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**Hypercubes, median graphs and products of graphs: Some  
algorithmic and combinational results**

**Jha, Pranava Kumar, Ph.D.**

**Iowa State University, 1990**

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**Hypercubes, median graphs and products of graphs: Some algorithmic  
and combinatorial results**

by

**Pranava Kumar Jha**

**A Dissertation Submitted to the  
Graduate Faculty in Partial Fulfillment of the  
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## 1. INTRODUCTION

In this chapter, we present (i) the basic terminology and essential notations, (ii) the definitions and simple characteristics of *hypercubes*, *median graphs*, *Cartesian product of graphs*, *Kronecker product of graphs* and *Strong product of graphs*, and (iii) an outline of the organization of the subsequent chapters of this dissertation.

### 1.1 Basic Terminology

This section introduces the important set- and graph-theoretical notation and terminology. For any undefined terms, we refer to [8].

Let  $S$  and  $T$  be sets. We write  $s \in S$  to indicate that  $s$  is an element of  $S$ . The cardinality of  $S$  is denoted by  $|S|$  and the empty set is written  $\emptyset$ .  $S \subseteq T$  indicates that  $S$  is a subset of  $T$  while  $S \subset T$  indicates proper inclusion. The union and intersection of  $S$  and  $T$  are denoted by  $S \cup T$  and  $S \cap T$  respectively.  $S \setminus T$  consists of those elements of  $S$  which are not in  $T$ , and is called the set difference. The *Cartesian product* (or simply *product*) of  $S$  and  $T$  is denoted by  $S \times T$  and is defined as the set of all ordered pairs  $(s, t)$  such that  $s \in S$  and  $t \in T$ . We also use the following logical connectives: negation ( $\neg$ ), implication ( $\implies$ ) and equivalence ( $\iff$ ).

A *graph*  $G = (V, E)$  consists of a set  $V$  of *vertices* and a set  $E$  of *edges*. Each edge consists of a pair of vertices called its *end points* or *end vertices*. All graphs

we consider are assumed to be finite, undirected and simple, i.e., without loops and multiple edges. For a graph  $G$ , we also use  $V(G)$  and  $E(G)$  to denote its vertex set and edge set respectively.

If  $e = \{u, v\}$  is an edge of  $G$ , then we say that  $e$  is *incident* on  $u$  and  $v$ , and that  $u, v$  are *adjacent* to each other, or  $u$  and  $v$  are *neighbors*. For a vertex  $v$  of  $G$ , we use  $\deg_G(v)$  to denote the *degree* of  $v$ , i.e., the number of neighbors of  $v$  in  $G$ . When the identity of the graph  $G$  is clear from the context, we will use  $\deg(v)$  instead of  $\deg_G(v)$ . By  $\Delta(G)$ , we will denote the *maximum degree* of a vertex of  $G$ . A vertex  $v$  is *isolated* if  $\deg(v) = 0$  and is a *terminal vertex* if  $\deg(v) = 1$ . We say that a graph  $G$  is *regular* if each vertex of  $G$  is of the same degree.

A *path* is a sequence of vertices  $P = v_1, \dots, v_n$ ,  $n \geq 1$ , such that  $\{v_i, v_{i+1}\} \in E(G)$ ,  $1 \leq i < n$ . The *length* of  $P$  is its number of edges.  $P$  is a *simple path* if it contains no repeated vertices, a *closed path* if  $v_1 = v_n$ , and a *cycle* if it is closed and all vertices except  $v_1$  and  $v_n$  are distinct. We say that  $G$  is *connected* if there is a path between any two distinct vertices of  $G$ . If  $u, v \in V(G)$ , then by a  $(u, v)$ -path we mean a simple (but not necessarily shortest) path between  $u$  and  $v$  in  $G$ .  $G$  is said to contain a *Hamiltonian path* (resp. *Hamiltonian cycle*) if there is a simple path (resp. cycle) in  $G$  which includes all vertices of  $G$ .

A graph is a *tree* if it is connected and acyclic. If any two (distinct) vertices of a graph  $G$  are adjacent, then  $G$  is said to be a *complete graph*. The complete graph on  $n$  vertices is denoted by  $K_n$ . We say that the one-vertex graph  $K_1$  is the trivial graph and a graph on two or more vertices is a non-trivial graph.

If the vertex set  $V$  of a graph  $G$  can be partitioned into two sets  $V_1$  and  $V_2$  such that every edge in  $G$  has one endpoint in  $V_1$  and the other in  $V_2$ , then  $G$  is *bipartite*, and  $V_1$  and  $V_2$  constitute a *bipartition* of  $V$ .  $G$  is a *complete bipartite graph* if each

vertex in  $V_1$  is adjacent to every vertex in  $V_2$ . The complete bipartite graph whose bipartition contains  $m$  and  $n$  vertices respectively is denoted by  $K_{m,n}$ .

A graph which is a simple path (resp. cycle) of length  $n$  is denoted by  $P_n$  (resp.  $C_n$ ). We assume that  $V(P_n) = \{1, \dots, n+1\}$ , where  $\{i, i+1\} \in E(P_n)$ ,  $1 \leq i \leq n$ . Similarly, we let  $V(C_n) = \{1, \dots, n\}$ , where  $\{1, n\}$  and  $\{i, i+1\} \in E(C_n)$ ,  $1 \leq i < n$ .

Let  $G$  be a connected graph. For  $u, v \in V(G)$ , the *distance* between  $u$  and  $v$  is denoted by  $d_G(u, v)$  (or simply,  $d(u, v)$  if the referenced graph is clear from the context) and is defined to be the length of a shortest (simple) path between  $u$  and  $v$  in  $G$ . Further, the *diameter* of  $G$  is defined to be the maximum distance between any two vertices of  $G$ .

Two graphs  $G$  and  $H$  are said to be *isomorphic*, written  $G \cong H$ , if there is a bijection between  $V(G)$  and  $V(H)$  which preserves adjacency. It is easy to see that the binary relation “is isomorphic to” is an equivalence relation on graphs. A graph property is said to be an *invariant* if it is shared by all graphs isomorphic to each other. For example, the number of vertices is (trivially) a graph invariant.

For a graph  $G$ , a graph  $H$  is said to be a *subgraph* of  $G$  if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ . For a vertex subset  $W$  of  $G$ , the *induced subgraph*  $\langle W \rangle$  has  $W$  as its vertex set, where two vertices are adjacent if and only if they are adjacent in  $G$ . A subgraph  $H$  of  $G$  is said to be an *isometric subgraph* of  $G$  if for  $u, v \in V(H)$ ,  $d_H(u, v) = d_G(u, v)$ . Clearly, an isometric subgraph must be vertex induced, and it is easy to see that the converse is false. A connected graph  $H$  is said to be *isometrically embeddable* in a graph  $G$  if there is an injective (one-to-one) mapping  $f : V(H) \rightarrow V(G)$  such that for all  $u, v \in V(H)$ ,  $d_H(u, v) = d_G(f(u), f(v))$ .

Let  $W$  be a vertex subset of a graph  $G$  such that  $W$  has a certain graphical property  $P$ . Then  $W$  is said to be *maximal w.r.t.  $P$*  if the following holds: if a vertex

subset  $W'$  of  $G$  has the property  $P$  and  $W \subseteq W'$ , then  $W = W'$ . Further,  $W$  is said to be *maximum w.r.t.  $P$*  if it is maximal and  $|W|$  is as large as possible.

A vertex  $v$  of a graph  $G$  is said to be an *articulation vertex* if removal of  $v$  (and the edges incident on it) from  $G$  results in a graph whose number of connected components is larger than that of  $G$ .  $G$  is said to be *biconnected* if it has no articulation vertex. A *biconnected component* of  $G$  is a subgraph of  $G$  which is itself biconnected and is maximal w.r.t. this property. The biconnected components of  $G$  are necessarily induced subgraphs; moreover, their edge sets partition the edge set of  $G$ .

For graphs  $G$  and  $H$ , a surjective (onto) mapping  $\phi : V(G) \rightarrow V(H)$  is said to define an *elementary contraction* of  $G$  if there exist adjacent vertices  $u$  and  $v$  of  $G$  such that the following conditions are satisfied:

1. for distinct  $x, y \in V(G)$ ,  $\phi(x) = \phi(y)$  if and only if  $\{x, y\} = \{u, v\}$ ,
2. for  $x, y \in (V(G) \setminus \{u, v\})$ ,  $\{x, y\} \in E(G)$  if and only if  $\{\phi(x), \phi(y)\} \in E(H)$ ,  
and
3. for  $w \in (V(G) \setminus \{u, v\})$ ,  $\{u, w\} \in E(G)$  or  $\{v, w\} \in E(G)$  if and only if  $\{\phi(u), \phi(w)\} = \{\phi(v), \phi(w)\} \in E(H)$ .

We say here that  $H$  is an *elementary contraction* of  $G$ , or that  $H$  is obtained from  $G$  by *identifying the adjacent vertices  $u$  and  $v$* , or that  $H$  is obtained from  $G$  by “collapsing” the edge  $\{u, v\}$ . Intuitively,  $H$  is obtained from  $G$  by deleting the edge  $\{u, v\}$ , identifying the vertices  $u$  and  $v$ , and discarding any multiple edges created through this identification process. In Figure 1.1, the graph  $H$  is an elementary contraction of the graph  $G$  obtained by identifying the adjacent vertices 2 and 5 of  $G$ .

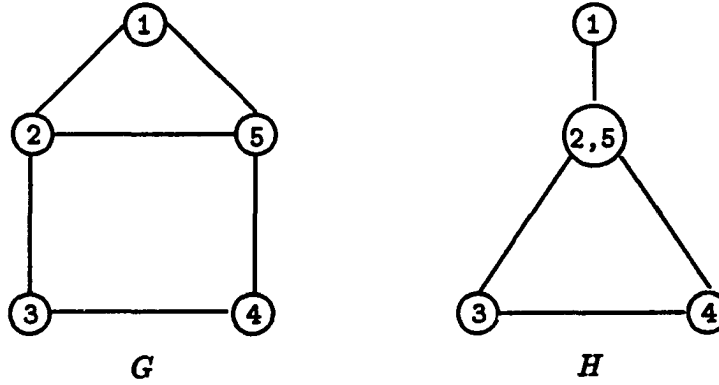


Figure 1.1: An example of an elementary contraction

If  $H$  is a graph obtainable from  $G$  by a sequence of elementary contractions, then we say that  $G$  is *contractable* to  $H$ . Further, a graph  $G'$  is said to be a *subcontraction* (or a *minor*) of  $G$  if a subgraph of  $G$  is contractable to  $G'$  [8, page 82].

For a graph  $G = (V, E)$ , let  $W \subseteq V$ . We say that  $W$  is an *independent set* of  $G$  if  $u, v \in W$  implies  $\{u, v\} \notin E$ . An independent set of  $G$  which is maximal w.r.t. the independence property is called a *maximal independent set* of  $G$ . A *minimum cardinality maximal independent set (mcmis)* of  $G$  is, as its name implies, a maximal independent set of smallest cardinality.  $W$  is said to be a *dominating set* of  $G$  if every vertex of  $G$  is either in  $W$ , or is adjacent to some vertex in  $W$ . It is easy to see that  $W$  is a maximal independent set of  $G$  if and only if  $W$  is an independent set as well as a dominating set of  $G$ . Further,  $W$  is said to be a *clique* if the corresponding induced subgraph  $\langle W \rangle$  is a complete subgraph of  $G$  and it is maximal w.r.t. this property, i.e., it is not properly contained in any other complete subgraph of  $G$ .

Finally, in this section, we define an  $r \times r$  *Latin square*  $M$  which is a square matrix over the set  $\{0, \dots, r-1\}$  such that every row and every column of  $M$  contains

every element of  $\{0, \dots, r-1\}$  exactly once, where  $r \geq 1$ . For instance, the following *cyclic matrix* is an example of a Latin square.

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & r \\ 1 & 2 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ r & 0 & 1 & \cdots & r-1 \end{pmatrix}$$

## 1.2 Hypercubes, Median Graphs and Products of Graphs

In this section, we present the definitions of hypercubes, median graphs and products of graphs, whose investigation will form the main topic of this dissertation. Also, we state some of their simple characteristics.

Let  $\{0,1\}^n$  be the set of all binary strings of length  $n$ . For  $x, y \in \{0,1\}^n$ , let  $d_H(x, y)$  denote the *Hamming distance* between  $x$  and  $y$ , i.e., the number of bit positions in which  $x$  and  $y$  differ.

**Definition 1.2.1** *An  $n$ -cube, or a hypercube of dimension  $n$ , is denoted by  $Q_n$  and is defined as the graph whose set of vertices is  $\{0,1\}^n$ , and where two vertices are adjacent if and only if their Hamming distance is exactly one. ■*

The three-cube, i.e.,  $Q_3$ , with vertices labeled by appropriate binary strings appears in Figure 1.2. The  $n$ -cube has been an object of study in Graph Theory, Coding Theory and related areas for a very long time [22,23,24]. Its regular structure has lent itself to several interesting characterizations in the literature [6,23,46,50] and has allowed simple characterizations of other types of graphs as transformations of



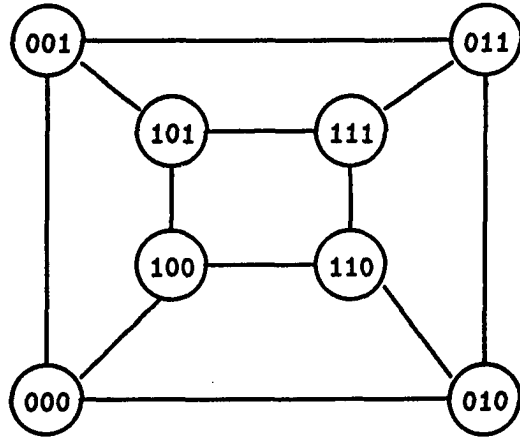


Figure 1.2: The hypercube of dimension three

hypercubes [3]. For a survey, see [31]. Recently, there is renewed interest in the structure of a hypercube because of its successful utilization as the underlying architecture of massively parallel computers [33]. We state some of the simple properties of the  $n$ -cube in the following lemma.

**Lemma 1.2.1**

1. *The  $n$ -cube is a regular bipartite graph of diameter  $n$ .*
2. *It has  $2^n$  vertices and  $n \cdot 2^{n-1}$  edges.*
3. *The distance between any two vertices of the  $n$ -cube is precisely the Hamming distance between the respective binary strings.*
4. *The  $n$ -cube contains a  $k$ -cube for all  $k \leq n$ .*
5. *[29] If the distance between two vertices  $u$  and  $v$  of the  $n$ -cube is  $d$ , then there are exactly  $d!$  distinct shortest paths between  $u$  and  $v$ . ■*

For three vertices  $u = u_1u_2 \cdots u_n$ ,  $v = v_1v_2 \cdots v_n$  and  $w = w_1w_2 \cdots w_n$  of  $Q_n$ , the *majority vertex*  $Maj(u, v, w) = m_1m_2 \cdots m_n$  is defined as follows:  $m_i = 0$  (resp. 1) if at least two of the values of  $u_i$ ,  $v_i$  and  $w_i$  are equal to 0 (resp. 1). A vertex subset  $X$  of  $V(Q_n)$  is said to be *majority closed* if for  $u, v, w \in X$ ,  $Maj(u, v, w) \in X$ . It is easily shown that for any triple of vertices of  $Q_n$ , the majority vertex coincides with the median vertex (defined below).

Let  $G$  be a graph. For a triple of vertices  $u, v, w$  of  $G$ , a vertex  $x$  of  $G$  is said to be a *median* of  $u, v, w$  if the following holds:

$$d(u, v) = d(u, x) + d(x, v),$$

$$d(v, w) = d(v, x) + d(x, w),$$

$$d(w, u) = d(w, x) + d(x, u);$$

in other words,  $x$  is a median of  $u, v$  and  $w$  if  $x$  lies simultaneously on suitably chosen shortest paths joining  $u$  and  $v$ ,  $v$  and  $w$ , and  $w$  and  $u$ , respectively.

**Definition 1.2.2**  $G$  is a *median graph* if  $G$  is connected and every triple of vertices of  $G$  admits a unique median. ■

All trees,  $n$ -cubes, and planar grids are examples of median graphs. If  $G$  is a median graph, then a connected, induced subgraph  $H$  of  $G$  is said to be a *median subgraph* of  $G$  if for any triple of vertices  $u, v, w$  of  $H$ , the median of  $u, v, w$  in  $G$  belongs to  $V(H)$ . Median graphs have a rich literature; [3,5,6,7,48,50,52,53] is a sample of papers on this subject. Some of their simple properties appear in the next lemma.

**Lemma 1.2.2**

1. *A median graph is a bipartite graph.*
2.  *$G$  is a median graph if and only if every biconnected component of  $G$  is a median graph in its own right.*
3. *[3] If  $G$  is a median graph, then a subgraph  $H$  of  $G$  is a median subgraph if and only if  $H$  is a median graph in its own right and  $H$  is isometric in  $G$ .*
4. *A median graph does not contain  $K_{2,3}$  as its subgraph. ■*

We next define the three graph products.

**Definition 1.2.3** *For graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ , the Cartesian product, Kronecker product and Strong product of  $G_1$  and  $G_2$  are respectively denoted by  $G_1 \square G_2$ ,  $G_1 \times G_2$  and  $G_1 \boxtimes G_2$ , and are defined as follows:*

$$V(G_1 \square G_2) = V(G_1 \times G_2) = V(G_1 \boxtimes G_2) = V_1 \times V_2$$

$$E(G_1 \square G_2) = \{ \{(x_1, x_2), (y_1, y_2)\} \mid \text{either } x_1 = y_1 \text{ and } \{x_2, y_2\} \in E_2 \text{ or } x_2 = y_2 \text{ and } \{x_1, y_1\} \in E_1 \}$$

$$E(G_1 \times G_2) = \{ \{(x_1, x_2), (y_1, y_2)\} \mid \{x_1, y_1\} \in E_1 \text{ and } \{x_2, y_2\} \in E_2 \}$$

$$E(G_1 \boxtimes G_2) = E(G_1 \square G_2) \cup E(G_1 \times G_2). \blacksquare$$

Note that  $E(G_1 \square G_2) \cap E(G_1 \times G_2) = \emptyset$ . Further,

$$|V(G_1 \square G_2)| = |V(G_1 \times G_2)| = |V(G_1 \boxtimes G_2)| = |V_1| \cdot |V_2|$$

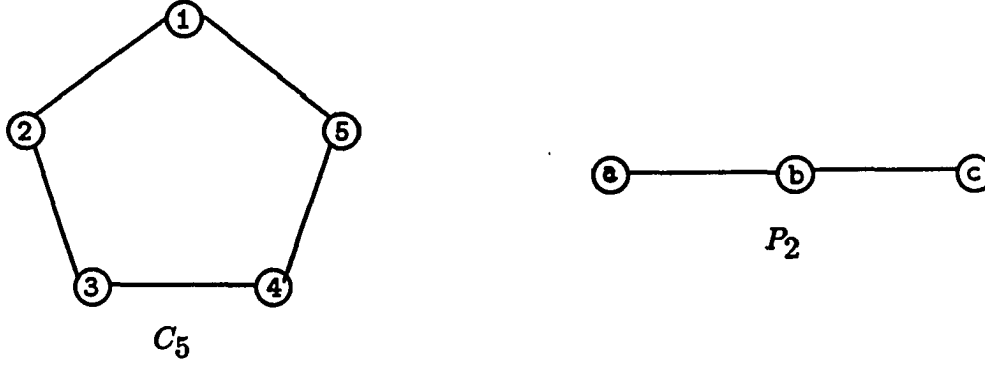


Figure 1.3: The graphs  $C_5$  and  $P_2$

$$|E(G_1 \square G_2)| = |V_1| \cdot |E_2| + |V_2| \cdot |E_1|$$

$$|E(G_1 \times G_2)| = 2 \cdot |E_1| \cdot |E_2| \text{ and}$$

$$|E(G_1 \boxtimes G_2)| = |E(G_1 \square G_2)| + |E(G_1 \times G_2)|.$$

Observe that we use ' $\times$ ' to denote the product of two sets as well as the Kronecker product of two graphs; we use the context to resolve any ambiguity. The graphs  $C_5$ , i.e., cycle of length five, and  $P_2$ , i.e., path of length two, appear in Figure 1.3 while the graphs  $C_5 \square P_2$ ,  $C_5 \times P_2$  and  $C_5 \boxtimes P_2$  appear in Figures 1.4, 1.5 and 1.6, respectively.

Graph products have found applications in a variety of areas of mathematics and computer science; [35,43,45,47,55,62] is a sample of papers dealing with some of them. The three products share the following important properties.

**Lemma 1.2.3** *Each of the graph products  $\square$ ,  $\times$  and  $\boxtimes$  is commutative and associative up to isomorphism.*

**Proof Sketch:** For each of the three graph products, the following canonical isomorphisms respectively establish commutativity and associativity:

1.  $(x_1, x_2) \leftrightarrow (x_2, x_1)$

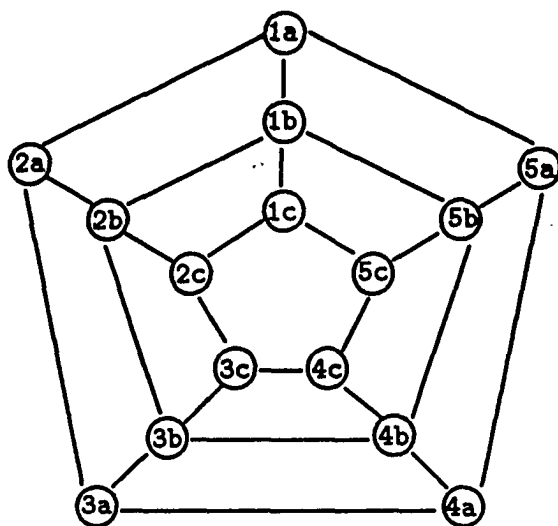


Figure 1.4: The graph  $C_5 \square P_2$

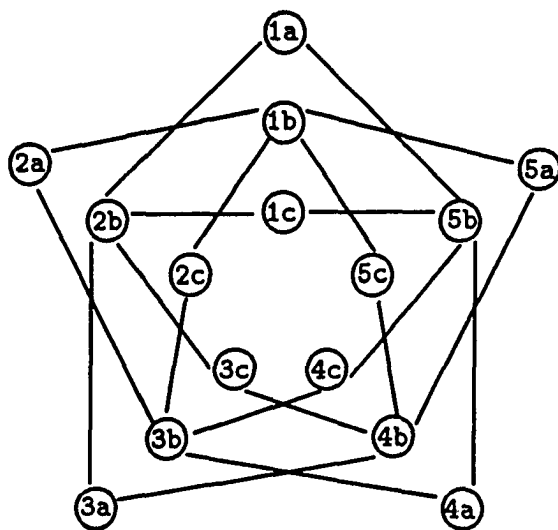
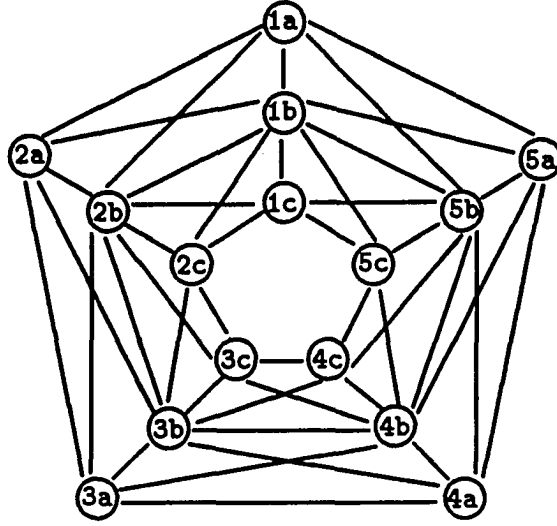


Figure 1.5: The graph  $C_5 \times P_2$

Figure 1.6: The graph  $C_5 \square P_2$ 

2.  $(x_1, (x_2, x_3)) \leftrightarrow ((x_1, x_2), x_3)$ . ■

It is interesting to note that (i) the  $n$ -cube  $Q_n$  is simply the  $\square$ -product of  $n$  copies of  $K_2$  and (ii) the class of median graphs is closed under the  $\square$ -product. Further, for any graph  $G$ ,  $G \square K_1 \cong G \cong G \boxtimes K_1$ , i.e., the one-vertex graph  $K_1$  acts as the two-sided identity for  $\square$ -product as well as  $\boxtimes$ -product. On the other hand,  $G \times K_1 \cong G$  if and only if  $E(G) = \emptyset$ . The following lemma gives a convenient way of obtaining the adjacency matrix of each of the three graph products in terms of the adjacency matrices of the factor graphs.

**Lemma 1.2.4** [68,69] *Let  $G_1$  and  $G_2$  be graphs with adjacency matrices  $A_1$  and  $A_2$  respectively. Then, the adjacency matrices of  $G_1 \square G_2$ ,  $G_1 \times G_2$  and  $G_1 \boxtimes G_2$  are respectively given by  $A_1 \otimes I_2 + I_1 \otimes A_2$ ,  $A_1 \otimes A_2$  and  $A_1 \otimes I_2 + I_1 \otimes A_2 + A_1 \otimes A_2$ ,*

where  $\otimes$  denotes the Kronecker product of matrices, and  $I_j$  is an identity matrix of the same order as  $A_j$ ,  $j = 1, 2$ . ■

The next lemma gives necessary and sufficient conditions for the connectedness of the  $\square$ -product and  $\times$ -product.

**Lemma 1.2.5**

1. [65] The graph  $G_1 \square G_2$  is connected if and only if both  $G_1$  and  $G_2$  are connected.
2. [68] For nontrivial graphs  $G_1$  and  $G_2$ , the graph  $G_1 \times G_2$  is connected if and only if both  $G_1$  and  $G_2$  are connected and either  $G_1$  or  $G_2$  is non-bipartite. ■

The following is a refinement of the second statement of the foregoing lemma.

**Corollary 1.2.6** [68] Let  $G_1$  and  $G_2$  be nontrivial, connected graphs. If  $G_1$  and  $G_2$  are both bipartite, then  $G_1 \times G_2$  has exactly two connected components, otherwise  $G_1 \times G_2$  is connected. ■

It is easy to see that the graph  $G_1 \boxtimes G_2$  is connected if and only if both  $G_1$  and  $G_2$  are connected. The next lemma gives a precise formula for the degree of a vertex in each of the three graph products; the proof is straightforward.

**Lemma 1.2.7** Let  $G_1$  and  $G_2$  be graphs. For  $x_1 \in V(G_1)$  and  $x_2 \in V(G_2)$ ,

$$\deg_{G_1 \square G_2}(x_1, x_2) = \deg_{G_1}(x_1) + \deg_{G_2}(x_2),$$

$$\begin{aligned} \deg_{G_1 \times G_2}(x_1, x_2) &= \deg_{G_1}(x_1) \cdot \deg_{G_2}(x_2) \text{ and} \\ \deg_{G_1 \boxtimes G_2}(x_1, x_2) &= \deg_{G_1 \square G_2}(x_1, x_2) + \deg_{G_1 \times G_2}(x_1, x_2). \blacksquare \end{aligned}$$

The following corollary is based on Lemma 1.2.7.

**Corollary 1.2.8**

1.  $G_1$  and  $G_2$  are regular graphs if and only if  $G_1 \square G_2$  is a regular graph if and only if  $G_1 \boxtimes G_2$  is a regular graph.
2. If  $G_1$  and  $G_2$  are regular graphs, then so is  $G_1 \times G_2$ .
3. Let  $G_1$  and  $G_2$  be graphs, each of which contains at least one edge. Then, if  $G_1 \times G_2$  is a regular graph, so are  $G_1$  and  $G_2$ . ■

It is straightforward to argue that if  $G_1$  and  $G_2$  are nontrivial connected graphs, then there are at least two vertex-disjoint paths between any two distinct vertices of  $G_1 \square G_2$ , and hence we have the following theorem.

**Theorem 1.2.9** [43] *The Cartesian product (and hence Strong product) of two non-trivial, connected graphs has no articulation vertex. ■*

On the other hand, conditions for the existence of articulation vertices in Kronecker product graphs are somewhat involved; see [43].



### 1.3 What Follows

We consider hypercubes, median graphs and the three graph products  $\square$ ,  $\times$  and  $\boxtimes$  from certain algorithmic and combinatorial points of view. The remainder of this dissertation is organized as follows:

- In Chapter 2, our most important result is a scheme, which for  $n = 2^k - 1$ ,  $k \geq 1$ , constructs a partition of the vertex set of the  $n$ -cube  $Q_n$  into  $n + 1$  sets, each of which is of size exactly  $\frac{2^n}{n+1}$  and is a minimum cardinality maximal independent set (mcmis) of  $Q_n$ . We accomplish this by an interesting application of a Latin square. For the case when  $n$  is not of the foregoing form, let  $r$  be the largest integer such that  $r < n$  and  $r = 2^k - 1$ ; we construct a partition of  $V(Q_n)$  into  $r + 1$  maximal independent sets of size  $\frac{2^n}{r+1}$  each. As a corollary, we obtain an upper bound on the cardinality of an mcmis of  $Q_n$  for all  $n$ . Our upper and lower bounds are within a factor of two, and for  $n$  of the form  $n = 2^k - 1$ , the two bounds coincide, and hence yield the optimal value. We further observe that these bounds are correct also for the domination number (i.e., the cardinality of a smallest dominating set) of  $Q_n$ .
- In Chapter 3, we present two algorithms, each of which recognizes median graphs in time  $O(n^2 \log n)$ . These are based on two different characterizations of median graphs given respectively by Bandelt [3] and Mulder [48]. We further comment on the possibility of extending these results to the problem of recognizing isometric subgraphs of hypercubes and quasimedial graphs.
- In Chapter 4, we present two  $O(n^2 \log n)$  algorithms, each of which obtains an isometric embedding of a given median graph in a hypercube of least possible dimension. These have structures similar to those of the respective recognition

schemes mentioned above. We further observe that if  $G$  is a graph known to be an isometric subgraph of a hypercube, then one of our algorithms will act (without any modification) on  $G$  to produce a similar embedding of  $G$  under the same time bound.

- In Chapter 5, we state and prove characterizations for the following: (i) planarity of the  $\boxtimes$ -product, (ii) outerplanarity of the  $\square$ -product and (iii) outerplanarity of the  $\times$ -product. For these, we employ known characterizations for the planarity and outerplanarity of general graphs, which have been given in terms of graph subcontraction [8,30,32,67].
- In Chapter 6, we discuss (i) bounds on the following topological invariants of the three graph products: chromatic numbers, independence numbers, domination numbers and clique numbers, and (ii) conditions for the existence of a Hamiltonian path/cycle in each of the three products. Our main contribution consists of an improved lower bound on the independence number of the  $\square$ -product and certain sufficient conditions for the existence of a Hamiltonian cycle in the  $\times$ -product.
- Finally, in Chapter 7, we summarize the results, discuss some related issues and state several open problems relating to hypercubes, median graphs and the three graph products.

