Investigating and representing inquiry in a college mathematics course

by

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ABSTRACT

Recent calls by the National Research Council and the National Science Foundation have stressed the need for excellence in undergraduate mathematics and science education with emphasis placed on inquiry learning. The purposes of this qualitative study include (1) the examination of the pursuit of inquiry in two collegiate mathematics classrooms incorporating methods of mathematical modeling and (2) the generation of a quantitative representation of characteristics of an inquiry environment.

Instructors and students in two classes of laboratory-based calculus for life sciences majors were observed. To capture descriptions of the environments and students' mathematical modeling skills, the classes surrounding three science investigations were audio or video recorded; interviews were conducted with one instructor and six students in the researcher's class; and copies of students' lab reports were obtained. Transcripts were coded using various scales to produce graphical images of the classroom environments.

The data were used to describe and document the effects of both classroom environments. Instructors' goals and time factors influenced the development of inquiry, mathematical modeling, symbol and language use, and the amount of reflection. In both classes when time was of minimal concern, the class pursued students' questions, developed students' modeling methods and notation, and consistently and frequently linked the mathematics and science contexts. When pressured by time to cover specific mathematical topics, the class pursued instructors' questions and methods and less frequently linked the mathematics and science contexts. Most students in both classes retained a process conception of mathematical modeling as they could apply the developed methods but relied on instructor prompts to relate the mathematics and science contexts.

The pictorial representations of the classroom environments illustrated that both classes had periods reflecting a constructivist inquiry environment. The graphs highlighted the classes' implementation of multiple cycles of inquiry, periods of consistency and inconsistency in connecting the mathematics and science, and intervals in which students' or instructor's ideas dominated discussion. Class observations suggested that the pictures lacked clarity in identifying periods of agreement or disagreement of the resonating concepts of students and instructors. Recommendations are made for future examination and representation of inquiry environments.

CHAPTER 1

INTRODUCTION

National organizations together with mathematics and science education researchers have called for the implementation of inquiry at all levels of education. The National Research Council (1996) urged colleges and universities across America to take action in improving the science, mathematics, and engineering education undergraduate students receive. One component of their call for improvement encouraged programs in which "all students have access to supportive, excellent undergraduate education in science, mathematics, engineering, and technology, and all students learn these subjects by direct experience with the methods and processes of inquiry" (NRC, 1996, p. 4). The National Science Foundation (NSF) reiterated this goal of inquiry in undergraduate education by imploring the opportunity for every student to be involved in inquiry and not just a "hands-on" experience (NSF, 1996). The National Council of Teachers of Mathematics in the Principles and Standards for School Mathematics (2000) encouraged the use of inquiry in mathematics classrooms since the processes of science can inspire an approach to solving mathematical problems.

The Standards-based reform efforts of both the mathematics and science education communities imply that all students need to learn more. Learning means more than to be shown or memorize or repeat (Romberg & Collins, 2000). "Learning involves asking, investigating, formulating, representing, and using strategies to solve problems, and then reflecting on the mathematics and the science being used" (p. 82). Revising the curriculum and class environment for "learning" to take place suggests that classrooms need to become settings in which hypotheses are made, arguments are presented, strategies are discussed, and discourse is exchanged regularly. Romberg and Collins call for research on the effects of reform in classroom environments supporting these characteristics.

The goals of this research study are to explore the implementation of an inquiry approach in a college mathematics course, represent the inquiry environment both qualitatively and quantitatively, explore what advances and what hinders the development of the inquiry environment in light of the instructors' goals, and suggest implications for future studies of inquiry in mathematics classrooms. This study will not and cannot prove that an inquiry oriented approach in the college mathematics setting provides the best learning environment for a college mathematics course (Hiebert, 1999; Kilpatrick & Silver, 2000; Schoenfeld, 2000). Instead, descriptions of the positive and negative consequences of the inquiry process pursued will be given with recommendations for those intending to implement inquiry in their classroom environments.

The Inquiry Process

Process Component

Inquiry in the mathematics or science classroom pertains both to the organizing structure or process as well as the philosophy behind the various stages. As a process, inquiry is a "multifaceted activity" involving the presentation of questions, assessment of prior knowledge, investigations to gather data to address the questions, interpretation and analysis of the data, communication and explanation of the results, refinement of previous questions, and consideration of new questions (National Research Council, 2000). Placed into four main stages, the process of inquiry displayed in Figure 1.1 consists of the prediction, experiment, analysis, and reflection stages.

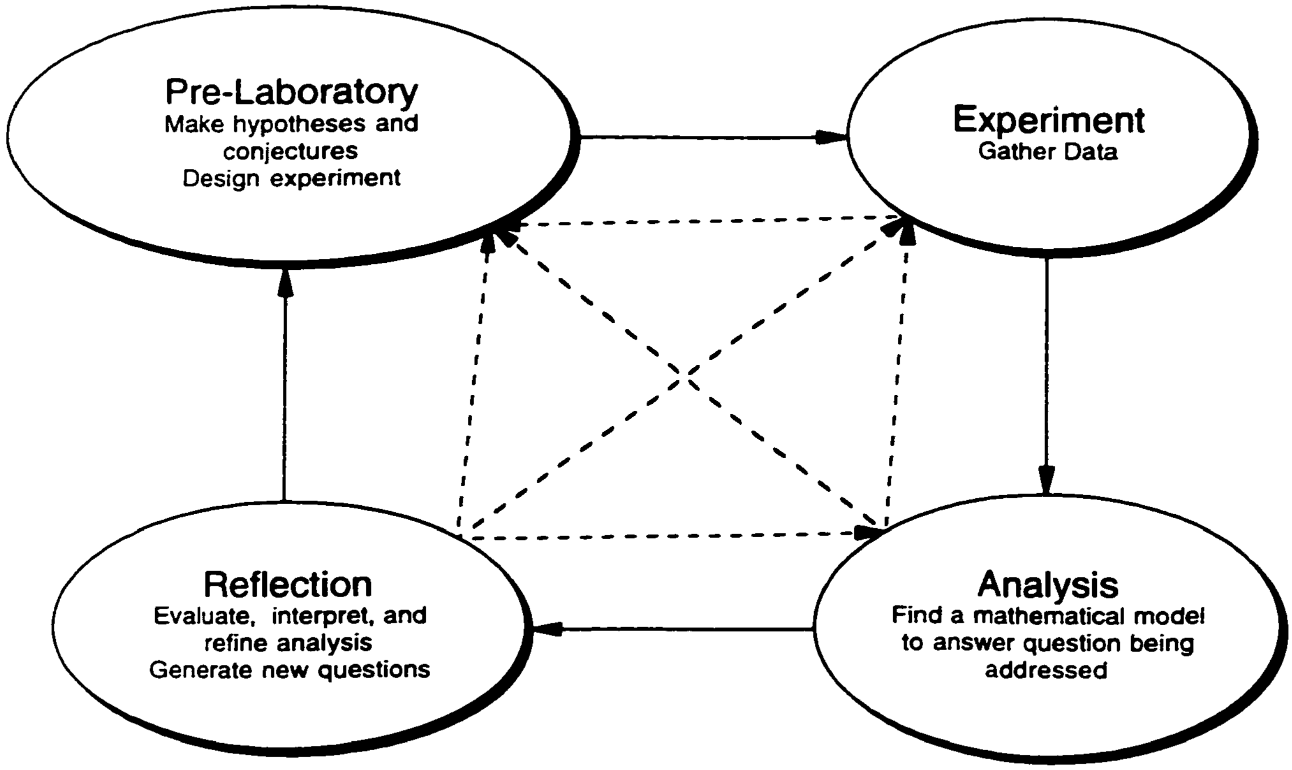

Figure 1.1. The inquiry process

As illustrated in Figure 1.1, the inquiry process may be implemented as a cycle. When a new problem or question is launched, the prediction and discussion stage provides students with the opportunity to make hypotheses about the concept or phenomena based on the problem situation and their prior knowledge. In addition, students plan an experiment to test their hypotheses. The experiment stage provides students with an opportunity to gather data to test their hypotheses. The data are analyzed mathematically to explain the phenomena and promote students' understanding of the mathematics embedded in the analysis. Students then elaborate on the procedures and generalize the mathematics. During the reflection stage, students assess what procedures have been implemented, how the original question or problem was answered, and how the procedures can be applied to other problems. New questions raised may be investigated in additional cycles of inquiry.

Philosophical Component

The philosophical component to inquiry emphasizes the behaviors, frames of mind, and attitudes behind the instructional strategies and process of inquiry in the classroom. The instructional strategies emphasize a student-centered environment in which students interact with the mathematics, the instructor, and other students. Stressed with student-centered techniques are issues of student-ownership of the problems and solutions. Hiebert et al. (1997) stress students' ownership of the mathematics indicating that a student must be challenged by a problem and must want to know the answer. The student must set a goal of resolving the problem. The goal might come from the student or be adopted by the student after listening to peers or the instructor. Due to the complexity of the relationship between teacher and learner (and between two learners) the mere transfer of a goal from one to another by command is simply not possible (Ernest, 1991). As Hiebert et al. state, the learner must adopt the goal.

To accomplish this goal setting and resolution, Zevin (1973) elaborates that the inquiry classroom should be characterized by behaviors stressing the extensive use of student ideas, questioning by both students and teachers with an emphasis on higher cognitive level questions, and wide and frequent student participation. Lecturing and recitation should be kept to a minimum in an inquiry environment, and students should raise varied ideas and questions representing the conceptual frames each holds in consciousness. As the inquiry process develops and evolves with reflection, students' questions should be resolved and their goals met.

The National Research Council (NRC) and National Council of Teachers of Mathematics (NCTM) identified several of the same components of the pedagogy involved in students' learning in an inquiry environment. The NRC (2000) classified essential features of classroom inquiry. These essential features include the learner's engagement of scientifically oriented questions, the priority to evidence in responding to questions, the formulation of explanations from evidence, the connection of explanations to scientific knowledge, and the communication and justification of explanations.

In their Principles and Standards for School Mathematics, NCTM (2000) proposed that students' understanding of mathematical ideas be built as they engage in tasks and experiences designed to deepen and connect their knowledge. The Standards are reflective of characteristics of an inquiry environment. NCTM calls for students' predictions and hypotheses with the students' development of questions, particularly when working with data. Students should "formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them" (p. 49). Analysis is encouraged as students should "represent and analyze mathematical situations and structures using algebraic symbols" (p. 38). As students analyze their mathematics, they should "monitor and reflect on the process of mathematical problem solving" (p. 54) and "develop and evaluate inferences and predictions" (p. 50). The Principles recommended by the NCTM encourage instructors to promote an environment of inquiry with student discussion and collaboration, student justification, and emphasis on developing mathematical argument, reasoning, and reflection. A classroom reflective of the NCTM Standards mirrors an inquiry environment.

Why Inquiry?

The choice to pursue the investigation of inquiry in mathematics education was founded primarily on three components. First, teaching a calculus class for life-sciences students led to the search for methods of instruction which accommodate students' constructions of mathematics. The course implemented data collection, and the question arose of how the methods scientists used in data collection could be applied in mathematics instruction. Secondly, the inquiry process is a potential application of constructivist learning theory. The association between the constructivist learning theory and the inquiry process opened exploration of how the mathematics and science components could be linked to promote symbolic reasoning, to support students' connections between the mathematics and science contexts, to encourage the resonating of compatible conceptual structures in students and instructor, to incorporate reflection, and to build community in the classroom. Thirdly, the realization was made that several claims are made by researchers and instructors of the effects of an inquiry environment or Standards-based classroom. The need for pictorial representations to compare and contrast the various environments could help to substantiate the claims.

The Need to Represent Inquiry

With the various calls for inquiry by national organizations and education researchers, different forms and levels of inquiry occur. In some classes, the procedural components may be emphasized with little attention given to the philosophical components. Or the philosophical components of inquiry may be stressed with execution of different structural components. Measures of different factors of inquiry are needed to illustrate the varying formats. Once constructed, the measures may be used to reflect the similarities and differences between the inquiry environments of different classrooms.

Constructed measures of inquiry could substantiate claims of a Standards-based or inquiry classroom. Lubienski's (2000) success in accomplishing a Standards-based classroom was judged through surveys and interviews with students. For example, a majority of her class of middle school students chose "Our teacher encourages us to figure it out for ourselves," with no student selecting "Our teacher tells us the answer." Huntley, Rasmussen, Villarubi, Sangtong, and Fey (2000) conducted interviews with high school instructors to suggest whether the classes were Standards-based or not. While surveys and interviews provide insight of participants' perspectives of the classroom environment, a better method of detecting differences between classes regarding inquiry might be to gather data while the class was in session.

With data gathered in the classroom, various attributes of inquiry could be identified and coded. The codes could be graphed to provide pictorial representations reflecting the form and degree to which inquiry was achieved. The representative pictures may then be examined to highlight similarities and differences in process and pedagogy. This study sought to develop methods to reflect the forms of inquiry which occurred in mathematics classrooms.

Constructivist Nature of Inquiry

The forms of inquiry vary according to the degrees of implementation of the procedural and philosophical components. In a similar fashion, numerous instructional strategies are applications of the constructivist learning theory. Future reference to "inquiry" in this study refers to the intersection of the set of inquiry methods with the set of

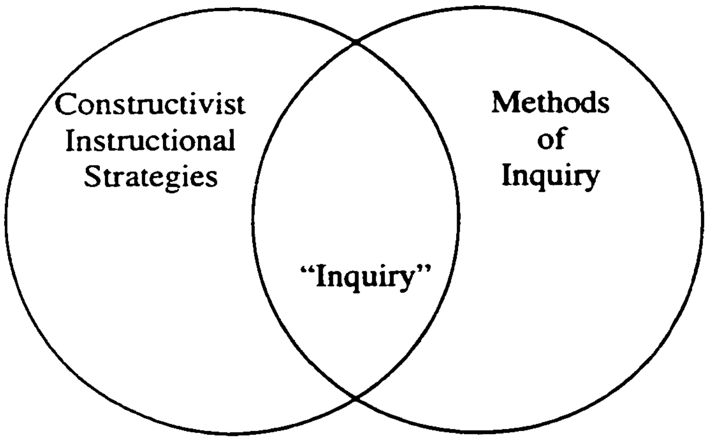

Figure 1.2. Use of the term "inquiry" for this study

constructivist instructional strategies. (See Figure 1.2.) Further discussion of the constructivist learning theory is now given in light of inquiry.

The constructivist learning theory maintains that an individual's knowledge is built upon existing knowledge. Learners must construct their own knowledge, individually and collectively (Davis, Maher, & Noddings, 1990). The constructivist learning theory emphasizes that concept formation is not a mechanical, passive process. Instead, a concept emerges and takes shape through the process of a complex operation aimed at the solution of some problem or question which resonates in the student (Vygotsky, 1934/1986).

Students construct their knowledge by interacting with others and making sense of their experiences in terms of their existing knowledge (Tobin, Tippins, & Gallard, 1994). This construction is a development of cognitive conceptual structures which does not proceed from a series of observations of contingent events or objects (Noddings, 1990). Instead, the construction is a response to challenges to refine or revise what is already known in order to cope with new situations (Smith, 1994).

When challenged with new concepts, students filter and interpret new information in terms of their existing knowledge and then assimilate the concept into an appropriate schema or conceptual structure (Skemp, 1987). Assimilation is not merely an absorption of the new information but a connection or appendage to existing knowledge. Hence, students cannot assimilate new information that is completely unfamiliar. The gap between the formal instruction of new information and a student's existing knowledge prevents assimilation. For this reason, instructors cannot transfer their knowledge to students in developed, orderly pieces of information since knowledge is built based on students' experiences. In light of assimilation, the purpose of the prediction phase in inquiry is to help students recall prior information related to the context and to make hypotheses about a new event.

The community in the classroom facilitates the assimilation of new information. The interactions students have with their instructor and their peers enhance the construction and inquiry processes. During interaction with others, students verbalize their thinking, explain or justify their solutions, and ask for clarifications. Attempts to resolve conflicts during the interactions provide opportunities for students to reformulate a problem and enlarge their conceptual framework of alternative solution methods (Koehler & Grouws, 1992). During the inquiry process students' predictions, data collection, analysis, and reflection improve as they first discuss their ideas with others before presenting the ideas to the class. Class examination of the predictions, analysis, and reflections highlights students' strengths and weakness of justifications.

Aided by interactions with others, an individual's construction of knowledge is advanced through reflection. In mathematics the reflective process is essential as the construct becomes the object of examination (Confrey, 1990). Reflection occurs in various ways such as mentally imagining the reversal of an action or solution process, explaining or justifying a solution method to others (Skemp, 1987), or self-questioning (Confrey, 1990) such as "What criteria allow me to make this step?" or "What am I doing and why am I doing it?" Reflection strengthens conceptual construction and promotes students' personal autonomy as they develop in their problem solving skills (Confrey, 1990 & Skemp, 1987).

Constructivism is evident in the various stages of the inquiry process. During the inquiry process, students actively engage in methods, interact with others, and reflect on the procedures used. Evident in the prediction stage, students' prior knowledge is brought to the forefront as students make hypotheses based on their existing knowledge and previous experiences. New knowledge is potentially added to the prior knowledge as students pose a question to be answered, gather pertinent data, and analyze the data. As students communicate their interpretations and reflect on the results, opportunity exists for new concepts to be interiorized and links between previously existing concepts to be strengthened, added, rearranged, or removed. This process of interiorizing and operating on one's own mental objects potentially develops students' cognitive conceptions.

Resonance

In the process of building onto existing conceptual structures, the components of a particular conceptual schema is activated by various input on which new concepts can be constructed. Resonance is the consciousness of a particular concept. Conle (1996) defines resonance "as a way of seeing one experience in terms of another" (p. 299). Skemp (1987) describes resonance as the activation of a schema, a conceptual structure stored in memory. "For each new situation we encounter, usually an appropriate schema is activated, and within this schema, the relevant concepts" (p. 118). Similar to the resonance used by radios and televisions to become sensitive to the frequency of particular wavelengths, resonance offers a model for the selective retrieval of a concept or schema into consciousness. In different people, different conceptual structures may be activated by the same input. In addition, at different times the same input can activate different structures in the same person leading to different interpretations. One goal in mathematics education is for the learners to develop compatible conceptual structures containing appropriate links between related concepts. When pertinent concepts are suitably linked and different individuals share similar conceptual structures, agreement in understanding is suggested. In light of construction, the need for compatible conceptual structures to resonate in individuals and a class of students is stressed.

With the importance placed on compatible conceptual structures resonating with students, methods to promote appropriate resonance are favored. Skemp (1987) acknowledges the importance of instructional methods in promoting conceptual construction through shared resonance:

> The mode of thinking available is partly a function of the degree to which the concepts have been developed...(learners) are largely dependent on the way material is presented to them. If the new concepts encountered are too far removed from any of their existing schemas, they may be unable to assimilate them, particularly if reconstruction (of an existing schema) is required. (p. 44)

Stated differently, schemas act as an attractor for incoming information. Sensory input is structured, interpreted, and understood in terms of whichever resonant structure is activated.

An example to illustrate what is meant by a resonating conceptual structure is appropriate. Davis and Vinner (1986) use the following example to illustrate compartmentalization of knowledge. The example is used here to demonstrate the activation or resonance of one conceptual structure over another:

Teacher:	How much is seven times seven?
Student:	14.
Teacher:	How much is seven *plus* seven? [italics included by Davis and Vinner]
Student:	Oh! It should be 49. (p. 284).

The initial conceptual structure resonating in the student's understanding is an "adding" schema. When the teacher asked the second question, the student's multiplication schema was activated and the correct answer to the first question was given. Desired during instruction is the activation of similar schemas across all students.

As an instructor, knowing the conceptual schemas resonating in students is impossible as conceptual schemas are internal objects. Some of what is internal is evident through students' statements, questions, symbolic manipulations, and other written work. When indications are given of a few students' resonating concepts, the instructor can easily make incorrect assumptions about all students based on the few students. Since resonance is an individual act, the resonance of one student is not necessarily a good indication of the resonance of other students in the class. Timing also is a consideration. Separated by one class session to the next, the instructor may not be able to pick up right where she ended assuming that the concepts that were resonating the previous class session continue to resonate or are ready to be activated again. Compatible resonance cannot necessarily be created between teacher and student or between student and student by a mere remark. Instead, interactive communication is needed in which each party reflexively influences each others' interpretations and actions (Cobb, 1991).

In light of the discussion about resonance, instructors need to know as much as possible about the students' ideas to understand students' thinking and to help students further develop their thinking (Maher & Davis, 1990). Attempts should be made to interpret what the students are doing and why they are doing it. This is achieved by observing and analyzing the students' constructions as they investigate solutions to problems. This knowledge of students' thinking allows for instructors to challenge, extend, and/or modify students' thinking.

Modification of students' resonance may be achieved through a process of communication and reflection on behalf of both instructors and students. As students communicate with the instructor and other students by sharing predictions, hypotheses, and questions, the concepts resonating in students' understanding are evident. Communicating to one another those concepts and reflecting on what others said promote the need to formulate common questions or problems to be resolved and the methods used to answer the questions.

Given the discussion of resonance and the encouragement for agreement in the resonating concepts for class participants, implementing the inquiry process seems to be an appropriate instructional method. In the inquiry process, emphasis is placed on building on students' prior knowledge from the start of the prediction phase with experimenting, analysis, and reflection enabling the construction of students' conceptual structures.

In this study, interactions in two collegiate mathematics classes will be examined to determine if and how agreement in resonating concepts was achieved. Attempts will be made to graphically represent the classroom environments to tag the occasions when agreement in resonance is or is not suggested.

Conceptual and Contextual Connections

Building conceptual constructions relies on the linking of appropriate concepts. Mathematics makes more sense and may be applied more easily when new knowledge is connected to existing knowledge in meaningful ways (Schoenfeld, 1988). Hiebert and

Carpenter (1992) highlighted the need for students to make multiple connections in learning mathematics. Understanding is present when a concept is linked to other concepts in multiple ways. This understanding promotes better retention of what is learned, reduces the amount that must be remembered, and enhances the transfer of knowledge as the concepts are well-connected to other concepts.

While connections between multiple representations within the mathematics promotes understanding, connecting the mathematics to other contexts, such as science, lends additional links between students' concepts. Methods of pursuing additional connections relating the mathematics and science contexts include mathematical modeling and linking symbols to science context.

Students do not often establish meaning for the symbols they use (Hiebert & Lefevre, 1986). Yet, symbols are the foundation for mathematical competence as the symbols act as handles with which to manipulate mental representations or concepts (Skemp, 1987). Symbols which make sense are more likely to be recalled and are more likely to activate the appropriate conceptual structures. Hiebert and Lefevre (1986) added that "it is easier to remember things that make sense, that are meaningful, (and) that are understood by their users" (p. 10-11). Thus, instructional methods which promote learners' sense-making ability of symbols are favored.

The inquiry process potentially promotes the connections between symbols and non-math contexts. In the analysis stage, students can create symbols to meaningfully represent the data and manipulations acted on the data. Students' meaning attached to the symbols advance future reflection and understanding of the mathematical relationships.

Mathematical modeling of data offers connections between the real-world context and mathematics as provisions are made for symbol association with context. The National Council of Teachers of Mathematics (2000) stresses the need for students to recognize and apply mathematics in contexts outside of mathematics. Provided in mathematical modeling, the students "use representations to model and interpret physical, social, and mathematical phenomena" (p. 70). Mathematical modeling also offers students opportunities to explain changes, to find regularities among changes, to develop tools in working with functions, and to apply their knowledge of functions (Sierpinska, 1992).

The development of a mathematical model need not be a linear process from the applied situation to the model and back to the situation. Huntley et al. (2000) emphasize the potential for modeling problems to provide a constant interplay between the applied problem situation and the mathematical representation. This constant interplay would suggest the strengthening of connections between the applied situation and mathematics, in turn strengthening understanding.

Lesh and Doerr (2000) discussed the implementation of mathematical modeling and the need to focus on the big ideas or main constructs and conceptual systems which are fundamental to the mathematics and science curriculum. Similar to the potential for interplay identified by Huntley et al. (2000), Lesh and Doerr saw the opportunity for rich real-life situations to be modeled in the mathematics classroom:

> For traditional textbook word problems, the problematic aspects tend to involve trying to make meaning out of symbolically stated questions; however when students use mathematics in real-life situations, the processes that are needed tend to emphasize the need to make symbolic descriptions of situations that are already meaningful. (p. 367).

The meaning and richness comes when the model is interpreted, constructed, modified, refined, or extended. Lesh and Doerr in turn, call for more research to be done on "the design of effective instructional sequences for learners" (p. 382).

Fennema, Sowder, and Carpenter (1999) emphasize the need for students to engage in tasks which stress problem situations. Problem-solving tasks in the mathematics classroom represent mathematics "worth learning" (p. 187) and promote students' engagement in the mental activities necessary to develop understanding. Engagement in mathematical modeling tasks with emphasis on the connections between symbols and context creates the problem-solving environment desired. Coupled with an inquiry process, the mathematical modeling classroom environment may promote students' conceptual constructions, foster links between multiple mathematics and contextualized concepts, and encourage agreement in the concepts resonating in students and instructor.

The Research Study

This study seeks to describe the use of inquiry and mathematical modeling to provide a rich environment for students to engage in mathematics with links to science contexts. In addition, means of representing the inquiry environment quantitatively will be pursued. In representing the components of inquiry, identification of the occurrences of agreement in resonance between instructor and students is desired. Pursuit of the representation will be characterized by the usefulness and limitations of tagging the following characteristics: the structure of inquiry, the sources of the goals and ideas, the roles of the contexts in the modeling activities, and the degrees of reflection. More information about these characteristics are provided in Chapters 2 and 3.

CHAPTER 2

REVIEW OF THE LITERATURE

The pursuit of inquiry in mathematics and science classrooms emphasizing diverse processes and philosophical components have been examined to varying degrees. While some studies highlight the process of inquiry, others address the philosophical and pedagogical issues. Those studies which directly inform the process of inquiry, the instructional practices, the implementation of mathematical modeling, the classroom interactions, the development of student understanding, and the construction of codes are reviewed.

Inquiry

Inquiry occurred in various forms in the research studies examined. Studies in the sciences implemented an inquiry process in which students completed a cycle of prediction, experiment, analysis, and development and application of theory. Inquiry studies in mathematics included students' exploration of open-ended questions, students' completion of projects, individual students meeting with an instructor to explore topics of interest, and students' attempts at mathematical modeling. The review of the following inquiry-related studies inform the structure of the inquiry process and students' development and reasoning through inquiry and mathematical modeling.

Inquiry in Science

Investigations of inquiry in science classes emphasized different components of the inquiry process. Some studies emphasized the structure of inquiry with prediction, data collection, and analysis phases. Others explored differences between open and closed formats for the laboratories. Those studies examining students' investigations indicated the use of instructor's scaffolding, a gradual decline in the instructor's support as students completed their investigations, with multiple cycles of the inquiry process. In the open-inquiry investigations negotiation of problem or task was recommended. Studies of the inquiry process in which more restrictions were placed on students' explorations illustrated a lesser degree of students' higher order thinking skills. In each of the studies described, inquiry was encouraged to develop students' thinking and reasoning skills with at least one study demonstrating benefits lasting beyond the inquiry-oriented course.

Cycles of Inquiry

White and Frederiksen (1998) implemented an inquiry approach in a middle school science curriculum. Pretest and posttest data measuring students' inquiry skills, physics knowledge, and attitudes about science were collected. Video recordings of class sessions and interviews with some students at the end of the curriculum were made. Students participated in scaffolded environments to learn about inquiry as they engaged in authentic scientific research of Newtonian physics. Students formulated questions, made hypotheses, conducted experiments using computer simulations, analyzed the data, and applied their "laws" and models to various situations. Application of the laws and models required reflection on the limitations of what they had learned and where additional knowledge was needed. Students evaluated their own and each other's work. Additional cycles were completed following reflection to address new questions. Each time the inquiry cycle was repeated some scaffolding was removed, so that eventually students conducted inquiry on their own questions. In addition, the complexity of the physics increased with multiple cycles.

The results of implementing the curriculum indicated that students incorporating reflective assessment had higher quality performance on project work than students in a control class. Pretest and posttest results indicated that students improved their scores on an inquiry test to examine students' abilities to develop hypotheses, experiments, data, and analysis when given a research question. The curriculum was effective in developing students' conceptual models for force and motion as students outperformed high school physics students on basic Newtonian physics problems.

In general, White and Fredericksen's study emphasized the process of inquiry in science investigation. Students became increasingly independent in addressing their investigative questions as scaffolding to assist students was removed. Other science investigations allowed students freedom in investigating their research questions. Roth (1993, 1995) demonstrated the implementation of inquiry in high school physics classes with emphasis placed on mathematical connections.

Open-Inquiry

Roth (1993) conducted a two-year interpretive research study in which students in his two-year high school physics class conducted primarily student-designed laboratories. Data consisted of video-taped science lessons and laboratories, students' lab reports, teacher's observations, notes, and recollections, students' essays on the nature of knowledge and learning, questionnaire data, interviews regarding the nature of knowledge of students, students' views on various aspects of learning physics, academic records of students, and results of two surveys administered to classify students' perception of the classroom environment and preferences for classroom environment.

Roth (1993) gave a case study of one student, Michael, who had struggled in his previous mathematics and science experiences. The students had been asked to investigate non-uniformly accelerated motion. Using a photo gate and a data collection program, Michael and his lab partner collected data of the motion of a cart pinned between two springs. Michael produced distance-time, velocity-time, and acceleration-time tables and graphs. Having difficulty interpreting the graphs in terms of the data, Michael returned to the lab outside of class to repeatedly run the experiment and construct understanding of the connections between the motion and the graphical and numerical representations. He later submitted a report describing his analysis of the connections between the graphs, function, and data.

Michael's report illustrated his ability to shift between representations and within representations. He demonstrated understanding of differentiation and integration through the physics applications before the topics were treated in his mathematics class. "Michael's involvement in a meaningful activity itself provided for the motivation necessary for its completion" (p. 120). His work evolved from a problem which the students framed for themselves in the context of a task to study non-uniform motion. "Michael made the problem his own,...he made the task problematic and constructed an understanding in his own terms beginning with what he knew" (p. 120). The inquiry environment provided the opportunity for Michael and other students to frame problems and construct understanding individually while working with group members.

In discussing the same study from various perspectives of inquiry, Roth (1995) described the questions his physics students investigated and the effects on class discussion. Students in their groups of two or three each framed a question they desired to investigate within a particular context. In one laboratory, students were to frame a question and investigation of the relationship between mass and acceleration in free fall. Students generally posed three different types of experiments which led to significantly different results: "A constant acceleration of about 10 m/s^2, a constant acceleration but less than 10 m/s^2, usually around 2 m/s^2 and a curvilinear relationship with an asymptote for large masses of about 10 m/s^2 and an intercept at 0 m/s^2" (p. 114-115). Students presented their results to the class and then discussed and defended their interpretations of the problem. Students focused discussion on the various uses of the term "free fall" and implications for the presented results. The class eventually came to consensus with the experiment and students' results. The discussion led to the next experiment in which students explored how the acceleration on an inclined airtrack changed with the angle of incline. Through class discussions of the presentations and students' defense of their positions and interpretations, students' understanding was advanced rather than hindered by the range of questions investigated. Roth emphasized that group members have to negotiate the nature of the problem which they are trying to resolve.

Roth (1995) described a similar study surrounding the need for shared understanding of the task in an inquiry environment. Students in an eighth grade biology class were to conduct investigations of classification schemes. In one described scenario, the teacher told students to

> come up with at least eight different classification zones...That means I want you to find the areas of the campus that are most similar to each other and indicate on the map where they are and then put a label to define that area and write out a description for it (p. 132).

The teacher thought his instructions were clear for students. Instead, the students constructed their own problems as they interacted with the teacher, other students, and the physical environment. Most students found exactly eight different categories and chose eight locations on campus and created eight different classifications. Some students chose the eight ball fields to demarcate their different classification zones, while others created new boundaries. Various responses grew from a single stated problem, illustrating that teachers' assumption that the clarity of the problem and that all students are working on the "same problem" can be questionable.

Roth (1993) called for research of the construction of meaning and community in classroom settings incorporating science and mathematics laboratories. In forming this community, additional evidence is needed to suggest implications in students' understanding when negotiation of the problem is and is not made.

The studies reviewed to this point have highlighted inquiry in environments where students were allowed a measure of freedom in their investigations, whether the process was initially scaffolded or whether students investigated the questions they posed. When in open laboratories, students developed in the content knowledge, inquiry process skills, and ability to make connections within the content and between the content and other contexts. Encouraged was the negotiation of the problem and task. Additional research suggests the types of thinking students engage when in closed or open labs as well as long-term effects of inquiry in the classroom.

Open vs. Closed Inquiry Labs

Related to the issue of negotiation of task or problem, Shepardson (1997) completed a comparative study of the types of thinking of twelve eighth-grade students in a life sciences class during laboratories. Observation data was gathered as six students conducted five confirmation laboratories and six other students conducted five open-inquiry laboratory activities. Designed by a textbook publisher and completed in groups of four, the confirmation laboratories were activities in which the research question, design, data collection techniques, and evaluation processes were given to the students to implement. The open-inquiry laboratories required students to form the research questions, design the experiment and data collection and evaluation techniques. Six observers were each matched with one of six students to gather data for five seconds every 30 seconds from the start to the finish of the laboratory. Observations were noted according to three components: laboratory activity structure, interactions, and thinking process. The laboratory activity component referred to the phase of inquiry whether the research question, research design, data gathering, data analysis, or evaluation. Interactions were classified as student-student, teacher-student, or self if the student was engaged in activity but the actions were not in response to or initiated toward others. Thinking processes were classified as to whether students' actions were focused on the questions or goals, information gathering, remembering, organizing, analyzing, generating, integrating, or evaluation.

In both open and closed laboratories, the frequency of students' thinking predominantly focused on information gathering. Based on multiple regression analysis, students in the confirmation laboratories often focused on the techniques to conduct the laboratory and interpret the instructions. In the open-inquiry labs, students engaged in thinking about the data and information gathered while interpreting and evaluating the results. Students in the confirmation labs did not relate their procedures and products. In the open-inquiry laboratories, student-student interactions prompted more thinking processes than in confirmation laboratories, with emphasis placed on focusing, gathering and sharing information and data, and integrating. In the confirmation laboratory, teacher-student interactions contributed to students' thinking processes emphasizing students' analysis and generalizations. Shepardson concluded that open-inquiry laboratories are more likely than confirmation laboratories to promote students' scientific thinking.

Zollar (1999) examined the effects of implementing higher-order cognitive skills-oriented chemistry teaching and assessment strategies in a small college chemistry class and a large lecture university chemistry class. The instructor asked inquiry questions with active involvement of students through assignments, exams, and peer grading. Emphasis was placed on the capabilities of problem solving, decision making, critical thinking, reflection, and evaluative thinking. Judging performance on the higher-order cognitive skills-oriented exams, the inquiry oriented discussions in large and small classes were feasible and effective.

In general, the research suggests that inquiry investigations promote students' thinking and cognitive constructions. Open laboratories, in particular, promote students' discussion of the problem and interpretation and evaluation of the results. Confirmation laboratories also promote students' thinking, though the thinking is more often focused on interpreting the question and instructions. The next study reviewed suggests that inquiry has long-lasting effects on students' analysis skills.

Lasting Effects of Inquiry

Scott (1977) examined the role of students' participation in an inquiry program during sixth and seventh grade classes on students' analysis abilities in algebra and geometry in their high school careers. The inquiry group and comparison group, students who had not had any training in an inquiry program, were comparable in socioeconomic and ability level during their seventh grade year as measured by the California Test of Mental Maturity. Data gathered included assessments of students' analytical "conceptual" style as measured on the Sigel Cognitive Style Test, course grades for students during the first-year geometry and second-year algebra studies, completed questionnaires of their recollections from sixth-grade science classes, and completed questionnaires of students' reactions to the effect of inquiry on their performance in geometry and algebra classes.

In a component of the sixth grade inquiry program, students were to analyze why for two pieces of wood, the larger piece floated and the smaller piece sunk. Students were to ask questions of the teacher to understand the resolution of the unbalanced forces in the situation. Responding with only a "yes" or "no," the instructor encouraged students to ask analytical questions about the situation. Those students who had completed the inquiry program had significantly higher grades in high school geometry and algebra classes than those students who had not completed the inquiry program. However, Pearson correlations between the analytical cognitive style data and the mathematics categories suggested that analytical styles between the inquiry group and comparison group were not significantly different. The majority of students in the inquiry group recalled learning different ways to approach and interpret problems as high school students, though none recalled the actual problem-solving activities. The survey data indicated that students did not necessarily believe that their success or difficulty in high school mathematics resulted from their involvement in elementary mathematics programs. Students did attribute the use of the inquiry program to develop their problem-solving skills, particularly in geometry. Scott (1977) suggested that the implementation of inquiry skills has long lasting effects on students' mathematical and analytical skills.

In summary, the studies on inquiry in science classrooms suggest that inquiry promotes students' development of their thinking skills and construction of scientific knowledge. Students' beliefs of their analytical skills are affected on a long-term basis when students receive inquiry training. Open-inquiry investigations tend to better provide for the development of students' interpretative and evaluative skills, while confirmation labs promote students' thinking of the task. In addition, studies which investigated students' questions and techniques suggest that scaffolding and negotiation of the problem and task is necessary to promoting students' understanding. Additional research is needed to indicate the consequences of the role of negotiation as well as the effects of inquiry in college mathematics classrooms.

Inquiry in Mathematics

Studies investigating inquiry in mathematics on a classroom level were few. More frequent was examination of students' behaviors when interacting one-on-one or two-on-one with a researcher. Even in working with one or two students, researchers acknowledged that true inquiry was difficult to achieve due to a lack of the instructor's control of what was going to happen next, the uncertainty of running with the student's ideas, and the changed roles of instructor and students (Arcavi & Schoenfeld, 1992; Borasi, 1992). In the examination of inquiry with the low teacher-student ratios, success in developing students' understanding of the mathematical process was frequently accomplished.

Borasi (1992) met with two high school female students for three weeks. The junior-level students had missed several class sessions at their alternative high school, and accommodations were made to explore the topic of mathematical definitions with the researcher as teacher. Through the mini-course, the researcher gathered qualitative data in written and audio form and analyzed the interactions and students' development in the inquiry and mathematics process. Borasi asked students to define various topics such as circle, polygon, and purple, and then had students test their definitions. The students created various mathematical objects and tested the objects against their definitions. The students analyzed and reflected on their definitions making refinements to correct errors and accommodating specific properties.

An example of the inquiry pursued by the students occurred after the students had generated the definition of a circle. Borasi had described taxicab geometry to the students, and they reevaluated the definition of a circle in light of the discovery that the set of points a given distance from a point in the "city" produced a "diamond" using the new geometry. The task provided evidence of the students' increased "understanding of the complexity of mathematical definitions and their growing creativity and critical stance within a mathematical learning context" (p. 90). The students on their own felt the need to create a definition, set the task, proceed in finding a solution, and evaluate the results.

At the end of the course, Borasi asked the students to define some of the terms developed at the start of the course. Rather than pursue the process of developing the definitions, one student tried to recall the definitions created previously. Borasi explained that the student did not have a purpose, context, or need for producing the definition other than being in a "test" situation. Borasi reemphasized the need for students to have a goal in pursuing inquiry.

Borasi encourages the implementation of inquiry at a classroom level. The researcher acknowledged that the instructor must deal with difficult decisions in allowing students to pursue genuine inquiry. She admonished that students working in small cooperative groups on a genuine inquiry problem would be significantly different from students working on a well-defined task by the instructor. In a class, students and instructor would have to negotiate the sharing of tasks, the direction of inquiry, the monitoring and evaluation of inquiry, and criteria for when a task is complete.

As recommended by Borasi, a component of the inquiry process is the degree to which instructors are willing to pursue students' ideas and methods above methods and ideas they intended to develop in the course. Arcavi and Schoenfeld (1992) acknowledged and encouraged the constructivist approach of "running with the student's ideas" in mathematics education. Arcavi and Schoenfeld examined transcripts from a tutoring session with an eighth grade student in light of the approach to run with students' ideas. The student attempted to guess the tutor's linear relationship when the student gave various input and the tutor responded with a particular output. When the student tried to find the equation using a method different than the tutor had planned, the tutor resorted to planned methods of intervention to help the student calculate slope and intercept. Had the tutor run with the student's ideas, the researchers found that quality mathematics would have emerged.

The researchers acknowledged that running with students' ideas makes demands on instructors' mathematical knowledge, pedagogical and cognitive-analytic abilities, and communication skills as they help students build on their current knowledge. A problem with the approach of "running with the student's ideas" is the difficulty of implementing this in a class of thirty or more students. In the current study, analysis of the class' pursuit in running with the class' ideas will be given.

Battista (1999) completed a case-study of three pairs of students within a fifth grade class as they completed an inquiry activity enumerating arrays of three-dimensional cubes.

Students were given grid diagrams of boxes with various dimensions. They were to predict the number of cubes needed to fill each box, then check their predictions by constructing and filling the box with cubes. Students were to check the answer with their prediction before proceeding with the next box. Within each pair, students were to collaborate on the predictions and reflections. Most students initially counted the number of face cubes to (incorrectly) determine the number of cubes to fit in the box before proceeding to methods of counting the number of cubes per layer. Those who relied on enumeration by mentally layering cubes cycled through a series of structuring and counting layers of cubes followed by reflection and abstraction. Battista highlighted the use of students' predictions and collaborations in constructing their conceptual understanding. Predictions were essential in prompting later reflection of their mental models and enumeration schemes. Battista emphasized that a minimum standard for collaboration was for students to attempt to establish consensus for their problem solving. Students needed to make a commitment to communicate their own ideas and understand others' ideas. Battista acknowledged that cognitive conflict resulted more from the mismatch between predictions and students' answers than from interpersonal conflict when filling the boxes. Battista encouraged inquiry promoting students' predictions and group interaction.

Results from inquiry based studies with individual students informs the inquiry at the classroom level, in particular that students on an individual level can pursue mathematical inquiry. Concerning the process of inquiry, emphasis is placed on the use of predictions to prompt later reflection. Concerning the pedagogy and philosophy, instructors are encouraged to run with students' ideas as quality mathematics and understanding evolves. In running with students' ideas negotiation and collaboration on the goal task is encouraged. Unknown is how inquiry proceeds when implemented as a class pursuit. In addition, questions arise of how inquiry proceeds when different students pursue different questions or when a class agrees to explore a single issue. In the current study, examination of both the pursuit of inquiry as a class and the pursuit of different questions in a single class will be given.

Mathematical Modeling through Context Connections

Studies examining students' abilities to mathematically model generally consisted of three types of studies: those examining students' translating abilities, those examining students' abilities to connect context to mathematics through the curriculum, and those examining students' abilities to mathematically model data with reliance on contextual and functional understanding. The translation studies demonstrated undergraduate students' difficulties in relating the context with symbols and equations. When classrooms implemented mathematical modeling, students' abilities to relate context and mathematics improved. Studies examining students abilities to develop a model using computation tools indicated students' difficulty to fully translate the context situation into mathematics.

Mathematical Modeling Based on Contextual and Functional Understanding

Zbiek (1992, 1998) examined 13 prospective secondary teachers as they relied on computing tools to mathematically model data for given real world problems. Students were enrolled in a class taught by the researcher which emphasized mathematics while addressing issues of pedagogy and learning. During the first three weeks of a 15 week semester, students explored their understanding of functions through previously proposed models of real-world situations and computing tools. The next eight weeks focused on more abstract issues of function and proof. The last 4 weeks of the semester surrounded a modeling unit in which students collected data, generated function models using computing tools, and discussed the real-world aspects of four open-ended modeling activities. One of the four activities was an individual modeling task asking students to describe which of the variables were related for data from the 50 states and the District of Columbia. Audiotapes of individual interviews with students and class interactions, written work from the interviews and class, instructor's notes of students' nonverbal communication and computing tool use, and teacher reflection notes from class session comprised the data for the study.

Zbiek identified four different approaches students generally used to model their data. With each approach students gave different types of justification for their choice of model. Most students relied on a fitted function selector and chose a model based on the goodness-of-fit values. A substantial number of students interpreted the graphs of the data when using a potential function generator. Few students used the scatter plot with a graphing tool to create and interpret graphs while linking the graph and algebraic rule. No students relied on unneeded or unused tools to examine ratios or formulas and to compare data and model values. Zbiek encouraged additional research to explore conditions under which students rely on the four approaches. Additional research is needed to observe if students with different mathematics backgrounds rely on the same four approaches and how a class evolves while emphasizing modeling.

Lanier (1999) conducted a case study to investigate three college students' understanding of linear modeling. In the course, non-math majors were taught methods of modeling data using a spreadsheet template. Lanier observed class sessions, audio-taped three students as they used a linear modeling spreadsheet template, audio-taped semi-structured, exit interviews with the three students, and analyzed students' project reports. When modeling data, students rarely deviated from the procedures given to find a model for different data sets using the spreadsheet. The students seldom reasoned about the linear model based on the graphical representation, but used the average error calculations to determine which linear equation was best. Students were able to reason to some extent when a model was not entirely appropriate. In one example, a student reasoned that a linear model of a decreasing population was limited in predicting future population as a population was not likely to be zero, and definitely would never be negative.

Lanier commented on students' reasoning skills when interpreting the meaning of the slope and y-intercept in terms of the context. Two of the three students correctly reasoned that the y-intercept represented the initial population. The third student acknowledge that the y-intercept occurred when x was zero. When interpreting the slope, two interpreted the slope as the amount of growth in population per year, while the third could only demonstrate how to find slope without giving an interpretation: "It's not like a percent or anything. I don't know how it tells like how much it increases. I just know it tells that it increases" (p. 44).

Though the modeling class implemented methods to connect mathematics to the real world, the procedural components of modeling were emphasized. As a result, not all students fully interpreted the relationship between the context and the mathematics. Lanier's recommendations included the need for investigations of students' mathematical understanding of linear modeling in a course where less emphasis was placed on procedures and more emphasis was placed on interpretation and sense-making. This current study intends to further examine the role of mathematical modeling when the instructors attempt to build students' understanding through modeling with sense-making.

Mathematical Modeling Based on Curriculum

Few studies implemented mathematical modeling to the degree in which the building of models evolved into the development of the topics covered in the course. Instead, a number of studies emphasized curriculum in which students related the mathematics with a real-world problem. Often, these problems were classified as mathematical modeling, though the modeling was used differently than intended in this study. The results from these studies are pertinent since insight of students' connections between mathematics and real-world context is gained. Those studies founded on the curriculum emphasizing mathematical modeling displayed that students' abilities to relate the context and mathematics was higher for those students given specific instruction in mathematical modeling than students who were given more traditional instruction.

Standards-Based and Calculus Curriculums

Huntley, Rasmussen, Villarubbi, Sangtong, and Fey (2000) tested the effects of Standards-based mathematics education on students' abilities in algebra. Students in six U.S. high schools who had completed three years of mathematics under the Core-Plus Mathematics Project (CPMP), a Standards-based curriculum, were compared with students of comparable mathematics ability who had been taught in a traditional mathematics program at the same schools or at neighboring schools. Interviews with each of the CPMP and control teachers generated data about instructional practices, use of calculators, and assessment practices. Three assessments of students' abilities in algebra were conducted. One focused on contextualized problem solving, typical of the Standards-based curricula. A second emphasized context-free symbolic manipulations of algebraic expressions and solutions of equations and systems. A third assessment required group work on open-ended contextualized problems The assessments occurred during two 50 minute class sessions. Graphing or scientific calculators were allowed on the first and third assessments.

