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INCOMPLETELY SPECIFIED RANDOM AND
MIXED MODELS

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by

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TABLE OF CONTENTS

	Page
I. INTRODUCTION	1
A. Incompletely Specified Random Models	1
B. Reduction of Mixed Models to Random Models	7
II. REVIEW OF LITERATURE	9
III. DERIVATION OF EXACT AND APPROXIMATE FORMULAS FOR POWER. COMPONENT OF VARIANCE MODEL	11
A. Mathematical Formulation of the Pooling Procedure	11
B. Integral Expressions for the Power	12
C. Exact Formulas	15
1. Series formulas	15
2. Recurrence formulas	26
D. Approximate Formulas for Large n_i	36
E. Theory of Reduction of Mixed Model to Random Model	39
F. Application of Derived Formulas	42
IV. DISCUSSION OF POWER AND SIZE CURVES AND COMPARISON OF TEST PROCEDURES	45
A. Type of Recommendations Attempted	45
B. Size Curves	46
C. Frequency of Pooling	50
D. Power Curves	51
V. ILLUSTRATION OF RECOMMENDED PROCEDURES WITH PRACTICAL EXAMPLES	57

	Page
A. Tests on Samples of Portland Cement	57
B. Porosity of Condenser Paper	59
VI. LITERATURE CITED	61
VII. ACKNOWLEDGMENTS	63
VIII. APPENDIX	64

I. INTRODUCTION

A. Incompletely Specified Random Models

Let us assume the component of variance model

$$x_{ijk} = \mu + a_i + b_{ij} + z_{ijk} \quad (1)$$

where

$$i = 1, 2, \dots, q; \quad j = 1, 2, \dots, r; \quad k = 1, 2, \dots, s;$$

$$a_i \text{ is } N(0, \sigma_a^2), \quad b_{ij} \text{ is } N(0, \sigma_b^2) \text{ and } z_{ijk} \text{ is } N(0, \sigma_z^2).$$

We wish to test a hypothesis concerning a_i . If $\sigma_b^2 \geq 0$, then

$$\left\{ \begin{array}{ll} x_{ijk} = \mu + a_i + b_{ij} + z_{ijk} & \text{for } \sigma_b^2 > 0, \\ x_{ijk} = \mu + a_i + z_{ijk} & \text{for } \sigma_b^2 = 0. \end{array} \right\} \quad (2)$$

In this case (1) is said to be an incompletely specified model. If,

$$\text{however, } \sigma_b^2 > 0, \quad x_{ijk} = \mu + a_i + b_{ij} + z_{ijk} \quad (3)$$

and (1) is completely specified. Similarly, if $\sigma_b^2 = 0$,

$$x_{ijk} = \mu + a_i + z_{ijk} \quad (4)$$

and again (1) is completely specified.

We wish to test the hypothesis $H_0: \sigma_a^2 = 0$ against the alternative $H_1: \sigma_a^2 > 0$. Now let us assume we have the completely specified model given by (3). Then $\sigma_b^2 > 0$ and we obtain the analysis of variance given in Table 1.

Table 1. Component of variance model with $\sigma_b^2 > 0$.
Analysis of variance

Source of variation	d. f.	Mean square	Exp. mean square
Between A	$n_3 = q-1$	V_3	$\sigma_z^2 + s\sigma_b^2 + rs\sigma_a^2$
Between B within A	$n_2 = q(r-1)$	V_2	$\sigma_z^2 + s\sigma_b^2$
Within B	$n_1 = qr(s-1)$	V_1	σ_z^2

Then it follows from the likelihood ratio principle that the appropriate test procedure is to calculate the test statistic

$$F_o = \frac{V_3}{V_2} \quad (5)$$

and to reject H_o if $F_o \geq F_\alpha(n_3, n_2)$, where α is the prescribed level of significance. This test is the never pool test.

Next let us assume the completely specified model given by (4). Now the expected mean squares of Table 1 no longer include the σ_b^2 component since $\sigma_b^2 = 0$. Application of the likelihood ratio test procedure to this model for the test of H_o gives us the test criterion

$$F_o = \frac{(n_1 + n_2)V_3}{n_1V_1 + n_2V_2}$$

and the rule to reject H_o if $F_o \geq F_\alpha(n_3, n_1 + n_2)$. This gives us the always pool test.

Finally, we assume $\sigma_b^2 \geq 0$, and hence are confronted with the incompletely specified model given by (2). Ordinarily this model (2) might be assumed when there exists some uncertainty as to whether $\sigma_b^2 = 0$ or $\sigma_b^2 > 0$. In such cases of incomplete specification attempts are often made to resolve the uncertainty by first testing the hypothesis that $\sigma_b^2 = 0$. The model finally chosen and hence the final test (test of H_0) depend upon the outcome of this original test. When the original and final tests are performed on the same set of data, the original test is referred to as a preliminary test of significance. In our example the preliminary test becomes the test of $H'_0: \sigma_b^2 = 0$ against $H'_1: \sigma_b^2 > 0$. Again a likelihood ratio test procedure is available for this preliminary test. The statistic $F_0 = V_2/V_1$ is calculated and H'_0 rejected at the level α_1 (usually different from α) if $F_0 \geq F_{\alpha_1}(n_2, n_1)$. If H'_0 is rejected the non-pooling test procedure indicated by (5) is used for the final test. If H'_0 is not rejected, the pooling procedure indicated by (6) is used for the final test. This entire testing procedure is called a sometimes pool procedure.

It should be noted that when the final test is carried out, the model is assumed to be completely specified, that is, to be either model (3) or model (4), according as the preliminary test is found to be significant or not significant, respectively.