## 2. MINIMUM CARDINALITY MAXIMAL INDEPENDENT SETS OF A HYPERCUBE

The main result of this chapter is a scheme, which for  $n = 2^k - 1$ ,  $k \geq 1$ , leads to a partition of the vertex set of the  $n$ -cube  $Q_n$  into  $n + 1$  sets, each of which is a minimum cardinality maximal independent set (mcmis) of  $Q_n$  of size exactly  $\frac{2^n}{n+1}$ . We accomplish this by a suitable application of a Latin square. The existence of such a partition is a known result [12,56]. Our contribution is in devising an interesting scheme for constructing such a partition. We also present an example to show that our method does *not* construct all possible partitions of  $V(Q_n)$  into mcmis's. For the case when  $n$  is *not* of the form  $n = 2^k - 1$ , let  $r$  be the largest integer such that  $r < n$  and  $r$  is of the form  $r = 2^k - 1$ ; we show that there is a partition of  $V(Q_n)$  into  $r + 1$  sets, each of which is a maximal independent set of  $Q_n$  of size  $\frac{2^n}{r+1}$ . As a by-product, we obtain an upper bound on the minimum cardinality of a maximal independent set of  $Q_n$  for all  $n$ ; our upper and lower bounds are within a factor of two, and for  $n$  of the form  $n = 2^k - 1$ , they coincide and hence yield the optimal value. We further observe that these bounds are correct also for the domination number (i.e., cardinality of a smallest dominating set) of  $Q_n$ .

In Section 2.1 below, we state some definitions and present two lemmas which will be useful in the sequel. The first lemma yields the theoretical lower bound on the cardinality of an mcmis of  $Q_n$ —we later show that the lower bound is achievable in

certain cases. Section 2.2 contains the main result and an example of a partition of the vertex set of  $Q_7$  which cannot be obtained by our scheme. Finally, in Section 2.3, we derive an upper bound on the cardinality of an mcmis of  $Q_n$ .

## 2.1 Preliminaries

Recall from Chapter 1 that if  $x$  and  $y$  are equal-length binary strings, then  $d_H(x, y)$  denotes the Hamming distance between  $x$  and  $y$ . Further, we have defined the  $n$ -dimensional hypercube  $Q_n$  to be the graph whose vertex set is the set of all binary strings of length  $n$ , and where two vertices are adjacent if and only if their Hamming distance is exactly one.

For two binary strings  $u$  and  $w$ , let  $u \cdot w$  denote their *concatenation*, and for two sets of strings  $U$  and  $W$ , let  $U \bullet W$  denote their concatenation defined as

$$U \bullet W = \{u \cdot w \mid u \in U \text{ and } w \in W\}.$$

For  $x \in V(Q_n)$ , let  $\bar{x}$  represent the vertex of  $V(Q_n)$  that is diametrically opposite to  $x$ ; in other words, if  $x \in V(Q_n)$ , then  $\bar{x}$  is the unique vertex in  $V(Q_n)$  such that  $d_H(x, \bar{x}) = n$ . Thus, if  $x = a_{n-1}a_{n-2} \cdots a_1a_0$ , then  $\bar{x} = \overline{a_{n-1}}\overline{a_{n-2}} \cdots \overline{a_1}\overline{a_0}$ , where  $\overline{0} = 1$  and  $\overline{1} = 0$ . A set  $S \subseteq V(Q_n)$  is said to be *closed* under *bitwise complementation* if  $x \in S \implies \bar{x} \in S$ .

**Lemma 2.1.1** *A maximal independent set of  $Q_n$  is of cardinality at least  $\frac{2^n}{n+1}$ .*

**Proof:** Note that a maximal independent set  $S$  of a graph  $G = (V, E)$  is also a dominating set of  $G$ , i.e., every vertex of  $G$  is either in  $S$  or adjacent to some member of  $S$ . In a hypercube  $Q_n$ , every vertex is adjacent to exactly  $n$  other vertices, and

hence it dominates a total of  $n+1$  vertices including itself. Thus, in order to dominate all  $2^n$  vertices of  $Q_n$ , we need to select at least  $\frac{2^n}{n+1}$  vertices. ■

**Lemma 2.1.2** *Let  $n = 2^k - 1$ ,  $k \geq 1$ . Let  $S$  be a vertex subset of  $Q_n$  such that  $|S| = \frac{2^n}{n+1}$ . Then  $S$  is an mcmis of  $Q_n$  if and only if for any two distinct vertices  $x$  and  $y$  of  $S$ ,  $d_H(x, y) \geq 3$ .*

**Proof:** Let  $n$ ,  $k$  and  $S$  be as in the statement of the lemma. First suppose that  $S$  is a vertex subset of  $Q_n$  such that  $d_H(x, y) \geq 3$  for any two distinct vertices  $x$  and  $y$  of  $S$ . Clearly  $S$  is an independent set of  $Q_n$ . Moreover, no two distinct vertices of  $S$  have any common neighbor. Consequently, every vertex in  $V(Q_n) \setminus S$  is adjacent to at most one vertex of  $S$ , and hence the vertices of  $S$  together dominate a total of  $|S| \times (n+1) = 2^n$  vertices of  $Q_n$ , i.e., all of them. It then follows that every vertex in  $V(Q_n) \setminus S$  is adjacent to some vertex in  $S$ , and hence by Lemma 2.1.1,  $S$  is an mcmis of  $Q_n$ .

Conversely, let  $S$  be an mcmis of  $Q_n$ . If for any two distinct vertices  $x$  and  $y$  of  $S$ ,  $d_H(x, y) < 3$ , (i.e.,  $d_H(x, y) = 2$ ) then the total number of vertices dominated by  $S$  will be strictly less than  $2^n$ ; consequently,  $S$  would not be a maximal independent set of  $Q_n$ . ■

## 2.2 Main Result

In this section, we state and prove our main result.

**Theorem 2.2.1** *Let  $n = 2^k - 1$ ,  $k \geq 1$ . Then there is a partition of  $V(Q_n)$  into  $n+1$  sets  $S_n^{(0)}, \dots, S_n^{(n)}$  of cardinality  $\frac{2^n}{n+1}$  each such that for all  $i$ ,  $S_n^{(i)}$  is an mcmis*

of  $Q_n$ .

**Proof:** (by induction on  $k$ ) For  $k = 1$ , we have  $n = 1$ , and  $V(Q_1) = \{0, 1\}$ . Consider the partition of  $V(Q_1)$  into  $\{0\}$  and  $\{1\}$ , each of which is an mcmis of  $Q_1$  and is of appropriate cardinality. Similarly, for  $k = 2$ , we have  $n = 3$  and the partition of  $V(Q_3) = \{0, 1\}^3$  into  $\{000, 111\}$ ,  $\{001, 110\}$ ,  $\{010, 101\}$  and  $\{011, 100\}$  has the required properties.

For some  $k \geq 2$ , let  $n = 2^k - 1$ . Assume that  $S_n^{(0)}, \dots, S_n^{(n)}$  are  $n + 1$  sets constituting a partition of  $V(Q_n) = \{0, 1\}^n$  such that for all  $i$ ,  $S_n^{(i)}$  is an mcmis of  $Q_n$  and  $|S_n^{(i)}| = \frac{2^n}{n+1}$ . In other words, every  $n$ -bit binary string is in exactly one  $S_n^{(i)}$  and any two distinct  $n$ -bit binary strings in  $S_n^{(i)}$  have a Hamming distance of at least three. Note that for  $i \neq j$ , if  $x \in S_n^{(i)}$  and  $y \in S_n^{(j)}$ , then  $x \neq y$  and hence  $d_H(x, y) \geq 1$ .

Let  $m = 2^{k+1} - 1 = 2n + 1$ . In what follows, we construct  $S_m^{(0)}, \dots, S_m^{(m)}$ . Let  $l = \frac{2^n}{n+1} - 1 = 2^{n-k} - 1$ . For  $i \in \{0, \dots, n\}$ , let

$$S_n^{(i)} = \{w_{i0}, w_{i1}, \dots, w_{il}\},$$

and define  $C_i$  and  $D_i$  as follows:

$$C_i = \{w_{i0} \cdot b_{i0}, w_{i1} \cdot b_{i1}, \dots, w_{il} \cdot b_{il}\}$$

$$D_i = \{w_{i0} \cdot \overline{b_{i0}}, w_{i1} \cdot \overline{b_{i1}}, \dots, w_{il} \cdot \overline{b_{il}}\}$$

where  $b_{ij}$  is the parity of  $w_{ij}$ , i.e.,  $b_{ij}$  is 0 if the number of 1's in  $w_{ij}$  is even and 1 otherwise. Thus, every string in  $C_i$  (resp.  $D_i$ ) is of even (resp. odd) parity. Consequently, if  $x \in C_i$  (resp.  $D_i$ ), and  $y \in C_j$  (resp.  $D_j$ ), where  $x \neq y$ , then

$d_H(x, y) \geq 3$  if  $i = j$ , and  $d_H(x, y) \geq 2$  otherwise. It is clear that every element of  $C_i$  and  $D_i$  is a binary string of length exactly  $n + 1$ . Note further that the sets  $C_0, \dots, C_n, D_0, \dots, D_n$  are mutually disjoint, because so are  $S_n^{(0)}, \dots, S_n^{(n)}$ .

Let  $T = (t_{ij})$  be an  $(n + 1) \times (n + 1)$  Latin square, i.e., a square matrix, every row and every column of which contains each element of  $\{0, \dots, n\}$  exactly once. For  $p \in \{0, \dots, m\}$ , we define  $S_m^{(p)}$  as follows:

$$\left. \begin{aligned} S_m^{(i)} &= \bigcup_{j=0}^n C_j \bullet S_n^{(t_{ij})} \\ S_m^{(n+1+i)} &= \bigcup_{j=0}^n D_j \bullet S_n^{(t_{ij})} \end{aligned} \right\} 0 \leq i \leq n.$$

The following four claims establish that for each  $p \in \{0, \dots, m\}$ ,  $S_m^{(p)}$  is an mcmis of  $Q_m$  of size  $\frac{2^m}{m+1}$  and the sets  $S_m^{(0)}, \dots, S_m^{(m)}$  constitute a partition of  $V(Q_m)$ .

1.  $|S_m^{(p)}| = \frac{2^m}{m+1}$ .

Let  $i \in \{0, \dots, n\}$ . It is clear that the sets  $C_0 \bullet S_n^{(t_{i0})}, \dots, C_n \bullet S_n^{(t_{in})}$  (respectively, the sets  $D_0 \bullet S_n^{(t_{i0})}, \dots, D_n \bullet S_n^{(t_{in})}$ ) are mutually disjoint and that they are of cardinality  $2^{n-k} \times 2^{n-k}$  each; consequently, their union, which is  $S_m^{(i)}$  (respectively,  $S_m^{(n+1+i)}$ ), is of cardinality  $(n + 1) \times 2^{n-k} \times 2^{n-k}$  which is equal to  $\frac{2^m}{m+1}$ . (Recall that  $n = 2^k - 1$  and  $m = 2n + 1$ .)

2. Every element of  $S_m^{(p)}$  is a binary string of length exactly  $m$ .

The length of every element of every  $C_j \bullet S_n^{(t_{ij})}$  (respectively,  $D_j \bullet S_n^{(t_{ij})}$ ) is  $2n + 1 = m$ ; consequently, the length of every element of  $S_m^{(p)}$  is exactly  $m$ .

3. For  $p \neq q$ ,  $S_m^{(p)} \cap S_m^{(q)} = \emptyset$ .

This follows from the choice of the matrix  $T$  and the facts that (a) the sets

$C_0, \dots, C_n, D_0, \dots, D_n$  are mutually disjoint, and (b) the sets  $S_n^{(0)}, \dots, S_n^{(n)}$  are mutually disjoint.

4. For distinct  $x, y \in S_m^{(p)}$ ,  $d_H(x, y) \geq 3$ .

For  $i \in \{0, \dots, n\}$ , let  $x, y$  be distinct elements of  $S_m^{(i)}$ . Then for some  $a, b, c, d \in \{0, \dots, n\}$ , we have

$$x \in C_a \bullet S_n^{(b)} \text{ and } y \in C_c \bullet S_n^{(d)},$$

where  $b = t_{ia}$  and  $d = t_{ic}$ . Thus  $x = x_1 \cdot x_2$  and  $y = y_1 \cdot y_2$ , where  $x_1 \in C_a$ ,  $x_2 \in S_n^{(b)}$ ,  $y_1 \in C_c$  and  $y_2 \in S_n^{(d)}$ . Since  $x$  and  $y$  are distinct, it cannot happen that  $x_1 = y_1$  and  $x_2 = y_2$ . If  $x_1 = y_1$  and  $x_2 \neq y_2$ , then we must have that  $a = c$ , whence  $b = d$ ; consequently,  $x_2$  and  $y_2$  are distinct elements of  $S_n^{(b)}$ , which means that  $d_H(x_2, y_2) \geq 3$  (by induction hypothesis) and hence  $d_H(x, y) \geq 3$ . By a similar argument, if  $x_1 \neq y_1$  and  $x_2 = y_2$ , then  $d_H(x_1, y_1) \geq 3$  and hence  $d_H(x, y) \geq 3$ . Finally, suppose that  $x_1 \neq y_1$  and  $x_2 \neq y_2$ . Then there are two possibilities:  $a = c$  or  $a \neq c$ . If  $a = c$ , then  $b = d$  and hence  $x_1, y_1$  are distinct elements of  $C_a$ , and  $x_2, y_2$  are distinct elements of  $S_n^{(b)}$ ; it then immediately follows that  $d_H(x, y) \geq 3$ . If  $a \neq c$ , then we must have that  $b \neq d$ ; consequently, by an earlier observation,  $d_H(x_1, y_1) \geq 2$  and  $d_H(x_2, y_2) \geq 1$  and hence  $d_H(x, y) \geq 3$ . The argument for  $S_m^{(n+1+i)}$  is similar. ■

**Corollary 2.2.2** Let  $n = 2^k - 1$ ,  $k \geq 2$ . For  $i \in \{0, \dots, n\}$ ,  $S_n^{(i)}$  defined in the proof of Theorem 2.2.1 is closed under bitwise complementation. Consequently, (a) half of the elements of  $S_n^{(i)}$  start with a 0 while the remaining half start with a 1 and



(b) half of the elements of  $S_n^{(i)}$  are of even parity while the remaining half are of odd parity.

**Proof:** (by induction on  $k$ ) Note that the statement holds for  $k = 2$ . For some  $n = 2^k - 1$ , where  $k \geq 2$ , assume that each of the sets  $S_n^{(0)}, \dots, S_n^{(n)}$  is closed under bitwise complementation, (i.e.,  $x \in S_n^{(i)}$  implies  $\bar{x} \in S_n^{(i)}$ ). Observe that every element of  $S_n^{(i)}$  is of length exactly  $n$ , which is an odd number. Consequently,  $x$  in  $S_n^{(i)}$  has even (resp. odd) parity if and only if  $\bar{x}$  in  $S_n^{(i)}$  has odd (resp. even) parity. It then follows from their definitions that the sets  $C_0, \dots, C_n, D_0, \dots, D_n$  are each closed under bitwise complementation. Finally, observe that if two sets  $X$  and  $Y$  are closed under the stated operation, then so are  $X \cup Y$  and  $X \bullet Y$ . ■

The next two corollaries are immediate.

**Corollary 2.2.3** For  $k \geq 2$ , the sets  $C_0, \dots, C_n, D_0, \dots, D_n$  as defined in the proof of Theorem 2.2.1 constitute a partition of  $\{0, 1\}^{n+1}$ . ■

**Corollary 2.2.4** Let  $k \geq 2$ ,  $n = 2^k - 1$  and  $m = 2^{k+1} - 1$ , as in the proof of Theorem 2.2.1. Let  $p \in \{0, \dots, m\}$ . Then for each  $S_m^{(p)}$  and each  $w \in \{0, 1\}^n$ ,  $w$  appears as the prefix (resp. suffix) of exactly  $\frac{2^n}{n+1} = 2^{n-k}$  elements of  $S_m^{(p)}$ . In particular, the set of  $n$ -bit prefixes (resp. suffixes) of the elements of each  $S_m^{(p)}$  is equal to  $\{0, 1\}^n$ . ■

To conclude this section, we present a partition of the vertex set of  $Q_7$  into mcmis's which cannot be obtained by our scheme. For convenience, we use decimal

(rather than binary) notation for the vertices of  $Q_7$ , i.e.,  $V(Q_7) = \{0, \dots, 127\}$ . The eight sets which constitute one such partition are the following:

$\{0, 11, 21, 30, 38, 45, 51, 56, 71, 76, 82, 89, 97, 106, 116, 127\}$   
 $\{1, 10, 20, 31, 39, 44, 50, 57, 70, 77, 83, 88, 96, 107, 117, 126\}$   
 $\{2, 9, 23, 28, 36, 47, 49, 58, 69, 78, 80, 91, 99, 104, 118, 125\}$   
 $\{3, 8, 22, 29, 37, 46, 48, 59, 68, 79, 81, 90, 98, 105, 119, 124\}$   
 $\{4, 15, 17, 26, 34, 41, 55, 60, 67, 72, 86, 93, 101, 110, 112, 123\}$   
 $\{5, 14, 16, 27, 35, 40, 54, 61, 66, 73, 87, 92, 100, 111, 113, 122\}$   
 $\{6, 13, 19, 24, 32, 43, 53, 62, 65, 74, 84, 95, 103, 108, 114, 121\}$   
 $\{7, 12, 18, 25, 33, 42, 52, 63, 64, 75, 85, 94, 102, 109, 115, 120\}$ .

It is straightforward to see that the above sets constitute a partition of  $V(Q_7)$  into mcmis's. That this partition cannot be obtained by our method follows from the observations that (i) there is a unique partition of  $V(Q_3)$  into mcmis's, i.e.,  $\{\{0, 7\}, \{1, 6\}, \{2, 5\}, \{3, 4\}\}$  and (ii) no  $4 \times 4$  Latin square coupled with this partition of  $V(Q_3)$  can yield a partition of  $V(Q_7)$  in which the vertex 0 appears along with the vertex 11 in the same set as in the partition above. It is interesting to note that the above partition of  $V(Q_7)$  satisfies Corollaries 2.2.2, 2.2.3 and 2.2.4.

### 2.3 An Upper Bound

In this section, we discuss the case when  $n$  is *not* of the form  $n = 2^k - 1$  and obtain an upper bound on the cardinality of an mcmis of  $Q_n$  for all  $n$ .

**Theorem 2.3.1** *Let  $n$  be a positive integer such that  $n \neq 2^i - 1$  for any  $i$ . Let  $k$  be the largest integer such that  $n > 2^k - 1$  and let  $r = 2^k - 1$ . Then there is a partition of  $V(Q_n)$  into  $r + 1$  sets, each of which is a maximal independent set of  $Q_n$  of cardinality  $\frac{2^n}{r+1} = 2^{n-k}$ .*

**Proof:** Let  $n$ ,  $k$  and  $r$  be as in the statement of the theorem. Obtain a partition  $\{S_r^{(0)}, \dots, S_r^{(r)}\}$  of  $V(Q_r)$  as in the proof of Theorem 2.2.1. In what follows, we construct a desired partition of  $V(Q_n)$ .

Let  $\{A_0, A_1\}$  be the partition of  $\{0, 1\}^{n-r}$  such that  $A_0$  (resp.  $A_1$ ) is the set of binary strings of even (resp. odd) parity and length  $n - r$ . Clearly,  $|A_0| = |A_1| = 2^{n-r-1}$ . We now define  $S_n^{(i)}$ ,  $0 \leq i \leq r$ , as follows:

$$\left. \begin{aligned} S_n^{(2j)} &= A_0 \bullet S_r^{(2j)} \cup A_1 \bullet S_r^{(2j+1)} \\ S_n^{(2j+1)} &= A_0 \bullet S_r^{(2j+1)} \cup A_1 \bullet S_r^{(2j)} \end{aligned} \right\} 0 \leq j \leq (r-1)/2.$$

For  $p, q \in \{0, \dots, r\}$ , the following are easy consequences and can be argued as in the proof of Theorem 2.2.1.

1.  $S_n^{(p)}$  is of cardinality exactly  $\frac{2^n}{r+1}$ .
2. Every element of  $S_n^{(p)}$  is a binary string of length exactly  $n$ .
3. For  $p \neq q$ ,  $S_n^{(p)} \cap S_n^{(q)} = \emptyset$ .
4. For distinct  $x, y$  in  $S_n^{(p)}$ ,  $d_H(x, y) \geq 2$ .
5. For every  $x \in \{0, 1\}^n$ , there is some  $y \in S_n^{(p)}$  such that either  $x = y$  or  $d_H(x, y) = 1$ .

It then follows that each  $S_n^{(p)}$  is a maximal independent set of  $Q_n$  of cardinality  $\frac{2^n}{r+1}$  and the sets  $S_n^{(0)}, \dots, S_n^{(r)}$  constitute a partition of  $V(Q_n)$ . ■

The following corollary is an analogue of Corollary 2.2.3.

**Corollary 2.3.2** *Let  $n, k$  and  $r$  be as in the proof of Theorem 2.3.1. For  $i \in \{0, \dots, r\}$ ,  $S_n^{(i)}$  defined in the proof of Theorem 2.3.1 is such that (a) half of the elements of  $S_n^{(i)}$  start with a 0 while the remaining half start with a 1 and (b) half of the elements of  $S_n^{(i)}$  are of even parity while the remaining half are of odd parity. ■*

We have thus shown that if  $n$  is not of the form  $n = 2^k - 1$ , then there is a maximal independent set of  $Q_n$  of cardinality  $\frac{2^n}{r+1}$ , where  $r$  is the largest integer such that  $r < n$  and  $r$  is of the form  $r = 2^k - 1$  (i.e.,  $k = \lfloor \log_2(n+1) \rfloor$ ). Using Lemma 2.1.1, Theorem 2.2.1 and Theorem 2.3.1, we have bounded the cardinality of an mcmis of  $Q_n$ , denoted here by  $\Lambda(Q_n)$ , as follows:

$$\frac{2^n}{n+1} \leq \Lambda(Q_n) \leq \frac{2^n}{2^{\lfloor \log_2(n+1) \rfloor}}.$$

Note that our upper and lower bounds are within a factor of two, and that for  $n$  of the form  $n = 2^k - 1$ , the two bounds coincide, and hence yield the optimal value. The exact determination of  $\Lambda(Q_n)$  in the general case would be very interesting.

Finally, observe that (i) a maximal independent set of a graph is also its dominating set and (ii) the lower bound on the cardinality of an mcmis of  $Q_n$  given by Lemma 2.1.1 is also a lower bound on the *domination number* (i.e., the cardinality of a smallest dominating set) of  $Q_n$ . Consequently, the above bounds on the minimum cardinality of a maximal independent set of  $Q_n$  are correct also for the domination number of  $Q_n$ .

### 3. EFFICIENT ALGORITHMS FOR RECOGNIZING MEDIAN GRAPHS

In this chapter, we have addressed the following question: “How efficiently can one determine whether a given graph  $G$  is a median graph or not?” In their paper “Dynamic Search in Graphs,” Chung, Graham and Saks [14, Theorem 3] have presented an algorithm for this purpose, which runs in time  $O(n^4)$ , where  $n = |V(G)|$ . Their algorithm is based on the decomposition algorithm of Graham and Winkler [28] and can be summarized briefly as follows.

First compute the distance matrix of  $G$  and use Graham–Winkler decomposition algorithm [28] to determine whether (and if so, how)  $G$  can be isometrically embedded into a hypercube. If such an embedding exists, then check whether the image of  $V(G)$  (under that embedding) is majority closed. (See Chapter 1, page 8, for the definition of majority closure.) If  $G$  passes all these tests, then  $G$  is proclaimed to be a median graph; see [14] for details. The time complexity of  $O(n^4)$  is contributed by the test for determining whether the image of  $V(G)$  is majority closed or not.

In what follows, we present two algorithms, each of which identifies median graphs in time  $O(n^2 \log n)$ . (Throughout this chapter,  $n$  will denote the number of vertices of a candidate graph  $G$ .) These are based on two different characterizations of median graphs given respectively by Bandelt [3] and Mulder [48].

Both algorithms are interesting in their own right. Some of their salient features

are the following.

- The first algorithm is based on the idea of retraction as used by Bandelt in [3] while the second is based on a completely different idea of convex-expansions (or more appropriately, convex-contractions) as used by Mulder in [48].
- The first algorithm starts with a subgraph isomorphic to  $K_2$  (i.e., an edge) of the input graph  $G$  and *attempts* to build a sequence of larger subgraphs of  $G$  such that the sequence eventually ends with  $G$ ; on the other hand, the second algorithm starts with (the input graph)  $G$  and *attempts* to build a sequence of smaller graphs (by appropriate contraction) such that a graph isomorphic to  $K_2$  is eventually obtained. Thus the two processes work their ways in opposite directions, but they are *not* “reverses” of each other in any obvious way.

In Section 3.1 below, we present the background material necessary for understanding the statements of the two characterizations while in Sections 3.2 and 3.3, we present and analyze the respective recognition algorithms. Finally, in Section 3.4, we summarize the results, comment on the relative characteristics of the two schemes, discuss the scope for possible improvements, and state some related issues.

### 3.1 Preliminaries

Let  $G$  be a connected graph. For a vertex  $v$  of  $G$ , let

$$N_G(v) = \{w \mid \{v, w\} \in E(G)\};$$

in other words,  $N_G(v)$  is the set of neighbors of  $v$  in  $G$ . For a triple of vertices  $u, v$  and  $w$  of  $G$ , let  $M_G(u, v, w)$  denote the set of medians of  $u, v$  and  $w$  in  $G$ . (Note that if  $G$  is a median graph, then  $M_G(u, v, w)$  must be a singleton set.)

We extend the definitions of  $N_G$  and  $M_G$  to nonempty vertex subsets  $S, S_1, S_2, S_3$  of  $V(G)$  as follows:

$$N_G(S) = \bigcup_{v \in S} N_G(v),$$

$$M_G(S_1, S_2, S_3) = \bigcup_{v_i \in S_i, 1 \leq i \leq 3} M_G(v_1, v_2, v_3).$$

If some  $S_i$  above is a singleton set, say  $S_i = \{v\}$ , then we will use  $v$  in place of  $S_i$  in the foregoing definitions. Also, if  $H$  is a subgraph of  $G$ , then we will occasionally use  $N_G(H)$  to denote  $N_G(V(H))$ . We will further omit the subscript  $G$  in these as well as in other definitions whenever the reference is clear from the context.

Given two connected graphs  $G$  and  $H$ , a mapping  $f : V(G) \rightarrow V(H)$  is said to be *edge-preserving* if  $\{x, y\} \in E(G)$  implies that  $\{f(x), f(y)\} \in E(H)$ . An edge-preserving mapping  $f : V(G) \rightarrow V(H)$  is said to be a *retraction* if there exists an edge-preserving mapping  $g : V(H) \rightarrow V(G)$  such that  $f \circ g$  is an identity on  $V(H)$ : we say that  $H$  is a *retract* of  $G$ . A common usage of the concept of retraction is when  $H$  is a subgraph of  $G$ . In that case,  $f : V(G) \rightarrow V(H)$  is a retraction just when  $f$  is edge-preserving and an identity on  $V(H)$ . If a subgraph  $H$  of a connected graph  $G$  is a retract of  $G$ , then it is easy to see that  $H$  is an isometric (hence vertex-induced) subgraph of  $G$ ; in other words, retraction implies isometry. The converse of the foregoing statement is false. To see this, consider the graph  $Q_3 - x$  (i.e., the graph obtained from the three-cube by deleting exactly one vertex), which is an isometric subgraph of  $Q_3$ . It is straightforward to see that the former is not a retract of the latter.

The following characterization of median graphs is given by Bandelt, and is based on the concept of retraction.

**Theorem 3.1.1** [9] *A graph is a median graph if and only if it is a retract of a hypercube. ■*

We will next prepare ground for the statement of another characterization of median graphs which has been given by Mulder [48].

Let  $G = (V, E)$  be a connected graph, and  $W \subseteq V$ .  $W$  is said to be a *convex subset* of  $V$  (and the corresponding induced subgraph  $\langle W \rangle$  a *convex subgraph* of  $G$ ) if for  $u, v \in W$ , all shortest  $(u, v)$ -paths in  $G$  lie entirely in  $\langle W \rangle$ . Clearly, a convex subgraph of a median graph must itself be a median graph. Further,  $W$  is said to be *2-convex* if for  $u, v \in W$  with  $d_G(u, v) = 2$ , every common neighbor of  $u$  and  $v$  is in  $W$ . Note that by Lemma 1.2.2(4), any two non-adjacent vertices in a median graph may have at most two common neighbors. For subsets  $S, T$  of  $V$ , let  $[S, T]$  denote the set of edges of  $G$  with one end vertex in  $S$  and the other in  $T$ .

Let  $W, W' \subseteq V$  be such that  $W \cup W' = V$ ,  $W \cap W' \neq \emptyset$ , and  $[W \setminus W', W' \setminus W] = \emptyset$ . The *expansion of  $G$  with respect to  $W$  and  $W'$*  is the graph  $G'$  constructed as follows:

- replace each vertex  $v \in W \cap W'$  by two vertices  $u_v, u'_v$ , which are joined by an edge;
- join  $u_v$  to the neighbors of  $v$  in  $W \setminus W'$  and  $u'_v$  to those in  $W' \setminus W$ ;
- if  $v, w \in W \cap W'$  and  $\{v, w\} \in E$ , then join  $u_v$  to  $u_w$  and  $u'_v$  to  $u'_w$ .

If  $W$  and  $W'$  are convex subsets of  $G$ , then  $G'$  will be called a *convex expansion* of  $G$ . An interesting and relevant feature of convex expansion is given in the following theorem of Mulder; see [48] for a proof.



**Theorem 3.1.2** [48] *Let  $G'$  be obtained from  $G$  by convex expansion. Then,  $G$  is a median graph if and only if  $G'$  is a median graph. ■*

Based on the foregoing theorem, Mulder obtained the following characterization of median graphs.

**Theorem 3.1.3** [48] *A graph is a median graph if and only if it can be obtained from a one-vertex graph by a sequence of convex expansions. ■*

As stated earlier, we have made the characterizations given by Theorems 3.1.1 and 3.1.3 as bases for our algorithms for recognizing median graphs. We conclude this section by stating a useful lemma due to R. L. Graham.

**Lemma 3.1.4** [26] *There exists a fixed, small  $c$  such that for any subgraph  $G = (V, E)$  of a hypercube,  $|E| \leq c \cdot |V| \log |V|$ . ■*

Note that by Theorem 3.1.1 and the remarks preceding it, every median graph is a subgraph of a hypercube, and hence the foregoing lemma applies to it.