Results varied across the three assessments. Overall, students in the CPMP courses performed better than control students on questions requiring problem-solving skills such as translating problem conditions into symbolic expressions, solving equations, and interpreting results (type 1 assessment) and on problems requiring integration of the same skills on core complex modeling tasks (type 3 assessment). On the type 2 assessment, students taught in classes using traditional mathematics curriculum performed better than the CPMP students on the symbolic manipulation tasks. The study highlighted that students learn more about the topics, whether modeling or symbolic manipulations, emphasized in class and less about the topics less emphasized. In other words, the curricular or instructor goals determined the emphases of the course content which influenced students' performance on the different types of tasks. In particular, the researchers noted that "if students are asked frequently to formulate mathematical models for situations and to interpret results of algebraic calculations, they develop greater understanding of and skill in those processes" (p. 354). The researchers could not determine from their study whether context cues were a strength or a disability to students, suggesting that more research is needed to indicate whether context cues are to students' advantage or disadvantage. Additional research is needed to indicate when and how context is advantageous to students' problem-solving abilities. This study intends to examine how the inter-linking of context and mathematics may be used to help students reason mathematically to find a model for real-world situations.

Strickland (1999), assuming the role of teacher-as-researcher, completed a case study of two female students enrolled in his college Project CALC (calculus as a laboratory course) section. The data was gathered through video recorded interactions of the respondents during classroom and computer laboratory activities, one-on-one and group interviews with the instructor-researcher, the students' weekly journals of their experiences in the classroom and computer lab, and the researcher's personal journal. Strickland found that his students had difficulty relating the mathematical symbols with their meanings in spite of the reform Project CALC curriculum which contained many real-world problems. The researcher observed the students' difficulty differentiating between symbols and their definitions and their use in equations. The researcher acknowledged students' differences in meaning when compared to his own meaning of the symbols. The differences were particularly noted when related to the difference quotient and interpreting the difference quotient in terms of a graph and derivative. Based on his study, Strickland recommends shying

> away from using just one approach to teach. Lecture only courses can be very ineffective in facilitating a student' understanding of concepts. The same can be said of a course where only group activities are used, or a course where only a computer lab is employed. (p. 149)

Strickland neither recommends nor suggests the form the collegiate mathematics classroom environment should take, supporting the examination of whether an inquiry approach aids the development of the desired concepts and connections for students.

Based on the studies in which modeling and real-world problems were incorporated into the curriculum, students generally did better than comparison students at interpreting results of algebraic calculations. Additional research is needed, however, to indicate how the instructional approaches which accompany the curriculum advance students' sense-making abilities and connections between the mathematics and contexts. In this study, the inquiry approach will be investigated when accompanying the modeling curriculum in the course.

Realistic Mathematics Education

An instructional theory related to mathematical modeling, Realistic Mathematics Education (RME) is founded on the premise that mathematics is a human activity in which students should learn by mathematizing subject matter from realistic situations. A realistic situation could consist of contextualized problems inside or outside the context of mathematics, but what is experientially real depends on an individual's background and experiences. In mathematizing, students are provided with the opportunity to reinvent mathematics. As students engage in generalizing their own mathematical activity, the formal mathematics emerge. Rasmussen and King (2000) conducted a developmental research study to design and analyze instructional tasks in a sequence based on the Realistic Mathematics Education instructional design heuristic of guided reinvention.

In Rasmussen and King's study, twelve students in a differential equations course were observed and video recorded in class sessions with copies of students' written work made. A subset of the twelve participated in video-recorded interviews. Core learning activities were developed in which the instructor introduced a context, students worked collaboratively in small groups on the activities and then the whole class discussed the activities. Before being supplied with methods or algorithms to approach the problem, students were asked to predict the population of fish in a pond for consecutive months, given the equation $dP/dt = k*P(t)$ and initial fish populations with time measured in months. After students had been introduced to and discussed the problem, students were to reflect on their work and describe using words and symbols how to approximate the future number of fish in a pond with the differential equation $dP/dt = P(t)$.

The work of one group of students illustrated the reasoning pursued when given the realistic situations. The students initially demonstrated a continuous conception of the situation while interpreting the d*P*/d*t* as an average rate of change. Students then realized that piecewise linear segments were appropriate. When giving the procedure to approximate the number of fish, the group did not distinguish between rate of change and change in population. One group member gave the equation $P_{(initial)} + dP/dt \rightarrow P_{(initial)}$ while another group member interpreted the equation as "You take the current population and add to it the change relative to time to establish the new population, then continue the process" (p. 169). To help the group distinguish between the concept of rate of change and change in population, the instructor asked students to approximate the number of fish in the pond in half-month increments. The students resolved their dilemma and created a correct symbolic description for Euler's method: $P_{(initial)} + (X\frac{dP}{dt})_{\substack{Fraction \\ of\ the\ month}} \rightarrow P_{(new)}$. Interaction within the group, with the instructor, and with graphical technology prompted the change in thinking for the students. Once students had created their own informal Euler's method, more conventional and formal methods of Euler's method emerged and were grounded in students' informal activity.

The process used by Rasmussen and King to develop an instructional strategy with implementation relates to this study as a form of inquiry. Students made predictions, experimented with methods to predict populations, analyzed those procedures while mathematizing the process, and upon reflection, made accommodations to model the fish population at half-month intervals rather than full month intervals.

Modeling on Homework and Projects

Galbraith and Clatworthy (1990) incorporated modeling into a senior-level high school course as students studied trigonometry, calculus, matrices, vectors, and particle mechanics. The modeling process was marked by seven stages in which students specified the real problem, made assumptions in the model, formulated the mathematical problem, solved the mathematical problem, interpreted the solution, validated the model, and used the model to explain, predict, and make decisions about the real world problem. Most of students' modeling was completed outside the class sessions in group sessions as homework or project assignments. Positive assessment of the success of the program was determined based on students' written and video reports, audio recordings of individual interviews during students' work on the projects, students' diary records, students' responses on a questionnaire, and class discussion. Based on students' work and responses on the questionnaire and interview, students indicated that they considered the modeling framework interesting and important. Students grew in confidence in approaching the problems. Students were successful in modeling the real world problems and submitting reports giving their analysis.

The studies illustrate that when context and modeling are emphasized in the curriculum, students perform better on algebraic problems connecting the mathematics to the context. However, students have difficulty constructing a mathematical model built from the context and apart from technology and taught procedures. When students make first attempts at modeling the data, students are better able to accommodate mathematical methods emerging from their experiences. Additional study is needed to suggest how the implementation of the inquiry process benefits students' mathematical modeling skills and ability to construct a model when they have the context.

Translation Studies

Student-Professor Problem

A number of studies (Rosnick & Clement, 1980; Clement, Lochhead, & Monk, 1981; Rosnick, 1981; Wollman, 1983) have documented college students' translation errors and difficulty on problems of the sort "write an equation for the statement 'There are six times as many students as professors at this university.'" Rosnick and Clement (1980) gave individual instruction to six students who had made reversal errors. Students made initial corrections to their errors, but the evidence demonstrated that the change was behavioral with little change in students' conceptual understanding. Wollman (1983) illustrated that students were successful in working with the statement when they were asked to perform computations, were required to give justifications, or were prompted with questions to compare sentence and equation. In light of the results of Wollman's study and a resonance perspective, students' difficulty with translation problems likely stems from the resonance or activation of the wrong schema rather than problems in students' conception of algebraic variables.

Varying Forms of Abstraction

Related to the translation studies, White and Mitchelmore (1996) examined first semester calculus students' abilities on problems ranging on four levels of abstraction. Abstract-apart problems had no attached contextual meaning while abstract-general problems were framed within a context. For each of the four levels of abstraction, four items related to rates of change and maximization were written. Forty students, divided into parallel groups of ten, were given a four question test containing one problem of each context and one of each level of abstraction. Tests were given before, during, immediately after, and six weeks after a six week period of instruction by White. Each test taking session, students attempted a different version of the test. In addition, four students from each group of ten were selected to be interviewed within three days of each of the four written data collections. The interviews provided a better opportunity to identify students' reasoning on their written responses.

When students were given the problems in which concepts had to be symbolized, students performed less successfully. When the problems were completely abstract, students had greater success completing the problem. Analysis of students' errors suggested that students based decisions about which procedure to apply on the given symbols and ignored the meaning behind the symbols. Researchers identified the main inhibiting factor of success as an underdeveloped concept of variable. The researchers suggested that "modeling a given situation using algebra may represent an even higher level of abstraction than the several abstract-general concepts that may be invoked" (p. 92).

The translation studies indicated students' difficulties in relating the mathematics and context apart from prompts or additional instruction. While the difficulty may stem from the need for appropriate conceptual schemas to resonate while completing the translation problems, few students successfully related the mathematics and context on their own. Additional research is needed to indicate how implementation of inquiry in the mathematics classroom while building on students' prior knowledge may help students in associating the symbols with the context.

Studies examining mathematical modeling and the interaction between context and mathematics illustrate students' varied reactions in relating the context with mathematics. Students' in several curricular domains, including science, had difficulty translating sentences into mathematical equations without instruction or prompts. Additional studies demonstrated that when curriculum emphasizes contextualized problems and mathematical modeling, on real-world problems, students outperform those students who have traditional instruction emphasizing symbolic manipulations. The studies by Zbiek and Lanier demonstrated that when given the liberty to model data, students rarely translated the mathematics in terms of the context or vice versa. The mixture of results suggest that additional research is needed to illustrate how students rely on the context when curriculum articulates the development of the mathematics from the science context.

Classroom Interactions Surrounding Mathematics and Physical Contexts

When instruction does emphasize the context and mathematics, the interactions in the classroom influence the types of connections made relating the mathematics and context. Bromme and Steinbring (1994) conducted an exploratory study analyzing the interactions and the role of context in two sixth grade classrooms. The researchers emphasized the role of the instructor in the development of mathematical meaning over time and presented a graphical form of the classroom interaction and context data. To form the graphs, transcriptions from the two classes were coded based on who was speaking; whether a question asked, explanation given, or statement made; and the classification of the elements, whether symbol, object, or relation. A symbol referred to the mathematical notation used to represent the concept. Object indicated the physical objects involved in the discussion, and relation regarded the linking between object and symbol and the associations connected to each.

The results of Bromme and Steinbring's study indicated that the two classes interacted differently, particularly regarding the achievement of stability of the relational level of the concept. In the "expert's" class, the relational level was of approximately the same rank as the symbol and object level with "soft transitions" between the levels. The lessons in the "non-expert's" class never stabilized at the relational level, but contained large blocks of object and symbol levels with sudden switches. Analyzed across a second lesson, the graphs maintained more similarity within the lessons taught by a single instructor than across instructors.

Bromme and Steinbring observed differences in how the instructors communicated with their students. The "expert" teacher interacted with a "collective student" which was constituted from the contributions of multiple individuals. Hence, there was a joint presentation of the material. The "non-expert" teacher interacted with individual learners, suggesting the discourse of the lesson was divided into subtopics of individual students with no consistent dialogue referring to connected topics.

Bromme and Steinbring hypothesize that the differences between the classes stemmed from the instructors' different subjective attitudes toward mathematics. They expressed that the "non-expert" instructor emphasized the manipulation of mathematical symbols and the equality of the object and symbol levels. The "expert" teacher accentuated the relationship between the object level and the symbol level. Additional research is needed at the college level to illustrate how interactions relating symbol, object, and relation inform the instructional techniques. Bromme and Steinbring's study informs this study concerning the influence of the instructors' beliefs and communication on class discussion of context. In addition, the study suggests the coding for symbol, object, and relation to illustrate where emphasis is placed in class discussion.

The instructors' goals and instructional techniques often dictate the role of context or object in relating the mathematics to context. When reviewed in light of the other studies investigating mathematical modeling, instructors' choices in curriculum and leadership in classroom interactions affect students' abilities to model and relate context and mathematics. Instructors are responsible to promote an environment in which compatible schemas resonate in students in order provide appropriate accommodation regarding modeling and solving real-world problems. The current study will draw on the current studies to indicate how to implement inquiry and interpret the results as students are taught to rely on the context to construct a mathematical model of data.

Levels of Mathematical Conceptions

Instructors' goals and students' mathematical conceptions of modeling are classified in this study according to levels of understanding sought and attained. Various researchers have identified similar classification structures which inform this study. While the classification system "APOS" will be the primary structure referred in this study, other classifications directly relate to and inform the interactions which occur.

APOS

Asiala, Brown, Devries, Dubinsky, Mathews, and Thomas (1996) composed a framework for mental constructions for learning concepts in mathematics. Students' concepts develop from one level to the next through reflective abstraction, a general coordination or drawing of properties from mental or physical actions at a particular level of thought (Dubinsky, 1991). Classified on four levels and characterized throughout this paper as "APOS," the four levels of conception are action, process, object, and schema. An individual has an action conception when the concept is perceived as external to the individual. One can enact the transformation only by reacting to external cues that give detail to the steps to take. When an individual has repeated the action sufficiently to the degree that reflection on the action can occur, the concept may be interiorized into a process. A process is perceived as being internal and under one's control as the individual may reflect on, describe, or reverse the steps of the transformation without actually performing the steps. A concept becomes an object when the process is viewed as a totality and actions can be performed on the process. Eventually, objects and processes may be interconnected and structured to form a schema. The schemas may then be treated as objects.

Structural vs. Operational

Similar to APOS, Sfard (1991, 1992) developed the descriptions of operational and structural approaches to instruction. Structural instruction refers to the teaching of mathematics emphasizing the structure of mathematics such as the building of new concepts by definitions and manipulations according to certain rules. An operational approach to instruction develops the mathematics as a computational process rather than as a static construct. Operational instruction is similar to an action or process conception whereas the structural would be similar to the object level of conception. Sfard (1992) formulated two principles which guide the operational approach to instruction: (i) "new concepts should not be introduced in structural terms" and (ii) "a structural conception should not be required as long as the student can do without it" (p. 69). Sfard recommended instruction that first used the operational approach to develop new mathematical concepts. The operational approach would eventually lead to a structural approach as the basis was established for a higher level concept. The recommendation followed the notion that "when a new concept is to be learned, an ability to think about it as a process should be expected in the majority of the students before an ability to consider it as an object has been acquired" (p. 70).

As part of a course on algorithms and computability, Sfard (1992) taught the concept of function operationally to four groups of undergraduates, ranging in size of 16 to 20 students. Initially, the term function was used almost synonymously with algorithm and then explained as the name of a product of an algorithm. An initial structural characteristic, the set of all input-output pairs, associated the function with computation. Different methods of constructing functions from other functions were discussed with multiple representations of functions given as well. Eventually explicit remarks and questions on the nature of functions as opposed to algorithms were stressed in the course. Finally, the question of existence of a nonalgorithmic, noncomputable function was posed. Class observations indicated that students did function operationally and struggled with a transition to the structural conception. Full reification of the function concept did not occur as there was evidence that students could not cope with the proof that noncomputable functions exist. Responses on a questionnaire about characteristics of functions further supported most students' operational rather than structural conceptions. Sfard (1992) acknowledged that students may need additional time and motivation to abstract the function concept in addition to proper instruction.

Additional research (Vinner & Dreyfus, 1989; Sierpinska, 1992; Dubinsky & Harel, 1992; Carlson, 1998) indicates students' operational conception of functions and their difficulty with a structural conception of functions. Carlson (1998) conducted a study with three groups of students who had just received A's in college algebra, second semester calculus, or first-year graduate mathematics courses. Students completed a written examination of their concept of function. Follow up interviews were conducted with fifteen students, five from each group. Students who had just finished college algebra had difficulty explaining what was meant by the statement "express *s* as a function of *t*." One student explained that the terminology meant one should find where *s* and *t* were equal. Another thought the roots of an equation were to be found. Carlson summarized that these students "are unable to translate a verbal function description into algebraic function representation" (p. 133-4).

Knuth (2000) examined high school students' reliance on the algebraic or graphical representation of function when solving a problem. More than three-fourths of the 178 subjects chose an algebraic approach as their primary solution method on a task for which a graphical representation seemed more appropriate. Knuth reasoned that students do not develop ability to flexibly employ, select, and move between algebraic and graphical representations.

Tool vs. Object

Duoady (1991) suggests a tool/object model for analyzing interactions in a mathematics classroom and the instructor's representations of the mathematics and mathematical activities in the classroom. Duoady contrasted the role of mathematical phenomena as receiving a status of tool or a status of object. A concept is attributed the status of "tool" when emphasis is placed on its use in solving a problem. "A tool is involved in a specific context, by somebody, at a given time. A given tool may be adapted to several problems, several tools may be adapted to a given problem" (p. 115). A concept acquires the status of "object" when "considered in a cultural dimension, as a piece of knowledge independent of any context, of any person, which has a place in the body of the socially recognized scientific knowledge" (p. 116). Duoady elaborates the object status when describing the allowance for the structuring of knowledge and the extension of the body of knowledge. When the teacher presents the mathematics in a contextualized setting and organized in an inter-setting dynamic to give meaning, the mathematics is used as a tool and decontextualized to acquire the object status.

Levels of Mathematical Understanding

Much like the other levels of classification, the classification scheme described by Gravemeijer, Cobb, Bowers, and Whitenack (1999) identifies students' level of mathematical understanding. A situational level of understanding is one in which interpretations and solutions depend on the understanding of how to act in the setting. At the referential level of understanding models-of are grounded in students' understanding of the experientially real settings. In the general level, focus is placed on interpretations and solutions apart from situation-specific imagery. Formal level of understanding is characterized by formal use of conventional notation and symbolization.

Making use of these levels of understanding, Rasmussen (1999) described his developmental research study in which he analyzed students' learning while developing, modifying, and refining initial conjectures of possible paths that students' learning might take. To develop the understanding of Euler's method, students were presented with the problem of how to use a rate of change equation to approximate future number of species at different time intervals when given an initial population. Students were challenged to use a rate of change equation for a quantity to inform them about the quantity itself. One student described his initial difficulty of knowing what to do with the value returned when plugging the initial population into the differential equation. When the student realized that the result was a rate of change for the population he assumed that the rate was constant over the time interval and found that he could calculate a rate at the next point. The student essentially developed his own informal Euler method. Initially, the student had a situational level of understanding as he determined how he was to act once he had a value from the equation. With a referential level of understanding, the student would refer back to the discrete approximations by acting if the slope field indicated a rate of change at every conceivable point. Eventually, students discussed results of individuals or groups of students to produce a general formula or procedure. The conventional means of symbolizing Euler's method evolved from students' reasoning and mathematical modeling.

Illustrated by research on students' conceptions of functions and development of Euler's method, the classification schemes are used to indicate students' levels of understanding of various concepts. References to each of the classifications schemes of mathematical conceptions are made in this study when describing students' levels of understanding of various concepts in the course and when describing instructors' goals for students' conceptions.

Reflection

Researchers who developed and use the concept classification schemes attribute students' upward movement in the classifications through reflection. In addition, many of the studies examining inquiry and mathematical modeling emphasize the role of reflection for students to connect the mathematics to the context. In many studies reflection was enhanced through student interaction and cooperative groups. In other studies, reflection was a component of the heuristics taught and recommended for students' use.

Reflection through Anticipation of Questions

At least two studies acknowledged the use of instructional techniques including students' anticipation of questions to prompt reflection. With this technique, students reflected on their work in anticipation of the instructor's or classmates' questions about their methods.

Tanner and Jones (1994) completed an action research study in which specific instructional tasks were structured into the class to promote students' reflection while mathematically modeling. Mathematics students in eight secondary schools were audio-recorded while participating in modeling tasks. Approximately 100 lessons were observed in order to capture "the creation of a modeling culture in the classroom" (p. 13). Researchers actively participated in the classrooms, interacting with the students to assess their perceptions and strategies during the modeling tasks.

Three different teaching approaches were observed in the creation of the modeling environment. "Start-Stop-Go" (p. 422) allowed for each student to think about the situation, generate ideas, and discuss those ideas with others. In the approach students spent time silently reading and planning, small groups discussed various approaches, the whole class brainstormed, and students returned to small group planning. At various times, students reported back to the class. Students monitored their progress in anticipation of the report back sessions. During times of reporting back, "internalization of scientific argument" (p. 423) occurred as students learned how to address various questions, first from the instructor and then from classmates. The questions emphasized the methods students used to collect and analyze the data. Again, students learned to anticipate the questions and developed skills to "argue with themselves" (p. 423) in order to prepare. Following the tasks, "encouraging reflection" occurred as students individually wrote and presented final reports to the class. In addition, students were to address the question, "If I were to do this investigation again, what would I do differently?" (p. 423-4). Students also were encouraged to complete self-assessments using frameworks provided by the researchers. The assessments validated the teacher's informal assessments and provided another opportunity for students to reflect on their work. Unclear in this study were the times at which the various types of assessments occurred and students developed in their ability to mathematically model due to the reflection.

Cobb, Wood, Yackel, and McNeal (1992) contrasted two elementary school mathematics classrooms and the types of discourse which arose in both. Emphasis was placed on the classroom environment and the meanings that students made of the completion of place-value numeration problems. In the more traditional classroom, students were expected to use the mathematical procedures and manipulations to determine the number associated with the number of tens and ones displayed in blocks. The teacher religiously asked students "How many tens do you see? How many ones do you see? What number is that?" (p. 585-6). This type of procedural instruction resulted in discussions in which students did not feel the need to explain or agree upon the types of justification for the solution. Emphasis had been placed on what students were "supposed to do" (p. 586) and that mathematics was a set of "fixed, objective rules" (p. 589). However, students learned to anticipate the questions the instructor asked and were prepared to respond with an alternative response should the first response not be acceptable.

The environment of the other second grade classroom had developed such that students were expected to explain their answers while the other students were to agree whether the solution was legitimate. Characteristics of this environment included students going to the board to illustrate their answers without being specifically asked to provide that explanation, other students offering alternative methods of solving the problem, the teacher's use of students' language, and the teacher's use of initiating and guiding the constitution of jointly understood mathematical interpretations. In both classes, students anticipated questions. In the first class students anticipated the teacher prompting a different answer if their first response was incorrect. In the second class students anticipated the need for additional support or justification asked by the teacher or other students.

In both studies, some reflection occurred as students' anticipated questions by others or anticipated the need to generate additional explanation. Such reflection promoted students' mathematical development. In this study, similar instructional techniques will be implemented to prompt reflection of the methods.

Reflection through Classroom Discourse

Cobb, Boufi, McClain, and Whitenack (1997) demonstrate the nature of discourse in an elementary mathematics classroom, the role of the instructor in promoting the discourse, and the development of reflection in the classroom. The researchers note how the class discussion surrounding the task shifted towards a higher level of activity through reflection. Students had been discussing the different possibilities for a given arithmetic problem. As the students listed the possibilities, one student, who previously demonstrated difficulty in using different representations in arithmetic, hypothesized that all the possibilities had been given. The teacher then asked if there was a way to be sure that all the possibilities were given. The teacher's question presented students with the opportunity to reflect on the work to this point and reorganize the activity mentally.

Cobb, et al (1997) observed the existing opportunity for all students in the class to reflect and objectify their prior activity, but noted that only those students who participated in the discourse were enabled to reflect. Comparing the acts of reflection and reorganization with the concept of resonance, all are individual acts or existence. Cobb et al. emphasize that though the classroom may promote a particular environment for reflection and reorganization to occur, the stipulation is for individuals to engage in the discourse. So, as a few students participate and indicate to their instructor that they are objectifying the concepts as intended, the teacher cannot assume that all students are objectifying the concepts. Important to emphasize, is Cobb et al.'s statement, "the discourse and the associated communal activity of collective reflection both support and are constituted by the constructive activities of individual children" (p. 266).

The researchers also noted the danger of a teacher to persist in initiating a shift in discourse when none of the students indicates the motion toward reflection on a prior activity. "The very real danger is, of course, that an intended occasion for reflective discourse will degenerate into a social guessing game in which students try to infer what the teacher wants them to say and do" (p. 269). The role of the teacher, as Cobb et al note, is to assess if students' tendency is toward objectification, to guide the development of reflective discourse, to ensure the interaction during such discourse, and to help students communicate notationally (symbolically) in the discourse.

Classroom discourse, as indicated, promotes the reflection and reorganization of conceptual structures in students' understanding. Advised, however, is the need for all students to engage in the discourse. Without engagement, little or no reflection, and as a result, little or no accommodation can occur. In this study, attempts will be made to engage all the students in discussion through inquiry. Evidence to support the existence or non-existence of such occurrences will be given.

Metacognition and Heuristics

Learning to solve problems and thinking mathematically requires reflection on the mathematical activity (Arcavi, Kessel, Meira, & Smith, 1998). To promote reflection the use of metacognition, thinking about one's thinking, and heuristics, rules of thumb for successful problem solving, were implemented in various studies emphasizing inquiry and mathematics education. In general, of the studies reviewed the use of reflection, metacognition, and heuristics advanced students' thinking when properly applied.

In a study of upper division science and mathematics majors, Schoenfeld (1985) examined whether problem solving experience was sufficient for students' acquisition of heuristic strategies or if specific instruction of the use of heuristics was beneficial. Each of seven students were given a pretest containing five problems. After the pretest, students were given twenty problems to work. Each student was given written and audio-taped solutions. Four of the seven students comprised the experimental group who received a list of five problem solving strategies in addition to the written and audio solutions. Their written and audio solutions also contained cues of how the heuristics were used in solving the twenty problems. Results on a five problem posttest indicated that students who received the heuristics training outperformed the students in the control group. Each student in the experimental group improved their performance from the pretest to the posttest, while only one student in the control group improved. Transcripts from students' posttest experiences indicated that students in the experimental group deliberately applied the heuristics taught and were successful in the use of three of the five strategies. Explicit instruction in heuristics made a difference in students' problem solving abilities.

Established Experiences vs. Plausible Reasoning

In spite of instruction of various heuristics, Lithner (2000) found that students frequently rely on established experiences rather than the heuristics and plausible reasoning. Plausible reasoning was classified as argumentation founded on mathematical properties of the components and intended "to guide towards what probably is the truth, without necessarily having to be complete or correct" (p. 167). Reasoning based on established experiences was considered the argumentation founded on notions and procedures in one's prior experiences from the learning environment and intended "to guide towards what probably is the truth, without necessarily having to be complete or correct" (p. 167). The reasoning based on established experiences concerns the transfer of properties from one familiar task situation to another task situation. The researcher examined undergraduate students' difficulties when trying to solve mathematical tasks. Three student volunteers, completing their first semester of their undergraduate study in mathematics, were videotaped as they individually worked two tasks, neither of which were purely routine nor non-routine. Both problems drew on students' calculus experiences as one was a maximization task and the other a graphical analysis task. The students' processes were then analyzed to determine the students use of plausible reasoning and reasoning based on established experiences.

Lithner found that students had the heuristics and resources they needed to correctly solve the tasks. Students limited their application of heuristics and reasoned through the problems based on their established experiences. Reasoning according to the established experiences created difficulty for the students when the familiar routines did not work for various reasons. This study suggests that additional methods are needed to better advance students' use of heuristics and plausible reasoning when problem solving.

Related to Lithner's study of established experiences, Rasmussen (1997) used a case study approach to explore students' understandings and difficulties with quantitative and qualitative numerical methods for analyzing differential equations and the factors which shaped the understandings and difficulties. Six students within a single section of an introductory differential equations course volunteered to complete semi-structured interviews with the researcher. Other data gathered were interviews with the instructor of the class, interviews with other mathematics faculty, classroom observations, collections of students' written work on four Mathematica problem sets, quizzes, and exams, and a questionnaire completed at the end of the semester. Four audio-taped interviews were conducted with each of the six students following students' submission of their Mathematica assignments. The interviews consisted of three parts: exploration of students' concept images of the topics presented in class, three to five problems to solve, and discussion of the role of Mathematica in their learning about differential equations. Interview tasks were classified as association, prediction, conceptualization, or modeling.

Class sessions included time for students to ask the instructor questions about homework or previous lectures. The remaining class time was spent as the instructor lectured on new course material. Students were able to ask questions during this time but few students did. All interaction was teacher-student oriented with no student-student interaction. No class time was spent discussing or demonstrating the use of Mathematica. The instructor regularly commented on the usefulness of the qualitative and numerical methods to learn about solutions of differential equations without having actual solutions. However, sixty percent of all textbook homework problems assigned emphasized analytic techniques with less than twenty percent emphasizing qualitative and numerical techniques for solving differential equations.

Student interviews demonstrated that students often relied on established experiences much like Lithner's (2000) discussion of students' reliance on established experiences rather than plausible reasoning. Rasmussen found that students tended to memorize graphical methods of analysis in isolation from other aspects of the problem. Students demonstrated their ability to complete the Euler's method algorithm and other procedural components but displayed little understanding of the symbols involved nor of the connections between the algorithms and the direction fields. Given the class and interview data, Rasmussen recommended an inquiry approach emphasizing discussion of students' interpretations, strategies, and ways of conceptualizations of the concepts and representations rather than procedures and final answers. Rasmussen also recommended incorporating mathematical modeling into the curriculum, with modeling launching discussion of the numerical and qualitative methods rather than discussion occurring as illustrations of the methods.

Though specific instruction in the use of heuristics proved beneficial for students, students in Lithner (2000) and Rasmussen's (1997) studies frequently relied on established experiences. Examination of the implementation of an inquiry process promoting the incorporation of modeling to launch mathematical discussion will be given in the current study.

Metacognition

While the use of heuristics in mathematics prove beneficial to students in mathematical problem solving, reliance on metacognition tended to be minimized by novice problem solvers. The following studies illustrate students' infrequency in implementing metacognition while demonstrating students' success when engaging in metacognition.

Stillman and Galbraith (1998) sought to explore female secondary students' metacognitive behaviors as they solved real-world, non-routine problems requiring the use of memory management techniques. In general, the senior level high school students spent most of their time on orienting and executing activities with little time on organization and verification. Students who were most successful spent less time on orientation components, engaged in a high number of organizational activities, and regulated their execution and evaluation activities. The researchers encouraged the development of teaching approaches which facilitate students' reliance on available cognitive and metacognitive resources through reflection and discussion.

Schoenfeld (1985, 1987) also found that novice problem solvers tended to spend less time organizing and regulating their activities and more time on exploring solutions. To display individuals' managerial and metacognitive behaviors and indicate models of reasonable problem solving behavior, Schoenfeld created a framework for the "macroscopic analysis of problem-solving protocols." Coding for the sequence and time spent on reading the problem, analyzing, exploring, planning, implementing, and verifying, Schoenfeld created maps of the time spent on the various components. Use of the codes and resulting maps in analyzing novice and expert problem solving abilities suggested that expert problem solvers demonstrated controlled behavior, curtailed "wild goose chases," frequently assessed the state of the solution, and spent more time thinking about the problem (analysis) than doing (exploring). Novice problem solvers failed to activate self-regulation and spent all of their time exploring and implementing a single method of solving the problem, considered by Schoenfeld to be a "wild goose chase."

Chin and Brown (2000) examined transcripts and written reports from eighth grade science students involved in inquiry process of learning a chemistry unit. The researchers identified two categories of approaches students applied in learning: deep and surface learning. The deep learners generated ideas more spontaneously and had more precise and elaborate responses. In addition, their questions focused on explanations, predictions, and resolving discrepancies. Deep learners relied on metacognition to regulate control of the on-going learning process and persisted in the follow-up of ideas and predictions. Surface learners gave explanations which referred only to what was visible and asked more factual, procedural questions. Surface learners focused on the procedural and observational levels of understanding. Chin and Brown recommended future research of instructional processes to encourage deep processing of students.

In general, when students relied on metacognitive and heuristics, they engaged in higher order thinking and were more successful in solving problems. When students did not rely on their metacognitive skills or heuristics and relied on their established experiences, students were often less successful in problem solving. In this study, the development of metacognitive skills and heuristics will be described as inquiry and mathematical modeling are pursued. Examination of students' reliance on reflection, metacognition, and heuristics will be completed.

Instructor Issues

Various issues surrounding the instructor's goals, directions, and interactions with the students influence the class environment and degree to which inquiry and mathematical modeling are implemented. Items directly influencing this study include, as mentioned previously, running with students' ideas, negotiation of problem or task, and overall structure and pedagogy. Other items needing to be addressed more fully include the authenticity of the instructor's beliefs and practices, classroom interactions, and interpretation of task.

Authenticity of Beliefs and Instructional Practices

Informing the classroom environments for this study and the methods used to develop representations of inquiry was a study on authenticity of beliefs and instructional practices. Brendefur (1999) related high school teachers' beliefs about student learning, pedagogy, and mathematics to the authenticity of their actual instructional practices. Surveys, class observations, interviews, and textual analysis helped to address the following research questions: (1) "What is the nature and relationship of high-school mathematics teachers' beliefs about student-learning, pedagogical practices, and mathematics related to authenticity of instruction? (2) What is the relationship between the authenticity of the high school mathematics teachers' practices and their beliefs?" (p. 8). Fifty-one teachers from six different high schools and one technical institution from across the nation completed pre- and post-surveys concerning beliefs about teaching. Of the 51, eight mathematics teachers were selected based on their involvement in curriculum-writing projects grounded in the NCTM Standards and connecting mathematics with a vocational or technical topic. The remaining 43 teachers formed the comparison group to determine if the project teachers represented the population of mathematics teachers and to evaluate the stability of teachers' beliefs over time. The eight project teachers participated in two interviews, allowed two sets of classroom observations, and wrote, pilot tested, and revised a curricular unit, and attended a summer workshop in addition to completing the two belief surveys. The eight teachers had volunteered to participate in the curriculum-writing project.

Brendefur classified the beliefs, classroom observations, and curriculum units according to three components of authenticity: Construction of Knowledge, Depth of Knowledge, and Value Beyond Instruction. Construction of Knowledge, grounded in higher order thinking, refers to the students' engagement of tasks where they make sense of a situation or phenomenon through discussions and exploration while relying on their prior knowledge and available resources. Depth of Knowledge relates to the belief that students learn mathematics by connecting mathematical ideas together. Value Beyond Instruction associates the belief that students learn and are engaged in mathematics when the problems have value to them or the real-world. Brendefur defined authentic pedagogy as

> teachers' deliberate actions to promote students' construction of knowledge through mathematical inquiries and problems that have personal or real-world significance. Pedagogy includes the curriculum, instructional activities and questions, and assessment tasks that teachers use in their classroom (p. 23).

Teacher's beliefs were characterized as being more authentic when they thought that students learn by making sense of the mathematics themselves. Teachers' beliefs were characterized as more traditional and less authentic when they believed that students learn by memorizing or absorbing the mathematics presented to them.

In relating high school teachers' beliefs toward authenticity with their instructional practices, Brendefur coded classroom observations according to various scales. The purpose of the scales was to gather numeric indicators of student experiences during instruction. The class sessions were observed by two independent observers. The scores from the two observers were then aggregated and discussed until agreement was achieved between the two observers. In coding the observations, Brendefur coded data in real-time rather than based on audio or video recordings. The researcher felt the use of written notes contain a person's reactions and other observations that cannot be captured on audio tape.

The scales used during the classroom observation were higher order thinking, depth of knowledge and student understanding, mathematical connections, cross-disciplinary connections, substantive conversation, and value beyond the class. Each scale had five ratings which could be attributed to class during observations. The higher order thinking scale ranged from "most students, for most of the time, are engaged in higher order thinking" to "students receive, recite, or perform routine procedures" (p. 250). Classroom activity was classified as a high depth of knowledge and student understanding when students' reasoning, explanations and arguments demonstrated fullness and complexity of understanding. The activity was classified as low when students applied algorithms with no attention to the underlying concepts. The scale for mathematical connections sought to address the extent to which the lesson connected topics across different areas of mathematics. Rated low was the isolated study of mathematics topics. The scale for cross-disciplinary connections rated whether mathematical topics were studied with or in isolation of other contexts.

In addition to classifying the level of thinking and mathematical content and context, Brendefur's observation scales accounted for the class environment. Brendefur examined the use of classroom discourse to promote shared understandings of mathematics. Rated high was "the creation of and maintenance of collective understandings" which could include the use of a common terminology and the careful negotiation of meanings. Rated low was

"Virtually no features of substantive conversation occur during the lesson" (p. 256). Value beyond the class was rated high when students worked on a topic directly connected to their personal experiences or actual contemporary public situations. In addition students were to recognize the connection between classroom knowledge and the current situation. Rated low was the lack of connection of the topic and activities to anything beyond itself.

Brendefur classified the teachers' beliefs based on their comments during interviews. Two of the eight instructors were rated as authentic as they consistently made comments regarding the making and discovering of connections within mathematics and between mathematics and other contexts. Four of the eight tended to make more traditional comments that mathematics is a set of tools developed in a linear fashion and that students learn by memorizing and using algorithms in trivial ways. The remaining two teachers made an equal number of traditional and authentic comments.

Seven of the eight instructional units were initially classified as traditional with one rated as a mix of traditional and authentic. After revisions, three remained traditional, three moved up in authenticity to "mostly traditional," and one moved to "mixed." The one "mixed" classification moved slightly towards the "mostly authentic" category.

Of the two instructors whose interviews suggested they had authentic beliefs, only one was rated as "mostly authentic" based on observations of the instruction. The other was rated as a mix of authentic and traditional classroom practices. The two instructors who gave mixed beliefs on the interviews demonstrated mixed instructional practices. The four instructors who rated as traditional during the interviews were mainly traditional in their use of instructional practices.

Positive correlations indicated a relationship between the authenticity of instruction and the teachers' belief regarding authenticity. In general, instructors' beliefs regarding authenticity tended to be rated higher than the ratings for instructional practice. Overall, changes in teachers' beliefs from the first to second implementation of the instructional units were found to be inconclusive.

In considering an inquiry environment, authenticity of beliefs and instructional practices strongly relates to the pedagogy implemented by the instructor and students. In this study, examination of the instructors' practices and beliefs will not be classified as authentic or traditional. However, the instructional practices will be examined in light of the instructors' beliefs and goals. In addition, when examining the classrooms for characteristics of inquiry, similar codes as used by Brendefur will be implemented, including connections across contexts and levels of mathematical understanding.

In their own study of "authenticity," Arcavi, Kessel, Meira, and Smith (1998) examined Schoenfeld's teaching practices during a semester of his course "Mathematical Problem Solving" to investigate how he creates a classroom community of problem solvers in which undergraduate students think and perform mathematically. The researchers found that Schoenfeld's lectures were infrequent and contained heuristics or other instruction students needed to make progress on problems. Instruction was shifted from the mathematical content to methods of how mathematics is done. Students worked mathematical problems, presented solutions to their problems, and engaged in small group work and whole class discussions. With emphasis placed on students' participation in the mathematical community, Schoenfeld's language reflected that of the community with "we," "you," and "Devon's question was...," and was informal and non-technical. In deemphasizing the formality of the written and oral mathematics, the process of how to do mathematics was reflected.

Both studies by Brendefur (1999) and Arcavi et al. (1998) suggested that language played a role in the authenticity of instruction or emphasis on the process of mathematics. Brendefur incorporated a code to suggest the use of student vs. teacher language in the mathematics classroom. Arcavi et al. indicated that Schoenfeld de-emphasized formal mathematics terminology to focus on the "how" of mathematics. Neither study gave classroom examples suggesting the advantages to using student language over more formal mathematical language. In light of Sfard's (1992) study, using student language may emphasize a more operational instructional method before developing the structural component of mathematics. As a secondary component, the use of language and degree of formality will be examined in this study.

Conceptual Orientation

Much like the authenticity of one's beliefs and instructional practices, classification of the instructor's conceptual orientations informs the goals, purposes behind a task or an assignment, and interactions held in the class. Clement (1999) examined the conceptual orientations of two instructors of a mathematics course for elementary education majors. In using innovative materials, the teachers were observed and video-taped during two instructional units. Observations across two units provided the researcher with the ability to address the stability of instructors' orientations across and within different mathematical topics. One instructor represented a calculational orientation while the other had a conceptual orientation. The instructor identified as having a conceptual orientation toward teaching expected students to share their thinking; focused on aspects of students' thinking associated with the underlying concepts; emphasized quantities over values in the problems; used diagrams and explanations to help develop students' reasonings and connections; and promoted students' thinking about their own and others' thinking about the mathematics. The instructor identified as having calculational oriented teaching practices demonstrated instructional techniques which emphasized mathematics as composed of procedures and facts. In the calculational instructor's class cooperative learning and diagrams were used to facilitate students' abilities to solve problems. Both instructors' orientations toward teaching remained stable throughout the course.

In a related study, Thompson and Thompson (1994) analyzed one teacher's interactions with one of his sixth grade students as the student developed in her understanding of distance, time, and speed. The researchers purposely intended to examine the discourse between the teacher and one student to relieve the teacher of all distractions and reduce the social complexity and demands of the whole-class interaction. In doing so, Thompson and Thompson sought to gain insight into the cognitive and attitudinal constraints teachers face when they attempt to influence children's thinking and conceptual construction.

The researchers had prepared a set of tasks for the student to work relying on a computer program "Over & Back" as needed to address issues of speed, distance, and time. The researchers worked with the teacher to help him be prepared for the student's difficulties with the tasks, but he more often relied on calculational understandings to explain to the student how to proceed in the problem. The discourse demonstrated the mismatch between the teacher and student's conceptualization of speed. The teacher's conceptualization was complex, but he was only able to use language reflective of the calculations required, rather than the everyday language the student used. The teacher also lacked a sense of clarity of the purpose of the tasks, other than completion.

Specific classifications of the instructors as conceptual or calculational or authentic or traditional will not occur in this study as it was not a goal of the study. However, examining the instructors' practices in light of their goals and beliefs will give justification behind some of the methods implemented. Identifying various aspects of the classroom interactions, language use, and instructor's beliefs regarding these studies of conceptual orientation and authenticity in instructional practices informs the nature of inquiry pursued in the classroom environments.

Interpretation of Task

Another component of the pedagogy is the negotiation of the problem or the interpretation of the task. Interpretation of task and explanations influence the inquiry in mathematics classrooms. Christiansen (1997) and Yackel (1995) both examined the role of interpretation in influencing classroom dynamics and inquiry in the class.

Christiansen (1997) conducted a qualitative study of a freshman high school mathematics class as students studied mathematical modeling of population growth using linear, exponential, and logistics equations. Christiansen audio-taped class sessions and focused observations on a group of five students. Students had been given a homework assignment to examine the given population data and were asked if the assumption that the data was linear was reasonable. One student argued against the rest of his group that based on the ability to predict future populations and on his common-sense knowledge about population, the data was not linear. The group sought the instructor's help to resolve their disagreement of whether the data was linear. The instructor encouraged the student to focus only on the data that was listed and to remember that a model did not have to be perfect, but reasonable. In the reminders, the instructor highlighted her intentions for the homework problem to use modeling as a tool to describe population before modeling became an object in itself. The instructor acknowledged the opposing student's judgment but indirectly gave her own judgment about the linearity of the data. In doing so, she advanced the exercise-oriented goal and potentially interfered with students' ownership of the goals for the activity. Christiansen stressed the importance for students to understand that an activity have a well-defined goal in order to know what kind of activity to engage.

Yackel (1995) examined second grade students' interactions during mathematics classes as students focused on instructional activities designed to enable students to create and coordinate arithmetical units of different ranks (Cobb, Yackel, Wood, 1995). The classroom was video recorded for a school year with ten weeks of class sessions analyzed by researchers. Yackel identified instances indicating when and why interaction broke down in the classroom. Three classifications included participants' differing interpretations though they were unaware of the differences; incompatibility of the interpretations of the immediate task and what activity was considered mathematics; and students' inability to explain their ideas so others could understand.

Both Yackel (1995) and Christiansen (1997) identified particular items which influence interactions in the mathematics classroom. Christiansen recognized the instructor's goals and comments in impacting students' interpretation of tasks and the intent of the assignments. Yackel identified that different interpretations among participants lead to breakdown of mathematical activity. Additional research is needed to extend these classifications to the breakdown of interaction and understanding of task goal in the collegiate mathematics classroom.

The interpretation of the task or problem as highlighted by Christiansen (1997) and Yackel (1995) as well as previous discussion of negotiation of problem in Roth (1995) and Borasi (1992) suggest the importance for students and instructor to agree on the nature of the task or problem under investigation. While not implying that all students have to investigate the same problem or question in an inquiry environment, understanding should exist of the purpose of the task and the means to assess whether the goals were attained. In this study, negotiation or interpretation of task will be examined as inquiry is pursued.

Summary

In summary of the literature reviewed, examination of inquiry and mathematical modeling in science and mathematics are well-documented. Several key points are noted which have a direct impact on the nature of this study. Studies indicate and support the implementation of inquiry in the classroom as students' higher order thinking is advanced, particularly in interpreting and evaluating results. When students have more freedom to pose and investigate their questions, their thinking is more often focused on the data and evaluation of the data and results rather than interpreting the instructions of the lab. When laboratories are more open, negotiation of task and purpose of the problem is encouraged. Investigations of inquiry in the college mathematics classroom demonstrating the interactions and students' reasoning are needed. In particular, this study will examine what happens when entire classes pursue inquiry, rather than instances in which inquiry is pursued in a situation with a low teacher-student ratio.

The studies relating context and mathematics indicate that students can better relate mathematics and context when given instruction and practice in mathematical modeling. More study is needed of how the context informs students' problem-solving abilities as some studies suggested students' reliance on established experiences and technological results rather than reasoning from the context. This study will seek to extend the literature on students' modeling abilities when relationships between context and mathematics are emphasized in class. As a component, the roles of reflection, metacognition, and heuristics will be addressed in the inquiry environment as students model and relate the context and mathematics.

Few studies reviewed indicated ways in which pictorial representations of an inquiry environment could be created. Various coding techniques by Bromme and Steinbring (1994), Brendefur (1999), and Schoenfeld (1985, 1987) informed the codes used in this study. Bromme and Steinbring highlighted the role of context and interactions in a mathematics class. Brendefur's codes indicated degrees of authenticity in instructors' beliefs and practices. Schoenfeld's codes emphasized individual student's metacognitive and self-regulatory skills in solving problems. In each of the codes components of inquiry such as phase of inquiry, cycles of inquiry, and indication of agreement or disagreement of resonance were missing. This study intends to examine the use of various codes to suggest inquiry in the classroom and the agreement and disagreement in resonance between class participants. Methods used to develop these codes and the inquiry and modeling topics of this study are discussed in Chapter 3.

CHAPTER 3

METHODOLOGY

The Unit of Analysis

The unit of analysis for this study consisted primarily of two sections of the freshman, collegiate level course Mathematics 181: Calculus for the Life Sciences. This course was selected based on the following reasons: the course already contained laboratory investigations, the instructors each had taught the course for several semesters, and the classroom environment promoted interaction.

A brief history of the origin and the development of the course is appropriate in describing the underlying philosophy and process of Mathematics 181 while informing the choice of the unit of analysis. The course originated in 1994 when a mathematics professor collaborated with a zoology professor, a second mathematics professor, and a mathematics instructor. The team determined that the mathematics and engineering sequence of calculus seemed to lack the application-oriented problems that life sciences students felt were relevant to their education. Relying on an NSF grant, they developed a two-course sequence and text containing biology and ecology based problems while emphasizing both the review of precalculus topics and discussion of rates of change. The team desired for students to have hands-on interaction with the mathematics and science problems.

The four originators of the course developed laboratories during which students would collect and analyze data as the mathematical topics unfolded. Prerequisites placed on the laboratories included the consistent gathering of data which accurately reflected specific mathematical relationships and a shortness of time required to gather data. The data needed to be collected within one 50 or one 85 minute class session. Catering to the life sciences students' mathematical needs and the prerequisites, the team developed laboratories focusing on properties and uses of linear, polynomial, exponential, and logarithmic equations as well as difference and differential equations. The labs were founded on life sciences topics such as population growth, pharmocokinetics, allometry, and other specie-specific properties of wildlife. When analyzing the data gathered in the laboratories, students used various technology, including graphing calculators, and wrote lab reports summarizing their findings. Each semester since the origins of the courses, the laboratories and text have been modified to correct mistakes, enhance the accuracy of the methods of gathering data, and better accommodate the needs of the students and instructors through the rearrangement of discussion of various topics.

The previously built-in laboratories in Mathematics 181 provided the potential for attaining process and philosophical components of an inquiry environment. Throughout the courses, laboratory investigations were completed both in the mathematics classroom and in a science laboratory. The class met for four 50 minute class sessions per week. During weeks in which students gathered data in the science laboratory, students attended one 85 minute lab session and three 50 minute math class sessions. During the laboratories, students often participated in the process component of inquiry as they made predictions, collected data, and analyzed the data. Students were encouraged to explore mathematical situations including data handling and mathematical modeling. The models were used to describe the relationships, patterns, and operations inherent in certain scientific relationships they explored. Similar to Schoenfeld's experiences (1987), the course allowed students to experience mathematics in a way that made sense much like the mathematics the mathematicians and scientists experience.

