable in this case. Finally, in the case $p, n > m$, $a > 1$, fix $\epsilon_1 > 0$ and let $w(x, \epsilon_1)$ be the stationary solution of as unstable branch. For $\epsilon \geq \epsilon_1$, $w(x, \epsilon_1)$ is a supersolution of (B). Hence it $u(x,t)$ is the solution of (B) for $\epsilon > \epsilon_1$ such that $u(x,0) = w(x, \epsilon_1)$,

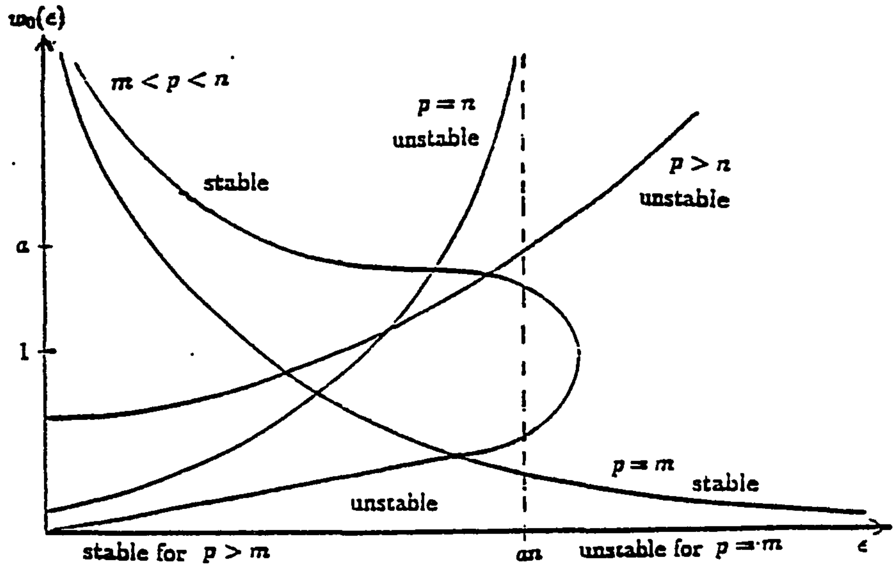

Figure 4.5: $w_0(\epsilon)$ where $-(w^m)_x(0) = aw^p(0)$, $1 \leq m < n$, $a > 1$, $w_0(0) = a^{1/(m-p)}$.

then by Lemma 4.5 it must be that $\lim_{t\to\infty} u(x,t) = 0$. In this manner we establish the stability of the trivial solution for $p, n > m$, $a > 1$, and $\epsilon > 0$.

The diagrams of these results are given below. We again remark that a better uniqueness and comparison theorem for (B) would allow one to continue in this manner and investigate the stability/instability of the remaining branches.

4.3 Blow-up Results

From the results of the previous section, we have that if $w(x)$ is the largest stationary solution of (A1) and $u(x,t)$ is a solution of (A1) with $u(x,t) \geq (1+\sigma)w(x)$, then $\| u(\cdot,t) \|_{L^\infty(0,1)}$ must become unbounded in finite or infinite time. We now show that there are some solutions for which this blow-up time is actually finite.

This section has three main parts. The first part follows some work of Levine and Sacks [23] to develop a blow up result for solutions of (A1) with $f(u) \equiv 0$. In the

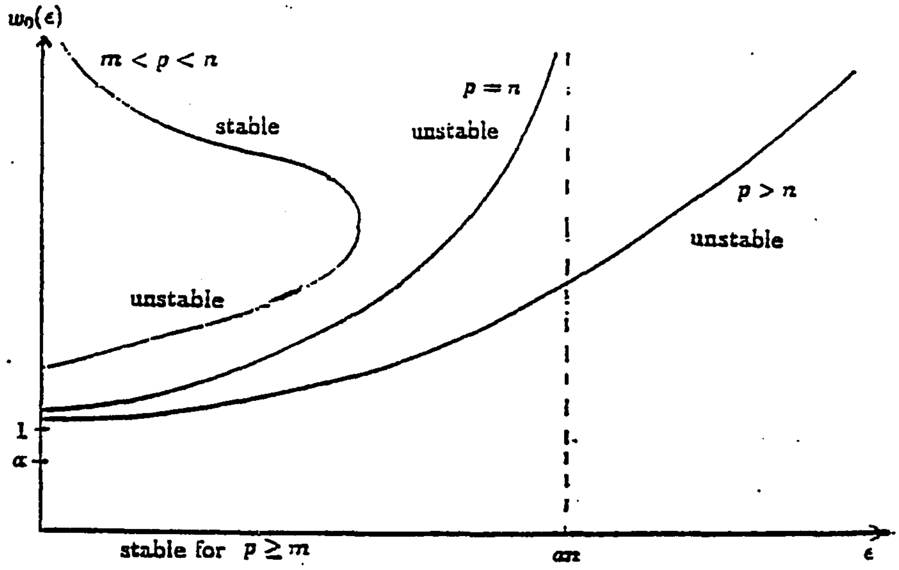

Figure 4.6: Same as above except $0 < a \leq 1$.

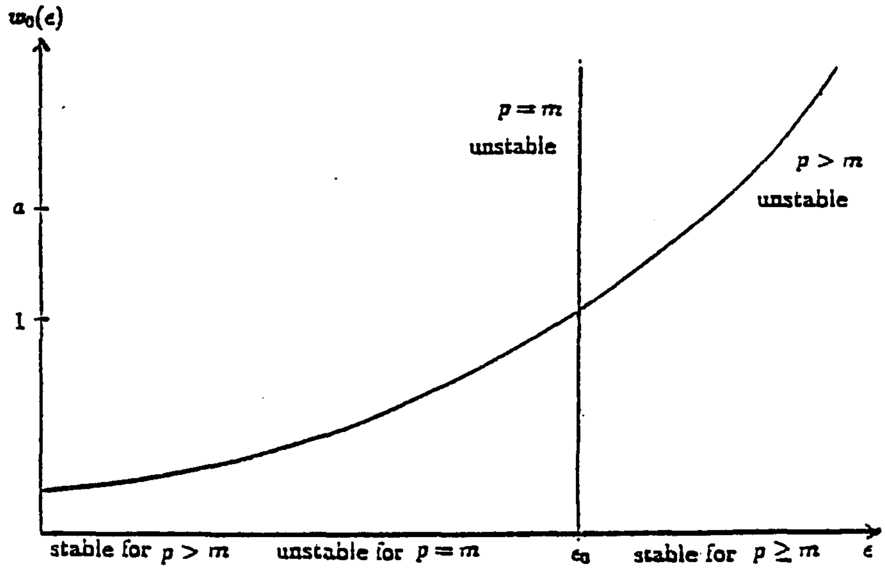

Figure 4.7: $w_0(\epsilon)$ where $-(w^m)_x(0) = aw^p(0)$, $1 \leq m = n$, $a > 1$.

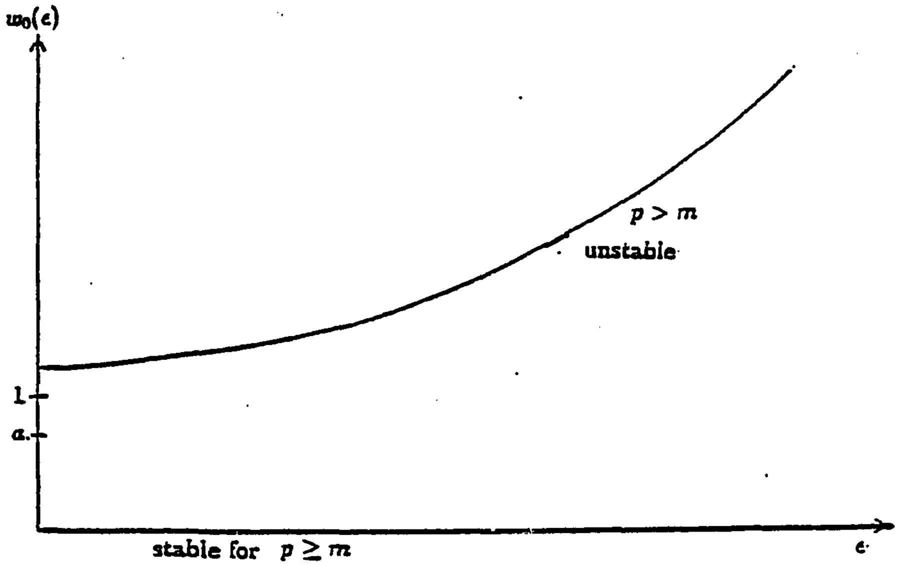

Figure 4.8: Same as above except $0 < a \leq 1$.

second part, this is used to prove such a theorem for solutions of (A1) with certain types of nontrivial f and monotone initial data. This same approach can be applied to (B1), but the resulting theorem does not apply to (B). Therefore, we follow the corresponding work in [19] to obtain a suitable blow up result for (B1). Finally these results are put to use in the third part to establish that some solutions of (A) do blow up in finite time.

The first theorem is a result which is parallel to Theorem 4.1 in [23]. We include a brief sketch of the proof for the convenience of the reader. First, recall some definitions from Chapter 2:

$$\Phi(u) \equiv \int_0^u \phi(v)dv, \text{ and } \Psi(u) \equiv \int_0^u \sqrt{\phi'(v)}dv.$$

Theorem 4.7 (A) *Suppose that u is a nontrivial solution of (A1) on Q_T with $f \equiv 0$*

and $\phi(u_0) \in H^1(0,1)$. *If there exists* $\kappa \in (0,1/2)$ *such that* Φ^κ *is convex on* R *and*

$$G(u) \equiv \int_0^u \phi'(s)g(s)ds$$

satisfies

$$G(u) \le \frac{1}{2}\phi(u)g(u)$$

on $[0,\infty)$, *then*

$$G(u_0(1)) < \frac{1}{2}\int_0^1 \phi(u_0)_x^2 dx + \frac{3-4\kappa}{(1-2\kappa)^2}\frac{1}{T}\int_0^1 \Phi(u_0)dx. \tag{4.1}$$

Proof:

As in [23], define

$$\alpha \equiv 1-2\kappa,\ \beta \equiv \frac{1}{\alpha^2 T}\int_0^1 \Phi(u_0)dx,\ \tau \equiv \alpha T,$$

and let

$$H(t) \equiv \int_0^t \int_0^1 \Phi(u(x,s))dxds + (T-t)\int_0^1 \Phi(u_0(x))dx + \beta(t+\tau)^2.$$

Using the techniques of [22] to handle the nonlinear boundary condition, Theorem 2.2(A), and the facts

$$\left(\Phi^\kappa\right)'' \ge 0 \text{ and } G(u) \le \frac{1}{2}\phi(u)g(u),$$

it can be shown that

$$\begin{aligned} & H(t)H''(t) - (\alpha+1)\left[H'(t)\right]^2 \\ \ge\ & H(t)\left\{2G(u_0(1)) - \int_0^1 \phi(u_0)_x^2 dx - 2\beta(1+2\alpha)\right\} \end{aligned}$$

for $t \in [0,T]$. Now if (4.1) fails, then the quantity in the curly brackets is nonnegative, and so

$$\left(H^{-\alpha}\right)''(t) \le 0.$$

Thus, on $[0,T]$,

$$H^{-\alpha}(t) \leq H^{-\alpha-1}(0)\left[H(0) - \alpha H'(0)t\right].$$

But, since

$$\frac{H(0)}{\alpha H'(0)} = (1-2\kappa)T,$$

this last inequality implies that $H^{-\alpha}$ vanishes for some $t_0 \in (0,T)$, i.e.,

$$\lim_{t\to t_0^-} H(t) = \infty,$$

which is a contradiction. □

Corollary 4.7 (A) *Let the assumptions of Theorem 4.3(A) hold, and let T denote the largest time for which $u(x,t)$ exists as a solution of (A1) on Q_T. If*

$$\frac{1}{2}\int_0^1 \phi(u_0)_x^2 dx < G(u_0(1)),$$

then

$$\limsup_{t\to T^-} \| u(\cdot,t) \|_{L^\infty(0,1)} = \infty.$$

To use these results, we obtain a comparison result for solutions of (A1) with $f \equiv 0$ and solutions of (A1) with $f' \geq 0$.