## 3.2 Recognition Scheme based on Bandelt's Characterization

The central idea of Bandelt's proof of Theorem 3.1.1 is the following neat proposition which is itself a characterization of median graphs.

### 3.2.1 Bandelt's proposition

*If  $G$  is a nontrivial, connected graph, then  $G$  is a median graph if and only if there exists a sequence of subgraphs  $H_1, \dots, H_r$  of  $G$  such that*

1.  $H_1 \cong K_2$ ,
2. every  $H_i$  is a median graph in its own right,
3.  $H_i$  is a proper retract (hence a proper isometric subgraph) of  $H_{i+1}$ ,  $1 \leq i < r$ ,  
and
4.  $H_r \cong G$ . ■

We will first review the essential construction in Bandelt's proof of his characterization. Let  $G$  be a median graph, and  $H$  a proper median subgraph of  $G$ . It is straightforward to see that a vertex of  $G$  not in  $H$  can be adjacent to at most two vertices of  $H$ . Let  $x \in N_G(H) \cap (V(G) \setminus V(H))$ . If  $x$  is adjacent to exactly one vertex of  $H$ , then the subgraph  $H + x$  of  $G$  (namely, the subgraph of  $G$  induced by  $V(H) \cup \{x\}$ ) must be a median subgraph of  $G$ . Alternatively, suppose that  $x$  has exactly two neighbors  $a$  and  $b$  in  $H$ . Then, by Lemma 1.2.2(3) and (4), there exists a unique vertex  $u$  of  $H$  which is a common neighbor of  $a$  and  $b$ . The following subsets of  $V(G)$  are relevant in that case:

$$A = \{v \in V(H) \mid d(a, v) < d(u, v)\}$$

$$B = \{v \in V(H) \mid d(b, v) < d(u, v)\}$$

$$U = N_H(A) \cap N_H(B)$$

$$Y = A \cap N_H(U)$$

$$Z = B \cap N_H(U)$$

$$X = M_G(x, Y, Z).$$

The next lemma states some useful properties of the foregoing sets and the corresponding induced subgraphs of  $G$ ; see [3] for a proof.

**Lemma 3.2.1 [3]**

1.  $A \cap B = \emptyset$ .
2.  $V(H) \cap X = \emptyset$ .
3.  $\langle Y \rangle \cong \langle U \rangle \cong \langle Z \rangle \cong \langle X \rangle$ .
4.  $H' = \langle V(H) \cup X \rangle$  is a median subgraph of  $G$ ; actually,  $H'$  is the least median subgraph of  $G$  containing  $H$  together with the vertex  $x$ . ■

The recognition algorithm RECOG-B based on Bandelt's characterization appears in Figure 3.1. It is an iterative procedure whose input is the candidate graph  $G$ . We assume that  $G$  (i) is non-trivial, connected and bipartite, (ii) satisfies the bound on  $|E(G)|$  as stated in Lemma 3.1.4, and (iii) has been split into its biconnected components. Note that these checks may be performed in time linear in the size of  $G$ . Our algorithm will act on the biconnected components of  $G$ . If at any time ACCEPT (resp. REJECT) is encountered, we terminate the algorithm and report success (resp. failure).

We will next discuss how the sets  $Y$ ,  $U$  and  $Z$  can be obtained in linear time.

**3.2.2 Constructing the sets  $Y$ ,  $U$  and  $Z$** 

Suppose that  $H$  is a proper subgraph of  $G$  such that  $H$  is a median graph in its own right, and satisfies the conditions stated in step 8 of the algorithm RECOG-B, viz., there are three distinct vertices  $a, u$  and  $b$  in  $H$  such that  $u$  is the (unique) common neighbor of  $a$  and  $b$  in  $H$ . Let  $A_0, \dots, A_p$  constitute a partition of  $V(H)$  such that for  $v \in A_i$ ,  $d_H(a, v) = i$ ,  $0 \leq i \leq p$ ; see [25, pp. 48–49] for an algorithm

*procedure* RECOG-B( $G$ );  
*begin*  
 1. let  $H \cong K_2$  be a subgraph of  $G$ ;  
 2. START: *if*  $|V(G)| = |V(H)|$  and  $|E(G)| = |E(H)|$ , *then* ACCEPT;  
       *if*  $|V(G)| = |V(H)|$  and  $|E(G)| \neq |E(H)|$ , *then* REJECT;  
 3. let  $x \in N_G(H) \cap (V(G) \setminus V(H))$ ;  
 4. *if*  $|N_G(x) \cap V(H)| > 2$ , *then* REJECT;  
 5. *if*  $|N_G(x) \cap V(H)| = 1$ , *then* let  $H := H + x$ , and *go to* START;  
 6. at this point,  $|N_G(x) \cap V(H)| = 2$ ; let  $N_G(x) \cap V(H) = \{a, b\}$ ;  
 7. *if*  $|N_G(a) \cap N_G(b)| \neq 2$ , *then* REJECT;  
 8. let  $N_H(a) \cap N_H(b) = \{u\}$ ;  
       note that at this point,  $a$  and  $b$  have exactly two common neighbors in  
        $G$ :  $u \in V(H)$  and  $x \notin V(H)$ ;  
 9. construct the sets  $Y$ ,  $U$  and  $Z$  as defined earlier; let  
        $Y = \{y_1, \dots, y_m\}$ ,  $U = \{u_1, \dots, u_m\}$  and  $Z = \{z_1, \dots, z_m\}$ ,  
       where  $y_1 = a$ ,  $u_1 = u$  and  $z_1 = x$ , and where  $y_i \leftrightarrow u_i \leftrightarrow z_i$  defines an  
       isomorphism among the induced subgraphs  $\langle Y \rangle$ ,  $\langle U \rangle$  and  $\langle Z \rangle$ ;  
 10. *for*  $i, j \in \{1, \dots, m\}$ , *verify* that (i)  $y_i$  and  $z_i$  have exactly two common  
       neighbors in  $G$ , one of which is  $u_i$  (in  $U \subseteq V(H)$ ) and denote the  
       other by  $x_i$  (not in  $V(H)$ ), where  $x_1 = x$ , and (ii) *if*  $i \neq j$ , *then*  $x_i \neq x_j$ ;  
       *if* the foregoing statement is false, *then* REJECT;  
 11. let  $X = \{x_1, \dots, x_m\}$ ;  
       *verify* that  $u_i \leftrightarrow x_i$  defines an isomorphism between  $\langle U \rangle$  and  $\langle X \rangle$ ;  
       *if* this is not the case, *then* REJECT;  
 12. let  $H := \langle V(H) \cup X \rangle$ ; accordingly, adjust  $V(H)$ ,  $E(H)$  and  $N_H(v)$   
       for every  $v \in V(H)$ , and *go to* START  
*end.* (\* RECOG-B \*)

Figure 3.1: The recognition algorithm RECOG-B

for obtaining such a partition in time  $O(|E(H)|)$ . Let  $U_0, \dots, U_q$ , and  $B_0, \dots, B_r$  be similar partitions of  $V(H)$  with respect to the vertices  $u$  and  $b$  respectively. Note that the following inequalities hold:

- $q \geq 1$  and  $p, r \geq 2$
- $|p - q|, |q - r| \leq 1$  and  $|p - r| \leq 2$ , and
- $\max\{p, q, r\} \leq \min\{p, q, r\} + 2$ .

Let  $s = \min\{p, q, r\} - 1$ . For  $i = 0, \dots, s$ , let

$$A'_i = A_i \cap U_{i+1}$$

$$U'_i = U_i \cap A_{i+1} \cap B_{i+1}$$

$$B'_i = B_i \cap U_{i+1}.$$

Let  $s'$  be the largest integer such that for all  $i$ ,  $1 \leq i \leq s'$ , each of  $A'_i, U'_i$  and  $B'_i$  is nonempty. We claim that  $s' \leq s = \min\{p, q, r\} - 1$ . It suffices to show that at least one of  $A'_{s+1}, U'_{s+1}$  and  $B'_{s+1}$  is empty. We know that any vertex of  $H$  is at a distance at most  $p$  from  $a$ . Consequently,  $A_{p+1}$  is empty. Similarly,  $U_{q+1}$  and  $B_{r+1}$  are empty. Now, if  $p = \min\{p, q, r\}$ , then  $U'_p$  must be empty because so is  $A_{p+1}$ . Similarly, if  $q$  (resp.  $r$ ) =  $\min\{p, q, r\}$ , then  $A'_q$  and  $B'_q$  (resp.  $U'_r$ ) must be empty. We conclude that at least one of  $A'_{s+1}, U'_{s+1}$  and  $B'_{s+1}$  is empty.

Note that even though the set  $A$  (resp.  $B$ ) is not explicitly constructed by our algorithm, the set  $\{A'_i\}$  (resp.  $\{B'_i\}$ ) constitute a partition of  $A$  (resp.  $B$ ) into “orbits” around  $a$  (resp.  $b$ ) in the graph  $H$ . Similarly,  $\{U'_i\}$  is a partition of  $U$  around  $u$ . The iterative procedure ITER-B shown in Figure 3.2 constructs the desired sets  $Y$ ,  $U$  and  $Z$  in time  $O(|E(H)|)$ .

```

procedure ITER-B;
begin
1.    $j := 0$ ;
2.   for  $i := 0$  to  $s'$  do
3.     for each  $v \in U_i'$  do begin
4.        $C := N_H(v) \cap A_i'$ ;
5.        $D := N_H(v) \cap B_i'$ ;
      (Note:  $C$  and  $D$  must be either empty or singleton [3])
6.       if  $C$  and  $D$  are both nonempty, then begin
7.          $j := j + 1$ ;
8.          $u_j := v$ ;
9.          $y_j :=$  the single member of  $C$ ;
10.         $z_j :=$  the single member of  $D$ 
      end (* if *)
    end (* for *)
end.

```

Figure 3.2: The iterative procedure ITER-B

Let  $m$  be the value of  $j$  at the end of the the procedure ITER-B. Then

$$Y = \{y_1, \dots, y_m\},$$

$$U = \{u_1, \dots, u_m\},$$

$$Z = \{z_1, \dots, z_m\},$$

where  $y_i \leftrightarrow u_i \leftrightarrow z_i$  defines an isomorphism among the induced subgraphs  $\langle Y \rangle$ ,  $\langle U \rangle$  and  $\langle Z \rangle$  of  $H$ ; see [3] for details. This concludes the discussion on the construction of the sets  $Y$ ,  $U$  and  $Z$ .

In the next two theorems, we respectively present the correctness proof and analysis of the algorithm RECOG-B.

**Theorem 3.2.2** *The recognition algorithm RECOG-B terminates and correctly determines whether a given graph is a median graph or not.*

**Proof:** We first establish that the algorithm will terminate. Observe that in every iteration, if the input graph  $G$  is not rejected, then the subgraph  $H$  of  $G$  will grow in size by at least one vertex and one or more edges. Since  $G$  is finite, this procedure will come to an end after a finite number of steps.

For correctness, it suffices to show that at all times,  $H$  is an isometric subgraph of  $G$  and every triple of vertices of  $H$  admits a unique median which is in  $H$ . We do this by induction on the number of iterations of the algorithm. Step 1 of the algorithm shows that the induction basis is (trivially) true.

Assume that at some point during execution of the algorithm,  $H$  is a proper isometric subgraph of  $G$ , every triple of vertices of  $H$  admits a unique median which is in  $H$ , and control has reached step 3 of the algorithm. Let  $x \in N_G(H) \cap (V(G) \setminus V(H))$  as in step 3. Clearly, such a vertex exists because  $H$  is a proper subgraph of (a connected graph)  $G$ . If  $x$  is adjacent to more than two vertices of  $H$ , then a median of any three such vertices of  $H$  will be  $x$ , contradicting the induction hypothesis that every triple of vertices of  $H$  admits a unique median which is in  $H$ ; thus step 4 is correct. If  $x$  is adjacent to exactly one vertex of  $H$ , then it is straightforward to see that  $H + x$  is an isometric subgraph of  $G$ , and every triple of vertices of  $H + x$  admits a unique median which is in  $H + x$ ; thus in such a case, step 5 correctly assigns  $H + x$  to  $H$ , and transfers control to step 2.

In what follows, assume that  $x$  has exactly two neighbors  $a$  and  $b$  in  $H$  as stated in step 6. At this point, note that since  $H$  is an isometric subgraph of  $G$  (by induction hypothesis),  $a$  and  $b$  must have a common neighbor, say  $u$ , in  $H$ . Note also that  $x$  and  $u$  are (distinct) common neighbors of  $a$  and  $b$  in  $G$ . If the number of common neighbors of  $a$  and  $b$  in  $G$  (which properly includes  $H$ ) is different from (hence greater

than) two, then  $G$  would contain  $K_{2,3}$ ; in such a case,  $G$  is correctly rejected in step 7—see Lemma 1.2.2(4). Thus, assuming that  $G$  is not rejected in step 7,  $a$  and  $b$  have exactly two common neighbors in  $G$ :  $u \in V(H)$  and  $x \in V(G) \setminus V(H)$ , as noted in step 8. Let  $Y, U$  and  $Z$  be subsets of  $V(H)$  as defined earlier. Recall that by Lemma 3.2.1, the induced subgraphs  $\langle Y \rangle$ ,  $\langle U \rangle$  and  $\langle Z \rangle$  of  $H$  are isomorphic to each other. To see the correctness of step 10, first observe that for  $i \in \{1, \dots, m\}$ , if  $|M_G(x, y_i, z_i)| \neq 1$ , then  $G$  cannot be a median graph. Alternatively, if  $x, y_i$  and  $z_i$  have a unique median vertex, then (i) it cannot be in  $V(H)$  (see Lemma 3.2.1(2)) and (ii) it must be a common neighbor of  $y_i$  and  $z_i$ . When such a median vertex exists in  $G$ , it is referred to by  $x_i$  in the algorithm. Next consider step 11. Suppose that  $u_i \leftrightarrow x_i$  does not define an isomorphism between  $\langle U \rangle$  and  $\langle X \rangle$ . Assume w.l.o.g. that  $u_i, u_j$  are adjacent, and  $x_i, x_j$  are not adjacent; then the triple  $y_i, z_i$  and  $x_j$  would have no median in  $G$ , in which case  $G$  is correctly rejected in step 11.

If control reaches step 12, then it follows from Bandelt's proof of Theorem 3.1.1 that the induced subgraph  $\langle V(H) \cup X \rangle (= H'$ , say) is an isometric subgraph of  $G$  and every triple of vertices of  $H'$  admits a unique median which is in  $H'$ ; see [3] for details. Thus the correctness proof is complete. ■

**Theorem 3.2.3** *The recognition algorithm RECOG-B runs in time  $O(n^2 \log n)$ .*

**Proof:** Since we assume that an input graph  $G = (V, E)$  satisfies Lemma 3.1.4, we have  $|E| \leq c \cdot |V| \log |V|$ , where  $c$  is a small, fixed constant. In every iteration of the algorithm, if  $G$  is not rejected, then  $H$  grows by at least one vertex and one or more edges. Thus the number of iterations is bounded by  $|V|$ . Next, the amount of work per iteration is governed by steps 9 through 12. We have earlier shown that the sets



$Y, U$  and  $Z$  in step 9 may be obtained in time  $O(|E(H)|)$ . Also, (i) verification of the conditions in step 10, (ii) testing for the isomorphism in step 11, and (iii) adjusting  $H$  in step 12 may each be done in time linear in the size of  $H$ . The theorem follows.

■

### 3.3 Recognition Scheme based on Mulder's Characterization

In this section, we present the recognition algorithm which is based on the idea of convex-expansions used by Mulder [48]. In the following subsection, we present the theoretical machinery necessary for the development of the algorithm.

#### 3.3.1 Theoretical background

The following lemma is a slight generalization of an observation of Bandelt and Mulder. It was suggested to us (together with a proof) by Bandelt [4], and will be useful in the sequel.

**Lemma 3.3.1** *Let  $G = (V, E)$  be a connected, bipartite graph in which every triple of vertices admits a (not necessarily unique) median, and let  $W \subseteq V$ . Then  $W$  is a convex subset of  $V$  if and only if  $W$  is 2-convex and  $\langle W \rangle$  is an isometric subgraph of  $G$ .*

**Proof:** [4] Let  $G$  and  $W$  be as in the statement of the lemma. The “only if” part of the lemma is obvious. For the “if” part, suppose that  $W$  is 2-convex and  $\langle W \rangle$  is an isometric (hence connected) subgraph of  $G$ . We show that  $W$  is a convex subset of  $V$ . We do this by induction on  $d(u, v)$ , where  $u, v \in W$ . For  $d(u, v) = 1, 2$ , the claim is immediate. Let  $d(u, v) \geq 3$ , and pick a shortest path  $P = u - w - \dots - v$  between

$u$  and  $v$  in  $G$ . If  $P$  is not included in  $W$ , then we may assume that  $w \notin W$ . Let  $t$  be a neighbor of  $u$  on a shortest path between  $u$  and  $v$  in (the isometric subgraph)  $\langle W \rangle$ . Let  $x$  be a median of  $v$ ,  $w$  and  $t$  in  $G$ . It is clear that  $x$  is a common neighbor of  $w$  and  $t$ , and  $x \neq u$ . Note that  $d(t, v) < d(u, v)$ ,  $t \in W$ , and  $x$  is on a shortest path between  $t$  and  $v$ . By the induction hypothesis,  $x \in W$ . Now,  $u, x \in W$ , and  $w$  is a common neighbor of  $u$  and  $x$  in  $G$ . By 2-convexity of  $W$ , we have  $w \in W$ , a contradiction. ■

Let  $G = (V, E)$  be a connected, bipartite graph. Fix an edge  $\{a, b\}$  of  $G$  and define the following sets:

$$\left. \begin{aligned} W_a &= \{w \in V \mid d(a, w) < d(b, w)\} \\ W_b &= \{w \in V \mid d(b, w) < d(a, w)\} \\ F &= [W_a, W_b] \\ U_a &= \{u \in W_a \mid u \text{ is the end vertex of an edge in } F\} \\ U_b &= \{u \in W_b \mid u \text{ is the end vertex of an edge in } F\}. \end{aligned} \right\} (*)$$

Note that  $\{W_a, W_b\}$  is a partition of  $V$ , and that  $\langle W_a \rangle$  and  $\langle W_b \rangle$  are connected subgraphs of  $G$ . The following theorem of Mulder states some important properties that these subgraphs have when  $G$  is a median graph. It will be very useful in the development of our algorithm.

**Theorem 3.3.2** [48] *Let  $G = (V, E)$  be a median graph. For an edge  $\{a, b\}$  of  $G$ , define the sets  $W_a, W_b, F, U_a$  and  $U_b$  as in (\*). Then*

1.  $U_a$  and  $U_b$  are convex subsets of  $V$ , and hence so are  $W_a$  and  $W_b$ .
2.  $F$  is a matching which defines an isomorphism between  $\langle U_a \rangle$  and  $\langle U_b \rangle$ .
3. If  $G'$  is the graph obtained from  $G$  by contracting the edges in  $F$ , then  $G'$  is a median graph, and  $G$  is the convex expansion of  $G'$  with respect to  $W_a$  and  $W_b$ . ■

Let  $\{V_0, \dots, V_r\}$  be a partition of  $W_a$  such that if  $x \in V_i$ , then  $d(a, x) = i$ ,  $0 \leq i \leq r$ . Define a similar partition of  $W_b$ . We call a vertex  $x$  of  $W_a$  (resp.  $W_b$ ) a *square vertex* if  $x \in U_a$  (resp.  $U_b$ ) and a *circle vertex* otherwise. Note that  $a$  and  $b$  themselves are square vertices. For  $j > i$ , if  $x \in V_i, y \in V_j$  and  $d(x, y) = j - i$ , then we say that  $x$  is a *predecessor* of  $y$ ; furthermore, if  $j - i = 1$ , then we say that  $x$  is an *immediate predecessor* of  $y$ . The following lemma states some important properties of the square and circle vertices that will be used by the algorithm. It is given in terms of  $U_a$  and  $W_a$  but an analogous statement holds for  $U_b$  and  $W_b$ .

**Lemma 3.3.3** *Let  $G = (V, E)$  be a connected, bipartite graph. For an edge  $\{a, b\}$  of  $G$ , define the sets  $W_a, W_b, F, U_a$  and  $U_b$  as in (\*). If  $U_a$  is a convex subset of  $W_a$ , then in  $W_a$ , every predecessor of a square vertex is a square vertex and a circle vertex has at most one immediate predecessor which is a square vertex.*

**Proof:** Suppose that  $U_a$  is a convex subset of  $W_a$ . Let  $w, x$  be square vertices and  $y$  a circle vertex. If  $y$  is a predecessor of  $w$ , then  $y$  must lie on a shortest  $(a, w)$ -path; since  $a, w \in U_a$ , convexity of  $U_a$  implies that  $y \in U_a$ . Next assume that  $w \neq x$  and

$w, x$  are immediate predecessors of  $y$ . Since  $G$  is a bipartite graph,  $w$  and  $x$  cannot be adjacent; thus  $y$  lies on a shortest  $(w, x)$ -path contradicting the hypothesis of the lemma. ■

The next theorem together with Theorem 3.3.2 and Lemma 3.3.3 provides the theoretical basis for the convex-expansions recognition algorithm.

**Theorem 3.3.4** *Let  $G = (V, E)$  be a connected, bipartite graph. For an edge  $\{a, b\}$  of  $G$ , define the sets  $W_a, W_b, F, U_a$  and  $U_b$  as in (\*). Assume that*

1.  $\langle U_a \rangle$  and  $\langle U_b \rangle$  are connected,
2.  $F$  is a matching which defines an isomorphism between  $\langle U_a \rangle$  and  $\langle U_b \rangle$ ,  
and
3. in  $W_a$  (resp.  $W_b$ ), every predecessor of a square vertex is a square vertex, and a circle vertex has at most one immediate predecessor which is a square vertex.

Let  $G'$  be the graph obtained from  $G$  by contracting the edges in  $F$ . Then, if  $G'$  is a median graph, so is  $G$ .

**Proof:** Let  $G$  and  $G'$  be as in the hypothesis of the lemma, and suppose that  $G'$  is a median graph. Recall that  $\{W_a, W_b\}$  is a partition of  $V(G)$ . Let  $V_a = W_a \setminus U_a$  and  $V_b = W_b \setminus U_b$ , whence  $W_a = V_a \cup U_a$  and  $W_b = V_b \cup U_b$ . Let  $U$  be the vertex subset of  $G'$  corresponding to the contraction of the edges in the matching  $F$ . Clearly,  $\langle U \rangle$  is isomorphic to each of  $\langle U_a \rangle$  and  $\langle U_b \rangle$ . We may write  $V(G') = V_a \cup U \cup V_b$ , where  $V_a, U$  and  $V_b$  are mutually disjoint. Observe that there is a natural bijection between the vertex subset  $V_a \cup U$  (resp.  $V_b \cup U$ ) of  $G'$  and the vertex subset  $W_a$

(resp.  $W_b$ ) of  $G$  such that the corresponding induced subgraphs are isomorphic to each other under that bijection. We maintain the distinction between square vertices and circle vertices in the graph  $G'$  as we did in  $G$ ; in other words, a vertex  $x$  in  $G'$  is a square vertex if  $x \in U$ , and a circle vertex otherwise. Note that the relative positions of square vertices and circle vertices in  $G'$  are similar to those in  $G$ , i.e., the natural isomorphism between  $\langle W_a \rangle$  (resp.  $\langle W_b \rangle$ ) and  $\langle V_a \cup U \rangle$  (resp.  $\langle V_b \cup U \rangle$ ) "preserves the square and circle vertices." It is clear that  $G$  is an expansion of  $G'$  with respect to  $V_a \cup U$  and  $V_b \cup U$ , and what remains to show is that it is a convex expansion, i.e.,  $V_a \cup U$  and  $V_b \cup U$  are convex subsets of  $V(G')$ . Since by hypothesis,  $G'$  is a median graph, it would follow from Theorem 3.1.2 that  $G$  is a median graph. To show that  $V_a \cup U$  and  $V_b \cup U$  are convex, it suffices to show that  $U$  is a convex subset of  $V(G')$ , and for this, using Lemma 3.3.1, it suffices to prove that  $U$  is 2-convex and  $\langle U \rangle$  is an isometric subgraph of  $G'$ .

Let  $c$  be the vertex of  $G'$  corresponding to the contraction of the edge  $\{a, b\}$  of  $G$ . Let  $\{V_0, \dots, V_r\}$  be the partition of  $V(G')$  such that if  $x \in V_i$ , then  $d(c, x) = i$ ,  $0 \leq i \leq r$ . This partition may be obtained easily from the earlier partitions of  $W_a$  and  $W_b$  with respect to the vertices  $a$  and  $b$  in the graph  $G$ . The definitions of predecessor and immediate predecessor remain as before. We first show that  $U$  is 2-convex. Let  $u, v$  be square vertices with  $d(u, v) = 2$ , where  $u \in V_i$ ,  $v \in V_j$  and  $j \geq i$ . Note that either  $j = i$  or  $j = i + 2$ . If  $j = i$ , then any common neighbor  $w$  of  $u$  and  $v$  will be either in  $V_{i+1}$  or in  $V_{i-1}$ , and hence  $w$  itself must be a square vertex, for otherwise statement 3 (of the hypothesis) will be violated. If  $j = i + 2$ , then any common neighbor  $w$  of  $u$  and  $v$  will be in  $V_{i+1}$ , in which case  $w$  will be an immediate predecessor of  $v$ , and hence a square vertex. We conclude that  $U$  is 2-convex with respect to  $V(G')$ .

We next establish that  $\langle U \rangle$  is an isometric subgraph of  $G'$ . Let  $u$  and  $v$  be square vertices, i.e.,  $u, v \in U$ . We claim that there is a shortest path between  $u$  and  $v$  in  $G'$  which contains only square vertices. The cases  $d(u, v) = 1, 2$  are easy. In what follows, suppose that  $d(u, v) \geq 3$ . If  $\langle U \rangle$  is not isometric in  $G'$ , then there exist two square vertices such that every shortest path between them contains at least one circle vertex. Among all such vertex pairs, let  $u, v$  be as close to each other as possible. As before, w.l.o.g. we assume that  $u \in V_i$ ,  $v \in V_j$  and  $j \geq i$ . Then every vertex (other than  $u$  and  $v$ ) on every shortest  $(u, v)$ -path in  $G'$  will be a circle vertex. We will derive a contradiction.

Let  $x$  be a neighbor of  $u$  on a shortest  $(u, v)$ -path. By assumption,  $x$  is a circle vertex, and hence cannot be a predecessor of  $u$ . Thus  $x \in V_{i+1}$ ; in other words,  $u$  is an immediate predecessor of  $x$ . Let  $z$  be the median of  $c, v, x$ . Since every shortest  $(c, v)$ -path contains only square vertices,  $z$  itself must be a square vertex. Also, since  $c$  is (trivially) a predecessor of  $x$ , so must be  $z$ . Furthermore, since every vertex (other than  $v$ ) on every shortest  $(x, v)$ -path is a circle vertex (and since  $z$  must be on some such path), we must have that  $z = v$ , i.e.,  $v$  is the median of  $c, v, x$ . This however means that  $z (= v)$  is a predecessor of  $x$  which together with  $j \geq i$  implies that  $v$  is an immediate predecessor of  $x$ . This contradicts condition 3 of the hypothesis of the theorem. ■

### 3.3.2 The convex-expansions algorithm

The recognition algorithm RECOG-M based on Mulder's characterization of median graphs appears in Figure 3.3. This is a recursive procedure whose input is the candidate graph  $G$ . As stated in Section 3.1, we assume that  $G$  (i) is non-

trivial, connected and bipartite, and (ii) satisfies the bound on  $|E(G)|$  as stated in Lemma 3.1.4. If at any time, "REJECT" is encountered, we terminate the algorithm and report failure; in the event of normal termination, we report success.

In the next two theorems, we respectively establish the correctness and bound on the time-complexity of the algorithm RECOG-M.

**Theorem 3.3.5** *The recognition algorithm RECOG-M terminates and correctly determines whether a given graph is a median graph or not.*

**Proof:** Termination is obvious. For correctness, first recall Lemma 1.2.2(2), viz.,  $G$  is a median graph if and only if every biconnected component of  $G$  is a median graph in its own right. Also note that  $K_2$  is trivially a median graph. Now, let  $G_i$  be a biconnected component of  $G$ , where  $G_i \not\cong K_2$ . It suffices to show that  $G_i$  is a median graph if and only if all four statements in step 5 of the algorithm are satisfied and  $G'$  (i.e., the graph obtained from  $G_i$  by appropriate contraction) is a median graph. This follows from Theorem 3.3.2, Lemma 3.3.3 and Theorem 3.3.4. ■

**Theorem 3.3.6** *The recognition algorithm RECOG-M runs in time  $O(n^2 \log n)$ .*

**Proof:** Let  $G = (V, E)$  be an input graph. As stated earlier, we assume that  $G$  (i) is nontrivial, connected and bipartite, and (ii) satisfies the condition stated in Lemma 3.1.4, i.e.,  $|E| \leq c \cdot |V| \log |V|$ , where  $c$  is a small, fixed constant. Clearly, these checks may be performed in time linear in the size of  $G$ .

Following step 1 of the algorithm, let  $G_1, \dots, G_r$  be the biconnected components of  $G$ . It is clear that obtaining these components takes linear time. Obviously,

```

procedure RECOG-M( $G$ );
begin
1. let  $G_1, \dots, G_r$  be the biconnected components of  $G$ ;
2. for  $i := 1$  to  $r$  do
3.   if  $G_i \not\cong K_2$  then begin
4.     fix an edge  $\{a, b\}$  of  $G_i$  and obtain the sets
        $W_a, W_b, U_a, U_b$  and  $F$ ;
5.     verify that
       (i)  $\langle U_a \rangle$  and  $\langle U_b \rangle$  are connected;
       (ii)  $F$  is a matching which defines an isomorphism between
            $\langle U_a \rangle$  and  $\langle U_b \rangle$ ;
       (iii) every immediate predecessor of a square vertex is a
           square vertex; and
       (iv) a circle vertex has at most one immediate predecessor
           which is a square vertex;
       if any one of the foregoing statements is false, then REJECT;
6.   let  $G'$  be the graph obtained from  $G_i$  by contracting the edges
       which are in  $F$ ;
7.   RECOG-M( $G'$ ) (* recursive call *)
       end (* if *)
   end. (* RECOG-M *)

```

Figure 3.3: The recognition algorithm RECOG-M



$r \leq |V|$ . Thus it suffices to show that each iteration of the “for” loop in step 2 takes linear time.