The experience of each of the instructors in teaching Mathematics 181 enhanced the choice of investigating inquiry in the course. Each of the instructors for the course, including the researcher, had many semesters of experience teaching the course. This experience allowed for confidence in the ability to teach the content and to apply different class and assessment activities which had been "tested." Each instructor often adjusted their methods based on what had worked well the previous semesters and what new methods, activities, or assessments they wanted to implement in the attempt to improve students' success in learning and understanding the material. Neither felt they had become "the expert" instructor, but each sought to improve upon their prior teaching experiences. For the teacher-as-researcher, the four semesters spent teaching Math 181 prior to the data collection period served the purposes to develop deep understanding of the environment and issues being examined and extended experience mulling over the issues under question. In addition, the time spent teaching allowed for growth as a researcher in learning how to identify and frame workable research problems--meaningful problems on which legitimate progress can be made in a reasonable amount of time (Schoenfeld, 1999)

The two sections of Mathematics 181 taught by two instructors of offered two different pursuits of inquiry in mathematics classrooms. In the two classroom environments, the researcher could identify characteristics or activities which promoted the inquiry process and those which hindered the inquiry process. The multi-sections provided the researcher with the possibility to determine if patterns in the characteristics or activities arose across the different sections and with different instructors.

Each Mathematics 181 class provided the opportunity to examine the philosophical component of inquiry. During class, students were often given time to discuss the data analysis and other mathematical tasks with their group members. Students in both sections of Math 181 were assigned seats and/or group members. Since the use of the graphing calculator often played a role in students work with the data, one instructor used students' graphing calculator type to arrange the assigned seating. The researcher also used graphing calculator type in addition to students' laboratory time, so students would be assured of working with the same students during the laboratory sessions.

The interactions among the students and between the instructor and students in the classroom forced certain issues to play out, in contrast to an environment in which the researcher strictly interacted in a one-on-one study of individuals learning in an inquiry environment. While studies consisting of one-on-one investigations of a researcher interacting with a single student in an inquiry environment add to the body of knowledge, particularly the metacognitive processes one incurs (Borasi, 1992; Schoenfeld, 1985, 1987; Arcavi & Schoenfeld, 1992; Thompson & Thompson, 1994), a classroom study provides additional information. One may argue that a classroom of many individuals as a unit of analysis establishes a greater potential for the unexpected, but a study of this nature more lends itself to the "real-world" environment for other collegiate mathematics instructors in pursuit of inquiry in their classrooms. Part of the question under investigation in this study is how inquiry can occur on a large scale. The interaction and laboratories conducted in Mathematics 181 provides the environment for the examination of this question.

Subjects

Students

Adult students in two sections of Math 181: Calculus for Life Sciences, the instructors for two sections, including the researcher, and a teaching assistant formed the participants for this study in the spring of 2000. The researcher taught one section of 31 students in Math 181 while the other section consisted of 33 students. Though Math 181 is a freshman level collegiate mathematics course, students at all undergraduate classification levels enrolled in the course. The students also had varied mathematics backgrounds. Some students had recently taken all or part of a high school or engineering calculus course. Others delayed fulfilling their mathematics requirements and had no mathematics instruction for two or more years. These characteristics are noted as a student's mathematics background potentially influences the types of questions, comments, and interactions made in class. One similarity among the students was a major in a life science or pre-medical field of undergraduate study.

Instructors

The instructor of section one of Mathematics 181 had master's degrees in both mathematics and molecular biology, providing her with a strong background in both the mathematics and sciences. Her knowledge enabled her to produce examples of how the mathematics and science interlinked. The instructor had also been a collaborator in writing the text and designing the experiments used in the course. As observed during the pilot study and confirmed during one-on-one interviews, the instructor felt the primary purpose of the laboratories was to develop contexts from which mathematical models could evolve. Once generated, the models launched further exploration and summary investigations of the family of functions produced and rates of change of the family of functions.

With a bachelor degree in mathematics with secondary certification, the researcher had taught Math 181 for four semesters prior to the data collection period. She often collaborated with the instructor to discuss methods of instruction, the laboratory investigations, and student assessment. The researcher used the laboratory experiences to lend to the development of the mathematical discussion particularly modeling and rates of change.

Teaching Assistant

An additional participant in this study was a science laboratory teaching assistant. For Susie, a graduate student in electrical and computer engineering and having prior experience in a veterinary medicine laboratory associated with the university, the Spring 2000 semester was her first as a teaching assistant for Math 181. Susie sat in almost all class sessions of section one of Math 181 and helped to answer students' questions. In addition, Susie led at least four laboratory experiences for this section in the science laboratory: an introductory lab session to familiarize students with the equipment and computer software, a motion experiment, a *vibrio* bacterial population growth lab, and the one-compartment model of the body's processing of a dose of penicillin. Susie also led the one-compartment model of the body's processing of a dose of penicillin for a portion of the researcher's students.

Design

The nature of this study was exploratory using both qualitative and quantitative methods as described by Guba and Lincoln (1994) to inform the nature of inquiry in a collegiate mathematics classroom. As Schoenfeld (2000) and others (Hiebert, 1999; Kilpatrick & Silver, 2000) have indicated, there are no proofs in mathematics education. The intent of this study was to explore and represent the nature of the inquiry process in a collegiate mathematics course. The goal was not to prove that a course taught using an inquiry approach is better than other methods of instruction, as what is "good" is inherent in the goals one has for a course (Schoenfeld). Though proof is not achievable, much like a study in life sciences, multiple pieces of evidence must be presented to determine if a discovery meaningfully illustrates or supports a given theory. Hence, triangulation of the data is essential.

In this study, multiple pieces of evidence were gathered with the intent to triangulate the data. The data consisted of audio or video taped class sessions surrounding the laboratories, copies of students' lab reports, copies of samples of the researcher's students' work during the laboratories, one-on-one interviews with six of the researcher's students, one-on-one interviews with the instructor, written observations of the instructor's class, and the researcher's class notes and journal.

Laboratories

Three laboratory investigations formed the context for the data gathered in Math 181. Two of the three laboratory investigations were conducted in the mathematics classroom: the water flow investigation and the light intensity investigation. The one compartment penicillin investigation was conducted in the science laboratory.

The water flow laboratory, performed within the first five weeks of the semester provided a context for quadratic functions and rates of change of quadratic functions. Hence, with completion of the lab early in the course, data from this laboratory produced information of the process of inquiry when students were first getting accustomed to the course. In the investigation, students gathered data by draining water through a cylindrical tube with a small hole drilled into the base of the tube. The data was then modeled with a quadratic function.

Midway through the course, the light intensity laboratory investigation was completed in the mathematics classroom. Gathering data at this stage of the course gauged the development of the inquiry process over time. During this investigation, students explored the depletion of light through increasing depths of water or increasing layers of tinted Plexiglas. This depletion was modeled by an exponential function.

To gain a sense of the use of the inquiry process after the majority of the course was completed, data were gathered in a third laboratory, a one compartment model. This laboratory followed on the heels of the light intensity investigation and was conducted in the science laboratory. This investigation mimicked how the kidneys in the human body eliminate a dose of penicillin in the blood and how the amount of penicillin in the blood changes as multiple doses are administered. The model implemented a simple wash-out of a given percentage of the fluid. Again exponential functions resulted in the modeling of the data, though in a different form than in the second laboratory.

Data Collection

Audio and Video Recording

To capture the developments of the laboratories, the instruction, students' interaction, and teacher and student interactions, each class session pertaining to the described laboratories was audio or video recorded. The researchers' class sessions were video recorded. The class sessions in the other section of Math 181 were audio recorded. Audio recording was selected by the researcher after discussion with the instructor. The instructor was willing to allow video recording in her class, but she preferred the audio recording. Audio recording was selected as this would be less intrusive to the classroom environment for the instructor and students. In addition, the researcher, a passive observer, recorded visual observations while sitting in the class sessions.

Video recording was selected for the researcher's class sessions after review of the audio recordings of pilot sessions. The audio recordings of the pilot sessions of the researcher's section failed to capture the accompanying visual observations informing the context of discussions. The researcher had not and could not physically teach and record many visual observations simultaneously. While a daily journal was kept to record the researcher's observations, thoughts, and questions during the laboratories, the video recordings were implemented with the attempt of capturing students' interactions and engagement outside the researcher's immediate attention in addition to lending documentation of the class developments. All recordings pertaining to the laboratories were transcribed.

During the transcription process, statements audible to the researcher were transcribed. In most cases, the researcher was able to identify the person who made the statement, and a pseudonym was typed with the comment. In those instances where the person speaking was identified but the words unclear, the pseudonym was recorded with "inaudible" marking the unclear statement or question. The occurrences in which the statement was recorded but the voice was unidentifiable, [?] was associated with the comment and eventually assigned a number during coding.

When the interactions of the small group discussions were recorded, the researcher attempted to record as much as possible of the conversations. However, in many cases, the instructor's voice was clearly identifiable lending to greater ease of transcription. In addition, the microphone was more effective at capturing the sound of the voices closest to it.

Though some students may have participated extensively in small and large group discussion, the transcriptions may not have fully represented the level of participation. As a result, though much student interaction occurred in both classes, the interaction was not fully reflected in the transcriptions due to the limitations of the microphones and the researcher's ability to decipher students' communication.

One exception in which the limitation of the microphones decreased occurred during the first four days of the light intensity laboratory in the researcher's class. Professional videographers recorded the investigation. The researcher's comments were captured through a cordless microphone while the interactions of a group of four students and students around this group were captured through a second microphone. The tape produced by the professionals was used for the transcription process. The use of the professionally produced tapes greatly increased the number of lines of transcription for these four days of the investigation and ultimately for the entire laboratory as better sound quality was obtained with the two microphones. Increase in sound quality and the transcriptions for this period influenced any resulting analysis emphasizing the number or type of student-to-student interactions. When reporting the qualitative results of the study, ellipsis points (...) designated where some comments were omitted from the discussion.

Students' Written Work

Lending to the information supplied by the transcriptions and classroom observations, additional evidence was gathered from students. In the researcher's section of Math 181, students were expected to regularly record their hypotheses, questions, data, analysis, and reflection questions in a "scientific notebook." Following the water flow laboratory, the students had not been held accountable for recording their observations. Thus, during the water flow investigation, students' questions about the laboratory and the analysis of the data were gathered as an incomplete component of the data. With the intent to be more systematic in capturing students' written work, during the light intensity and penicillin investigations, the researcher developed worksheets which students were to complete throughout the investigations. Copies of students' "scientific notebooks" containing their questions, hypotheses, data, initial analysis, and reflection questions were made.

Written work gathered of students in the other class took slightly different forms than the work gathered in the researcher's section. In section one of Math 181, data from select students were recorded by the researcher according to what the students wrote in their notebooks, on the board, or on overhead transparencies.

Across all sections and each laboratory investigation, photocopies of students' lab reports were made. Students wrote lab reports following the class discussion of the analysis of the data. The lab reports demonstrated to the instructors the connections students made between the mathematics and science, the interpretations generated, the accuracy of students' work using the methods discussed in class, and students' use of multiple representations in communicating the mathematics and science. In the researcher's section of Math 181, students completed the water flow and light intensity lab reports in their groups while the penicillin reports were completed individually. Students in the other section of Math 181 were given the option of submitting reports individually or with other classmates. In section one of Math 181, students had worked with their light intensity data on daily homework assignments which were not submitted for evaluation. To gain understanding of students' application of the methods developed when analyzing the light intensity data, the instructor collected students' *Vibrio natrigens* bacteria growth lab reports. The bacterial growth laboratory produced data of a population which grew exponentially as opposed to the data gathered from the light intensity laboratory which exponentially decayed. The researcher photocopied these reports, and they assumed the same role as light intensity lab reports.

Interviews

One-on-one interviews with the researcher were conducted to gather additional information about the instructors (see Appendix A) and a select number of students (see

Appendix B). The instructor of section one of Math 181 was interviewed before and after each laboratory. The purpose of the pre-interviews served to understand the instructor's goals for the laboratory investigation, her thoughts of how the class sessions would unfold including her anticipations of students' questions and hypotheses, and the instructor's perspectives of how students' growth in the inquiry process would be evident in the investigation. The purpose of the post-interviews served to record the instructor's observations made of students during the investigation, her reflections on how the goals for the laboratory were or were not met, and the instructor's perceptions of students' growth as the course progressed. Each interview was audio recorded and transcribed.

Six students from the researcher's section of Math 181 were selected to participate in a single, task-oriented, video-recorded interview held at the end of the course. These six students were selected based on the data the researcher had from the transcriptions, the students' high attendance records throughout the course, particularly during the laboratories, and students' achievement throughout the course. The researcher desired to have a range of student success in the grades for the course, with a goal to capture differences in students' reasoning on the task.

The semi-structured interview (see Appendix B) began with the question, "Have you ever noticed how quickly a cup of hot liquid, like coffee, tea, or cocoa, cools over time?" After a student gave a response, he or she was asked, "Using methods similar to what we've done in class, what would you do to try to understand this phenomena?" Once a plan was given, the researcher gave the student a list of ten *time* and *temperature* data points and instructed, "Here is some data I have from a cup of hot water cooling over time. Temperature readings were taken every 6 minutes. Show me what you would do with this data." During the sixty minute interview, students were asked to "talk aloud" as they worked the problem. When a student was quiet for several seconds at a time, the researcher prompted, "What are you thinking?" Those times when students were obviously frustrated and did not know what to do next, the researcher intervened. Comments the researcher interjected included: "What did you graph?" "You originally thought this was what kind of graph?" "What's your end goal?" "What are you looking for?" or "What did we do in class?" Data generated from the student interviews were used to demonstrate students' individual inquiry skills as well as their conceptual, reasoning, and reflective skills following their experiences during a semester of Math 181.

Researcher's Journal

The researcher maintained a daily journal containing her daily plans, reflections on the class sessions, improvements to be made, and her observations of students' work and comments. This data served the same purposes as the instructor interviews. In addition, notes received in the form of relevant emails from the instructor and students, and conversations with instructor and students not captured on tape were recorded.

Limitations of the Data

As mentioned previously, multiple pieces of evidence were obtained to develop different illustrations of the development of the inquiry process throughout the semester and achieve triangulation. Areas in which the study was limited due to a lack of triangulation primarily resulted from a failure to obtain adequate pieces of students' daily class work, in the form of written questions, comments, or reflections. Failure to do so limits the researcher's ability to suggest agreement and or disagreement between the instructors' and students' resonating conceptual schemas. Skemp (1987) indicated that schemas cannot be observed directly but can only be inferred by individuals' responses. Hence, the fewer the records of students' and the instructors' behaviors, statements, and written responses, the greater the limitations in inferring what conceptual schemas are resonating as well as whether two parties' schemas are compatible.

While the collection of written work was limited, the researcher understood that demanding the other instructor to daily collect samples of students' work, questions, or reflections would potentially interfere with the development of the instructor's desired inquiry in her classroom environment and daily plans. Hence, the researcher did not require that a daily artifact be gathered from each student for the purposes of the research study. The transcriptions of the class sessions provide some image of the agreement and disagreement of the resonating schemas. The use of the transcriptions are also limited as not all the students spoke during class, and if they did, the microphone did not always capture what was said.

Methods of Analysis

Pilot Study

The nature of this study was exploratory in the sense that data was gathered and used as an opportunity to gain new insights about the development of the inquiry process in a collegiate mathematics course (Bromme & Steinbring, 1994). Though exploratory, data collection and analysis was conducted in various forms in an on-going pilot study begun in the spring of 1999. Methods of recording, transcribing, and coding were applied as part of the development and focusing of the research question. In addition, interviews with students and copies of students' lab reports were gathered to examine students' development in mathematical modeling and the inquiry process. The following sections describe the tools used and not used to analyze the data gathered in this study as a result of the pilot investigation.

Transcriptions and Coding

Recorded class sessions pertaining to the laboratory investigations and interviews were transcribed with line numbers assigned to each line of transcription. The transcripts were analyzed for common themes regarding the development of inquiry; the role of the context when discussing the mathematics; reflection and metacognition; students' use of symbols and mathematical language; evidence of students' procedural and conceptual understanding; the source, ownership, and level of the questions asked; and characteristics of interactions between the instructors and students and among students. Snapshots illustrating each of these items were gathered to give a larger picture of the nature and development of the inquiry process.

Adding to the big picture were quantitative descriptors of the class occurrences. Coding students' problem-solving behaviors and classroom environment is not a new technique. Schoenfeld (1985) cited others (Kilpatrick, 1967; Lucas, 1980; Kantowski, 1977) who created coding schemes to identify and objectify students' problem solving behaviors in order to explore problem solving success and frequency of occurrences of particular problem-solving process. Brendefur (1999) coded overall class behaviors and instructional units established in high school mathematics courses. Unlike the prior implementations of coding, this study sought to quantify interactions occurring at a classroom level, rather than a one-on-one level, and sought to identify and examine characteristics of classroom practices as they affected the interactions.

Each line of transcript was encoded using six scales which were variations of Cheffers' Adaptation to Flanders' Interaction Analysis Scales (Cheffers & Mancini, 1989; Flanders, 1970). The lines were classified according to the micro source of the statement, the macro source of the statement, who was speaking, the nature of the comment, the level of comment according to Bloom's Taxonomy, the phase of the inquiry process, the context of the comment, and the level of mathematical thought . Additional items were tagged including who prompted the periods of reflection which occurred. An idea spoken by one individual whose transcription extended beyond one line was encoded as one concept, unless a noticeable change in intent or concept occurred.

Source of the Idea

Intending to capture illustrations of the concepts resonating in students and instructors, in each segment of transcription, the source of the idea was acknowledged. The scale given in Table 3.1 classified the source of the idea discussed. When the discussion suggested the development of an idea, the line of transcription was assigned a "2" or "5" depending if the idea originated with the instructor or with the student, respectively. Each line of transcription was classified twice using this scale: once for the source of the idea at a micro level and a second time for the source of the idea on a macro level. The micro source of the idea indicated the purpose of the speaker's comment or question on the more local idea being discussed. The macro source of the idea identified the larger, more global purpose which prompted the smaller questions or statements. The macro source of the idea was intended to illustrate whose ideas, students' or instructors', drove the investigations and analysis. The scale for the micro source of the idea was developed to show how questions and comments occurring more locally affected discussion and the resonance of participants in the discussion. The sources of idea were grouped further with items one through three being assigned as the teacher's ideas and items four through six being assigned as the students' ideas. The groupings provided for simpler graphical uses while acknowledging the source of the idea.

Level of Question or Statement

Each segment of transcription was encoded according to Bloom's Taxonomy as given in Table 3.2. The classification of a statement or question was determined to a degree by the

Table 3.1. Micro and macro source of idea

Teacher	1. Give direction, praise or criticism
	2. Develop teacher idea - use, develop and clarify teacher's idea
	3. Cite teacher idea - acknowledge a teacher's idea
Student	4. Give direction, praise or criticism
	5. Develop student idea - use, develop, and clarify student's idea
	6. Cite student idea - acknowledge a student's idea

statements or questions surrounding the segment. The nature of a response indicated the depth of thought needed to respond, the prior knowledge recalled, or new knowledge being built. As noted by Mills, Rice, Berliner, and Rosseau (1980) the interaction between two parties, including student and instructor, does not always produce agreement in the level of responses. Hence, the use of Bloom's Taxonomy could suggest when agreement in resonance was not attained if questions and responses occurred at different levels.

Table 3.2. Level of question or statement classified on Bloom's Taxonomy

1. Knowledge – recall
2. Comprehension - uses idea without relating it to other ideas or seeing fullest meaning
3. Application - use generalizations in new and concrete situations
4. Analysis - break down material into its parts and determine relations or organization among parts
5. Synthesis - put together parts into a new, unified whole
6. Evaluation - judges the value of ideas, procedures, methods using appropriate criteria

Classroom Speakers

Not originally coded in the pilot study, questions and comments by individual students were coded in this investigation. In so doing, the codes could suggest the level of participation with the interaction of many class members or a few students. Students, instructors, and TA's were given pseudonyms to protect their identity. (See the Human Subjects form in Appendix E.) The pseudonyms were assigned a number based on the order of alphabetized pseudonyms. These numbers were then used to identify who was speaking when working with just the codes. The instructor for any given section was assigned the number "0." When a student's voice could not be identified on the recording, the number "34" or "35" was assigned. Two numbers designated this role as there were occasions in which one unidentified voice followed another. The number "36" was used to designate when several students of the class gave the same comment. The teaching assistant who worked with the section was assigned the number "37," and the researcher as an observer in section one of Math 181 was assigned "38."

While specific numbers were assigned to indicate the speaker, a more general number was assigned to each segment to indicate whether the instructor or a student was speaking and whether a statement was made or a question asked. This scale, given in Table 3.3, also distinguished between the types of statements including a response or initiation of an idea or procedure.

Table 3.3. Who is speaking?

Teacher	1. Responds - reaction or reply to a prior question, comment, or procedure is given.
	2. Solicits - question is asked or participation of another is invited.
	3. Initiates - idea or procedure is suggested.
Student	4. Responds - reaction or reply to a prior question, comment, or procedure is given.
	5. Solicits - question is asked or participation of another is invited.
	6. Initiates - idea or procedure is suggested.

Phase of Inquiry

Not originally acknowledged in the codes in the pilot study, coding for the phase of inquiry indicated which phases were implemented in the laboratory investigations, time emphases placed on the various phases, and the existence of multiple cycles of the inquiry process. The four phases of inquiry identified were prediction, experiment, analysis, and reflection. The phase descriptions are given in Table 3.4. Both students' behaviors and recorded comments suggested the classification to be used. When a non-laboratory comment or question was made, the corresponding transcription was encoded with "X" representing "non-lab." Characteristics which were classified as "non-lab" included generic "how-to" calculator discussions, comments related to exams or assignments separate from the laboratory investigations, statements such as "Did everyone hear that?" or "small-talk" including comments made in jest.

Table 3.4. Phase of inquiry

Prediction	Hypotheses are made about the context. Questions to be investigated are determined. Plans for data collection are generated and discussed.
Experiment	Data are collected.
Analysis	Data are mathematically modeled and used to answer the questions generated during the prediction phase. Abstraction of the mathematics occurs here.
Reflection	The process is reviewed and discussed emphasizing the degree to which the original questions were answered, the new questions which arose, and general heuristics to follow when presented with new and similar future situations.
Non-Lab	Topics discussed are not immediately relevant to the laboratory investigation.

Context

Separate from the phase in which the comments and questions occurred, each segment of transcription was encoded according to the context emphasized whether mathematics, science, a link between mathematics and science, or other. (See Table 3.5.) Much like Bromme and Steinbring's (1994) classification of symbol, object, or relation, categorization of the context was used to indicate when switches in the nature of the discussion occurred surrounding the mathematics and science. The context scale was added to the scales implemented in the pilot study. The other scales had not addressed when links

Table 3.5. Context

Mathematics	Mathematics including symbolic representations, concepts, procedures, and patterns forms the center of discussion.
Science	The science component or "physical objects" of the laboratory investigation are discussed.
Link	The mathematics and science are linked as the symbols, equations, graphs, and numbers are directly related to the scientific interpretation.
Other	A context other than mathematics or science is discussed.

were made between the mathematics and science. Being a course in which the mathematics was founded in the science discussion, a measure was needed to indicate the emphases placed on each context and the relationships between contexts.

Level of Mathematical Thinking and Understanding

Described by Rasmussen (1999) and Gravemeijer, Cobb, Bowers, and Whitenack (1999), the level of mathematical thinking and understanding was classified for each segment of transcription encoded as having a mathematical context. The categories of understanding were labeled as situational, referential, general, and formal (Table 3.6). Not originally coded in transcriptions from the pilot study, classification of the level of mathematical thinking and understanding is intended to suggest whether students had procedural or conceptual understanding. Situational understanding referred to how to act mathematically. Typically, more procedural based comments were assigned to this category. Referential understanding indicated that students' understandings were of paradigmatic, experientially real settings, suggesting some conceptual understanding of the links between the mathematics and the context being represented. General understanding referred to the focus on the interpretation and solutions independent of a situation or specific context. Formal use of conventional

Table 3.6. Level of mathematical understanding

Situational	Understanding is based on how to act mathematically.
Referential	Understanding is grounded in paradigmatic, experientially real settings.
General	Understanding focuses on interpretation and solutions independent of a specific context or situation.
Formal	Understanding is demonstrated with conventional notation and inscriptions.

notation and inscriptions was classified as formal understanding. A sample of the transcripts with the assigned codes are included in Appendix C.

Graphical Representations

One goal of the study was to produce a quantitative representation of inquiry process and the ability to discern differences beyond qualitative observations. To achieve this, once the transcriptions are encoded, various graphs will be examined to illustrate the interactions which occurred in the classroom, the cycling of the phases of the inquiry process, the sources of the ideas, the role of the context, and the interactions between the various codes. The graphs will be analyzed to determine if new information about the courses is generated in addition to supporting the conclusions made with the qualitative data.

Particular graphs will be examined to indicate the development of the structural components of the class. Graphs of the phases of inquiry across time will be constructed to illustrate what phases are incorporated into the laboratories as well as the time spent on each phase. This type of graph would suggest if multiple phases of inquiry occurred and the variations in the cycles when multiple cycles did occur. Graphing the context against time would suggest the emphases placed on the mathematical, science, and relational components and when and how often discussion of the various contexts occurred. Also intended was the use of the context graph to inform the qualitative data of when and how the classes discussed methods of mathematical modeling. For instance, depending on the emphasis on each of the contexts, how were students' modeling abilities affected? While graphs of the context cannot answer the question apart from the qualitative data, indications of the roles of the contexts in the lab discussions would point to reasons for students' successes or difficulties.

Aggregates of the coded data will be used to point to characteristics of the classroom not attained through examination of the separate scales. An aggregate of the context with the micro sources of ideas could point to whose ideas were most influential in discussion of the contexts. For instance, since the students were life sciences majors, did students' ideas dominate science discussion? Whose ideas, students or instructor, tended to promote discussion of the links between science and the mathematics? An aggregate of the phase of inquiry and sources of ideas could indicate similar information. Which source tended to produce ideas in each of the phases, particularly the analysis and reflection phases? If the instructor's ideas were the main source during the analysis phase, was agreement in resonating conceptual schemes attained? These characteristics could inform the interpretation of students' achievements in light of the classroom environments.

Statistical Comparisons

The coded data provided the opportunity to examine various statistical comparisons. The statistical comparisons could be used to indicate additional characteristics of the classrooms and determine if patterns regarding the various interactions across sections and across laboratories were significant. The intent of the statistics was not to indicate which section was better but to better inform the nature of inquiry and offer a different lens with which to view the data and classroom environments.

Research Goals and Analysis

One goal of this study was to examine attempts to implement inquiry in a college mathematics course. In light of the literature reviewed, key points of interest in pursuing and studying ways in which inquiry was achieved include the role of negotiation and interpretation of the tasks and problems explored, the methods of mathematical modeling used in class, the development and use of symbols and language in the course, students' levels of understanding, and the degree to which reflection, metacognition, and heuristics play a role in the classes. These items will be used to indicate points when agreement in the resonance of appropriate schemas was or was not likely occurring in the minds of class participants. Chapter 4 will highlight these issues in terms of the classroom environments. Chapter 5 will address the effects of the environments on classroom interactions and students' understanding with these issues in mind.

A second goal in this study was to develop and apply methods to represent inquiry in the mathematics classrooms. Coding class transcripts for the phase of inquiry, context of discussion, sources of ideas, level of mathematical understanding, level of comment and question, and speaker, will quantify classroom characteristics. Snapshots of the environments will suggest attributes of the environments, time spent on various phases and contexts, and the periods in which students' ideas dominated discussion. Accompanied by the qualitative data, the graphs will be used to point to the role of the above characteristics in promoting or hindering agreement in resonance. Times in which agreement in resonance was or was not occurring evidenced by the qualitative data but not by the quantitative representations will be mentioned. Suggestions for future codes which may enhance the characteristics tagged by the existing scales will be given. Graphs of the codes which illustrate the structural components of the environment will be displayed. More complex graphs of the aggregation of various codes will be shown in Chapter 5 to indicate the interactions of the class affecting students' mathematical development and understanding.

CHAPTER 4

INQUIRY ENVIRONMENTS

Two sections of Mathematics 181, Calculus for the Life Sciences, were observed during class sessions related to three laboratory investigations. The intent of this research project was to examine the nature of inquiry in the classroom environments during the laboratories, classify the degree to which certain characteristics of an inquiry environment were achieved in the different sections, and represent the characteristics quantitatively. In this chapter the structural components of the two class environments will be described. As part of the structural components, instructors' goals and intents for the classes will be discussed with similarities and differences noted. Illustrations will be given of how the class interactions highlighted the intended goals.

Instructors' Goals for Mathematical Modeling

Some of the greatest influences on the nature of inquiry in the classroom environments were the goals set by the instructors. Before any of the laboratories were conducted, each instructor set goals regarding the nature of inquiry, the role of mathematical modeling, the importance of context, and the types of mathematics to be pursued in the course. Both instructors intended the classroom environment to promote inquiry and placed emphasis on the use of context to bring real world application to the mathematics.

The instructors had different goals for the role of modeling in the course. The instructor of section one of Math 181 intended for mathematical modeling to be developed as a tool in the course, much like other mathematical procedures the students were to learn. The modeling would be used as a tool to bridge the context and the mathematics. The researcher and instructor of the other section of Math 181 intended for mathematical modeling to provide the motivation and the foundation for most of the mathematical discussion in the course. The instructors' goals for the laboratories were described in the interviews prior to the investigations, course objectives, and daily lesson plans.

Tools-Focused Class

The instructor of section one of Math 181 was an experienced instructor who had taught Math 181 several times in previous semesters. For the instructor, mathematical modeling was a tool used to take the students from the context of the scientific setting to the mathematics. Once reaching the model, the family of functions containing the particular model would be discussed. The mathematics would be abstracted and additional methods for solving mathematics problems would be developed. The instructor's desire, as highlighted during the class sessions, was for students to see that many mathematical problems could be solved using a variety of different methods.

The instructor communicated her goals for mathematical modeling prior to the laboratories. Before the water flow laboratory, her goals were:

> To have (a) context to do more math; to have the students revisit the quadratic equation and how to work with that; to have the students have another application where they are able to talk about what kinds of models might be appropriate.

Before the light intensity investigation, the instructor reiterated the role of the laboratories when she said that the goal was "To have a meaningful context for (the) exponential, in this case for exponential decay." She elaborated:

> So my goals are to come up with a good context and then to use that context to develop, to talk about difference equations which we haven't talked about, and generating equations. So we'll have that tool to use later on.

While mathematical modeling played an important role for students in the course, the process more often led to the mathematics from which the main goals of the course could be met.

These main course goals included discussing different families of functions, methods to solve problems, and concepts of rates of change and derivatives. With tools as the emphasis of this instructor's goals for the course, the class will be referred to as the "tools-focused-class."

The instructor's goals for the penicillin laboratory differed slightly from the previous laboratories. The analysis of the data would consist of two parts. In the first part of the laboratory, analysis of the data representing the wash-out of the penicillin would be used to assess students' abilities in modeling data indicative of exponential decay:

> I want to see if they understand that it's an exponential decay, and if they come up with that themselves. That they'll be comfortable enough with the ideas to do the first part. So part of it is an assessment for that.

A second part of the lab was to develop methods to model data representing the amounts of penicillin in the blood when new doses were administered. The instructor's goals for the second part was to proceed with instruction of additional tools which could be used to model the data:

> And then part of it I'm going to go ahead and do the algebraic method to figure out where we're headed. Then I'm going to ask them to plot, once we know where the limit is, to plot the differences on semilog paper and then to see if we can tie it in that way.

The penicillin lab was in part to be used as an assessment and to form a context for new instruction to be tied with previous methods. A portion of the lab maintained the instructor's goals to develop and extend mathematical tools for the students.

Modeling-Focused Class

The researcher served as the instructor for section two of Math 181 and had different goals for the course, the laboratories, and the role of mathematical modeling in the development of the mathematical concepts. In addition to providing links between the science and mathematics context, the instructor intended for mathematical modeling to be an object obtained by students. Modeling framed the development of the course.

The instructor's intents were noted in the course syllabus, in her lesson plans for the laboratories, and in the class discussions. In the course syllabus, the instructor wrote:

> With an emphasis on inquiry and modeling, students will acquire and apply mathematical tools and demonstrate understanding of the underlying mathematical concepts of functions and derivatives. Laboratories will be conducted in which students will make hypotheses, conduct experiments to gather data, and then analyze the data. During the analysis and reflection, the mathematics will evolve.

On the first day of the semester, the students were informed of the inquiry process and the modeling process. As students prepared for their first laboratory, separate from the laboratories discussed in this study, the class generated a list of items to keep in mind as they conducted experiments. Highlighted on the list was the need for justification of their work and the numbers they generated when deriving a formula or a model.

Throughout each of the subsequent laboratories, students were reminded to document the process, to justify the development of their model(s), and to use appropriate variable names in their model. In her lesson plans prior to the water flow investigation, the instructor recorded her intentions for the first assignment in working with the data. Each student was to record the "observations and the methods you use to analyze the data." The instructor explained to students that "analyze the data" meant to calculate the flow rate, find an equation relating the flow rate and height of water and justify the equation. In addition, students were to "present their analysis with justification for their calculations and interpret their results in light of data collection. Each group of students should demonstrate how they predict the flow rate based on a height of water."

The light intensity lab followed the water flow investigation. When launching the light intensity laboratory, the instructor encouraged students to reflect on the procedures from the previous laboratories, specifically: "What mathematical relationships were found? What were the primary variables involved in the water flow laboratory?...What are you going to do once you have data? What graphs will you examine once you have data? With the water flow data, what did you do to find the equation?" The researcher intended for students to "objectify" the modeling process by recording their hypotheses, observations, questions, and methods. Reflecting on procedures applied in previous laboratories was intended to promote the objectification. This class will be referred to as the "modeling-focused-class."

The instructors' intents for mathematical modeling in the course rated differently on the APOS scale. For the instructor of the tools-focused class, mathematical modeling would be classified as an action or a process, while the instructor for the modeling-focused class intended her students to encapsulate modeling as an object. Using similar ideas as posed by Duoady (1991), the tools-focused class would be structured with modeling being a tool used in the solving of problems. The modeling-focused class would be structured with mathematical modeling as the overriding organization scheme for the course. Students in the modeling-focused class were to gain a modeling perspective while developing other tools to solve the different types of problems faced.

The differences in the instructors' goals resulted in different manifestations of working with the data and other mathematical problems addressed in the course. These differences were slight at times and changed with laboratories within class sessions. Included in the differences were the role of context, language and symbol use, and the amounts of reflection. The differences did not suggest that one section was better than the other. Proving that one section was better was never an intent of this research study, nor did the results point to any one section as higher achieving. The focus was placed on how the differences in the instructors' goals influenced the interactions, the developments of the laboratory investigations and other mathematical problems, and ultimately the role of inquiry in the course throughout the semester.

Timing of the Laboratories

One way in which the two sections of Math 181 differed as a result of the instructors' goals was the timing of the three laboratories throughout the semester. The two sections of Math 181 acted on independent time schedules when conducting the laboratories. With 59 class sessions in a fifteen week time period, Table 4.1 indicates when each of the laboratories began and the number of class sessions spent discussing the laboratories. Only those class sessions which led directly to methods students could implement in their lab reports were counted. If discussion of the lab occurred in a portion of a class session, but not an entire class session, the class session counted as a whole session toward the number of class sessions.

Table 4.1. The beginning and number of class sessions used for each lab

	Tools-Focused		Modeling-Focused	
	Began at Class Session	Number of Sessions	Began at Class Session	Number of Sessions
Water Flow Lab	3	8	7	11
Light Intensity Lab	28	6	37	10
Penicillin Lab	44	5	47	4

The contents of Table 4.1 do not suggest that spending more time on the laboratories was better but support the difference in focus between the two classes. For each of the water flow and light intensity laboratories, the modeling-focused class spent at least three more class sessions over the tools-focused class in developing and discussing the contents and results of the laboratories. The additional class sessions suggest that greater emphasis was placed on the development of the inquiry process or that the modeling-focused class was slower in reaching the desired content.

A switch occurred with the penicillin laboratory. The tools-focused class discussed the laboratory over five class sessions with the modeling-focused discussing the laboratory in just four class sessions. In both classes the penicillin lab proceeded differently than the previous two laboratories. In the tools-focused class, as intended by the instructor, during the first part of the laboratory, students modeled the exponential decay branches of the "penicillin wash-out." Students' work was completed outside the class sessions and submitted for assessment. The instructor had intended for the second part of the lab to be a period of instruction on procedures to find a model for the amounts of penicillin when new doses were administered. Instead, the students worked on a similar "wash-out" problem. (See Appendix D.) Rather than giving immediate instruction, students worked with others in class to generate a discrete equation. Previously in the water flow and light intensity investigations, the instructor had more often had students present their data and discuss differences in graphs before instructing students of tools to use to model the data. In the penicillin lab, students were to first develop a model before additional instruction occurred.

In the modeling focused class, as intended for the water flow and light intensity laboratories, the students presented their initial analyses of the data. Students hypothesized various models and raised additional questions targeting the need for more mathematical procedures to justify and confirm whether their mathematical equations were adequate. Students' questions led to class discussion and instruction of additional means of modeling. The penicillin lab occurred differently. As the table indicates, only four class sessions were devoted to the penicillin lab. The class sessions were not consecutive with most discussion occurring during the 52nd and 53rd class sessions due to an absence by the instructor and a scheduled exam. With the ending of the semester, less time was devoted to the laboratory as additional topics needed to be covered before the semester ended. So less time was permitted to the development of the inquiry process. Students did not present their initial attempts of analysis, and the instructor proceeded with explanations of modeling methods. The tools-focused class had fewer additional topics to cover in the remaining three weeks as some topics the modeling-focused class needed to address had been previously discussed in the tools-focused class.

As examined in this portion and throughout the remainder of this chapter, the instructors' intents for the laboratories differed. In general, the instructor of the tools-focused class intended for the laboratories to provide the underlying context on which to build the mathematical tools. The instructor of the modeling-focused class used inquiry and modeling as the overriding structure to promote discussion of the mathematical procedures. Some changes did occur within the classes. Coverage concerns influenced the degree to which inquiry occurred, evident in this portion and in the remainder of the chapter.

The Inquiry Process

The two instructors for the course pursued an inquiry environment in their classrooms. Striving for constructivist environments, the instructors intended to build on students' prior knowledge, promote interactive classrooms, and engage students with scientific contexts of mathematics. In their pursuits, many similarities and differences existed. The differences included structural components of the class and the instructors' goals for modeling. The appearances of the classroom environments were similar in spite of the goal and structure differences.

Cycling in the Inquiry Process

In both classroom environments, multiple cycles of inquiry were pursued. Each class accomplished multiple cycles. The developments of the cycling differed in the classes while the appearance of the cycles were similar.

Intents for Multiple Cycles in the Modeling-Focused Class

The instructor for the modeling-focused class intended inquiry to proceed in a cyclical fashion. The lesson plans for the laboratories demonstrated the instructor's intent for multiple cycles of the inquiry process across the laboratories. Each cycle was to include prediction, experiment, analysis, and reflection phases. In her lesson plans, the instructor described how the process would unfold.

Each phase had characteristic components. During the prediction phase, a context would be presented, questions would be asked of the context, hypotheses would be made, and an experiment would be planned. The experiment phase would proceed as students conducted their experiment(s). Once data was collected, students would analyze the data by finding a mathematical model using methods they felt appropriate. The analysis phase would continue as students presented their data and analysis. Following the presentations, students would reflect on the methods of analysis and ask new questions about the data and methods of analysis.

Once one cycle was completed, the instructor intended to pursue at least a second and possibly a third cycle of inquiry. The additional cycles would focus more fully on the mathematics. The predictions would emphasize the patterns within the mathematics and a particular family of functions. Explorations of the patterns, analysis of the mathematical context and modeling methods, and further reflection comprised the experiment, analysis, and reflection phases of the additional cycles.

Inquiry Cycles in the Modeling-Focused Class

The modeling-focused class did proceed in a cyclical fashion during two of the three laboratories. Graphs of the coded transcripts demonstrated the phases within the cycles for the laboratories. Figure 4.1 displays a graph of the phases of inquiry across the number of lines of transcription for the water flow lab in the modeling-focused class. The line numbers were indicative of time. Typical of the first two labs in this class, at least two full cycles of prediction, experiment, analysis, and reflection were evident in the water flow investigation. During the second and third cycles greater emphasis was placed on the mathematics in the analysis phase as the instructor intended.

The penicillin lab proceeded in a slightly different fashion from the water flow and light intensity laboratories. The penicillin lab had followed on the heels of and may be considered an extension of the light intensity investigation. Forms of exponential equations resulted from both the light intensity and penicillin labs. Different factors influenced the cycling of inquiry in the penicillin lab. These factors included a break in the discussion of the results due to an absence by the instructor and the need to address other course topics with few class sessions remaining. The factors constrained the degree to which open inquiry was pursued and occurred. In particular, students did not give formal

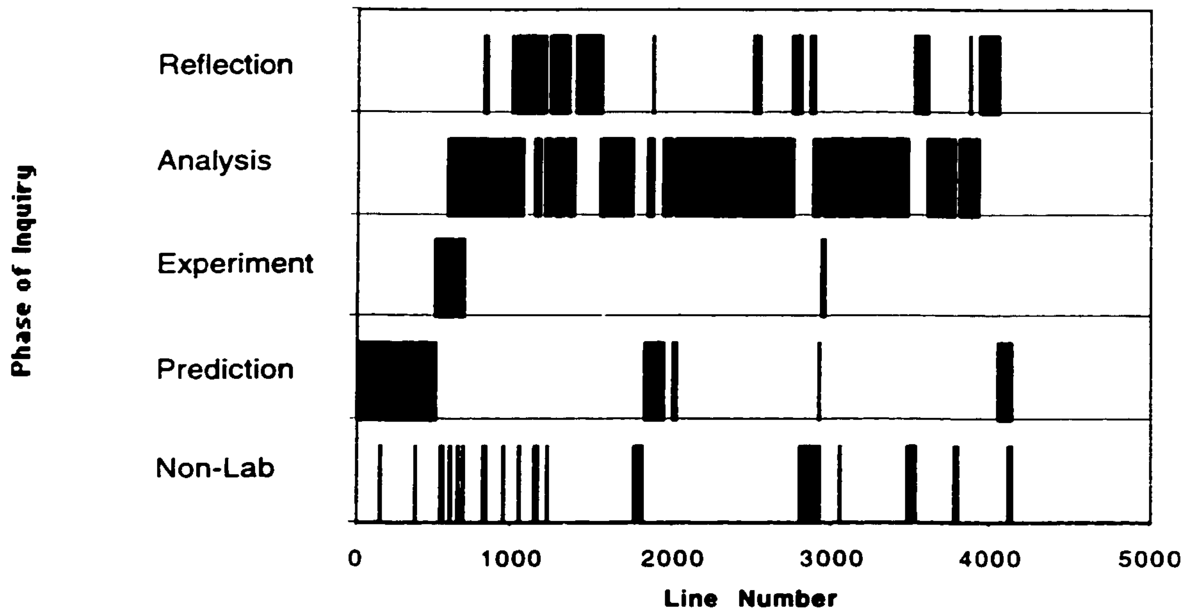

Figure 4.1. Inquiry phases for the water flow lab in the modeling-focused class

presentations of their initial analyses. Hence, a separate reflection phase during which students could pose new questions for investigation did not occur. As shown in Figure 4.2, only one cycle of inquiry occurred in the penicillin lab with solid prediction, experiment, and analysis phases. Emphasis was again placed on the analysis phase.

Intents for Multiple Cycles in the Tools-Focused Class

The instructor of the tools-focused class also intended to implement the various phases of inquiry in the investigations in her class. During the interviews before the investigations the instructor stated that her goals were for students to "hypothesize a graph...and then plan the experiment, and then do the experiment and then come back and compare the data, (and) talk about analyzing it." Implied by her statement, students would

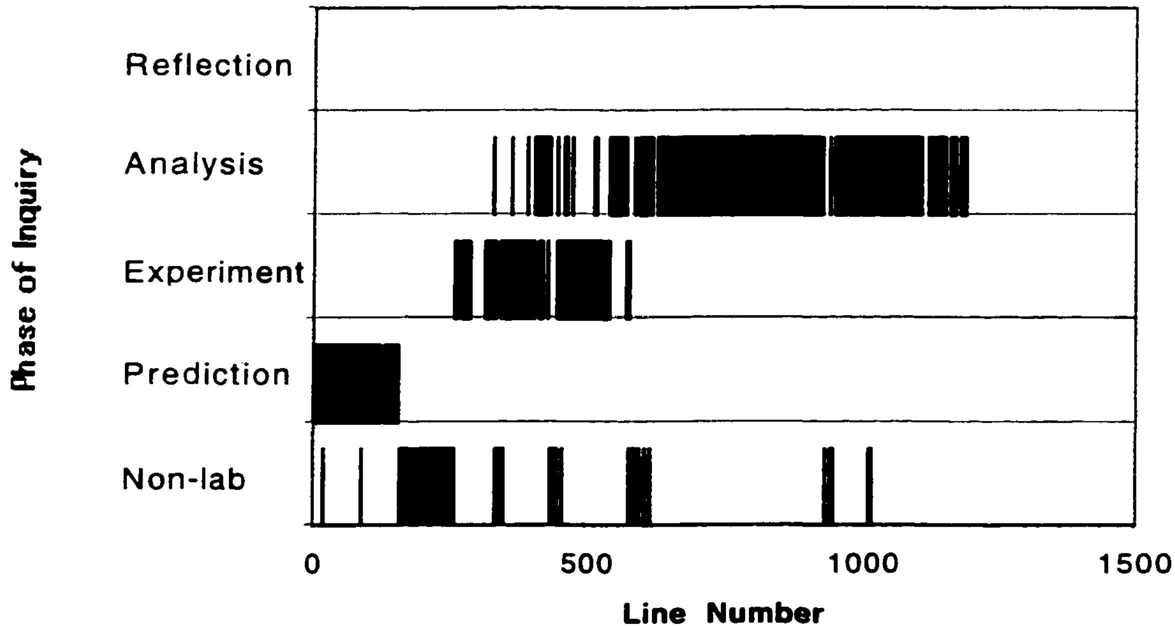

Figure 4.2. Inquiry phases for the penicillin lab in the modeling-focused class

complete prediction, experiment, and analysis phases. Thus the instructors both intended for students to engage in similar phases of inquiry.

Some differences were evident in the instructors' intents. One difference was that students in the tools-focused class would present their data and then discuss the analysis as a class. The students in the modeling-focused class would first attempt analysis before presenting the data and analysis. A second difference was the intent for reflection phases in the modeling-focused class. The instructor for the tools-focused class did not mention her intention for a separate reflection phase.

Similar to the instructor of the modeling-focused class, the instructor for the tools-focused class intended for multiple cycles during the investigations. The instructor's goal for multiple cycles could be inferred when she described working with the mathematical models.

In the interview before the light intensity investigation she stated that the context would be used to:

> talk about difference equations...and generating equations....And then to give them some experience with playing with a model because they are going to graph the differences versus the intensity and the differences versus the time and...look at data a variety of different ways and see if some insights come from it or help you to build a model.

The instructor desired multiple cycles of inquiry during the investigations. A second cycle would be used to "play" with the different graphs and models suggestive of additional experiment and analysis phases.

Inquiry Cycles in the Tools-Focused Class

Graphical evidence of the coded transcripts illustrated that two cycles of the phases were present during the water flow and light intensity laboratories in the tools-focused class. (See Figure 4.3.) Representative of the first two labs in the class, two distinct cycles of inquiry were assumed in the water lab investigation. Similar to the graph for the modeling-focused class, the second cycle was weak in the experiment phase and strong in the more mathematical analysis phase. The graph also demonstrated that some formal reflection occurred.