Lemma 4.6 (A) *Suppose $f', g \geq 0$ on $[0,\infty)$. In addition, assume that there exists a $\delta > 0$ for which either $f' > 0$ or $f \equiv 0$ on $(0,\delta)$. Let $u(x,t)$ be the solution of (A1), and let $v(x,t)$ be the solution of the same problem (A1) except with $f \equiv 0$. If $u(x,0)$ is nondecreasing and $u(x,0) \geq v(x,0)$ a.e. on $(0,1)$, then $u(x,t) \geq v(x,t)$ for as long as both solutions exist.*

Proof: To prove this we rely heavily on the existence proof for (A1), where it is shown that $u(x,t)$ and $v(x,t)$ are obtained as pointwise limits of solutions of the associated regularized problems, $u_{\kappa,n}$ and $v_{\kappa,n}$, respectively. We shall construct a regularizing sequence, f_n, for which $f_n' \geq 0$ and additional sequences, u_{0n} and v_{0n}, for which $u_{0n} \geq v_{0n}$ and $u_{0n}' \geq 0$. (See section 5.1 for clarification of this notation.) With such constructions, maximum principles will be applied to give $(u_{\kappa,n})_x \geq 0$ and then $u_{\kappa,n} \geq v_{\kappa,n}$. The desired result will follow upon taking limits.

To this end, fix $\kappa \in (0, \delta/4)$ and choose ϕ_n and g_n as in section 5.1. Observe that if $x \in [\kappa/2, 2\kappa]$, then

$$\begin{aligned} J_n * f'(x) &= \int_{-1/n}^{1/n} \rho(z) f'\left(x - \frac{z}{n}\right) dz \\ &\geq \min_{[\kappa/4, 4\kappa]} f'(x) \\ &> 0 \end{aligned}$$

provided $\kappa/4 \leq \kappa/2 - 1/n$ and $2\kappa - 1/n \leq 4\kappa$. Thus, we may choose $r_n \in C^\infty(R)$ such that

$$\begin{aligned} r_n \equiv 0 &\quad \text{on } (-\infty, 0] \cup [2\kappa, \infty), \\ 0 \leq r_n \leq 1 &\quad \text{on } [0, 2\kappa], \\ r_n \to 0 &\quad \text{uniformly as } n \to \infty, \\ r_n' \geq 0 &\quad \text{on } [0, \kappa/2], \\ -\min_{[\kappa/4, 4\kappa]} f'(x) \leq r_n'(x) \leq 0 &\quad \text{on } [\kappa/2, 2\kappa], \text{ and} \\ r_n'(\kappa) = -J_n * f'(\kappa). \end{aligned}$$

Then

$$f_n'(x) = J_n * f'(x) + r_n(x) \geq 0$$

for all x and all sufficiently large n. (If $f' \equiv 0$ on $(0,\delta)$, then let $r_n \equiv 0$.) Furthermore, choose $u_{0n} \in C^{\infty}([0,1])$ such that $u'_{0n} \geq 0$, $u_{0n} \to u_0$ in $L^2(0,1)$, $\| u_{0n} \|_{L^\infty(0,1)} \leq \| u_0 \|_{L^\infty(0,1)} + 1$, $u_{0n}(0) = u''_{0n}(0) = 0$, $u_{0n}(1) = \lim_{x\to 1^-} u_0(x)$, and $\phi'_n(u_{0n}(1)+\kappa)u'_{0n}(1) = g_n(u_{0n}+\kappa)$. By arguments in [9], the solution, $u_{\kappa,n}$, of $(A_{\kappa,n})$ is smooth enough so that the equation can be differentiated in x. Applying the maximum principle to the problem for $(u_{\kappa,n})_x$, we have $(u_{\kappa,n})_x \geq 0$.

Construct v_{0n} as above such that $v_{0n} \leq u_{0n}$ on $[0,1]$. Then $w \equiv u_{\kappa,n} - v_{\kappa,n}$ is a solution of

$$\begin{aligned} &w_t = a w_{xx} + b w_x + c w + d && \text{on } Q_T \\ &w(0,t) = 0 && \text{on } (0,T) \\ &a w_x(1,t) + h w(1,t) = 0 && \text{on } (0,T) \\ &w(x,0) \geq 0 && \text{on } [0,1], \end{aligned}$$

where

$$\begin{aligned} a(x,t) &\equiv \phi'(u_{\kappa,n})(x,t), \\ b(x,t) &\equiv \phi''(u_{\kappa,n}(u_{\kappa,n}+v_{\kappa,n})_x(x,t), \\ c(x,t) &\equiv \phi''(\cdot)(v_{\kappa,n})_{xx}(x,t) + \phi'''(\cdot)(v_{\kappa,n})_x^2(x,t), \\ d(x,t) &\equiv f'_n(u_{\kappa,n})(u_{\kappa,n})_x(x,t), \\ h(t) &\equiv \phi''(\cdot)(v_{\kappa,n})_x(1,t) - g'(\cdot). \end{aligned}$$

Finally, $d \geq 0$ on Q_T, so $w \geq 0$ on Q_T by the maximum principle. □

We now have the following blow-up result for (A1).

Theorem 4.8 (A) *Let $u(x,t)$ be the solution of (A1) with initial data $u_0(x)$ such that $\phi(u_0) \in H^1(0,1)$. If the assumptions of Theorem 4.7(A) hold and u_0 is nonde-*

creasing with

$$\frac{1}{2}\int_0^1 \phi(u_0)_x^2 dx < G(u_0(1)),$$

then $\| u(\cdot,t) \|_{L^\infty(0,1)}$ *becomes unbounded in finite time.*

In the case of (A), the above result applies provided there exists a nondecreasing function $u_0(x)$ such that $u_0^m \in H^1(0,1)$ and

$$\frac{1}{2}\int_0^1 [v_0'(x)]^2 dx < \frac{am}{m+p} v_0^{(m+p)/m}(1),$$

where $p, n \geq m > 1$ and $v_0(x) \equiv u_0^m(x)$. To see that such functions exist, we follow [19, Example 3.1] and set

$$v_0(x) = A[(r^2 - (\alpha - x)^2)^{1/2} - (r^2 - \alpha^2)^{1/2}]$$

where α, r, A are positive constants to be chosen later with $r > \alpha > 1$. Observe that for v_0 so defined

$$\frac{1}{2}\int_0^1 [v_0'(x)]^2 dx = \frac{A^2}{2}\left\{\frac{r}{2}\ln\left[\frac{(r-\alpha+1)(r+\alpha)}{(r+\alpha-1)(r-\alpha)}\right]\right\},$$

and

$$\frac{am}{m+p} v_0^{(m+p)/m}(1) = \frac{am}{m+p} A^{(m+p)/m}[(r^2-(\alpha-1)^2)^{1/2}-(r^2-\alpha^2)^{1/2}]^{(m+p)/m}.$$

Since $(m+p)/m > 2$, it is clear that for any $r > \alpha > 1$, A can be chosen sufficiently large so that the inequality of Theorem 4.8(A) is satisfied. Note that finding suitable initial states is easier here than in [19] because we do not need the compatibility condition $\phi(u_0)'(1) = g(u_0)(1)$. However, with a little more work, initial states which satisfy the hypotheses of the theorem and this compatibility condition can be found. The analysis is exactly as in [19].

In the direction of developing a corresponding blow up result for (B1), let us first remark that when $f \equiv 0$, problems (A1) and (B1) are essentially the same. Hence Theorem 4.7(A), with obvious modifications, applies to (B1). However, the comparison result in Lemma 4.6(A) relied on the fact that $(u_{\kappa,n})_x \geq 0$ in Q_T. For (B1), the boundary condition $u(1,t) = 0$ causes this inequality to reverse for suitable initial states, which ultimately reverses the inequality in the corollary. Because of this fact, no conclusion about blow-up it possible for (B) using such an approach. Therefore, we must, as in [19], obtain a suitable result directly for the problem (B1) with possibly nonzero f. This is the content of the following results.

Lemma 4.7 (B) *Let $f'(u) \geq 0$ for $u > 0$, and let $\phi(u_0) \in H^1(0,1)$. Suppose that for each $\kappa \in (0,1)$ there exist functions $u_{0n} \in C^\infty([0,1])$ which satisfy*

$$\lim_{n\to\infty} \| u_{0n} - u_0 \|_{L^2(0,1)} = 0,$$

and

(i) $u'_{0n} \leq 0$ *on* $[0,1]$,

(ii) $\phi n(u_{0n} + \kappa)'' + fn(u_{0n} + \kappa)' \geq 0$ *for* $x \in (0,1)$,

(iii) $-\phi n(u_{0n} + \kappa)'(0) = gn(u_{0n} + \kappa)(0)$,

(iv) $u_{0n}(1) = u''_{0n}(1) = 0$,

for all sufficiently large n. Then the solution, $u(x,t)$, of (B1) with $u(x,0) = u_0(x)$ satisfies

$$\int_0^t \int_0^1 \Psi(u)_s dx ds \leq -\frac{1}{2} \int_0^1 \phi(u)_x^2 \Big|_{s=0}^{s=t} dx + \int_{u_0(0)}^{u(0,t)} \phi'(s) g(s) ds.$$

Proof:

Recall, from previous calculations, that if $u_{\kappa,n}$ is the solution of $(\mathrm{A}_{\kappa,n})$ with $u_{\kappa,n}(x,0) = u_{0n}(x) + \kappa$, then $v = u_{\kappa,n}$ has

$$\begin{aligned}&\int_0^t \int_0^1 \phi_n(v)_s v_s dx ds \\ &= -\frac{1}{2}\int_0^1 \phi_n(v)_x^2 \Big|_{s=0}^{s=t} dx + \int_{v(0,0)}^{v(0,t)} \phi_n'(s) g_n(s) ds + \int_0^t \int_0^1 \phi_n(v)_s f_n(v)_x dx ds.\end{aligned}$$

By (i) - (iv) and standard applications of the maximum principles, $-v_t, v_x \leq 0$ on $\overline{Q_T}$. Hence,

$$\int_0^t \int_0^1 \phi_n(v)_s v_s dx ds \leq -\frac{1}{2}\int_0^1 \phi_n(v)_x^2 \Big|_{s=0}^{s=t} dx + \int_{v(0,0)}^{v(0,t)} \phi_n'(s) g_n(s) ds,$$

from which the result follows upon taking limits. □

Theorem 4.9 (B) *Let u_0 and f be as in the previous lemma, and define*

$$H(t) \equiv \int_0^t \int_0^1 \Phi(u)(x,s) dx ds + (T-t)\int_0^1 \Phi(u_0)(x) dx + \beta(t+\tau)^2$$

where $\tau, \beta > 0$. Assume that Φ^κ is convex for some $\kappa \in (0, 1/2)$ and that $\lambda \geq 1 - 2\kappa$. Then

$$\begin{aligned}&H(t)H''(t) - (\lambda+1)[H'(t)]^2 \\ &\geq H(t)\left\{\phi(u)[g(u) - f(u)](0,t) - \left(\frac{\lambda+1}{1-\kappa}\right) G(u(0,t)) - 2\beta(2\lambda+1)\right. \\ &\left. + \left(\frac{\lambda+1}{1-\kappa}\right)\left[G(u_0(0)) - \frac{1}{2}\int_0^1 \phi(u_0)_x^2 dx\right]\right\}.\end{aligned}$$

This is established exactly as the previous blow-up result. The parallel result for (B1) is now contained in the following corollary.

Corollary 4.8 (B) *Let $u_0, f,$ and ϕ be as in the previous theorem, and suppose that there exists $\alpha \geq 2$ such that*

$$\phi(v)[g(v) - f(v)] - \alpha G(v) \geq 0$$

for $v \geq 0$. If

$$\frac{1}{2}\int_0^1 \phi(u_0)_x^2 dx < G(u_0(0)),$$

then

$$\limsup_{t \to T^-} \| u(\cdot, t) \|_{L^\infty(0,1)} = \infty$$

for some finite $T > 0$.