Let  $H = (X, D)$  be a biconnected component of  $G$ . Clearly, the check whether  $H \cong K_2$  or not may be performed in  $O(1)$  time. Suppose that  $H \not\cong K_2$ . Let  $\{a, b\} \in D$ . Let  $\{A_0, \dots, A_p\}$  be a partition of  $X$  such that for  $v \in A_i$ ,  $d(a, v) = i$ ,  $0 \leq i \leq p$ . Note that such a partition may be obtained in time  $O(|D|)$ ; for example, see [25, pages 48–49]. Let  $\{B_0, \dots, B_q\}$  be a similar partition of  $X$  with respect to  $b$ . Note that  $A_0 = \{a\}$ ,  $B_0 = \{b\}$ ,  $p, q \geq 1$  and  $|p - q| \leq 1$ . Let  $r$  be the largest integer such that  $A_r \cap B_{r+1} \neq \emptyset$ , and let  $s$  be the largest integer such that  $B_s \cap A_{s+1} \neq \emptyset$ . Note that  $r \leq p$  and  $s \leq q$ . Let  $C_i = A_i \cap B_{i+1}$ ,  $0 \leq i \leq r$ , and  $D_j = B_j \cap A_{j+1}$ ,  $0 \leq j \leq s$ . It is clear that  $W_a = \bigcup_{i=0}^r C_i$ ,  $W_b = \bigcup_{j=0}^s D_j$  and that  $\{C_0, \dots, C_r\}$  and  $\{D_0, \dots, D_s\}$  are respectively partitions of  $W_a$  and  $W_b$  such that the following distance constraints hold:  $v \in C_i$  if and only if  $d(a, v) = i$ ,  $0 \leq i \leq r$ , and  $v \in D_j$  if and only if  $d(b, v) = j$ ,  $0 \leq j \leq s$ . Observe that  $W_a$ ,  $W_b$  and their partitions (as defined above) may be obtained in time  $O(|D|)$ .

The iterative procedure *ITER-M* in Figure 3.4 constructs the sets  $U_a$ ,  $U_b$  and the matching  $F$  in time  $O(|D|)$ ; also, it marks the various vertices of  $H$  as square and circle vertices. We have assumed that  $r \leq s$ . (Note that if  $r = 0$ , then the “for” loop in step 3 is not executed at all.)

We maintain a representation of  $H$  based on the partitions of  $W_a$  and  $W_b$ , and the marking of the various vertices. Based on this representation, we may now verify conditions (i) and (ii) of step 5 of the algorithm in time  $O(|D|)$ . Also, since conditions (iii) and (iv) are local, they too are verifiable in time  $O(|D|)$ . The claim follows. ■

```

procedure ITER-M;
begin
1.  first mark  $a$  and  $b$  as square vertices and then mark every other
    vertex of  $W_a$  and  $W_b$  as a circle vertex;
2.   $U_a := \{a\}; U_b := \{b\}; F := \{\{a, b\}\};$ 
3.  for  $i := 1$  to  $r$  do
4.    for  $v \in C_i$  do
5.      if  $v$  has a neighbor, say  $w$ , in  $D_i$  then begin
6.         $U_a := U_a \cup \{v\};$ 
7.         $U_b := U_b \cup \{w\};$ 
8.         $F := F \cup \{\{v, w\}\};$ 
9.        Mark  $v$  (resp.  $w$ ) as a square vertex in  $W_a$  (resp.  $W_b$ )
        (* This overrides the earlier marking of  $v$  and  $w$  *)
      end (* if *)
    end
end.

```

Figure 3.4: The iterative procedure ITER-M

### 3.4 Concluding Remarks

In this chapter, we have presented two algorithms, each of which recognizes median graphs in time  $O(n^2 \log n)$ . Both schemes are completely different and are interesting in their own right. However, the algorithm RECOG-M (Figure 3.3) based on Mulder's characterization of median graphs [48] is conceptually simpler than the algorithm RECOG-B (Figure 3.1) which is based on Bandelt's characterization [3]. In the next chapter, we present two algorithms, each of which obtains an isometric embedding of a given median graph in a hypercube of least possible dimension under the same time bound, viz.,  $O(n^2 \log n)$ . Both algorithms have structures similar to those of the respective recognition algorithms of this chapter.

Median graphs have received renewed attention recently. For example, Chung, Graham and Saks [14] have obtained several new characterizations of median graphs, one of which is the following:  $G$  is a median graph if and only if  $\text{windex}(G) = 2$ , where

windex is a graph invariant related to some dynamic search problems in graphs. The graphs of finite windex are natural generalizations of median graphs, and Chung, Graham and Saks have proved in a subsequent paper [15] that they are precisely the *retracts of the Cartesian products of complete graphs* (cf. Theorem 3.1.1). These graphs were earlier studied by Mulder [51] who called them *quasimedial graphs*. It is interesting to note that median graphs are exactly the bipartite quasimedial graphs [70]. Issues like this lend greater importance to the design of efficient recognition algorithms.

We have attempted to improve the performances of the algorithms presented in this chapter, and we will outline our ideas here. We first consider the possible improvement of the algorithm RECOG-B (Figure 3.1). Let  $G$  be a connected, bipartite graph, and  $H$  a proper, isometric subgraph of  $G$  such that  $H$  is a median graph in its own right. Let  $x \in V(G) \setminus V(H)$ . If  $|N_G(x) \cap V(H)| \geq 3$ , then  $G$  cannot be a median graph for reasons stated in Section 3.2. Thus, assuming that the previous statement is not true, the number of neighbors of  $x$  in  $H$  may be (i) none, (ii) one, or (iii) two. This defines a natural partition of  $V(G) \setminus V(H)$  into (at most) three subsets, say  $V_0$ ,  $V_1$  and  $V_2$ , where if  $v \in V_i$ , then  $|N_G(v) \cap V(H)| = i$ . If  $G$  is a median graph, then there are some interesting properties of such a partition; for example, (i) no vertex in  $V_1$  is adjacent to any vertex in  $V_2$ , (ii)  $\langle V_1 \rangle$  consists of  $K_1$ 's and  $K_2$ 's, and (iii) a vertex in  $V_0$  has at most two neighbors in  $V_1$ . However, we could not exploit these observations to further improve the asymptotic complexity of the algorithm RECOG-B. This remains an interesting open problem.

One possible approach for improving the performance of the recognition algorithm RECOG-M (Figure 3.3) is that instead of choosing an arbitrary edge of the input graph (step 4, Figure 3.3), we might look for an edge for which the correspond-

ing matching  $F$  is as large as possible. Such a strategy will speed up the recognition procedure in many cases, since (i) we always split the graph into its biconnected components (step 1, Figure 3.3), and (ii) if a biconnected component is simply  $K_2$ , then we do not execute the main body of the algorithm. However, we could not implement this observation to further lower the asymptotic bound.

Recall from Theorem 3.1.1 and the discussion preceding it that the class of isometric subgraphs of hypercubes properly includes the class of median graphs. A natural question to ask is whether it is possible to construct a similar efficient recognition algorithm for graphs isometrically embeddable in hypercubes. We should mention here that Wilkeit [71] has obtained an  $O(n^3)$  algorithm that recognizes graphs that are isometrically embeddable in Cartesian products of complete graphs. Her algorithm can be easily specialized for identifying isometric subgraphs of hypercubes.

In what follows, we will briefly discuss the problems we encountered in our attempts to adapt our convex-expansions recognition algorithm (Figure 3.3) to resolve the foregoing question. Let  $G$  be a connected bipartite graph. Fix an edge  $\{a, b\}$  of  $G$  and obtain the sets  $W_a$ ,  $W_b$ ,  $F$ ,  $U_a$  and  $U_b$  as in (\*) on page 40. If  $W_a$  and  $W_b$  were guaranteed to be convex subsets of  $V(G)$ —see page 30 for the definition of convex sets of a graph—then we could construct an algorithm for recognizing isometric subgraphs of hypercubes by using the following two observations: (i)  $F$  is a matching under which  $\langle U_a \rangle$  and  $\langle U_b \rangle$  are isomorphic, and (ii) the graph  $G'$  obtained from  $G$  by contracting the edges belonging to  $F$  is isometrically embeddable in a hypercube if and only if  $G$  is. Thus, in order to construct an efficient recognition algorithm, we must have an efficient (say, linear time) way of testing the convexity of  $W_a$  and  $W_b$ . Now, Lemma 3.3.1 does not apply here, since isometric subgraphs of a hypercube need not have a median for every triple of vertices. Furthermore,

convexity of  $W_a$  and  $W_b$  does not imply the convexity of  $U_a$  and  $U_b$  (in the case of median graphs,  $U_a$  and  $U_b$  are convex, which implies the convexity of  $W_a$  and  $W_b$ —recall Theorem 3.3.2). Therefore, our scheme for testing convexity in median graphs does not carry over to a similar test here. Thus an important open problem is to construct an efficient convexity test: doing it via distance matrix of the graph will not lead to any improvement.

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#### 4. ISOMETRIC EMBEDDING OF MEDIAN GRAPHS IN THE HYPERCUBE

In this chapter, we present two  $O(n^2 \log n)$  algorithms, each of which obtains an isometric embedding of a given median graph  $G$  in a hypercube of least possible dimension, where  $n = |V(G)|$ . That such an embedding exists is a known result; for example, see [3,50]. (We will further argue this point in Section 4.1 below.) Our contribution is in providing efficient algorithms and achieving the minimality in the dimension. The structures of our embedding algorithms are quite similar to those of the respective recognition algorithms of the previous chapter.

In Figure 4.1, we have shown inclusion relationships among the following classes of graphs: (i) Bipartite Graphs ( $\mathcal{B}$ ), (ii) Subgraphs of Hypercubes ( $\mathcal{S}$ ), (iii) Isometric Subgraphs of Hypercubes ( $\mathcal{I}$ ), and (iv) Median Graphs ( $\mathcal{M}$ ). By the earlier remark that every median graph is an isometric subgraph of a hypercube and the fact that a hypercube is a bipartite graph, it follows that  $\mathcal{M} \subseteq \mathcal{I} \subseteq \mathcal{S} \subseteq \mathcal{B}$ . Further, we have shown a graph in each of the four regions of the Venn diagram to highlight the point that each inclusion is proper.

It is remarkable to note that the problem of deciding membership in class  $\mathcal{S}$  is NP-complete [1,42]. In other words, whether or not a given graph is a subgraph (not necessarily isometric) of a hypercube is NP-complete. Quite recently, it has further been shown that the problem of determining for a given tree  $T$  and a positive integer

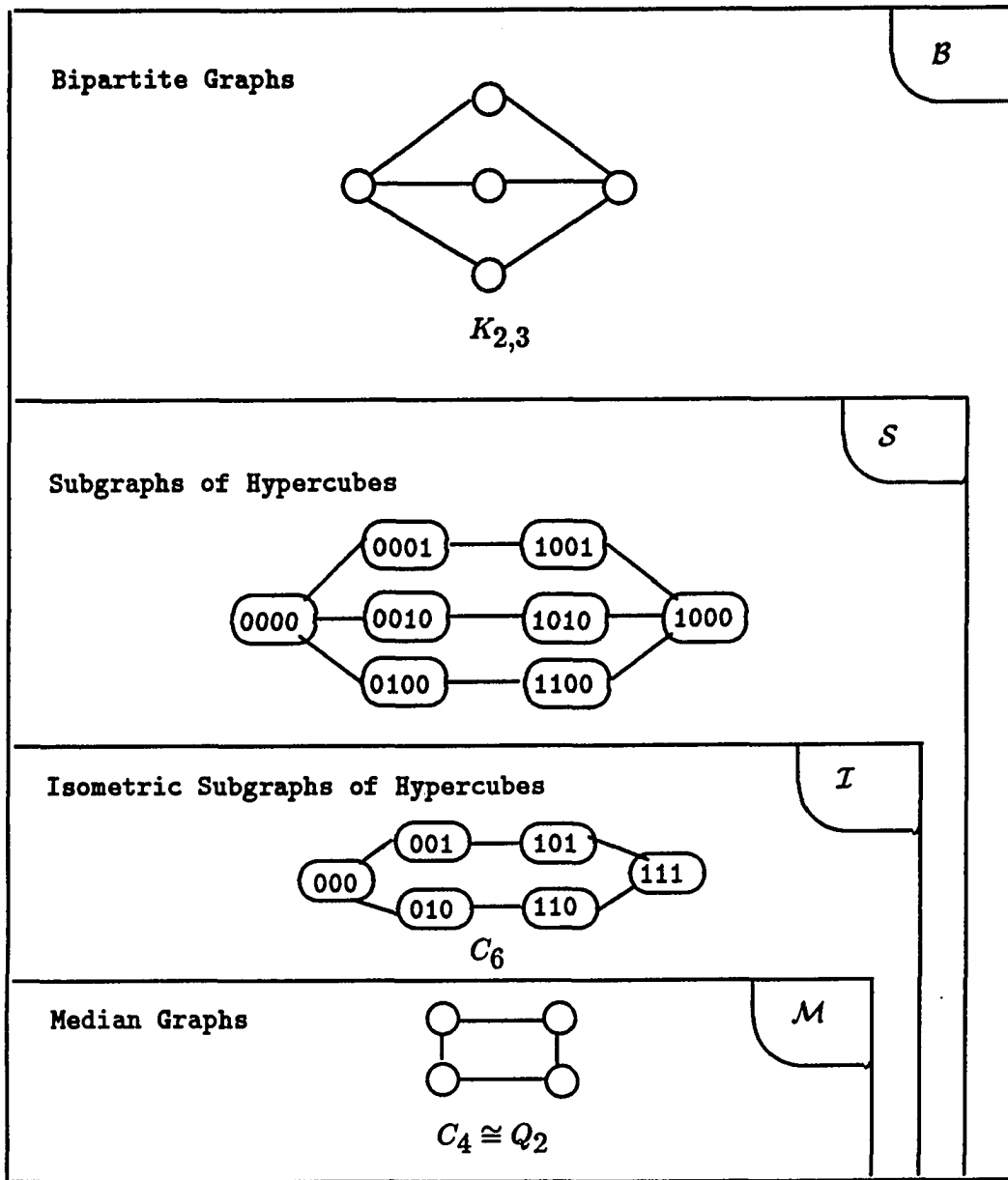


Figure 4.1: Inclusion relationships among certain classes of graphs

$k$ , whether or not  $T$  is a subgraph of  $Q_k$  is NP-complete [66]. Note that every tree is (a median graph and is) necessarily a subgraph of a sufficiently large hypercube. (We will subsequently argue that if  $T$  is a tree on  $r + 1$  vertices, then the smallest hypercube in which  $T$  has an isometric embedding is  $Q_r$ .)

We will make use of a more general theory of isometric embedding of graphs developed by Graham and Winkler [28]. In Section 4.1, we present the essential definitions and constructions of that theory while in Sections 4.2 and 4.3, we present and analyze the respective embedding algorithms. Finally, in Section 4.4, we comment on the relationships between the recognition schemes of the previous chapter and the corresponding embedding schemes of the present chapter, and discuss some related issues.

#### 4.1 Theoretical Background

Let  $G = (V, E)$  be a nontrivial, connected graph. Define a binary relation  $\theta$  on the edge set  $E$  of  $G$  as follows: if  $e = \{x, y\}$  and  $e' = \{x', y'\}$  are in  $E$ , then  $e \theta e'$  if and only if

$$d(x, x') + d(y, y') \neq d(x, y') + d(x', y).$$

This relation is easily seen to be reflexive and symmetric, but in general, it is not transitive. To see this, consider the edge set of the graph  $C_5$ . Let  $\hat{\theta}$  be the transitive closure of  $\theta$ . Then  $\hat{\theta}$  is an equivalence relation. Let  $E_1, \dots, E_r$  be the corresponding equivalence classes, which we refer to as  $\hat{\theta}$ -classes (or simply  $\theta$ -classes whenever  $\theta$  is transitive). Note that  $r \leq |E|$ . The following is a key lemma in this theory.

**Lemma 4.1.1** [28] *For  $x, y \in V$ , let  $P$  be the set of edges on a shortest path between  $x$  and  $y$ , and let  $Q$  be the set of edges on any other path between  $x$  and  $y$ . Then for*



any  $i$ ,  $1 \leq i \leq r$ ,  $|P \cap E_i| \leq |Q \cap E_i|$ . ■

It follows from Lemma 4.1.1 that for any  $i \in \{1, \dots, r\}$ , removing the edges of  $E_i$  from  $G$  will result in a disconnected graph. The reason is that if  $\{x, y\} \in E_i$ , then every path between  $x$  and  $y$  will contain at least one edge of  $E_i$  and thus the removal of the edges belonging to  $E_i$  from  $G$  will disrupt all paths between  $x$  and  $y$ . For each  $E_i$ , we construct a graph  $G_i$  from  $G$  as follows. First remove from  $G$  all the edges belonging to  $E_i$ . Let  $\{C_1, \dots, C_{m_i}\}$  denote the set of connected components of the resulting graph. Define  $G_i = (V_i, E_i)$ , where  $V_i = \{C_1, \dots, C_{m_i}\}$  with  $C_j$  and  $C_k$  adjacent if and only if some edge in  $E_i$  joins a vertex in  $C_j$  to a vertex in  $C_k$  in the original graph  $G$ . For every  $G_i$ , define a mapping  $\alpha_i : V \rightarrow V_i$  by setting  $\alpha_i(v) = C_j$  if and only if  $v \in C_j$ . Next, define an embedding  $\alpha$  of  $G$  into the Cartesian product of  $G_1, \dots, G_r$  given by  $v \mapsto (\alpha_1(v), \dots, \alpha_r(v))$ . This embedding is called the *canonical embedding* of  $G$ . Graham and Winkler prove the following important property of a canonical embedding.

**Theorem 4.1.2** [28] *The canonical embedding is an isometric embedding.* ■

The graphs  $G_1, \dots, G_r$  are called the *canonical factors* of  $G$  and  $r$  is called the *isometric dimension* of  $G$ , denoted by  $\dim_I(G)$ .

In the remainder of this section, we state some useful characterizations of graphs isometrically embeddable in hypercubes. Djokovic was the first to obtain such a characterization, which we state in the following theorem.

**Theorem 4.1.3** [17] *A graph  $G$  is isometrically embeddable in  $Q_r$  for some  $r$  if and only if  $G$  is bipartite and for every edge  $\{a, b\}$  of  $G$ , the vertex subsets  $W_a$  and*

$W_b$  of  $G$  are convex. ■

See page 30 for the definition of convex sets of a graph and page 40 for the definition of the sets  $W_a$  and  $W_b$ . Another characterization of the isometric subgraphs of hypercubes has been given by Graham and Winkler which we state in the next theorem.

**Theorem 4.1.4** [28] *A graph  $G$  is isometrically embeddable in  $Q_r$  for some  $r$  if and only if  $G$  is bipartite and the relation  $\theta$  defined on  $E(G)$  is transitive. ■*

As noted by Djokovic [17] and Graham [27], the smallest  $r$  satisfying the foregoing theorem is equal to  $\dim_I(G)$ , i.e., the number of  $\theta$ -classes of  $E$  (note that if  $\theta$  is transitive, then  $\theta = \hat{\theta}$ ). By Theorems 3.3.2 and 4.1.3, it follows that every median graph is isometrically embeddable in a hypercube and hence the relation  $\theta$  is transitive for (the edge set of) a median graph. (Recall that an isometric subgraph of a hypercube need not be a median graph.)

## 4.2 Embedding Scheme based on Bandelt's Characterization

The isometric embedding algorithm that we present in this section is based on Bandelt's characterization [3] of median graphs (Theorem 3.1.1, page 29) The following lemma will be useful in the development and analysis of the algorithm.

**Lemma 4.2.1** *If  $G$  is a connected graph and  $G_1, \dots, G_r$  are the biconnected components of  $G$ , then  $\dim_I(G) = \sum_{i=1}^r \dim_I(G_i)$ .*

**Proof:** Let  $G$  and  $G_1, \dots, G_r$  be as in the statement of the lemma. Let  $e = \{u, v\} \in E(G_i)$  and  $f = \{x, y\} \in E(G_j)$ ,  $i \neq j$ . It suffices to show that  $e(\neg\theta)f$ , i.e.,  $e$  and  $f$  are not related by  $\theta$ . It is clear that there is an articulation vertex  $w$  in  $G$  such that every path between a vertex of  $G_i$  and a vertex of  $G_j$  goes via  $w$ . It then follows that  $d(u, x) = d(u, w) + d(w, x)$ ,  $d(u, y) = d(u, w) + d(w, y)$  etc. Consequently,  $d(u, x) + d(v, y) = d(u, y) + d(v, x)$  and hence  $e(\neg\theta)f$ . ■

The isometric embedding algorithm called **EMBED-B**, which is based on Bandelt's characterization of median graphs, appears in Figure 4.2. As we stated earlier, it is quite similar to the recognition algorithm **RECOG-B** (Figure 3.1, page 34), which we have presented in the previous chapter. It is an iterative procedure in which the input graph  $G$  is assumed to be a nontrivial, median graph. The algorithm constructs a required isometric embedding by assigning a label  $L(v)$  (which is a binary string) to every vertex  $v$  of  $G$ .

**Theorem 4.2.2** *The embedding algorithm EMBED-B terminates and correctly embeds a given median graph  $G$  in a hypercube  $Q_r$  with  $r$  as small as possible.*

**Proof:** Termination follows by an argument as in the proof of Theorem 3.2.2 in the previous chapter. For correctness, let  $G$  be an input (nontrivial, median) graph. We claim that the labels assigned to the vertices of  $G$  by the algorithm defines an isometric embedding of  $G$  in  $Q_r$  with  $r$  as small as possible. We prove our claim by induction on the number of iterations of the algorithm. Step 1 of the algorithm shows that the induction basis is trivially true.

*procedure* EMBED-B( $G, L$ );

*begin*

1. let  $H \cong K_2$  be a subgraph of  $G$ ; label the two vertices, say  $a$  and  $b$  of  $H$ , by (the binary strings) 0 and 1, i.e.,  $L(a) = 0$  and  $L(b) = 1$ ;
2. START: if  $|V(G)| = |V(H)|$ , then STOP;
3. let  $x \in V(G) \setminus V(H)$  be such that  $x$  is adjacent to some vertex of  $H$ ;  
(Note that  $x$  may be adjacent to at most two vertices of  $H$ )
4. if  $x$  is adjacent to exactly one vertex of  $H$ , say  $u$ , then in the subgraph  $H + x$ , label  $x$  by  $0L(u)$  and label every other vertex  $v$  by  $1L(v)$ , where  $L(w)$  denotes the label of  $w$  in the graph  $H$ ;  
let  $H := H + x$  and go to START;
5. at this point  $x$  is adjacent to exactly two vertices of  $H$ , say  $a$  and  $b$ ;  
let  $u$  be the (unique) common neighbor of  $a$  and  $b$  in  $H$ ;
6. let  $Y, U, Z$  and  $X$  be the vertex subsets of  $G$  as defined during the construction of the recognition algorithm RECOG-B; observe that  $Y, U, Z \subseteq V(H)$ ,  $X \subseteq V(G) \setminus V(H)$ , and we may write

$$Y = \{y_1, \dots, y_m\}, U = \{u_1, \dots, u_m\},$$

$$Z = \{z_1, \dots, z_m\}, X = \{x_1, \dots, x_m\},$$

where  $y_1 = a$ ,  $u_1 = u$ ,  $z_1 = b$  and  $x_1 = x$  such that

$y_i \leftrightarrow u_i \leftrightarrow z_i \leftrightarrow x_i$  defines an isomorphism among the induced subgraphs  $\langle Y \rangle$ ,  $\langle U \rangle$ ,  $\langle Z \rangle$  and  $\langle X \rangle$ , and that  $y_i$ ,  $u_i$ ,  $z_i$  and  $x_i$  are (in that order) the four vertices of an induced four-cycle in  $G$ ; note that the labels assigned to  $y_i$ ,  $u_i$  and  $z_i$  (in the subgraph  $H$ ) must be such that  $L(u_i)$  will differ from  $L(y_i)$  (resp.  $L(z_i)$ ) in exactly one bit position, say  $j$  (resp.  $k$ ); in the induced subgraph  $H' = \langle V(H) \cup X \rangle$  of  $G$ , let  $L(x_i)$  be such that it differs from  $L(y_i)$  in the  $k$ -th bit and from  $L(z_i)$  in the  $j$ -th bit; the labels assigned to all other vertices remain unchanged;

7. let  $H := H'$ ; accordingly, adjust  $V(H)$ ,  $E(H)$  and  $N_H(v)$  for every vertex  $v$  of  $H$ , and go to START

*end.* (\* EMBED-B \*)

Figure 4.2: The isometric embedding algorithm EMBED-B

Assume that at some point during execution of the algorithm,  $H$  is a proper median subgraph of  $G$  and control has reached step 3 of the algorithm. Thus the vertices of  $H$  have been labeled by uniform binary strings of length, say  $r$ . Let  $x \in V(G) \setminus V(H)$  as in step 3. First suppose that  $x$  is adjacent to exactly one vertex  $u$  of  $H$  as stated in step 4. Note that in the (median) subgraph  $H+x$  of  $G$ , the edge  $\{u, x\}$  constitutes a biconnected component by itself; in other words,  $u$  is an articulation vertex in  $H+x$ . By Lemma 4.2.1, it follows that  $\dim_I(H+x) = \dim_I(H) + 1$ . Since the labeling of the vertices of  $H$  corresponds to an isometric embedding of  $H$  in a hypercube of smallest possible dimension (by induction hypothesis) and since our scheme is such that the length of a label increases by exactly one in step 4, we conclude that the labeling of the vertices of  $H+x$  in step 4 corresponds to a valid isometric embedding of  $H+x$  in a hypercube of least possible dimension as required.

Next suppose that the condition in step 4 is false and control has reached step 5 of the algorithm. Let  $a, b$  and  $u$  be vertices of  $H$  as stated in step 5. The fact that the vertex subsets  $Y, U, Z$  and  $X$  appear in  $G$  with properties stated in step 6 follows from Lemma 3.2.1, and from the discussion on the correctness and analysis of the recognition algorithm RECOG-B in the previous chapter. It is straightforward to see that the labels assigned to the vertices of (the median subgraph)  $H' = \langle V(H) \cup X \rangle$  in step 6 correspond to a valid isometric embedding of  $H'$  in a hypercube. Since the length of a label does *not* increase in this step, we conclude that the minimality of the dimension of the hypercube is (trivially) maintained. ■

By an analysis as in Theorem 3.2.3, we have the following result.

**Theorem 4.2.3** *The embedding algorithm EMBED-B runs in time  $O(n^2 \log n)$ . ■*

### 4.3 Embedding Scheme based on Mulder's Characterization

The isometric embedding algorithm that we present in this section is based on Mulder's characterization [48] of median graphs (Theorem 3.1.3, page 31). In the remainder of this section,  $G$  and  $G'$  are arbitrary but fixed graphs with characteristics stated below.

Let  $G = (V, E)$  be a nontrivial median graph. Fix an edge  $\{a, b\}$  of  $G$  and define the sets  $W_a, W_b, F, U_a$  and  $U_b$  as in (\*) on page 40 in the previous chapter. By Theorem 3.3.2, we may write  $F = \{\{a_1, b_1\}, \dots, \{a_m, b_m\}\}$ , where  $U_a = \{a_i \mid 1 \leq i \leq m\}$  and  $U_b = \{b_i \mid 1 \leq i \leq m\}$  for some  $m \geq 1$ . Let  $G'$  be the graph obtained from  $G$  by contracting the edges of  $F$ . Let  $V_a = W_a \setminus U_a$  and  $V_b = W_b \setminus U_b$ , whence  $W_a = V_a \cup U_a$  and  $W_b = V_b \cup U_b$ . Let  $U = \{c_i \mid 1 \leq i \leq m\}$  be the vertex subset of  $G'$  corresponding to the contracted edges of  $F$ , i.e.,  $c_i$  corresponds to the edge  $\{a_i, b_i\} \in F$ . Observe that  $a_i \leftrightarrow b_i \leftrightarrow c_i$  defines an isomorphism among  $\langle U_a \rangle$ ,  $\langle U_b \rangle$  and  $\langle U \rangle$ . Thus we may write  $V(G') = V_a \cup U \cup V_b$ , where  $V_a, U$  and  $V_b$  are mutually disjoint. We will use  $V_a, V_b$  (resp.  $\langle V_a \rangle, \langle V_b \rangle$ ) as vertex subsets (resp. induced subgraphs) of both  $G$  and  $G'$ , and use the context to resolve any ambiguity. As stated in Theorem 3.3.2,  $G$  is the convex expansion of  $G'$  with respect to the vertex subsets  $V_a \cup U$  and  $V_b \cup U$ .

The next two lemmas are useful in the development and analysis of the embedding algorithm.

**Lemma 4.3.1** *Let  $e \in F, e' \in E(G)$ . Then,  $e \theta e' \iff e' \in F$ .*

**Proof:** Let  $e = \{a_i, b_i\} \in F, e' \in E(G)$ . First suppose that  $e' \in F$  and let  $e' = \{a_j, b_j\}$ . By Theorem 3.3.2,  $F$  is a matching which defines an isomorphism between the (convex) subgraphs  $\langle U_a \rangle$  and  $\langle U_b \rangle$  of  $G$ . Therefore,  $d(a_i, a_j) =$

$d(b_i, b_j) = d(a_i, b_j) - 1 = d(a_j, b_i) - 1$ . Thus  $d(a_i, a_j) + d(b_i, b_j) \neq d(a_i, b_j) + d(a_j, b_i)$ , whence  $e \theta e'$ .

Conversely, suppose that  $e' = \{x, y\} \notin F$ . W.l.o.g. assume that  $x, y \in W_a$ . Since  $W_a$  is convex (recall Theorem 3.3.2) and  $G$  is bipartite, it follows that  $d(a_i, x) = d(b_i, x) - 1$  and  $d(a_i, y) = d(b_i, y) - 1$ . Consequently,  $d(a_i, x) + d(b_i, y) = d(a_i, y) + d(b_i, x)$ , whence  $e(-\theta)e'$ . ■

**Lemma 4.3.2**  $\dim_I(G) = \dim_I(G') + 1$ .

**Proof:** By the previous lemma,  $F$  constitutes one  $\theta$ -class of  $E(G)$  and hence contributes one dimension to the (minimum dimensional) hypercube in which (the median graph)  $G$  has a canonical embedding. Thus, it suffices to show that the number of  $\theta$ -classes of  $E(G')$  is equal to the number of  $\theta$ -classes of  $E(G)$  minus one.