The model (1) we have assumed is that of a hierarchal classification. Similar situations of incomplete specification and hence analogous test procedures arise for numerous other random models. For example, if we assume the model

$$x_{ijk} = \mu + a_i + b_j + (ab)_{ij} + z_{ijk} \quad (6)$$

where $i = 1, 2, \dots, r$; $j = 1, 2, \dots, s$; $k = 1, 2, \dots, t$; a_i is $N(0, \sigma_a^2)$, b_j is $N(0, \sigma_b^2)$, $(ab)_{ij}$ is $N(0, \sigma_{ab}^2)$ and z_{ijk} is $N(0, \sigma_z^2)$; we wish to test the hypothesis $H_0: \sigma_a^2 = 0$, against the alternative $H_1: \sigma_a^2 > 0$, but are uncertain as to whether $\sigma_{ab}^2 = 0$ or $\sigma_{ab}^2 > 0$; then we have a situation analogous to the one described above. Analogous situations of incompletely specified models also arise with the mixed model discussed in the following section.

Situations frequently arise in experimental design where the model is not completely specified. Furthermore, with the wider application of analysis of variance to operational research and to the study of routine data, analyses are often made of data which have not resulted from a designed experiment; and in these situations the model is often incompletely specified. In such cases of uncertainty, preliminary tests have often been used in the past as an aid to choosing an appropriate model.

Pauli (1948) describes the following operational experiment on the sources of variability in the determination of the protein content of wheat by different laboratories. Out of a large population of laboratories in Canada, three were selected at random and each of them asked to analyze 10 subsamples of the same sample of wheat, making five protein determinations on each of two days. Let x_{tij} denote the protein determination from the j -th sample analyzed on the i -th day at the t -th laboratory. Then it is assumed that the data are adequately described

by the model

$$x_{tij} = \mu + \ell_t + d_{ti} + z_{tij} \quad (7)$$

where the laboratory variables, ℓ_t , the day variables, d_{ti} , and the test variables, z_{tij} , are assumed to be random samples from the respective normal populations $N(0, \sigma_\ell^2)$, $N(0, \sigma_d^2)$ and $N(0, \sigma_z^2)$. The analysis of variance based on the above model is shown below.

Table 2. Component of variance model example.
Analysis of variance

Source of variation	d. f.	Mean square	Exp. mean square
Between laboratories	$n_3 = 2$	V_3	$\sigma_3^2 = \sigma_z^2 + 5\sigma_d^2 + 10\sigma_\ell^2$
Between days within laboratories	$n_2 = 3$	V_2	$\sigma_2^2 = \sigma_z^2 + 5\sigma_d^2$
Within days	$n_1 = 24$	V_1	$\sigma_1^2 = \sigma_z^2$

The main interest of the experiment lies in testing whether the laboratories introduce any variability additional to that encountered within each laboratory, that is, in testing the null hypothesis whether $\sigma_\ell^2 = 0$ against the alternative $\sigma_\ell^2 > 0$. It is well known (and indicated by the above table) that the appropriate F statistic to use is V_3/V_2 , unless $\sigma_d^2 = 0$, in which case we should pool V_2 and V_1 and use the statistic

$$\frac{(n_1 + n_2) V_3}{n_1 V_1 + n_2 V_2}.$$

In the present example the three mean squares V_1 , V_2 and V_3 , with which the pooling procedure is concerned, have arisen from what is commonly known as a hierarchical (or nested) analysis of variance with laboratories as primary units, days as secondary units and test results as the final observations. Moreover, we have assumed a component of variance model (Model II) consisting entirely of normal effect variates. We may summarize the consequences of such a model by three essential features:

(i) The three mean squares V_i are independently distributed as

$\chi^2_i \sigma_i^2 / n_i$ where χ^2_i is the (central) χ^2 statistic for n_i degrees of freedom.

(ii) The main purpose of the analysis is to test the null hypothesis

$\sigma_3^2 = \sigma_2^2$ against the alternative $\sigma_3^2 > \sigma_2^2$.

(iii) The error mean square V_2 has an expectation σ_2^2 which is greater than or equal to the expectation, σ_1^2 , of the doubtful error mean square V_1 , which may or may not be pooled.

It is clear that the above nested (or hierarchical) classification is not the only analysis of variance situation giving rise to the above conditions, (i), (ii) and (iii) which we take to define the Model II pooling procedure.

There are numerous other analysis of variance tables resulting in the same test situation. As an example we may quote the two way classification with both factors random and cell repetition. Here V_1 would play the part of the within cell mean square while V_2 would be represented by the residual in the two-way analysis.

B. Reduction of Mixed Models to Random Models

The preceding section has been devoted to random models only.

Another frequently occurring type of model is the mixed model in which one of the factors is fixed and the other factors are random, and the hypothesis of interest is concerned with the fixed factor. A typical example of an experiment giving rise to this type of model is a randomized block experiment in which k rations are each fed to a pen of m animals in each of n replicates. Then a suitable model for these data is given by

$$x_{tij} = \mu + a_t + b_i + d_{ti} + \overset{z}{\sigma}_{tij}, \quad (8)$$

where the replicate variates b_i , error variates d_{ti} and within pen error variates z_{tij} are assumed to be random samples from the respective normal populations $N(0, \sigma_b^2)$, $N(0, \sigma_d^2)$ and $N(0, \sigma_z^2)$, while the ration means a_t are fixed parameters. The analysis of variance based on this model is shown below.

Table 3. Mixed model example. Analysis of variance

Source of variation	d. f.	Mean square	Exp. mean square
Between replicates	$n-1$		
Between rations	$n_3 = k-1$	V_3	$\sigma_3^2 = \sigma_z^2 + m\sigma_d^2 + mn\theta(a)$
Reps x rations	$n_2 = (k-1)(n-1)$	V_2	$\sigma_2^2 = \sigma_z^2 + m\sigma_d^2$
Within pens	$n_1 = nk(m-1)$	V_1	$\sigma_1^2 = \sigma_z^2$

Here $\theta_{(a)} = \sum (a_t - \bar{a})^2 (k-1)$.

The main purpose of the experiment is to test the equality of the ration means, that is, to test the hypothesis $H_0: \theta_{(a)} = 0$ against the alternative $H_1: \theta_{(a)} > 0$. The present situation is therefore identical with that of the component of variance model except that the treatment sum of squares, $n_3 V_3$, is distributed as $\chi'^2 \sigma_2^2$, where χ'^2 is the non-central χ^2 statistic with n_3 degrees of freedom.* However, we shall show that recommendations based upon the results obtained for the random model can be made for the present model. The main reasons for this are as follows:

- (i) In the case of the null hypothesis of no treatment differences the mixed model becomes identical with the random model; hence all our results on size curves are directly applicable.
- (ii) In the case of the alternative hypothesis we can apply Patnaik's approximation for the non-central χ'^2 statistic, and, therefore, we can apply our power results to the present situation by making certain transformations.