Like the modeling-focused class, the penicillin lab in the tools-focused class consisted of just one cycle of inquiry with no separate reflection phase. As suggested by Figure 4.4, the analysis phase dominated the discussion. The instructor intended the penicillin lab to provide an assessment for students' understanding of exponential decay. Rather than discuss methods focused more specifically on the analysis of the data gathered in lab, students were expected to analyze the data outside class sessions while focusing in class on a related problem examining the "wash-out" of pollution in a lake. (See Appendix D.) Little direct

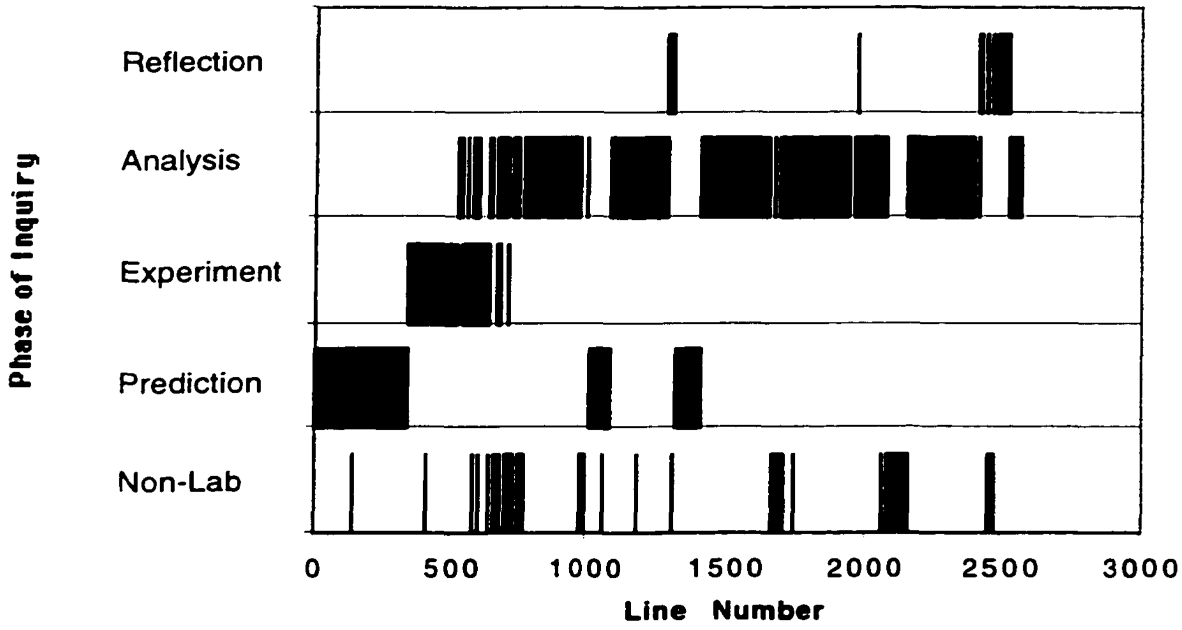

Figure 4.3. Inquiry phases for the water flow lab in the tools-focused class

instruction occurred as students collaborated with classmates on methods to solve the problem before presenting solutions. More discussion of students' work on this problem is given in Chapter 5.

Inquiry Cycle Comparisons between the Two Classes

The two classes had more similarities than differences when considering the phases of inquiry implemented in the class sessions, though the instructors' goals for the structure of the classes slightly differed. One similarity concerned the non-lab component. The two classes had several non-lab items displayed on the graphs. Non-lab items included interactions surrounding basic calculator operations, assessment issues such as upcoming tests, project reports, and grading policies, and remarks within class or group discussions not related to the laboratories including "small talk," sarcasm, and campus events. The non-lab

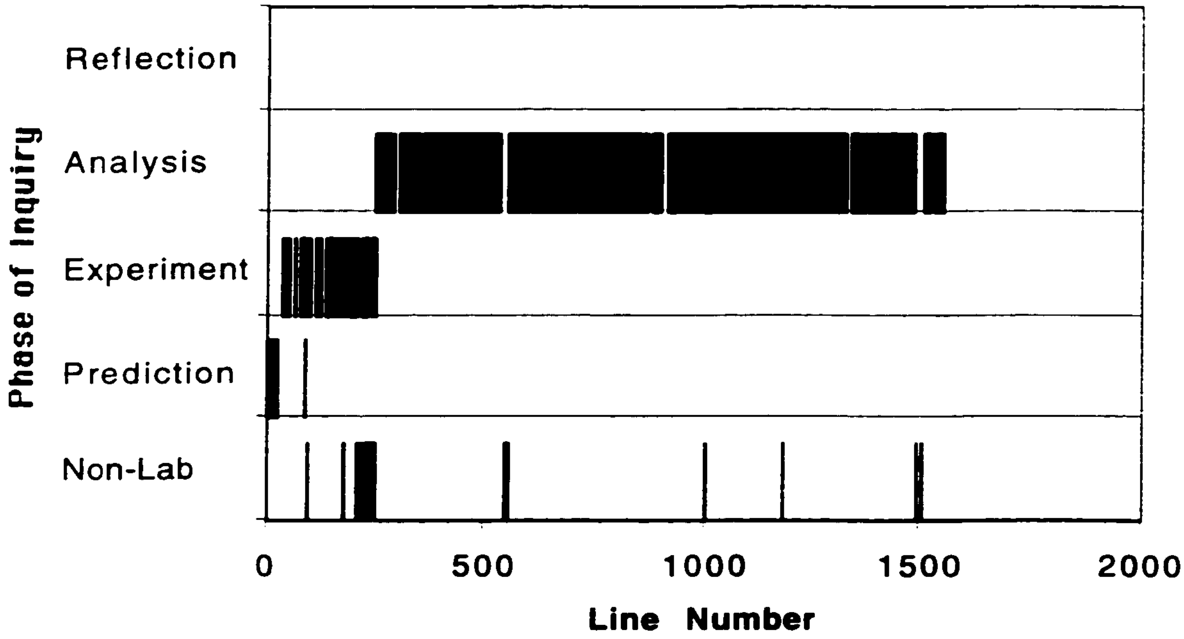

Figure 4.4. Inquiry phases for the penicillin lab in the tools-focused class

items in the class sessions may have interrupted the flow of the class sessions, though the graphs suggest that the items did not hinder the flow of the cycles of inquiry.

More often, both instructors intended and provided for the accomplishment of multiple cycles of the inquiry process in their classes. In each class, emphasis was placed on prediction, experiment, and analysis phases during the first cycle with weaker experiment and stronger analysis phases occurring during the additional cycles. Differences between the two classes included methods of accomplishing the various phases, including reflection. The graph of the phases for the modeling-focused class indicated more reflection than held in the tools-focused class. For the modeling-focused class, the significant portion of reflection occurred in the first half of the investigation, suggesting that the reflection was due to the structure of the course as students' presented and then generated new questions based on the presentations. Amounts of reflection in the second halves of the investigations were more comparable to the reflection which occurred in the tools-focused class.

The differences in class structure and instructor goals resulted in few overall differences in the appearance of the inquiry phases. Both classes completed multiple cycles of inquiry with similarities in emphasis placed on each phase.

Questioning

In the inquiry oriented environments the questions under investigation played an important role in the structure and development of the content in both classes. In examining the types of questions under investigation and the number of questions asked by students and instructors, the classes differed. In the modeling-focused class, students' questions determined the direction of the investigations and the manner in which the content was covered. In the tools-focused class questions from both the instructor and students influenced the direction of the laboratories. In both classes, much hinged on who posed the question for investigation.

Who Posed the Question?

Forman and Steen (2000) noted "experience with rich contexts helps students recognize that asking questions is often as important as finding answers" (p. 148). Such contexts offer variations causing the stimulation of mathematical habits of mind and propellant of students to deep understanding. One essential feature of classroom inquiry as given by the National Research Council (2000) was student engagement in scientifically oriented questions. The variations of this essential feature were presented on a scale ranging from more to less student self-direction with less to more direction from teacher and materials. Having the greatest amount of student direction was "learner poses a question"

(p. 29). Having the least student direction was "learner engages in question provided by teacher, materials, or other source" (p. 29). The two Math 181 environments differed in their ranking on the scales for the laboratories.

<u>Questions in the tools-focused class.</u> The level of student engagement on scientifically oriented questions in the tools-focused class varied across laboratories. Both the instructor and students posed questions to be investigated. Most often, the instructor posed the initial questions for investigation.

During the water flow investigation, the instructor initially posed the question for investigation. At the start of the laboratory, the instructor launched the investigation by giving a context emphasizing the structure of dams with intake towers and asking:

> My mom was skiing down here at the dam and she lost her wedding ring. So they're going to drain the lake to get that wedding ring back for her....I wondered how long that would take....They're going to drain the lake. The question is how long will it take?

The instructor posed the question for investigation, "How long will it take to drain the lake as a function of depth?"

During students' discussion of hypothesized graphs in the prediction phase, a second question was raised, "How does the depth of water affect how fast the flow is?" In the second two class sessions, students verified that both questions were being addressed. Some students thought the question under investigation was "how long it takes (water) to drain" while others interpreted "how fast the depth of the water makes it affect how fast the flow is" as the main question. In the water flow laboratory, questions posed by both the students and instructor were raised and investigated.

The other two laboratories began much like the water flow laboratory, with a variety of instructor and student input towards the development of the questions. For the light intensity lab both the instructor and students operated together to formulate the question for investigation. For the penicillin lab, once establishing the context, the instructor asked, "Would you draw me the graphs of what that penicillin in her body will look like?" before stating, "We are doing a one compartment model…to model what's happening with penicillin in your body." Overall, the instructor strongly influenced the questions for investigation with some input by students.

Questions in the modeling-focused class. The environment for the investigations in the modeling-focused class ranked high in the level of student direction. The instructor intended for students' questions during the prediction and reflection phases to give direction for how the mathematical content would be covered. During the investigations, students posed the questions for exploration. As part of the prediction phase, students were presented with a context, students listed questions related to the context, and the class decided which questions to address.

Typical of the laboratories in the modeling-focused class, the launch of the water flow investigation demonstrated how students posed and settled on the question(s) for exploration. On the first day of the investigation, an overhead transparency described the Hoover Dam, the role of intake towers, and a sample flow rate of water for a particular lake level. Students were asked to rewrite the following sentence as a question to be investigated: "A tower is capable of emptying approximately 3800 cubic feet per second at the present lake elevation." The students were encouraged to think about the situation, specifically, what the statement said about water flowing.

Most students posed questions of the form "How does lake elevation affect the flow rate?" Slight variations of the question included "At *x* lake elevation, how many cubic feet of water does the dam expel for a second?" and "How does the rate that water flows a function of lake elevation?" An additional question, "How does changing the diameter in relation to the height affect diameter?" was disregarded due to the equipment limitations with all the tubes having the same diameter. Once students reported their questions and realized most had the same type of question, the class agreed to answer the question "How does lake elevation affect flow rate?" Students were to be prepared to respond to the statement, "Given a height or depth or elevation, predict the flow rate."

The modeling-focused classroom environment ranked high on the level in which the students engaged in scientifically oriented questions. Students posed the questions for investigation, evidence of student self direction.

The classes varied across sections in terms of who posed the questions for investigations, whether students or instructor. In the modeling-focused class, the prediction phase was structured so that students posed the questions for exploration. In the tools-focused class, both students and instructor influenced the questions for investigation, with the instructor more often initiating the question(s).

<u>Who Asked More Questions?</u>

The number of questions asked in the classes lent additional descriptive evidence of the inquiry environment in each class. Various components contributed to the numbers of questions asked in the laboratories. Any time an instructor asked, "Are there any questions?" or "Does everyone agree with the number _?" or "Is that okay?" the item was coded as a question and contributed to the numbers in the table. Questions related to non-lab items also figured into the calculations. Table 4.2 gives the percent of occurrences of questions in the two classes and the percents across the laboratories. Chi-square statistics generated from two-by-two contingency tables were used to test differences across sections and laboratories. According to the percents of questions asked during the investigations, both instructors asked more questions than students ($p < .01$).

Initially, few differences existed between the classes. During the water flow lab, the tools-focused instructor asked a fewer percentage of questions than the modeling-focused

Table 4.2. Percent of occurrences of questions across sections and laboratories

	Instructor	Students	Occurrences of Questions
Water Flow Lab			
Tools-focused class	67.70	32.30†	610
Modeling-focused class	71.57	28.43**	837
Light Intensity Lab			
Tools-focused class	70.02*	29.98	527
Modeling-focused class	60.44*	39.56**	1034
Penicillin Lab			
Tools-focused class	58.14	41.86†	344
Modeling-focused class	59.46	40.54	296

*$p < .001$. **$p < .001$. †$p < .005$

instructor, but not a significant amount. Students in the modeling-focused class asked a significantly higher percentage of questions ($p < .001$) during the light intensity lab and the penicillin lab than during the water flow lab. Students in the tools-focused class did not ask a significantly different percentage of questions between the water flow lab and the light intensity lab. For the tools-focused class there was a significant difference in the occurrences of questions between each of the first two laboratories and the penicillin lab ($p < .005$). For the light intensity lab the instructor of the modeling-focused class asked a significantly smaller percentage of questions ($p < .001$) than the instructor of the tools-focused class. There was no significant difference in the percentages of questions asked by the instructors for the penicillin lab. Students' familiarity with the class set-up and gains in confidence in the structure of the laboratories likely contributed to the significant difference between the number of questions asked as the semester progressed. In addition, as students inquired more, the classrooms likely reflected an inquiry environment to a greater extent.

The role of questioning in both classes was important. In both classes instructors asked high percentages of questions in the investigations. In the tools-focused class, the instructor and students both posed questions for investigation, with greater influence by the instructor. Little change occurred from the water flow lab to the light intensity lab of which party asked more questions. When the penicillin lab occurred, students asked a significantly greater proportion of questions than they had during the previous laboratories. In the modeling-focused class, the students posed the questions for investigation. As the semester developed, the modeling-focused students asked more questions during the investigations than initially. Based on the occurrences of questions and the progression and building of the questions for investigation, the classes developed in the implementation of inquiry.

Context of the Discussions

The context under discussion, much like the role of questioning, played a significant role in the development of inquiry in both classes. Bromme and Steinbring (1994) stated that "students are able to understand and remember concepts for the subject matter if it has meaning for them" (p. 217). Hiebert and Lefevre (1986) also noted the importance for students to make meaning of the symbols and equations developed. The National Research Council (2000) acknowledged as essential features of inquiry the formulation of explanations from evidence and connections between sources of knowledge. Using an inquiry process to facilitate connections between mathematics and science contexts, both instructors tried to promote an environment rich in connections to further students' mathematical understanding.

Context Goals for the Tools-Focused Class

The goals set for the laboratories in the tools-focused class emphasized the role of context in the development of the mathematics. The instructor intended for students to reason about the mathematics from the science background. In reasoning from the contexts, mathematics was to be the primary focus of the course.

During interviews prior to the water flow and light intensity investigations, the instructor affirmed her goals regarding the science context. She stated that her goal for the laboratories was "to have a context to do more math." The instructor described her plans for students to wrestle with the kinds of data to collect and the accuracy of the measurements. With the reliance on context, the instructor acknowledged the constraints of time, and she asked, "At what point are we trying to teach science and at what point are we trying to teach math? And it's the math we're trying to teach." The instructor knew the science context brought to students experiential grounding in the mathematics, but she desired for the class focus to be on the mathematics.

Contexts in the Tools-Focused Class

The tools-focused class maintained an environment in which the science informed the mathematics of the course. Discussion of the science context was strong during the prediction and experiment phases. During these two phases, relations made to mathematics were most often in the form of predicted graphical relationships or rates of change. Once students reached the analysis phase, science discussion was minimized with emphasis on the mathematics and some discussion linking the science and mathematics.

Various class activities promoted discourse relating the mathematics and science. For the water flow laboratory, link discussion emphasized the relationship between gravity and the mathematics of a falling object. Once the quadratic relationship was established, class discussion focused on the mathematics and methods of working with quadratic data and equations. For the light intensity laboratory, the symbols and equations were interpreted in terms of the light being absorbed by the depths of water. Discussion linking the mathematics and the science continued through much of the analysis. During one class session, groups of students wrote and debated the accuracy of statements interpreting the recursion relationship $I_{t+1} = .82I_t$ developed during analysis:

T: Are they equivalent? If you think they are equivalent, raise your hand. If you think they are not all equivalent, raise your hand. Yes, why not?

Amber: Umm, because some of them don't say at depth 20 percent [inaudible] Like Ruby's, "For every filter added, light is inhibited by 18 percent."....The way I take hers is that for every filter added, there's a total of 18 percent, total of the other too?

T: So if she added 3 filters, what would you expect that sentence to mean?

Amber: 18 percent times three.

T: That it would be decreased by 18 percent times three, so you're reading a linear relationship in there....If we agree there's a possible - another alternate explanation of what this equation means, is there one of these sentences that seems to get around that?

Ruby: That one.

Rita: Yeah, the one that says by the previous.

T: This one.

Rita: Yes.

T: Does that look like a good explanation of what we're talking about? "For each filter added the light received by the detector is 82 percent of the previous one"....So we're saying that talking about the ocean, talking about the sea lions, that if we are going down every 10 feet or going down to another level, that there's a certain percent of light that is being absorbed in every 10 feet.... Decreased by 18 percent, 18 percent of the light is being absorbed, or 82 percent of the light gets through whichever way you want to look it.

The mathematics informed the science followed by additional discussion where the science informed the mathematics.

Context Graphs for the Tools-Focused Class

Figures 4.5, 4.6, and 4.7 display graphs of the discussed contexts in the water flow, light intensity, and penicillin laboratories for the tools-focused class. The graphs illustrate the different emphases placed on the contexts for the three laboratories. In the first two laboratories, emphasis was placed on the science component early in the investigation as the context was presented and experiment conducted. The mathematics component dominated the remaining class time and discussion. The dominance of the mathematics discussion demonstrated that the instructor's goal emphasizing the mathematics was accomplished. During the penicillin laboratory more links between the science and mathematics discussion occurred in discussion reflecting the instructor's goal for students to reason about the mathematics from the science context.

Context Goals in the Modeling-Focused Class

In the modeling-focused class, the instructor's goals for the role of the context emphasized frequent interpretation of the mathematics in terms of the science context.

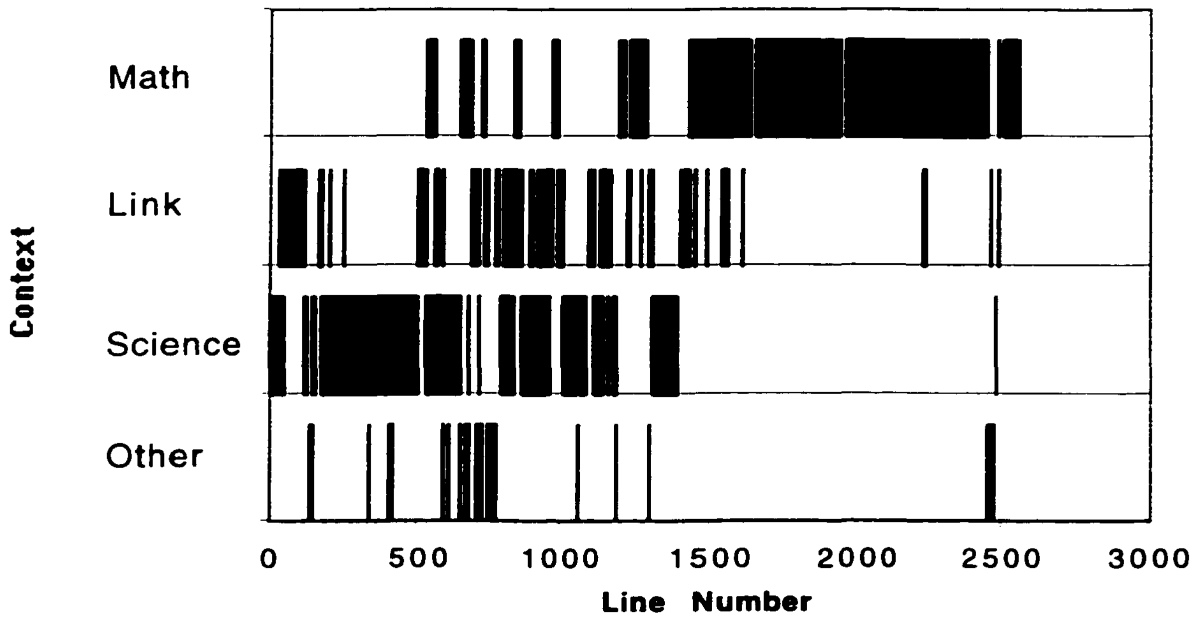

Figure 4.5. Contexts in the water flow lab in the tools-focused class

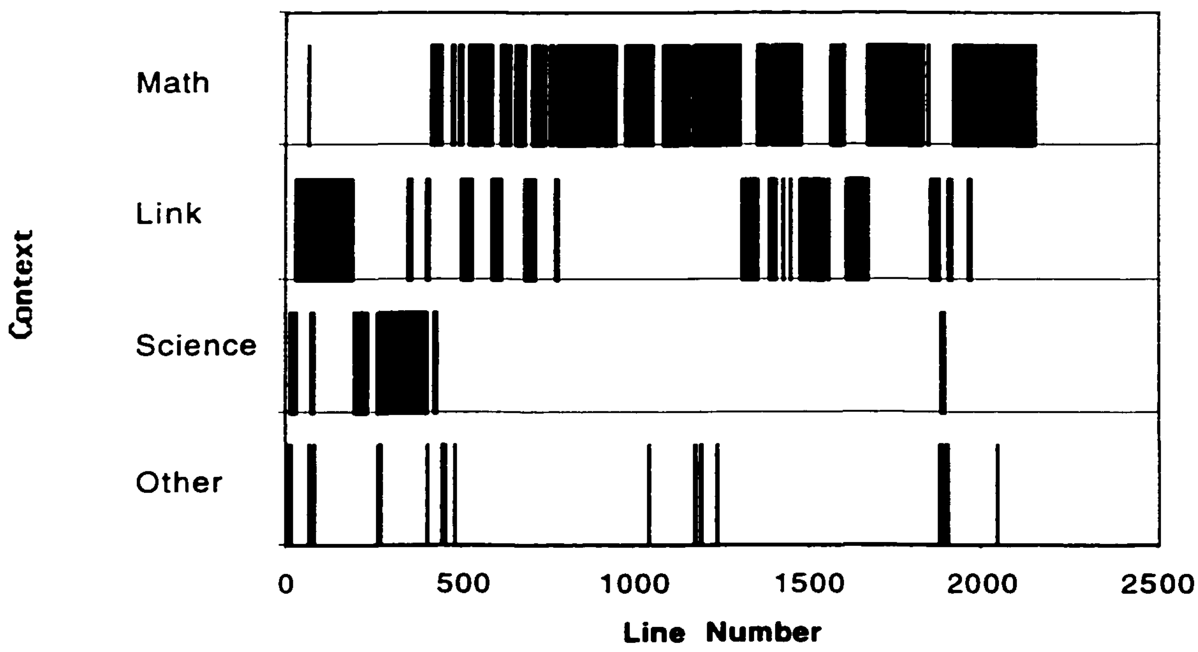

Figure 4.6. Contexts in the light intensity lab in the tools-focused class

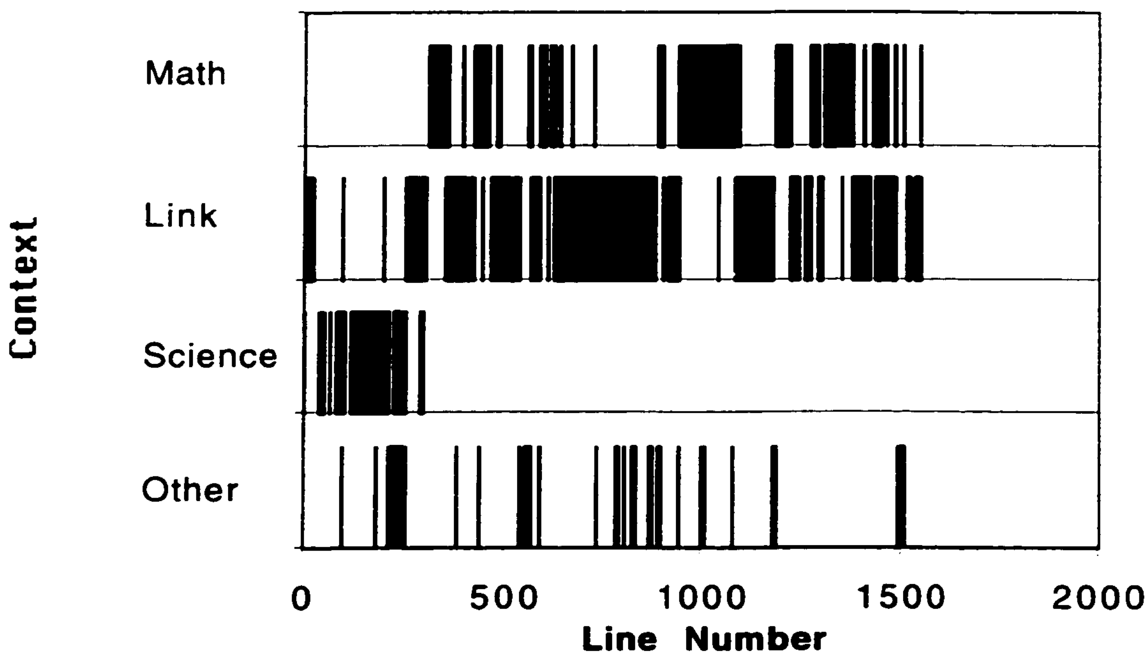

Figure 4.7. Contexts in the penicillin lab in the tools-focused lab

Before the investigations, the instructor affirmed her goals about the context. Prior to the water flow investigation, the instructor stated that she wanted the class to use the context to give justification for mathematical relationships. Specifically, students were to explain why the science suggested a quadratic relationship between the heights of water and draining times. Students were then to relate the calculated flow rates to the methods of data collection. During the light investigation, the researcher intended, "Students will make sense of the equations, interpreting the equations and the signs and size of the numbers in the equations while keeping the perspective of the physical act of the light being absorbed."

Contexts in the Modeling-Focused Class

The instructor communicated to the students the importance of relating the mathematical equations to the science context. Students were instructed to include interpretations of the equations in terms of the data and methods of data collection in their lab reports. During the analysis of the data, she encouraged students to reason about the equations in light of the science context. During the water flow investigation, students had reasoned that a quadratic function would model the height and draining times due to the effect of gravity on the water. Before linearizing the data, students used the context to support their hypothesis that the vertex for the height and time relationship would be the origin.

R: Do we know anything about a, the l, or the k? [Referring to the equation $h = a(t-l)^2 + k$]
Matt: l and k are 0.
R: l and k are 0.
Jamie: Ohh
Dave: You are assuming.
Jamie: That's assuming.
R: You've got some argument here. Can you tell us why you think it should be (0,0)?...
Matt: When the water drains all the way,
Will: You can't have negative slope. The water is not going to drain up [inaudible]
R: The water is not going to drain up so that means you can't have a negative -
Will: You can't have anything below 0.
R Okay. So their argument is that you can't have a vertex down here. Because you are not going to have a negative height because it's not going to drain up....
Kiene: We decided that the only thing you can say for sure is that a is positive....
Martha: If there's no water, there's nothing to drain, so there won't be any time for it to drain.
Jamie: I understand that.
R: So, that's an argument that (0,0) is definitely a data point....
Laurie: I was just going to say if the vertex were (1,2) or something, well then why do you get that it's possible to fill the tube up to .9 and then what happens, does it just sit there and not flow out even though the hole is open?
Jamie: Well, then, (0,0) is the vertex.

Students reasoned about the mathematics based on the context.

Discussion linking the mathematics and science during class was encouraged. The instructor also encouraged students to ask mathematical questions during the reflection phases. Students had a tendency to ask questions regarding how data was collected and how improvements in data collection could improve the results. While encouraging students' comments, the instructor reminded students that in a math class, they should be asking mathematical questions in addition to the science questions related to data collection. The instructor desired the mathematics to remain the focus of the course.

Context Graph for the Modeling-Focused Class

Figures 4.8, 4.9, and 4.10 display the graphs of the context discussion for the three investigations in the modeling-focused class. The frequency of interactions in the math component indicated the emphasis on mathematics as desired by the instructor. The interactions in the link context illustrated that discussion relating the mathematics and science occurred through much of the investigation while the mathematics was performed. Based on the graphs, the environment promoted reasoning between mathematics and science. Both classes were successful in achieving environments which emphasized the relationships between the mathematical and science contexts as intended by the instructors. Class interactions demonstrated how the science informed the mathematics and how the mathematics informed the science. Also noted was the emphasis on the mathematical component in both classes, while discussion solely focusing the science context ended early in the investigations.

Comparison between the Two Classes

The contexts in the laboratories as displayed in Figures 4.5, 4.6, 4.7, 4.8, 4.9, and 4.10 illustrated that most interaction focused on the science and the links or on the mathematics and the links between mathematics and science. Rarely in either class did discussion jump from the science to the mathematics or the mathematics to the science without some linkage or relational meaning made. According to Bromme and Steinbring's (1994) work, the

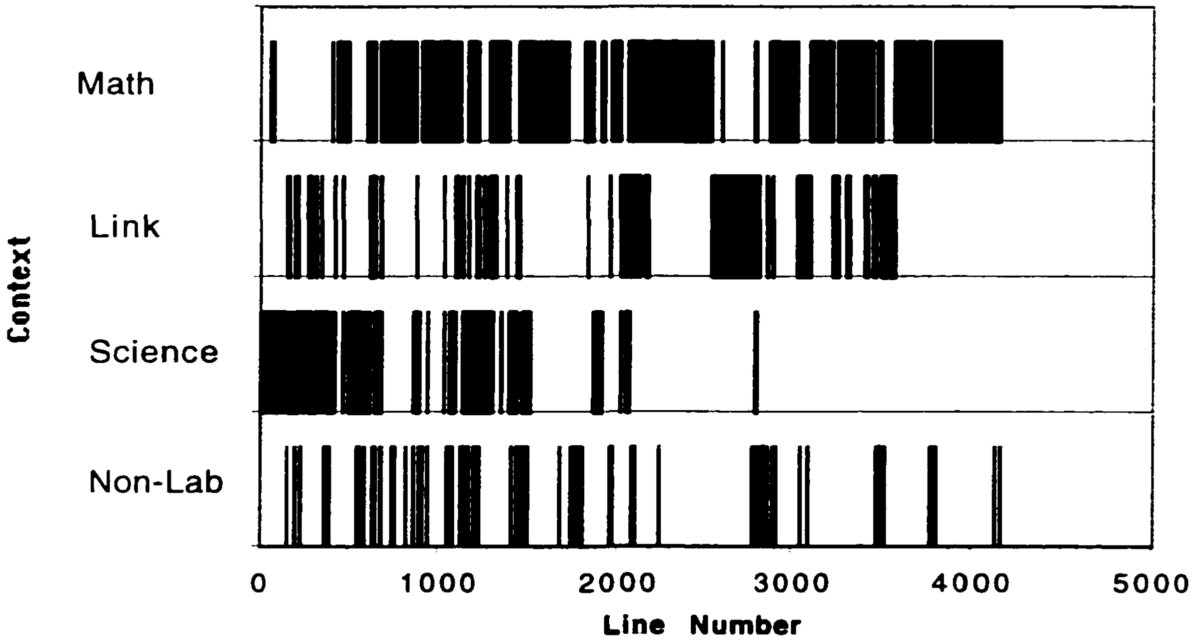

Figure 4.8. Contexts in the water flow lab in the modeling-focused class

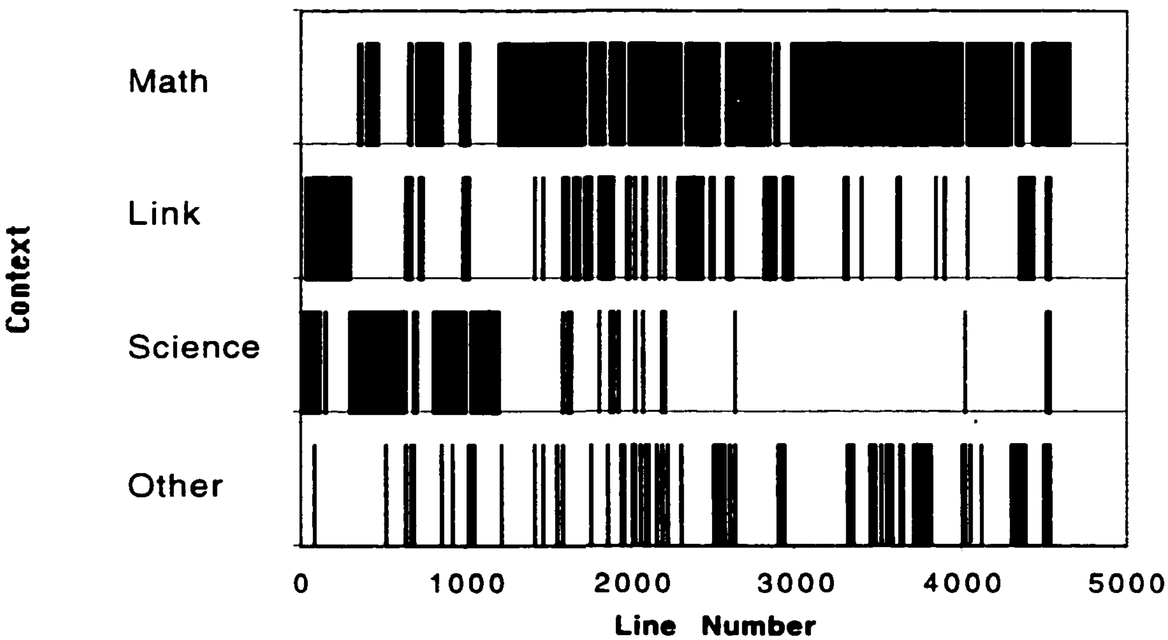

Figure 4.9. Contexts in the light intensity lab in the modeling-focused class

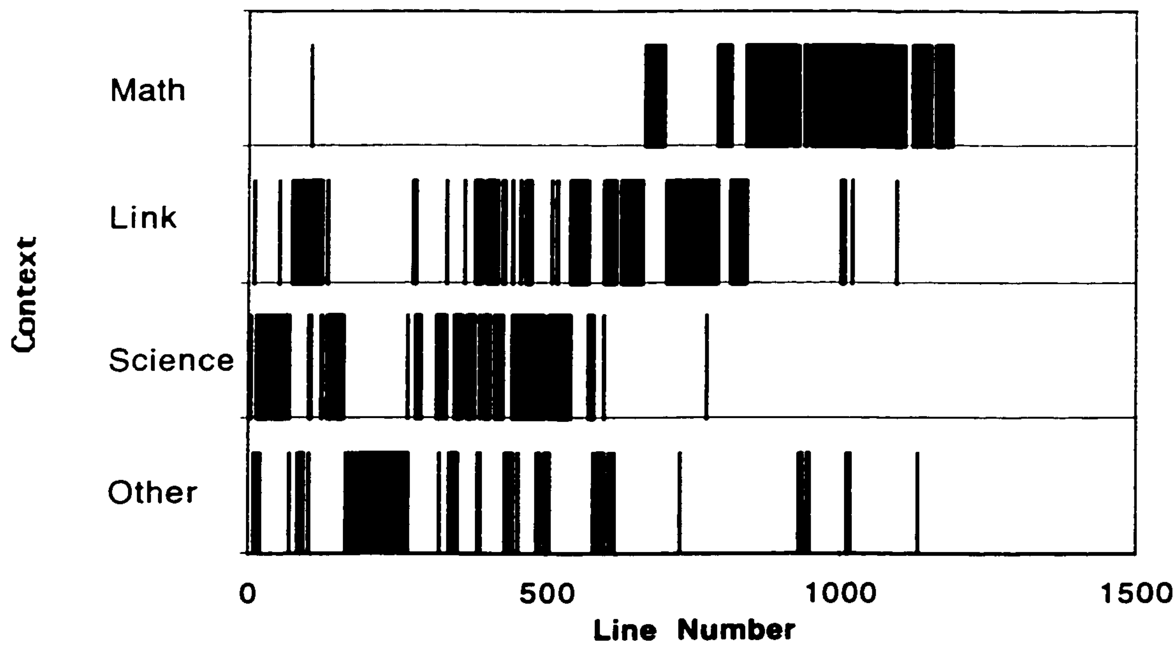

Figure 4.10. Contexts in the penicillin lab in the modeling-focused class

context component in both classes supported meaning-making within the inquiry process. With both classes having a balance between symbol, object, and relational discussion, both instructors behaved more closely to the expert instructor in Bromme and Steinbring's study.

Some distinctions between Figures 4.5, 4.6, 4.7, 4.8, 4.9, and 4.10 demonstrated that slight differences occurred within and between the classes. These distinctions were evident in the link context. In the tools-focused class during the water flow and light intensity investigations, the links between the mathematics and science tended to occur in blocks of discussion. From the water flow lab (Figure 4.5) to the light intensity lab (Figure 4.6) to the penicillin lab (Figure 4.7), the link discussion changed from occurring in the first two-thirds of the investigation to most and then all of the investigation. In the modeling-focused class, the number of occurrences of links appeared to be fewer than the number of link occurrences in the tools-focused class. In the modeling-focused class the links seemed more consistent across the entire investigations for the water flow and light intensity labs, rather than occurring in blocks. Discussion during the penicillin lab (Figure 4.10) seemed to reflect more of a sequence of science to link to mathematics contexts.

Differences in the graphs may be due to a lesser degree to the differences between classes as much as time pressures to cover specific content topics. As previously mentioned, the instructor of the tools-focused class commented on the need to focus on the mathematics due to a time concern. The modeling-focused class was under time constraints to cover the desired content during the penicillin lab. In these cases, as evidenced in Figures 4.5, 4.6, and 4.10, discussion tended to occur in a sequence of science to link to mathematics with little overlap and infrequent return to the science or link contexts. When timing and coverage of topics was less of a concern, link discussion tended to occur consistently and all throughout the laboratories in both classes as seen in Figures 4.7, 4.8, and 4.9.

Another distinction between the two classes was the number of "other" classifications for the context of discussion. Items classified as "other" were issues such as directions and assignments not related to the laboratory, an instructor calling on a student, responses including, "I wasn't going to say anything," and classroom dynamics like, "Did everyone hear that?" Small talk in the classroom would also be considered as "other." More "other" context items were classified in the modeling-focused class than the tools-focused class. The number of "other" classifications suggest that the modeling-focused classroom may have been an environment in which the students felt comfortable expressing opinions and raising "other" issues with times of being side-tracked and losing focus on the tasks at hand.

Students in the tools-focused class tended to remain more focused on the tasks being discussed.

More similarities than differences occurred between the two classes when relating the context discussion. Both classes demonstrated "expert" levels of instruction by linking the science and mathematics without making sudden switches between the math and science components. The mathematics component received the greatest emphasis, while the science context received lesser emphasis. Differences in the link discussion in the laboratories in the tools-focused class, connections between the science and math initially seemed to develop in blocks of discussion and occurred more consistently as the semester progressed. The link discussion in the modeling-focused class seemed to touch on the links through much of the laboratory with reduced link discussion during the penicillin lab.

Micro Sources of Ideas

Siegel and Borasi (1994) noted the importance of inquiry teachers to listen to students. When consistently talking or engaged in teacher-student discussion patterns, teachers inhibit the inquiry process and students' generation of ideas. At various times, the instructors of Math 181 were more successful to let students' ideas lead the discussion.

Micro Sources of Ideas in the Modeling-Focused Class

Indication that students' or instructors' ideas were the immediate focus of discussion was noted by the micro sources of ideas. The instructors' intents for the origin of the ideas were related to their intents for who posed the questions for investigation. For the modeling-focused class, the instructor desired that students' ideas proactively contribute to the direction of the class discussions. The instructor's intents were evident in her lesson plans and in the class structure. In her lesson plans for the water flow investigation, the instructor made the note, "Do not push the content. Let students do the analysis." During the class with the instructor's guidance, students first asked the questions for investigation and asked additional questions during reflection phases. Students' questions and ideas gave direction to the manner of development of the mathematical methods.

In the modeling-focused class, students' ideas were discussed and influenced the direction of the class. One example of how students' ideas motivated the analysis discussion occurred during the light intensity investigation. The instructor was developing a method used to model exponential data. The class had developed the equation relating the rates of change with the intensities.

R: You have y equals -.4288 x + .0002. [R writes y =-.4288x + .0002].....All right. Since I was real big on appropriate variable names, I don't really care for y and x. What can we put in their place? What was on the y?
Student: Rate
Student: Rate
R: Let's go ahead and put that in. What was on the x?
Laurie: Intensity.
R: A couple days ago Ellen asked an important question....She asked, "What does the rates graph say? What are we supposed to grasp from the graphs?"
Jamie: Can we get rid of the .0002?
Brett: Yeah.
R: What do you think?
Brett: I like that idea.
R: Can we get rid of it just to get rid of it?...What does the y-intercept mean in this case?

The instructor was the initial source of the micro idea as evidenced by her prompts. She urged students to use appropriate variable names in the equation to build associations between the science and mathematics and to promote interpretation. Before reaching the interpretation of the graph and equation, one student prompted a slight change in class discussion when he asked, "Can we get rid of the .0002?" His question noted a change in micro source of idea. Others in the class agreed with his desire to rid the equation of the

.0002, promoting discussion about the size of .0002, the role of .0002 as the y-intercept, and the connection between the y-intercept and science context.

The class example demonstrated the use of micro sources of ideas and the impact of students' ideas on class discussion in the modeling-focused class. The instructor intended for much of the direction of the class instruction to be dominated by students' ideas.

Micro Sources of Ideas in the Tools-Focused Class

The instructor of the tools-focused class recognized the impact of students' ideas and questions on the direction class sessions would take. During interviews the instructor noted her plans for the class sessions and commented that the plans would change based on students' questions, contributions, and challenges. The instructor's ideas often initiated discussion, and her plans changed as students offered their questions, contributions, and challenges.

An example of how a student's question changed the instructor's plans for the tools-focused class session occurred at the beginning of a class session during the light intensity investigation. Students were to have completed an assignment of modeling data for the removal of a dose of penicillin from the body. The methods of analysis were to be similar to those completed for the light intensity data.

T: Please take out your penicillin homework and pass it that way. I'm really anxious to see those.

Rita: Wait, umm, the numbers they gave us, had like an intercept of 6.789 and [inaudible]....

T: Okay. And the question was how do you handle that 6.789?

Rita: Right. We were supposed to add that into the equation, right? Because in our other example the intercept was zero and in this case it's 6.789. Do you add that onto the end of the equation?

T: Good question. How will we handle it? You've got a lot of company. You've got a lot of company. How many - does everybody understand the question? Okay, good.

> How did you handle that remainder? How did you handle that intercept?...Okay. Let's look at this.

The instructor had not intended to talk about the role of the y-intercept. Her plans were changed when a student asked how to handle the intercept of size 6.789. Over half of the class session was then devoted to addressing the student's question and related issues, issues relevant to modeling data. Students' ideas and questions in the tools-focused class contributed to the direction of the class discussion, at times different than planned by the instructor.

The instructors' intents for whose micro sources dominated the class discussions were similar to their intents for the posing of questions in the class. The modeling-focused class encouraged students' ideas to give direction to the mathematics and science discussion. The tools-focused class followed students' ideas and questions as they arose in the context of other methods and mathematical discussion.

<u>Language</u>

Supportive of the instructors' goals and reflective of student authority and sources of ideas in the classrooms, language use indicated another area in which the two classrooms differed. In the tools-focused class, language use reflected the instructor's goal to teach the mathematics and keep the mathematics as the focus of the class. Hence, language reflective of the mathematical concepts was emphasized. In the modeling-focused class, less emphasis was placed on formal mathematical terminology as students' language was used to describe the scientific and mathematical processes. As the need arose for the more formal mathematical terminology, the modeling-focused instructor introduced the terms to accompany the previously discussed concepts.

Language in the Tools-Focused Class

An example of the use of language in the tools-focused class occurred during the launch of the water flow investigation. Students had been told about the Hoover Dam and the desire to examine the relationship between the depth of water in the lake and the time needed to drain the lake. The instructor asked,

> I want to see a graph of what you think it would look like of how long it would take to drain the lake as a function of how deep that water is. What do you think that graph would look like?

As students drew axes to sketch their hypotheses for the graph relating the depth and drainage time, discourse arose as students questioned what the instructor meant by "as a function of." Students didn't understand which variable went on which axis. This resulted in the instructor answering several of students' questions of which variable she wanted on which axis. She eventually addressed the issue to the class as a whole:

> I used mathematical language when I expressed the question. And the question I asked was, ... How long it took to drain the lake as a function of how deep the lake is. Let me write that down. We haven't talked about this, but the question said, as a function of how deep the water is. That question has an implicit assumption about which is the independent and which is the dependent variable. Is this question assuming that how long it takes determines how deep the water is? Or is the question assuming that how deep the water is determines how long it takes to drain?

Note that the instructor clarified the language "as a function of" by asking students which of the two relationships made more sense, and proceeded to give the two possible relationships of one variable depending on the other. Kieran (1993) cited researchers Freudenthal, Davis, Shuard and Neill who strongly emphasized the use of functional dependencies in the development of function understanding in mathematics, particularly since the dependency promotes "pedagogical accessibility." To help her students quickly become reestablished in the context of the problem as well as understand "as a function of," the instructor relied on this dependency approach.

The instructor's use of the terms "as a function of" was fundamentally warranted in the class. Students met the prerequisites for the course, having math through trigonometry. Technically, students should have been familiar with function terminology as well as the connection between the terminology and the process of graphing functions. As well documented (Sfard, 1991, 1992; Vinner & Dreyfus, 1989; Sierpinska, 1992; Dubinsky & Harel, 1992; Carlson, 1998), students in mathematics up to and including the first year calculus courses have difficulty with function notation and terminology. Comparable in mathematics background to the college algebra students in Carlson's study, the tools-focused students had difficulty explaining what was meant by "express *s* as a function of *t*."

The use of the terminology "as a function of" occurred early in the course as students adjusted to the methods, terminology, and symbols used in the course. During the launch of the light intensity laboratory, most students had better understanding of the relationship suggested by "as a function of." The light intensity laboratory began in a similar manner as the water flow laboratory with the establishment of the context and the request for students to graph one variable as a function of the other. Most students correctly responded as to which variable was the independent variable. The language used in the tools-focused class highlighted the instructor's emphasis on the mathematics and the accompanying precise mathematical terminology.

Language in the Modeling-Focused Class

In the modeling-focused class less emphasis was placed on formal mathematical terminology with more emphasis given to students' terminology in describing the mathematical concepts. The instructor's purpose for this difference of emphasis may be summarized by Sfard's (1992) statement, "a structural conception should not be required as long as the student can do without it" (p. 69). One example where less emphasis was placed on formal mathematical terms surrounded the issue of functional relations. Neither the use of the term function, nor function notation was strongly emphasized. The instructor's goal for modeling to be the focused object led to the emphasis of dependency relationships as Kieran (1993) described.

A small group of students chose to use the language of functions both in class and in their lab reports. The times that these students used the function notation were appropriate and meaningful for these students. Two main instances in which students chose to express mathematical relationships using "as a function of" occurred during the prediction and reflection phases. During these phases, students asked how one variable was a function of the other or specifically asked how to verify the suspected mathematical relation for data thought to be modeled by a given family of functions.

When the modeling-focused class initially discussed rates of change, language use emphasized the mathematical concepts before formal terms of "instantaneous rates of change" and "derivative" were introduced. Students put forth the terminology of average rates of change when comparing groups' water flow experiments. Some groups collected data by filling a cylindrical tube to a given mark, recording the drainage time, filling the tube to a lower mark, and recording the drainage time. Other groups measured the amount of water that flowed out the tube for ten seconds as more water was added at the top to maintain a constant water level in the tube. Students noted that some groups calculated "average rates of change" while the others calculated a "constant rate of change," closely related to an instantaneous rate of change. The class retained the use of the terms "average" and "constant" to denote the rates of change until the class developed the concept of instantaneous rates of change graphically and numerically.