The above result suggests that some solutions of (B1), and hence (B), do blow up in finite time. However, the complicated conditions of Lemma 4.7(B) prohibit the construction of an explicit example as was previously done for (A). The conditions of this lemma are simply ones which allow us to prove the evergy inequality. With this in mind, we now formulate a weaker version of our blow up result for problem (B1) by adding the energy inequality to the hypotheses. This is the point of view taken in [23], for example.

Corollary 4.9 (B) *Assume that Φ_κ is convex for some $\kappa \in (0, 1/2)$ and that there exists $\alpha \geq 2$ such that*

$$\phi(v)[g(v) - f(v)] - \alpha G(v) \geq 0$$

for $v \geq 0$. Let $u(x,t)$ be the solution of (B1) with $u(x,0) = u_0(x)$, and suppose that u satisfies the energy inequality,

$$\int_0^t \int_0^1 \Psi(u)_s dx ds \leq -\frac{1}{2}\int_0^1 \phi(u)_x^2 \,\Big|_{s=0}^{s=t} dx + \int_{u_0(0)}^{u(0,t)} \phi'(s) g(s) ds,$$

for each $t \geq 0$ *such that* u *is defined on* Q_t. *If*

$$\frac{1}{2}\int_0^1 \phi(u_0)_x^2 dx < G(u_0(0)),$$

then

$$\limsup_{t \to T^-} \| u(\cdot, t) \|_{L^\infty(0,1)} = \infty$$

for some finite $T > 0$.

5 PROOF OF THE THEOREMS FROM CHAPTER 2

5.1 Introduction

In this chapter we present the proofs of the results which appeared in Chapter 2. The proof of the existence and continuation theorem will be presented in Section 2; in Section 3 we will prove the increased regularity results.

In the theory of PDEs an equation of the form

$$u_t = a(x,t)u_{xx} + b(x,t,u,u_x)$$

is called uniformly parabolic if there exists a positive constant, ν, for which $a \geq \nu$. For such equations there is a well established theory. See [18] or [9], for instance. In light of (H1), we have not ruled out the possibility that $\phi'(0) = 0$, i.e., $\phi(u) = u^m$ for $m \geq 1$. Because of this, the PDE

$$u_t = \phi'(u)u_{xx} + \phi''(u)u_x^2 + f(u)_x$$

is often called degenerate parabolic. To prove an existence theorem for such an equation, there are basically three approaches that are used. One is the previously mentioned semigroup theory approach. Another is to apply the existing theory to the equations

$$u_t = [\phi_n'(u) + \kappa]u_{xx} + \phi_n''(u)u_x^2 + f_n(u)_x,$$

where the functions ϕ_n and f_n are smoothings of ϕ and f, respectively. One gets a sequence of solutions to these problems for each $\kappa > 0$ and hopes to be able to pass to the limits $n \to \infty$ and $\kappa \to 0^+$ in order to obtain a solution of the original problem. This idea has been used by Sacks [29,28] and has the advantage of usually giving one a continuous solution for any L^∞ data regardless of its sign. The third approach is to leave the equation alone and change the boundary conditions. For instance, one would replace boundary conditions of the form $u(0,t) = 0$ with $u(0,t) = \kappa > 0$. In doing this, the problem is made nondegenerate, not by eliminating the degeneracy of the equation, but by preventing the solution from taking on values where the equation is degenerate. This method has been used in [10,2,5], to name a few, and has the advantage of a simpler passage to the limit. However, such an approach yields an existence theorem only for nonnegative data.

We shall prove the existence theorem via a hybrid of the last two of these three methods. Our main steps will follow those of the latter method, but we will borrow the idea of regularizing the functions ϕ and f from the second. Such an approach was mentioned by Gilding on the last page of [10], but it is not clear at the outset how the flux boundary conditions should be handled. We will outline the existence proof more thoroughly throughout the course of the actual proof. At this time some additional function spaces which will be needed later are defined.

$C^{2,1}(\overline{Q_T})$ will simply be the space of all functions on $\overline{Q_T}$ which are twice continuously differentiable in x and once continuously differentiable in t. For $\alpha \in (0,1)$,

$$H^{(\alpha)}(\overline{Q_T}),\ H^{(1+\alpha)}(\overline{Q_T}),\ H^{(2+\alpha)}(\overline{Q_T}),$$

etc. will denote the usual Hölder spaces. See [18, pages 7- 8] for precise definitions.

The increased regularity results are virtually the same as their counterparts in

[23]. In fact it is our goal to obtain blow up results parallel to those in [23] which motivates us to develop these theorems in their present form.

The proofs given in this chapter are all one-dimensional. However, it is straightforward to show that all of these arguments can be suitably generalized to apply to problems in higher dimensions. One could replace the results in Appendix A with those of Amann [1] or those of Lieberman [24] depending on whether or not the boundary is disconnected or connected. The rest of the proofs carry through with little change.

5.2 Proof of Local Existence and Continuation

In order to establish the existence and continuation theorem the following problems will be needed.

$$(\mathrm{A}_\kappa)\qquad \begin{aligned} &u_t = \phi(u)_{xx} + f(u)_x && \text{on } Q_T \\ &u(0,t) = \kappa && \text{on } (0,T) \\ &\phi(u)_x(1,t) = g(u)(1,t) - g(\kappa) && \text{on } (0,T) \\ &u(x,0) = u_0(x) + \kappa && \text{on } [0,1], \end{aligned}$$

and

$$(\mathrm{B}_\kappa)\qquad \begin{aligned} &u_t = \phi(u)_{xx} + f(u)_x && \text{on } Q_T \\ &-\phi(u)_x(0,t) = g(u)(0,t) - g(\kappa) && \text{on } (0,T) \\ &u(1,t) = \kappa && \text{on } (0,T) \\ &u(x,0) = u_0(x) + \kappa && \text{on } [0,1]. \end{aligned}$$

Under the hypotheses (H1) - (H3) we have the following theorem:

Theorem 5.1 *For each $\kappa \in (0,1)$ there exists a $T > 0$, which depends only on $\| u_0 \|_{L^\infty(0,1)}$, and a weak solution (defined as for problem (A1)), u_κ, of (A_κ), alternatively of (B_κ), on Q_T such that*

$$0 < \sigma \leq \kappa < 1 \Longrightarrow u_\kappa \geq u_\sigma \geq 0 \text{ on } Q_T.$$

The existence theorem now appears as a corollary to this theorem.

Corollary 5.1 *Let $u(x,t) \equiv \lim_{\kappa \to 0^+} u_\kappa(x,t)$. Then $u(x,t)$ is a weak solution of (A1), alternatively of (B1), on Q_T.*

Proof:(of Theorem 5.1)

We shall first consider the case of problems (A_κ). To this end, fix $\kappa \in (0,1)$. We begin by constructing smooth approximations of ϕ, f, g, and u_0 possessing certain properties which will be needed later. Let $\{J_n\}_{n=1}^{\infty}$ denote a sequence of symmetric mollifiers with supp $J_n \subset [-\frac{1}{n}, -\frac{1}{n}]$. For definiteness we will take $J_n(x) \equiv n\rho(nx)$, where

$$\rho(x) \equiv \begin{cases} 0 & , \text{if } \mid x \mid \geq 1 \\ c\, exp[(x^2-1)^{-1}] & , \text{if } \mid x \mid < 1, \end{cases}$$

and c is chosen so that $\int_{-\infty}^{\infty} \rho(x)dx = 1$. Define

$$\phi_n(x) \equiv J_n * \phi(x) + k_n(x),$$

where "$*$" denotes the usual convolution,

$$J_n * \phi(x) \equiv \int_{-\infty}^{\infty} J_n(x-y)\phi(y)dy,$$

and $k_n \in C^\infty(R)$ is chosen such that $k_n(x) = 0$ for $x \leq 0, k_n \to 0$ uniformly on $(0,\infty), k_n' = 0$ for $x \geq 1, k_n' \geq 0$ on R, and $k_n''(\kappa) = -(J_n * \phi'')(\kappa)$. Similarly define

$$f_n(x) \equiv J_n * f(x) + r_n(x),$$

where $r_n \in C^\infty(R)$ is chosen such that $r_n(x) = 0$ for $x \in R \setminus (0,1), 0 \leq r_n \leq 1$ on $[0,1], r_n \to 0$ uniformly on $(0,1)$, and $r_n'(\kappa) = -J_n * f'(\kappa)$, and define

$$g_n(x) \equiv J_n * g(x) - J_n * g(\kappa),$$

Finally, let $u_{0n}(x)$ be chosen such that $u_{0n} \in C^\infty([0,1])$, supp $u_{0n} \subset (0,1), 0 \leq u_{0n} \leq$

$\| u_0 \|_{L^\infty(0,1)} + 1$ on $[0,1]$, and $u_{0n} \to u_0$ in $L^2(0,1)$. With these constructions we now turn our attention to the "regularized" problems

$$(\mathrm{A}_{\kappa,n}) \qquad \begin{array}{ll} u_t = \phi_n(u)_{xx} + f_n(u)_x & \text{on } Q_T \\ u(0,t) = \kappa & \text{on } (0,T) \\ \phi_n(u)_x(1,t) = g_n(u)(1,t) & \text{on } (0,T) \\ u(x,0) = u_{0n} + \kappa & \text{on } [0,1], \end{array}$$

and try to establish existence for them. (Once this is done it will be shown that the solutions of $(\mathrm{A}_{\kappa,n})$ converge to a solution of (A_κ) as $n \to \infty$.) However, Theorem 7.1 is not applicable because of the various boundedness assumptions. To rectify this situation we will truncate the functions ϕ_n, f_n, and g_n outside a set of the form $\kappa/2 \leq u \leq M \equiv \| u_0 \|_{L^\infty(0,1)} + 3$ in such a way that these boundedness assumptions hold. The motivation to do this comes from observing that if u is a classical solution of $(\mathrm{A}_{\kappa,n})$, then it should be the case that $\kappa \leq u \leq M - 1$. Thus $(\mathrm{A}_{\kappa,n})$ and the corresponding "truncated" problem are equivalent. The actual application of this idea will occupy us for most of what remains of this proof. In order that this general idea not be lost in the shuffle, we now give a brief outline of the steps which follow.

I Definition of the truncated functions, ϕ_n^M, f_n^M, g_n^M.

II Application of Theorem 7.1 to the truncated problems

$$(\mathrm{A}^M_{\kappa,n}) \qquad \begin{array}{ll} u_t = \phi^M_n(u)_{xx} + f^M_n(u)_x & \text{on } Q_T \\ u(0,t) = \kappa & \text{on } (0,T) \\ \phi^M_n(u)_x(1,t) = g^M_n(u)(1,t) & \text{on } (0,T) \\ u(x,0) = u_{0n}(x) + \kappa & \text{on } [0,1] \end{array}$$

III Maximum principles which will give a time $T^* > 0$ such that if u is a solution of $(\mathrm{A}^M_{\kappa,n})$, then $\kappa \le u \le M-1$ on Q_{T^*}.

At the conclusion of these steps we will have established the existence of a classical solution, $u_{\kappa,n}$, of $(\mathrm{A}_{\kappa,n})$ on Q_{T^*}. (T^* will turn out to depend only on M and not on κ or n.)

Before defining such truncations, first recall the definition of M:

$$M \equiv \| u_0 \|_{L^\infty(0,1)} + 3.$$

Define $\phi^M_n \in C^\infty(R)$ so that

$$\begin{array}{ll} \phi^M_n(u) = \phi_n(u) & \text{for } \frac{\kappa}{2} \le u \le M, \\ (\phi^M_n)'(u) = 1 & \text{on } (-\infty, \frac{\kappa}{4}] \cup [M+1, \infty), \\ (\phi^M_n)'(u) \ge \frac{1}{2}\min\left\{1, \phi_n'(\frac{\kappa}{2})\right\} & \text{on } [\frac{\kappa}{4}, \frac{\kappa}{2}], \text{ and} \\ (\phi^M_n)'(u) \ge \frac{1}{2}\min\left\{1, \phi_n'(M)\right\} & \text{on } [M, M+1]. \end{array}$$

Define $f^M_n \in C^\infty(R)$ so that

$$\begin{array}{ll} f^M_n(u) = f_n(u) & \text{for } \frac{\kappa}{2} \le u \le M, \\ f^M_n(u) = 0 & \text{on } (-\infty, \frac{\kappa}{4}] \cup [M+1, \infty), \text{ and} \\ | f^M_n(u) | \le 2\max_{[\frac{\kappa}{2}, M]} | f_n(u) | & \text{on } [\frac{\kappa}{4}, \frac{\kappa}{2}] \cup [M, M+1]. \end{array}$$

Finally, let $g_n^M(u)$ be defined in the same manner as f_n^M.