* The appropriate non-centrality parameter will be given in Subsection E of Section III.

II. REVIEW OF LITERATURE

The earliest work involving preliminary tests of significance was that of Bancroft (1944), who investigated the bias, variance and mean square error of a variance estimator obtained after performing a preliminary test of the equality of two variances.

In the same paper Bancroft studied the bias in the estimator b_1 of the regression coefficient β_1 in the model

$$Y = \beta_1 x_1 + \beta_2 x_2 + e,$$

b_1 being dependent upon a preliminary test of significance of b_2 , made to decide whether or not to retain the variable x_2 .

Later Bancroft (1950) reported on biases in estimates of variance due to the omission of several independent variables in the multiple regression equation analysis.

Mosteller (1948) has examined the effect of using preliminary tests of significance as an aid to deciding when to pool two sample means, in estimating a population mean.

Kitagawa (1950) derived the distribution function and the moments for the estimator obtained by the rule of procedure studied by Bancroft in the variance estimation problem. He also derived the distribution and moments of a pooled estimator of a mean based on a preliminary test assuming unknown variance.

Bennett (1952) extended the studies of Mosteller and Kitagawa to situations where preliminary tests are performed for both homogeneity

of variances and equality of means prior to estimating the mean or testing hypotheses about the mean. He derived the distribution functions, the biases, and the mean square errors for various cases, depending upon what assumptions were made concerning the parameters of the associated normal distributions.

Paull (1948) studied the size and power for the incompletely specified component of variance model described in Section I. However, he was able to express the size and power in closed form only for the case $n_3 = 2$, so that all comparisons made by him are restricted to that value of n_3 .

A similar investigation was undertaken by Bechhofer (1951) for the incompletely specified linear hypothesis model.

The object of the present study is to provide the necessary extension of Paull's investigation to cover all of the important degrees of freedom combinations occurring in the analyses of variance under discussion. This extension was made possible by

- (i) the development of the power integrals as series formulas, for even values of the degrees of freedom n_1 , n_2 and n_3 ;
- (ii) the derivation of recurrence formulas for the power, for even values of n_1 , n_2 and n_3 ; and
- (iii) the development of approximate formulas valid for large degrees of freedom, for even values of n_1 , n_2 and n_3 .

III. DERIVATION OF EXACT AND APPROXIMATE FORMULAS FOR POWER. COMPONENT OF VARIANCE MODEL

A. Mathematical Formulation of the Pooling Procedure

We now derive formulas for the power and size of the pooling procedure applied to the component of variance model described in Section I. Let us first state the procedure in mathematical terms. We are given an analysis of variance table as shown below.

Table 4. Component of variance model. Analysis of variance

Source of variation	Mean square	d. f.	Exp. mean square
Treatments	V_3	n_3	σ_3^2
Error	V_2	n_2	σ_2^2
Doubtful error	V_1	n_1	σ_1^2

We are interested in testing the hypothesis $H_0: \sigma_3^2 = \sigma_2^2$ against the alternative $H_1: \sigma_3^2 > \sigma_2^2$ when it is known that $\sigma_2^2 \geq \sigma_1^2$. We assume the sums of squares $n_i V_i$ are independently distributed as $\chi^2_i \sigma_i^2$ where χ^2_i is the central χ^2 statistic based on n_i degrees of freedom. The test procedure with sometimes pooling V_2 and V_1 is then as follows:

Reject H_0 if

$$\text{either } V_2/V_1 \geq F_{n_2, n_1}(\alpha_1) \text{ and } V_3/V_2 \geq F_{n_3, n_2}(\alpha_2) \quad (9)$$

$$\text{or } V_2/V_1 \leq F_{n_2, n_1}(\alpha_1) \text{ and } V_3/V \geq F_{n_3, n_1+n_2}(\alpha_3)$$

where $V = (n_1 V_1 + n_2 V_2)/(n_1 + n_2)$ and $F_{n_i, n_j}(\alpha)$ is the upper 100 α % point of the F distribution with numerator d.f. = n_i and denominator d.f. = n_j .

The probability, P , of rejecting H_0 , which in general is the power of the test procedure, is a function of the degrees of freedom, n_1 , n_2 and n_3 , the ratios, $\theta_{32} = \sigma_3^2/\sigma_2^2$ and $\theta_{21} = \sigma_2^2/\sigma_1^2$, and the levels of significance employed, α_1 , α_2 and α_3 . In the special case when $\theta_{32} = 1$ this power is equal to the size of the test, that is, to the type one error. In general the power P is obtained as the sum of two components corresponding to the mutually exclusive alternatives headed by either and or in its definition above, namely,

$$P_1 = \text{Pr. } \left\{ V_2/V_1 \geq F_{n_2, n_1}(\alpha_1) \text{ and } V_3/V_2 \geq F_{n_3, n_2}(\alpha_2) \right\}, \quad (10)$$

$$P_2 = \text{Pr. } \left\{ V_2/V_1 \leq F_{n_2, n_1}(\alpha_1) \text{ and } V_3/V \geq F_{n_3, n_1+n_2}(\alpha_3) \right\}. \quad (11)$$