The use of language in the two classes reflected the instructors' goals for the course. The instructor of the tools-focused class emphasized formal mathematical terminology as the mathematics and tools were developed. The instructor of the modeling-focused class allowed the mathematical terminology to arise as the concepts were built in the modeling process.

Symbols

Related to the issue of language use and development of terminology in the class, symbol development and usage also differed in the two classes. New symbolic notation in the tools-focused class was given and explained by the instructor as needed for the various mathematical tools. In the modeling-focused class, the instructor allowed students to develop notation to represent the mathematical concepts and procedures.

One area in which the development of symbols was particularly evident in both classes involved subscript notation. During the light intensity laboratory, subscript notation was used in both classes to distinguish the intensity at the "next" depth from the intensity at the "current" depth. Previous experiences in the pilot studies suggested students' difficulty with subscript notation. Between the two classes, differences existed in how the subscript notation originated.

Symbol Development in the Tools-Focused Class

In the tools-focused class, the instructor and the textbook were the originators of the subscript notation. Students had come to class with their calculations of the differences in intensities and graphs of the *intensities* versus *depth*, *change in intensities* versus *depth*, and *change in intensities* versus *intensities*. After discussing properties of the graphs, students were asked to turn to a page in their book which contained additional intensity data.

T: Now, let's come back up and look at the headings on those tables. I_d What does I_d represent?
Pete: Light intensity.
T: Light intensity. And what is $I_{d+1} - I_d$? Just be quiet for a second. Can you express that in words? What does $I_{d+1} - I_d$ mean? Bret, what do you think? What does that mean?
Bret: The change? Change in absorption?
T: The change in absorption from what?
Bret: From the one point to the other.
T: From one point to the other point. And what about the next? What about the other point?
Bret: It's one below?
T: It's one below it. It's one below it. The change in intensity between two adjacent readings. Between one reading and the next reading. Which is what this $d+1$ is talking about. Is everybody okay with the notation? We just sort of slid into this notation. We're kind of using it without talking about it a lot. Any problems with the notation? Some people have a lot more experience with this type of notation than other people. So if you have any questions please feel free to ask.
Amy: What is I_d?
T: I_d means the intensity at a particular depth or at a particular number of filters. If this is the intensity at a particular depth, this is the intensity at the next depth. [The instructor points to the numbers in the table on the transparency.]

The class continued discussing how the equation for the *change in intensities* versus *intensities* could be expressed with the subscript notation. The subscript notation was a tool used in the development of additional tools: methods for modeling data thought to be exponential.

Symbol Development in the Modeling-Focused Class

In the modeling-focused class, students developed the subscript notation. Students had come to the class with calculations of the ratios for the light intensity data. Students observed that the ratios of the "next intensity" to the "current intensity" remained fairly constant for consecutive pairs of intensities. The instructor asked students to develop notation to represent the calculations they had performed. Each group decided on a representation and presented their notation on the board. The class examined and discussed which representation(s) seemed to succinctly and precisely indicate the calculations:

R: Let's look at Dan's. [$\frac{I+1}{I}$] ... Could you take that formula and could you do a test and does that work?
Kiene: No.
Ben: Nope.
R: Some are saying yes and some are saying no. Let's hear from the group that said no.
Ben: We explained it to you because that was our first equation and then we realized - ...
Kiene: You can't do it because like for *I* you put in .81 over whatever your second value was but that would give you 1.81 over your second value. It's close.
Dan: Okay, well, what we meant I guess you have to know what we meant by representing. That doesn't mean add one to that point, it means your second point.
R: Okay.
Dan: It means your first one divided by - well the second one divided by the first one.
Ben: See then you could do it.
Kiene: You just need a legend.
Dan: I understand. Actually, I thought about it right here, I do-
R: Okay. What about Kiene, x_2 divided by x_1? Does that - could you do your calculations looking at that formula?
Kiene: Once again, we'd have to make a legend.
R: And what would your legend be?
Kiene: Because your points would change every time. It'd be -
Jamie: Go down
Kiene: We would have went with the *n*, *n* - 1 method if we were redoing it again. [referring to $\frac{Light\_Int_n}{Light\_Int_{n-1}}$ or $\frac{INTEN(n)}{INTEN(n-1)} \times 100$]

Others in the class decided that both $\frac{Light\_Int_n}{Light\_Int_{n-1}}$ or $\frac{INTEN(n)}{INTEN(n-1)} \times 100$ were slightly faulty since the substitution of zero for *n* would result in a negative one as the subscript for the intensity in the denominator.

When asked which of the notations could be used with no legend or minimal additional information given in a legend, the class opted for $\frac{Y_{n+1}}{Y_n}$ or $\frac{I_{n+1}}{I_n}$. As the class discussed the two options, one student argued that the use of Y was useful since the notation could be used for any situation where Y represented the dependent variable, not just intensity. A student countered that for the same reason I was more useful since the I would indicate more precisely what variable was involved, intensity in this case.

Building on the class discussion, the instructor reminded students to use appropriate variable names. Students were given the heuristic to "use appropriate variable names" during the water flow laboratory to help in the process of modeling. With the reminder, the instructor recommended the use of $\frac{I_{n+1}}{I_n}$ and the replacement of n with d to represent depth. The development of the subscript notation supported the modeling-focused environment since students modeled the procedure of calculating ratios and attaching meaning to the symbols.

In both classes, the functions of the symbols as described by Hiebert and Carpenter (1992) were evident. The symbols were a record to share and communicate what was already known and were used to organize and manipulate ideas. The development and organization of symbols differed in the two classes though they reflected and promoted the instructors' goals for the class. In the tools-focused class, symbols were the handles of the tools used for modeling data and other mathematical procedures in the course. In the modeling-focused class, both the development and use of the symbols reflected the process of modeling data.

Heuristics and Reflection

Throughout the course, when addressing methods of modeling data and building on the contexts, the classes developed heuristics. The heuristics were intended to help students to think about their thinking, also known as metacognition. The use of the heuristics was to promote students' abilities to communicate and justify their explanations when modeling data, an essential feature of classroom inquiry as noted by the National Research Council (2000). The heuristics were also intended to prompt reflection in students. The tools-focused class developed several heuristics, each tailored more closely to the types of problems they could be used to solve. The modeling-focused class, for the most part, emphasized one heuristic used in modeling data.

Heuristics in the Tools-Focused Class

In the tools-focused class new heuristics were developed or new steps were added to existing heuristics as new categories of problems were confronted in the class. The heuristics helped students move forward in various kinds of problems by looking back at what they had completed. The instructor explained the heuristics and encouraged students to record them in their notebooks and to use them when attacking various problems.

One development of a heuristic occurred across laboratories. Early in the semester, students gathered data to examine the mathematical relationship between an adult's height and stride length. When the class analyzed the data and sought to find a mathematical relationship, the instructor informed the students of Occam's Razor: "when you have two competing theories which make exactly the same predictions, the one that is simpler is the better" (http://www.weburbia.com/physics/occam.html). When the class mathematically modeled the height and drainage time data from the water flow experiment, the instructor added to the heuristic the need for prior knowledge. Students needed to use the simplest model informed by prior knowledge based on mathematics applied to the context. In the case of the water flow lab, this prior knowledge concerned students' knowledge of acceleration due to gravity and equations relating the height of an object to the time taken for an object to reach the ground.[1] Once prior knowledge was established, a mathematical model was suggested and tested using linearization techniques.

Other heuristics were developed throughout the course. Additional heuristics, acting as tools, included how to proceed using the difference method when modeling exponential data and how to plot data and find equations for semi-logarithmic and log-log plots. A key component in several of the written procedures emphasized students' reflection on the created plots of the data and careful recording of the calculations which had been performed.

Heuristics in the Modeling-Focused Class

In the modeling-focused class, one primary heuristic was used to assist students in the process of modeling data. The one heuristic acted much like the multiple heuristics in the tools-focused class. Like the students in the tools-focused class, students in the modeling-focused class were encouraged to rely on the context and linearization in the development of their explanations and justifications of their models.

Students were encouraged to apply the heuristic "use appropriate variable names." When students had data to be modeled, students were to graph the data and suggest a model based on the shape of the graph and on the underlying scientific context. To both verify the

[1] The instructor realized that more advanced physics is needed to fully explain the relationship between the height of water in a cylindrical tube and the time required to drain the tube. Students readily propose and affirm the role of gravity in draining the water, thereby suggesting a quadratic relationship.

appropriateness of the model and to find the equation of the model, students were taught to linearize the data. Linearized data was data that was transformed such that if the model was appropriate, a line would be evident on the plot. Once the equation of the line was found, students were advised to use appropriate variables to represent what transformed data was graphed on the *x* and *y* axes. By identifying what was graphed and writing the terms using meaningful symbols, students reflected on what they had done, linked the mathematics to the science context, remembered what they anticipated the equation would be, and potentially had the direction needed to rearrange the terms to a more common form.

Students in both classes were encouraged to use heuristics to develop mathematical models for data and to solve problems. Students in the tools-focused class were taught several heuristics to apply to various problems. The multiple heuristics had underlying themes of using prior knowledge and correctly representing what variables were graphed. Students in the modeling-focused class were primarily taught the one heuristic to use appropriate variable names which applied to modeling data in several different contexts.

Reflection

Key components of the heuristics in both classes was the use of reflection. Both instructors recognized what researchers Hiebert and Carpenter (1992) noted: that without reflection, symbol manipulation would unlikely stimulate the relationship construction leading to understanding. Mathematical modeling may be attributed as a form of symbol manipulation. During modeling, students were encouraged to reflect on the context of the problem and relate the context to their prior knowledge. Reflection was prompted when students were asked to represent what was graphed on the *x* and *y* axes. Instructors in both classes often modeled the reflection process and prompted the students to reflect on the procedures to promote advancement on the problems students worked. Examples from each class illustrate the instructors' prompts toward reflection.

Prompted reflection in the tools-focused class. In the tools-focused class, students were taught heuristics to develop mathematical models for exponential data. The instructor emphasized the multiple ways of finding the same type of equation and exemplified how to reflect on what was graphed.

T: Ron, tell me what we did. Tell me yesterday or on Tuesday, please
Ron: Well we found the equation of our line.
T: And what were we plotting with the line?
Ron: The light intensity versus change in intensity.
T: Okay, we plotted the intensity versus the change in intensity. And what do we get when we plotted that?
Ron: We got a bunch a dots.
T: A whole bunch of dots. What was the relationship between those dots - if there was any pattern?
Ron: It was linear.
T: Okay, it was linear. We got a line. So the first thing we do is the plot. Then what did we do?
Ron: The equation of our linear regression line.
T: Amy, what do we do next?
Amy: Once you find the line, then you have to do the $I_d - I_{d+1}$.
T: You have to do the $I_d - I_{d+1}$. What do you mean by that?
Amy: Well you have to put that into your equation.
T: Put that into equation. Okay. Robin, does that make sense?
Robin: Uh-huh.
T: What did Amy mean when she said put the I_d and the I_{d+1} in the equation?
Robin: What do you mean what did she mean? The I_d is like the first point and then the I_{d+1} is the point right above that. And you set them to equal each other. Oh, wait. I mean, I don't know what you mean.
T: It's real hard to say in words what we did. What did we do?
Robin: Well we took $I_{d+1} - I_d$ equals the slope times I_d.
T: So this $I_{d+1} - I_d$ we found the equation of the line, my x corresponded to the intensity right? Which is for - some people used I_d and some people used I_{d+1} because there was a mismatch with the intensities and the change in intensities. There was one more intensity in that list than there was change in intensities. So some people ignored the top number. Some people ignored the bottom number. It doesn't matter, but which one you do will effect which one shows up here. And then the y was the change, so it was either $I_{d+1} - I_d$ or it was $I_d - I_{d+1}$ depending on how you decided to

subtract. It doesn't matter what your decisions are as long as your equation reflects that perfectly. This is not a place to hurry.

The instructor prompted reflection on what occurred during the previous class session and modeled how to reflect what variables should be plugged into the equation in place of *x* and *y*. Demonstrating the instructor's goal for multiple tools to arise in the course, multiple methods of modeling the data were encouraged as she emphasized, "It doesn't matter...as long as your equation reflects that." The instructor advocated the use of reflection on what was graphed to prompt appropriate modeling of data.

Prompted reflection in the modeling-focused class. Similar to the tools-focused class, the instructor of the modeling-focused class encouraged reflection on students' calculations, graphing, and the context to promote proper symbolic representation in the equation modeling the data. When developing an exponential equation for the light intensity data, students had graphed the rates versus the intensity. As previously discussed, students questioned whether the *y*-intercept should be zero. The class reasoned that the intercept should be zero, given the context. Students were then encouraged to use appropriate variable names to replace the *x* and *y*.

R: So our equation is rate equals -.43 times Intensity. $Rate = -.43Int$....I'd like you to write down in our notation of I_d or I_{d+1} or whatever, write a formula for how you calculated the rate. You calculated rate - how do you calculate a rate?

Alison: Change in Intensity over change in depth

R: You calculated rate by taking change in intensity divided by change in depth. Rewrite change in intensity divided by change in depth using I_d and I_{d+1}. How did you calculate - write a formula for how you calculated rate. We want to use the same notation that we had been using....What was your change in depth each time when you were calculating the rates?

Several respond: one

R: one. Change in intensity - how can you use that notation to rewrite your change in intensity? Okay, that's your assignment – that's part of your assignment for tomorrow. For tomorrow, come up with a formula to write change in intensity over change in depth using this notation.

The instructor encouraged students' reflection on the calculations performed and the data plotted. Students were led through stages of how to transform the equation in x and y into an equation using symbols more meaningful to the context. The use of the new notation would lead to the development of an exponential equation using discrete methods. Through reflection and the use of appropriate variable names, the exponential model developed.

In both classes, reflection was encouraged as students mathematically modeled data. The reflection was often prompted by the instructors rather than originating with students. Related to the phases of inquiry as displayed in Figures 4.1, 4.2, 4.3 and 4.4, overall in both classes, long periods of reflection were weak. The instructors attempted to aid in reflection during analysis phases with emphasis on connections between the mathematics and science contexts. Figures 4.5, 4.6, 4.7, 4.8, 4.9, and 4.10 illustrated that the links were not always consistent nor frequent. Reflection, though present in both classes, did not exist in the strength as recommended by researchers.

Conclusions

The similarities and differences in the classroom environments for the tools-focused class and the modeling-focused class were described in this chapter. Both instructors pursued inquiry through the conduction of experiments and mathematical modeling in the classes. Some of the highlighted similarities and differences are reviewed.

The instructors differed in their goals for the role mathematical modeling would play in the course. The tools-focused class sought to develop mathematical models reasoning from the context. Once the data was modeled, the mathematics surrounding the model was emphasized. Observations of the class and evidence in the graphs of Figures 4.3, 4.4, 4.5, 4.6, and 4.7 illustrated that the instructor's goals for the environment were accomplished. In the pursuit of an environment in which the mathematical tools were the emphasis, certain course developmental characteristics were evident. More often the instructor posed the questions for investigation, used more traditional mathematical language when discussing the relationships, and introduced new notation for the mathematical modeling.

The instructor for the modeling-focused class sought to achieve an environment focused on the process of modeling. With the modeling emphasis, the class posed the questions for investigation, used less formal mathematical terminology initially, and developed the symbols needed for proper representation of the data.

Observations of the classes and graphs illustrated that few differences existed in the appearance of the modeling-focused class and the tools-focused class. Both classes pursued multiple cycles of inquiry with prediction, experiment, and analysis phases. In both classes, reflection on the analytical methods and context was encouraged. In light of the encouragement, reflection in both the tools-focused class and the modeling-focused class seemed weak. Each section emphasized the link of the science context with the mathematics, relied on heuristics in mathematical modeling, and sought agreement in the questions under investigation. When coverage of content was a concern, open-inquiry was limited with contextualized discussion following a sequence of science to link to mathematics.

In light of the characteristics presented, both classes achieved a degree of inquiry in the classroom environments. In the next chapter, the effects of the characteristics on students' mathematical modeling skills will be demonstrated. The affects will be used to further assess the accomplishments of the inquiry environments.

CHAPTER 5

CONSEQUENCES OF CLASSROOM ENVIRONMENTS

The instructors' goals regarding the structure of inquiry for the classroom and mathematical modeling resulted in various outcomes for students. Overall, the structure of the laboratories led to differences in students' responses toward the questions under investigation, students' reliance on the science context to inform the mathematics, students' use of symbols when modeling data, and the forms of reflection in the laboratories. Aside from the differences in class and laboratory structure, students' abilities in modeling were often similar when applying new methods of modeling taught by the instructors. In particular, when coverage was a concern for instructors, then inquiry was somewhat restricted. In both classes, when students' generation of ideas and methods was emphasized, inquiry occurred more consistently.

As described in Chapter 4, in the modeling-focused class, the instructor intended for students to objectify the modeling process. To develop students' abilities in inquiry and modeling, students posed questions for investigation, collected data, analyzed data by developing mathematical models and presented their models while in small groups. After the presentations, students refined and revised their questions before further discussion of mathematical models as a class with the instructor. The class did not maintain this structure during the penicillin laboratory due to time constraints and coverage concerns.

In the tools-focused class, the instructor intended to use the context from the different experiments to be the underlying foundation on which to develop the mathematics for the course. While the instructor desired for students to reason from the context, she maintained that the focus of the class was the development of the mathematics. In addition, she aimed for multiple perspectives, tools, and methods to be applied in solving the various problems. Students in the tools-focused class made predictions, conducted experiments, presented the data, and discussed means to mathematically model the data as a class. The structure of the laboratories in both classes influenced the characteristics of inquiry and the effects on students' modeling skills and learning.

Agreement of Question

As stated in Chapter 4, the two classes differed in who initiated the questions for investigation. In the modeling-focused class, the students settled on the question to be addressed. In the tools-focused class, the instructor more often initiated the questions for investigation. Additional questions were raised as students discussed hypotheses. Throughout the investigations, students in the two classes differed in their understanding of what questions were being addressed. Students in the modeling-focused class were more often in agreement of the questions under investigation. During laboratories in the tools-focused class, students varied in their agreement of the questions under investigation.

Agreement of the Question in the Modeling-Focused Class

Students in the modeling focused class generated and agreed upon the questions to be investigated in the laboratories. During the laboratories and in their lab reports, students demonstrated their continued agreement of the questions being answered. When students presented their data and initial analyses, each group of students addressed the common questions initially set to be answered. For example, during student presentations of the water flow data, the groups demonstrated how they addressed the question, "How does lake elevation affect flow rate?" and proved how they would predict a flow rate given a height or volume measurement of water. Students were similarly prepared after initial analyses of the light data to predict a light intensity reading given a depth.

Assessment of Question in Class

Later in the investigation, when the instructor taught additional modeling methods to be used, she assessed what questions students perceived were being addressed. The instructor asked students to record "What questions are we trying to answer?" Frequently, students responded with questions similar to the questions posed and agreed upon as a class during previous class sessions. Students' responses suggested that students and instructor were in agreement on the questions being addressed. Additional questions from students indicated that students did not always understand the methods used to answer the questions though they did understand what questions were of importance.

Assessment of Question in Lab Reports

Students' lab reports for the experiments demonstrated agreement in the questions addressed during the investigations. In the reports, students presented their analysis for the data. Though some students' analysis contained errors and indicated misunderstanding of some of the methods to address the question, agreement of the question was evident. Using the water flow lab reports as an example, most students stated the purpose of the experiment and their initial hypotheses of the relationships between the amount of water and the flow rate. Typical of students' statements and reflective of the class discussion which occurred during the laboratory, one group of students wrote "we wanted to find a relationship between the elevation of a body of water and it's rate of flow. Our hypothesis was that as the height of a body of water decreased, so would the rate of flow." This group proceeded to explain how they collected data, what assumptions were made about the role of gravity, and what justified their development of the square root relationship between the height and flow rate.

In the modeling-focused class, students posed and agreed on the questions to be investigated. Throughout the investigation and on lab reports, students demonstrated their agreement of the questions under investigation. Most often students responded with the question agreed upon with related questions and hypotheses.

Agreement of the Question in the Tools-Focused Class

In the tools-focused class, the instructor more often initiated the questions for investigation. In the light intensity and penicillin investigations, students demonstrated agreement of the question(s) being addressed in class and on lab reports. During the water flow investigation, students had different interpretations of what questions were being addressed. The interpretations of the questions under investigation were noted during class and on students' lab reports.

Assessment of Questions in Class

In the tools-focused class, the instructor often assessed question agreement to prompt reflection. The instructor asked individuals at the start of class sessions, "What did we do yesterday?" or "Would you tell me what we're doing?" During the water flow investigation, when asked what was done the previous day, one student responded that the class had worked with water to address "how long it takes to drain." The student noted that some hypothesized drainage with a constant rate with increasing depth while others said that the rate would decrease as the height of water decreased. The following class session, another student was asked what experiment was being performed. She replied, "We're timing how fast the depth of the water makes it affect how fast the flow is." Both students interpreted slightly different goals for the laboratory.

Students' different interpretations of the questions under investigation led to different types of data to be gathered and reported. The instructor encouraged and supported the various representations of the data gathered. When presenting new methods to model the data, the instructors' methods emphasized one representation of data and how to model the relationship between the time needed to drain a given depth of water. Some students needed to modify their data to apply the instructor's methods of modeling to their data.

Assessment of Questions in Lab Reports

Most students did not change their data, while proceeding to use the methods taught by the instructor. These students overlooked the difference in the instructor's goal in modeling data and their own goal in modeling the data. One example occurred when a group stated their objective was to explore how "depth of water affects rate of flow." Little discussion of flow rates was given as students modeled their time and depth data. These students had data reflecting the time needed to drain two inches of water at varying heights of water. This group proceeded to model the relationship between time and depth using methods given by the instructor without adjusting their data.

On the light intensity and penicillin laboratories, students and instructor agreed on the questions under investigation. Class discussion and lab reports reflected the agreement to model the light intensity data and the amounts of penicillin at various times for the respective laboratories. Students' work demonstrated some difficulty in finding models for the data, though all seemed to have the common goal to model the same types of data with the same types of methods.

In the two classes, with the mix of results in students' interpretations of the questions under investigation, the laboratory equipment and procedures likely influenced the agreement between students and instructor of the question(s) being addressed in the laboratories. Data in the water flow laboratory could be gathered a variety of ways resulting in an assortment of graphs of the data. In the penicillin and light intensity investigations, there was less variation in the types of data which could be gathered due to restrictions placed by the equipment. The equipment and procedures likely contributed to the agreement in questions in addition to how the questions for investigation were introduced.

Overall, the two classes differed in the introduction of the questions for investigation. The differences likely contributed to differences in students' understanding of the questions being investigated. In each of the laboratories in the tools-focused class, the instructor initiated the questions for investigation. In the water flow investigation, students sought to explore issues surrounding rates in addition to the instructor's goal question to relate the draining time with depth. Different interpretations of the objectives and data led to mixed results on students' lab reports. During the other investigations, students agreed with the questions the instructor posed. In the modeling-focused class, students posed the questions and demonstrated agreement of the questions under investigation during class and on their reports.

Discussion Linking Mathematics and Science Contexts

In both classes, the instructors intended for the various contexts to play a role in how students answered the questions under investigation. The instructors desired that students rely on the science context to inform their mathematical procedures and insights. In the modeling-focused class, the instructor's goals for students to objectify the modeling process and use appropriate variable names were not fully achieved. Students demonstrated their abilities to apply methods of modeling and assumed consistency in their use of notation. However, students lacked reflective skills of the meaning of the variables when learning and applying new methods of modeling. In the tools-focused class, students attained an action or process level of conception of modeling as they infrequently relied on the science context and assumed inconsistencies in symbolic representations. The students were successful in implementing various tools as the instructor desired. Observations surrounding the roles of the contexts, symbols, and reflection support the level of mathematical modeling conceptions achieved by students in both classes.

Context Connections in the Modeling-Focused Class

In the modeling-focused class, students' use of the context to inform and interpret the mathematics differed depending on the methods of modeling they applied. When students modeled data using methods that made sense to them, students relied on the context to interpret their results. When students modeled data using methods taught by the instructor, students relied less on the science context to inform and interpret the mathematics.

Context Connections When Students Applied Their Modeling Methods

When students were free to model data using methods of their choosing, students interpreted and adjusted their results based on the science context. Students in the modeling-focused class analyzed their data before presenting the results of their experiments. During these times, students mostly relied on regression features of their calculators to find an equation for their data. When analyzing their data from the light intensity lab, students discussed how the calculator-produced quadratic function fit the data very well. These students soon realized that the equation insufficiently modeled the data for an extended domain. The graph of the parabola reached a minimum and slowly increased. Reasoning from the science context, students understood that for increasing depths of water, the light intensity would never increase but would approach zero. The students proceeded to consider alternative equations which better fit the data and the science context.

Context Connections When Students Applied Instructor's Modeling Methods

When using modeling methods taught by the instructor, students less frequently connected the mathematics and science contexts unless prompted by the instructor. Students applied the new mathematical methods, but made few connections to the scientific context. One example in which students failed to rely on the context and faultily modeled data occurred during the water flow laboratory. Students had been reminded of the motivation behind the choice of a quadratic model and the vertex in the equation relating the drain time and height for a column of water. For additional practice using the methods of linearization with quadratic data, the instructor asked students to model data for a falling ball. The class discussed how to model the data.

R: Okay, let's quick look at a graph of this data [of a dropped ball]. [See Figure 5.1.]

R: ...What can we do next? If we want to find a quadratic equation for that using this method of linearizing data, what could we do? I'm hearing people say "square the time."

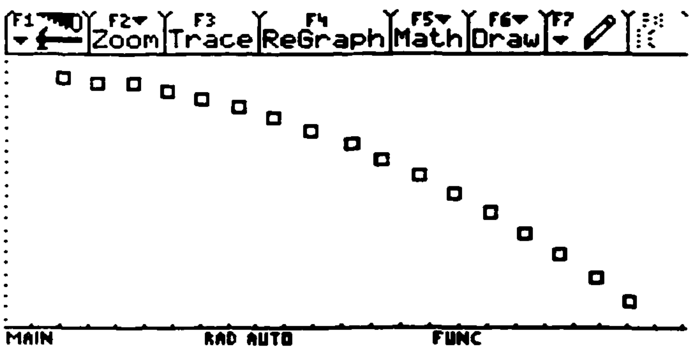

Figure 5.1. Graph of the height vs. time for the ball data

Dan: Take your graph and height
R: And make a graph of height vs. time squared.
Jake: Just like we did.
R: Just like we did. So you want to graph of height vs. time squared. All right, let's do it. My times are in C1 [column 1] and I'm now going to square them. And I'm going to graph C3 [column 3]. I'm going to graph the time squared on the *x*, and the heights on the *y*. [See Figure 5.2.]

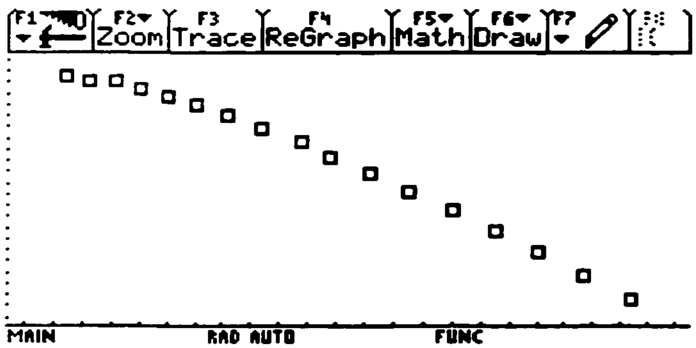

Figure 5.2. Graph of the height vs. time-squared for the ball data

R: Did we get a line?
[?]: Nope.
Brett: Wouldn't we take the square root because isn't - the line is going the other way compared to the last one?
R: So you want to take the square root of what?
Brett: Umm, let's try time.
R: Okay. And your justification for that is?
Brett: slope
R: It's going the other way?
Brett: The other graph went like this, and it was square root. And this graph went like this. [The instructor took Brett's advice and graphed height vs. the square root of time. See Figure 5.3.]
Brett: So why not be embarrassed? Nope.
R: This is why I wrote down the steps. What assumptions did we make?
Dan: We forgot and assumed that ball was at (0,0).
R: Is the vertex at (0,0)?
[Several]: No.
R: Where is it?
Jamie: Wherever you were holding it.
R: Wherever we were holding it, right?

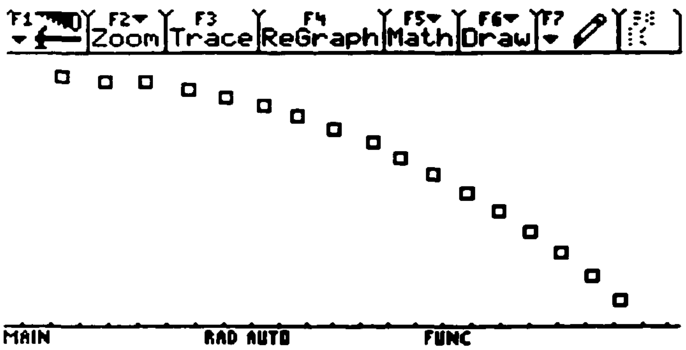

Figure 5.3. Graph of the height vs. square root of time for the ball data

When students applied new methods of modeling data, students failed to rely on the context to inform the mathematics. As a result, they told the instructor to apply incorrect steps to model the data.

Graphical Support of the Context Connections

In Figures 5.4, 5.5, and 5.6 graphs of the context with the micro sources of ideas demonstrate the interactions between instructor and students in discussing the contexts. When the micro source of ideas was the instructor, a mark in the bottom half of the relevant context was made. When a student was the source of the idea, a mark in the top half of the relevant context was made. The marks across time reflected the interactions which occurred in the class and the sources of immediate ideas which influenced discussion.

Sources of ideas in the modeling-focused class. At various times and within certain contexts, students' ideas played a larger role in discussion when compared with instructors' ideas. The figures illustrate that students ideas, marked in the top half of the context bands, dominated the science discussions and in both laboratories. The students were life sciences

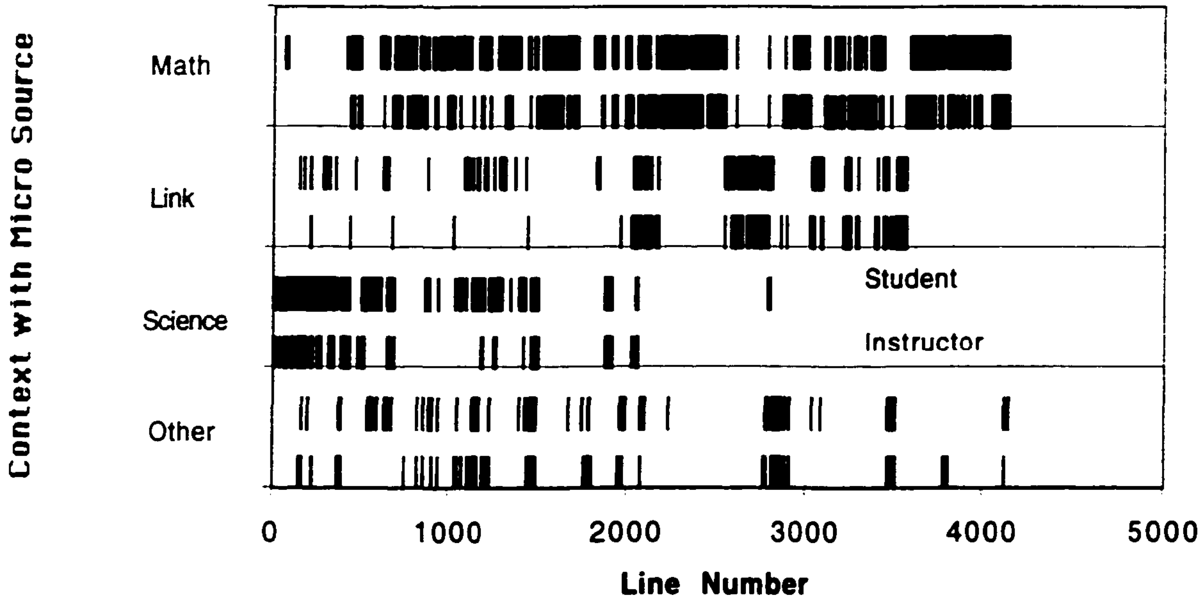

Figure 5.4. Context with the micro sources of ideas for the water flow lab in the modeling-focused class

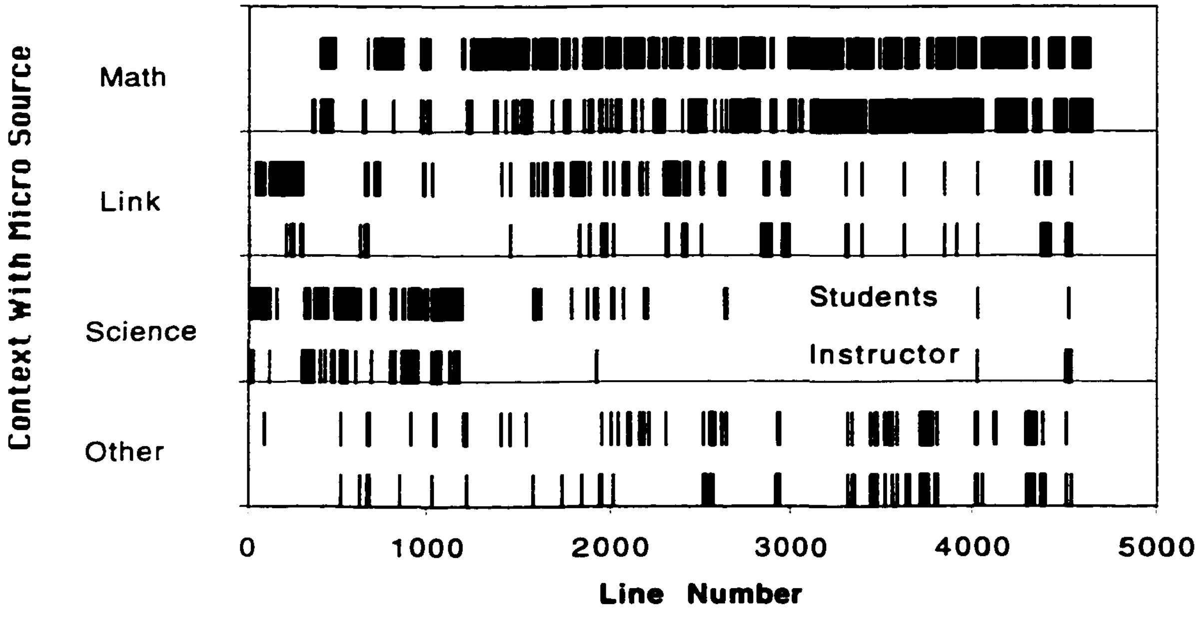

Figure 5.5. Context with the micro sources of ideas for the light intensity lab in the modeling-focused class

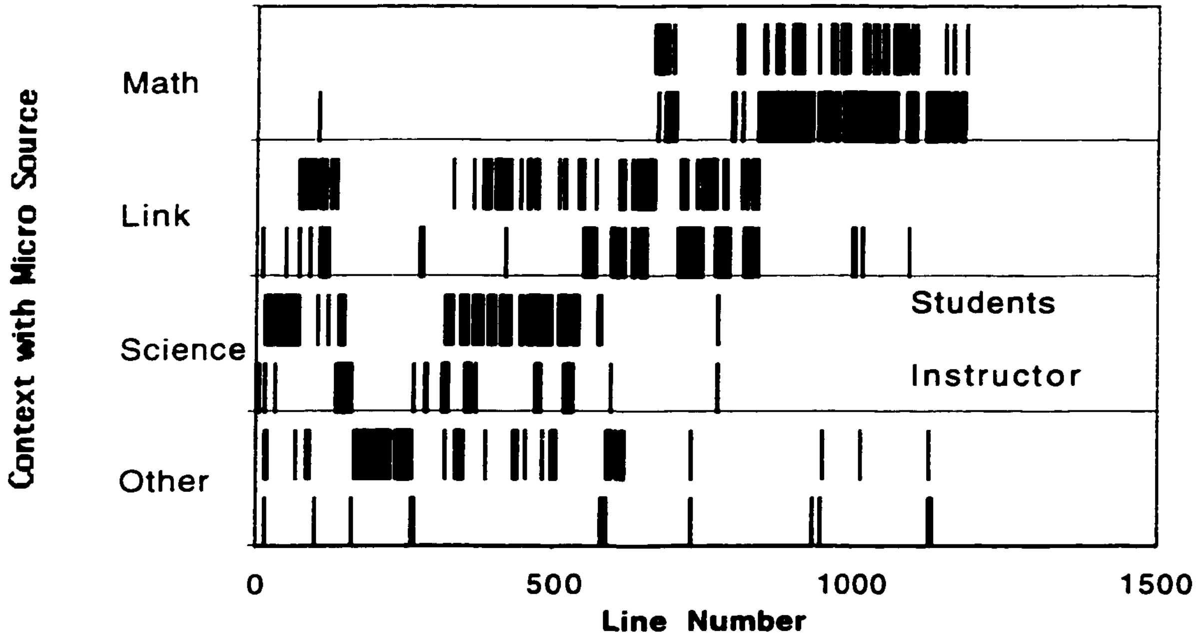

Figure 5.6. Context with the micro sources of ideas for the penicillin lab in the modeling-focused class

majors, so most students felt comfortable discussing and generating ideas about the science contexts and issues surrounding data collection.

Sources of ideas during discussion in the link and mathematics contexts were mixed between instructors and students. In the modeling-focused class, the first half of the investigations were dominated by discussion of students' ideas. In each of Figures 5.4, 5.5, and 5.6, the number of black segments in the top half of the link and mathematics bands were more frequent when compared to the number of black segments in the bottom half. The comparison of the number of these segments within the link and math contexts illustrate that students' ideas were more frequent than instructors' ideas in the modeling-focused class for the first half of the investigation. During the second half of the investigation, the instructor's ideas outweighed students' ideas discussed concerning the mathematics and the links with science.

Informed by the multiple cycles of inquiry for the water flow and light intensity investigations (Figure 4.1), in the modeling-focused class the first half of the investigation corresponded to the first cycle of inquiry, while the second half corresponded to the additional cycle(s). The sources of ideas supported the instructor's goal for students to discuss their predictions, experiments, and initial analyses before a second cycle emphasized mathematics with instruction of modeling methods. The first halves of the investigations gave students the opportunity to explore personal choices of methods to model data including regression features on students' calculators. Students had more segments in the link context than the instructor initially. Students reflected on the context as they sought a model. In the second halves of the investigations, students and instructor had similar numbers of segments in the link context. The similarity suggests that students and instructor interacted at a common level about the connections between the mathematics and science contexts with neither party dominating the discussion.

The graphs in Figures 5.4, 5.5, and 5.6 illustrate differences in the structure of the laboratories. During the water flow and light intensity investigations, the class proceeded with the multiple cycles of inquiry as students presented their models, reflected on their work, and revised the questions being addressed. In each of the laboratories, ideas discussed linking the mathematics and science contexts were consistent across the entire investigations. In the penicillin laboratory, as illustrated in Figure 5.6, discussion proceeded in a terminal sequence of science, links, and mathematics. The discussion of the links did not exist throughout the entire laboratory investigation. Pressures due to time and the need to cover remaining topics in the course influenced the time spent on the investigation and the time devoted to the links between the science and mathematics contexts.

Starting sources of ideas in the modeling-focused class. The graphs in Figures 5.7, 5.8 and 5.9 illustrate who's ideas initiated discussion within a given context. Once discussion began by students or instructor was coded as a particular context, subsequent comments were coded as having the same source until the discussion switched to another context. Figures 5.4, 5.5, and 5.6 demonstrated the interactions which occurred. Figures 5.7, 5.8, and 5.9 give related information of which party began the particular discussion.

For the first halves of the investigations, students' ideas began the link discussions. For the second halves of the investigations, the instructor's ideas prompted the link discussion. When students generated models on their own in the first half of an investigation, they connected the science and mathematics contexts. When interaction occurred of the new methods of modeling, the instructor prompted the connections relating the mathematics and science. Students infrequently made the connections on their own.

Context Connections in Students' Written Work in the Modeling-Focused Class

Students' lab reports reflected the infrequency with which students connected the mathematics and the science contexts. Students highlighted the main connections emphasized in class, but failed to mention other connections. For the water flow lab reports, most students mentioned the role of gravity in causing the water to fall, thereby suggesting a quadratic relationship for the height and drain time relationship. Most also reasoned from the science context that the vertex for the quadratic relationship would be (0,0). For the light intensity reports, most students indicated why the context justified the y-intercept could be considered zero in the rates versus intensity relationship. Few students made additional connections such as the interpretation of the recursion relationship or the scientific significance of the values of a and r in the exponential equation $y = a \cdot r^t$.

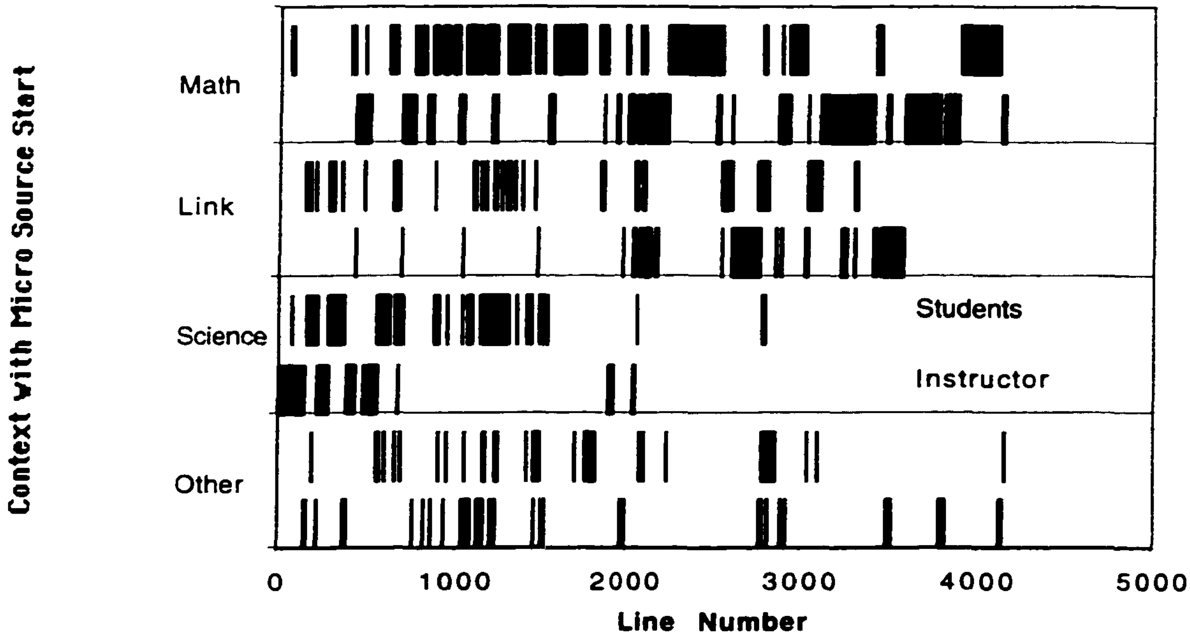

Figure 5.7. Context with the micro source starts for the water flow lab in the modeling-focused class

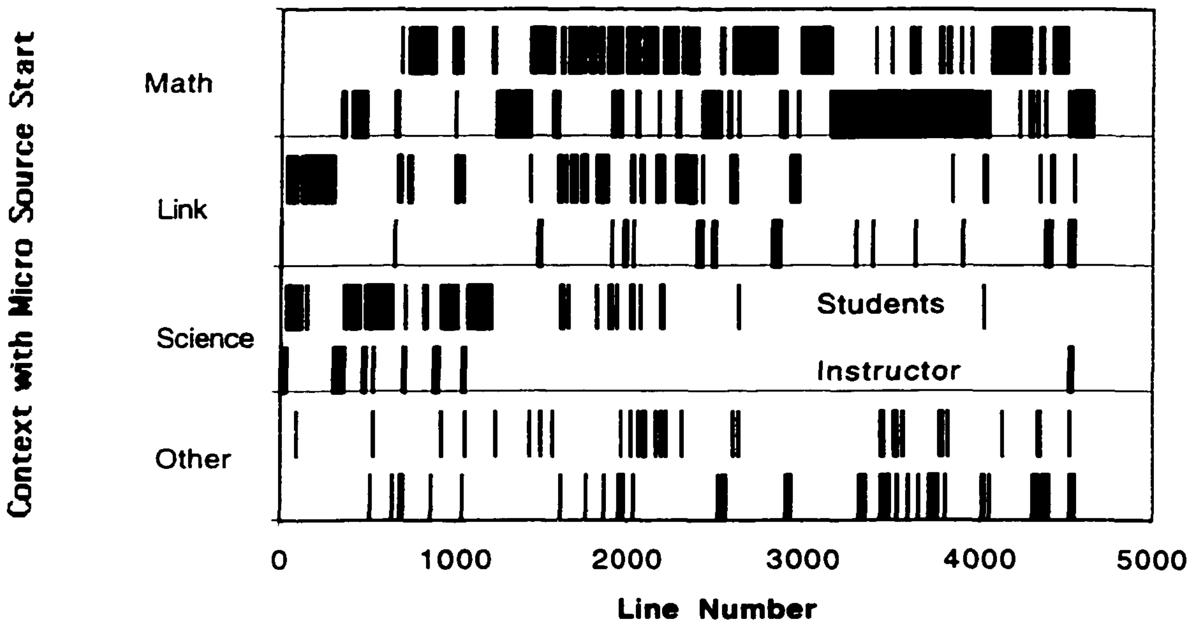

Figure 5.8. Context with the micro source starts for the light intensity lab in the modeling-focused class

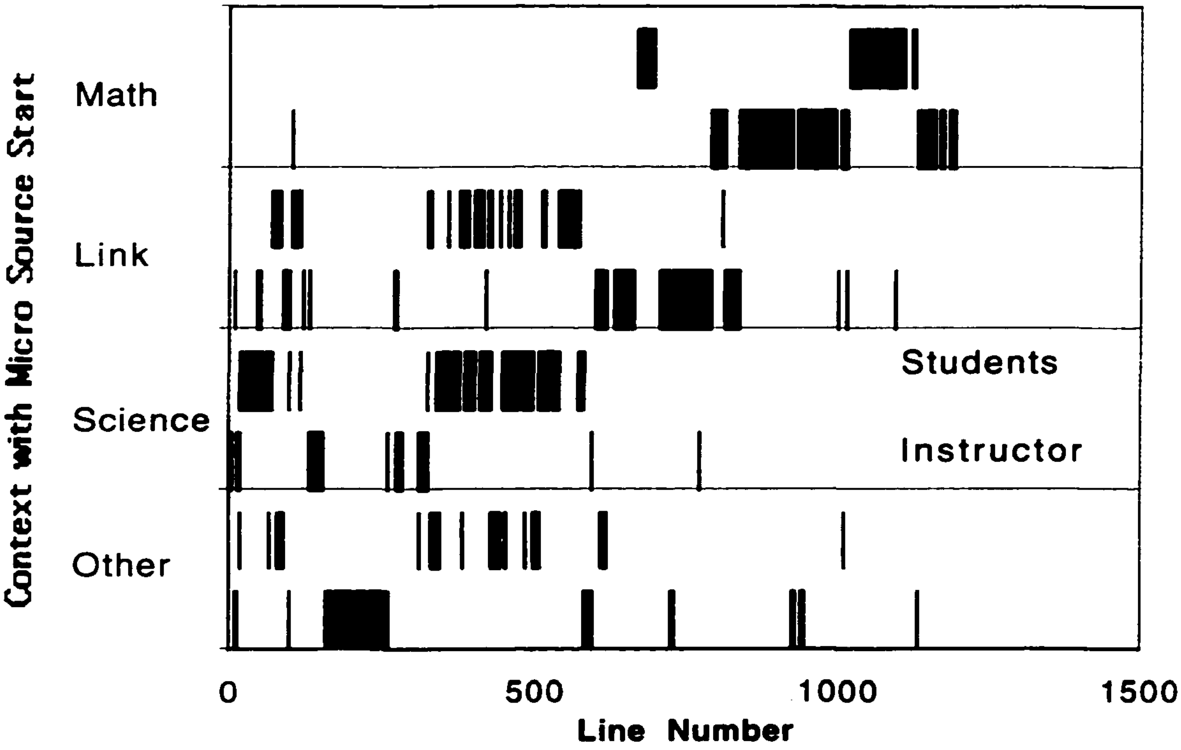

Figure 5.9. Context with the micro source starts for the penicillin lab in the modeling-focused class

Students in the modeling-focused class demonstrated mixed occurrences in linking the mathematics and science contexts. When students developed their own methods for modeling the data during the first halves of the investigations, students regularly reasoned about the model from the science component. When students applied modeling methods taught by the instructor occurring in the second halves of the investigations, students more often had to be prompted to relate the mathematics and science contexts. Graphs of the micro sources of ideas during context discussion supported the consistent interactions between students and instructor about the science, mathematics, and connections. Graphs of the starting sources of ideas indicated that students frequently began link discussion when they first attempted modeling the data, while the instructor began link discussions when she gave instruction on modeling methods.