With the above definitions we now claim that Theorem 7.1 applies to problem $(\mathrm{A}_{\kappa,n}^M)$. To see that this is the case, set

$$\begin{aligned}
&a(u) = (\phi_n^M)'(u),\\
&-b(u,u_x) = (\phi_n^M)''(u)u_x^2 + (f_n^M)'(u)u_x,\\
&-\psi(u) = g_n^M(u),\\
&k = \kappa, \text{ and}\\
&\psi_0(x) = u_{0n}(x) + \kappa.
\end{aligned}$$

Clearly $a, \psi \in C^2(R), b \in C^1(R \times R), \psi_0 \in H^{(2+\beta)}([0,1])$. Also for all $u, p \in R$ and all $x \in [0,1]$ it is the case that

$$a(u) \geq \min\left\{1, \frac{1}{2}\min\left[1, \phi_n'(\tfrac{\kappa}{2})\right], \frac{1}{2}\min\left[1, \phi_n'(M)\right], \min_{[\frac{\kappa}{2}, M]} \phi_n'(u)\right\} > 0,$$

$$a(u) \leq \max\left\{1, \max_{[\frac{\kappa}{4}, M+1]} (\phi_n^M)'(u)\right\},$$

$$-ub(u,0) = 0 \leq b_1 u^2 + b_2 \quad \text{with} \quad b_1 = b_2 = 0,$$

$$\begin{aligned}
&-ub(u + \psi_0(x), p + \psi_0'(x))\\
&= u(\phi_n^M)''(u + u_{0n}(x) + \kappa)(p + u_{0n}'(x))^2\\
&\quad + u(f_n^M)'(u + u_{0n}(x) + \kappa)(p + u_{0n}'(x))\\
&\leq M \max_{0 \leq v \leq M+1} |(\phi_n^M)''(v)| \left[p^2 + \| u_{0n}' \|_{L^\infty(0,1)}\right]\\
&\quad + 2M \max_{0 \leq v \leq M+1} |(f_n^M)'(v)| \left[|p| + \| u_{0n}' \|_{L^\infty(0,1)}\right],
\end{aligned}$$

$$-b(\kappa,p)=\phi_n''(\kappa)p^2+f_n'(\kappa)p=0,$$

$$-u\psi(u+\psi_0(x))=ug_n^M(u+u_{0n}(x)+\kappa)\le 2M\max_{0\le v\le M+1}|g_n^M(v)|,$$

$$\begin{aligned}
&|b_p(u+\psi_0(x),p+\psi_0'(x))|(1+|p|)+|b_u(u+\psi_0(x),p+\psi_0'(x))|\\
&= |2(p+u_{0n}'(x))(\phi_n^M)''(u+u_{0n}+\kappa)+(f_n^M)'(u+u_{0n}(x)+\kappa)|(1+|p|)\\
&\quad+|(p+u_{0n}'(x))^2(\phi_n^M)'''(u+u_{0n}(x)+\kappa)|\\
&\quad+|(p+u_{0n}'(x))(f_n^M)''(u+u_{0n}(x)+\kappa)|\\
&\le 2\max_{0\le v\le M+1}|(\phi_n^M)''(v)|\left[(|p|+\|u_{0n}'\|_{L^\infty(0,1)})(1+|p|)\right]\\
&\quad+\max_{0\le v\le M+1}|(\phi_n^M)'''(v)|\left[p^2+\|u_{0n}'\|^2_{L^\infty(0,1)}\right]\\
&\quad+\max_{0\le v\le M+1}|(f_n^M)''(v)|\left[|p|+\|u_{0n}'\|_{L^\infty(0,1)}\right],
\end{aligned}$$

and

$$\begin{aligned}
|b(u,p)| &= |(\phi_n^M)''(u)p^2+(f_n^M)'(u)p|\\
&\le p^2\max_{0\le v\le M+1}|(\phi_n^M)''(v)|+|p|\max_{0\le v\le M+1}|(f_n^M)'(v)|,
\end{aligned}$$

from which assumptions (ii) and (iii) of Theorem 7.1 follow. Finally,

$$\begin{aligned}
\psi_0(0)=u_{0n}(0)+\kappa &= \kappa,\\
\psi_0''(0)=u_{0n}''(0) &= 0, \text{ and}\\
a(\psi_0(1))\psi_0'(1)+\psi(\psi_0(1)) &= (\phi_n^M)'(\kappa)u_{0n}'(1)-g_n^M(\kappa)\\
&= 0.
\end{aligned}$$

Therefore, for any n and any $T>0$, there exists a unique solution, $u_{\kappa,n}\in C^{2,1}(\overline{Q_T})$, of $(\mathrm{A}^M_{\kappa,n})$.

It will now be established that $u_{\kappa,n}(x,t) \geq \kappa$ on $\overline{Q_T}$; this is accomplished via a standard maximum principle argument which we include here for the convenience of the reader.

Lemma 5.1 *For every $\kappa \in (0,1)$ and every $n = 1, 2, \cdots$,*

$$u_{\kappa,n} \geq \kappa$$

on $\overline{Q_T}$.

Proof:(of Lemma 5.1)

Define

$$w(x,t) \equiv (u_{\kappa,n}(x,t) - \kappa)e^{\theta x + \lambda t},$$

where $\theta, \lambda < 0$ are chosen so as to satisfy

$$\frac{g_n^M(v+\kappa)}{v(\phi_n^M)'(v+\kappa)} + \theta < 0,$$

for any $v < 0$, and

$$\lambda + \theta^2(\phi_n^M)'(v+\kappa) + \theta\left[(\phi_n^M)''(v+\kappa)v_x + (f_n^M)'(v+\kappa)\right] < 0,$$

for all $(x,t) \in Q_T$. To see that such a choice of θ is possible recall that $g_n^M(\kappa) = g_n(\kappa) = 0, g_n^M \in C^\infty(R)$, and $(g_n^M)'(x) = 0$ for $x \in R \setminus [\frac{\kappa}{4}, M+1]$. So there exists a constant $C > 0$ such that $\mid (g_n^M)'(x) \mid \leq C$ on R. Furthermore, $g_n^M(v+\kappa) = g_n^M(\kappa) + v(g_n^M)'(\xi)$ for some ξ between κ and $v+\kappa$, and hence

$$\mid \frac{g_n^M(v+\kappa)}{v(\phi_n^M)'(v+\kappa)} \mid \leq \frac{C}{\nu},$$

where $(\phi_n^M)' \geq \nu > 0$. Therefore, we may choose $\theta = -2C/\nu$. With θ and λ so chosen observe that w satisfies

$$\begin{aligned}
w_t &= (\phi_n^M)'(v+\kappa)w_{xx} \\
&\quad + \left[2\theta(\phi_n^M)'(v+\kappa) + (\phi_n^M)''(v+\kappa)v_x + (f_n^M)'(v+\kappa)\right] w_x \\
&\quad + \left\{\lambda + \theta^2(\phi_n^M)'(v+\kappa) + \theta\left[(\phi_n^M)''(v+\kappa)v_x + (f_n^M)'(v+\kappa)\right]\right\} w, \\
&\quad \text{on } Q_T \\
w(0,t) &= 0, \text{ on } (0,T) \\
w_x(1,t) &= \left[\frac{g_n^M(v+\kappa)}{v(\phi_n^M)'(v+\kappa)}(1,t) + \theta\right] w(1,t), \text{ on } (0,T) \\
w(x,0) &= u_{0n}(x)e^{\theta x}, \text{ on } [0,1].
\end{aligned}$$

If $\min_{\overline{Q_T}} w(x,t) < 0$, then the minimum must occur either in $Q_T \cup ((0,1) \times \{T\})$ or on the set $\{1\} \times (0,T]$. The first of these cases is ruled out by the PDE. For at such a point we would have $w_t > (\phi_n^M)'(v+\kappa)w_{xx} \geq 0$ which is a contradiction. The latter case is ruled out by the boundary condition at $x = 1$. Therefore, $w(x,t) \geq 0$ on $\overline{Q_T}$, and it follows that

$$\min_{\overline{Q_T}} u_{\kappa,n}(x,t) = \kappa.$$

We now need to obtain an upper bound on $u_{\kappa,n}$. For this a different approach is taken since a certain type of uniformity in κ and n is needed which cannot be obtained from the work above. The following lemma works in this direction; its proof is simply a modification of some standard arguments. See e.g., [29].

Lemma 5.2 *Let $a, b, c \in C(R)$ be such that there exists $k_0 \geq 1$ and $\theta, \alpha > 0$ for which*

$a(u) \geq 2\theta$ for $|u| > k_0$, and

$| b(u) |, | c(u) | \leq \alpha$ for all $u \in R$.

If $u \in C^{2,1}(\overline{Q_T})$ is a classical solution of

$$\begin{aligned} &u_t = (a(u)u_x + b(u))_x && \text{on } Q_T \\ &u(0,t) = \kappa && \text{on } (0,T) \\ &a(u)u_x(1,t) = c(u)(1,t) && \text{on } (0,T) \\ &u(x,0) = \psi_0(x) && \text{on } [0,1] \end{aligned}$$

with $k_0 \geq \max\left\{\kappa, \| \psi_0 \|_{L^\infty(0,1)}\right\}$, then there exists a constant, C, which depends only on θ and α, such that

$$u(x,t) \leq k_0 + C\left[1 + T^{\frac{1}{6}}\right] T^{\frac{1}{3}}$$

for all $(x,t) \in \overline{Q_T}$.

Proof:

Let $k > k_0$, and make the following definitions:

$$(r)^+ \equiv \max(r, 0).$$

$$A_k(t) \equiv \{x \in [0,1] \mid u(x,t) > k\}.$$

$$\mu(k) \equiv \int_0^T \int_{A_k(t)} dxdt.$$

Then for any $T^* \in [0,T]$ we use the PDE to derive that

$$\begin{aligned} \frac{1}{2}\int_0^1 (u-k)^{+2}(x,T^*)dx &= \int_0^{T^*}\int_0^1 (u-k)^+ u_t dxdt \\ &= -\int_0^{T^*}\int_0^1 (u-k)^+_x \left(a(u)u_x + b(u)\right) dxdt \\ &\quad + \int_0^{T^*} (u-k)^+ [c(u) + b(u)](1,t)dt \\ &\leq \int_0^{T^*}\int_{A_k(t)} \left\{-\theta(u-k)^{+2}_x + \frac{10\alpha}{\theta}\right\} dxdt \end{aligned}$$

Hence

$$
\begin{aligned}
|\,(u-k)^{+}\,|^{2}_{V_2(Q_T)} &\equiv \text{ess sup}_{0\le t\le T} \parallel (u(\cdot,t)-k)^{+} \parallel^{2}_{L^2(0,1)} \\
&\quad + \parallel (u-k)^{+}_{x} \parallel^{2}_{L^2(Q_T)} \\
&\le \frac{2}{\min(\frac{1}{2},\theta)}\left(\frac{10\alpha^2}{\theta}\right)\mu(k) \\
&\le C_1 T^{\frac{1}{2}} \mu(k)^{\frac{1}{2}},
\end{aligned}
$$

where

$$
C_1 \equiv \frac{2}{\min(\frac{1}{2},\theta)}\left(\frac{10\alpha^2}{\theta}\right).
$$

Now by a result from [18, page 77], there exists a constant β, which does not depend on T, such that

$$
\parallel (u-k)^{+} \parallel_{L^6(Q_T)} \le \left[2\beta + T^{\frac{1}{6}}\right] |\,(u-k)^{+}\,|_{V_2(Q_T)}\,.
$$

Combining these results, it follows that for $\delta > 0$ and $h > k \ge k_0 + \delta$,

$$
\begin{aligned}
(h-k)^2 \mu(h)^{\frac{1}{3}} &\le \parallel (u-k)^{+} \parallel^{2}_{L^6(Q_T)} \\
&\le C_1 \left[2\beta + T^{\frac{1}{6}}\right]^2 T^{\frac{1}{2}} \mu(k)^{\frac{1}{2}},
\end{aligned}
$$

and so

$$
\mu(h) \le \left\{ C_1 \left[2\beta + T^{\frac{1}{6}}\right]^2 \right\}^3 T^{\frac{3}{2}} \frac{\mu(k)^{\frac{3}{2}}}{(h-k)^6}.
$$

Since $\mu(\cdot)$ is nonincreasing in k, we may apply a lemma from [17, page 63], to get $\mu(d + k_0 + \delta) = 0$, where

$$
d = \left\{ \left[C_1 \left(2\beta + T^{\frac{1}{6}}\right)^2 \right]^3 T^{\frac{3}{2}} 2^{18} \mu(k_0+\delta)^{\frac{1}{2}} \right\}^{\frac{1}{6}}
$$

$$\leq \; C\left(1+T^{\frac{1}{6}}\right)T^{\frac{1}{3}}.$$

Therefore, $u(x,t) \leq k_0 + \delta + C\left(1+T^{\frac{1}{6}}\right)T^{\frac{1}{3}}$ for all $(x,t) \in \overline{Q_T}$ and for any $\delta > 0$. The lemma now follows by letting $\delta \to 0^+$.