B. Integral Expressions for the Power

Definite integrals for P_1 and P_2 will now be derived. The joint density of V_1 , V_2 and V_3 is given by

$$c_1 V_1^{\frac{n_1}{2}-1} V_2^{\frac{n_2}{2}-1} V_3^{\frac{n_3}{2}-1} \exp \left\{ -\frac{1}{2} \frac{n_1 V_1}{\sigma_1^2} + \frac{n_2 V_2}{\sigma_2^2} + \frac{n_3 V_3}{\sigma_3^2} \right\},$$

where c_1 is a constant independent of V_1 , V_2 and V_3 . By introducing new variates,

$$u_1 = \frac{n_2 V_2}{n_1 V_1}, \quad u_2 = \frac{n_3 V_3}{n_2 V_2}, \quad w = \frac{n_1 V_1}{n_3},$$

and integrating out w as a gamma function, we obtain the joint density of u_1 and u_2 as

$$f(u_1, u_2) = \frac{c_2 u_1^{\frac{n_2+n_3}{2}-1} u_2^{\frac{n_3}{2}-1}}{(0_{21} 0_{32} + 0_{32} u_1 + u_1 u_2)^{\frac{1}{2}(n_1+n_2+n_3)}},$$

$$\text{where } c_2 = \frac{0_{21}^{\frac{n_1}{2}} 0_{32}^{\frac{n_1+n_2}{2}}}{B(\frac{n_1}{2}, \frac{n_2}{2}) B(\frac{n_3}{2}, \frac{n_1+n_2}{2})}, \quad 0_{21} = \frac{\sigma_2^2}{\sigma_1^2} \text{ and } 0_{32} = \frac{\sigma_3^2}{\sigma_2^2}. \quad (12)$$

Let us now make the transformation

$$u = \frac{u_1}{0_{21}}, \quad v = \frac{u_2}{0_{32}}.$$

The Jacobian of this transformation is $0_{21} 0_{32}$, and the joint density of u and v becomes

$$f(u, v) = \frac{k u^{\frac{n_2+n_3}{2}-1} v^{\frac{n_3}{2}-1}}{(1+u+uv)^{\frac{n_1+n_2+n_3}{2}}}, \quad (13)$$

where
$$k = \frac{1}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right) B\left(\frac{n_3}{2}, \frac{n_1+n_2}{2}\right)} . \quad (14)$$

The probability of rejecting the hypothesis H_0 is obtained by integrating $f(u, v)$ over the two ranges of variation of u and v which correspond to the two alternatives either and or of definition (9). These ranges are respectively given by

either
$$\frac{u_1^0}{\theta_{21}} \leq u < \infty, \quad \frac{u_2^0}{\theta_{32}} \leq v < \infty ; \quad (15)$$

or
$$0 \leq u \leq \frac{u_1^0}{\theta_{21}}, \quad \frac{u_3^0 (1 + \theta_{21} u)}{\theta_{32} \theta_{21} u} \leq v < \infty ;$$

where
$$u_1^0 = \frac{n_2}{n_1} F_{n_2, n_1}(a_1), \quad u_2^0 = \frac{n_3}{n_2} F_{n_3, n_2}(a_2) , \quad (16)$$

and
$$u_3^0 = \frac{n_3}{n_1 + n_2} F_{n_3, n_1+n_2}(a_3) .$$

Hence the formulas for the two power components become

$$P_1 = \int_a^\infty \int_d^\infty f(u, v) dv du , \quad (17)$$

and

$$P_2 = \int_0^a \int_{\frac{c(1+\theta_{21}u)}{u}}^\infty f(u, v) dv du , \quad (18)$$

where $a = \frac{u_1^0}{\theta_{21}}$, $c = \frac{u_3^0}{\theta_{21}\theta_{32}}$, and $d = \frac{u_2^0}{\theta_{32}}$. (19)

C. Exact Formulas

1. Series formulas

From (13) and (17),

$$P_1 = k \int_a^\infty \int_d^\infty \frac{u^{\frac{n_2+n_3}{2}-1} v^{\frac{n_3}{2}-1}}{(1+u+uv)^{\frac{n_1+n_2+n_3}{2}}} dv du ,$$

where $k = \frac{1}{B(\frac{n_1}{2}, \frac{n_2}{2}) B(\frac{n_3}{2}, \frac{n_1+n_2}{2})}$. (20)

Let $y = \frac{v}{1+u+uv}$; then it is easily verified that

$$v = \frac{(1+u)y}{1-uy}, \quad dv = \frac{(1+u)dy}{(1-uy)^2},$$

and $P_1 = k \int_a^\infty \int_{x_5}^{\frac{1}{u}} \frac{y^{\frac{n_3}{2}-1} (1-uy)^{\frac{n_1+n_2}{2}-1} (u)^{\frac{n_2+n_3}{2}-1}}{(1+u)^{\frac{n_1+n_2}{2}}} dy du ,$

where $x_5 = \frac{d}{1+u+du}$. (21)

Let $z = 1-uy$, $dz = -u dy$.

Then $P_1 = k \int_a^\infty \int_0^{x_6} \frac{z^{\frac{n_1+n_2}{2}-1} (1-z)^{\frac{n_3}{2}-1} u^{\frac{n_2}{2}-1}}{(1+u)^{\frac{n_1+n_2}{2}}} dz du$,

where $x_6 = \frac{1+u}{1+u+du}$. (22)

The binomial expansion of $(1-z)^{\frac{n_3}{2}-1}$ gives us

$$P_1 = k \int_a^\infty \int_0^{x_6} \frac{u^{\frac{n_2}{2}-1} f(z)}{(1+u)^{\frac{n_1+n_2}{2}}} dz du,$$

where

$$\begin{aligned} f(z) = & z^{\frac{n_1+n_2}{2}-1} - \binom{\frac{n_3}{2}-1}{1} z^{\frac{n_1+n_2}{2}} + \binom{\frac{n_3}{2}-1}{2} z^{\frac{n_1+n_2}{2}+1} \\ & - \binom{\frac{n_3}{2}-1}{3} z^{\frac{n_1+n_2}{2}+2} + \dots + (-1)^j \binom{\frac{n_3}{2}-1}{j} z^{\frac{n_1+n_2}{2}+j-1} \\ & + \dots + (-1)^{\frac{n_3}{2}-1} z^{\frac{n_1+n_2+n_3}{2}-1}. \end{aligned}$$