Define a mapping  $f : V(G) \rightarrow V(G')$  as follows:  $f(a_i) = c_i = f(b_i)$  for  $a_i \in U_a$ ,  $b_i \in U_b$ , and  $f(x) = x$  for  $x \in V_a \cup V_b$ . Consider edges  $e_1, e_2$  in  $E(G) \setminus F$  and let  $e_i = \{x_i, y_i\}$ ,  $i = 1, 2$ . It is clear that  $x_i, y_i$  are either both in  $W_a$  or both in  $W_b$ . Let  $f(x_i) = x'_i$ ,  $f(y_i) = y'_i$ . Then  $e'_i = \{x'_i, y'_i\} \in E(G')$ ,  $i = 1, 2$ . It is straightforward to see that  $e_1 \theta e_2$  in  $G$  if and only if  $e'_1 \theta e'_2$  in  $G'$ . It then follows that the number of  $\theta$ -classes of  $E(G')$  is exactly one fewer than the number of  $\theta$ -classes of  $E(G)$ . ■

The isometric embedding algorithm called EMBED-M, which is based on Mulder's characterization of median graphs, appears in Figure 4.3. It is a recursive procedure which has a structure similar to that of the recognition algorithm RECOG-M (Figure 3.3) in the previous chapter.  $G$  is the input (nontrivial, median) graph. The procedure outputs a set  $L$  of labels for the vertex set  $V(G)$ , where each label is an

$r$ -bit binary string corresponding to a canonical embedding of  $G$  in  $Q_r$  for some  $r \geq$

1. As stated earlier, we use  $V_a$  and  $V_b$  to denote the vertex subsets of  $G$  as well as  $G'$ . This point is particularly evident in steps 7 and 8 of the algorithm.

**Theorem 4.3.3** *The embedding algorithm EMBED-M terminates and correctly embeds a given median graph  $G$  in  $Q_r$ , where  $r$  is as small as possible.*

**Proof:** Termination is obvious. For correctness, let  $G$  be an input (nontrivial, median) graph. We claim that the set  $L$  of labels assigned to the vertices of  $G$  by the algorithm defines an isometric embedding of  $G$  in  $Q_r$  where  $r$  is as small as possible. We prove our claim by induction on the number  $i$  of recursive calls to the algorithm. For  $i = 0$ , the claim is trivially true.

It suffices to show that if (the median graph)  $G'$  obtained from  $G$  (as in the algorithm) satisfies the claim, then so does  $G$ . Observe that if  $u, v \in W_a$  (resp.  $W_b$ )  $\subseteq V(G)$ , and  $u', v'$  are the corresponding vertices in  $V_a \cup U$  (resp.  $V_b \cup U$ )  $\subseteq V(G')$  under the natural bijection between  $W_a$  and  $V_a \cup U$  (resp.  $W_b$  and  $V_b \cup U$ ), then  $d_G(u, v) = d_{G'}(u', v')$ . Also, if  $u \in W_a, v \in W_b$ , and  $u', v'$  are the corresponding vertices in  $V_a \cup U$  and  $V_b \cup U$  respectively in  $G'$ , then  $d_G(u, v) = d_{G'}(u', v') + 1$ . It is straightforward to see from our construction (statements 7 through 11) that if  $G'$  has an isometric embedding in  $Q_r$ , then the set  $L$  of labels assigned to the vertices of  $G$  by the algorithm defines an isometric embedding of  $G$  in  $Q_{r+1}$ . That the embedding thus produced is in the smallest possible hypercube follows from the induction hypothesis and Lemma 4.3.2. ■

By an analysis similar to that of the recognition algorithm RECOG-M (Theorem 3.3.6) in the previous chapter, we obtain the following result.



```

procedure EMBED-M( $G, L$ );
begin
1. if  $G \cong K_2$  then label the two vertices, say  $x$  and  $y$ , of  $G$  by
   (the binary strings) 0 and 1 respectively, i.e.,  $L(x) = 0$  and  $L(y) = 1$ 
   else begin
2. fix an edge  $\{a, b\}$  of  $G$  and obtain the sets  $W_a, W_b, F, U_a$  and  $U_b$ ;
   suppose that  $F = \{\{a_1, b_1\}, \dots, \{a_m, b_m\}\}$ , where
    $U_a = \{a_1, \dots, a_m\}$  and  $U_b = \{b_1, \dots, b_m\}$ ;
3. let  $V_a := W_a \setminus U_a$  and  $V_b := W_b \setminus U_b$ ;
4. let  $G'$  be the graph obtained from  $G$  by contracting the edges in  $F$ ;
5. let  $U := \{c_1, \dots, c_m\}$  be the vertex subset of  $G'$ , where  $c_i$ 
   corresponds to the contracted edge  $\{a_i, b_i\}$ ,  $1 \leq i \leq m$ ;
6. EMBED-M( $G', L'$ ) (* recursive call *)
7. for  $v \in V_a$  do  $L(v) := 0L'(v)$ ;
8. for  $v \in V_b$  do  $L(v) := 1L'(v)$ ;
9. for  $i := 1$  to  $m$  do begin
10.  $L(a_i) := 0L'(c_i)$ ;
11.  $L(b_i) := 1L'(c_i)$ 
   end (* for *)
   end; (* else *)
end. (* EMBED-M *)

```

Figure 4.3: The isometric embedding algorithm EMBED-M

**Theorem 4.3.4** *The embedding algorithm  $EMBED-M$  runs in time  $O(n^2 \log n)$ . ■*

#### 4.4 Concluding Remarks

In this chapter, we have presented two algorithms  $EMBED-B$  and  $EMBED-M$ , each of which obtains an isometric embedding of a given median graph  $G$  in a hypercube of least possible dimension in time  $O(n^2 \log n)$ , where  $n = |V(G)|$ . These have structures similar respectively to those of the algorithms  $RECOG-B$  and  $RECOG-M$  (presented in the previous chapter), each of which recognizes median graphs under the same time bound, i.e.,  $O(n^2 \log n)$ . It is fairly straightforward to see that the recognition algorithm  $RECOG-B$  (resp.  $RECOG-M$ ) and the isometric embedding algorithm  $EMBED-B$  (resp.  $EMBED-M$ ) may be combined in that order to yield a single  $O(n^2 \log n)$  algorithm which will first tackle the recognition problem, and then produce an appropriate isometric embedding if the given graph is found to be a median graph. Alternatively, it is also possible to carry out the embedding procedure in conjunction with (i.e., as a part of) the recognition procedure so that if the given graph is found to be a median graph, then at the termination of the algorithm, its vertices will be suitably labeled by uniform binary strings corresponding to an appropriate isometric embedding. The resulting algorithm will have the same asymptotic complexity, viz.,  $O(n^2 \log n)$ . However, it will be somewhat complicated.

Because of the structural similarity between the recognition schemes and the embedding schemes, it is easy to see that any improvement in the recognition procedure (presented in the previous chapter) will lead to a similar improvement in the corresponding embedding procedure of this chapter.

In regard to the isometric embedding procedure, it is remarkable to note that

the size of the hypercube (in which the isometric embedding is prescribed) may be exponential in the size of the given median graph. To see this, consider a tree  $T$  (which is known to be a median graph) on  $r + 1$  vertices. It is straightforward to see that the smallest hypercube in which  $T$  has an isometric embedding is  $Q_r$ , whose size ( $|V(Q_r)| = 2^r$  and  $|E(Q_r)| = r \cdot 2^{r-1}$ ) is clearly exponential in the size of  $T$ . However, our embedding algorithm produces just the appropriate labeling of the vertices of the given median graph, and does not explicitly construct the hypercube.

It is worthwhile to point out here that if  $G$  is a graph known to be isometrically embeddable in a hypercube and  $G'$  is derived from  $G$  as in Section 4.3, then Lemmas 4.3.1 and 4.3.2 will still hold, and our embedding algorithm **EMBED-M** will work to produce an isometric embedding of  $G$  in the smallest possible hypercube under the same time bound.

## 5. CHARACTERIZATIONS FOR THE PLANARITY AND OUTERPLANARITY OF PRODUCT GRAPHS

In this chapter, we discuss necessary and sufficient conditions for the planarity and outerplanarity of each of the three product graphs in terms of the factor graphs. We observe that (i) characterizations for the planarities of the  $\square$ -product and  $\times$ -product already appear in the literature and (ii) for any two graphs  $G_1$  and  $G_2$ , each containing at least one edge, the graph  $G_1 \boxtimes G_2$  is always non-outerplanar. For the sake of completeness, we will present complete statements of these results later in appropriate places. Our contribution consists of characterizations for the following: (i) planarity of the  $\boxtimes$ -product, (ii) outerplanarity of the  $\square$ -product and (iii) outerplanarity of the  $\times$ -product. Thus, together with our results, we now have a complete characterization for the planarity and outerplanarity of each of the three products.

In Section 5.1, we state characterizations for the planarity and outerplanarity of general graphs, which have been given in terms of graph subcontraction and are useful in the sequel. Section 5.2 contains (i) statements of the known results concerning planarities of the  $\square$ -product and  $\times$ -product, and (ii) statement and proof of our characterization for the planarity of the  $\boxtimes$ -product. Section 5.3 deals with the outerplanarity of each of the  $\square$ -product and  $\times$ -product. Finally, in Section 5.4, we summarize the results of this chapter and discuss some related issues.

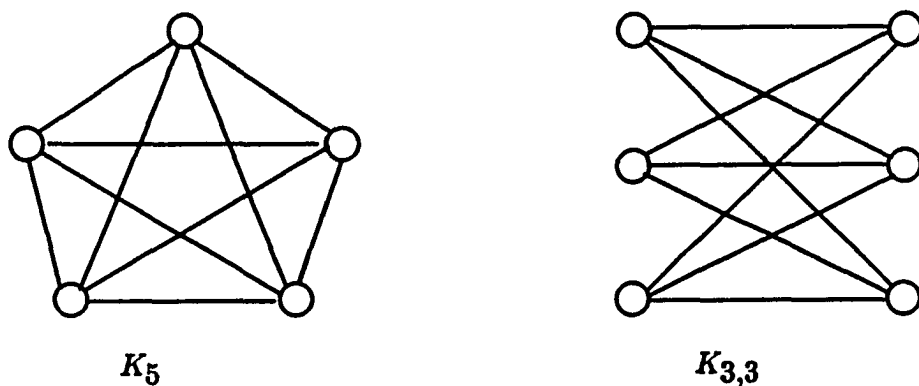


Figure 5.1: The graphs  $K_5$  and  $K_{3,3}$

### 5.1 Preliminaries

We will first state characterizations of planar and outerplanar graphs in terms of graph subcontraction, which will be useful in the sequel. Recall from Chapter 1 (page 4) that if  $G$  is a graph, then the graph obtained from  $G$  by identifying any two adjacent vertices of  $G$  is said to be an elementary contraction of  $G$ . Further, a graph  $H$  is said to be a subcontraction (or, a minor) of  $G$  if  $H$  is obtainable from a subgraph of  $G$  by a sequence of elementary contractions. Obviously, planar as well as outerplanar graphs are closed under the operation of subcontraction.

**Theorem 5.1.1** [90,92,67] *A graph is planar if and only if neither  $K_5$  nor  $K_{3,3}$  is a subcontraction of  $G$ . ■*

The graphs  $K_5$  and  $K_{3,3}$ —which are forbidden for planar graphs—appear in Figure 5.1. The next theorem is an analogue of Theorem 5.1.1 for outerplanar graphs [8, page 89].

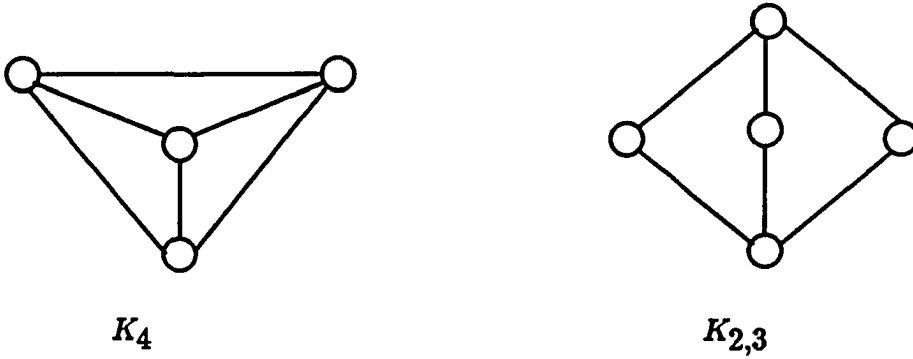


Figure 5.2: The graphs  $K_4$  and  $K_{2,3}$

**Theorem 5.1.2** *A graph is outerplanar if and only if neither  $K_4$  nor  $K_{2,3}$  is a subcontraction of  $G$ . ■*

The graphs  $K_4$  and  $K_{2,3}$ —which are forbidden for outerplanar graphs—appear in Figure 5.2. The following theorem shows that while dealing with the planarity (resp. outerplanarity) of the  $\square$ -product and  $\boxtimes$ -product graphs, it suffices to consider only those factor graphs which are themselves planar (resp. outerplanar).

**Theorem 5.1.3** *Let  $G_1$  and  $G_2$  be connected graphs. If one of  $G_1$  and  $G_2$  is nonplanar (resp. non-outerplanar), then so is each of  $G_1 \square G_2$  and  $G_1 \boxtimes G_2$ .*

**Proof Sketch:** Each of  $G_1 \square G_2$  and  $G_1 \boxtimes G_2$  contains  $G_1$  as well as  $G_2$  as a subgraph. The claim then follows from Theorems 5.1.1 and 5.1.2. ■

We will later show that for nontrivial, connected graphs  $G_1$  and  $G_2$ , if one of  $G_1$  and  $G_2$  is nonplanar (resp. non-outerplanar), then so is  $G_1 \times G_2$ .

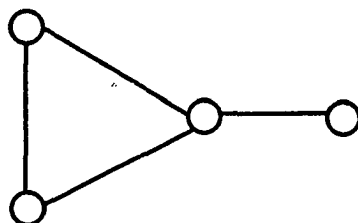


Figure 5.3: The graph  $K_3 + x$

## 5.2 Planarity

We observe that characterizations for the planarity of each of the  $\square$ -product and  $\times$ -product already appear in the literature. The following theorem of Behzad and Mahmoodian gives necessary and sufficient conditions for the planarity of the  $\square$ -product of graphs.

### Theorem 5.2.1 [9]

1. *If  $G_1$  and  $G_2$  are nontrivial, connected graphs, each different from  $K_2$ , then  $G_1 \square G_2$  is planar if and only if both  $G_1$  and  $G_2$  are paths or one is a path and the other is a cycle.*
2. *If  $G$  is a nontrivial, connected graph, then  $G \square K_2$  is planar if and only if  $G$  is outerplanar. ■*

By a *1-contraction* of a graph  $G$  we mean the removal from  $G$  of each vertex of degree one. Let  $K_3 + x$  denote the graph shown in Figure 5.3 and let  $K_4 - e$  denote the graph obtained from  $K_4$  by deleting exactly one edge. The next theorem, given by Farzan and Waller, characterizes the planar  $\times$ -product graphs.

**Theorem 5.2.2 [20]**

1. *Let  $G_1$  and  $G_2$  be connected graphs with more than four vertices each. Then,  $G_1 \times G_2$  is planar if and only if one of the following holds:*
  - (a) *one of the graphs is a path and the other is 1-contractable to a path or a cycle;*
  - (b) *one of the graphs is a cycle and the other is 1-contractable to a path.*
2. *Each of the graphs  $K_4 \times G$  and  $(K_4 - e) \times G$  is planar if and only if  $G \cong K_2$ .*
3.  *$(K_3 + x) \times G$  is planar if and only if  $G$  is a path.*
4.  *$K_{1,3} \times G$  is planar if and only if  $G$  is a path or a cycle.*
5.  *$C_4 \times G$  is planar if and only if  $G$  is a tree.*
6.  *$C_3 \times G$  is planar if and only if  $G$  is a path or 1-contractable to a path. ■*

It follows from the above characterization that if  $G$  is a nonplanar graph, then so is  $G \times K_2$ . Consequently, if  $G_1$  and  $G_2$  are nontrivial, connected graphs, one of which is nonplanar, then  $G_1 \times G_2$  is nonplanar (cf. Theorem 5.1.3). In the next section, we will derive an analogous result for non-outerplanar  $\times$ -product graphs. In the remainder of this section, we present a characterization for the planarity of the  $\boxtimes$ -product of graphs.

**5.2.1 Planarity of the Strong product**

The following three lemmas provide the basis for our characterization.

**Lemma 5.2.3** *If  $G$  is a tree, then  $G \boxtimes K_2$  is a planar graph.*



**Proof:** (by induction on  $|V(G)|$ ) Let  $G = (V, E)$  be a tree. If  $|V| = 1$ , then  $G \boxtimes K_2 \cong K_2$  which is trivially planar. For  $|V| = 2$ , we have  $G \cong K_2$ , in which case  $G \boxtimes K_2 \cong K_4$  which is known to be planar.

Let  $|V| = n > 2$ . We will denote a vertex of  $G \boxtimes K_2$  by  $(x, i)$ , where  $x \in V$  and  $i \in \{1, 2\}$ . Let  $v$  be a terminal vertex (i.e., a vertex of degree one) of  $G$ , and assume that  $u$  is the neighbor of  $v$  in  $G$ . Let  $G - v$  denote the tree obtained from  $G$  by deleting the vertex  $v$  from  $G$ . Note that

$$V(G \boxtimes K_2) = V((G - v) \boxtimes K_2) \cup \{(v, 1), (v, 2)\}, \text{ and}$$

$$E(G \boxtimes K_2) = E((G - v) \boxtimes K_2) \cup E', \text{ where}$$

$$E' = \{(u, 1), (v, 1)\}, \{(u, 1), (v, 2)\}, \{(u, 2), (v, 1)\}, \{(u, 2), (v, 2)\}, \{(v, 1), (v, 2)\}.$$

Observe further that the subgraph of  $G \boxtimes K_2$  induced by the vertex subset  $\{(u, 1), (u, 2), (v, 1), (v, 2)\}$  is simply  $K_4$ . By induction hypothesis,  $(G - v) \boxtimes K_2$  is a planar graph. So consider a planar embedding of  $(G - v) \boxtimes K_2$ , and choose a face in that embedding which contains the edge  $\{(u, 1), (u, 2)\}$ . Introduce the vertices  $(v, 1), (v, 2)$  and the edges of  $E'$  in that face so that the planarity is maintained. Since  $K_4$  is known to be a planar graph, it is clear that the foregoing statement is valid and we have a planar embedding of  $G \boxtimes K_2$ . ■

Recall that we use  $C_n$  (resp.  $P_n$ ) to denote a cycle (resp. path) of length  $n$ , where  $V(C_n) = \{1, \dots, n\}$  and  $V(P_n) = \{1, \dots, n + 1\}$  with adjacencies defined in

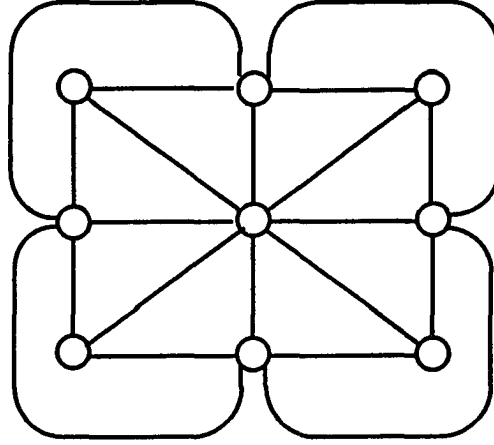


Figure 5.4: A planar embedding of the graph  $P_2 \square P_2$

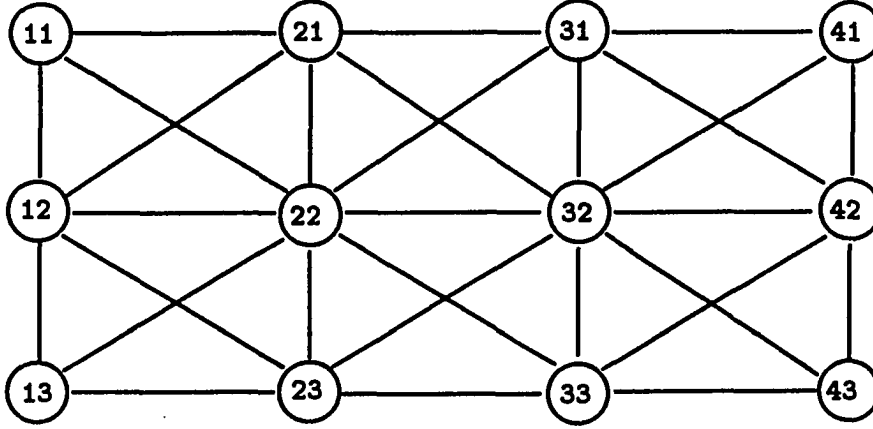
the natural way (see Chapter 1).

**Lemma 5.2.4** *For  $n \geq 3$ ,  $C_n \square K_2$  is a nonplanar graph.*

**Proof:** (by induction on  $n$ ) For  $n = 3$ , we have  $C_n \square K_2 \cong K_6$ , which is known to be a nonplanar graph. So let  $n > 3$ . Assuming that 1 and 2 are the two (adjacent) vertices of  $K_2$ , we may denote the vertex set of  $C_n \square K_2$  by  $\{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq 2\}$ , where adjacencies are defined as in the definition of the  $\square$ -product. Consider the graph  $H$  obtained from  $C_n \square K_2$  by contracting the edges  $\{(1, 1), (2, 1)\}$  and  $\{(1, 2), (2, 2)\}$ . It is easy to see that  $H \cong C_{n-1} \square K_2$ . By induction hypothesis and Theorem 5.1.1, the lemma follows. ■

**Lemma 5.2.5** *For  $n \geq 1$ ,  $P_n \square P_2$  is a planar graph if and only if  $n \leq 2$ .*

**Proof:** It suffices to show that  $P_2 \square P_2$  is a planar graph while  $P_3 \square P_2$  is a nonplanar graph. A planar embedding of  $P_2 \square P_2$  appears in Figure 5.4.

Figure 5.5: The graph  $P_3 \boxtimes P_2$ 

The graph  $P_3 \boxtimes P_2$  appears in Figure 5.5. As stated earlier, we may assume that  $V(P_n) = \{1, \dots, n+1\}$  where vertex  $i$  is adjacent to  $i+1$ ,  $1 \leq i \leq n$ . Thus  $V(P_3 \boxtimes P_2) = \{(i, j) \mid 1 \leq i \leq 4, 1 \leq j \leq 3\}$  with adjacency as in the definition of the Strong product. Now contract the following edges of  $P_3 \boxtimes P_2$ :  $\{(3, 1), (4, 2)\}$ ,  $\{(1, 2), (2, 1)\}$  and  $\{(1, 2), (2, 3)\}$ . It is straightforward to see that the resulting graph is such that the subgraph induced on  $\{(2, 1), (3, 1), (2, 2), (3, 2), (3, 3)\}$  is  $K_5$ . By Theorem 5.1.1,  $P_3 \boxtimes P_2$  must be nonplanar. ■

We will now state and prove a characterization for the planarity of the Strong product of graphs. Recall from Theorem 5.1.3 that there is no loss of generality in assuming that the factor graphs are themselves planar.

**Theorem 5.2.6** *Let  $G_1$  and  $G_2$  be nontrivial, connected (planar) graphs. Then,  $G_1 \boxtimes G_2$  is planar if and only if one of the following holds:*

1. *one graph is a tree and the other is  $K_2$ ;*

2. both graphs are  $P_2$ .

**Proof:** The “if” part of the theorem follows from Lemmas 5.2.3 and 5.2.5. For the “only if” part, assume that none of the conditions 1 and 2 (in the statement of the theorem) is satisfied; we will show that  $G_1 \boxtimes G_2$  is nonplanar. First suppose that one of the graphs is  $K_2$ . Then the other cannot be a tree, and hence must contain a cycle. By Lemma 5.2.4,  $G_1 \boxtimes G_2$  is nonplanar. Next suppose that none of the graphs is  $K_2$ . Then one graph will contain  $P_2$  as a subgraph while the other will contain one of the following as a subgraph:  $C_3$ ,  $P_3$ ,  $K_{1,3}$ . (Recall that both  $G_1$  and  $G_2$  are nontrivial and connected.) By Lemmas 5.2.4 and 5.2.5,  $P_2 \boxtimes C_3$  and  $P_2 \boxtimes P_3$  are nonplanar graphs. Further, by Theorem 5.2.1,  $P_2 \square K_{1,3}$  and hence  $P_2 \boxtimes K_{1,3}$  is a nonplanar graph. It then follows that if none of the conditions 1 and 2 is satisfied, then  $G_1 \boxtimes G_2$  is a nonplanar graph. ■

### 5.3 Outerplanarity

We first observe that  $K_2 \boxtimes K_2 \cong K_4$ , which is a non-outerplanar graph. It then follows that *for all graphs  $G_1$  and  $G_2$  containing at least one edge each, the graph  $G_1 \boxtimes G_2$  is non-outerplanar*. We, therefore, concentrate only on the outerplanarities of the  $\square$ -product and  $\times$ -product.

#### 5.3.1 Outerplanarity of the Cartesian product

The following two lemmas provide a basis for our characterization of the outerplanarity of  $\square$ -product.

**Lemma 5.3.1** *For  $n \geq 3$ ,  $C_n \square K_2$  is a non-outerplanar graph.*

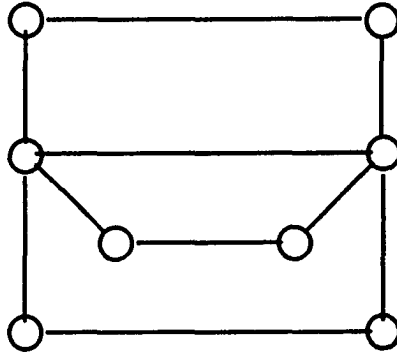
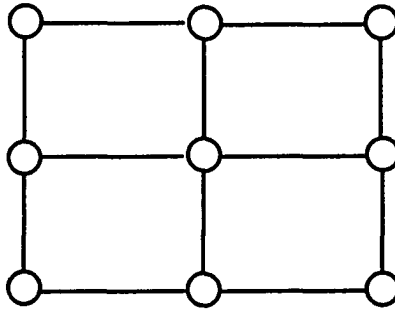
**Proof:** (by induction on  $n$ ) Let  $V(C_n) = \{1, \dots, n\}$  with adjacency defined in the natural way. Further, let 1 and 2 be the two (adjacent) vertices of  $K_2$ .

For the basis, let  $n = 3$  and consider the graph  $C_3 \square K_2$ . It is straightforward to see that contraction of the edges  $\{(1, 1), (2, 1)\}$ ,  $\{(2, 1), (3, 1)\}$  and  $\{(3, 1), (1, 1)\}$  results in a graph which is simply  $K_4$ . By Theorem 5.1.2,  $C_3 \square K_2$  must be non-outerplanar. For the induction, consider the graph  $C_n \square K_2$ , where  $n \geq 4$ . It is easy to see that contraction of the edges  $\{(1, 1), (2, 1)\}$  and  $\{(1, 2), (2, 2)\}$  results in a graph which is (isomorphic to)  $C_{n-1} \square K_2$ . The lemma follows from the induction hypothesis and Theorem 5.1.2. ■

We will now state and prove a characterization for the outerplanarity of the  $\square$ -product of two graphs. Observe that by Theorem 5.1.3, there is no loss of generality in assuming that the factor graphs are themselves outerplanar.

**Theorem 5.3.2** *The Cartesian product of two nontrivial, connected (outerplanar) graphs is outerplanar if and only if one graph is a path and the other is  $K_2$ .*

**Proof:** It is straightforward to see that the  $\square$ -product of a path and  $K_2$  is an outerplanar graph. For the converse, let  $G_1$  and  $G_2$  be nontrivial, connected graphs, and assume that condition of the lemma is not satisfied; we will show that  $G_1 \square G_2$  is non-outerplanar. If one graph is  $K_2$ , then the other cannot be a path, and hence must contain either a cycle  $C_n$ ,  $n \geq 3$ , or  $K_{1,3}$  as a subgraph. By Lemma 5.3.1,  $C_n \square K_2$  is a non-outerplanar graph. The graph  $K_{1,3} \square K_2$  appears in Figure 5.6, from which it is clear that  $K_{2,3}$  is a subcontraction of  $K_{1,3} \square K_2$ . By Theorem 5.1.2,  $K_{1,3} \square K_2$  must be a non-outerplanar graph. Alternatively, if none of  $G_1$  and  $G_2$

Figure 5.6: The graph  $K_{1,3} \square K_2$ Figure 5.7: The graph  $P_2 \square P_2$ 

is  $K_2$ , then each must contain  $P_2$  as a subgraph. The graph  $P_2 \square P_2$  appears in Figure 5.7, from which it is clear that  $K_{2,3}$  is a subcontraction of  $P_2 \square P_2$ . The theorem follows. ■

### 5.3.2 Outerplanarity of the Kronecker product

In this subsection, we discuss necessary and sufficient conditions for the outerplanarity of the  $\times$ -product. For this, we first introduce the concept of an *almost*

*bipartite graph.*

Let  $G = (V, E)$  be a connected graph. Recall from Chapter 1 that if  $x, y \in V$ , then by an  $(x, y)$ -*path* we mean a simple (but not necessarily shortest) path between the vertices  $x$  and  $y$  in  $G$ . Let  $C_m$  be a cycle of  $G$ . As usual, we assume that the vertex set of  $C_m$  is  $\{1, \dots, m\}$ , where  $\{1, m\}$  and  $\{i, i+1\} \in E$ ,  $1 \leq i \leq m-1$ . We will use  $C_m$  also to denote its vertex set. We say that  $C_m$  is a *minimal cycle* of  $G$  if no proper vertex subset of  $C_m$  induces a smaller cycle in  $G$ . Further, we define a *minimal odd cycle* (resp. *minimal even cycle*) as a minimal cycle which is of odd (resp. even) length. It is interesting to note that if a graph contains an odd cycle, then it necessarily contains a minimal odd cycle while the same is not true of an even cycle. To see this, consider the graph  $K_4$  which contains an even cycle, viz.,  $C_4$ , but no minimal even cycle. We say that  $G$  is an *almost bipartite graph* (or an *a-b graph*) if it contains a unique minimal odd cycle. Figure 5.8 shows an example of an *a-b* graph. Note that if  $G$  is just an odd cycle, then it is trivially an *a-b* graph. In the following lemma, we state and prove a useful property of such a graph.