To apply Lemma 5.2 to problem $(\mathrm{A}^M_{\kappa,n})$, set $a(u) = (\phi^M_n)'(u), b(u) = f^M_n(u)$, $c(u) = g^M_n, \psi_0(x) = u_{0n}(x) + \kappa$, and $k_0 = \| u_0 \|_{L^\infty(0,1)} + 2$. Then for $| u | > k_0$ it follows that

$$a(u) \leq \min\left\{\frac{1}{2}\min\{1, \phi'_n(M)\}, \min_{[k_0,M]} \phi'_n(u)\right\}.$$

Now

$$\begin{aligned}
\min_{[k_0,M]} \phi'_n(u) &= \min_{[k_0,k_0+1]} \int_{-1}^{1} \rho(z)\phi'\left(u - \frac{z}{n}\right) dz \\
&\geq \min_{[k_0-1,k_0+2]} \phi'(u) \\
&\geq \min_{[1,M+1]} \phi'(u),
\end{aligned}$$

so it follows that if $| u | > k_0$, then

$$a(u) \geq \frac{1}{2}\min\left\{1, \min_{[1,M+1]} \phi'(u)\right\} \equiv 2\theta > 0.$$

Furthermore,

$$\begin{aligned}
| b(u) | &\leq 2 \max_{[0,k_0+3]} | f_n(u) | \\
&\leq 2\left\{\max_{[-1,M+5]} | f(u) | + 1\right\},
\end{aligned}$$

and

$$| c(u) | \leq 3 \max_{[-1,M+5]} | g(u) |,$$

so α can be taken to be the larger of these upper bounds. Therefore, by Lemma 5.1,

$$u_{\kappa,n}(x,t) \leq M - 1 + C\left(1 + T^{\frac{1}{6}}\right)T^{\frac{1}{3}},$$

for all $(x,t) \in \overline{Q_T}$.

Thus, for $T^* > 0$ such that $C(1 + (T^*)^{\frac{1}{6}})(T^*)^{\frac{1}{3}}) < 1$, we have $\kappa \leq u_{\kappa,n}(x,t) < M$ on Q_{T^*}. It follows that $u_{\kappa,n}$ is actually a classical solution of $(\mathrm{A}_{\kappa,n})$. Note that T^* does not depend on n or κ. So what we have shown up to this point is that there exists a $T^* > 0$ such that for every $n = 1, 2, \ldots$ and for every $\kappa \in (0,1)$, $(\mathrm{A}_{\kappa,n})$ has a unique solution, $u_{\kappa,n} \in C^{2,1}(\overline{Q_{T^*}})$, with

$$\kappa \leq u_{\kappa,n}(x,t) < \| u_0 \|_{L^\infty(0,1)} + 3$$

on $\overline{Q_{T^*}}$. From this point on the star on T^* will be dropped and this time will be referred to as simply T.

We now take a moment to outline the final steps in the completion of the proof of Theorem 5.1.

IV Hölder estimates of $u_{\kappa,n}$ which are uniform in n for fixed $\kappa \in (0,1)$. Hence, for fixed $\kappa \in (0,1)$, the set $\{u_{\kappa,n}\}_{n=1}^{\infty}$ is equicontinuous.

V Invoke the Arzela - Ascoli Theorem to obtain a convergent subsequence, u_{κ,n_k}, the limit of which will turn out to be a weak solution of (A_κ).

VI Estimate of $\| \phi_n(u_{\kappa,n})_x \|_{L^2(Q_T)}$ which is uniform in κ and n. Thus, invoking weak compactness arguments and taking a further subsequence of $\{u_{\kappa,n_k}\}_{k=1}^{\infty}$, if necessary, it follows that $u_\kappa \equiv \lim_{k\to\infty} u_{\kappa,n_k}$ has $\phi(u_\kappa)_x \in L^2(Q_T)$. Subsequently, u_κ is a weak solution of (A_κ).

VII Monotonicity result for the problems (A_κ): $0 < \sigma \le \kappa < 1 \Longrightarrow u_\sigma \le u_\kappa$.

The Hölder estimates referred to in IV are contained in Theorems 1.1 [18, page 419] and 7.1 [18, page 478]. These bounds are uniform in n for fixed $\kappa \in (0,1)$, and hence $\{u_{\kappa,n}\}_{n=1}^{\infty}$ is a uniformly bounded, equicontinuous collection. It follows that there exists a subsequence, $\{u_{\kappa,n_k}\}$, which converges uniformly to a limit function, u_κ, on $\overline{Q_T}$.

Let $v(x,t) \equiv u_{\kappa,n}(x,t)$. Multiplying the PDE of problem ($A_{\kappa,n}$) by $\phi_n(v)$ and integrating over Q_T, we obtain

$$\begin{aligned}\int_0^T\int_0^1 \phi_n(v)v_t\,dxdt &= -\int_0^T\int_0^1 \phi_n(v)_x\left[\phi_n(v)_x + f_n(v)\right]dxdt\\ &\quad + \int_0^T \phi_n(v)\left[\phi_n(v)_x + f_n(v)\right]\Big|_{x=0}^{x=1} dt.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{2}\int_0^T\int_0^1 \phi_n(v)_x^2\,dxdt &\le -\int_0^1 \Phi_n(v)(x,t)\Big|_{t=0}^{t=T} dx + \frac{1}{2}\int_0^T\int_0^1 f_n(v)^2\,dxdt\\ &\quad + \int_0^T \phi_n(v)\left[g_n(v) + f_n(v)\right](1,t)dt\\ &\quad - \int_0^T \phi_n(\kappa)\left[\phi_n'(\kappa)v_x(0,t) + f_n(\kappa)\right]dt,\end{aligned}$$

where

$$\Phi_n(u) \equiv \int_0^u \phi_n(v)dv.$$

Since $v(0,t) = \kappa \le v(x,t)$ on $\overline{Q_T}$, it must be that $v_x(0,t) \ge 0$ on $(0,T)$. Therefore,

$$\begin{aligned}\int_0^T\int_0^1 \phi_n(v)_x^2\,dxdt &\le -2\int_0^1 \Phi_n(v)(x,t)\Big|_{t=0}^{t=T} dx + \int_0^T\int_0^1 f_n(v)^2\,dxdt\\ &\quad + \int_0^T \phi_n(v)\left[g_n(v) + f_n(v)\right](1,t)dt,\end{aligned}$$

and so we have a uniform bound on $\| \phi_n(u_{\kappa,n})_x \|_{L^2(Q_T)}$. By the weak compactness of bounded sets in the Hilbert space, $L^2(Q_T)$, see [8], taking a further subsequence of $\{u_{\kappa,n}\}$, if necessary, will give a sequence $\{\phi_{n_k}(u_{\kappa,n_k})\}$ which converges weakly in $L^2(Q_T)$. Moreover, since for any $\xi \in C^\infty(Q_T)$,

$$\begin{aligned}\int_0^T\int_0^1 \phi_{n_k}(u_{\kappa,n_k})_x\xi dxdt &\equiv -\int_0^T\int_0^1 \phi_{n_k}(u_{\kappa,n_k})\xi_x dxdt \\ &\overset{k\to\infty}{\longrightarrow} -\int_0^T\int_0^1 \phi(u_\kappa)\xi_x dxdt,\end{aligned}$$

it follows that $\phi(u_\kappa)_x \in L^2(Q_T)$, and $\phi_{n_k}(u_{\kappa,n_k})_x$ converges to $\phi(u_\kappa)_x$ weakly in $L^2(Q_T)$.

Now let $\xi \in P_{\{0\}}(\overline{Q_T})$. Then for any $t \in [0,T]$, we have

$$\begin{aligned}&\int_0^1 u_{\kappa,n_k}(x,s)\xi(x,s)\,|_{s=0}^{s=t}\,dx \\ &= \int_0^t\int_0^1 \left\{u_{\kappa,n_k}\xi_s - \left[\phi_{n_k}(u_{\kappa,n_k})_x + f_{n_k}(u_{\kappa,n_k})\right]\xi_x\right\}dxds \\ &\quad + \int_0^t \left[g_{n_k}(u_{\kappa,n_k}) + f_{n_k}(u_{\kappa,n_k})\right]\xi(1,s)ds.\end{aligned}$$

Passing to the limit in this equality, via the Lebesgue dominated convergence theorem, yields

$$\begin{aligned}\int_0^1 u_\kappa(x,t)\xi(x,t)dx &= \int_0^1 (u_0(x)+\kappa)\xi(x,0)dx + \int_0^t [g(u_\kappa) - g(\kappa) + f(u_\kappa)]\xi(1,s)ds \\ &\quad + \int_0^t\int_0^1 \{u_\kappa\xi_s - [\phi(u_\kappa)_x + f(u_\kappa)]\,\xi_x\}\,dxds,\end{aligned}$$

from whence it follows that u_κ is a weak solution of (A_κ) on Q_T.

The monotonicity result is a consequence of the following two lemmas.

Lemma 5.3 *If $0 < \sigma \le \kappa < 1$, then*

$$u_{\sigma,n} \le u_{\kappa,n}$$

on $\overline{Q_T}$ for each $n = 1, 2, \cdots$.

Lemma 5.4 *The weak solution of (A_κ) is unique when further restricted to the class of functions*

$$\left\{u : u \geq \kappa \text{ on } \overline{Q_T}\right\}.$$

Once these lemmas have been proved, it will follow from Lemma 5.4 and our previous work that

$$\lim_{n\to\infty} u_{\kappa,n} = u_\kappa$$

for each $\kappa \in (0,1)$. Hence the monotonicity is concluded from Lemma 5.3. We now take up the proofs of these results.

Proof:(of Lemma 5.3)

This is another standard maximum principle argument much like that used to establish Lemma 5.1. Let

$$w(x,t) \equiv \left(u_{\kappa,n} - u_{\sigma,n}\right) e^{\theta x + \lambda t},$$

where $\theta, \lambda < 0$ are chosen so that

$$\max_{\sigma \leq v \leq M} \frac{| g_n'(v) | + | \phi_n''(v) | \| (u_{\sigma,n})_x \|_{L^\infty(Q_T)}}{\phi_n'(v)} + \theta < 0$$

and

$$\lambda + A\theta^2 + B\theta + C < 0,$$

where

$$A \equiv \max_{\sigma \leq v \leq M} \phi_n'(v),$$

$$\begin{aligned} B \equiv & \max_{\sigma \leq v \leq M} | \phi_n''(v) | \left[\| u_{\sigma,n} \|_{L^\infty(Q_T)} + \| (u_{\sigma,n})_x \|_{L^\infty(Q_T)}\right] \\ & + \max_{\sigma \leq v \leq M} | f_n'(v) |, \end{aligned}$$

and

$$
\begin{aligned}
C \equiv & \max_{\sigma\le v\le M} |\, \phi_n'''(v) \,|\| \, (u_{\sigma,n})_x^2 \, \|_{L^\infty(Q_T)} \\
&+ \max_{\sigma\le v\le M} |\, \phi_n''(v) \,|\| \, (u_{\sigma,n})_{xx} \, \|_{L^\infty(Q_T)} \\
&+ \max_{\sigma\le v\le M} |\, f_n' \,|\| \, (u_{\sigma,n})_x \, \|_{L^\infty(Q_T)} \, .
\end{aligned}
$$

Proceed as in Lemma 5.1 to conclude that $w \geq 0$ on Q_T. The lemma now follows.