Context Connections in the Tools-Focused Class

When compared to students in the modeling-focused class, students in the tools-focused class demonstrated similar conceptions of connections between the mathematics and science components. Students in the tools-focused class generally made infrequent links between the mathematics and science contexts unless prompted by the instructor.

Context Connections When Students Applied Their Modeling Methods

The tools-focused class was structured such that in the water flow and light intensity investigations, students presented their data before most analysis occurred. The structure provided students little opportunity to explore possible models individually or in small groups prior to instruction of methods of modeling. As a result, little evidence was known of how students reason from and about the science context when pursuing a mathematical model using their own methods. The evidence that was generated during the penicillin lab indicated that students infrequently related the science context to the mathematics.

The penicillin laboratory differed slightly in format than the water flow and light intensity laboratories. As stated in an interview, the instructor intended for the penicillin lab to provide an assessment of how students model exponential decay data and to generate discussion of modeling exponential data whose asymptote differs from the x axis. Analysis during the penicillin laboratory offered students some opportunity to explore models on their own.

Students examined the data from the penicillin lab in two parts. The first part required students to model the data of a "wash-out" of a single dose of penicillin. Modeled by an exponential equation, the instructor was pleased that students relied on several different methods to demonstrate that the data was exponential. Students demonstrated their use of various tools including graphing the concentrations versus time on semilog graph paper, graphing the logarithms of concentration versus time, and graphing change in concentration versus concentration. The instructor was disappointed, however, that students had not reasoned from the context of the problem. "None of those people that came in said, 'from the context of the problem I know it should be exponential.'" Students reasoned from the graph and other mathematical modeling "tools," but they had not relied on the context to inform the mathematics.

During the second portion of the penicillin lab, students were to find a difference equation to represent the change occurring in the concentration from one dose of penicillin to the next when considering five dilutions between doses. Students worked on a related lake pollution problem to develop methods to find a model for concentration of penicillin when the new doses were administered. [See Appendix D for the problem students worked in class.] On the lake pollution problem, students in groups spent the majority of two class sessions discussing and attempting methods to solve the problem. Several students had difficulty with the units and translating the words to equations. In addition, many students initially assumed that they needed to find an equation generating the amount of pollution in the lake after *n* number of days rather than the difference in amounts for a given pair of days. Others translated the problem differently in terms of when new pollutant was added, resulting in various equations to represent the situation.

Some students were successful in finding a difference equation to model the change in the amount of pollution from one day to the next. One group that was successful in finding a difference equation consistently relied on the science context of the problem. A representative of the group explained the group's solution:

Meg: [For the equation $d(t+1) = d(t) + 100 - \frac{d(t)+100}{2000}$] I initially thought about this as what pollution was leaving during - over the course of the day minus how much was there to start with. And we were taking our readings at the end of the day instead of the beginning, so this was like, how much pollution is left at the end of the day? This is what you started with at the beginning of the day, so that would make up the solution the next day, that was there. And we got the 2000 by dividing the total volume of the lake by the flow rate so that's how we got the numbers. So basically at the end of the day, at the first day you would have everything else would be 0 so you would have 100 kg divided by 2000 because one two-thousandths of the pollution is leaving that day. So this is the total that was there to begin with minus one 2000th the first day. And that's how much you have left over.

T And the 2000th came from simplifying that fraction.

Meg Yeah.

When students relied on the context and translated how to express the science context in mathematical notation, students were successful in modeling the situation.

A portion of the class searched and found the problem and solution in the textbook. When asked to explain where the numbers in the equation came from, some students lacked understanding of how the equation connected to the science context:

[On the board, Amy wrote the equation $W_{t+1} - W_t = 100 - \frac{10000}{20000000} W_t$ which was found in the book.]

T: Would you explain exactly where these numbers in the equation are coming from then?

Amy: All right... Umm, W_t is the waste that comes in the chemical dumping, plus one, minus the W_t which is the [inaudible] plus one equals the waste for that day and then minus the 100 is the amount of chemical added that day, and divided by the amount removed which is the little lake divided by the volume. And then -

T: Craig, what does W_{t+1} mean?

Craig: I don't know. It's like the waste plus a day. That's the way I took it.

T: The waste plus a day.

Craig: Yeah. Another waste, I don't know.

[?]: The amount of waste.

Craig: Yeah, the amount of waste.
[?]: From one day to the next.
T: So the amount of waste for one day and the next day. So then you don't take the waste and add one, it's the waste on the next day.
Amy: Yeah.
T: Okay.
Amy: So it's kind of like the change.
T: And why are we looking at 10000 over 2 million, 20 million?
Amy: Umm, this is the volume. This is the little river going through, and then this is the volume of the lake, the depth times the area. And so then that is how much clean water is in there.

When students relied on the textbook solution, students had some difficulty relating the mathematics to the science contexts. Students demonstrated better understanding of the meaning and purpose behind the equations when they reasoned from the context and developed their own models rather as opposed to reading a textbook solution.

Context Connections When Students Applied Instructor's Modeling Methods

When students in the tools-focused class applied methods of modeling taught by the instructor or included in the book, students less frequently related the science and mathematics context unless prompted by the instructor. In the example given above, students were specifically asked how the context and mathematics were related. Students did not make immediate connections nor did the connections come easily for the students. An additional example, described in Chapter 4, occurred during the light intensity laboratory when the instructor asked students to explain how the recursion relation $I_{t+1} = .82I_t$ related to the light and the filters of tinted Plexiglas. Students were prompted to relate the mathematics and science and as illustrated, some misunderstood how the equation informed the science. These examples reflected the instances in which students were prompted by the instructor to relate the science and mathematics contexts.

<u>Graphical Support of the Context Connections</u>

Figures 5.10, 5.11, and 5.12 graphically illustrate the interactions between the instructor and students on the various contexts discussed during each of the investigations. These graphs present the micro sources of ideas across time in the separate contexts. Figures 5.13, 5.14, and 5.15 display whose ideas initiated the contexts discussions. The graphs support the observations made about the students and instructor interactions, the origins of the context discussions, and the degree to which each context was discussed.

<u>Sources of ideas in the tools-focused class.</u> As evidenced in Figures 5.10, 5.11, and 5.12, within each lab and within each context, students and instructor interactions were frequent. In the tools-focused class, the micro sources of ideas were mixed in the link and mathematics contexts. During the first halves of the investigations, students' ideas slightly outweighed the instructor's ideas during the link discussion. When link discussion occurred during the second halves of the investigations, more frequently, the instructor was the source of the ideas. During the second half of the penicillin lab students continued to be the source of ideas when discussing links between mathematics and science. The frequency of students as the source of ideas stemmed from students' high involvement in discussing, interpreting, and questioning other groups' methods as they presented their analysis of the lake pollution problem. In the mathematics contexts in all investigations, the instructor's ideas occurred more regularly than students' ideas and tended to dominate class discussion.

Informed by the graphs of the phases across time (Figure 4.3, 4.4), the tools-focused class progressively developed from the prediction and experiment phases to the analysis phase. Corresponding to the micro sources of ideas, most of the ideas during the prediction and experiment phases were contributed by students with the analysis phase dominated by

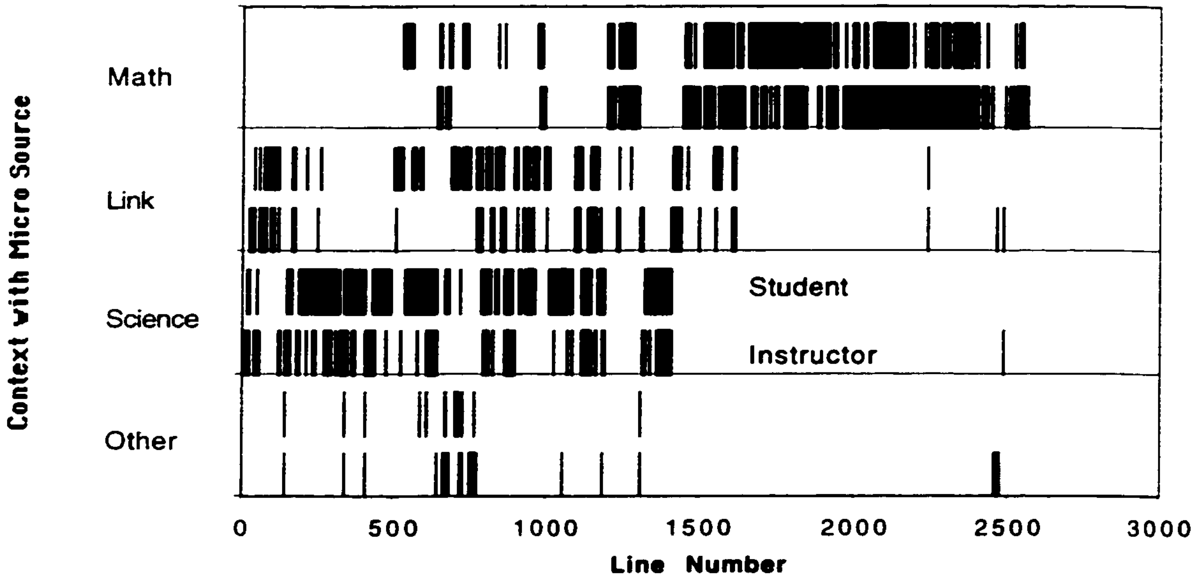

Figure 5.10. Context with the micro sources of ideas for the water flow lab in the tools-focused class

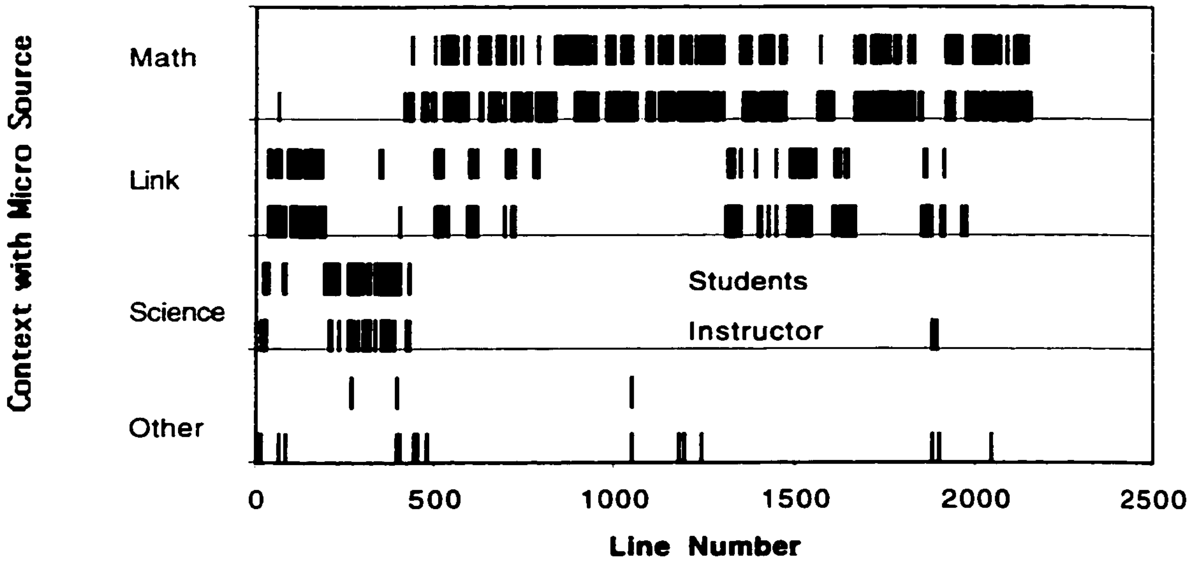

Figure 5.11. Context with the micro sources of ideas for the light intensity lab in the tools-focused class

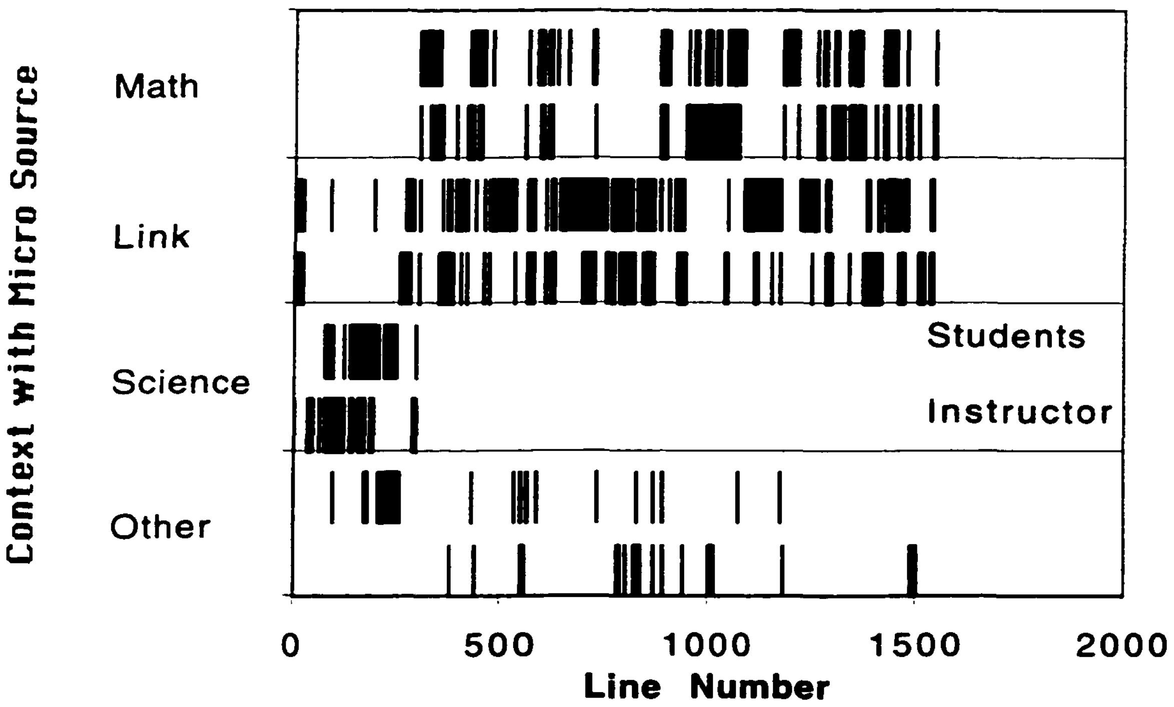

Figure 5.12. Context with the micro sources of ideas for the penicillin lab in the tools-focused class

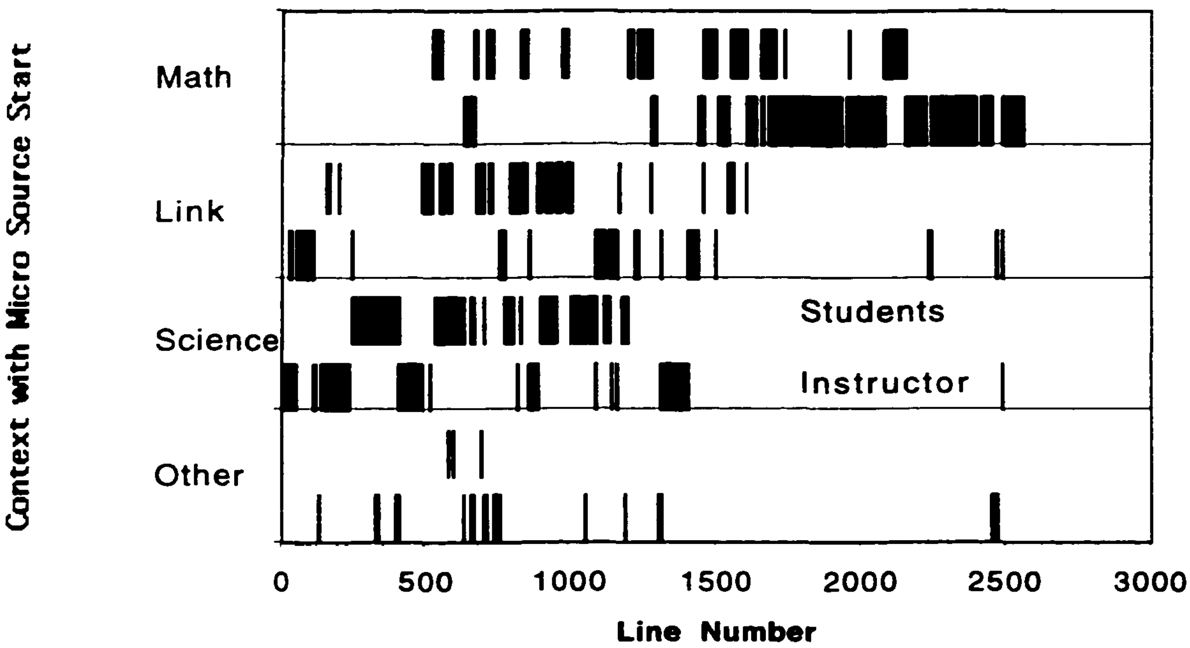

Figure 5.13. Context with the micro source starts for the water flow lab in the tools-focused class

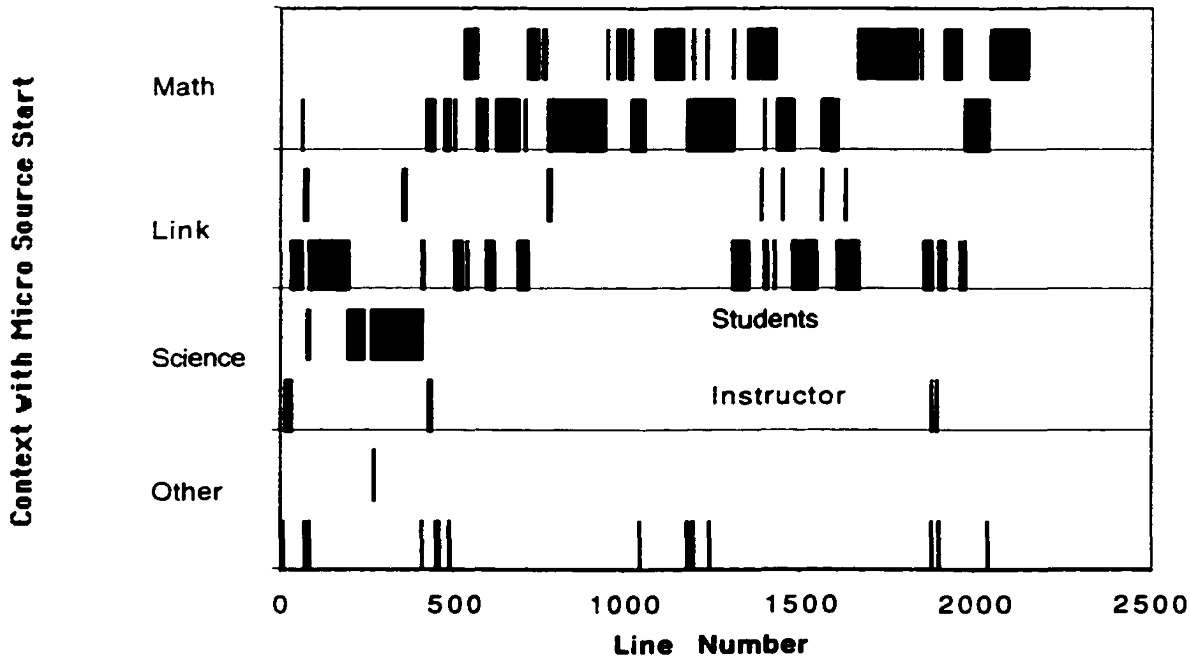

Figure 5.14. Context with the micro source starts for the light intensity lab in the tools-focused class

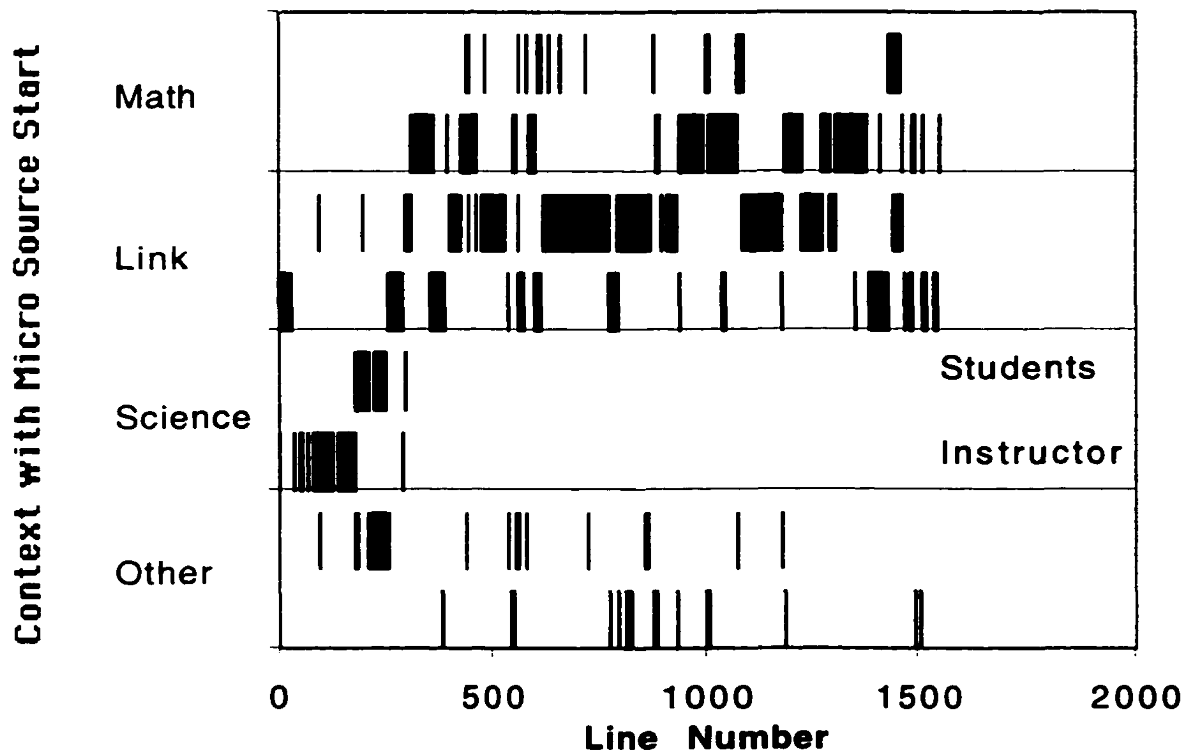

Figure 5.15. Context with the micro source starts for the penicillin lab in the tools-focused class

the instructor's ideas with much student involvement. The graphs of the phases associated with the graphs of the contextualized micro sources of ideas demonstrated achievement of the instructor's goals to develop the mathematics founded on the contexts. In the water flow and light intensity investigations (Figures 5.10, 5.11), once the mathematical component was reached, discussion more often emphasized the mathematical properties and methods of solving problems. In the water flow lab, discussion of the links between the mathematics and science did not continue throughout the entire investigation. Instead, the discussion followed the sequence of science, link, and mathematics. In the light intensity lab, discussion returned to the relationship between the science and mathematics following a period of time in which just the mathematics was discussed. In the penicillin lab (Figure 5.12), the links dominated much of the discussion throughout the entire investigation. When little time was allowed for students to develop their methods for modeling the situation, the link discussion was limited and segmented. When students were given the opportunity to pursue models of the situation, the link discussion extended through most of the investigation.

Starting sources of ideas in the tools-focused class. Figures 5.13, 5.14, and 5.15 display the originators of the discussions across the various contexts for the tools-focused class. Results were mixed across the three laboratories. In both the water flow and the light intensity laboratories, students' ideas most often began discussion of the science context. In the penicillin lab, the instructor initiated the science discussion by launching the laboratory. Following the launch, instruction of the laboratory procedures were given. For the first half of the water flow investigation, students' ideas frequently began the conversations within the link and mathematics contexts. The instructor's ideas usually initiated discussions in the latter halves of the investigations with significant periods of discussion in the mathematics context. In the light intensity lab, students' ideas infrequently launched conversation in the link context. Almost all link discussion originated with the instructor, supporting the class observation that students infrequently made connections between the mathematics and science unless prompted.

As stated above, link discussion in the penicillin lab proceeded differently than in the water flow and light intensity laboratories. Figure 5.15 further supports the distinction between the labs as the link discussion in the second half of the penicillin investigation often originated with the students. Figures 5.12 and 5.15 suggest that when students are given the opportunity to develop mathematical models on their own, they more often generate ideas of how the science and mathematics components are linked and how one context informs the other context.

Context Connections in Students' Written Work in the Tools-Focused Class

Across each investigation students' lab reports contained discussion on the science and mathematics components of the investigations but contained few connections of the two contexts. Most often students described the experiment performed and generated the mathematical model for the data. With some exceptions, generally students did not describe how the experiment or science informed the mathematics nor how the mathematics informed the science. Students' lab reports for the first portion of the penicillin lab supported the instructor's observations that students reasoned about the mathematical equations from their graphs but did not reason from the context. Most students described the methods used to conduct the experiment but did not associate the dilution factor with a recursion relation or with the ratio of the "next" concentration to the "current" concentration. Students described methods of finding the equations but did not relate how the mathematics and science contexts were connected.

In the tools-focused class, most discussion relating the mathematics and science components in the water flow and light intensity investigations was prompted by the instructor. The penicillin lab occurred differently in that students were successful in using a variety of mathematical tools and methods taught in class to generate new models for the first component of the penicillin laboratory. For the second component of the penicillin lab, those students who relied on the science context to build a model were successful in their model construction.

Comparisons between the Two Classes

Students in both the tools-focused class and modeling-focused class behaved similarly when relating the mathematics and science contexts. In class, students infrequently specified the relationship between the mathematics and the science contexts without instructor prompts. Most link discussion on taught methods of modeling originated with the instructors. On lab reports, students most often explained the science context when describing the experiment and described their mathematical procedures. The mathematical procedures most often were those taught by the instructors. When elaborating on both the mathematics and science, students made few connections between the mathematics and science.

The structural component of the classes gave reason for most differences between students' own initiation of link discussion. During the water flow and light intensity investigations in the tools-focused class, students rarely proposed models before the class discussed methods of modeling. During the penicillin lab in the modeling-focused class, students were not given time to explore and discuss potential models in their groups. In these instances, students rarely relied on the context to inform and interpret implementation of methods. In the tools-focused class during the penicillin lab and in the modeling-focused class during the water flow and light intensity investigations, students developed models for their data prior to instruction of new methods. Students more frequently reasoned about the appropriateness of the model from a science perspective. Students rarely generated the exact form(s) of equations warranted by the science context, but they were successful in eliminating other choices of models by considering graphs and contextual interpretation of such graphs. In addition, students' inquiries of models and methods that would fit the data and science context were heightened. Overall, when students were given the opportunity to construct possible models for the data and time pressures to cover content were of less concern, discussion relating the science and mathematics continued consistently throughout the investigations.

Symbols

Closely associated to students' connections between the mathematics and science contexts was students' use of symbols. Noted in Chapter 4, the modeling-focused class placed emphasis on the development of symbols, particularly subscript notation, and stressed the use of appropriate variable names. The tools-focused class stressed the use of subscript notation in various methods for solving problems after the notation was introduced and explained in the book and by the instructor. Many students in the tools-focused class switched notation when working problems discussed in their lab reports. Most students in the modeling-focused class were consistent in their use of notation, but did not fully implement the heuristic to use appropriate variable names.

Symbol Use and Understanding by Tools-Focused Students

On lab reports and during class, students in the tools-focused class demonstrated inconsistencies in their symbolic representations of the data and mathematical models. Students wrote lab reports applying the methods of modeling exponential data to population growth data gathered in ten minute intervals in the science laboratory. Students had worked with their light intensity data in class and on homework assignments. The bacterial growth data provided students with additional practice in applying the modeling methods.

Many students switched notation when developing the mathematical model for the data. In several reports, students implemented one type of notation, such as d for time and I for the amount of bacteria and $d + 1$ to denote the "next" time. After using the subscript notation with d and I, several expressed their exponential equation in the form $y = a \cdot r^{x/10}$. The switch in notation suggested that students had initially followed the example of modeling the light intensity data in class. Students then rewrote the independent variable as time and accounted for the ten minute interval by dividing by ten in the exponent. Students did not reason that the ten should also be represented in the "d+1" subscript notation.

The instructor observed students' difficulty with the symbols and acknowledged during the post-interview for the light intensity laboratory how students were inconsistent in their use of notation:

> Notationally, they still need to work. They have a problem with "How do we handle notation?" I saw lab reports that said $I_{d+1} = .7 \cdot 1.67^n$ and that wasn't what I_{d+1} was. That would be I_n. So just sort of thinking about what their equations means instead of following the model slavishly.

Students displayed similar difficulties with notation during discussion of the lake pollution problem and in their penicillin lab reports. Noted previously in this chapter regarding the book's solution of the lake pollution problem, one student did not understand the notation W_t and W_{t+1}, thinking that W_{t+1} was "waste plus a day" rather than the amount of waste on the next day.

Students in the tools-focused class demonstrated inconsistencies in applying the subscript notation in their light intensity lab reports, penicillin lab reports, and lake pollution problem. Students' work illustrated that they acted at an action or process level of conception when applying the methods of modeling.

Symbol Use and Understanding by Modeling-Focused Students

Students in the modeling focused class were more consistent in their use of symbols when modeling data. Most students correctly modeled their data for each of the three investigations: water flow, light intensity, and the penicillin labs. For the most part, students did not mix variables to represent the same concept. For example, on students' light intensity lab reports, most developed a recursion relation of the form $I_{d+1} = r \cdot I_d$ from the linear equation for the (average) rates of change versus intensity graph. They then generated an exponential equation of the form $I_d = r^d \cdot I_0$ through induction. Much of the students' success on the lab reports was attributed to the multiple problems worked in class and on homework assignments which were similar in the modeling process.

Additional evidence in the course demonstrated that students had not objectified the modeling process as they did not fully master the "use of appropriate variables" when modeling data. On a problem assigned in class, students were to model data of a population of bacteria with an exponential equation using the discrete methods developed in class. The difference between this data set and previous data sets modeled by exponential equations was the readings were taken every 16 minutes rather than each minute. Students correctly calculated rates of change, but committed slight, "fatal" errors when they expressed the recursion relation and inductively developed the exponential equation. A minority of students presented correct solutions using proper subscript notation to represent the 16 minute intervals. The majority of the class failed to continue to represent the 16 minute intervals when applying induction. One group of students presented the following solution. Other groups presented similar solutions.

$$Rate = .0398(D)$$
$$Rate = \frac{D_{d+16} - D_d}{16} = .0398(D)$$
$$D_{d+16} - D_d = .6368(D)$$
$$D_{d+16} = .6368(D_d) + D_d$$
$$D_{d+16} = 1.6368(D_d)$$
$$D_0 = .022$$
$$D_1 = 1.6368(D_0)$$
$$D_2 = 1.6368(D_1) = 1.6368^2(.022)$$
$$D_n = 1.6368^n(D_0)$$

Students' work suggested a process conception of modeling as they generally understood the modeling procedure used previously and did not require external prompts. Students lacked an object conception of modeling and the use of appropriate variables. In the students' work, the *D* represented population density while *d* was assumed to represent time, the independent variable. Students replicated the implementation of the variable, *d*, which had been used as the independent variable in the light intensity investigation. The students did not continue to use appropriate variable names through the induction process with the use of D_1 rather than D_{16} for the first implementation of the recursion relation. As a result, the exponential equation did not account for the 16 minute intervals. A switch in notation from *d* to n was displayed as well with no reason given for the change. The switch did occur in both the subscript and superscript, so the inconsistency from d to n was minor.

Graphical Evidence of the Use of Symbols

Figures 5.4 – 5.15 indicate the micro sources of ideas and whose ideas began the discussions within the various contexts. Examination of these graphs in terms of the mathematics context sheds understanding on the observations made in the classes. Figures 5.4, 5.5, 5.6, 5.10, 5.11 and 5.12 demonstrate that students and instructor in both classes engaged in frequent interactions about the mathematics. The lack of solid blocks of marks in either the instructor or students regions suggests that no one party in either class completely dominated discussion about the mathematics. Thus, students were highly engaged in the conversation surrounding the mathematics including the use of symbols in modeling data.

Figures 5.7, 5.8, 5.9, 5.13, 5.14, and 5.15 further inform the mathematics discussion. These graphs suggest whose ideas launched discussion within a given context. In each of the laboratories in both classes except the light intensity lab in the tools-focused class (Figure 5.14), the graphs indicate that math discussion most often began with the instructor's ideas in the second halves of the investigations. Most discussion about the methods students were to apply to the data in the lab reports occurred during the latter halves of the investigations. These graphs suggest that instructors' ideas began the discussion about the methods and symbols to be used in the reports.

Figure 5.14 displays the origin of the micro sources of ideas for the light intensity lab in the tools-focused class. Marked in the last third of the graph the math context has solid blocks suggesting that students' ideas began the mathematics discussion on three significant occasions. Examination of the transcripts provided the reference of these blocks and the ideas prompting the switches from a previous context. The ideas being discussed during this time marked by the blocks did not entail subscript notation as the discussion of subscript notation occurred earlier in the laboratory. Instead, these three blocks referred to three separate issues surrounding exponential equations. The issues included a student's idea of the form of an exponential equation, a second student's request for help from the instructor, and a third student's representation of the exponent for data whose intervals are greater than one.

Class observations indicate that some differences occurred between classes in the use of symbols. Students in the tools-focused class were inconsistent in their use of notation when modeling data. Students in the modeling-focused class remained consistent in their use of notation when modeling data, but did not fully master the use of appropriate variable names. Most students in both classes did not objectify the use of symbols when modeling data but remained on an action or process level conception.

The graphs of the micros sources of ideas across contexts demonstrated that students in both classes and in each laboratory were highly engaged in the mathematics discussion. The graphs of the starts of the micro sources of ideas illustrated that much of the discussion on the methods the instructors desired students to use began with the instructors. As a result, symbolic discussion most likely began with the instructors and influenced students' work on lab reports.

Reflection

Reflection in the classes occurred in small degrees on a structural level and on levels related to contexts and mathematical procedures. Much informed by the observations of the connections between contexts and the symbol use in the classes, reflection was weak in both classes. In both classes the periods of reflection concerning the methods taught to students were most often prompted and dominated by the instructors' ideas.

Reflection in the Modeling-Focused Class

In the modeling focused class, most periods of reflection were structured into the inquiry development process. As described in chapter 4, after students presented their data and methods of analysis, the class reflected on the experiment(s) and analyses. New questions about the mathematics were generated. The reflection which occurred at these times of the investigations were more significant than at other periods of reflection in the investigations. Discussion during the structured reflection phases was usually dominated by students' ideas.

Figures 5.16, 5.17, and 5.18 display the graphs of the phases of inquiry with the micro sources of ideas for the three investigations. In Figures 5.16 and 5.17, corresponding to the water flow and light intensity investigations, the segments recorded in the reflection phases in the first halves of the investigations refer to the reflection which occurred after student presentations. In the graph associated with the water flow laboratory, the last portions in the reflection phase also refer to students' questions which followed additional presentations relating rates of change and heights. During the segments in which students reflected on what was presented, students' ideas dominated the discussion. The other periods of reflection, not directly related to issues stemming from students' presentations, were frequented more often by the instructor's ideas. These periods of reflection most often pertained to the methods of modeling taught by the instructor. In the penicillin lab, no separate stages of reflection occurred. As described previously, the penicillin lab transpired

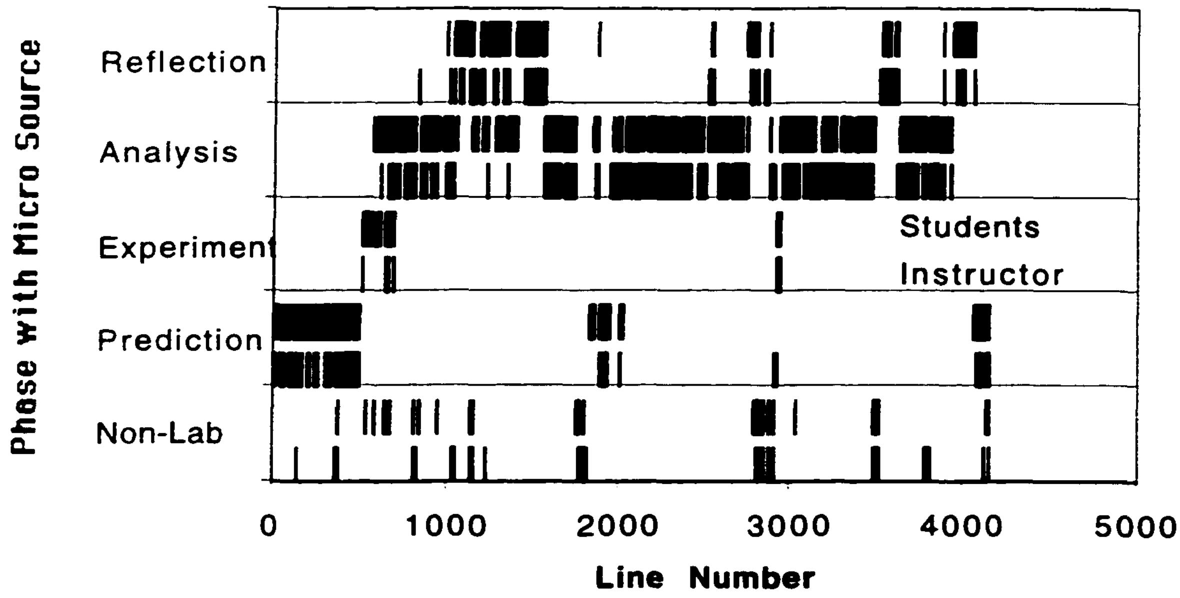

Figure 5.16. Phase of inquiry with the micro sources of ideas for the water flow lab in the modeling-focused class

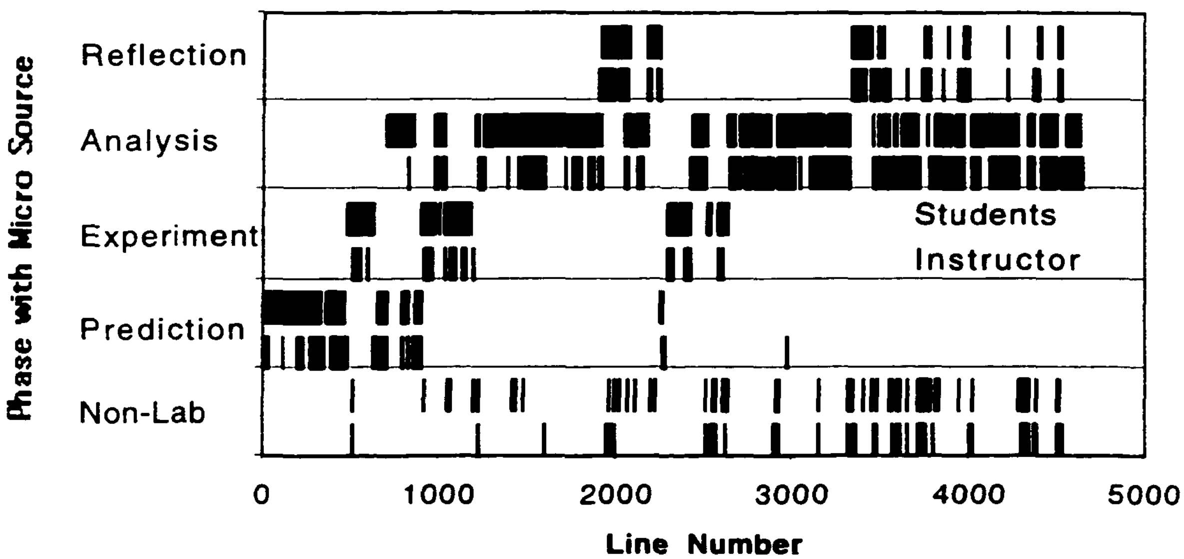

Figure 5.17. Phase of inquiry with the micro sources of ideas for the light intensity lab in the modeling-focused class

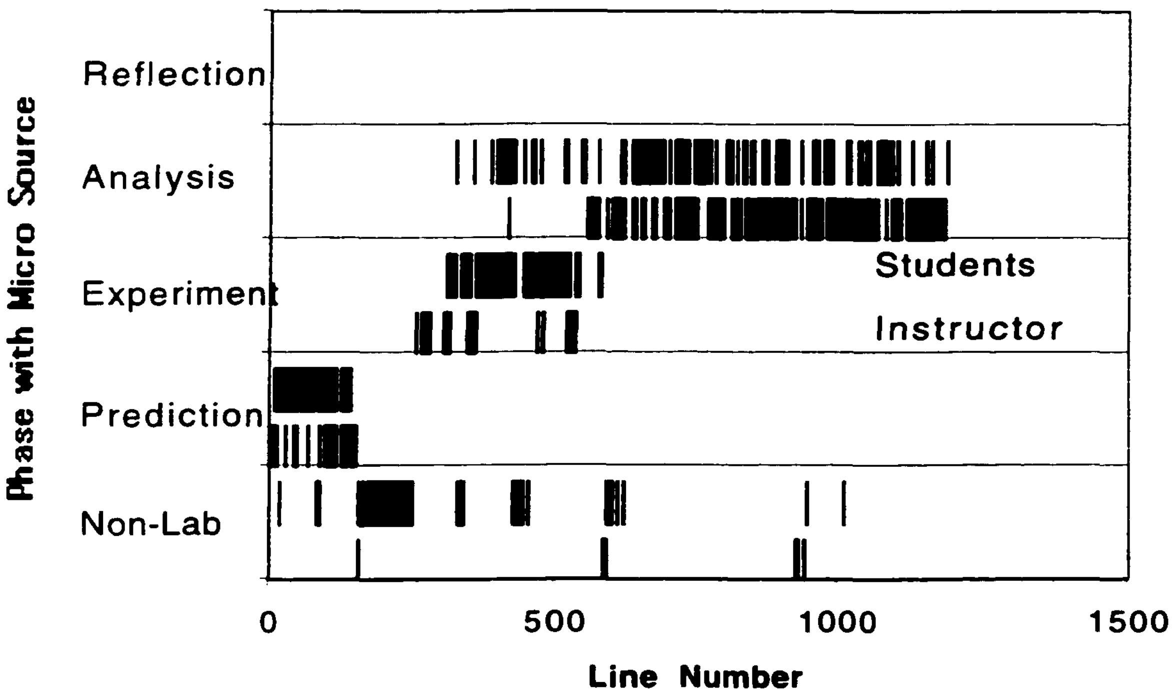

Figure 5.18. Phase of inquiry with the micro sources of ideas for the penicillin lab in the modeling-focused class

over segmented class sessions toward the end of the semester. Students were not given the opportunity to present group methods of modeling data nor reflect on methods developed.

Who started the reflection? As stated previously, students were prompted to connect the mathematics and science contexts when the instructors' methods were applied. Figures 5.19 and 5.20 generate further support that when instructor's methods were taught during the second halves of the investigations, the reflection which occurred was more often prompted by the instructor.

The graphs illustrate that reflection in the modeling-focused class was weak when related to the instructed methods of modeling. Stronger periods of student-generated reflection periods occurred following students' presentations of their analysis methods.

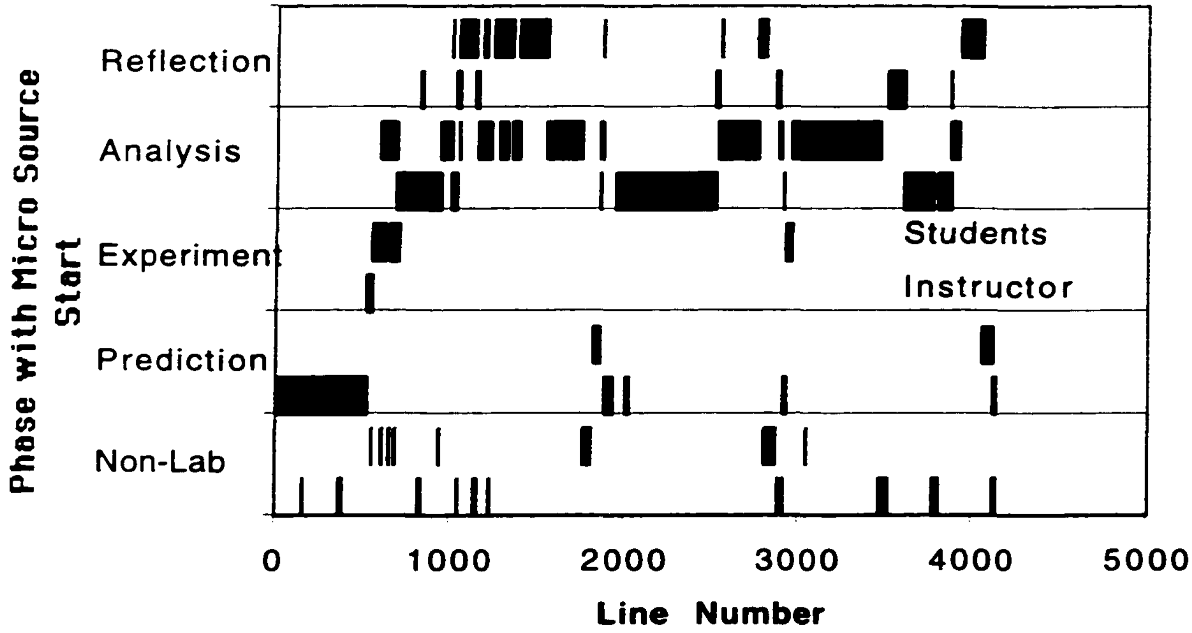

Figure 5.19. Phase of inquiry with the micro source starts for the water flow lab in the modeling-focused class

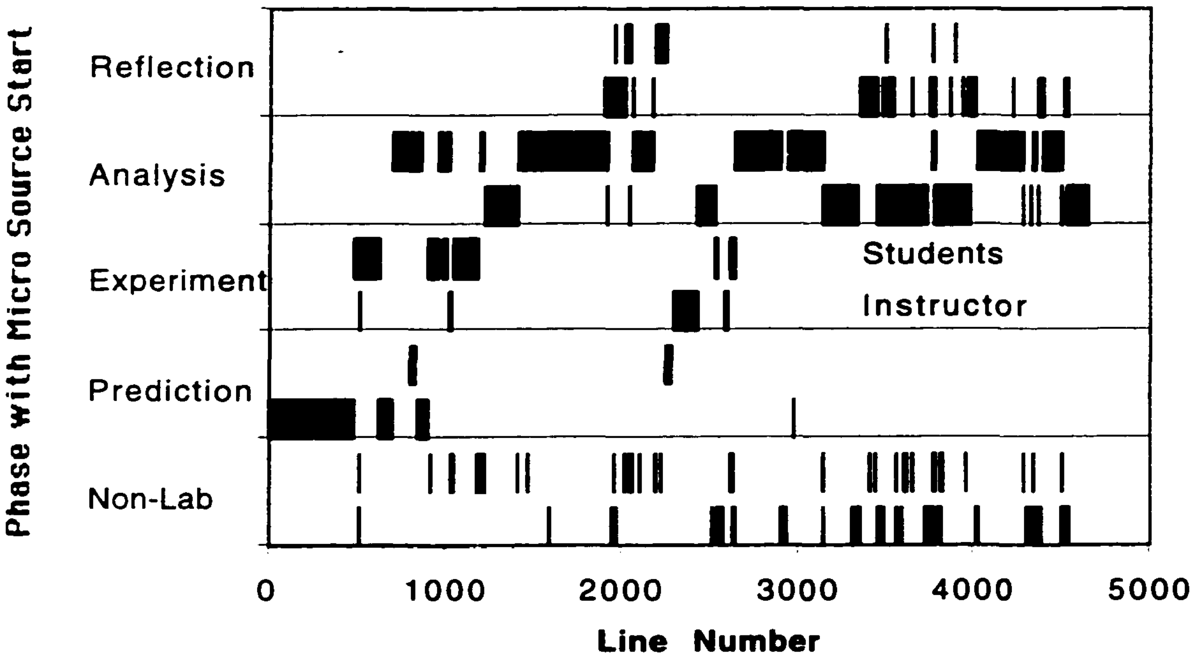

Figure 5.20. Phase of inquiry with the micro source starts for the light intensity lab in the modeling-focused class

These reflection periods were structured into the class. Time pressures likely influenced the existence of reflection periods in the investigation, particularly when the need to cover additional content influenced the class procedures.