**Lemma 5.3.3** *Let  $G$  be an almost bipartite (a-b) graph with  $C_{2k+1}$  as its unique minimal odd cycle, and let  $W \subseteq V(G)$ . Then,  $\langle W \rangle$  is a bipartite graph in its own right if and only if  $C_{2k+1} \not\subseteq W$ .*

**Proof:** Let  $G$ ,  $C_{2k+1}$  and  $W$  be as in the statement of the lemma. Obviously, if  $C_{2k+1} \subseteq W$ , then  $\langle W \rangle$  cannot be bipartite. Conversely, if  $\langle W \rangle$  is non-bipartite, then it contains an odd cycle and hence a minimal odd cycle. Since  $G$  has a unique minimal odd cycle,  $C_{2k+1} \subseteq W$ . ■

Let  $G$  and  $C_{2k+1}$  be as in Lemma 5.3.3. For  $i \in C_{2k+1}$ , let

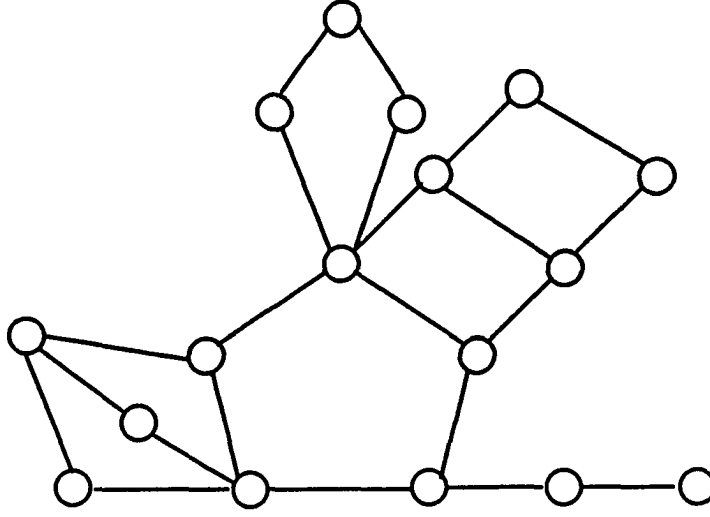


Figure 5.8: An almost bipartite graph

$A_i = \{x \in V(G) \mid x \neq i \text{ and for all } j \in C_{2k+1}, i \text{ appears on every } (x, j) - \text{path} \}$ .

Note that the set  $A_i$  may be empty for some  $i$ , and that if  $A_i$  is nonempty, then the induced subgraph  $\langle A_i \rangle$  is bipartite in its own right.

Next, let  $v$  be a vertex of  $G$  such that (i)  $v \notin A_i$  for any  $i$ , (ii)  $v \notin C_{2k+1}$ , and (iii) for some distinct  $i, j \in C_{2k+1}$ , there are  $(v, i)$ - and  $(v, j)$ -paths, none of which contains any other vertex of  $C_{2k+1}$ . We claim that  $\{i, j\} \in E(C_{2k+1})$ . Assume otherwise. Let  $w$  be a vertex which is common to a  $(v, i)$ -path and a  $(v, j)$ -path such that there exist a  $(w, i)$ -path and a  $(w, j)$ -path which are vertex-disjoint (except for  $w$ , of course). By Lemma 5.3.3, every cycle of  $G$  which does not include all vertices of  $C_{2k+1}$  is even. Consequently, every cycle consisting of (i) a  $(w, i)$ -path, (ii) an  $(i, j)$ -path along  $C_{2k+1}$ , and (iii) a  $(j, w)$ -path must be even, since (by our assumption) it does not include all of  $C_{2k+1}$ . However, this condition cannot



always be satisfied as there are two paths between the vertices  $i$  and  $j$  along the cycle  $C_{2k+1}$ , one of which is of even length while the other is of odd length. This contradiction shows that  $\{i, j\} \in E(G)$  as claimed. Based on the foregoing argument, for every edge  $e = \{i, j\}$  of  $C_{2k+1}$ , we define the set

$$B_e = \{x \in V(G) \setminus C_{2k+1} \mid \text{for every } m \in C_{2k+1} \setminus \{i, j\}, \text{ there is an } (x, m) - \text{path} \\ \text{in which } i \text{ appears but } j \text{ does not, and an } (x, m) - \text{path} \\ \text{in which } j \text{ appears but } i \text{ does not}\}.$$

Note again that  $B_e$  may be empty for some  $e$  and that if  $B_e$  is nonempty, then the induced subgraph  $\langle B_e \rangle$  is bipartite in its own right. It is clear that the sets  $A_i$ 's and  $B_e$ 's are all mutually disjoint. It is further easy to see that a vertex of  $G$  is in exactly one of the following sets: (i)  $C_{2k+1}$ , (ii)  $A_i$  for some  $i$ , and (iii)  $B_e$  for some  $e$ . Therefore, there is a natural partition of  $V(G)$  into the foregoing sets. We now state and prove an interesting lemma which will be useful in the sequel.

**Lemma 5.3.4** *If  $G$  is an almost bipartite ( $a$ - $b$ ) graph, then  $G$  is a subcontraction of  $G \times K_2$ .*

**Proof:** Let  $G$  be an  $a$ - $b$  graph with  $C_{2k+1}$  as its unique minimal odd cycle. We will outline the construction of (a graph isomorphic to) a subgraph of  $G \times K_2$ , and then contract certain edges of it to obtain the graph  $G$ .

For  $i \in C_{2k+1}$ , and  $e \in E(C_{2k+1})$ , let  $A_i$  and  $B_e$  be the vertex subsets of  $G$  as defined in the discussion preceding the statement of this lemma. Obviously, the corresponding induced subgraphs  $\langle A_i \rangle$  and  $\langle B_e \rangle$  are bipartite, and hence the

graph  $G \times K_2$  will contain two disjoint copies of each of  $\langle A_i \rangle$  and  $\langle B_e \rangle$  (see also [61]).

Let  $u$  and  $v$  be the two (adjacent) vertices of  $K_2$  so that the vertex set of  $G \times K_2$  is simply  $V(G) \times \{u, v\}$ . Note that corresponding to the (unique, minimal) odd cycle  $C_{2k+1}$  of  $G$ , the graph  $G \times K_2$  contains the even cycle  $C_{4k+2}$ , and that  $\{i, j\}$  is an edge of  $C_{2k+1}$  if and only if  $\{(i, u), (j, v)\}$  and  $\{(i, v), (j, u)\}$  are (antipodal) edges of  $C_{4k+2}$ .

We now outline the construction of a subgraph of  $G \times K_2$  of which  $G$  will be a subcontraction. First include the even cycle  $C_{4k+2}$  whose vertices are labeled  $(i, u)$  or  $(j, v)$  as stated above. Next, for a nonempty vertex subset  $A_i$  of  $G$  (where  $i \in C_{2k+1}$ ), let  $v_1, \dots, v_m$  be the vertices of  $A_i$  such that  $\{i, v_p\} \in E(G)$ ,  $1 \leq p \leq m$ . "Prepare and attach" one copy of  $\langle A_i \rangle$  to  $C_{4k+2}$  as follows: if  $i$  is odd (resp. even), then introduce an edge between the vertex  $(i, u)$  (resp.  $(i, v)$ ) of  $C_{4k+2}$  and each of  $v_1, \dots, v_m$  of  $\langle A_i \rangle$ . (Note that in the graph  $G \times K_2$ , there is a copy of  $\langle A_i \rangle$  attached to the "diametrically opposite" vertex of  $C_{4k+2}$ , but we do not include that in our subgraph.) Similarly, for a nonempty vertex subset  $B_e$  of  $G$ , where  $e = \{i, j\} \in E(C_{2k+1})$ , let  $v_1, \dots, v_m$  and  $w_1, \dots, w_n$  be the vertices of  $B_e$  such that  $\{i, v_p\}, \{j, w_q\} \in E(G)$ ,  $1 \leq p \leq m, 1 \leq q \leq n$ . Prepare a copy of  $\langle B_e \rangle$  and attach it to  $C_{4k+2}$  as follows: (i) if  $i = 1$  and  $j = 2$ , then introduce an edge between the vertex  $(1, u)$  of  $C_{4k+2}$  and each of the vertices  $v_1, \dots, v_m$  of  $\langle B_e \rangle$ , and an edge between the vertex  $(2, v)$  and each of  $w_1, \dots, w_n$ , (ii) if  $i = 1$  and  $j = 2k + 1$ , then do a similar attachment of  $\langle B_e \rangle$  to the (adjacent) vertices  $(2k + 1, u)$  and  $(1, v)$  of  $C_{4k+2}$ , and (iii) if  $i, j \neq 1$ , then assume that  $j = i + 1$  and for odd (resp. even)  $i$ , do an analogous attachment of a copy of  $\langle B_e \rangle$  to the adjacent vertices  $(i, u)$  and  $(j, v)$  (resp.  $(i, v)$  and  $(j, u)$ ) of the even cycle  $C_{4k+2}$ .

(Note again that in the graph  $G \times K_2$ , there is an identical copy of  $\langle B_e \rangle$  attached to the antipodal edge of  $C_{4k+2}$ , but we do not include that in our subgraph.) We perform the foregoing operations for all nonempty vertex subsets  $A_i$  and  $B_e$  of  $G$ . It is clear that the graph  $H$  thus obtained is (isomorphic to) a subgraph of  $G \times K_2$ .

Finally, we contract the following  $2k + 1$  edges of (the cycle  $C_{4k+2}$  of)  $H$ :  $\{(1, v), (2, u)\}, \{(2, u), (3, v)\}, \dots, \{(2k + 1, v), (1, u)\}$ , whence the vertices  $(1, u)$  and  $(1, v)$  get identified. It is straightforward to see that the graph thus obtained is isomorphic to  $G$ . We have thus accomplished the proof of the theorem. ■

**Example:** We have illustrated the construction in the proof of Lemma 5.3.4 in Figure 5.9, which contains (i) an  $a$ - $b$  graph  $G$ , whose unique minimal odd cycle is  $C_5$  (with vertices  $1, \dots, 5$ ), (ii) graphs  $A_1, B_{\{2,3\}}, A_4$  and  $B_{\{1,5\}}$  which are subgraphs of  $G$ , and (iii) the graph  $H$  obtained from  $G$  as outlined in the proof of Lemma 5.3.4. ■

**Lemma 5.3.5** *If a connected graph  $G$  contains at least two distinct, minimal odd cycles, then  $G \times K_2$  is a non-outerplanar graph.*

**Proof Sketch:** It suffices to show that if  $G$  is a connected graph in which the number of distinct, minimal odd cycles is exactly two, then  $G \times K_2$  is a non-outerplanar graph. We need to consider the following three cases: the two minimal odd cycles (i) are vertex-disjoint, (ii) share exactly one vertex and (iii) share one or more edges. It is easy to see that in each case,  $K_{2,3}$  is a subcontraction of  $G \times K_2$ . ■

**Example:** Let  $H', H''$  and  $H'''$  be graphs, shown in Figure 5.10, which respectively satisfy the three conditions mentioned in the proof of Lemma 5.3.5. Let  $u$  and  $v$  be the

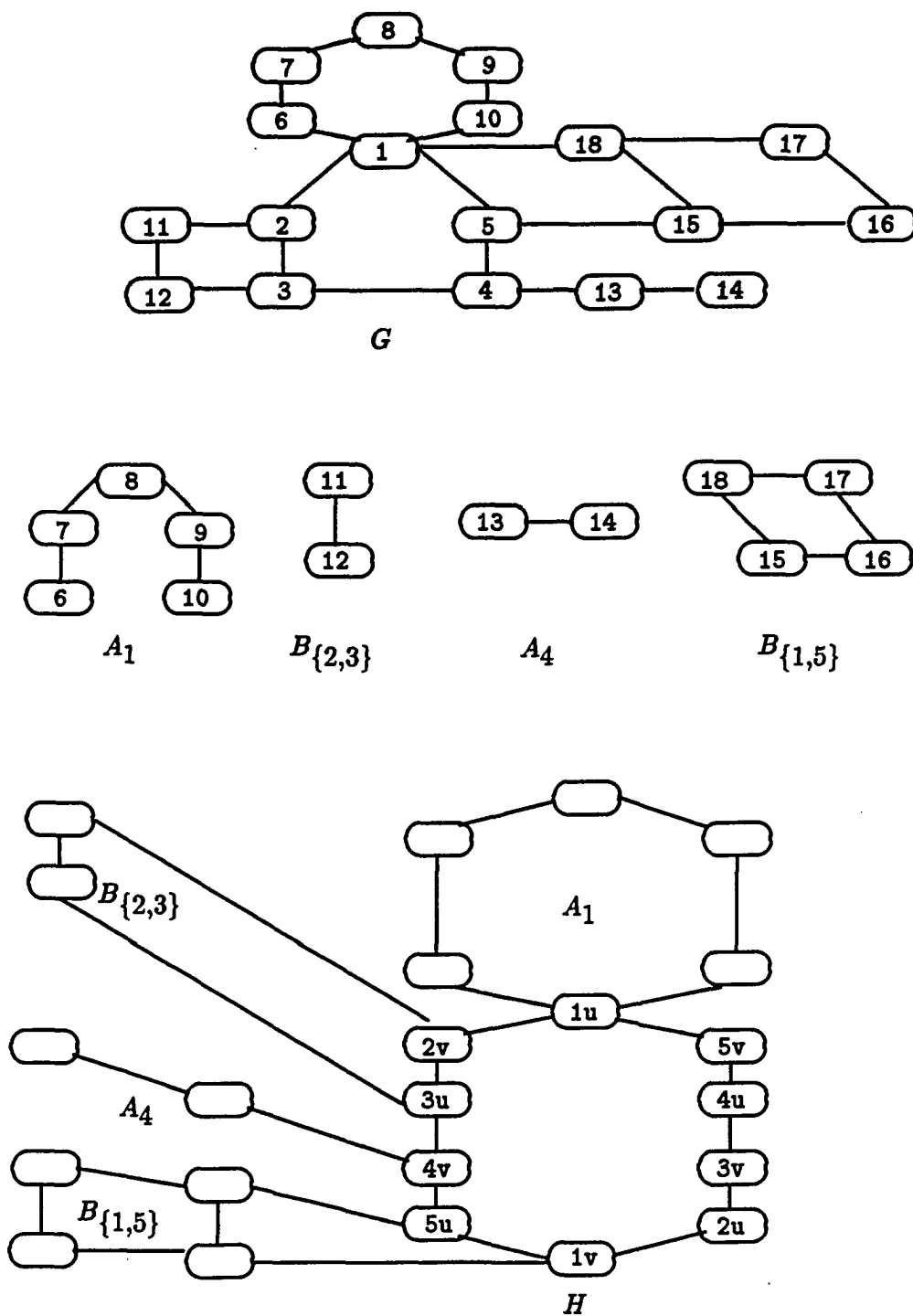
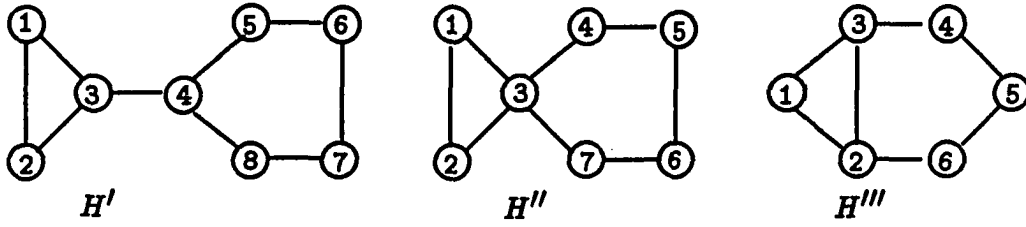
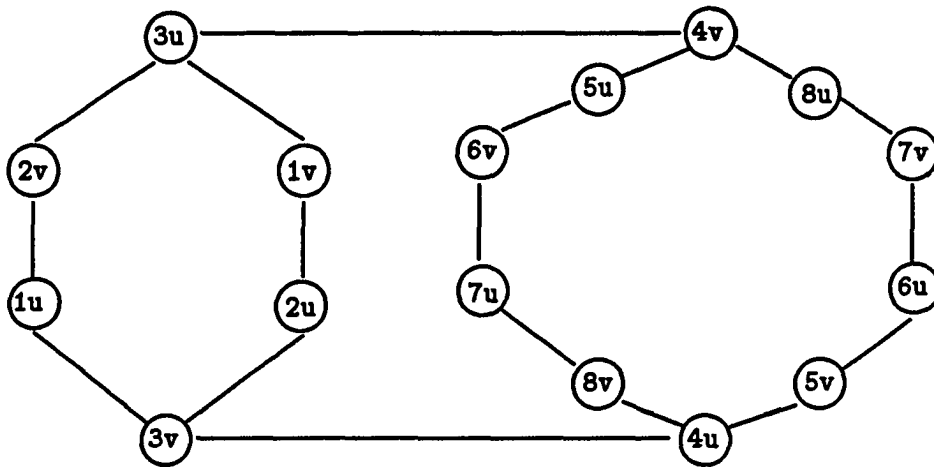


Figure 5.9: Graphs illustrating the construction in the proof of Lemma 5.3.4

Figure 5.10: Graphs  $H'$ ,  $H''$  and  $H'''$ Figure 5.11: The graph  $H' \times K_2$ 

two (adjacent) vertices of  $K_2$ , and observe the graphs  $H' \times K_2$ ,  $H'' \times K_2$  and  $H''' \times K_2$  which appear in Figures 5.11, 5.12 and 5.13 respectively. It is straightforward to see that each of these graphs has a subgraph contractable to  $K_{2,3}$  and hence is non-outerplanar. ■

In the following theorem, we show that while dealing with the outerplanarity of the  $\times$ -product of graphs, it suffices to consider only those factor graphs which are themselves outerplanar (cf. Theorem 5.1.3 and the remarks following the statement

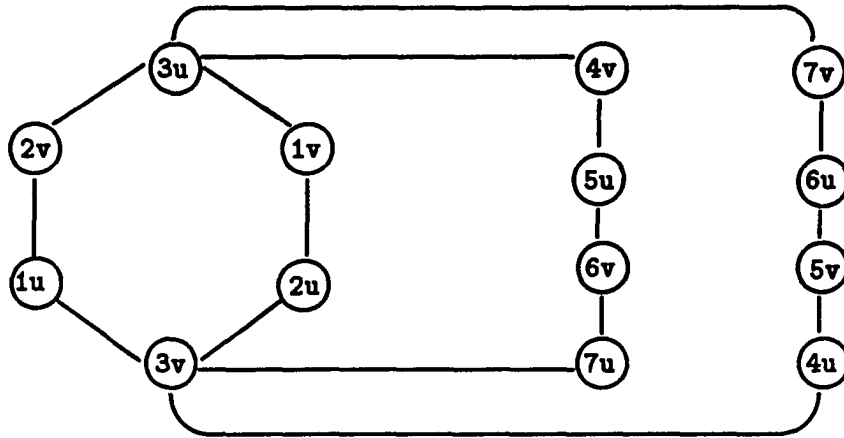


Figure 5.12: The graph  $H'' \times K_2$

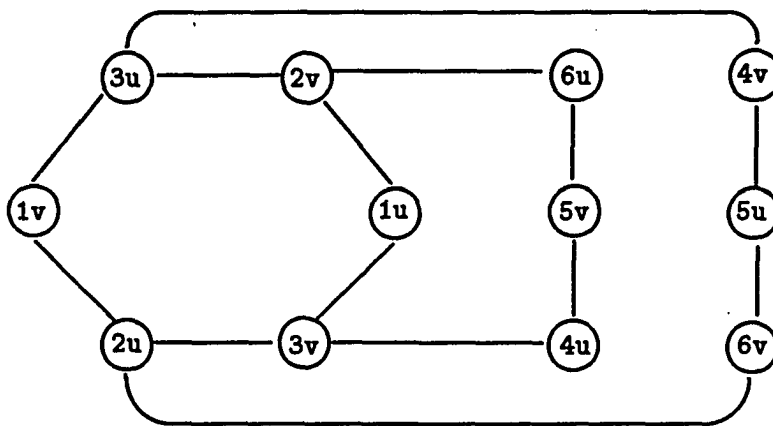


Figure 5.13: The graph  $H''' \times K_2$

of Theorem 5.2.2).

**Theorem 5.3.6** *If  $G_1$  and  $G_2$  are nontrivial, connected graphs, one of which is non-outerplanar, then the graph  $G_1 \times G_2$  is non-outerplanar.*

**Proof:** It suffices to show that if  $G$  is a non-outerplanar graph, then so is  $G \times K_2$ . So assume that  $G$  is a connected, non-outerplanar graph. If  $G$  is bipartite, then  $G \times K_2$  contains exactly two disjoint copies of  $G$ , in which case we are done. On the other hand, if  $G$  is non-bipartite, then there are two cases: (i)  $G$  is an  $a$ - $b$  graph, i.e., it contains exactly one minimal odd cycle, and (ii)  $G$  contains two or more minimal odd cycles. In the former case, the claim follows from Lemma 5.3.4 and Theorem 5.1.2 while in the latter case, the claim follows from Lemma 5.3.5. ■

The following is one of the key lemmas of this subsection.

**Lemma 5.3.7** *If  $G$  is a connected, outerplanar  $a$ - $b$  graph, then  $G \times K_2$  is an outerplanar graph.*

**Proof:** First observe that if  $G$  is itself an odd cycle, say  $C_{2m+1}$ , then  $G \times K_2$  is simply  $C_{4m+2}$  (i.e., a cycle twice as large) which is trivially outerplanar. So consider a connected, outerplanar  $a$ - $b$  graph  $G$  which contains exactly one minimal odd cycle, say  $C_{2k+1}$ , as a proper subgraph. Recall that every subgraph of  $G$  which does not include all vertices of  $C_{2k+1}$  must be a bipartite graph in its own right.

Our construction will be somewhat similar to that in the proof of Lemma 5.3.4. Note that every vertex of  $G$  is a member of one of the following sets: (i)  $C_{2k+1}$ , (ii)  $A_i$ , where  $i \in C_{2k+1}$ , and (iii)  $B_e$ , where  $e \in E(C_{2k+1})$ . (Definitions of the sets  $A_i$  and  $B_e$  appear just before the statement of Lemma 5.3.4.) Let  $u$  and  $v$  be the two (adjacent) vertices of  $K_2$  so that the vertex set of  $G \times K_2$  is simply  $V(G) \times \{u, v\}$ .

In what follows, we outline an outerplanar embedding of  $G \times K_2$  based on an outerplanar embedding of  $G$ . First embed the even cycle  $C_{4k+2}$  as an outerplanar graph corresponding to the cycle  $C_{2k+1}$  of  $G$ . Next, for a vertex  $i$  of  $C_{2k+1}$ , if the set  $A_i$  is nonempty, then prepare two copies of (the induced subgraph)  $\langle A_i \rangle$ , and “attach and embed” the first copy to  $C_{4k+2}$  through the vertex  $(i, u)$  and the second copy through the “diametrically opposite” vertex  $(i, v)$  in exactly the same manner as  $\langle A_i \rangle$  is connected to the vertex  $i$  in  $G$  (see also the proof of Lemma 5.3.4). Similarly, for an edge  $e = \{i, j\}$  of  $C_{2k+1}$ , if the set  $B_e$  is nonempty, then prepare two copies of  $\langle B_e \rangle$ , and attach and embed the first copy to  $C_{4k+2}$  through the edge  $\{(i, u), (j, v)\}$  and the second copy through (the antipodal edge)  $\{(i, v), (j, u)\}$ —again in exactly the same manner as  $\langle B_e \rangle$  is connected to  $e = \{i, j\}$  in  $G$  (see the proof of Lemma 5.3.4). We perform the foregoing operations for all nonempty sets  $A_i$  and  $B_e$ .

It is easy to see that we have accounted for all vertices and edges of  $G \times K_2$ . Further, our embedding of  $G \times K_2$  closely “mimics” an outerplanar embedding of  $G$ . In other words, the relative positions of a copy of  $\langle A_i \rangle$  or  $\langle B_e \rangle$  in  $G \times K_2$  w.r.t. its neighbors are exactly similar to those in the graph  $G$ . It then follows that we have accomplished an outerplanar embedding of  $G \times K_2$ . ■

We will state and prove two more lemmas before presenting our characterization of outerplanar  $\times$ -product graphs.

**Lemma 5.3.8** *For all  $m, n \geq 1$ , the graph  $P_m \times P_n$  is outerplanar if and only if either  $m \leq 3$  or  $n \leq 3$ .*

**Proof:** For the “only if” part, it suffices to show that  $P_4 \times P_4$  is a non-outerplanar graph. This graph (which has two connected components) appears in Figure 5.14,



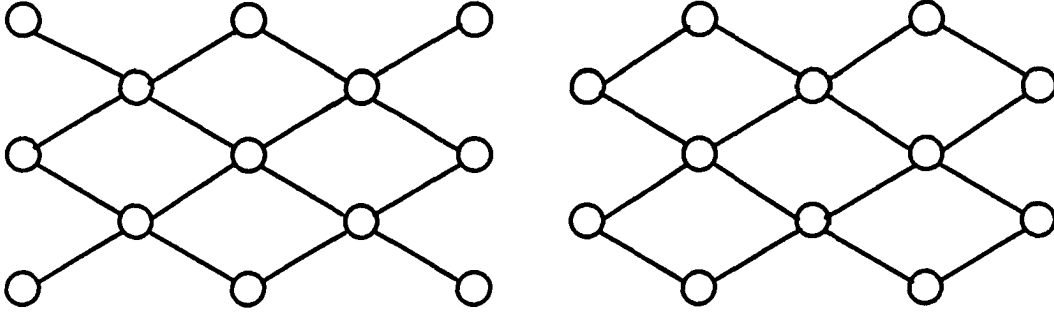


Figure 5.14: The graph  $P_4 \times P_4$

from which it is clear that  $K_{2,3}$  is a subcontraction of  $P_4 \times P_4$ . By Theorem 5.1.2, the “only if” follows.

For the “if” part, note that  $P_m \times P_3$  is (disconnected and is) made up of two identical grid-like components, each of length  $m$  and width three. It is easy to see that each component is outerplanar. For example, an outerplanar embedding of the graph  $P_6 \times P_3$  appears in Figure 5.15. ■

**Lemma 5.3.9** *For  $n \geq 3$ , the graph  $C_n \times P_2$  is non-outerplanar.*

**Proof Sketch:** For odd  $n \geq 3$ , it is easy to see that the graph  $C_n \times P_2$  is just a sequence of  $n$   $C_4$ ’s connected in a cycle. To illustrate our point, we have shown the graphs  $C_3 \times P_2$  and  $C_5 \times P_2$  in Figure 5.16. For even  $n \geq 4$ , the graph  $C_n \times P_2$  is made up of two identical connected components, each of which consists of a sequence of  $\frac{n}{2}$   $C_4$ ’s connected in a cycle. For example, the graphs  $C_4 \times P_2$  and  $C_6 \times P_2$  appear in Figure 5.17. It is easy to see that for all  $n \geq 3$ ,  $C_n \times P_2$  is a non-outerplanar graph. (A formal proof will proceed by induction on  $n$  and will make use of Theorem 5.1.2.) ■

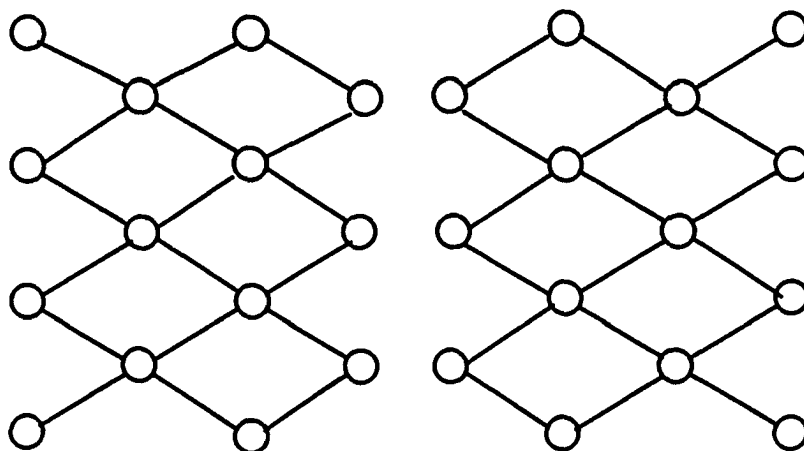
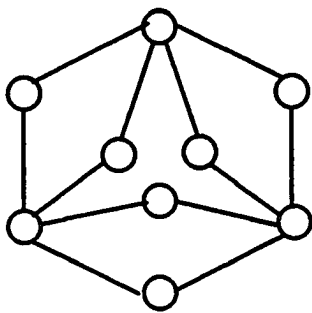
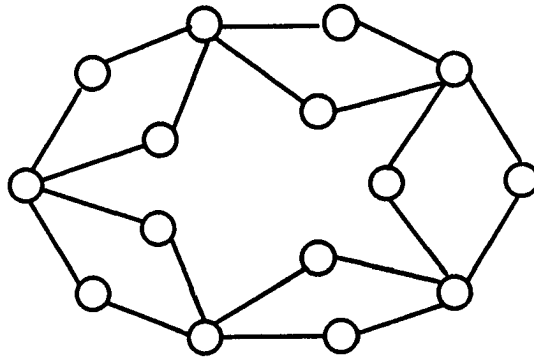
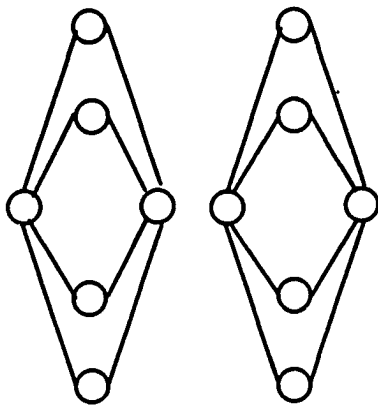
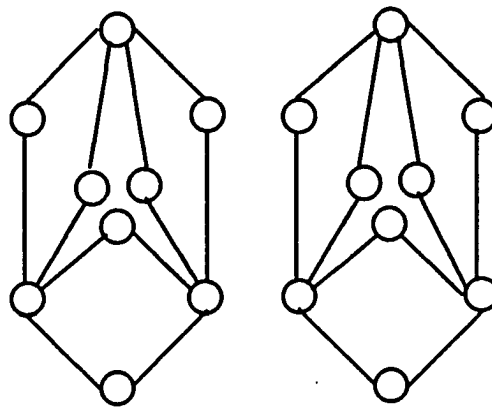


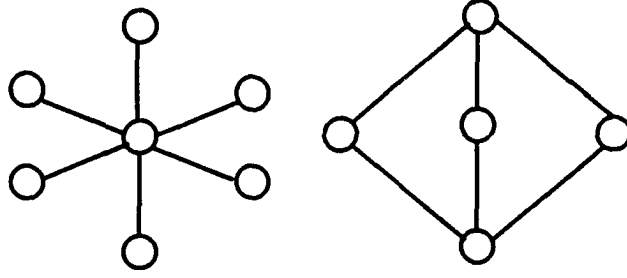
Figure 5.15: An outerplanar embedding of the graph  $P_6 \times P_3$

We will now state and prove a characterization for the outerplanarity of the  $\times$ -product of graphs. Recall that by Theorem 5.3.6, there is no loss of generality in assuming that the factor graphs are themselves outerplanar.