Proof:(of Lemma 5.4)

Suppose u_K and v_K are both weak solutions of (A$_K$) with $u_K \geq \kappa$ and $v_K \geq \kappa$ on $\overline{Q_T}$. Then we have

$$
\begin{aligned}
\int_0^1 u_K(x,t)\xi(x,t)dx = & \int_0^1 (u_0(x)+\kappa)\xi(x,0)dx + \int_0^t [g(u_K) - g(\kappa) + f(u_K)]\xi(1,s)ds \\
&+ \int_0^t \int_0^1 \{u_K\xi_s - [\phi(u_K)_x + f(u_K)]\,\xi_x\}\, dxds,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^1 v_K(x,t)\xi(x,t)dx = & \int_0^1 (u_0(x)+\kappa)\xi(x,0)dx + \int_0^t [g(v_K) - g(\kappa) + f(v_K)]\xi(1,s)ds \\
&+ \int_0^t \int_0^1 \{v_K\xi_s - [\phi(v_K)_x + f(v_K)]\,\xi_x\}\, dxds,
\end{aligned}
$$

for every $t \in [0,T]$ and every $\xi \in P_{\{0\}}(\overline{Q_T})$. Thus, if $\xi \in C^{2,1}(\overline{Q_T}) \cap P_{\{0\}}(\overline{Q_T})$, then an integration by parts gives

$$
\begin{aligned}
\int_0^1 u_K(x,t)\xi(x,t)dx = & \int_0^1 (u_0(x)+\kappa)\xi(x,0)dx \\
&+ \int_0^t \int_0^1 \{u_K\xi_s + \phi(u_K)\xi_{xx} - f(u_K)\xi_x\}\, dxds \\
&+ \int_0^t \{[g(u_K) + f(u_K)]\,(1,s)\xi(1,s) - \phi(u_K)\xi_x(1,s)\}\, ds \\
&+ \int_0^t \{\phi(\kappa)\xi_x(0,s) - g(\kappa)\xi(1,s)\}\, ds
\end{aligned}
$$

and a similar expression for v_K. Subtracting these equalities, we have

$$\begin{aligned}
&\int_0^1 (u_K - v_K)\xi(x,t)dx \\
&= \int_0^t \int_0^1 \{(u_K - v_K)\xi_s + (\phi(u_K) - \phi(v_K))\xi_{xx} - (f(u_K) - f(v_K))\xi_x\}\, dxds \\
&\quad + \int_0^t \{[g(u_K) + f(u_K) - g(v_K) - f(v_K)]\xi(1,s) - [\phi(u_K) - \phi(v_K)]\xi_x(1,s)\}\, ds,
\end{aligned}$$

for each such ξ. Define

$$\Phi(x,t) \equiv \int_0^1 \phi'(\theta u_K(x,t) + (1-\theta)v_K(x,t))d\theta,$$

$$F(x,t) \equiv \int_0^1 f'(\theta u_K(x,t) + (1-\theta)v_K(x,t))d\theta,$$

and

$$H(t) \equiv \int_0^1 g'(\theta u_K(1,t) + (1-\theta)v_K(1,t))d\theta + F(1,t).$$

Then we have

$$\begin{aligned}
\int_0^1 (u_K - v_K)\xi(x,t)dx \;\leq\; & \int_0^t \int_0^1 (u_K - v_K)\{\xi_s + \Phi\xi_{xx} - F\xi_x\}\, dxds \\
& + \int_0^t (u_K - v_K)(1,s)\{H(t)\xi(1,s) - \Phi(1,s)\xi_x(1,s)\}\, ds
\end{aligned}$$

for each $t \in [0,T]$ and every $\xi \in C^{2,1}(\overline{Q_T}) \cap P_{\{0\}}(\overline{Q_T})$. Now proceeding as in the proof of the uniqueness result in Chapter 4, let $\Phi_n, F_n \in C^\infty(\overline{Q_T})$ be functions such that

1. $\Phi_n \to \Phi$ and $F_n \to F$ as $n \to \infty$ in $L^2(Q_T)$,
2. $0 < \nu \leq \Phi_n$ and $\Phi_n, | F_n | \leq \mu$ on $\overline{Q_T}$ for constants ν and μ which are independent of n, and
3. $F_n(0,t) = 0$.

Further, let $\psi \in C^{\infty}([0,T])$, and let $\chi \in C^{\infty}([0,1])$ be such that $\text{supp}\xi \subset (0,1)$ and $0 \le \xi \le 1$. With $\xi = \xi_n$ denoting the solution of the "adjoint problem"

$$\text{(AP)} \qquad \begin{aligned} &\xi_s + \Phi_n \xi_{xx} - F_n \xi_x = 0 && \text{on } Q_t \\ &\xi(0,s) = 0 && \text{on } (0,t) \\ &\xi_x(1,s) = \psi(s)\xi(1,s) && \text{on } (0,t) \\ &\xi(x,t) = \chi(x) && \text{on } [0,1], \end{aligned}$$

we thus have

$$\begin{aligned} &\int_0^1 (u_K - v_K)\xi(x,t)dx \\ &\le \int_0^t \int_0^1 (u_K - v_K)\left\{[\Phi - \Phi_n]\xi_{xx} - [F - F_n]\xi_x\right\} dxds \\ &\quad + \int_0^t (u_K - v_K)\xi(1,s)[H(s) - \Phi(1,s)\psi(s)]ds \\ &\le C \parallel \Phi - \Phi_n \parallel_{L^2(Q_T)} \parallel \xi_{xx} \parallel_{L^2(Q_t)} + C \parallel F - F_n \parallel_{L^2(Q_T)} \parallel \xi_x \parallel_{L^2(Q_t)} \\ &\quad + C \parallel \xi(1,\cdot) \parallel_{L^\infty(0,t)} \int_0^t \mid H(s) - \Phi(1,s)\psi(s) \mid ds. \end{aligned}$$

By the results of Appendix B, letting $n \to \infty$ in this inequality yields

$$\int_0^1 (u_K - v_K)(x,t)\chi(x)dx \le C \int_0^t \mid H(s) - \Phi(1,s)\psi(s) \mid ds,$$

where here the constant C depends on $\parallel \psi \parallel_{L^\infty(0,T)}$.

To complete the proof we first consider a limit of such functions ψ which converge a.e. on $(0,t)$ to the function $H/\Phi(1,\cdot)$ and then a limit of functions χ which converge a.e. on (0,1) to the function which is given by

$$\begin{cases} 0 & \text{if } u_K(x,t) - v_K(x,t) \le 0 \\ 1 & \text{if } u_K(x,t) - v_K(x,t) > 0 \end{cases}.$$

Thus

$$\int_0^1 (u_\kappa - v_\kappa)^+(x,t)dx \leq 0,$$

for each $t \in [0,T]$, from which it follows that $u_\kappa \leq v_\kappa$ on $\overline{Q_T}$. Reversing the above argument now completes the proof.

This concludes the proof of Theorem 5.1.

Corollary 5.1A now follows immediately by observing that our estimate on $\| \phi_n(u_{\kappa,n})_x \|_{L^2(Q_T)}$ was uniform in both κ and n, and hence by letting $\kappa \to 0^+$ in

$$\begin{aligned}\int_0^1 u_\kappa \xi(x,t)dx &= \int_0^1 (u_0(x)+\kappa)\xi(x,0)dx + \int_0^t [g(u_\kappa) - g(\kappa) + f(u_\kappa)]\xi(1,s)ds \\ &\quad + \int_0^t \int_0^1 \{u_\kappa \xi_s - [\phi(u_\kappa)_x + f(u_\kappa)]\,\xi_x\}\,dxds\end{aligned}$$

the conclusion follows.

The above process could be repeated to give similar results for problems $(B^M_{\kappa,n})$, $(B_{\kappa,n})$, (B_κ), and, ultimately, (B1). However it is simpler to observe that if $u_{\kappa,n}$ is the solution of $(A_{\kappa,n})$ with f_n and g_n replaced by $-f_n$ and g_n, respectively, then $v_{\kappa,n}(x,t) \equiv u_{\kappa,n}(1-x,t)$ is the solution of $(B_{\kappa,n})$. Moreover, the results established for $u_{\kappa,n}$ allow one to easily complete the proof.

In regards to the asserted continuation, we remark that this follows from the observation that the time T^* which was constructed above depended only on $\| u_0 \|_{L^\infty(0,1)}$.

5.3 Increased Regularity Results

In the present section the proof of Theorem 2.2 is given. This proof relies heavily on the constructions from Section 5.1, and so the notation from that section will be used freely here. Our proof also makes use of the fact that bounded sets in a Hilbert space are weakly precompact, a result which follows from the reflexivity of Hilbert spaces and Alaoglu's theorem. (Such facts can be found in any good modern analysis text. See e.g., [8].)

To begin, define

$$\Phi_n(u) \equiv \int_0^u \phi_n(v)dv \text{ and } \Psi_n(u) \equiv \int_0^u \sqrt{\phi_n'(v)}dv.$$

Also, to keep the number of subscripts from becoming unruly and confusing, we shall continue to write $v(x,t)$ in place of $u_{\kappa,n}(x,t)$. Now, by multiplying the PDE in $(\mathrm{A}_{\kappa,n})$ by v and integrating over Q_T, we obtain

$$\begin{aligned} \frac{1}{2}\int_0^1 v^2(x,t)\,\Big|_{t=0}^{t=T} dx &= \int_0^T\int_0^1 vv_t\,dxdt \\ &= \int_0^T\int_0^1 v\left[\phi_n(v)_x + f_n(v)\right]_x dxdt. \end{aligned}$$

An integration by parts thus yields

$$\begin{aligned} &\frac{1}{2}\int_0^1 v^2(x,t)\,\Big|_{t=0}^{t=T} dx \\ &= -\int_0^T\int_0^1 v_x\left[\phi_n(v)_x + f_n(v)\right] dxdt \\ &\quad + \int_0^T \left\{v\left[g_n(v) + f_n(v)\right](1,t) - \kappa\left[\phi_n'(\kappa)v_x(0,t) + f_n(\kappa)\right]\right\} dt, \end{aligned}$$

which upon rearrangement becomes

$$\begin{aligned} &\frac{1}{2}\int_0^1 v^2(x,t)\,\Big|_{t=0}^{t=T} dx + \int_0^T\int_0^1 \Psi_n(v)_x^2\,dxdt \\ &\qquad + \int_0^T \kappa\left(\phi_n'(\kappa)v_x(0,t) + f_n(\kappa)\right) dt \\ &= -\int_0^T\int_0^1 f_n(v)v_x\,dxdt + \int_0^T v\left[g_n(v) + f_n(v)\right](1,t)dt \\ &= \int_0^T \left\{\left[v(g_n(v) + f_n(v)) - F_n(v)\right](1,t) + F_n(\kappa)\right\} dt, \end{aligned}$$

where

$$F_n(u) \equiv \int_0^u f_n(v)dv.$$

Recall $v_x(0,t) \geq 0$. From this and the last inequality above it follows that $\| \Psi_n(u_{\kappa,n})_x \|_{L^2(Q_T)}$ is bounded uniformly in n and κ.