Reflection in the Tools-Focused Class

Figures 5.21, 5.22, and 5.23 give the graphs of the phases of inquiry with the micro sources of ideas for the tools-focused class. The infrequency of segments by both students and instructors support the observation that periods of reflection were not structured into the tools-focused class. The reflection which did occur more often was generated by the instructor. Like the modeling-focused class, Figure 5.23 illustrates that no separate reflection phase occurred in the tools-focused class during the penicillin lab. Again, time pressures and needs to cover specific topics likely contributed to the lack of reflection.

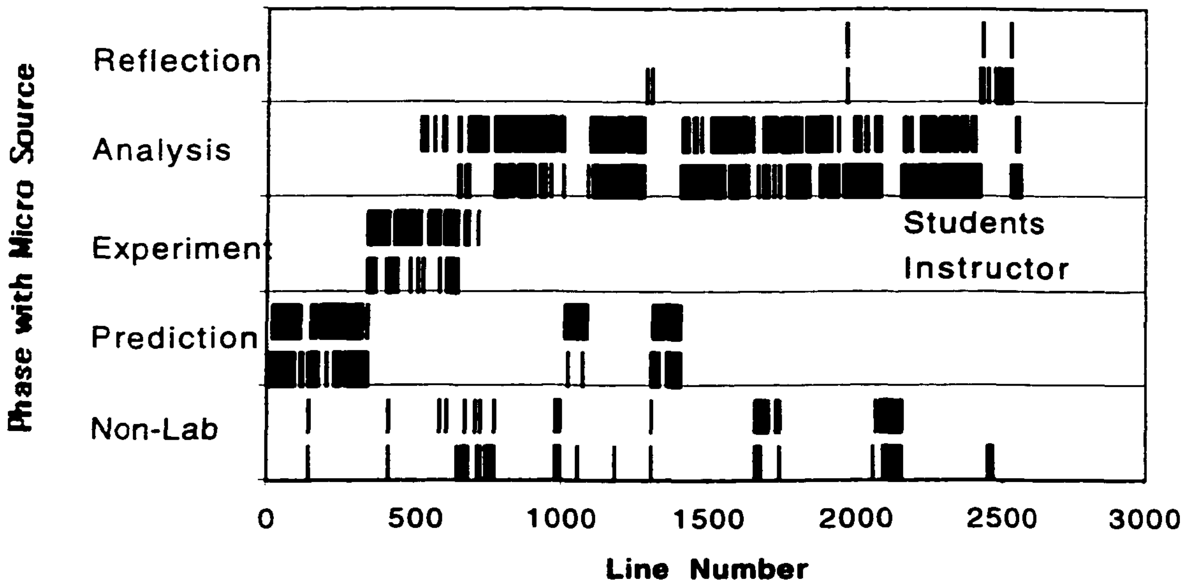

Figure 5.21. Phase of inquiry with the micro sources of ideas for the water flow lab in the tools-focused class

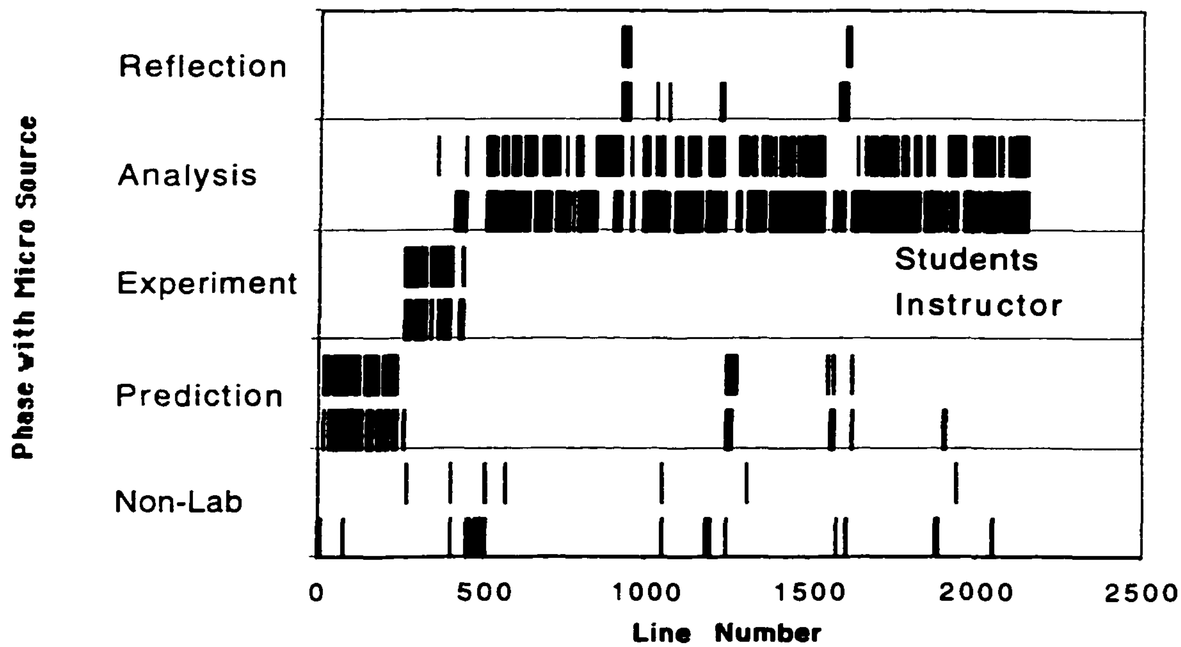

Figure 5.22. Phase of inquiry with the micro sources of ideas for the light intensity lab in the tools-focused class

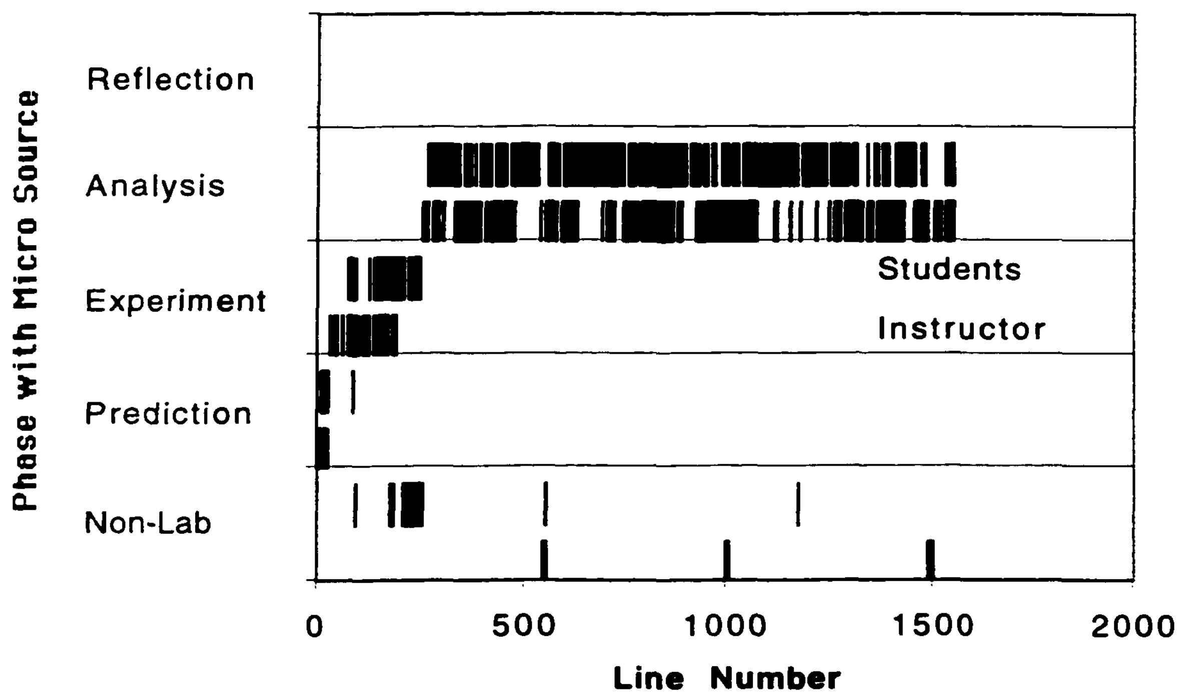

Figure 5.23. Phase of inquiry with the micro sources of ideas for the penicillin lab in the tools-focused class

Who started reflection? Based on the graphs in Figures 5.24 and 5.25, the periods of reflection in the tools-focused class were rarely initiated by students' ideas. Given the few number of reflection periods, however, more information about students' reflection was needed to support the observations. Students' reflections on the connections between the mathematics and science contexts and the use of the symbols, as discussed previously, suggested that students needed to be prompted to reflect on the procedures taught by the instructor.

Significant periods of reflection in both the tools-focused class and the modeling-focused class were weak. The modeling-focused class had slightly more periods of reflection with students' contributions structured as part of the inquiry process. When the reflection focused on the methods taught by the instructor, students' ideas less frequently began the reflective discussion. The tools-focused class had periods of reflection though reflection was not specifically mentioned as a goal by the instructor prior to the laboratories. Supported by the observations in the context and symbol discussions in both classes, students' reflections were few unless prompted by the instructor.

Student Interviews

Interviews with six students from the modeling-focused class at the end of the semester gave further support of the observations recorded previously in this chapter. Provided with a context and data (Appendix B.), students were to demonstrate and share their thinking as they modeled the data using methods similar to those used in class. Results from the interviews illustrated that most students had achieved a process rather than object conception of mathematical modeling. One of the six students correctly found a mathematical model for the temperature data. The other five students committed various

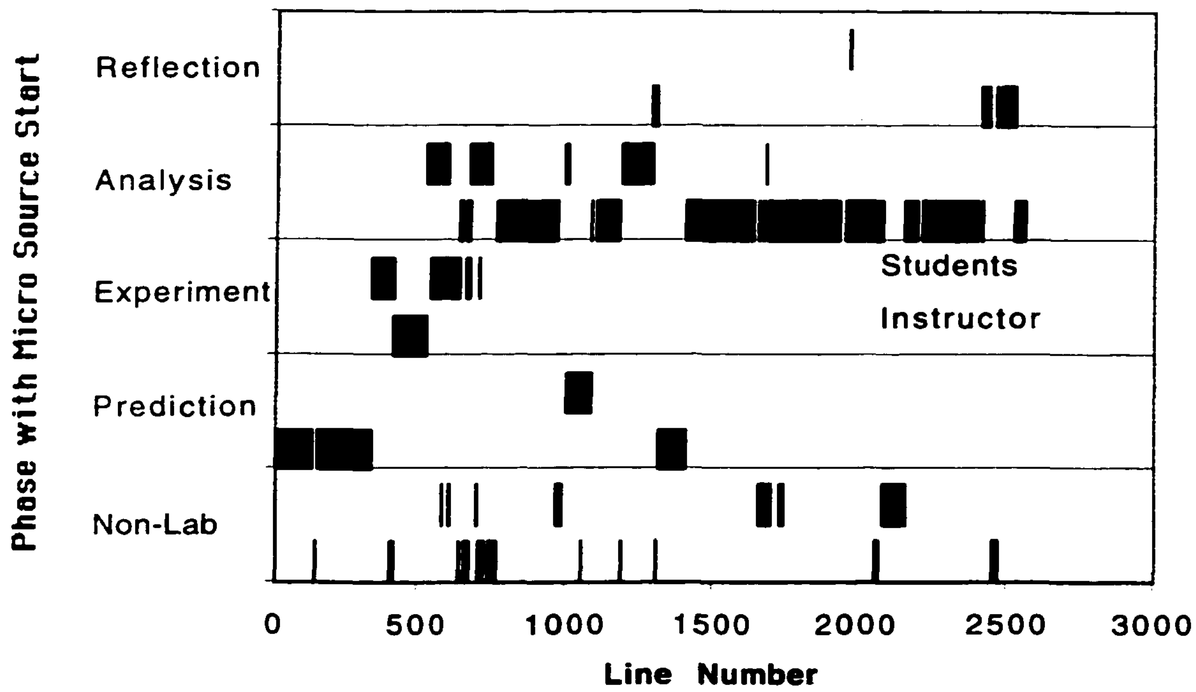

Figure 5.24. Phase of inquiry with the micro source starts for the water flow lab in the tools-focused class

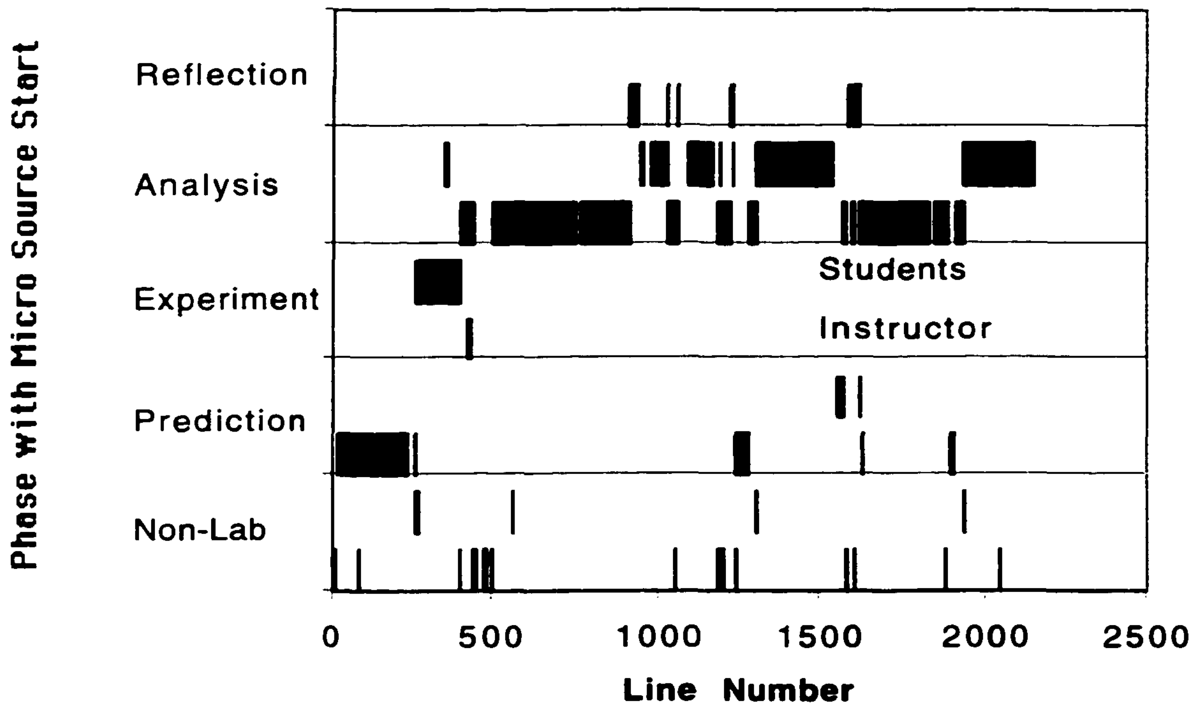

Figure 5.25. Phase of inquiry with the micro source starts for the light intensity lab in the tools-focused class

errors in the representation of the context. The students often lacked the ability to consistently use appropriate variables and often did not fully rely on the science context to reason about the mathematics.

Interview with Brett

Brett found a mathematical model for the data of a hot liquid cooling over time. During the interview, he often stopped his work and asked, "What am I doing?" He reflected on his work, reminded himself of the context, and gauged his performance in light of his goal. Brett relied little on specific modeling methods developed in class. He depended on the graphs of the data, rates of change, the science context, and different regression options on his calculator.

Brett pursued different "wild goose chases" during the interview. He began by "zero-zeroing" the data, meaning he wanted the data to begin at (0,0) and increase rather than begin at (0, 59.50) and decrease. Though the graph was not what he expected, he realized the "zero-zeroed" graph made sense in terms of what he had predicted about the liquid cooling to room temperature. He estimated the rates of change and recognized that the rates were nonlinear with time. Calculating the rates of change of the rates of change, he indicated that the new rates were again nonlinear with time. His calculations supported his hypothesis that the data was exponential. After several "wild goose chases" Brett shifted the temperature values so that they approached zero rather than room temperature. He applied exponential regression to generate a model for the translated values and shifted the graph up to produce a model for the original temperature data. After checking his work by testing a time value, he was confident that he had found an appropriate model for the data.

Brett successfully relied on the context to reason about the temperatures and the rates at which the liquid cooled. He attempted several different methods and consistently gauged his performance in light of his goal to find a model. While Brett did not rely on specific modeling techniques taught in class, he had objectified modeling in the sense of monitoring his progress, relying on the context, and interpreting and testing his results.

Interview with Jewel

Five other students demonstrated during the interviews that they had not fully objectified mathematical modeling into their conceptual schemas. Most students hypothesized that the data would be exponential since the temperatures would cool until they reached room temperature. After the hypothesis, students seldom referred to the science context. Instead, they depended on their experiences and class notes of the instructed procedures to find a model of the data. These five students stated their desire to linearize the data. They resorted to combinations of methods used in class and failed to reason about their methods in light of the context. Students frequently used inappropriate variable names causing additional problems in their modeling attempts. Since in-depth analysis of individual student's modeling techniques was not a primary goal of this research study, the work of only one of the five remaining students will be described here. Each of the five students pursued slightly different avenues in modeling the data, and Jewel's techniques fairly represent the major issues arising in the other interviews.

Jewel failed to reason about the mathematics from the science context and inappropriately represented the relationships she graphed. Jewel calculated the rates and examined the *rates* versus *time* graph. Since the temperatures went down, the negative rate values made sense. To linearize the data, Jewel graphed the (average) *rates* versus *time-*

squared. Though the graph was not linear, she knew she wanted a recursive equation to evolve from a rate equation. She used linear regression to produce an equation for *rate* versus *time*. She ignored the y-intercept and replaced the x and y variables with Tem_t and $\frac{Tem_{t-6} - Tem_t}{6}$, respectively. Jewel's representation of the *rate*, change in *temperature* over change in *time*, was off by a negative sign, and the x was replaced by a temperature variable rather than the time variable. Jewel used her equation $\frac{Tem_{t-6} - Tem_t}{6} = .048\, Tem_t$ to develop the recursion relation $Tem_{t-6} = 1.288 Tem_t$.

Jewel occasionally paused to consider her work, but did not fully assess her methods and the relation to the science context. She recognized that her work did not tell her if the data was exponential: "It just tells me that if you know this one you can find that temperature....I can find the exponential though." Working with the equation and assuming that the subscript on the left of the equation was t+6, she inductively generated the exponential equation $T_{tim} = 1.288^{t/6}(59.50)$, keeping the representation for the six minute time intervals. Jewel numerically checked her equation against her data and realized that her work was incorrect. The instructor indicated to her that she needed to examine the graph of the *rates* versus *temperatures* rather than the graph of *rates* versus *time*. She graphed the *rates* versus *temperatures*, saw a line, and applied linear regression to produce the linear relationship $y = -.0656x + 1.605$. She again ignored the y-intercept and developed an exponential equation $Temp = .61^{t/6}(59.50)$. The exponential equation took into account the six minute intervals, but overlooked the effect of the room temperature on the y-intercept.

Graphical Support of Interview Observations

Figures 5.26 and 5.27 display the graphs of the phases of inquiry for the interviews for Brett and Jewel. Students proceeded through each of the phases of inquiry. Jewel stopped and considered her mathematics as indicated in the reflection phase. Figures 5.28 and 5.29 indicate students' uses of the context. These graphs may be more informative of students' success in solving the problem. Brett regularly linked his work with the context as illustrated in Figure 5.28. These links were not indicated in the reflection phase of inquiry in Figure 5.26. Jewel linked the science and mathematics occasionally, but less frequently than Brett. Figures 5.27 and 5.29 also demonstrate greater instructor influence in Jewel's interview when Jewel did not know how to continue or correct the work she had done.

The student interviews supported the observations from the class transcripts and students' lab reports for the modeling-focused class. Few students reflected on the relationship linking the mathematics and science contexts unless prompted by the instructor. Few students consistently applied the use of appropriate variable names. Students were more successful in modeling the data when they regularly linked the science and mathematics contexts. In general, few students objectified the modeling conception.

Conclusions

Graphical evidence and interviews with modeling-focused students supported the observations of the two classes. The differences that existed within and across the sections of Math 181 may be attributed to the structural components of the laboratories. In each class, the emphasis placed on the process of inquiry varied with the laboratories and influenced the classroom discussion and students' interactions with the mathematical modeling.

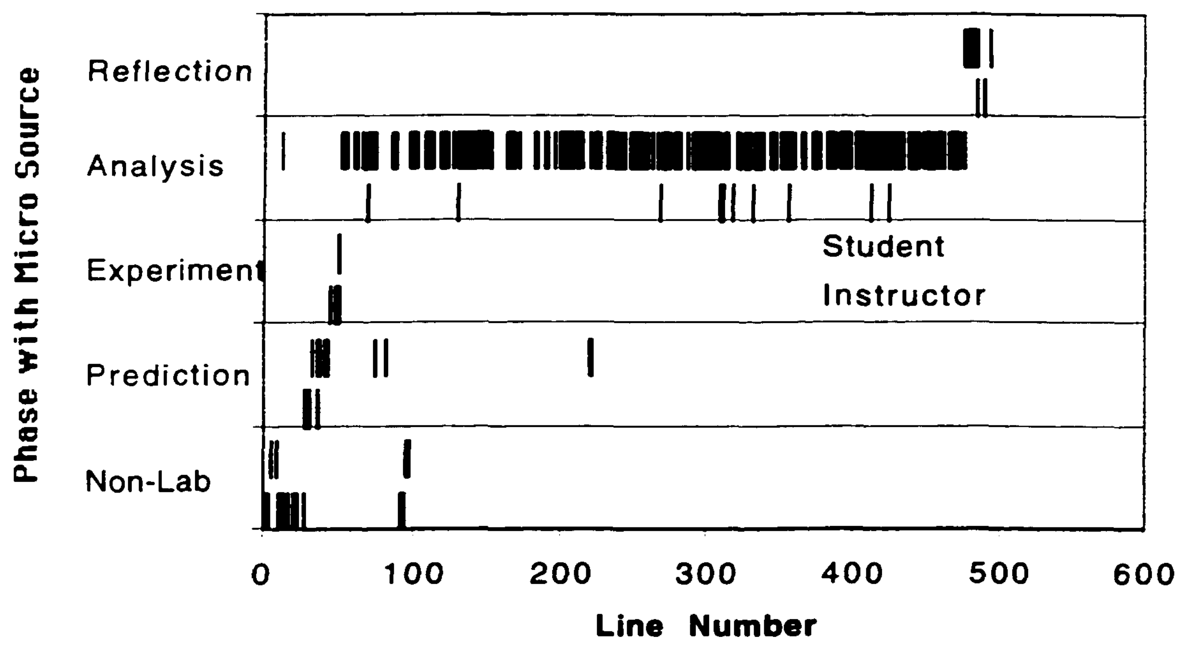

Figure 5.26. Phase of inquiry with the micro sources of ideas in Brett's interview

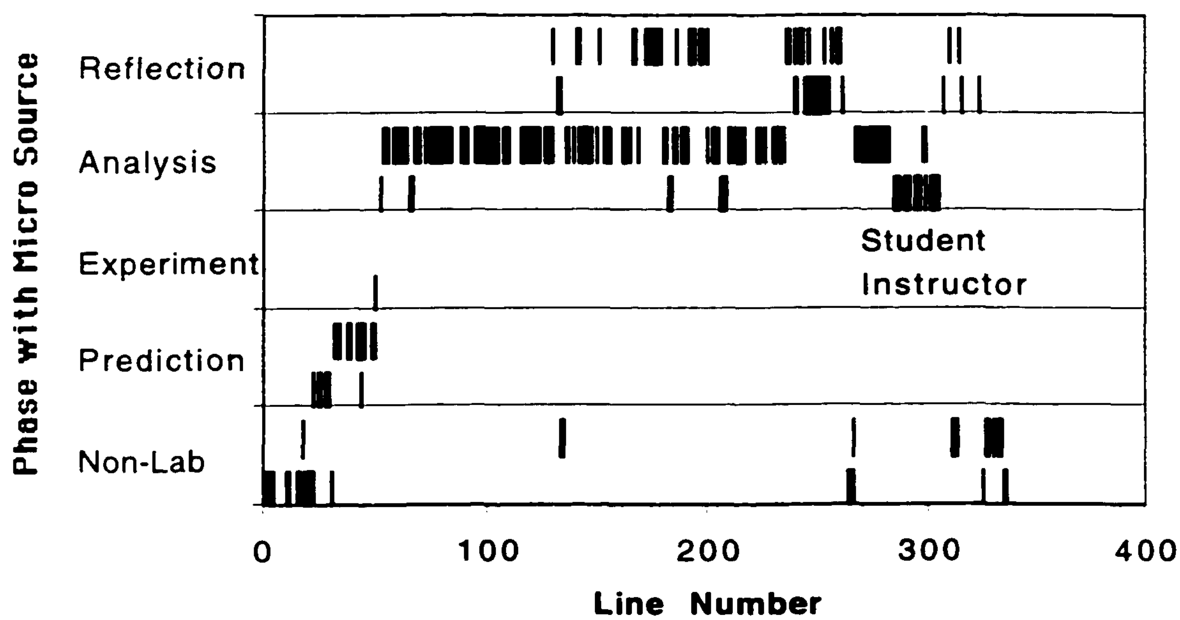

Figure 5.27. Phase of inquiry with the micro sources of ideas in Jewel's interview

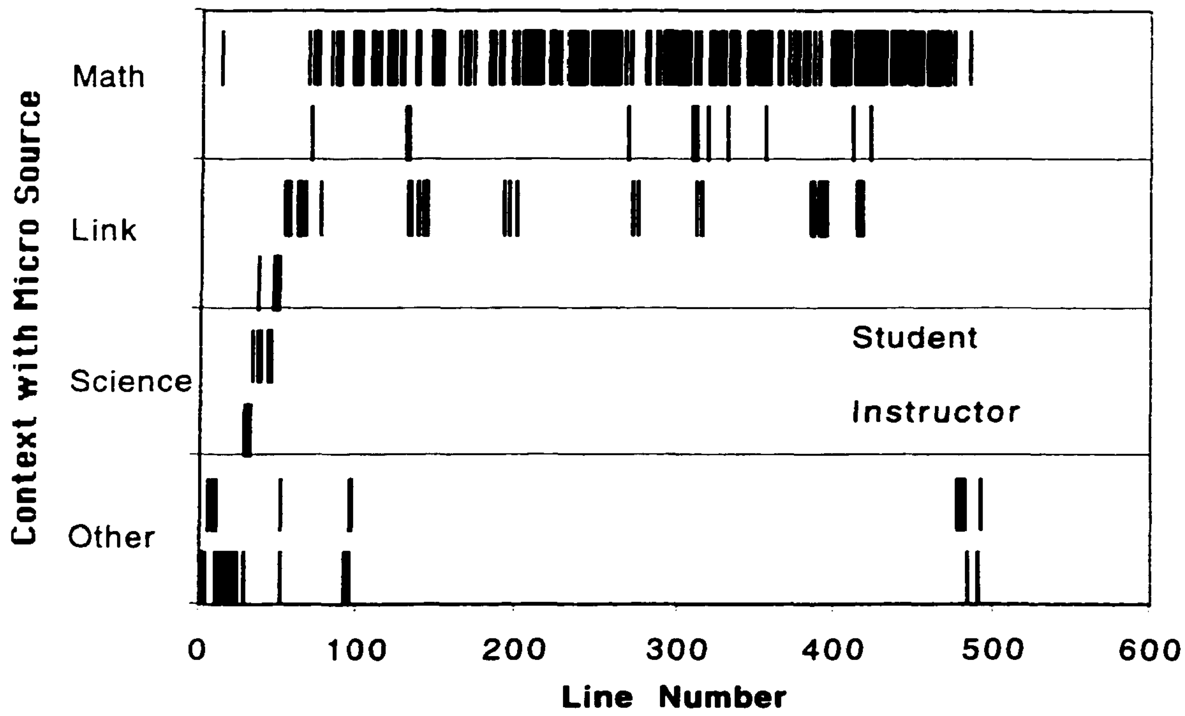

Figure 5.28. Context with the micro sources of ideas in Brett's interview

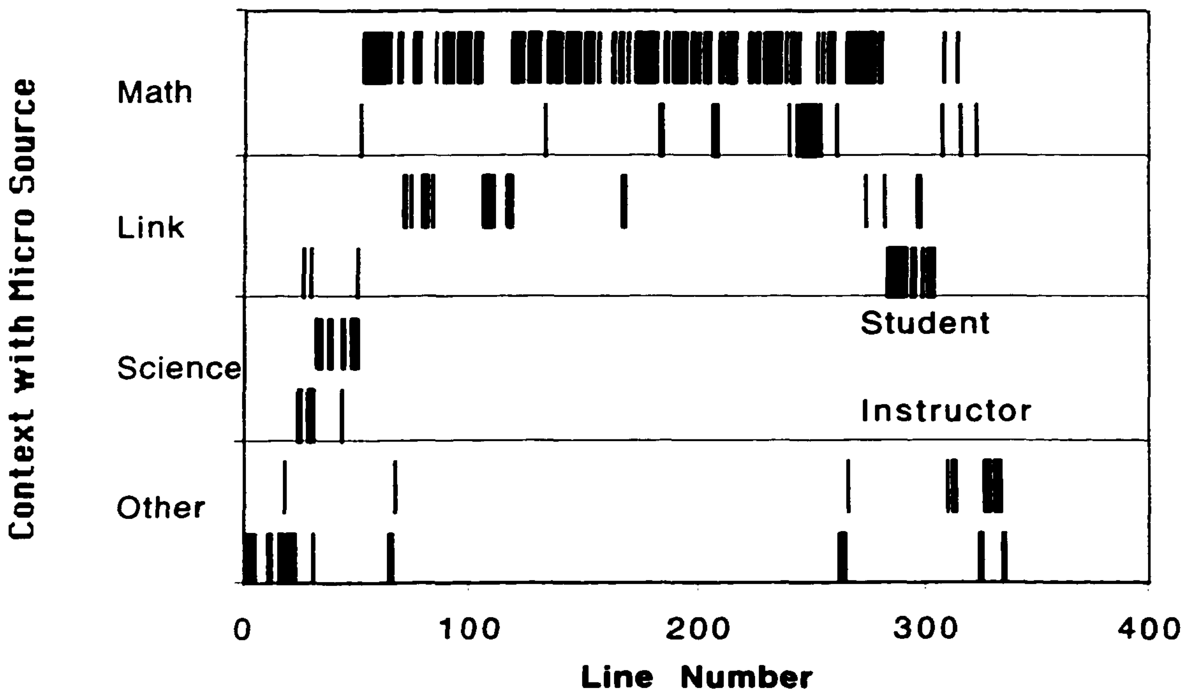

Figure 5.29. Context with the micro sources of ideas in Jewel's interview

The party which posed the question(s) for investigation coupled with the equipment and procedures used in the laboratories affected students' interpretations of the question(s) being addressed. Students in the modeling-focused class pursued and achieved agreement in the questions under investigation. The tools-focused class investigated questions posed by the instructor. During the water flow investigation in the tools-focused class, the equipment promoted a variety of means to investigate the instructor's question and address other questions raised by students. The multiple interpretations of the questions led to varied results on students' lab reports when applying the methods of modeling. In the light intensity and penicillin labs the equipment and procedures limited the pursuit of multiple questions, helping to maintain agreement on the questions addressed.

Pressures to cover specific topics in a limited amount of time influenced the types of inquiry which occurred. Coverage seemed to be a concern during the penicillin lab in the modeling-focused class and during the water flow and light intensity investigations in the tools-focused class. During these times, link discussion did not occur consistently throughout the entire lab. When the coverage of specific topics was less of a concern, link discussion occurred throughout the entire lab. This was seen during the water flow and light intensity investigations in the modeling-focused class and during the penicillin lab in the tools-focused class.

In both classes, when students had the opportunity to pursue methods of modeling data before new, instructed methods were given, they more frequently referred to the science context to inform the mathematics. When applying new and instructed modeling methods, students rarely related the science and mathematics contexts unless prompted by the instructor. The effects were evident in students' use of notation. When time was devoted to the development and meaning of the notations students were more consistent in their use and interpretation of the notation. In the tools-focused class, students were inconsistent in their use of subscript notation to represent the concepts. Students in the modeling-focused class were more consistent in their use of variables within a given problem, though they did not to regularly attend to the use of appropriate variable names.

In both classes, periods of discussion were dominated by different parties in light of when they occurred. Students' ideas regularly dominated and initiated discussion during the first halves of the investigations as the first cycle of inquiry occurred. During the second halves of the investigations, instructors' ideas more often initiated discussion as instruction of new methods of modeling and other mathematics were given. Periods of reflection were rather weak in both halves of the investigations and in both classes. The reflection which did occur was generally structured into the lab format.

CHAPTER 6

CONCLUSIONS

This study sought to explore and represent the implementation of inquiry in two mathematics classroom. Chapters 4 and 5 described the environments and the effects on students' modeling skills. Provided below is a summary of the findings, implications for future pursuits of inquiry in collegiate mathematics classrooms, and implications for future research studies examining issues of representing inquiry in the classroom.

Was Inquiry Achieved in Both Classes? Implications for Future Pursuits of Inquiry

This study adds to and extends the current body of knowledge concerning inquiry in collegiate mathematics education. Studies examining inquiry (Roth, 1995; Galbraith & Clatworthy, 1990) allowed for and encouraged students to explore open-ended questions in open-inquiry environments. The instructors of the two classes in this study set goals to cover specific content areas which did not allow for full inquiry investigation of all the subject matter to be covered. Both classes structured the kinds of questions to be investigated with limitations placed by context, equipment, and time. In structuring the classes, some components of inquiry were more frequent than others such as the structural stages of inquiry and student and teacher interaction in all contexts in both classes. Other components such as reflection and consistent linking of the science and mathematics contexts were less frequent in both classes across all investigations. The pursuit and indication of inquiry in both classes provide implications for future pursuits of inquiry in collegiate mathematics classrooms and future studies of inquiry.

Cycles of Inquiry

On a structural level, inquiry occurred in cyclic processes in both classes much as described by White and Fredericksen (1998). Students made predictions about a context, collected data, analyzed the data, generalized the results, and reflected on the processes. In White and Fredericksen's study, as students repeated the cycles of inquiry, students pursued deeper content questions and were less aided by the instructor as scaffolding was removed. In both classes in this study, multiple cycles of inquiry were pursued with deeper mathematical questions addressed. However, "scaffolding" frequently increased rather than decreased in the second and third cycles. This fact was evident in the graphs of the micro sources of ideas and the starts of the micro sources of ideas (Figures 5.16-5.25). The instructors' ideas were more frequent and often started the sequences of discussion during the second halves of the investigations in the analysis phases.

The greatest factor for the increased influence by the instructors in the development of the concepts in the second halves of the investigation was time constraints. Both instructors acknowledged the need to cover planned topics within the semester long time period since the course was a prerequisite for the subsequent calculus course and other science courses within the students' majors. The instructor of the tools-focused class admitted the restrictions due to time: "I can afford a class period. I can't afford two class periods to work on experiment protocol." The instructor of the modeling-focused class agreed stating, "We don't have much time to keep going back and refining like scientists do....Sometimes the semester just doesn't seem long enough." The instructors wanted students to pursue their questions and experiments in the multiple cycles, but the time restraints limited the amount of time students could investigate questions on their own without some instructor input of additional methods and mathematics students were to learn.

Implications for future studies of inquiry in collegiate mathematics involve time and curricular issues. As indicated in the study by Huntley, Rasmussen, Villarubi, Sangtong, and Fey (2000), students acquire different types of mathematical skills based on the methods of instruction. In particular, students perform better computationally when instruction emphasizes symbol-manipulation routines. Those students who are regularly asked to mathematically model and interpret their results in inquiry oriented environments develop greater understanding of those skills and processes. Collegiate mathematics educators must decide as a profession which types of skills, whether inquiry-oriented or computational, are a priority for their students and determine the methods to assess and measure students' acquisition of the desired skills.

If pursuing an inquiry environment, an instructor must decide the level of involvement or the amount of scaffolding to be structured into the inquiry cycles. If students are having their first experiences in an inquiry oriented environment and the course is a single semester, some scaffolding of the inquiry process will likely need to be maintained for students for the duration of the course. If the course continues into an additional semester, the scaffolding may be removed depending on students' and instructor's success in pursuing the curricular items needed.

Posing and Interpreting the Question under Investigation

In inquiry environments, negotiation of the goal of the task should be understood by students and instructor. If the investigation is to be open-ended, then students should understand what guidelines will be used to judge success as they run with their ideas. If the instructor has targeted goals for the investigation, formulation of the goals can occur as a class and agreement of the goals can be carried through the investigation. Techniques which help to focus students' questions and maintain agreement include those recommended by Tanner and Jones (1994) in which students can be prepared to present results to address a particular question.

For inquiry to occur, the students do not need to pose all the questions for investigation, but agreement of interpretation of the question(s) and of the tasks for investigation should be pursued (NRC, 2000; Hiebert et al., 1997; Roth, 1995; Borasi, 1992). The two classes initially showed differences in student and instructor agreement of the interpretation of the problems pursued. Yackel's (1995) and Christiansen's (1997) research indicated that breakdown of interactions occur when the different participants have different interpretations of the task or of what constitutes mathematics. In the tools-focused class, the different interpretations in the goal of the water flow investigation between students and instructor led to misapplication of the instructed methods of analysis in students' lab reports. Students misaligned the analysis for their data with their goals. Subsequent investigations in the tools-focused class and investigations in the modeling-focused class demonstrated the agreement between students and instructors on the interpretation of the goal(s) of the investigations.

Reasons for agreement or disagreement in the interpreted goals for the investigations may be attributed to factors described in prior research studies. As Borasi (1992) recommended, the modeling-focused class, negotiated the direction of inquiry and the monitoring and evaluation of inquiry. At the launch of each investigation in the modeling-focused class, the students posed and agreed on the question being pursued. In addition, the students were to be prepared to present their data and analysis, giving explanation and justification for their work and conclusions. For example, in the water flow investigation, students were to be prepared to predict a flow rate given a height or volume of water. Students anticipated questions and were prepared to address the questions as encouraged in the Tanner and Jones (1994) study. Applying the strategy for students to be prepared to present coupled with students' generation of new questions promoted agreement of the questions, focused the task for students, and caused specific issues related to the course content to be raised.

Reasons for agreement or disagreement in the interpreted goals for the investigations in the tools-focused class are less clear. None of the strategies given by Borasi (1992) and Tanner and Jones (1994), were directly applied in the investigations for the tools-focused class. Yet agreement in interpretation of the goals for the light intensity and penicillin labs was attained between students and instructor. The agreement was noted as the students recorded their goals in their lab reports and demonstrated their abilities to model the exponential data. The students' methods agreed with their goals to find equations which represented the data and the characteristics of the lab procedures. One possible reason for agreement include the students' familiarity with the class procedures and instructor. The light intensity and penicillin labs occurred at eight and twelve weeks of the semester, respectively. By this time, students were more familiar with the process of conducting experiments and writing the lab reports. The water flow investigation began at the end of the first week of the semester. Students were adjusting to the class procedures.

A second possible reason for agreement on the interpretation of the goals for the light intensity and penicillin labs was the limitations placed by laboratory equipment. In these two investigations, the equipment allowed for little variation in the kinds of data gathered. In the light intensity lab, the procedures and equipment used to gather depth and intensity readings were demonstrated. In the penicillin lab, descriptions of the conductivity probes and their use to collect conductivity readings at each step were given. During the water flow investigation, students used the tubes, water, and stop watches to gather data relating height and time variables. Different groups used the same equipment and gathered different types of data. Some gathered the time to drain entire heights of water. Others collected the height paired with the time to drain a small amount of water, such as five centimeters, at that height. Some paired time values which increased in equal increments with the heights of the draining water as the times were announced. The variations in the kinds of data collected in the water flow investigation likely contributed to the different interpretations of the questions under investigation and the kinds of analysis to include in the lab reports.

In general, agreement of the intents and tasks of the investigation whether open-ended or focused needs to be attained in inquiry environments. When the agreement is promoted due to the limited variability of the kinds of data allowed by the equipment, scaffolding by the instructor is increased thereby limiting the openness of inquiry. When the agreement of the question or task is not achieved, misalignment of instructor and student goals promotes confusion and misunderstanding of application of the modeling methods. To encourage agreement of the questions under investigation and of the tasks while maintaining an open-inquiry environment, strategies may be applied to help students to focus on particular issues in their investigations. Strategies similar to those given by Tanner and Jones (1994), in which students anticipate particular questions when presenting their analysis and results, advance both the goals of inquiry and the curricular goals set by the instructor.

Running with Students' Ideas

Arcavi and Schoenfeld (1992) encouraged instructors to pursue the approach of "running with students' ideas" in mathematics problem solving sessions. Interpreted as a component of inquiry, the graphs of the micro source starts of ideas (Figures 5.7, 5.8, 5.9, 5.13, 5.14, & 5.15) illustrate a degree to which students' ideas were pursued. In both classes instructors ran with students' ideas in varying manners across the laboratories. In the modeling-focused class, students' ideas consistently and frequently started discussion in the science, link, and mathematics contexts and across the various phases for the first halves of the investigations. (See Figures 5.7, 5.8, & 5.9.) During the second halves of the investigations, the instructor's ideas more frequently launched discussion in a phase or context. In the multiple cycles of inquiry, the instructor may have been willing to run with students' ideas, but the class understood that new modeling methods were being presented, so students' likely felt less freedom to offer ideas.

In the tools-focused class, the same measure shows mixed results. During the first half of the water flow investigation (Figure 5.13), students' ideas often started discussions in each of the contexts. For the second half, the instructor's ideas regularly started the discussions. The graph suggests, like the modeling-focused class, the class ran with the students' ideas during the first half of the investigation. In the light intensity investigation (Figure 5.14), the graph suggests that the instructor's ideas initiated the discussion across most of the lab suggesting that less emphasis was placed on running with students' ideas. During the penicillin laboratory, students' ideas often initiated discussion for most of the laboratory including the discussion concerning the links in the second half of the laboratory.

In both classes, results were mixed when considering the approach to run with students' ideas. In most investigations, the class frequently ran with students' ideas during the initial stages of the investigations with greater instructor initiation during the latter parts. When the first portion of the laboratory was specifically set aside for students to develop and run with their ideas, as in the water flow and light intensity labs for the modeling-focused class and in the water flow and penicillin labs in the tools-focused class, students remained focused on the task while running with their ideas. Implementing techniques into second and third cycles of inquiry in which students can remain focused on the mathematics while resolving issues of their ideas with others' ideas promotes the richness of mathematics to evolve as in Roth's (1995) physics classroom. Additional study should examine methods to accomplish continual running with students' ideas while yet accomplishing curricular objectives.

Context

The role of context was an influencing factor in both classroom environments. In both classrooms, the role of the science and link contexts was associated with the time constraints placed on the laboratories. During the water flow and light intensity investigations in the tools-focused class (Figures 5.10 & 5.11) and during the penicillin lab (Figure 5.6) in the modeling-focused class, the graphs of the contexts over time indicated a sequence of science to link to mathematics discussion. In other words, when time spent on the laboratory seemed to be limited, the inquiry process seemed to be limited. Once the mathematics context was attained in discussion, little science or links between the mathematics and science existed. During the light intensity lab in the tools-focused class, the class did return to a period of link discussion following a period of sole mathematics discussion (Figure 5.11). During the penicillin lab in the tools-focused class and during the water flow and light intensity labs in the modeling-focused class, discussion of the links between science and mathematics tended to occur regularly and consistently throughout the laboratory. During these labs, students were given more opportunity to pursue their own methods as the instructors seemed less stressed for time. While modeling the data from these labs, students relied on the context to inform the modeling process.

Overall, this study adds to the body of literature concerning students' difficulty in relating mathematics with science or real world contexts (Rasmussen & King, 2000; White & Mitchelmore, 1996; Strickland, 1999). The role of the context influenced students' linking of the science and mathematics. When the context was used as a tool to build the mathematics ideas, students less frequently relied on the context to give direction to their mathematical procedures and less frequently reflected back on their procedures in light of the science context. When the context was incorporated to direct the mathematics, students more often used symbols which reflected the context, and interpreted some of the mathematics in light of the science context.

Evaluation of the transcripts from the student interviews suggest the kinds of contextual qualities which may be preferred in an inquiry environment incorporating mathematical modeling. The graph of the phase with micro source of ideas for Jewel's interview (Figure 5.27) indicated that she periodically reflected on the mathematics she performed. The level of reflection lacked depth as Jewel inconsistently linked the science and mathematics contexts, and Jewel was unsuccessful in solving the problem (See Figure 5.29.). Examination of Brett's graphs illustrate no periods of reflection until the end of the interview (Figure 5.26). However, regular and consistent links were made between the mathematics and science contexts as portrayed in Figure 5.28. Brett successfully solved the modeling task. Based on the student interviews, when mathematically modeling data, regular and consistent periods of relating the mathematics and science contexts may be of greater benefit than separate and specific periods of reflection.

One question to pursue in future studies is "Can the collegiate mathematics classroom attain regular and consistent links between the science and mathematics contexts with the students as the micro sources of the link ideas?" If the answer is yes, then the natural and more important follow-up question is "Are the students in a 'linking,' inquiry environment successful in the mathematical modeling of data?"

The graph of the context with micro sources of ideas for the light intensity lab in the modeling-focused class (Figure 5.5) and the graph for the penicillin lab in the tools-focused class (Figure 5.12) reflected classrooms which were close to achieving regular and consistent links with the students as the sources of ideas. Figure 5.8 with the micro source starts and contexts for the light intensity lab in the modeling-focused class suggests that the students' link ideas followed the instructor's link ideas in the latter third of the investigation. The instructor's link ideas prompted the students' link ideas, calling into question the accuracy in comparing the graphs to Figure 5.28. Figure 5.15 with the micro source starts and contexts for the penicillin lab in the tools-focused class suggests that the links in the latter half of the investigation were initiated by the students with some instructor initiation. Though the graph indicates student initiation, comparing student links in this penicillin lab with Brett's links in the student interview (Figure 5.28) may also be questionable as many students were unable to create a model for the lake pollution problem while Brett was successful in finding a model for the temperature data.

Class observations, Figures 5.5, 5.8, 5.12, 5.15, and 5.28, and results of students' lab reports highlight the need for mathematics classrooms to pursue environments incorporating the regular and consistent occurrence of students' link ideas, students' link ideas which launch discussion, and many or all students interacting and generating the link ideas. Consistent linking of contexts influences the appropriate use of symbols in modeling data and solving problems, in general. Consistent link is not enough as evidenced by Figure 5.15 and the need for more success in students' modeling abilities. Additional research is needed to identify codes and graphs which suggest when success in modeling and in relating the mathematics and science contexts was achieved.