**Theorem 5.3.10** *Let  $G_1$  and  $G_2$  be nontrivial, connected (outerplanar) graphs.*

1. *If  $G_1$  and  $G_2$  are paths of lengths  $m$  and  $n$  respectively, then  $G_1 \times G_2$  is outerplanar if and only if either  $m \leq 3$  or  $n \leq 3$ .*
2. *If  $G_1$  and  $G_2$  are both bipartite and  $G_1$  is not a path, then  $G_1 \times G_2$  is outerplanar if and only if  $G_2 \cong K_2$ .*
3. *If  $G_1$  is non-bipartite, then  $G_1 \times G_2$  is outerplanar if and only if  $G_1$  is an  $a$ - $b$  graph (i.e., contains exactly one minimal odd cycle) and  $G_2 \cong K_2$ .*

 $C_3 \times P_2$  $C_5 \times P_2$ Figure 5.16: The graphs  $C_3 \times P_2$  and  $C_5 \times P_2$  $C_4 \times P_2$  $C_6 \times P_2$ Figure 5.17: The graphs  $C_4 \times P_2$  and  $C_6 \times P_2$

Figure 5.18: The graph  $K_{1,3} \times P_2$ 

**Proof:** (1) follows from Lemma 5.3.8 while (3) follows from Lemmas 5.3.5, 5.3.7 and 5.3.9. For (2), let  $G_1$  and  $G_2$  be (nontrivial, connected and) bipartite graphs, where  $G_1$  is different from a path. First observe that if  $G_2 \cong K_2$ , then  $G_1 \times G_2$  consists of simply two disjoint copies of  $G_1$ , and hence outerplanarity of  $G_1 \times G_2$  follows from that of  $G_1$ . For the converse, assume that  $G_2 \not\cong K_2$ . Then  $P_2$  must be a subgraph of  $G_2$ , and since  $G_1$  is not a path, it must contain either  $K_{1,3}$  or an even cycle as a subgraph. The graph  $K_{1,3} \times P_2$  appears in Figure 5.18, which shows that  $K_{1,3} \times P_2$  is non-outerplanar as it contains  $K_{2,3}$ . Further, by Lemma 5.3.9,  $C_{2n} \times P_2$  is non-outerplanar. It follows that  $G_1 \times G_2$  is non-outerplanar and (2) is established. ■

An interesting observation which is related to the outerplanarity of  $\times$ -product graphs is the following. For some  $n \geq 1$ , consider the odd cycle  $C_{2n+1}$ . Introduce exactly  $n - 1$  chords in it so that the resulting graph  $G$  satisfies the following conditions:

1.  $G$  is outerplanar and

2. exactly one minimal cycle of  $G$  is  $C_3$  while all other minimal cycles are  $C_4$ .

Then,  $G \times K_2$  is an outerplanar graph; moreover, if any additional chord is introduced, then the  $\times$ -product of the resulting graph  $G'$  and  $K_2$  is non-outerplanar. The reason for this is that  $G'$  will have at least two distinct (minimal)  $C_3$ 's.

## 5.4 Concluding Remarks

In this chapter, we have discussed necessary and sufficient conditions for the planarity and outerplanarity of each of the three graph products. Characterizations for the planarities of the  $\square$ -product and  $\times$ -product already appear in the literature and we have just stated them. We have further observed that if  $G_1$  and  $G_2$  are graphs which contain at least one edge each, then  $G_1 \boxtimes G_2$  is a non-outerplanar graph.

We have stated and proved characterizations for the following: planarity of the  $\boxtimes$ -product, outerplanarity of the  $\square$ -product, and outerplanarity of the  $\times$ -product. For these, we have made use of known characterizations of planarity and outerplanarity of general graphs in terms of graph subcontraction.

While dealing with the outerplanarity of the  $\times$ -product, we have defined a minimal cycle of a graph, and have introduced an interesting class of graphs called almost bipartite ( $a$ - $b$ ) graphs (see page 77), which are connected graphs containing a unique minimal odd cycle. We have shown (Lemma 5.3.4) that if  $G$  is an  $a$ - $b$  graph, then it is a subcontraction of the graph  $G \times K_2$ . (In the case of a bipartite graph, the analogous statement is trivially true.) We *conjecture* that every connected graph  $G$  is a subcontraction of the graph  $G \times K_2$ . If this is indeed the case, then Lemma 5.3.4 and Theorem 5.3.6 would follow immediately. To support this conjecture, we quickly show that for  $n \geq 3$ ,  $K_n$  is a subcontraction of  $K_n \times K_2$ . Let  $1, \dots, n$  be the vertices

of  $K_n$ , and let  $u, v$  be the vertices of  $K_2$ . Consider the graph  $K_n \times K_2$  whose vertex set is  $\{1, \dots, n\} \times \{u, v\}$  and contract the following  $n$  edges of this graph:  $\{(1, u), (2, v)\}$ ,  $\{(2, u), (3, v)\}$ ,  $\dots$ ,  $\{(n-1, u), (n, v)\}$ , and  $\{(n, u), (1, v)\}$ . It is easy to see that the resulting graph is isomorphic to  $K_n$ , and hence  $K_n$  is a subcontraction of  $K_n \times K_2$ . It can further be shown that if  $G$  consists of exactly two cliques, then too  $G$  is a subcontraction of  $G \times K_2$ .

## 6. TOPOLOGICAL INVARIANTS OF PRODUCT GRAPHS

For a graph  $G$ , let

$\chi(G)$  = *chromatic number of  $G$ , i.e., the least number of colors required to color the vertices of  $G$  so that adjacent vertices receive different colors;*

$\alpha(G)$  = *independence number of  $G$ , i.e., the cardinality of a largest independent set of  $G$ ;*

$\beta(G)$  = *domination number of  $G$ , i.e., the cardinality of a smallest dominating set of  $G$ ;*

$\gamma(G)$  = *clique number of  $G$ , i.e., the cardinality of a largest clique of  $G$ .*

It is easy to see that each of the foregoing parameters is a graph invariant, and that  $\gamma(G) \leq \chi(G)$  and  $\beta(G) \leq \alpha(G)$ . In this chapter, we discuss bounds on these invariants for each of the three graph products in terms of the invariants of the factor graphs. In addition, we discuss certain conditions for the existence of a Hamiltonian path/cycle in the product graphs.

A major portion of this chapter consists of a survey of important known results. Our important contributions are the following: an improved lower bound on the independence number of the  $\square$ -product which we present in Section 6.2, and certain conditions which ensure the existence of a Hamiltonian cycle in the  $\times$ -product which we present in Section 6.5. The first four sections deal with the bounds on the above four invariants in that order while the fifth section is concerned with the Hamiltonian

property. Finally, in Section 6.6, we summarize the results of this chapter, make some relevant observations and discuss certain related issues.

## 6.1 Chromatic Numbers

Throughout this section, by a coloring of a graph  $G$  we will mean a coloring of the vertices of  $G$  so that adjacent vertices receive different colors.

For the  $\square$ -product of graphs, Vizing obtained the following theorem, which gives an exact value of  $\chi(G_1 \square G_2)$  in terms of  $\chi(G_1)$  and  $\chi(G_2)$

**Theorem 6.1.1** [65]  $\chi(G_1 \square G_2) = \max\{\chi(G_1), \chi(G_2)\}$ . ■

For the other two products, only certain bounds are known. The following theorem gives an upper bound for  $\chi(G_1 \times G_2)$ ; the proof is fairly straightforward.

**Theorem 6.1.2**  $\chi(G_1 \times G_2) \leq \min\{\chi(G_1), \chi(G_2)\}$ .

**Proof:** Let  $\chi(G_1) = r$ . Then there is a partition of  $V(G_1)$  into  $r$  color classes  $W_1, \dots, W_r$  such that each  $W_i$  is an independent set of  $G_1$ . Note that in the graph  $G_1 \times G_2$ , the sets  $W_1 \times V(G_2), \dots, W_r \times V(G_2)$  constitute a partition of  $V(G_1 \times G_2)$  into  $r$  independent sets. Thus  $\chi(G_1 \times G_2) \leq r = \chi(G_1)$ . Similarly,  $\chi(G_1 \times G_2) \leq \chi(G_2)$ . The theorem follows. ■

Hedetniemi [34] conjectured that equality holds in the foregoing theorem. Whether this is, in general, valid or not remains a major open problem. However, for several special cases, the conjecture has been found to be true, which means that the bound itself is sharp. We state some interesting results in the next four theorems.



**Theorem 6.1.3** *Let  $G_1$  and  $G_2$  be connected graphs with  $\chi(G_1) = \chi(G_2) = n$ .*

1. [34] *If  $n = 2, 3$ , then  $\chi(G_1 \times G_2) = n$ .*
2. [19] *If  $n = 4$ , then  $\chi(G_1 \times G_2) = n$ . ■*

**Theorem 6.1.4** [29] *For any graph  $G$ ,  $\chi(G \times K_n) = \min\{\chi(G), n\}$ . ■*

**Theorem 6.1.5** [13] *Let  $G_1$  be an  $(n+1)$ -chromatic graph in which every vertex is contained in an  $n$ -clique. Then, for every  $(n+1)$ -chromatic graph  $G_2$ ,  $\chi(G_1 \times G_2) = n + 1$ . ■*

**Theorem 6.1.6** [18] *Let  $G_1$  and  $G_2$  be two connected  $(n+1)$ -chromatic graphs each containing an  $n$ -clique. Then  $\chi(G_1 \times G_2) = n + 1$ . ■*

The following theorem of Vesztergombi deals with the chromatic number of the Strong product.

**Theorem 6.1.7** [63] *If  $G_1$  and  $G_2$  are graphs, each having at least one edge, then  $\max\{\chi(G_1), \chi(G_2)\} + 2 \leq \chi(G_1 \boxtimes G_2) \leq \chi(G_1) \cdot \chi(G_2)$ . ■*

That the upper bound above is correct follows by a simple argument. Note further that if  $G_1$  and  $G_2$  are nontrivial, connected bipartite graphs, then the lower bound and the upper bound given by the foregoing theorem coincide, and hence yield the optimal value, viz., 4. Also, the upper bound is achieved if  $\chi(G_1) = \gamma(G_1)$  and

$\chi(G_2) = \gamma(G_2)$  [63] (see also Corollary 6.4.4, page 108). In particular, the upper bound is achieved if  $G_1$  and  $G_2$  are complete graphs: this is because the Strong product of complete graphs is a complete graph. However, in general, there is a gap between the lower bound and the upper bound.

Vesztergombi proved the lower bound in Theorem 6.1.7 by showing that if  $G$  is a graph containing at least one edge, then  $\chi(G \boxtimes K_2) \geq \chi(G) + 2$ . We will next present a theorem which is a generalization of Vesztergombi's result; our proof is similar to his.

**Theorem 6.1.8** *For a nontrivial, connected graph  $G$  and  $n \geq 2$ ,  $\chi(G \boxtimes K_n) \geq \chi(G) + n$ .*

**Proof:** Let  $G$  and  $n$  be as in the statement of the theorem, and let  $\chi(G) = k$ . It suffices to show that if (the vertices of)  $G \boxtimes K_n$  can be colored with  $k + n - 1$  colors, then (the vertices of)  $G$  can be colored with  $k - 1$  colors. So assume that  $1, 2, \dots, k + n - 1$  are the colors corresponding to a valid coloring of  $G \boxtimes K_n$ . For a vertex  $u$  of a graph, we will use  $c(u)$  to denote the color assigned to  $u$  in a proper coloring of that graph. Let  $v_1, \dots, v_n$  be the  $n$  vertices of  $K_n$ . Note that for all  $u \in V(G)$ , the vertices  $(u, v_1), \dots, (u, v_n)$  of  $G \boxtimes K_n$  are mutually adjacent. Consequently, (under the assumption that  $\chi(G \boxtimes K_n) = k + n - 1$ ) we have  $\min\{c(u, v_1), \dots, c(u, v_n)\} \leq k$  for all  $u \in V(G)$ . We now prescribe a coloring (of the vertices) of the graph  $G$ : for  $u \in V(G)$ , let

$$c(u) = \begin{cases} \min\{c(u, v_1), \dots, c(u, v_n)\}, & \text{if this minimum is smaller than } k \\ k - 1, & \text{otherwise.} \end{cases}$$

Let  $x$  and  $y$  be adjacent vertices of  $G$ . We will show that  $c(x)$  (i.e., the color assigned to  $x$  as above) is different from  $c(y)$ . Observe that the vertices  $(x, v_1), \dots, (x, v_n), (y, v_1), \dots, (y, v_n)$  are mutually adjacent in the graph  $G \boxtimes K_n$ . Therefore, the sets  $\{c(x, v_1), \dots, c(x, v_n)\}$  and  $\{c(y, v_1), \dots, c(y, v_n)\}$  are disjoint, and hence  $\min\{c(x, v_1), \dots, c(x, v_n)\} \neq \min\{c(y, v_1), \dots, c(y, v_n)\}$ . From this, it is easy to see that  $c(x) \neq c(y)$ . We have thus accomplished the proof of the theorem. ■

The following corollary follows from the above theorem.

**Corollary 6.1.9** *If  $G$  and  $H$  are nontrivial, connected graphs and  $\gamma(H) = n$ , then  $\chi(G \boxtimes H) \geq \chi(G) + n$ . ■*

In the next corollary, we obtain a sequence of inequalities based on the results of Theorems 6.1.1, 6.1.2 and 6.1.7.

**Corollary 6.1.10** *If  $G_1$  and  $G_2$  are graphs, each having at least one edge, then  $\chi(G_1 \times G_2) \leq \min\{\chi(G_1), \chi(G_2)\} \leq \max\{\chi(G_1), \chi(G_2)\} = \chi(G_1 \square G_2) < \max\{\chi(G_1), \chi(G_2)\} + 2 \leq \chi(G_1 \boxtimes G_2) \leq \chi(G_1) \cdot \chi(G_2)$ . ■*

It is interesting to note that the graph  $G_1 \times G_2$  may, in general, be much denser than the graph  $G_1 \square G_2$  (in the sense of the number of edges), but  $\chi(G_1 \times G_2)$  is always smaller than or equal to  $\chi(G_1 \square G_2)$ .

## 6.2 Independence Numbers

### 6.2.1 Bounds on $\alpha(G_1 \square G_2)$

Vizing obtained the following theorem which gives bounds on  $\alpha(G_1 \square G_2)$ .

**Theorem 6.2.1 [65]** *Let  $\alpha_i = \alpha(G_i)$  for graphs  $G_i = (V_i, E_i)$ ,  $i = 1, 2$ . Then,  $\alpha_1 \cdot \alpha_2 + \min\{|V_1| - \alpha_1, |V_2| - \alpha_2\} \leq \alpha(G_1 \square G_2) \leq \min\{\alpha_1 \cdot |V_2|, \alpha_2 \cdot |V_1|\}$ . ■*

It is interesting to note that graphs exist for which the foregoing lower bound and upper bound coincide and hence yield the optimal value. This is particularly so if both  $G_1$  and  $G_2$  are complete graphs. However, in general, there is a gap between the two bounds. We next present a scheme which leads to an improvement on Vizing's lower bound.

**6.2.1.1 An improved lower bound on  $\alpha(G_1 \square G_2)$**  We present a greedy algorithm which constructs a maximal independent set of  $G_1 \square G_2$ . In the process, we obtain an improvement on Vizing's lower bound on  $\alpha(G_1 \square G_2)$ . We further remark that our scheme does *not* necessarily produce a maximum independent set of  $G_1 \square G_2$ .

Recall that if  $W$  is a vertex subset of a graph  $G$ , then  $\langle W \rangle$  denotes the subgraph of  $G$  induced by  $W$ . Figure 6.1 shows a recursive procedure which constructs a maximal independent set of  $G_1 \square G_2$  while Figure 6.2 is an iterative version of the same procedure. Note that these procedures are nondeterministic in that the sets  $W_1$  and  $W_2$  in step 1 of the inner loop may be any largest independent sets of  $G_1$  and  $G_2$  respectively. Also note that our scheme is such that we obtain a maximal independent set of the graph  $G_1 \square G_2$  without actually constructing  $G_1 \square G_2$ . In other

words, we derive a maximal independent set of  $G_1 \square G_2$  from the factor graphs  $G_1$  and  $G_2$ .

It is straightforward to see that for any graphs  $G_1$  and  $G_2$ , each of the algorithms R-INDEP and I-INDEP will terminate. For the remainder of the discussion on the foregoing algorithm, let  $G_1$  and  $G_2$  denote arbitrary but fixed graphs, and let  $S$  denote a subset of  $V(G_1 \square G_2)$  obtained at the termination of the algorithm. In the discussion below, we will refer mainly to the iterative version (Figure 6.2) of the algorithm. Lemma 6.2.2 shows that  $S$  is a maximal independent set of  $G_1 \square G_2$  while Lemma 6.2.3 shows that  $|S|$  is greater than or equal to Vizing's lower bound.

**Lemma 6.2.2**  *$S$  is a maximal independent set of  $G_1 \square G_2$ .*

**Proof:** It suffices to show that  $S$  is an independent set as well as a dominating set of  $G_1 \square G_2$ . We first show that it is an independent set. Let  $(x_1, x_2)$  and  $(y_1, y_2)$  be distinct elements of  $S$ . Clearly, either  $x_1 \neq y_1$  or  $x_2 \neq y_2$ . If  $x_1 \neq y_1$  and  $x_2 \neq y_2$  then  $(x_1, x_2)$  and  $(y_1, y_2)$  must be independent vertices of  $G_1 \square G_2$ . Suppose that  $x_1 \neq y_1$  and  $x_2 = y_2$ . Then  $(x_1, x_2)$  and  $(y_1, y_2)$  must have been added to  $S$  during the same iteration of the algorithm. This means that  $x_1$  and  $y_1$  are independent in  $G_1$ , and hence so must be  $(x_1, x_2)$  and  $(y_1, y_2)$  in  $G_1 \square G_2$ . The case when  $x_1 = y_1$  and  $x_2 \neq y_2$  is similar. Thus  $S$  is an independent set of  $G_1 \square G_2$ .

We next show that  $S$  is a dominating set of  $G_1 \square G_2$ . Let  $X_1$  be the projection of the first co-ordinates of  $S$ , i.e.,  $X_1 = \{x_1 \in V(G_1) \mid (x_1, x_2) \in S \text{ for some } x_2 \in V(G_2)\}$ . Define  $X_2$  similarly. Let  $(x_1, x_2) \in V(G_1 \square G_2) \setminus S$ . It suffices to prove that  $(x_1, x_2)$  is adjacent to some vertex in  $S$ . Note that either  $X_1 = V(G_1)$  or  $X_2 = V(G_2)$ . Assume w.l.o.g. that  $X_1 = V(G_1)$ , which means that  $x_1 \in X_1$ . If  $x_2 \notin X_2$ , then  $x_2$  must be adjacent (in the graph  $G_2$ ) to some vertex in  $X_2$ , which

```

procedure R-INDEP( $G_1, G_2, S$ );
Comment:  $G_1$  and  $G_2$  are graphs.  $S$  is a set assumed to be initialized to  $\emptyset$  by the
           calling program. At the termination of this procedure,  $S$  will be a maximal
           independent set of  $G_1 \square G_2$ .
begin
  if both  $G_1$  and  $G_2$  are nonempty graphs then begin
    (* Assume that  $V(G_1) = V_1$  and  $V(G_2) = V_2$ . *)
    1. Let  $W_1$  and  $W_2$  be largest independent sets of  $G_1$  and  $G_2$  respectively;
    2.  $S := S \cup W_1 \times W_2$ ;
    3. R-INDEP( $\langle V_1 \setminus W_1 \rangle, \langle V_2 \setminus W_2 \rangle, S$ ); (* recursive call *)
  end (* if *)
end. (* R-INDEP *)

```

Figure 6.1: The recursive procedure R-INDEP

```

procedure I-INDEP( $G_1, G_2, S$ );
Comment:  $G_1$  and  $G_2$  are graphs.  $S$  is a set assumed to be initialized to  $\emptyset$  by the
           calling program. At the termination of this procedure,  $S$  will be a maximal
           independent set of  $G_1 \square G_2$ .
begin
  while both  $G_1$  and  $G_2$  are nonempty graphs do begin
    (* Assume that  $V(G_1) = V_1$  and  $V(G_2) = V_2$ . *)
    1. Let  $W_1$  and  $W_2$  be largest independent sets of  $G_1$  and  $G_2$  respectively;
    2.  $S := S \cup W_1 \times W_2$ ;
    3.  $G_1 := \langle V_1 \setminus W_1 \rangle$ ;
    4.  $G_2 := \langle V_2 \setminus W_2 \rangle$ ;
  end (* while *)
end. (* I-INDEP *)

```

Figure 6.2: The iterative procedure I-INDEP

means that  $(x_1, x_2)$  is adjacent to some vertex in  $S$ , and the claim follows. So assume that  $x_2 \in X_2$ . Suppose that elements of  $S$  of the form  $(x_1, y_2)$  (resp.  $(y_1, x_2)$ ) were added to  $S$  in the  $i$ th (resp.  $j$ th) iteration of the algorithm. Since  $(x_1, x_2) \notin S$ , we must have that  $i \neq j$ . Assume that  $i < j$ . Then, for some  $(x_1, y_2) \in S$ ,  $x_2$  must be adjacent to  $y_2$  in the graph  $G_2$ . Consequently,  $(x_1, x_2)$  is adjacent to  $(x_1, y_2) \in S$  in the graph  $G_1 \square G_2$ . The lemma follows. ■

**Lemma 6.2.3**  $\alpha(G_1) \cdot \alpha(G_2) + \min\{|V(G_1)| - \alpha(G_1), |V(G_2)| - \alpha(G_2)\} \leq |S|$ .

**Proof:** It is clear that the number of elements added to  $S$  during the first iteration is exactly  $\alpha(G_1) \cdot \alpha(G_2)$ . Thus it suffices to show that the number of elements added to  $S$  during the subsequent iterations is at least  $\min\{|V(G_1)| - \alpha(G_1), |V(G_2)| - \alpha(G_2)\}$ .

Define  $X_1$  and  $X_2$  as in the proof of the preceding lemma, and assume that  $X_1 = V(G_1)$ . Then every vertex of  $G_1$  appearing after the deletion of  $\alpha(G_1)$  vertices in the first iteration must appear as the first co-ordinate of some element of  $S$ . Thus the number of elements added to  $S$  in the second and subsequent iterations must be at least  $|V(G_1)| - \alpha(G_1)$ . The lemma follows. ■

Recall that there is a nondeterministic choice in the selection of the sets  $W_1$  and  $W_2$  in step 1 of the inner loop of the algorithm. Therefore, it is conceivable that  $|S|$  is sensitive to this choice. Let  $\alpha^*(G_1, G_2)$  denote the cardinality of a largest maximal independent set of  $G_1 \square G_2$  over all executions of our algorithm. Lemma 6.2.4 below shows that graphs exist for which our lower bound is strictly greater than that of Vizing. It also shows that our lower bound is *not* optimal.

**Lemma 6.2.4** *There exist graphs  $G_1$  and  $G_2$  such that*

$$\alpha(G_1) \cdot \alpha(G_2) + \min\{|V(G_1)| - \alpha(G_1), |V(G_2)| - \alpha(G_2)\} < \alpha^*(G_1, G_2) < \alpha(G_1 \square G_2).$$

**Proof:** Let  $G_1 = G_2 = C_5$ , i.e., a cycle of length five. It is clear that  $\alpha(C_5) = 2$ . Thus  $\alpha(C_5) \cdot \alpha(C_5) + \min\{|V(C_5)| - \alpha(C_5), |V(C_5)| - \alpha(C_5)\} = 7$ . A simple trace of our algorithm shows that  $\alpha^*(C_5, C_5) = 9$ . Further, assuming that  $\{1, 2, 3, 4, 5\}$  is the vertex set of  $C_5$  with adjacency defined in the natural way, it is easy to see that the following set, which is of cardinality ten, is an independent set of  $C_5 \square C_5$ :  $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (2, 4), (3, 5), (4, 1), (5, 2)\}$ . By Theorem 6.2.1, this is in fact a maximum independent set, and hence  $\alpha(C_5 \square C_5) = 10$ . The lemma follows. ■

### 6.2.2 Bounds on $\alpha(G_1 \times G_2)$

In the following theorem, we offer bounds on  $\alpha(G_1 \times G_2)$ .

**Theorem 6.2.5** *Let  $\alpha_i = \alpha(G_i)$  and  $\gamma_i = \gamma(G_i)$  for graphs  $G_i = (V_i, E_i)$ ,  $i = 1, 2$ . Then,  $\max\{\alpha_1 \cdot |V_2|, \alpha_2 \cdot |V_1|\} \leq \alpha(G_1 \times G_2) \leq |V_1| \cdot |V_2| - \gamma_1 \cdot \gamma_2 + \max\{\gamma_1, \gamma_2\}$ .*

**Proof:** Let  $G_1, G_2, \alpha_1, \alpha_2, \gamma_1$  and  $\gamma_2$  be as in the statement of the theorem. W.l.o.g. let  $\{1, \dots, \alpha_1\}$  and  $\{1, \dots, \alpha_2\}$  be largest independent sets of  $G_1$  and  $G_2$  respectively. Note that each of  $\{1, \dots, \alpha_1\} \times V_2$  and  $V_1 \times \{1, \dots, \alpha_2\}$  is an independent set of  $G_1 \times G_2$ . This settles the lower bound.

For the upper bound, assume w.l.o.g. that  $\{1, \dots, \gamma_1\}$  and  $\{1, \dots, \gamma_2\}$  are largest cliques of  $G_1$  and  $G_2$  respectively. Observe that in the vertex subset  $\{1, \dots, \gamma_1\} \times \{1, \dots, \gamma_2\}$  of the graph  $G_1 \times G_2$ , there are at most  $\max\{\gamma_1, \gamma_2\}$  vertices which are



mutually independent. Thus a maximum independent set of  $G_1 \times G_2$  may be of cardinality at most  $|V_1| \cdot |V_2| - \gamma_1 \cdot \gamma_2 + \max\{\gamma_1, \gamma_2\}$ . ■

Note that if both  $G_1$  and  $G_2$  are complete graphs, then the lower and upper bounds of the foregoing theorem coincide, and hence yield the optimal value.

### 6.2.3 Bounds on $\alpha(G_1 \boxtimes G_2)$

**Theorem 6.2.6** *For graphs  $G_1$  and  $G_2$ ,  $\alpha(G_1) \cdot \alpha(G_2) \leq \alpha(G_1 \boxtimes G_2) \leq \alpha(G_1 \square G_2)$ .*

**Proof Sketch:** For the lower bound, note that if  $S_1$  and  $S_2$  are maximum independent sets of  $G_1$  and  $G_2$  respectively, then  $S_1 \times S_2$  is a maximal independent set of  $G_1 \boxtimes G_2$ . The upper bound follows trivially. ■

By a theorem of Shannon [62], it follows that if  $G_1$  and  $G_2$  are even cycles of the same length, i.e.,  $G_1 \cong G_2 \cong C_{2n}$ , then the foregoing lower bound is achieved. On the other hand, if they are odd cycles of the same length, then the lower bound is not achieved [45,58,62]. In particular,  $\alpha(C_5) = 2$  while  $\{(1, 1), (2, 3), (3, 5), (4, 2), (5, 4)\}$  is a maximum independent set of  $C_5 \boxtimes C_5$ , and hence  $\alpha(C_5 \boxtimes C_5) = 5$ . In the following corollary, we obtain a sequence of inequalities involving the bounds, presented in this section, on the independence numbers of the  $\square$ -product and  $\times$ -product.

**Corollary 6.2.7** *Let  $\alpha_i = \alpha(G_i)$  and  $\gamma_i = \gamma(G_i)$  for graphs  $G_i = (V_i, E_i)$ ,  $i = 1, 2$ . Then,  $\alpha_1 \cdot \alpha_2 + \min\{|V_1| - \alpha_1, |V_2| - \alpha_2\} \leq \alpha^*(G_1, G_2) \leq \alpha(G_1 \square G_2) \leq \min\{\alpha_1 \cdot |V_2|, \alpha_2 \cdot |V_1|\} \leq \max\{\alpha_1 \cdot |V_2|, \alpha_2 \cdot |V_1|\} \leq \alpha(G_1 \times G_2) \leq |V_1| \cdot |V_2| - \gamma_1 \cdot \gamma_2 + \max\{\gamma_1, \gamma_2\}$ . ■*

Note again that the graph  $G_1 \times G_2$  may, in general, be much denser than the graph  $G_1 \square G_2$  (in the sense of the number of edges), but  $\alpha(G_1 \times G_2)$  is always greater than or equal to  $\alpha(G_1 \square G_2)$ .

### 6.3 Domination Numbers

Vizing obtained the following upper bound on  $\beta(G_1 \square G_2)$ .

**Theorem 6.3.1** [65] *Let  $\beta_i = \beta(G_i)$  for graphs  $G_i = (V_i, E_i)$ ,  $i = 1, 2$ . Then,  $\beta(G_1 \square G_2) \leq \min\{\beta_1 \cdot |V_2|, \beta_2 \cdot |V_1|\}$ . ■*

Vizing conjectured that  $\beta(G_1) \cdot \beta(G_2) \leq \beta(G_1 \square G_2)$ . Whether this is, in general, valid or not remains a major open problem. However, for some special cases, the conjecture has been found to be true. We state some interesting results in the following theorem of Jacobson and Kinch.

**Theorem 6.3.2** [36,37]

1.  $\beta(P_n \square K_2) = \lceil \frac{n}{2} \rceil + 1$ .
2.  $\beta(P_n \square P_2) = n - \lfloor \frac{n-1}{4} \rfloor$ .
3. For all  $n$ ,  $\beta(P_n \square P_3) = n + 1$  if  $n \in \{1, \dots, 6, 9\}$ , and  $n$  otherwise.
4. For all graphs  $G$  and any tree  $T$ ,  $\beta(G) \cdot \beta(T) \leq \beta(G \square T)$ .
5. For almost all trees  $T_1$  and  $T_2$ ,  $\beta(T_1) \cdot \beta(T_2) < \beta(T_1 \square T_2)$ . ■

Recall that  $\Delta(G)$  denotes the maximum degree of a vertex of  $G$ . The next theorem gives a lower bound on  $\beta(G_1 \square G_2)$ .

**Theorem 6.3.3** [97] For graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ ,  
 $\max\{\frac{|V_1|}{\Delta(G_1)+1} \cdot \beta(G_2), \frac{|V_2|}{\Delta(G_2)+1} \cdot \beta(G_1)\} \leq \beta(G_1 \square G_2)$ . ■

The following theorem shows that if  $G_1$  and  $G_2$  are graphs without isolated vertices, then Vizing's upper bound on  $\beta(G_1 \square G_2)$  (cf. Theorem 6.3.1) is correct also for  $\beta(G_1 \times G_2)$ .

**Theorem 6.3.4** Let  $\beta_i = \beta(G_i)$  for graphs  $G_i = (V_i, E_i)$ ,  $i = 1, 2$ . If none of  $G_1$  and  $G_2$  contains any isolated vertex, then  $\beta(G_1 \times G_2) \leq \min\{\beta_1 \cdot |V_2|, \beta_2 \cdot |V_1|\}$ .