The integration with respect to z gives us, upon simplification,

$$\begin{aligned} P_1 = k \int_a^\infty g(u) du, \quad \text{where} \\ g(u) = \frac{2u^{\frac{n_2}{2}-1}}{(n_1+n_2)(1+u+du)^{\frac{n_1+n_2}{2}}} - \frac{2 \binom{\frac{n_3}{2}-1}{1} (1+u) u^{\frac{n_2}{2}-1}}{(n_1+n_2+2)(1+u+du)^{\frac{n_1+n_2}{2}-1}} \end{aligned}$$

$$\begin{aligned}
& + \frac{2 \binom{\frac{n_3}{2}-1}{2} (1+u)^2 u^{\frac{n_2}{2}-1}}{(n_1+n_2+4)(1+u+du)^{\frac{n_1+n_2}{2}+2}} - \frac{2 \binom{\frac{n_3}{2}-1}{3} (1+u)^3 u^{\frac{n_2}{2}-1}}{(n_1+n_2+6)(1+u+du)^{\frac{n_1+n_2}{2}+3}} \\
& + \dots + \frac{(-1)^j 2 \binom{\frac{n_3}{2}-1}{j} (1+u)^j u^{\frac{n_2}{2}-1}}{(n_1+n_2+2j)(1+u+du)^{\frac{n_1+n_2}{2}+j}} + \dots + \frac{(-1)^{\frac{n_3}{2}-1} 2 (1+u)^{\frac{n_3}{2}-1} u^{\frac{n_2}{2}-1}}{(n_1+n_2+n_3-2)(1+u+du)^{\frac{n_1+n_2+n_3}{2}-1}}.
\end{aligned}$$

Before proceeding to integrate these terms, let us consider a general case.

Let
$$H = \int_a^\infty \frac{u^m}{(1+u+du)^n} du.$$

Let us make the transformation

$$y = \frac{1}{1+u+du};$$

then
$$u = \frac{1-y}{(1+d)y}, \quad du = \frac{-1}{(1+d)y^2} dy.$$

Upon simplification, we obtain

$$H = \frac{1}{(1+d)^{m+1}} \int_0^{x_7} y^{n-m-2} (1-y)^m dy$$

$$= \frac{B(n-m-1, m+1) I_{x_7}(n-m-1, m+1)}{(1+d)^{m+1}}, \quad (23)$$

where $x_7 = \frac{1}{1+a+ad}$. (24)

Now let us indicate the successive terms of P_1 as $P_{11}, P_{12}, P_{13}, \dots$.

$P_{1 \frac{n_3}{2}}$. From (23) it follows that

$$\begin{aligned} P_{11} &= \frac{2k}{n_1+n_2} \int_a^\infty \frac{u^{\frac{n_2}{2}-1}}{(1+u+du)^{\frac{n_1+n_2}{2}}} du = \frac{2k B(\frac{n_1}{2}, \frac{n_2}{2}) I_{x_7}(\frac{n_1}{2}, \frac{n_2}{2})}{(n_1+n_2)(1+d)^{\frac{n_2}{2}}} \\ &= \frac{I_{x_7}(\frac{n_1}{2}, \frac{n_2}{2})}{(\frac{n_1+n_2}{2}) B(\frac{n_3}{2}, \frac{n_1+n_2}{2})(1+d)^{\frac{n_2}{2}}} \end{aligned}$$

upon substitution of k from (14).

$$P_{12} = c_{12} \int_a^\infty \frac{u^{\frac{n_2}{2}-1}}{(1+u+du)^{\frac{n_1+n_2}{2}+1}} du + c_{12} \int_a^\infty \frac{u^{\frac{n_2}{2}}}{(1+u+du)^{\frac{n_1+n_2}{2}+1}} du,$$

where

$$c_{12} = - \frac{\left(\frac{n_3}{2} - 1\right) k}{\frac{n_1+n_2}{2} + 1}. \quad (25)$$

$$\begin{aligned}
P_{12} &= \frac{c_{12} B\left(\frac{n_1}{2}+1, \frac{n_2}{2}\right) I_{x_7}\left(\frac{n_1}{2}+1, \frac{n_2}{2}\right)}{(1+d)^{\frac{n_2}{2}}} + \frac{c_{12} B\left(\frac{n_1}{2}, \frac{n_2}{2}+1\right) I_{x_7}\left(\frac{n_1}{2}, \frac{n_2}{2}+1\right)}{(1+d)^{\frac{n_2}{2}+1}} \\
&= - \frac{\binom{\frac{n_3}{2}-1}{1}}{\left(\frac{n_1+n_2}{2}\right) B\left(\frac{n_1}{2}, \frac{n_2}{2}\right) B\left(\frac{n_3}{2}, \frac{n_1+n_2}{2}\right) (1+d)^{\frac{n_2}{2}}} \left[B\left(\frac{n_1}{2}+1, \frac{n_2}{2}\right) I_{x_7}\left(\frac{n_1}{2}+1, \frac{n_2}{2}\right) \right. \\
&\quad \left. + \frac{B\left(\frac{n_1}{2}, \frac{n_2}{2}+1\right) I_{x_7}\left(\frac{n_1}{2}, \frac{n_2}{2}+1\right)}{1+d} \right], \text{ from application of (23), and}
\end{aligned}$$

upon substitution of c_{12} from (25) and of k from (14).

Using the procedure indicated in evaluating P_{11} and P_{12} , formulas for the remaining terms can be similarly derived. In general, the j -th term ($1 \leq j \leq \frac{n_3}{2}-1$) is seen to be the product of two expressions,

$$\begin{aligned}
P_{1j} &= \left[\frac{(-1)^{j-1} \binom{\frac{n_3}{2}-1}{j-1}}{\left(\frac{n_1+n_2}{2} + j-1\right) B\left(\frac{n_1}{2}, \frac{n_2}{2}\right) B\left(\frac{n_3}{2}, \frac{n_1+n_2}{2}\right) (1+d)^{\frac{n_2}{2}}} \right] \\
&\quad \times \left[\sum_{r=0}^{j-1} \frac{\binom{j-1}{r} B\left(\frac{n_1}{2}+j-1-r, \frac{n_2}{2}+r\right) I_{x_7}\left(\frac{n_1}{2}+j-1-r, \frac{n_2}{2}+r\right)}{(1+d)^r} \right].
\end{aligned}$$