Implications for Future Research

One intention of this study was to produce means to quantitatively represent inquiry in the mathematics classroom to illustrate when inquiry does and does not occur. The measures presented in this study portray images of the inquiry environments. Some existing characteristics of the environment remain blurred. Future studies intending to examine inquiry in the mathematics classroom need to focus on those traits not fully clarified in this study to produce a clearer reflection of the inquiry environment.

Attributes Portrayed with Clarity

The codes and graphs produced in this study were successful in isolating and reflecting the phase, context, micro sources of ideas, the starts of those ideas, and the interactions between instructors and students observed in the two classes. Future studies intending to represent the degree of inquiry in the classroom in light of these components would do well to continue in their use.

Phases of Inquiry

For both classes, coding for the phase of inquiry proved useful in illustrating the structural components of the class. Graphs of the phases over time revealed that multiple cycles of inquiry were attained with greater emphasis placed on the prediction and experiment phases during the first cycle and on the analysis phase during the additional cycle(s). In both classes, the graphs demonstrated that periods of reflection were infrequent unless structured into the course. The graphs portrayed support of the instructors' goals to pursue inquiry with additional cycles focused more on the analytical and mathematical components of the investigations.

When the phases were graphed with the micro sources of ideas, students' influences were identified in the inquiry process. The codes and graphs depicted the students' high engagement during the first half of the investigation as their ideas frequented the prediction, experiment, and analysis phases. The instructors' ideas were more frequent during the second halves of the investigations. The micro sources of ideas paired with the phases also indicated how the reflection phase was frequented more by instructors' ideas excluding the periods in which reflection was structured into the process.

Context

Coding to identify the context of the discussion proved useful in classifying the emphases of mathematics, science, and the links in class discussion. The classifications distinguished differences in the classes regarding the regularity and consistency in recognizing the relationships between the science and mathematics. The distinctions between the two classes regarding the role of context and the times at which the emphases occurred aided in the interpretation of students' in-class and written responses surrounding symbolic notation. The role of context gave indication of a form of reflection or metacognition which occurred separate from the reflection phase as connections of how the science informed the mathematical procedures were made.

Associating micro sources of ideas with the context informed whose ideas frequented each context. The graphs illustrated high engagement of students in the class when discussing the various contexts in both classes. Implementing the starts of the micro sources of ideas suggested whose ideas launched discussion in each of the context and aided in understanding when the instructors potentially prompted discussion within a given context.

In this study the graphs of the interactions between context and micro source and between phase and micro source illustrated the emphases placed on context, the participants which most influenced the various phases or contexts, and the structural components of the classes. Incorporating each of phase, context, and micro source of idea gives additional illustration of the same influences described in Chapters 4 and 5 with portrayal of the context pursued within each phase. Figures 6.1 and 6.2 give the graphs of the three interactions for the light intensity investigations for both the tools-focused class and the modeling-focused class. These figures portray the same types of information as given in Figures 5.11 and 5.22 for the tools-focused class and Figures 5.5 and 5.17 for the modeling-focused class with the additional component of the context across the phases.

Figures 6.1 and 6.2 demonstrate which contextual discussion dominated the different phases of inquiry with the sources of ideas within the contexts and phases. To be expected, the science and link contexts dominate the prediction and experiment phases while the mathematics and links dominate the analysis and reflection phases. Also evident is the emphasis placed on the mathematics and links for each phase during the second cycle of

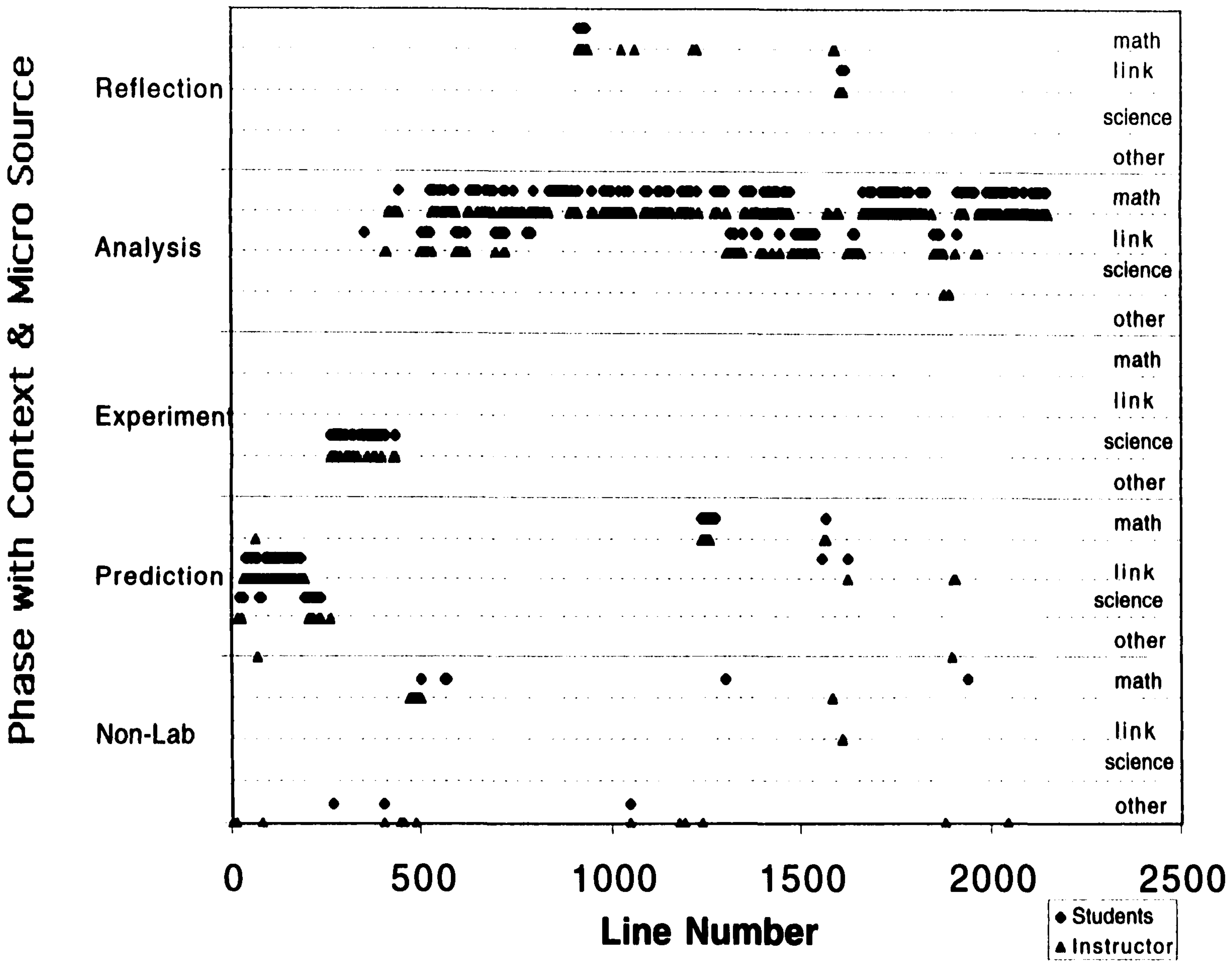

Figure 6.1. Phase with context and micro sources of ideas for the light intensity lab in the tools-focused class

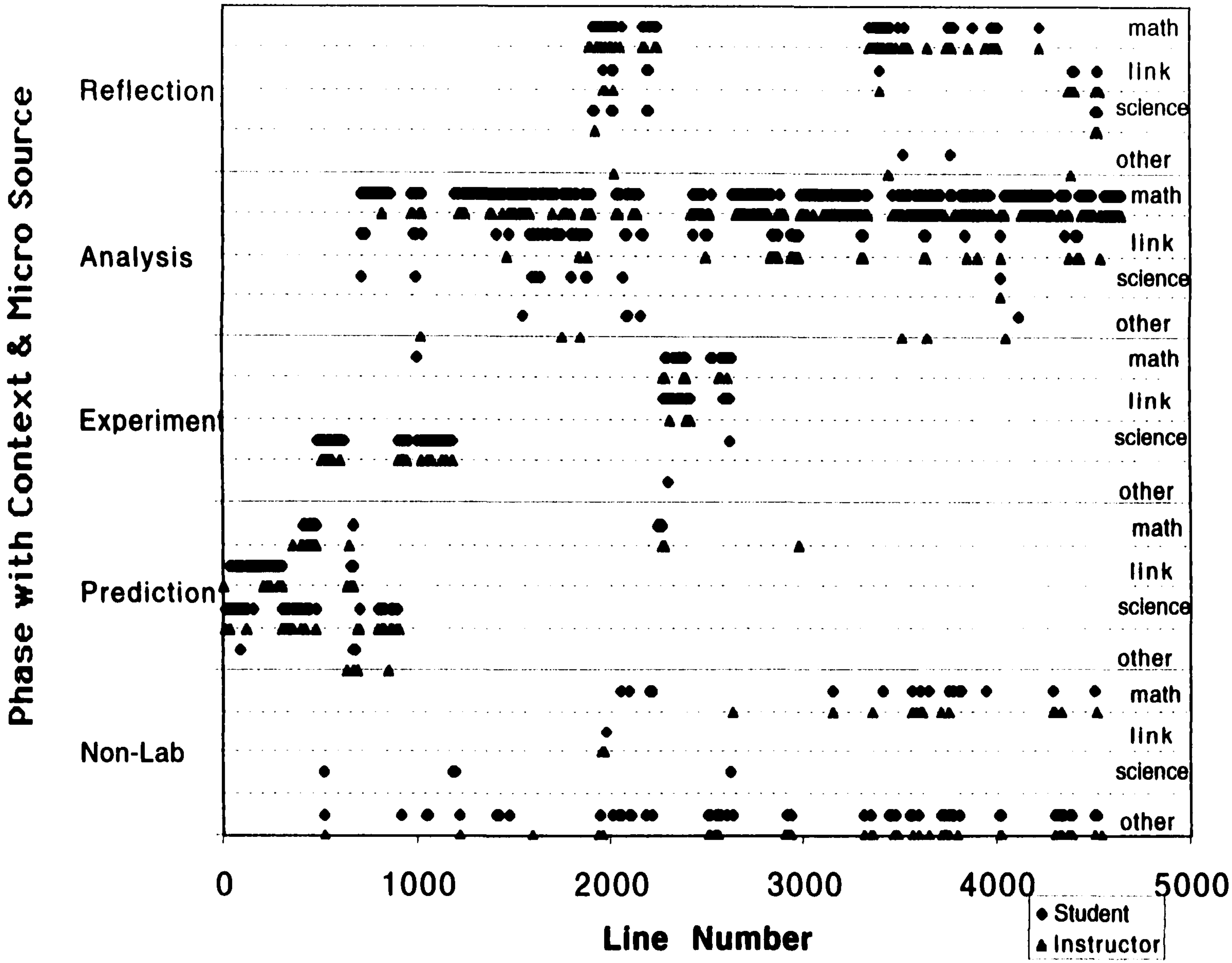

Figure 6.2. Phase with context and micro sources of ideas for the light intensity lab in the modeling-focused class

inquiry. Relatively little link discussion occurred in the reflection phases of either class, justifying the use of link as a separate portrayal of another type of reflection which occurred in the classes.

Overall, the codes and graphs of phase, context, and micro sources of ideas clarify the structural and procedural emphases of the inquiry environment. Coding for these three components in future studies is recommended.

Attributes Not Portrayed with Clarity

The codes and resulting graphs in this study did not fully capture some traits of the classroom. Particularly noticeable were those instances in which the transcripts and classroom observations suggested cognitive conflict, disagreement in interpretations, or general confusion in the classroom. Periods of dissonance and agreement or resonance of other periods observed in the classroom, were not often distinguishable on the graphs produced using the existing scales. Particular examples given in previous chapters were items in which new language or symbols were introduced before students were structurally prepared to use the language or symbols. Additional examples were the agreement in interpretations of questions under investigation or of mathematical arguments given for the class.

The scales for micro sources of ideas and macro sources of ideas were intended to capture evidence suggesting the compatibility of resonating conceptual schemas. Observations obtained by the scale for the micro source of idea were those identifying the origin of the idea on a fairly narrow level of conversation. Observations detected by the macro source of idea were broad as the instructors were frequently the originators of most of the overriding ideas as the instructors often set in motion the context of the investigations and mathematical methods. In the current forms, the observations captured by the line-by-line coding for the sources of idea lacked clarity in distinguishing the dissonance or resonance which occurred in the classroom. Factors in addition to the sources of idea should be considered when identifying resonance and dissonance.

Possible factors to consider when attempting to identify resonance or dissonance in participants' conceptions being discussed include language source, symbol source, and interpretation factor. These factors are discussed with possible methods to implement them into future studies.

Language Use

Coding for language use and shared understanding on an interaction level would identify the type of terminology being used or referred. Classification for whether students and instructor agreed in the use and understanding of the terms could suggest agreement or disagreement in the resonance of compatible conceptual schemas. In the present study, identification of whose language and the appropriateness of the language was not attributed. Instances during which instructor or students' language was being used were described in Chapter 4. Conflict caused by the use of structural, formal language such as the term "function" early in the class was not evident on the graphs. In the modeling-focused class, the class' use of student terminology or more action-oriented language was not captured by the graphs. Coding for the types of language used and whether shared understanding was achieved could filter through those occurrences and suggest a degree to which agreement in language use is pursued.

Developing a new code for the use of language could isolate those instances where disagreement or agreement occurred in language use. Building on Sfard's (1992) research on structural and operational instruction, those times during which structural terms or descriptions are incorporated into class discussion could be contrasted with those instances in which the operational terminology is used in the course. Associated with who was speaking, student or instructor, a joint coding could indicate who introduces new terminology and would identify those instances in which the instructor used structural instruction before students had fully developed operationally in their understanding of the concept. Building also on research by Brendefur (1999) and Arcavi, Kessel, Meira, and Smith (1998), identification of the creation and maintenance of collective understandings while including common terminology could suggest when resonance of conceptual schemas of class participants was compatible.

Symbol Use

Much related to language use is symbol use. In a code, identification of new notation, proper use of notation, or inappropriate uses of notation by students could indicate where conflict or agreement was attained in the classroom when mathematical methods are discussed. The observations captured by the current codes and graphs in this study did not distinguish when procedures were applied correctly or incorrectly, nor did they suggest when questions arose of particular notation. The scale for start of micro source of idea potentially suggested whose ideas prompted discussion for new notation or ideas. However, the start of micro source of idea did not identify those times students used the notation properly or improperly. Coding for symbolic use on a person-to-person interaction level could identify those instances when new symbols were introduced, who developed and/or introduced them, and the appropriateness in the use of the symbols.

Slightly different from the language use, distinguishing between a structural or operational use of symbols would be difficult. An additional scale to code for the agreement or disagreement in the use of the symbol with concept could prove useful in filtering the instances in which symbol use was or was not appropriate. Building on Lithner's (2000) study of established experiences and plausible reasoning, a person's symbol use could be coded to identify when symbol use matched the new context or problem. For example, if a student is given a new context but relied on established experiences or previous examples to model the data or solve the problem, the notational use would be assigned a code to represent old notation and old application. The student did not fully take into account new notation to represent the new context nor were decisions in manipulating the symbols made based on the new context. To be attempted in future studies, identification of the uses of notation could prove useful in distinguishing differences in class environments.

Interpretation in Discussion

The class observations revealed occasions in both classes where agreement or disagreement in interpretations of the questions under investigation or the mathematical arguments used in addressing the questions transpired. Particular instances in the tools-focused class occurred when agreement existed in the interpretation of the goal question(s) within groups as discussed in class and on lab reports for the water flow investigation. Disagreement in interpretation of the overriding questions occurred during the water flow investigation across groups or between particular groups and the instructor. In the modeling-focused class, agreement and disagreement transpired within groups, across groups, and between students and instructor. When analyzing the light intensity data some members examined the quality of fit based on the graph of the function with the actual data points.

Disagreement arose as another reasoned that the function's end behavior compared with potential additional data readings suggested a poor fit. Codes are needed to identify when the agreements and disagreements occur.

The instances of agreement or disagreement in interpretation were not captured in the graphs in this study. Coding for the existence or nonexistence in the agreement of interpretation at both macro and micro level may filter those occurrences to be evident on graphs. A macro level of agreement in interpretation would suggest agreement across groups or between students and instructor. A micro level of agreement in interpretation would suggest agreement within groups. Accounting for Yackel's (1995) identification of causes of interaction break-down could be attributed in coding scales to distinguish interpretation differences.

Filtering the data through additional codes of language use, symbol use, and interpretation at a micro and macro level may help to graphically indicate the occurrences when compatible conceptual structures are resonating or are not resonating between participants. Coding for the structural and operational language use would suggest those times when students were able and willing to use the structural language or when they resorted to operational language though the instructor encouraged structural language. Coding for the use of notation would help to identify instances when symbols were developed and resulting interactions surrounding the context and phases of inquiry. In addition, occurrences could be tagged to suggest students' use of established experiences or plausible reasoning as described by Lithner (2000). Coding for interpretation point to the instances when instructors and students had different goals for a task and suggest where the interaction broke down. Each of language use, symbol use, and interpretation are characteristics to tag in transcripts in future studies to produce a clearer graphical image of the environment and to point to item which advance and hinder the inquiry process.

Certain items were coded according to the transcripts but were not included in the graphs to represent the data. The reason some items were not incorporated into the graphs included the lack of distinctive information provided by the scales. Other items provided too many differences to produce meaningful and interpretable graphs. Items not represented in the graphs of the study include which student was speaking; the distinction of whether statements were in the form of a question or comment; level of question or comment classified according to Bloom's Taxonomy; and level of mathematical understanding. Discussion of these items is given.

Which and How Many Students Engage in Discussion

Cobb et al. (1997) emphasized the individual components of reflection. One may engage in reflective discourse supporting and enabling individual reflection on and reorganization of prior activity, but the actual engagement in discussion does not cause, determine, or generate the reflection or reorganization. While engagement does not guarantee reflection, Cobb et al. emphasized the initial need for students' engagement to encourage the reflection. Instructors should realize that though one or few students are engaged in discussion and reflection, no assumptions may be made about the reflection or concepts of the entire class of students based on what few students are contributing. The graphs and discussion presented in this study indicated that throughout the investigations in both classes some students were highly engaged and contributing their ideas. The graphs also suggest when students' engagement was highest, particularly in the first halves of the investigations. Caution must be taken to attribute success or failure of the class environment based on a few students' contributions or based on a few students' lab reports.

Coded but not represented in the graphs was identification of which students were engaged in discussion and reflection. The transcripts were coded for who was speaking, but to generate the graphs the codes were reduced to identify teacher or student. When including graphs with the thirty or more class members, the graphs were complicated and patterns relating the phases of inquiry, context, and sources of ideas were difficult to identity. Overall, examining individual students' contributions was not a goal of this study. Representing the classroom environment including interactions which occurred was a goal. Researchers wishing to track individual students' contributions and their effects on the classroom environment should consider incorporating a representation of which and how many students engaged in discussion.

An item to note when considering which students engaged in discussion concerns the use of the recording devices in the classroom. The voices clearest on the recordings were most often the loudest and closest voices to the microphone in the classroom. Any study implementing similar methods of data collection and attempting to represent which and how many students were involved in class interaction would need to keep in mind the limitations of the recording devices.

Bloom's Taxonomy

Comments and questions made in the class were coded based on Bloom's Taxonomy. The classifications were not represented in the graphs since most of the questions and comments were coded at knowledge or comprehension levels with no distinctive patterns evident. Carlsen (1991) reviewed studies on classroom questioning. He noted that most of the research on level of questioning and the effects on students' achievement is not meaningful apart from the context of the questions and the comments leading to the particular question. For example, a question asked at the beginning of a study of new content could be classified as a high level question. The same question asked the following class session could be considered as a lower level recall question. In this study, the context surrounding the questions and comments was attributed. In accounting for the context, no distinctive patterns were observed. Future studies may wish to examine the component of the level of question and comment more intently with possibly a different set of codes.

Mathematical Level of Understanding

The class transcripts were coded for mathematical levels of understanding. Much like the codes for level of question or comment, the codes for the mathematical level of understanding produced minimal distinctions. Almost all classifications were made at the situational level as most class discussions and interactions focused on how to act in given problem situations. Future use of this scale is not recommended. Coding for language and symbol use as well as understanding would likely prove more useful to identify distinctions within and between the classes.

In general, items which were coded but not incorporated into the graphs provided too little distinctive information or provided too much distinctive information. Future studies emphasizing levels of questions and comments according to Bloom's Taxonomy or mathematical levels of understanding would need to further filter the scales to produce distinctive information. Studies examining inquiry in light of individual students' contributions and reflections would need to examine means to portray the level of individual students' engagement while maintaining the presentation of other characteristics of the inquiry process.

Summary

In summary, this qualitative study sought to examine and represent the implementation of inquiry methods into a collegiate mathematics course. When coupled with exploration and instruction of mathematical modeling, the following results were evident with implications for future pursuits of inquiry environments and future studies of inquiry.

1. Coding for phase of inquiry, context, and micro sources of ideas generated pictorial representations of the structure of the classes, emphases placed on the different phases of inquiry, the importance given to contexts, and the frequency with which students or instructors' ideas dominated discussion.

2. Graphs of the codes involving phase of inquiry, context, and micro sources of ideas did not capture instances in which students' and instructor's ideas were or were not resonating with compatibility. Future studies seeking to distinguish the resonance or dissonance of compatible conceptual schemas of class participants may find coding for language use, symbol use, and interpretation components useful.

3. An instructor's goals and tasks influence the amount of time spent on mathematics, science, and the links between the mathematics and science. In particular, when the coverage of specific mathematical topics was a goal, a sequence of science to links to mathematics discussion was evident contextually. When there was less pressure to cover specific topics, more focus was placed on inquiry and consistent and frequent linking of the mathematics and science contexts.

4. Students' regular and consistent connections made between the context and mathematics promoted students' mathematical reasoning and appropriate use of symbols in mathematical modeling and other mathematical procedures.

5. When the prediction phase of inquiry incorporated negotiation of task and the goal(s) of the investigation were clear to students and instructor, agreement of question and task purpose was promoted and better maintained throughout the investigation and instruction.

6. When instruction of new mathematical modeling methods was given, students needed to be more frequently prompted to connect the mathematics and science components.

APPENDIX A

INSTRUCTOR INTERVIEW QUESTIONS AND PROCEDURES

Questions prior to the Water Flow Laboratory:

1. What are some of your goals for this laboratory?

2. Describe how you envision the lab will unfold from the introduction of the lab to the completion of the lab reports.

3. What are some of the questions you plan to ask? What do you anticipate students' responses to be?

Interview Questions after Day 5 of the Water Flow Laboratory:

1. What observations did you make as students collected data? as students graphed their data and explained their graphs to you and to the class?

2. Where will you go from here?

3. In the first interview you mentioned wanting to set the stage for the rest of the class in terms of having a sense of the laboratories and what they are doing and then analyzing the data. Given the classes' experiences from this lab and the stride and height lab, how will you begin the next lab? What types of things will you continue to do and what will you do differently?

Questions prior to the Light Intensity Laboratory:

1. What are some of your goals for this light intensity laboratory?

2. Describe how you think this light intensity lab will unfold from the introduction to the lab reports?

3. What are some questions you will ask and what do you anticipate students' responses to be?

4. One of my goals in gathering data in Math 181 is to document students' growth in the inquiry process. What are some ways that you feel growth might be evident from the water flow lab to this light lab? [or from the beginning of the class to this stage?]

Questions after the Light Intensity Laboratory:

1. What observations have you made as students made hypotheses, collected data, graphed their data, and modeled their data in this light intensity lab?

2. What are your perceptions of how students have grown from the first of the semester to this midterm? What observations made during the laboratory indicate growth or suggest that more growth is needed?

3. For those areas that you mentioned where more growth is needed, what types of things do we need to be doing in Math 181 to achieve that growth?

4. In the previous interview you mentioned having 2 main goals for this lab:

   have a good context and use the context to develop and talk about difference and generating equations and examine data a variety of ways and see if some insights come from it or help build a model.

   Do you feel these goals were met? Why or why not?

5. Where will you go from here?

Questions prior to the Penicillin, One-Compartment Model Laboratory:

1. What are some of your goals for this One-Compartment Laboratory?

2. Describe how you think this one-compartment lab will unfold from the introduction of the lab reports?

3. What are some questions you will ask and what do you anticipate students' responses to be?

4. What are some ways that you feel growth might be evident from the beginning of the semester to this one-compartment lab?

5. What would you like me to watch for during the laboratory and discussion?

Questions after the Penicillin, One-Compartment Model Laboratory:

1. In the penicillin lab reports that students wrote, did they interpret the mathematical equations as desired? Did they detail the relationship between what's happening mathematically and experimentally? Is there a particular example that sticks out in your mind which illustrates this?

2. One of my goals in gathering data during this semester is to observe and document the growth that students demonstrate in the inquiry process after a semester of Math 181.

   What are your perceptions of how students grew throughout the semester of Math 181 and what observations made during this laboratory indicate growth or suggest that more growth is needed?

APPENDIX B

STUDENT INTERVIEW QUESTIONS AND PROCEDURES

Student Interview

Have you ever noticed how quickly a cup of hot liquid (like coffee, tea, or cocoa) cools over time?

1. Using methods similar to what we've done in class, what would you do to try to understand this phenomena.

2. Here is some data I have from a cup of hot water cooling over time. Temperature readings were taken every 6 minutes. Show me what you would do with this data.

Time (min)	Temp (°C)
0	59.50
6	45.36
12	37.41
18	32.51
24	29.36
30	27.21
36	25.81
42	24.74
48	24.09
54	23.66

3. Now I have a few additional questions I would like to ask you.

   For this (*Time*, *Temperature*) graph, give a sketch of the rate of change vs. time graph. Explain why it has the shape that it does.

   Explain the difference between an average rate of change and an instantaneous rate of change.

APPENDIX C

SAMPLE TRANSCRIPTS AND CODES

		Micro	Macro	Bloom	Speaker	Inquiry Phase	Context	Math Level
Cale	Wouldn't it let some light pass all the way through?	6	5	2	5	P	S	S
Brett	Yeah. Because -	6	5	1	4	P	S	S
Cale	It would absorb certain wavelengths -	5	5	2	4	P	S	S
R	Okay it sounds like another hypothesis.	6	5	1	1	P	S	S
Cale	You couldn't actually measure just because you're never going to have the water like that.	5	5	3	4	P	S	S
Brett	But when you chemically you can make it, but we're not going to do that. And so I'm just, what I was thinking-	5	5	1	4	P	S	S
R	That's great that you're coming up with a plan. Is there - do you have a plan of action once you have data?	1	5	1	1	P	S	S
		2	2	3	2	P	L	S
Brett	What do you mean?	3	2	1	5	P	L	S
Cale	For Thursday?	3	2	1	5	P	L	S
R	Like what are you going to graph, What are you going to do with the data?	2	2	3	2	P	L	S
Cale	We know what we're going to do with the experiment.	5	2	1	4	P	S	S
R	Okay. Do you know what you are going to do with the data once you have data? What graphs are you going to look at?	2	2	3	2	P	L	S
Cale	Probably going to look at depth.	3	2	1	4	P	L	S
Brett	Depth vs. intensity.	5	2	1	4	P	L	S
Cale	Yeah, we could look at depth vs. intensity.	6	2	1	4	P	L	S

APPENDIX D

LAKE POLLUTION PROBLEM

This problem was presented to the tools-focused students with the intention to help develop methods of modeling the penicillin "peak" data.

A pristine lake of area 2 km^2 and average depth of 10 meters has a river flowing through it at a rate of 10,000 m^3 per day. A factory is built beside the river and begins releasing 1000 kg of chemical waste into the lake per day.

1. Write a mathematical model that describes the change in the amount of chemical waste in the lake each day after the factory begins production.

2. Write a difference equation that describes the amount of chemical waste in the lake t days after the factory begins production.

3. What will be the concentration of chemical waste in the lake after one year?

APPENDIX E

HUMAN SUBJECTS APPROVAL FORM

Information for Review of Research Involving Human Subjects

Iowa State University

(Please type and use the attached instructions for completing this form)

1. Title of Project The Nature and Development of Systematic Inquiry and Resonances in the College Mathematics Classroom

2. I agree to provide the proper surveillance of this project to insure that the rights and welfare of the human subjects are protected. I will report any adverse reactions to the committee. Additions to or changes in research procedures after the project has been approved will be submitted to the committee for review. I agree to request renewal of approval for any project continuing more than one year.

Heather A. Thompson — Typed name of principal investigator
10/25/99 — Date
Heather A. Thompson — Signature of principal investigator

Mathematics — Department
400 Carver Hall — Campus address

294-1752 — Phone number to report results

OCT 25 1999

3. Signatures of other investigators: Brian Keller — Date: 10/25/99 — Relationship to principal investigator: Major Professor

4. Principal investigator(s) (check all that apply)
☐ Faculty ☐ Staff ☒ Graduate student ☐ Undergraduate student

5. Project (check all that apply)
☐ Research ☒ Thesis or dissertation ☐ Class project ☐ Independent Study (490, 590, Honors project)

6. Number of subjects (complete all that apply)

adults, non-students: 3 # minors under 14: ____ # minors 14 - 17: ____

ISU students: 98 other (explain): ____

7. Brief description of proposed research involving human subjects: (See instructions, item 7. Use an additional page if needed.)

(A) Three classes consisting of both sections of Math 181 and one section of Math 182 in Spring 2000 will be observed to document the nature and development of systematic inquiry as an instructional technique in mathematics, what concepts resonate in students during inquiry, and the effects on students' understanding of mathematical concepts. This study will report the details of the inquiry process, what's resonating in students and instructors suggested by questions, comments, and tasks performed, and the effects on students' conceptual understanding evident by students' performance on laboratory reports, written responses to questions and student interviews. To gather data, students and instructors of both sections of Math 181, Spring 2000 will be observed during three laboratory experiments and analysis, each lasting 1-2 weeks in length. One section of Math 181 will be taught by the researcher, while the other will be taught by an instructor experienced in teaching Math 181. Students and the instructor in one section of Math 182, Spring 2000 taught by the researcher's major professor will be observed during one laboratory experiment. Class sessions pertaining to the laboratory exercises will be video taped. (continued on fourth page)

(Please do not send research, thesis, or dissertation proposals.)

8. Informed Consent: ☒ Signed informed consent will be obtained. (Attach a copy of your form.)
☐ Modified informed consent will be obtained. (See instructions, item 8.)
☐ Not applicable to this project.

9. Confidentiality of Data: Describe below the methods you will use to ensure the confidentiality of data obtained. (See instructions, item 9.)

Actual names of participants in the study will not be reported in any written or oral form. Names of participants on written information gathered as data will be removed and replaced with pseudonyms. Video tapes of class sessions containing the names of participants will be transcribed using pseudonyms. The file containing the match of the participants' names with the pseudonyms will be kept in an encrypted file accessible only by password on the computer used primarily by the researcher and to be deleted at the completion of the study. Video tapes will be kept in locked cabinet in the researcher's office and accessible only to the researcher.

10. What risks or discomfort will be part of the study? Will subjects in the research be placed at risk or incur discomfort? Describe any risks to the subjects and precautions that will be taken to minimize them. (The concept of risk goes beyond physical risk and includes risks to subjects' dignity and self-respect as well as psychological or emotional risk. See instructions, item 10.)

Data gathered will be based on observations, transcriptions from video taped class sessions and one-on-one interviews with the researcher, written samples of students' work, and instructors' journal entries. Since the observations and taping are conducted in the class with focus on no one student in particular, this should not cause discomfort. Collected written work is standard in the course which should lend no additional risks or discomfort than being enrolled in the course. Any discomfort felt by students during interviews would stem from discomfort in their level of understanding of the material and their ability to explain their methods. To minimize this, the researcher will ensure the students that their thoughts and comments are of interest and that they are not being evaluated. Students' responses on the interviews will not affect their grades for the course, so no risk is involved academically. No evaluation report will be made for participating instructors, so no pressure should be felt in their job performance.

11. **CHECK ALL** of the following that apply to your research:

☐ A. Medical clearance necessary before subjects can participate
☐ B. Administration of substances (foods, drugs, etc.) to subjects
☐ C. Physical exercise or conditioning for subjects
☐ D. Samples (blood, tissue, etc.) from subjects
☐ E. Administration of infectious agents or recombinant DNA
☐ F. Deception of subjects
☐ G. Subjects under 14 years of age and/or ☐ Subjects 14 - 17 years of age
☐ H. Subjects in institutions (nursing homes, prisons, etc.)
☐ I. Research must be approved by another institution or agency (Attach letters of approval)

If you checked any of the items in 11, please complete the following in the space below (include any attachments):

Items A–E Describe the procedures and note the proposed safety precautions.

Items D–E The principal investigator should send a copy of this form to Environmental Health and Safety, 118 Agronomy Lab for review.

Item F Describe how subjects will be deceived; justify the deception; indicate the debriefing procedure, including the timing and information to be presented to subjects.

Item G For subjects under the age of 14, indicate how informed consent will be obtained from parents or legally authorized representatives as well as from subjects.

Items H–I Specify the agency or institution that must approve the project. If subjects in any outside agency or institution are involved, approval must be obtained prior to beginning the research, and the letter of approval should be filed.

Last name of Principal Investigator Thompson

Checklist for Attachments and Time Schedule

The following are attached (please check):

12. ☒ Letter or written statement to subjects indicating clearly:
    a) the purpose of the research
    b) the use of any identifier codes (names, #'s), how they will be used, and when they will be removed (see item 17)
    c) an estimate of time needed for participation in the research
    d) if applicable, the location of the research activity
    e) how you will ensure confidentiality
    f) in a longitudinal study, when and how you will contact subjects later
    g) that participation is voluntary; nonparticipation will not affect evaluations of the subject

13. ☒ Signed consent form (if applicable)

14. ☐ Letter of approval for research from cooperating organizations or institutions (if applicable)

15. ☒ Data-gathering instruments

16. Anticipated dates for contact with subjects:

First contact	**Last contact**
01/10/2000	05/05/2000
Month/Day/Year	Month/Day/Year

17. If applicable: anticipated date that identifiers will be removed from completed survey instruments and/or audio or visual tapes will be erased:

12/31/2000
Month/Day/Year

18. Signature of Departmental Executive Officer	Date	Department or Administrative Unit
M. M	10/25/99	Mathematics

19. Decision of the University Human Subjects Review Committee:
☒ Project approved ☐ Project not approved ☐ No action required

Name of Human Subjects in Research Committee Chair	Date	Signature of Committee Chair
Patricia M. Keith	10/28/99	PMKeith

7. (continued) The video tapes will be transcribed and analyzed. Students' written work in the form of brief, written, and in-class responses to instructors' questions (see attached page for example questions), and students' written lab reports detailing the methods and mathematics applied in the laboratories will be collected (see attached page for lab report format). These written components of students' work are a normal part of the courses. Names on written work will be removed and replaced with a pseudonym. Instructors' written responses to the same questions answered by students, daily journals by the researcher and major professor, and video taped pre- and post-laboratory interviews with the second Math 181 instructor conducted by the researcher will be gathered. (See attached page for interview questions.) Six students chosen at random from the researcher's section of the course will complete one-on-one, video taped interviews with the researcher.

(B) The instructors in the study will be the researcher, an instructor experienced in teaching Math 181, and the researcher's major professor. Students in both sections of Math 181, Spring 2000 and in the researcher's major professor's section of Math 182 will be observed during the data collection periods. These students are life-sciences majors and complete the courses as a requirement for their majors. Students will be involved as they ask and respond to questions, communicate ideas, and practice mathematical processes in class. Six students from the researcher's section selected at random will be asked to participate in one 45-60 minute interview following the second laboratory. Interview questions will consist of mathematical questions. (See attached page.)

The only incentive offered to student participants in the study is the offer of an additional resource to help answer questions regarding the course material. Students' written, in-class responses to the instructors' questions and lab-reports will be used as a part (25%) of students' course grades, but these are standard components of students' grades which should not be affected by the nature of the study. Participating instructors will not be given any incentive to participate in the study.

Dear Math 181 or 182 student:

In Mathematics 181 and 182, Calculus and Differential Equations for the Life Sciences, an inquiry approach is often pursued in the laboratories as students make hypotheses, collect data, analyze the data, generalize the mathematics, and reflect on the process. Throughout the process different ideas and concepts are discussed. I am conducting a research study to document the inquiry process in the college mathematics classroom and what ideas and concepts are generated and discussed. To create an instrument to measure the degree to which inquiry occurs and the ideas or concepts which resonate, I will collect data during laboratories throughout the semester.

Throughout the laboratories I will video tape the class sessions. The tapes will be transcribed and coded using various scales regarding the inquiry process. When your voice is recorded on tape, a pseudonym will be used in place of your name. This pseudonym will be attached to any written work submitted for use as data. I will be the only one who knows which pseudonym goes with which student. When the research study is complete, the tapes will be erased. This is expected to occur by December 31, 2000.

In addition to video taped class sessions and the use of written course work as part of the data, six students will be selected at random to complete a one-on-one, video taped interview with me. The purpose of the interview is to discern your understanding of key mathematical concepts of the course. This information will be related to the inquiry process and mathematical ideas discussed during the laboratories. While completion of the interview is not intended to influence your final course grade, this will be an opportunity to ask questions on the course material.

Your participation in this research study is voluntary. Should you not consent to the use of your quotations and written work as being a part of this study, though your voice or image may be captured on tape and your work collected as part of the course, these items will not be included in any report of the data. Any report of the data will occur in my dissertation, at an educational research conference, or in an educational research journal.

Thank you for considering being a part of this research study. If you have any questions, please contact me.

Sincerely,

Heather Thompson
Department of Mathematics
489 Carver Hall
294-1752
hathomps@iastate.edu

The Nature and Development of Inquiry and Resonances in Math 181, 182
Participant Consent Form

You are invited to be in a research study to explore the implementation of the inquiry process in Math 181, 182 and measure its effects on students and instructor interactions. You were selected as a participant because you registered for Mathematics 181 or 182, Calculus and Differential Equations for the Life Sciences. Please read this form and ask any questions before agreeing to be in this study.

The purpose of this study is to explore how the inquiry process is implemented in Math 181 and 182 and the effects on what resonates in students. What is resonating is suggested by the questions asked, ideas communicated, and mathematics performed. Your instructor has allowed the researcher to be an observer in the class sessions and to video tape class sessions related to laboratories you will conduct.

In the course of the study:

1. Your name will not be reported in any written or oral form. Names on papers or lab reports will be removed and replaced with pseudonyms. Names recorded on the video tapes will be transcribed using pseudonyms.

2. The records of this study will be kept confidential. Only the researcher and her major professor will access the research records including the video tapes and the transcriptions of the video tapes.

3. There are no risks to participants in this study. No payment or reimbursement will be given to students. One benefit of this study includes the opportunity for you to ask the researcher questions about the course material since she is experienced in teaching mathematics.

4. Your decision to participate in this study will not affect your current or future status in Math 181, in Math 182, nor in any other class at Iowa State University. If you decide to participate, you are free to withdraw at any time without influencing your relationship with your instructor nor any other relationship at Iowa State University. Should you choose not to participate or withdraw from this study, your comments and actions recorded on video tape or by the researcher will not be included in any report of the data.

If you agree to participate in this study, please check (☒) the following boxes, and print and sign your name below:

- ☐ I agree that my questions and statements recorded on video tape or by the researcher may be transcribed for the study.
- ☐ I agree that the transcriptions may be used in a report of this research in the researcher's dissertation, at educational research conferences, or in educational research journals.
- ☐ I agree that my written work as a part of the course may be used in a report of this research in the researcher's dissertation, at educational research conferences, or in educational research journals.
- ☐ I agree that portions of the video tapes recording my comments and/or displaying my image may be used in a report of this research for educational purposes.

______________________________ ____________________
Signature Date

____
Print Name

The researcher conducting this study is Heather A. Thompson under the guidance of Dr. Brian A. Keller. For questions, please contact Heather in 489 Carver Hall, by phone at 294-1752, or by email hathomps@iastate.edu.

The Nature and Development of Inquiry and Resonances in Math 181
Interview Participant Consent Form

You were selected at random from the students in this section of Math 181 to participate in one 45-60 minute interview with your instructor. The interview will be scheduled to occur after the completion of the analysis of the light intensity data in class. This intent of this interview is to examine your conceptual understanding of major mathematical concepts of this course in an effort to explore how an inquiry approach in teaching accompanied by an emphasis on students' resonances affects students' understanding. Please read this form and ask any questions about the nature of the study before agreeing to participate.

In the course of the study:

1. Your name will not be reported in any written or oral form. Names on papers will be deleted and replaced with a pseudonym. Names recorded on the video tapes will be transcribed using a pseudonym.
2. The records of this study will be kept confidential. Only the researcher and her major professor will access the research records including the video tapes and the transcriptions of the video tapes.
3. There are no risks to participants in this study. No payment or reimbursement will be given to students. One benefit of completing the interview includes the opportunity for you to ask the researcher questions about the course material in a one-on-one setting.
4. Your decision to participate in an interview will not affect your current or future status in Math 181, in Math 182, nor in any other class at Iowa State University. If you decide to participate, you are free to withdraw from completing the interview at any time without influencing your relationship with your instructor nor any other relationship at Iowa State University.

If you agree to be a participant in this study, please check (☒) the following boxes, and print and sign your name:

- ☐ I agree that my interview may be recorded on video tape and may be transcribed for the study.
- ☐ I agree that the transcriptions may be used in a report of the research in the researcher's dissertation, at educational research conferences, or in educational research journals.
- ☐ I agree that my graphs, equations, and notes written during the interviews may be used in a report of the research in the researcher's dissertation, at educational research conferences, or in educational research journals.
- ☐ I agree that portions of the video taped interview may be used in a report of this research for educational purposes.

______________________________ ________________________

Signature Date

____

Print Name

The researcher conducting this study is Heather A. Thompson under the guidance of Dr. Brian A. Keller. For questions, please contact Heather in 489 Carver Hall, by phone at 294-1752, or by email hathomps@iastate.edu.

Dear Math 181 or 182 Instructor:

I am conducting a research study to document the inquiry process in the college mathematics classroom and what ideas and concepts are generated and discussed throughout the process. To create an instrument to measure the degree to which an inquiry environment occurs and the ideas or concepts which resonate, I will collect data during laboratories throughout the semester.

I would like your permission to observe and video tape your section of Math 181 or 182 during laboratories. While I observe, I will record student-student interactions and student-instructor interactions with an emphasis on the questions asked and ideas communicated. The video tapes will be transcribed and encoded. The codes will be examined and used to develop an instrument which quantifies the degree to which an inquiry environment is achieved and the nature of the concepts which resonate.

To help triangulate the data, your written responses to the same daily in-class questions asked of the students are requested. In addition, a daily journal kept during the laboratory or six one-on-one video taped interviews with me are requested. This data should address the goals of each class session, your plans to implement an inquiry process, and your methods to be aware of students' questions and ideas as well as your assessment of these goals and procedures.

To help maintain confidentiality, all video tape transcriptions and written work will contain a pseudonym in place of your name. Also, only the researcher and her major professor will handle the data. I anticipate that the research will be completed by December 31, 2000, and all video tapes will be erased at that time.

Your participation in this research study is voluntary. Should you not consent to the use of your quotations in this study, though your voice may be captured on tape, they will not be included in any report of the data including in my dissertation, at educational research conferences, or in educational research journals.

Thank you for your help in this study. If you have any questions or comments about the nature of this study, please contact me.

Sincerely,

Heather A. Thompson
Department of Mathematics
489 Carver Hall
294-1752
hathomps@iastate.edu

The Nature and Development of Inquiry and Resonances in Math 181, 182
Instructor 181 Consent Form

You are invited to be in a research study to explore the implementation of the inquiry process in Math 181, 182 and measure its effects on students and instructor interactions. You were selected as a participant because you are an instructor for Mathematics 181, Calculus and Differential Equations for the Life Sciences. Please read this form and ask any questions before agreeing to be in this study.

The purpose of this study is to explore how the inquiry process is implemented in Math 181 and 182 and the effects on what resonates in students evident by the questions asked, ideas communicated, and mathematics performed.

In the course of the study:

1. Your name will not be reported in any written or oral form. Names recorded on video tapes will be transcribed using a pseudonym.

2. The records of this study will be kept confidential. Only the researcher and her major professor will access the research records including the video tapes and the transcriptions of the tapes.

3. There are no risks to participants in this study. No payment or reimbursement will be given to students.

4. Your decision to participate in this study will not affect your current or future status at Iowa State University. If you decide to participate, you are free to withdraw at any time without influencing your relationships at Iowa State University. Should you choose not to participate or withdraw from this study, your comments and actions recorded on video tape or by the researcher will not be included in any report of the data.

If you agree to participate in this study, please check (☒) the following boxes, and print and sign your name below:

- ☐ I agree that the researcher may observe and video tape the class sessions related to the water flow laboratory, the light intensity laboratory, and the one-compartment wash-out laboratory.
- ☐ I agree to complete one-on-one, video taped interviews with the researcher before and after each laboratory to assess my goals for the laboratory, my intent for the role of inquiry in the laboratories, and my plan to consider what students are thinking and the questions students have.
- ☐ I agree that my questions and statements recorded on video tape or by the researcher may be transcribed for the study.
- ☐ I agree that the transcriptions may be used in a report of the research in the researcher's dissertation, at educational research conferences, or in educational research journals.
- ☐ I agree that my responses to the brief, written, in-class questions posed to students may be reported in the researcher's dissertation, at educational research conferences, or in educational research journals.

______________________________ ________________________
Signature Date

____
Print Name

The researcher conducting this study is Heather A. Thompson under the guidance of Dr. Brian A. Keller. For questions, please contact Heather in 489 Carver Hall, by phone at 294-1752, or by email hathomps@iastate.edu.

The Nature and Development of Inquiry and Resonances in Math 181, 182
Instructor 182 Consent Form

You are invited to be in a research study to explore the implementation of the inquiry process in Math 181, 182 and measure its effects on students and instructor interactions. You were selected as a participant because you are an instructor for Mathematics 182, Calculus and Differential Equations for the Life Sciences. Please read this form and ask any questions before agreeing to be in this study.

The purpose of this study is to explore how the inquiry process is implemented in Math 181 and 182 and the effects on students' resonances evident by the questions asked, ideas communicated, and mathematics performed.

In the course of the study:

1. Your name will not be reported in any written or oral form. Names recorded on the video tapes will be transcribed using a pseudonym.

2. The records of this study will be kept confidential. Only the researcher and her major professor will access the research records including the video tapes and the transcriptions of the video tapes.

3. There are no risks to participants in this study. No payment or reimbursement will be given to students.

4. Your decision to participate in this study will not affect your current or future status at Iowa State University. If you decide to participate, you are free to withdraw at any time without influencing your relationships at Iowa State University. Should you choose not to participate or withdraw from this study, your comments and actions recorded on video tape or by the researcher will not be included in any report of the data.

If you agree to participate in this study, please check (☒) the following boxes, and print and sign your name below:

- ☐ I agree that the researcher may observe and video tape the Math 182 class sessions related to the two-compartment wash-out laboratory.
- ☐ I agree to complete daily journal entries during the laboratory.
- ☐ I agree that my questions and statements recorded on video tape or by the researcher may be transcribed for the study.
- ☐ I agree that the transcriptions may be used in a report of the research in the researcher's dissertation, at educational research conferences, or in educational research journals.
- ☐ I agree that my responses to the brief, written, in-class questions posed to students may be reported in the researcher's dissertation, at educational research conferences, or in educational research journals.
- ☐ I agree that the portions of the video tapes containing my statements and/or image may be used in a report of the research for educational purposes.

______________________________ __________________________
Signature Date

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Print Name

The researcher conducting this study is Heather A. Thompson under the guidance of Dr. Brian A. Keller. For questions, please contact Heather in 489 Carver Hall, by phone at 294-1752, or by email hathomps@iastate.edu.

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