**Proof:** Let  $S \subseteq V_2$  be a dominating set of  $G_2$ . We claim that  $V_1 \times S$  is a dominating set of  $G_1 \times G_2$ . Let  $(x, y) \in V_1 \times V_2 \setminus V_1 \times S$ . Since  $S$  is a dominating set of  $G_2$  and  $x$  is not an isolated vertex, there exist  $x' \in V_1$  and  $y' \in S$  such that  $\{x, x'\} \in E_1$  and  $\{y, y'\} \in E_2$ . Consequently,  $(x', y')$ , which is a member of  $V_1 \times S$ , dominates  $(x, y)$ . Since  $S$  was an arbitrary dominating set of  $G_2$ , we have  $\beta(G_1 \times G_2) \leq \beta_2 \cdot |V_1|$ . By symmetry,  $\beta(G_1 \times G_2) \leq \beta_1 \cdot |V_2|$ . The theorem follows. ■

In the following corollary, we present a sequence of inequalities between  $\beta(G_1 \times G_2)$  and  $\alpha(G_1 \times G_2)$ . It makes use of (i) Theorem 6.3.4, (ii) the fact that  $\beta(G) \leq \alpha(G)$  for every graph  $G$  and (iii) the lower bound on  $\alpha(G_1 \times G_2)$  stated in Theorem 6.2.5.

**Corollary 6.3.5** Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be graphs, none of which contains any isolated vertex, and let  $\alpha_i = \alpha(G_i)$ ,  $\beta_i = \beta(G_i)$ ,  $i = 1, 2$ . Then  
 $\beta(G_1 \times G_2) \leq \min\{\beta_1 \cdot |V_2|, \beta_2 \cdot |V_1|\} \leq \min\{\alpha_1 \cdot |V_2|, \alpha_2 \cdot |V_1|\}$   
 $\leq \max\{\alpha_1 \cdot |V_2|, \alpha_2 \cdot |V_1|\} \leq \alpha(G_1 \times G_2)$ . ■

Note that it trivially follows that  $\beta(G_1 \boxtimes G_2) \leq \min\{\beta(G_1 \square G_2), \beta(G_1 \times G_2)\}$ .

The following theorem gives another interesting upper bound on  $\beta(G_1 \boxtimes G_2)$ .

**Theorem 6.3.6**  $\beta(G_1 \boxtimes G_2) \leq \beta(G_1) \cdot \beta(G_2)$ .

**Proof:** Let  $S_1$  and  $S_2$  be dominating sets of  $G_1$  and  $G_2$  respectively. It suffices to show that  $S_1 \times S_2$  is a dominating set of  $G_1 \boxtimes G_2$ . Let  $(x_1, x_2)$  be a vertex of  $G_1 \boxtimes G_2$  which is not in  $S_1 \times S_2$ ; we will show that  $(x_1, x_2)$  is adjacent to some vertex of  $S_1 \times S_2$ . First suppose that  $x_1 \in S_1$  and  $x_2 \notin S_2$ . Since  $S_2$  is a dominating set of  $G_2$ , there is some  $y_2 \in S_2$  such that  $\{x_2, y_2\} \in E(G_2)$ . This shows that  $(x_1, x_2)$  is adjacent to  $\{x_1, y_2\}$  which is in  $S_1 \times S_2$ . The case when  $x_1 \notin S_1$  and  $x_2 \in S_2$  is similar. Next suppose that  $x_1 \notin S_1$  and  $x_2 \notin S_2$ . Then there are  $y_1$  and  $y_2$  such that  $y_i \in S_i$  and  $\{x_i, y_i\} \in E(G_i)$ ,  $i = 1, 2$ . Consequently,  $(x_1, x_2)$  is adjacent to  $(y_1, y_2)$  which is in  $S_1 \times S_2$ . The lemma follows. ■

In the following corollary, we obtain a sequence of inequalities based on Theorems 6.3.1, 6.3.4 and the remark preceding Theorem 6.3.6 above.

**Corollary 6.3.7** *Let  $\beta_i = \beta(G_i)$  for graphs  $G_i = (V_i, E_i)$ ,  $i = 1, 2$ , and assume that none of  $G_1$  and  $G_2$  contains any isolated vertex. Then*

$$\beta(G_1 \boxtimes G_2) \leq \min\{\beta(G_1 \square G_2), \beta(G_1 \times G_2)\} \leq \max\{\beta(G_1 \square G_2), \beta(G_1 \times G_2)\} \leq \min\{\beta_1 \cdot |V_2|, \beta_2 \cdot |V_1|\}.$$

## 6.4 Clique Numbers

In this section, we derive exact values for the clique numbers of all three products in terms of the clique numbers of the factor graphs. Unless otherwise stated, by a

clique we mean a maximal clique. For any clique  $Q$  of a graph, we will use  $Q$  also to denote its vertex set.

**Theorem 6.4.1**  $\gamma(G_1 \square G_2) = \max\{\gamma(G_1), \gamma(G_2)\}$ .

**Proof:** Since each of  $G_1$  and  $G_2$  is a subgraph of  $G_1 \square G_2$ , it is clear that  $\max\{\gamma(G_1), \gamma(G_2)\} \leq \gamma(G_1 \square G_2)$ . We will establish the reverse inequality. For this, it suffices to show that if  $G_1 \square G_2$  contains a triangle, then the three vertices of the triangle are constant either in the first co-ordinate or in the second co-ordinate. So consider a triangle in  $G_1 \square G_2$  and assume w.l.o.g. that two vertices of the triangle are  $(u, x_2)$  and  $(u, y_2)$ . Let  $(v_1, v_2)$  be the third vertex of the triangle. If  $v_1 \neq u$ , then  $v_2 = x_2$  and  $v_2 = y_2$ . But that is impossible, since  $x_2 \neq y_2$ . ■

**Theorem 6.4.2**  $\gamma(G_1 \times G_2) = \min\{\gamma(G_1), \gamma(G_2)\}$ .

**Proof:** Let  $Q_1 = \{1, \dots, m\}$  and  $Q_2 = \{1, \dots, n\}$  be largest cliques of  $G_1$  and  $G_2$  respectively. Assume that  $m \leq n$ , and observe that the set  $Q = \{(i, i) \mid 1 \leq i \leq m\}$  is a (not necessarily maximal) clique of the graph  $G_1 \times G_2$ . This implies that  $m = \min\{\gamma(G_1), \gamma(G_2)\} \leq \gamma(G_1 \times G_2)$ . For the reverse inequality, consider a largest clique, say  $Q$ , of the graph  $G_1 \times G_2$ . Note that if  $(x_1, x_2)$  and  $(y_1, y_2)$  are distinct elements of  $Q$ , then  $x_1$  and  $y_1$  (resp.  $x_2$  and  $y_2$ ) must be adjacent, hence distinct vertices in the graph  $G_1$  (resp.  $G_2$ ). Let  $Q_1 = \{x_1 \in V(G_1) \mid (x_1, x_2) \in Q \text{ for some } x_2 \in V(G_2)\}$  and define the set  $Q_2$  analogously. Clearly,  $Q_1$  and  $Q_2$  must be (not necessarily maximal) cliques in the graphs  $G_1$  and  $G_2$  respectively. Hence  $\gamma(G_1 \times G_2) = |Q| = |Q_1| = |Q_2| \leq \min\{\gamma(G_1), \gamma(G_2)\}$ . The theorem follows. ■

**Theorem 6.4.3** *Let  $G_1$  and  $G_2$  be nontrivial, connected graphs. If  $Q_1$  and  $Q_2$  are cliques of  $G_1$  and  $G_2$ , then  $Q_1 \times Q_2$  is a clique of  $G_1 \boxtimes G_2$ . Conversely, if  $Q$  is a clique of  $G_1 \boxtimes G_2$ , then there exist cliques  $Q_1$  and  $Q_2$  of  $G_1$  and  $G_2$  such that  $Q = Q_1 \times Q_2$ .*

**Proof:** Let  $G_1$  and  $G_2$  be as in the statement of the theorem. It is straightforward to see that if  $Q_1$  and  $Q_2$  are cliques of  $G_1$  and  $G_2$ , then  $Q_1 \times Q_2$  is a clique of  $G_1 \boxtimes G_2$ . For the converse, consider a clique  $Q$  of  $G_1 \boxtimes G_2$  and let  $Q_1 = \{u \in V(G_1) \mid (u, v) \in Q \text{ for some } v \text{ in } V(G_2)\}$  and  $Q_2 = \{v \in V(G_2) \mid (u, v) \in Q \text{ for some } u \text{ in } V(G_1)\}$ . Note that  $Q_1$  and  $Q_2$  must be cliques of  $G_1$  and  $G_2$  respectively, and that  $Q \subseteq Q_1 \times Q_2$ . Further, if  $Q$  were properly contained in  $Q_1 \times Q_2$ , then maximality of  $Q$  would be violated. ■

The next corollary follows immediately from Theorem 6.4.3.

**Corollary 6.4.4** *Let  $G_1$  and  $G_2$  be nontrivial, connected graphs.*

1.  $\gamma(G_1 \boxtimes G_2) = \gamma(G_1) \cdot \gamma(G_2)$ .
2. *The order of every clique of  $G_1 \boxtimes G_2$  is a composite number, and hence  $4 \leq \gamma(G_1 \boxtimes G_2)$ .* ■

In the following corollary, we obtain a sequence of inequalities based on the results of Theorems 6.4.1, 6.4.2 and Corollary 6.4.4.

**Corollary 6.4.5**  $\gamma(G_1 \times G_2) = \min\{\gamma(G_1), \gamma(G_2)\} \leq \max\{\gamma(G_1), \gamma(G_2)\} = \gamma(G_1 \square G_2) \leq \gamma(G_1) \cdot \gamma(G_2) = \gamma(G_1 \boxtimes G_2)$ . ■

Observe again that the graph  $G_1 \times G_2$  may be much denser than the graph  $G_1 \square G_2$ , but the clique number of  $G_1 \times G_2$  is always less than or equal to the clique number of  $G_1 \square G_2$ .

### 6.5 Hamiltonian Paths/Cycles

The following theorem of Vizing (see also [9]) gives conditions that are sufficient for the existence of a Hamiltonian path/cycle in the  $\square$ -product, and also states that these conditions are not necessary.

**Theorem 6.5.1** [65] *Let  $G_1$  and  $G_2$  be graphs, each of which has a Hamiltonian path. Then,  $G_1 \square G_2$  has a Hamiltonian path. If, in addition, at least one graph has a Hamiltonian cycle or an even number of vertices, then  $G_1 \square G_2$  has a Hamiltonian cycle. The converses of the foregoing statements are false. ■*

For the  $\square$ -product of a connected graph and a cycle, Rosenfeld and Barnette (see also [2]) obtained the following result.

**Theorem 6.5.2** [59] *For a connected graph  $G$ , the graph  $G \square C_n$  has a Hamiltonian cycle if and only if  $\Delta(G) \leq n$ . ■*

The next theorem gives sufficient conditions for the existence of a Hamiltonian cycle in the  $\times$ -product. It also shows that these conditions are not necessary.

**Theorem 6.5.3** *Let  $G_1$  and  $G_2$  be nontrivial, connected graphs. If  $|V(G_1)|$  and  $|V(G_2)|$  are relatively prime and each of  $G_1$  and  $G_2$  contains a Hamiltonian cycle, then  $G_1 \times G_2$  contains a Hamiltonian cycle. The converse is false.*

**Proof:** It suffices to show that if  $m$  and  $n$  are relatively prime integers and  $m, n \geq 3$ , then the graph  $C_m \times C_n$  contains a Hamiltonian cycle. So assume that  $m$  and  $n$  are integers satisfying these conditions. For  $k = m, n$ , let  $V(C_k) = \{0, \dots, k-1\}$ , where  $\{0, k-1\}, \{i, i+1\} \in E(C_k)$ ,  $0 \leq i \leq k-2$ . Thus  $V(C_m \times C_n) = \{(i, j) \mid 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$  with adjacency as in the definition of the  $\times$ -product. Note that at least one of  $m$  and  $n$  must be odd, and hence the graph  $C_m \times C_n$  is necessarily connected (see Lemma 1.2.5 and Corollary 1.2.6).

We construct a sequence of  $m \cdot n$  elements, indexed from 0 through  $m \cdot n - 1$ , where each element is a vertex of  $C_m \times C_n$  as follows: the  $k$ th element is a pair  $(i, j)$  where  $i = k \bmod m$  and  $j = k \bmod n$ . Since  $m$  and  $n$  are relatively prime, it is straightforward to see that every vertex of the graph  $C_m \times C_n$  appears exactly once in this sequence. We claim that the sequence thus constructed constitutes a Hamiltonian cycle of  $C_m \times C_n$ . It is obvious that the (first and last elements of this sequence, i.e.,) vertices  $(0, 0)$  and  $(m-1, n-1)$  are adjacent. Now consider two consecutive vertices  $(i, j)$  and  $(p, q)$  of this sequence. It is clear that either  $i = m-1$  and  $p = 0$  or  $p = i+1$ . Similarly, either  $j = n-1$  and  $q = 0$  or  $q = j+1$ . In each case, it is easy to see that  $\{i, p\} \in E(C_m)$  and  $\{j, q\} \in E(C_n)$ . Consequently,  $\{(i, j), (p, q)\} \in E(C_m \times C_n)$  as required.

To see that the converse is false, note that for all  $n \geq 1$ ,  $C_{2n+1} \times K_2 \cong C_{4n+2}$ . In other words, the existence of a Hamiltonian cycle in the  $\times$ -product of two graphs does not necessarily imply that each factor graph contains a Hamiltonian cycle. ■

To illustrate the construction in the proof of Theorem 6.5.3, we have shown the graphs  $C_4$ ,  $C_5$  and a Hamiltonian cycle in the graph  $C_4 \times C_5$  in Figure 6.3.



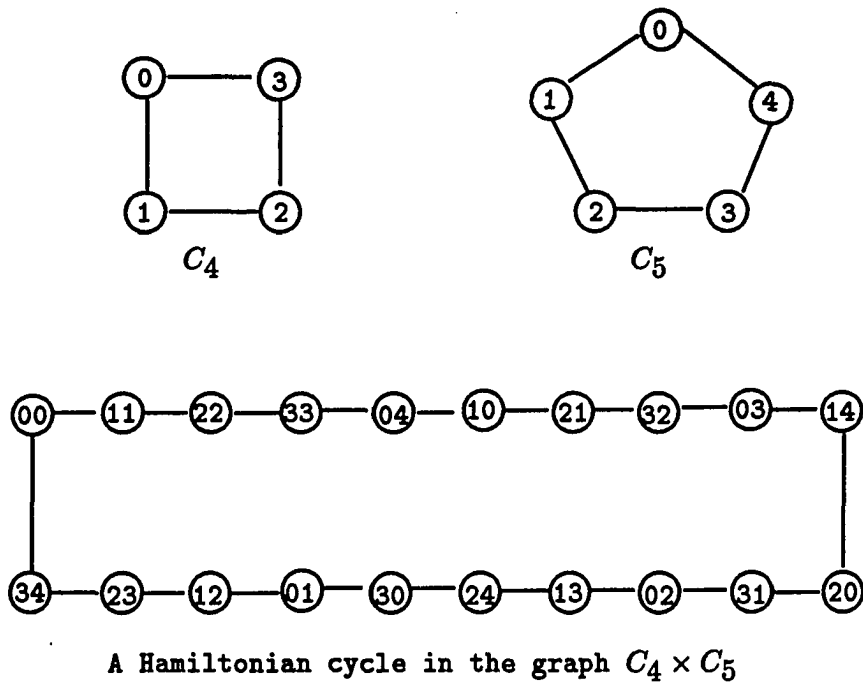


Figure 6.3: An illustration of the construction in the proof of Theorem 6.5.3

For the  $\boxtimes$ -product, note that since  $G_1 \square G_2$  as well as  $G_1 \times G_2$  is a subgraph of  $G_1 \boxtimes G_2$  on the same set of vertices, the conditions which ensure the existence of a Hamiltonian path/cycle in the  $\square$ -product or  $\times$ -product are correct also for the  $\boxtimes$ -product. However, as stated in the following theorem, a slightly weaker condition ensures the existence of a Hamiltonian cycle in  $G_1 \boxtimes G_2$ .

**Theorem 6.5.4 [10]** *If  $G_1$  and  $G_2$  have Hamiltonian paths, then  $G_1 \boxtimes G_2$  has a Hamiltonian cycle. ■*

As noted by Vizing [65], the graph  $K_{1,3} \square C_3$  contains a Hamiltonian cycle. Therefore, the graph  $K_{1,3} \boxtimes C_3$  contains a Hamiltonian cycle, and hence the conditions of Theorem 6.5.4 are not necessary for the existence of a Hamiltonian cycle in the  $\boxtimes$ -product of graphs.

We conclude this section by stating an interesting result w.r.t. Hamiltonian cycles in  $\boxtimes$ -products.

**Theorem 6.5.5 [10]** *For any connected graph  $G$  with at least two vertices, there exists an integer  $k$  such that  $G \boxtimes \cdots \boxtimes G$  ( $k$  factors) has a Hamiltonian cycle. ■*

## 6.6 Concluding Remarks

In this chapter, we have discussed (i) bounds on the chromatic numbers, independence numbers, domination numbers and clique numbers of the three graph products in terms of the invariants of the factor graphs, and (ii) certain conditions which are sufficient (but not necessary) for the existence of Hamiltonian paths/cycles in the three products.

Recall that for graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ ,  $|E(G_1 \square G_2)| = |V_1| \cdot |E_2| + |V_2| \cdot |E_1|$  while  $|E(G_1 \times G_2)| = 2 \cdot |E_1| \cdot |E_2|$ . This implies that if  $G_1$  and  $G_2$  are connected, dense graphs (in the sense of the number of edges), then the graph  $G_1 \times G_2$  may be much denser than the graph  $G_1 \square G_2$ . However, it is interesting to note (Corollaries 6.1.10, 6.2.7 and 6.4.5) that (i)  $\chi(G_1 \times G_2) \leq \chi(G_1 \square G_2)$ , (ii)  $\alpha(G_1 \times G_2) \geq \alpha(G_1 \square G_2)$ , and (iii)  $\gamma(G_1 \times G_2) \leq \gamma(G_1 \square G_2)$ . It is further our experience that the problem of obtaining bounds on the invariants of the  $\times$ -product are, in general, involved compared to the analogous problems for the other products.

For the clique numbers, we have obtained exact values for all three products. Among the remaining (numerical) invariants, exact value is known for only the chromatic number of the  $\square$ -product. For the Hamiltonian paths/cycles, only certain sufficient (but not necessary) conditions are known. Therefore, several open problems remain which merit investigation. We have stated some of them in the next chapter.

## 7. DISCUSSION AND OPEN PROBLEMS

In this chapter, we summarize the results of this dissertation, discuss some related issues and state several open problems which have direct relevance to the topics of the preceding chapters.

In Chapter 2, we have presented certain schemes for partitioning the vertex set of the  $n$ -cube  $Q_n$  into certain equal-size maximal independent sets. This includes an interesting application of a Latin square. As a by-product, we have obtained the following bounds on the minimum cardinality of a maximal independent set of  $Q_n$ , denoted by  $\Lambda(Q_n)$ :

$$\frac{2^n}{n+1} \leq \Lambda(Q_n) \leq \frac{2^n}{2^{\lfloor \log_2(n+1) \rfloor}}.$$

It is clear that our upper and lower bounds are within a factor of two, and that for  $n$  of the form  $n = 2^k - 1$ , the two bounds coincide, and hence yield the optimal value. We have further noted that the foregoing bounds are correct also for the domination number of  $Q_n$ . An immediate question that arises is the following.

**Question 1** *Determine the exact value of  $\Lambda(Q_n)$  for all  $n$ . ■*

In fact, there are several numerical invariants of the  $n$ -cube for which determination of the exact values remains an open problem. In [31], Harary, Hayes and Wu

have catalogued the best known bounds on many such invariants of  $Q_n$ , and have posed the problems of improving those bounds.

In Chapter 3, we have presented two algorithms—RECOG-B (Figure 3.1, page 34) and RECOG-M (Figure 3.3, page 46)—each of which recognizes median graphs in time  $O(n^2 \log n)$ . (As usual,  $n$  denotes the number of vertices of the candidate graph.) Our results have led to an improvement over the previously best-known bound of  $O(n^4)$  of an algorithm of Chung, Graham and Saks [14]. Both of our algorithms are interesting in their own right. The first is based on the concept of retraction used by Bandelt [3] in his characterization of median graphs while the second is based on the concept of convex-expansion used by Mulder [48] for the same purpose. We have observed that the latter scheme is conceptually simpler.

It is easy to see that the theoretical lower bound on the time complexity of recognizing median graphs is  $O(n \log n)$ . This follows from the facts that (i) a nontrivial graph property cannot be decided in sublinear time by a sequential algorithm and (ii) a necessary condition for a graph to be a median graph is that its number of edges be bounded by a constant multiple of  $n \log n$ , since every median graph is a subgraph of a hypercube (cf. Lemma 3.1.4, page 31). It is, therefore, clear that there is still a gap between the lower bound and our upper bound of  $O(n^2 \log n)$ . So a natural question that arises is how to bridge (or, at least narrow) the gap between the two bounds. As in any computational problem, one might obtain an asymptotically better recognition scheme, or establish a better lower bound. We state this as a research problem below.

**Question 2** *Improve the upper bound of  $O(n^2 \log n)$  for recognizing median graphs, or establish a lower bound better than  $O(n \log n)$ . ■*

As we stated earlier, Mulder's characterization of median graphs (Theorem 3.1.3, page 31) has been given in terms of convex expansions. Suppose that we drop the condition of convexity and define a class  $\mathcal{G}$  of graphs obtainable from the one-vertex graph by a sequence of (not necessarily convex) expansions. (See page 30 for the definition of an expansion of a graph.) It is easy to see that (i) every member of  $\mathcal{G}$  is a bipartite graph and (ii) the class of median graphs is properly contained in  $\mathcal{G}$ . An interesting problem related to this class of graphs is the following.

**Question 3** *Obtain a simple characterization for  $\mathcal{G}$  and/or (i) examine the relationship between graphs in  $\mathcal{G}$  and the subgraphs of hypercubes, and (ii) determine whether the membership problem for  $\mathcal{G}$  is efficiently decidable or not. ■*

At the end of Chapter 3, we have noted that the problem of recognizing the isometric subgraphs of hypercubes has a certain similarity to the analogous problem for median graphs. Further, the best-known upper bound for the foregoing problem is  $O(n^3)$  of an algorithm of Wilkeit [71]. We have also stated the problems we encountered in adapting our convex-expansions algorithm RECOG-M (Figure 3.3) for recognizing isometric subgraphs of hypercubes. We state this problem as follows.

**Question 4** *Construct an algorithm for recognizing isometric subgraphs of hypercubes in less than cubic time, or improve the theoretical lower bound of  $O(n \log n)$ . ■*

We have further stated at the end of Chapter 3 that the so-called quasimedial graphs defined by Mulder [51] are natural generalization of median graphs and that they are precisely the retracts of the  $\square$ -product of complete graphs [15]. In fact,

Mulder characterizes these graphs in terms of what he calls *quasimedial expansion* [51]. A problem whose solution appears promising to us is to extend the construction of our recognition algorithms for median graphs to obtain similar efficient scheme(s) for recognizing quasimedial graphs. In fact, Chung, Graham and Saks [15] have obtained several other characterizations of these graphs, and it may be possible to adapt their proofs for constructing alternative schemes for the same problem. We state this problem as follows.

**Question 5** *Construct an efficient algorithm, say  $O(n^2 \log n)$ , for recognizing quasimedial graphs. ■*

In Chapter 4, we have presented two  $O(n^2 \log n)$  algorithms—EMBED-B (Figure 4.2, page 58) and EMBED-M (Figure 4.3, page 63)—each of which obtains an isometric embedding of a given median graph in a hypercube of least possible dimension. The structures of the two algorithms are quite similar to those of the respective recognition algorithms of Chapter 3. We have also observed that any improvement in our recognition scheme will lead to a similar improvement in the corresponding isometric embedding scheme. We have further noted that if a graph  $G$  is known to be an isometric subgraph of a hypercube, then the algorithm EMBED-M will act on  $G$  to produce a similar isometric embedding of  $G$  in a hypercube of least possible dimension. This is in sharp contrast to the non-applicability of any of our recognition schemes (for median graphs) for identifying isometric subgraphs of hypercubes.

In Chapter 5, we have discussed necessary and sufficient conditions for the planarity and outerplanarity of each of the three product graphs in terms of the factor graphs. We have observed that (i) characterizations for the planarities of the  $\square$ -product and  $\times$ -product already appear in the literature and (ii) for any two graphs

$G_1$  and  $G_2$ , each containing at least one edge, the graph  $G_1 \boxtimes G_2$  is non-outerplanar. Our contribution consists of characterizations for the following: (i) planarity of the  $\boxtimes$ -product, (ii) outerplanarity of the  $\square$ -product and (iii) outerplanarity of the  $\times$ -product. We have made use of known characterizations for the planarity and outerplanarity of general graphs which have been given in terms of graph contractions. Note that together with our results (of Chapter 5), we now have a complete characterization for the planarity and outerplanarity of each of the three graph products.

While dealing with the outerplanarity of the  $\times$ -product of graphs, we have defined a minimal cycle of a graph and have introduced an interesting class of graphs called almost bipartite ( $a$ - $b$ ) graphs (see page 77). We have shown that if  $G$  is an  $a$ - $b$  graph, then  $G$  is a subcontraction of  $G \times K_2$ . (If  $G$  is a bipartite graph, then  $G \times K_2$  consists of two disjoint copies of  $G$ , and hence, in that case,  $G$  is trivially a subcontraction of  $G \times K_2$ .) We have further stated at the end of Chapter 5 that if  $G$  contains at most two cliques, then too  $G$  is a subcontraction of  $G \times K_2$ . Whether or not this is true of all graphs appears to be a very interesting problem. We state it as follows.

**Question 6** *Prove or disprove: every connected graph  $G$  is a subcontraction of the graph  $G \times K_2$ . ■*

Note that in the case of an affirmative answer to the foregoing question, it will follow that if  $G_1$  and  $G_2$  are nontrivial, connected graphs, then each of  $G_1$  and  $G_2$  is a subcontraction of  $G_1 \times G_2$ .

In Chapter 6, we have discussed (i) bounds on the chromatic numbers, independence numbers, domination numbers and clique numbers, and (ii) conditions for the existence of Hamiltonian paths/cycles, in the three graph products. (For a graph  $G$ ,



we use  $\chi(G)$ ,  $\alpha(G)$ ,  $\beta(G)$  and  $\gamma(G)$  to denote respectively the chromatic number, independence number, domination number and clique number of  $G$ .) For the clique numbers, we have obtained exact values for all three products (see Theorems 6.4.1, 6.4.2 and 6.4.3). For the other invariants, exact value is known for only the chromatic number of the  $\square$ -product (see Theorem 6.1.1 due to Vizing).

In Corollary 6.1.10, we have presented a sequence of inequalities involving bounds on  $\chi(G_1 \square G_2)$ ,  $\chi(G_1 \times G_2)$  and  $\chi(G_1 \boxtimes G_2)$ . A natural question that arises is the following.

**Question 7** *Improve the bounds on  $\chi(G_1 \times G_2)$  and  $\chi(G_1 \boxtimes G_2)$ . In particular, obtain a nontrivial lower bound on  $\chi(G_1 \times G_2)$ . ■*

In Theorem 6.2.6, we have stated bounds on the independence number of the  $\boxtimes$ -product, i.e.,  $\alpha(G_1 \boxtimes G_2)$ , while in Corollary 6.2.7, we have presented a sequence of inequalities involving bounds on  $\alpha(G_1 \square G_2)$  and  $\alpha(G_1 \times G_2)$ . For the independence numbers of the three products, we ask the following question.

**Question 8** *Obtain a tight upper bound on  $\alpha(G_1 \boxtimes G_2)$  and improve the bounds on  $\alpha(G_1 \square G_2)$  and  $\alpha(G_1 \times G_2)$ . In particular, obtain a tight upper bound on  $\alpha(G_1 \times G_2)$  in terms of  $\alpha(G_1)$  and  $\alpha(G_2)$ . ■*

Theorems 6.3.1, 6.3.4 and 6.3.6 respectively state upper bounds on  $\beta(G_1 \square G_2)$  (due to Vizing),  $\beta(G_1 \times G_2)$  and  $\beta(G_1 \boxtimes G_2)$  while Corollary 6.3.7 states a sequence of inequalities involving bounds on these invariants. An immediate question that arises is the following.

**Question 9** Obtain nontrivial lower bounds on  $\beta(G_1 \square G_2)$ ,  $\beta(G_1 \times G_2)$  and  $\beta(G_1 \boxtimes G_2)$ .

■

It is a longstanding *conjecture* of Vizing [65] that  $\beta(G_1) \cdot \beta(G_2) \leq \beta(G_1 \square G_2)$ . We pose the problem of determining the validity of this conjecture along with two other conjectures as follows.

**Question 10** For graphs  $G_1$  and  $G_2$ , let  $\beta_i = \beta(G_i)$ ,  $i = 1, 2$ .

*Prove or disprove:*

1.  $\beta_1 \cdot \beta_2 \leq \beta(G_1 \square G_2)$
2.  $\beta(G_1 \square G_2) \leq \beta(G_1 \times G_2)$  and
3.  $\beta_1 \cdot \beta_2 \leq \beta(G_1 \times G_2)$ . ■

As we mentioned earlier, we have established exact values for the clique numbers of all three products (Theorems 6.4.1, 6.4.2 and 6.4.3) in terms of the clique numbers of the factor graphs.

Theorems 6.5.1, 6.5.3 and 6.5.4 respectively state conditions that are sufficient (but not necessary) for the existence of Hamiltonian cycles in the three products. We, therefore, ask the following question.

**Question 11** Obtain necessary and sufficient conditions for the existence of Hamiltonian paths/cycles in each of the three product graphs. ■

Theorem 6.5.5 states that for every connected graph  $G$ , there exists an integer  $k$  such that  $G \boxtimes \cdots \boxtimes G$  ( $k$  factors) has a Hamiltonian cycle. It would be interesting to determine whether or not analogous statements are true for the other two products. We state this problem as follows.

**Question 12** *Prove or disprove: for every connected graph  $G$  with at least two vertices, there exists an integer  $k$  such that the graph  $G \boxtimes \cdots \boxtimes G$  (resp.  $G \times \cdots \times G$ ) ( $k$  factors) has a Hamiltonian cycle. ■*

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Finally, thanking the wife in public is not a Hindu custom. I will, therefore, keep quiet.

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