If we now multiply the PDE in $(\mathrm{A}_{\kappa,n})$ by $\phi_n(v)_t$, integrate over Q_T, and integrate by parts, then

$$\begin{aligned}\int_0^T\int_0^1 \phi_n(v)_t v_t dxdt &= \int_0^T\int_0^1 \left\{-\phi_n(v)_{xt}\phi_n(v)_x + f_n'(v)\phi_n'(v)v_t v_x\right\} dxdt \\ &\quad + \int_0^T \phi_n(v)_t g_n(v)(1,t)dts \\ &= -\int_0^1 \frac{\phi_n(v)_x^2}{2} \Big|_{t=0}^{t=T} dx + \int_0^T\int_0^1 f_n'(v)\Psi_n(v)_t\Psi_n(v)_x dxdt \\ &\quad + \int_0^T \frac{\partial}{\partial t}\left\{\int_{u_{0n}(1)}^{v(1,t)} \phi_n'(s)g_n(s)ds\right\} dt.\end{aligned}$$

Observe that $| f_n'(v) | \leq \max_{[-1,M+1]} | f'(v) |$. Thus by Cauchy's inequality we find

$$\begin{aligned}\frac{1}{2}\int_0^T\int_0^1 \Psi_n(v)_t^2 dxdt &\leq -\int_0^1 \frac{\phi_n(v)_x^2}{2} \Big|_{t=0}^{t=T} dx + C\int_0^T\int_0^1 \Psi_n(v)_x^2 dxdt \\ &\quad + \int_{u_{0n}(1)}^{v(1,T)} \phi_n'(s)g_n(s)ds.\end{aligned}$$

If $\phi(u_0)_x \in L^2(0,1)$, then we claim that $\| \phi_n(u_{0n})_x \|_{L^2(0,1)}$ is bounded uniformly in n. This is actually a consequence of the Principle of Uniform Boundedness, see [8], and the fact that $\phi_n(u_{0n})_x \rightarrow \phi(u_0)_x$ weakly in $L^2(0,1)$. Therefore, there exists a constant C such that $\| \Psi_n(u_{\kappa,n})_t \|_{L^2(Q_T)} \leq C$. Using a method similar to the way we established $\phi(u)_x \in L^2(Q_T)$ in the previous section, it follows that $\Psi(u)_t \in L^2(Q_T)$.

In the above manner, it is also shown that, for fixed $t \in [0,T]$,

$\| \phi_n(u_{\kappa,n}(\cdot,t))_x \|_{L^2(0,1)}$ is bounded uniformly in n and hence has a subsequence which converges weakly to $\phi(u)(\cdot,t)_x$ in $L^2(0,1)$. Therefore, if $f \equiv 0$, then passing to the limit in the above inequality (with T replaced by t) yields

$$\int_0^t \int_0^1 \Psi(u)_s^2 dxds \leq -\frac{1}{2}\int_0^1 \phi(u)_x^2(x,s) \Big|_{s=0}^{s=t} dx + \int_{u_0(1)}^{u(1,t)} \phi'(s)g(s)ds,$$

which is the "energy inequality."

That

$$\phi(u)_t, (u\phi(u))_t, \Phi(u)_t \in L^2(Q_T)$$

and

$$\Phi(u)_t = \frac{\phi(u)}{\sqrt{\phi'(u)}}\Psi(u)_t$$

now follows exactly as in [23, pages 155–156]. Finally, we see that $\phi(u) \in P_{\{0\}}(\overline{Q_T})$, and so Theorem 2.2A(ii) follows by setting $\xi = \phi(u)$.

The results for Theorem 2.2B are proved exactly as above.

6 BIBLIOGRAPHY

[1] H. Amann. "Quasilinear parabolic systems under nonlinear boundary conditions." Arch. Rat. Mech. Anal., 92(1986), 153-192.

[2] D. Aronson, M. G. Crandall, L. A. Peletier. "Stabilization of solutions of a degenerate nonlinear diffusion problem." University of Wisconsin - Madison, Mathematics Research Center, Technical Summary Report No. 2220(1981).

[3] P. Benilan. Evolution Equations and Accretive Operators. Unpublished notes(written by S. Lenhart), 1981. Available from P. Sacks, Dept. of Mathematics, Iowa State University, Ames, Iowa 50011.

[4] M. G. Crandall. "Nonlinear semgroups and evolution governed by accretive operators." Proceedings of Symposia in Pure Math., 45(1986).

[5] J. I. Diaz and R. Kersner. "On a nonlinear degenerate parabolic equation in infiltration or evaporation through a porous medium." Journal of Diff. Eqns., 69(1987), 368-403.

[6] E. DiBenedetto. "Continuity of weak solutions to a general porous medium equation." Indiana Univ. Math. Journal, 32(1983), 83-118.

[7] L. C. Evans. Estimates for Second Order Partial Differential Equations. Unpublished notes. Available from P. Sacks, Dept. of Mathematics, Iowa State University, Ames, Iowa 50011.

[8] A. Friedman. Foundations of Modern Analysis. New York, New York: Holt, Rinehart, and Winston, 1970.

[9] A. Friedman. Partial Differential Equations of Parabolic Type. Englewood Cliffs, New Jersey: Prentice Hall, Inc., 1964.

[10] B. H. Gilding. "Improved theory for a nonlinear degenerate parabolic equation." Tech. Hoge. Twente, Mem. Nr. 587(1986).

[11] B. H. Gilding. "A nonlinear degenerate parabolic equation." Ann. Scola Norm. Sup. Pisa, 4(1977), 393-432.

[12] B. H. Gilding. "Properties of solutions of an equation in the theory of infiltration." Archive for Rat. Mech. and Anal., 65(1977), 203-225.

[13] B. H. Gilding and R. Kersner. "Instantaneous shrinking in nonlinear diffusion-convection." University of Twente, Mem. No. 715, 1988.

[14] M. W. Hirsch. "Differential equations and convergence almost everywhere in strongly monotone semiflows." Contemporary Mathematics, 17(1983), 267-285.

[15] F. John. Partial Differential Equations. Fourth edition. New York, New York: Springer- Verlag, 1982.

[16] R. Kersner. "Filtration with absorption: Necessary and sufficient condition for the propagation of perturbations to have finite velocity." Journal of Math. Anal. and Appl., 90(1982), 463-479.

[17] D. Kinderlehrer and G. Stampacchia. An Introduction to Variational Inequalities and their Applications. New York, New York: Academic Press, 1980.

[18] O. A. Ladyzenskaja, V. A. Solonikov, N. N. Ural'ceva. Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs 23. Providence, Rhode Island: American Mathematical Society, 1968.

[19] H. A. Levine. "Stability and instability for solutions of Burgers' equation with a semilinear boundary condition." SIAM Journal on Mathematical Analysis, 19(1988), 312-336.

[20] H. A. Levine and L. E. Payne. "Nonexistence of global weak solutions for classes of nonlinear wave and parabolic equations." Journal of Math. Anal. and Appl., 55(1976), 329-334.

[21] H. A. Levine and L. E. Payne. "Some nonexistence theorems for initial boundary value problems with nonlinear initial constraints." Proc. AMS, 46(1974), 277-284.

[22] H. A. Levine and L. E. Payne. "Nonexistence theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backward in time." Journal of Diff. Eqns., 16(1974), 319-334.

[23] H. A. Levine and P. E. Sacks. "Some existence and nonexistence theorems for solutions of degenerate parabolic equations." Journal of Diff. Eqns., 52(1984), 135- 161.

[24] G. M. Lieberman. "Mixed boundary value problems for elliptic and parabolic differential equations of second order." Journal Math. Anal. Appl., 113(1986), 422-440.

[25] H. Matano. "Asymptotic behavior and stability of solutions of semilinear diffusion equations." Publ. RIMS, Kyoto Univ., 15(1979), 401-454.

[26] L. A. Peletier. "The porous medium equation," in: Application of Nonlinear Analysis in the Physical Sciences (eds. H. Amman, N. Bazley, and K. Kirchgassner), 229-241. Boston, Massachusetts: Pitman, 1981.

[27] M. H. Protter and H. F. Weinberger. Maximum Principles in Differential Equations. Englewood Cliffs, New Jersey: Prentice Hall, Inc., 1967.

[28] P. E. Sacks. "Continuity of solutions of a singular parabolic equation." Nonlinear Analysis, Theory, Methods, and Applications, 7(1983), 387-409.

[29] P. E. Sacks. "Existence and regularity of solutions of inhomogeneous porous medium type equations." University of Wisconsin - Madison, Mathematics Research Center, Technical Summary Report No. 2214, 1981.

[30] K. R. Symon. Mechanics. Third edition. Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1971.

[31] N. Wolanski. "Flow through a porous column." Journal of Math. Anal. and Appl., 109(1985), 140-159.

[32] N. Wolanski. "Degenerate nonlinear parabolic equations comparison and existence theorems." Trabajos de Math., 72(1984), 1-34.

[33] X. Xu. "Semigroup approach to the Cauchy problem for the equations $u_t - \alpha(u)_{xx} + \beta(u)_x \ni f$ with nonlinear boundary conditions." Preprint. Available from Xiangsheng Xu, Dept. of Mathematics, Texas A & M University, College Station, Texas 77843, U.S.A.

7 APPENDIX A. GLOBAL EXISTENCE FOR NONLINEAR, UNIFORMLY PARABOLIC EQUATIONS WITH MIXED BOUNDARY CONDITIONS

Consider the boundary value problem

$$
\text{(NBVP)} \qquad \begin{array}{ll}
u_t - a(u)u_{xx} + b(u, u_x) = 0 & \text{on } Q_T \equiv (0,1) \times (0,T) \\
u(0,t) = k & \text{on } (0,T) \\
[a(u)u_x + \psi(u)](1,t) = 0 & \text{on } (0,T) \\
u(x,0) = \psi_0(x) & \text{on } [0,1]
\end{array}
$$

where the following assumptions will be made.

(i) $a, \psi \in C^2(R)$; $b \in C^1(R \times R)$; $\psi_0 \in H^{2+\beta}([0,1])$ for some $\beta \in (0,1)$; k is a given constant.

(ii) There exist constants μ_1 , ν_1 and b_i ,$c_j \geq 0$, $(i = 1,2; j = 1,2,3,4)$ such that $0 < \nu_1 \leq a(u) \leq \mu_1$ (uniform parabolicity), $-ub(u,0) \leq b_1 u^2 + b_2$,

$$-ub(u + \psi_0(x), p + \psi_0'(x)) \leq c_0 p^2 + c_1 u^2 + c_2,$$

$b(k,p) = 0$, and $-u\psi(u + \psi_0(x)) \leq c_3 u^2 + c_4$ for every $u, p \in R$ and every $x \in [0,1]$.

(iii) Given a constant $M \geq 0$ there exists a constant $\mu \geq 0$ such that

$$| b_p(u+\psi_0(x), p+\psi_0'(x)) | (1+ | p |) + | b_u(u+\psi_0(x), p+\psi_0'(x)) | \leq \mu(1+p^2),$$

and $| b(u,p) | \leq \mu(1+ | p |)^2$ for every $| u | \leq M$, $p \in R$, $x \in [0,1]$.

(iv) $\psi_0(0) = k$, $\psi_0''(0) = 0$, and $a(\psi_0(1))\psi_0'(1) + \psi(\psi_0(1)) = 0$.

The purpose of this appendix is to sketch the proof of the following global existence result:

Theorem 7.1 *Under the assumptions (i) - (iv) and for any $T > 0$ there exists a unique solution, $u \in H^{2+\beta,1+\frac{\beta}{2}}(\overline{Q_T})$, of (NBVP).*

This theorem is really just an adaptation of results in Chapter 5 of [18]. However, due to the great amount of generality of [18] which causes a certain degree of difficulty in a first reading, we feel the need to discuss its proof to the degree of highlighting the main ingredients and referencing the appropriate theorems from [18] when they are needed. Although existence theorems for problems such as (NBVP) are known to follow from classical results, we have been able to find few authors, such as Amann [1] and Lieberman [24], who explicitly state theorems for problems with mixed boundary conditions. We have chosen to follow [18] for the sake of being able to get the greatest number of necessary results from the same source. Other good references on this topic include [9,17]. A good introduction to some of the main topics is [7].

The proof of Theorem 7.1 is really an application of the Leray - Schauder theorem which, for the convenience of the reader, we now state in the form found in [9, page 189].