Upon substituting for $B(\frac{n_1}{2}, \frac{n_2}{2})$, $B(\frac{n_3}{2}, \frac{n_1+n_2}{2})$ and $B(\frac{n_1}{2} + j-1-r, \frac{n_2}{2} + r)$ in each term and collecting terms, we obtain

$$\begin{aligned}
 P_1 = & \frac{1}{(\frac{n_1+n_2}{2}) B(\frac{n_3}{2}, \frac{n_1+n_2}{2})(1+d) \frac{n_2}{2}} \left\{ I_{x_7} \left(\frac{n_1}{2}, \frac{n_2}{2} \right) \right. \\
 & - \frac{(\frac{n_3}{2}-1)}{1!(\frac{n_1+n_2}{2}+1)} \left[\frac{n_1}{2} I_{x_7} \left(\frac{n_1}{2}+1, \frac{n_2}{2} \right) + \frac{\frac{n_2}{2} I_{x_7} \left(\frac{n_1}{2}, \frac{n_2}{2}+1 \right)}{(1+d)} \right] \\
 & + \frac{(\frac{n_3}{2}-1)(\frac{n_3}{2}-2)}{2!(\frac{n_1+n_2}{2}+1)(\frac{n_1+n_2}{2}+2)} \left[\frac{n_1}{2}(\frac{n_1}{2}+1) I_{x_7} \left(\frac{n_1}{2}+2, \frac{n_2}{2} \right) + \frac{2(\frac{n_2}{2})(\frac{n_1}{2}) I_{x_7} \left(\frac{n_1}{2}+1, \frac{n_2}{2}+1 \right)}{(1+d)} \right. \\
 & \quad \left. + \frac{(\frac{n_2}{2}+1, \frac{n_2}{2}) I_{x_7} \left(\frac{n_1}{2}, \frac{n_2}{2}+2 \right)}{(1+d)^2} \right] \\
 & - \frac{(\frac{n_3}{2}-1)(\frac{n_3}{2}-2)(\frac{n_3}{2}-3)}{3!(\frac{n_1+n_2}{2}+1)(\frac{n_1+n_2}{2}+2)(\frac{n_1+n_2}{2}+3)} \left[\frac{n_1}{2}(\frac{n_1}{2}+2)(\frac{n_1}{2}+1)(\frac{n_1}{2}) I_{x_7} \left(\frac{n_1}{2}+3, \frac{n_2}{2} \right) \right. \\
 & + \frac{3(\frac{n_1}{2}+1)(\frac{n_1}{2})(\frac{n_2}{2}) I_{x_7} \left(\frac{n_1}{2}+2, \frac{n_2}{2}+1 \right)}{(1+d)} + \frac{3(\frac{n_2}{2}+1)(\frac{n_2}{2})(\frac{n_1}{2}) I_{x_7} \left(\frac{n_1}{2}+1, \frac{n_2}{2}+2 \right)}{(1+d)^2} \\
 & \left. + \frac{(\frac{n_2}{2}+2)(\frac{n_2}{2}+1)(\frac{n_2}{2}) I_{x_7} \left(\frac{n_1}{2}, \frac{n_2}{2}+3 \right)}{(1+d)^3} \right] + \dots \left. \right\}.
 \end{aligned}$$

We now consider P_2 . From (13) and (18),

$$P_2 = k \int_0^a \int_{x_8}^{\infty} \frac{u^{\frac{n_2+n_3}{2}-1} v^{\frac{n_3}{2}-1}}{(1+u+uv)^{\frac{n_1+n_2+n_3}{2}}} dv du ,$$

where $x_8 = \frac{c(1 + \theta_{21} u)}{u} = \frac{c + bu}{u} ,$

where $b = \frac{u_3^0}{\theta_{32}} .$ (26)

Now let $z = \frac{1+u}{1+u+uv} ;$

then $v = \frac{(1+u)(1-z)}{uz} , \quad dv = - \frac{(1+u)}{uz^2} dz .$

After performing the necessary simplification, we obtain

$$P_2 = k \int_0^a \int_0^{x_9} \frac{z^{\frac{n_1+n_2}{2}-1} (1-z)^{\frac{n_3}{2}-1} u^{\frac{n_2}{2}-1}}{(1+u)^{\frac{n_1+n_2}{2}}} dz du ,$$

where $x_9 = \frac{1+u}{(1+c+u(1+b))} .$ (27)

The binomial expansion of $(1-z)^{\frac{n_3}{2}-1}$ gives us

$$P_2 = k \int_0^a \int_0^{x_9} \frac{u^{\frac{n_2}{2}-1} f(z)}{(1+u)^{\frac{n_1+n_2}{2}}} du dz , \quad (28)$$

where

$$\begin{aligned}
 f(z) = & z^{\frac{n_1+n_2}{2}-1} - \binom{\frac{n_3}{2}-1}{1} z^{\frac{n_1+n_2}{2}} + \binom{\frac{n_3}{2}-1}{2} z^{\frac{n_1+n_2}{2}+1} \\
 & - \binom{\frac{n_3}{2}-1}{3} z^{\frac{n_1+n_2}{2}+2} + \dots + (-1)^j \binom{\frac{n_3}{2}-1}{j} z^{\frac{n_1+n_2}{2}+j-1} \\
 & + \dots + (-1)^{\frac{n_3}{2}-1} z^{\frac{n_1+n_2+n_3}{2}-1}.
 \end{aligned}$$

The integration with respect to z gives us, upon simplification,

$$P_2 = k \int_0^a h(u) du,$$

where

$$\begin{aligned}
 h(u) = & \frac{2u^{\frac{n_2}{2}-1}}{(n_1+n_2) [1+c+u(1+b)]^{\frac{n_1+n_2}{2}}} - \frac{2 \binom{\frac{n_3}{2}-1}{1} u^{\frac{n_2}{2}-1} (1+u)}{(n_1+n_2+2) [1+c+u(1+b)]^{\frac{n_1+n_2}{2}+1}} \\
 & + \frac{2 \binom{\frac{n_3}{2}-1}{2} u^{\frac{n_2}{2}-1} (1+u)^2}{(n_1+n_2+4) [1+c+u(1+b)]^{\frac{n_1+n_2}{2}+2}} - \frac{2 \binom{\frac{n_3}{2}-1}{3} u^{\frac{n_2}{2}-1} (1+u)^3}{(n_1+n_2+6) [1+c+u(1+b)]^{\frac{n_1+n_2}{2}+3}}
 \end{aligned}$$

$$\begin{aligned}
& + \dots + \frac{(-1)^j 2^{\left(\frac{n_3}{2} - 1\right)} u^{\frac{n_2}{2} - 1} (1+u)^j}{(n_1+n_2+2j) [1+c+u(1+b)]^{\frac{n_1+n_2}{2} + j}} + \dots \\
& + \frac{(-1)^{\frac{n_3}{2} - 1} 2u^{\frac{n_2}{2} - 1} (1+u)^{\frac{n_3}{2} - 1}}{(n_1+n_2+n_3-2) [1+c+u(1+b)]^{\frac{n_1+n_2+n_3}{2} - 1}} .
\end{aligned}$$

Let us now consider the general case

$$G = \int_0^a \frac{u^m}{[1+c+u(1+b)]^n} du .$$

If we make the transformation

$$y = \frac{u(1+b)}{1+c+u(1+b)} ,$$

$$\text{then } u = \frac{y(1+c)}{(1-y)(1+b)} , \quad du = \frac{(1+c) dy}{(1-y)^2 (1+b)} ,$$

and it is easily verified that

$$G = \frac{1}{(1+c)^{n-m-1} (1+b)^{m+1}} \int_0^{x_8} y^m (1-y)^{n-m-2} dy , \quad (30)$$

where

$$x_8 = \frac{a(1+b)}{1+c+a(1+b)} \quad (31)$$

Therefore,

$$G = \frac{B(m+1, n-m-1) I_{x_8} (m+1, n-m-1)}{(1+b)^{m+1} (1+c)^{n-m-1}} \quad (32)$$

Now let the successive terms of (28) be denoted by $P_{21}, P_{22}, P_{23}, \dots, P_{2, \frac{n_3}{2}-1}$. From (32),

$$\begin{aligned} P_{21} &= \frac{2k}{n_1+n_2} \int_0^a \frac{u^{\frac{n_2}{2}-1}}{(1+c+u(1+b))^{\frac{n_1+n_2}{2}}} du \\ &= \frac{2k B(\frac{n_2}{2}, \frac{n_1}{2}) I_{x_8}(\frac{n_2}{2}, \frac{n_1}{2})}{(n_1+n_2)(1+b)^{\frac{n_2}{2}} (1+c)^{\frac{n_1}{2}}} \\ &= \frac{2 I_{x_8}(\frac{n_2}{2}, \frac{n_1}{2})}{(n_1+n_2) B(\frac{n_3}{2}, \frac{n_1+n_2}{2})(1+b)^{\frac{n_2}{2}} (1+c)^{\frac{n_1}{2}}} \end{aligned}$$

after substituting for k from (14) and simplifying. The remaining terms of P_2 can be obtained similarly. The j -th term ($1 \leq j \leq \frac{n_3}{2}-1$)

becomes the product of two expressions,

$$P_{2j} = \left[\frac{(-1)^{j-1} \binom{\frac{n_3}{2} - 1}{j-1}}{\left(\frac{n_1+n_2}{2} + j-1\right) B\left(\frac{n_1}{2}, \frac{n_2}{2}\right) B\left(\frac{n_3}{2}, \frac{n_1+n_2}{2}\right) (1+b) \frac{n_2}{2} \frac{n_1}{2} (1+c)} \right]$$

$$\times \left[\sum_{r=0}^{j-1} \frac{j-1-r}{(1+b)^r (1+c)^{j-1-r}} I_{x_8} \left(\frac{n_2}{2} + r, \frac{n_1}{2} + j-1-r\right) I_{x_8} \left(\frac{n_2}{2} + r, \frac{n_1}{2} + j-1-r\right) \right].$$

After substituting for $B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$, $B\left(\frac{n_3}{2}, \frac{n_1+n_2}{2}\right)$ and $B\left(\frac{n_2}{2} + r, \frac{n_1}{2} + j-1-r\right)$ in each term and simplifying, we obtain

$$P_2 = \frac{1}{\left(\frac{n_1+n_2}{2}\right) B\left(-\frac{n_3}{2}, -\frac{n_1+n_2}{2}\right) (1+b) \frac{n_2}{2} \frac{n_1}{2} (1+c)} \left\{ I_{x_8} \left(\frac{n_2}{2}, \frac{n_1}{2}\right) \right. \\ - \frac{\left(\frac{n_3}{2} - 1\right)}{1! \left(-\frac{n_1+n_2}{2} + 1\right)} \left[\frac{\left(\frac{n_1}{2}\right) I_{x_8} \left(\frac{n_2}{2}, \frac{n_1}{2} + 1\right)}{(1+c)} + \frac{\left(\frac{n_2}{2}\right) I_{x_8} \left(\frac{n_2}{2} + 1, \frac{n_1}{2}\right)}{(1+b)} \right] \\ + \frac{\left(\frac{n_3}{2} - 1\right) \left(-\frac{n_3}{2} - 2\right)}{2! \left(-\frac{n_1+n_2}{2} + 1\right) \left(-\frac{n_1+n_2}{2} + 2\right)} \left[\frac{\left(\frac{n_1}{2}\right) \left(\frac{n_1}{2} + 1\right) I_{x_8} \left(\frac{n_2}{2}, \frac{n_1}{2} + 2\right)}{(1+c)^2} + \frac{2 \left(\frac{n_1}{2}\right) \left(\frac{n_2}{2}\right) I_{x_8} \left(\frac{n_2}{2} + 1, \frac{n_1}{2} + 1\right)}{(1+c)(1+b)} \right]$$