Theorem 7.2 (Leray - Schauder) *Let X denote a Banach space with norm $\|\cdot\|$, and consider a transformation $T : X \times [a,b] \longrightarrow X$. Assume:*

(i) *$T(x,k)$ is defined for all $x \in X, a \leq k \leq b$.*

(ii) *For any fixed k, $T(x,k)$ is continuous in X.*

(iii) *For x in bounded sets of X, $T(x,k)$ is uniformly continuous in k, i.e., for any bounded set $X_0 \subset X$ and for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in X_0$, $| k_1 - k_2 | < \delta$, and $a \leq k_1, k_2 \leq b$, then $\| T(x,k_1) - T(x,k_2) \| < \epsilon$.*

(iv) *For any fixed k, $T(x,k)$ is a compact transformation, i.e., it maps bounded subsets of X in compact subsets of X.*

(v) *There exists a (finite) constant M such that every possible solution x of*

$$x - T(x,k) = 0$$

$(x \in X, k \in [a,b])$ satisfies $\| x \| \leq M$.

(vi) *The equation $x - T(x,a) = 0$ has a unique solution in X.*

Then there exists an $x \in X$ such that $x - T(x,b) = 0$.

To apply this theorem, define the transformation $T(v,\tau)(x,t) = w(x,t)$, where $w(x,t)$ is the solution of

$$(\mathrm{L}_\tau) \qquad \begin{aligned} & u_t = [(1-\tau) + \tau a(v+\psi_0)]u_{xx} + \tau b(v+\psi_0, v_x+\psi_0') && \text{on } Q_T \\ & u(0,t) = 0 && \text{on } (0,T) \\ & \{[(1-\tau) + \tau a(v+\psi_0)]u_x + \tau\psi(v+\psi_0, v_x+\psi_0')\}(1,t) = 0 && \text{on } (0,T) \\ & u(x,0) = 0 && \text{on } [0,1]. \end{aligned}$$

We will consider

$$T : H^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q_T}) \times [0,1] \longrightarrow H^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q_T})$$

where $\alpha \in (0,1)$ will be chosen later. The hypotheses, (i) - (iv) and (vi), of the Leray - Schauder theorem will follow from the following corollary of [18, Theorem 5.3, page 320].

Corollary 7.1 *Let $l \in (0,1)$. If*

(i) $a \in H^{l,\frac{l}{2}}(\overline{Q_T})$, $0 < \nu \leq a(x,t) \leq \mu$ *in* Q_T, $a(1,\cdot) \in H^{\frac{1+l}{2}}([0,T])$,

(ii) $f \in H^{l,\frac{l}{2}}(\overline{Q_T})$, $f(0,t) = 0$,

(iii) $b(1,\cdot), g(1,\cdot) \in H^{\frac{1+l}{2}}([0,T])$,

(iv) $u_0 \in H^{2+l}([0,1]), u_0(0) = u_0''(0) = 0, a(1,0)u_0'(1) + b(1,0)u_0(1) = g(1,0)$,

then there exists a unique solution, $u \in H^{2+l,1+\frac{l}{2}}(\overline{Q_T})$, *of*

$$(LP) \qquad \begin{array}{ll} u_t = a(x,t)u_{xx} + f(x,t) & \text{on } Q_T \\ u(0,t) = 0 & \text{on } (0,T) \\ a(1,t)u_x(1,t) = g(1,t) & \text{on } (0,T) \\ u(x,0) = u_0(x) & \text{on } [0,1]. \end{array}$$

Furthermore,

$$|\,\hat{u}\,|_{R_T}^{(l+2)} \leq C\left\{|\,\hat{f}\,|_{R_T}^{(l)} + |\,\hat{u_0}\,|_{[-1,1]}^{(l+2)} + |\,g(1,\cdot)\,|_{[0,T]}^{(l+1)}\right\},$$

where $\hat{u}, \hat{f}$,and $\hat{u_0}$ are the odd extensions in x (about $x = 0$) of $u, f,$ and u_0, respectively, to $R_T \equiv [-1,1] \times [0,T]$.

This corollary is established by simply reflecting all the functions in problem (LP) about $x = 0$ to arrive at a new problem which is handled by Theorem 5.3 of [18]. Note that this theorem along with the Arzela - Ascoli theorem establishes (i) - (iv), and (vi) of Theorem 7.2 for the transformation $T(x, k)$. Hence it only remains to show that (v) is true. This is usually the most difficult task when applying the Leray - Schauder theorem. We must prove the existence of a constant, M, such that if $u = T(u, \tau)$, then

$$| u |_{Q_T}^{(1+\alpha)} \leq M.$$

The first step in this direction is an $L^{\infty}(Q_T)$ estimate of u which, in the present situation, follows from the maximum principle, such as [18, Theorem 7.3, page 487]. Next we need bounds on

$$\| u_x \|_{L^{\infty}(Q_T)} \text{ and} \langle u_x \rangle_{Q_T}^{(\alpha)} .$$

For this there are the following theorems from [18]:

Chapter 5 1.1, 3.1, 4.1, 5.3, 7.1, 7.2 .

Chapter 6 3.2 .

With such bounds established, it follows that there exists a $u \in H^{1+\alpha, \frac{1+\alpha}{2}}(\overline{Q_T})$ such that $u = T(u, 1)$. (The actual choice of α is dictated by the above quoted theorems.) The uniqueness follows from maximum principles. (See [18, Theorem 2.2, page 16].)

Finally, observe that if $T(v, 1) = v$, then $u(x, t) \equiv v(x, t) + \psi_0(x)$ is a solution of (NBVP). In fact (NBVP) and the corresponding problem for $v(x, t)$ are equivalent.

8 APPENDIX B. ESTIMATES FOR SOLUTIONS OF THE ADJOINT PROBLEM

We now present a proof of the following theorem.

Theorem 8.1 *Consider the linear boundary value problem*

$$(AP) \qquad \begin{array}{ll} \xi_t + a\xi_{xx} - b\xi_x = 0 & \text{on } Q_T \\ \xi(0,t) = 0 & \text{on } (0,T) \\ \xi_x(1,t) = c(t)\xi(1,t) & \text{on } (0,T) \\ \xi(x,T) = \chi(x) & \text{on } [0,1], \end{array}$$

where $a, b \in C^\infty(\overline{Q_T})$; $0 < \nu \leq a(x,t)$ *on* $\overline{Q_T}$; $a, | b |, | c | \leq \mu$ *on* $\overline{Q_T}$; $c \in C^\infty([0,T])$;

$\xi \in C^\infty([0,1])$; $0 \leq \xi \leq 1$ *on* $[0,1]$; *supp*$\xi \subset (0,1)$; $b(0,t) = 0$.

a.) *There exists a unique solution,* $\xi \in H^{(3+\alpha)}(\overline{Q_T})$, *of (AP).*

b.) *Let* $\theta = -2\mu$ *and* $\lambda = -2(4\mu^3 + 2\mu^2)$. *Then*

$$0 \leq \xi(x,t) \leq e^{-\theta - \lambda T} \text{ on } \overline{Q_T}.$$

c.) $\| \xi_x \|_{L^\infty(Q_T)} \leq \max\left\{ \| \chi' \|_{L^\infty(0,1)}, \| c(\cdot)\xi(1,\cdot) \|_{L^\infty(0,T)} \right\}$.

d.)

$$
\begin{aligned}
&\| \xi_{xx} \|^2_{L^2(Q_T)} \\
&\leq \frac{2}{\nu}\left\{\frac{2\mu^2}{\nu}\int_0^T\int_0^1 \xi_x^2 dxdt + \frac{1}{2}\int_0^1 (\chi')^2 dx\right\} \\
&\quad + \left\{\| \xi^2 \|_{L^\infty(Q_T)}\left[\| c' \|_{L^\infty(0,T)} + \mu\right]\right\}.
\end{aligned}
$$

Proof:

We first note that (formally) $\xi(x,t)$ is a solution of (AP) iff $\eta(x,t) \equiv \xi(x,T-t)$ is a solution of

$$
\text{(AP.1)}\qquad
\begin{aligned}
&\eta_t(x,t) = a(x,T-t)\eta_{xx}(x,t) - b(x,T-t)\eta_x(x,t) && \text{on } Q_T \\
&\eta(0,t) = 0 && \text{on } (0,T) \\
&\eta_x(1,t) = c(T-t)\eta(1,t) && \text{on } (0,T) \\
&\eta(x,0) = \chi(x) && \text{on } [0,1]\ .
\end{aligned}
$$

The existence of a solution, $\eta \in H^{(3+\alpha)}(\overline{Q_T})$, of (AP.1) follows from [18, Theorem 5.3, page 320] via the same reflection technique used to establish Corollary 7.1. To prove (b.) we use the standard trick of defining a function $\rho(x,t) = e^{\lambda t + \theta x}\eta(x,t)$, for appropriately chosen values of θ and λ, and then applying maximum principles to it. To establish (c.) define $\zeta(x,t) = \eta_x(x,t)$. Then ζ is a solution of

$$
\text{(AP.2)}\qquad
\begin{aligned}
&\zeta_t(x,t) = (a(x,T-t)\zeta_x(x,t) + b(x,T-t)\zeta(x,t))_x && \text{on } Q_T \\
&a(0,T-t)\zeta_x(0,t) + b(0,T-t)\zeta(0,t) = 0 && \text{on } (0,T) \\
&\zeta(1,t) = c(1,T-t)\xi(1,T-t) && \text{on } (0,T) \\
&\zeta(x,0) = \xi'(x) && \text{on } [0,1]\ .
\end{aligned}
$$

Fix

$$m > \max\left\{\| \chi' \|_{L^\infty(0,1)}, \| c(\cdot)\xi(1,\cdot) \|_{L^\infty(0,T)}\right\}.$$

It is a straightforward calculation to show that

$$(\zeta - m)^+(x,t) \equiv \max\{\zeta(x,t) - m, 0\}$$

is an absolutely continuous function and, moreover,

$$(\zeta - m)^+_x \equiv \frac{\partial}{\partial x}(\zeta - m)^+$$

vanishes a.e. on $\{(x,t) \in Q_T \mid \zeta(x,t) \leq m\}$. Thus for $s \in [0,T]$

$$\frac{1}{2}\int_0^1 (\zeta - m)^{+2}(x,s)dx = \int_0^s \int_0^1 (\zeta - m)^+ \zeta_t dxdt.$$

Using (AP.2), we substitute in for ζ_t in this equality and integrate by parts to get

$$\frac{1}{2}\int_0^1 (\zeta - m)^{+2}(x,s)dx =$$
$$-\int_0^s \int_0^1 \left[a(x,T-t)(\zeta - m)^{+2}_x + b(x,T-t)(\zeta - m)^+(\zeta - m)^+_x\right] dxdt.$$

Now using the given bounds on a and b and Cauchy's inequality on the right hand side of this last equality we conclude that

$$\int_0^1 (\zeta - m)^{+2}(x,s)dx \leq \frac{4\mu^2}{\nu}\int_0^s \int_0^1 (\zeta - m)^{+2} dxdt.$$

By Gronwall's inequality it now follows that

$$\int_0^1 (\zeta - m)^{+2}(x,s)dx = 0$$

for all $s \in [0,T]$. Therefore, $\zeta(x,t) \leq m$ on $\overline{Q_T}$ for all such m. By observing that $-\zeta(x,t)$ also solves a problem much like (AP.2), one can repeat the argument above to get $-\zeta(x,t) \leq m$ on $\overline{Q_T}$. (c.) now follows.

Finally to prove (d.) we multiply the PDE in (AP) by ξ_{xx}, integrate over Q_T, and integrate by parts to get

$$\int_0^T \int_0^1 a\xi_{xx}^2 dxdt = \int_0^T \int_0^1 b\xi_x\xi_{xx} dxdt + \int_0^1 \frac{\xi_x^2}{2} \Big|_{t=0}^{t=T} dx + \int_0^T c\left(\frac{\xi^2}{2}\right)_t (1,t)dt.$$

Working on the right hand side of this equation, we use Cauchy's inequality on the volume integral and integrate by parts in the boundary integral. The given bounds on a and b now allow us to establish (d.).