$$\begin{aligned}
& + \frac{\left(\frac{n_2}{2}\right)\left(\frac{n_2}{2}+1\right) I_{x_8}\left(\frac{n_2}{2}+2, \frac{n_1}{2}\right)}{(1+b)^2} \Bigg] \\
& - \frac{\left(\frac{n_3}{2}-1\right)\left(\frac{n_3}{2}-2\right)\left(\frac{n_3}{2}-3\right)}{3!\left(\frac{n_1+n_2}{2}+1\right)\left(\frac{n_1+n_2}{2}+2\right)\left(\frac{n_1+n_2}{2}+3\right)} \left[\frac{\left(\frac{n_1}{2}\right)\left(\frac{n_1}{2}+1\right)\left(\frac{n_1}{2}+2\right) I_{x_8}\left(\frac{n_2}{2}, \frac{n_1}{2}+3\right)}{(1+c)^3} \right. \\
& + \frac{3\left(\frac{n_1}{2}\right)\left(\frac{n_1}{2}+1\right)\left(\frac{n_2}{2}\right) I_{x_8}\left(\frac{n_2}{2}+1, \frac{n_1}{2}+2\right)}{(1+c)^2(1+b)} + \frac{3\left(\frac{n_2}{2}\right)\left(\frac{n_2}{2}+1\right) I_{x_8}\left(\frac{n_2}{2}+2, \frac{n_1}{2}+1\right)}{(1+c)(1+b)^2} \\
& + \left. \frac{\left(\frac{n_2}{2}+2\right)\left(\frac{n_2}{2}+1\right)\left(\frac{n_2}{2}\right) I_{x_8}\left(\frac{n_2}{2}+3, \frac{n_1}{2}\right)}{(1+b)^3} + \dots \right] .
\end{aligned}$$

2. Recurrence formulas

From (13) and (17), we know that

$$P_1 = k \int_a^\infty \int_d^\infty \frac{u^{\frac{n_2+n_3}{2}-1} v^{\frac{n_3}{2}-1}}{(1+u+uv)^{\frac{n_1+n_2+n_3}{2}}} dv du . \quad (33)$$

Integrating by parts with respect to v , we obtain

$$P_1(n_3) = k \int_a^\infty u^{\frac{n_2+n_3}{2}-2} \left[\frac{-v^{\frac{n_3}{2}-1}}{\frac{n_1+n_2+n_3}{2}-1} \cdot \frac{1}{(1+u+uv)^{\frac{n_1+n_2+n_3}{2}-1}} \right]_d^\infty du$$

$$\begin{aligned}
& + \frac{\frac{n_3}{2} - 1}{\frac{n_1+n_2+n_3}{2} - 1} k \int_a^\infty \int_d^\infty \frac{v^{\frac{n_3}{2}-1} u^{\frac{n_2+n_3}{2}-2}}{(1+u+uv)^{\frac{n_1+n_2+n_3}{2}}} dv du, \\
& = \frac{(d)^{\frac{n_3}{2}-1}}{\frac{n_1+n_2+n_3}{2} - 1} k \int_a^\infty \frac{u^{\frac{n_2+n_3}{2}-2}}{(1+u+ud)^{\frac{n_1+n_2+n_3}{2} - 1}} du \quad (34)
\end{aligned}$$

$$+ \frac{(\frac{n_3}{2} - 1) k}{\frac{n_1+n_2+n_3}{2} - 1} P_1(n_3 - 2) . \quad (35)$$

To integrate (34), which we designate by $P_{11}(n_1, n_2, n_3)$, let $u = \frac{1-z}{z} \cdot \frac{1}{1+d}$, $du = -\frac{1}{(1+d)z^2} dz$.

Then, upon simplification, we obtain

$$P_{11}(n_3) = c_3 \int_0^{x_0} \frac{z^{\frac{n_1}{2}-1}}{(1-z)^{\frac{n_2+n_3}{2}-2}} dz ,$$

$$\text{where } c_3 = \frac{(d)^{\frac{n_3}{2}-1} k}{\left(\frac{n_1+n_2+n_3}{2} - 1\right)(1+d)^{\frac{n_2+n_3}{2} - 1}} \quad (36)$$

$$\text{and } x_0 = \frac{1}{1+a(1+d)} . \quad (37)$$

It follows that

$$P_{11}(n_3) = c_3 B\left(\frac{n_1}{2}, \frac{n_2+n_3}{2} - 1\right) I_{x_0}\left(\frac{n_1}{2}, \frac{n_2+n_3}{2} - 1\right). \quad (38)$$

Finally, by substituting for k from (14) and for c_3 from (36), and performing the necessary simplification, we obtain the recurrence formula for $P_1(n_3)$,

$$P_1(n_3) = \frac{(d)^{\frac{n_3}{2}-1} I_{x_0}\left(\frac{n_1}{2}, \frac{n_2+n_3}{2} - 1\right)}{\left(\frac{n_3}{2} - 1\right) B\left(\frac{n_3}{2} - 1, \frac{n_2}{2}\right)(1+d)^{\frac{n_2+n_3}{2}-1}} + P_1(n_3 - 2). \quad (39)$$

For the special case $n_3 = 2$, it follows from (33) that

$$\begin{aligned} P_1(2) &= k \int_a^\infty \int_d^\infty \frac{u^{\frac{n_2}{2}}}{(1+u+uv)^{\frac{n_1+n_2}{2}+1}} dv du \\ &= -c_4 \int_a^\infty \frac{u^{\frac{n_2}{2}-1}}{(1+d+du)^{\frac{n_2+n_2}{2}}} du, \end{aligned} \quad (40)$$

$$\text{where } c_4 = \frac{2k}{n_1 + n_2}. \quad (41)$$

It is easily verified that (40) is merely a special case of (34) with n_3 replaced by 2. Hence for the set of initial values at $n_3 = 2$,

$$P_1(2) = \frac{I_{x_0} \left(\frac{n_1}{2}, \frac{n_2}{2} \right)}{\frac{n_2}{(1+d)^{\frac{2}{2}}}} \quad (42)$$

We now turn to P_2 . From (13) and (18), we know that

$$P_2 = k \int_0^a \int_{\frac{c(1+\theta_{21}u)}{u}}^{\infty} \frac{u^{\frac{n_2+n_3}{2}-1} v^{\frac{n_3}{2}-1}}{(1+u+uv)^{\frac{n_1+n_2+n_3}{2}}} dv du \quad (43)$$

If we make the transformation

$u = \frac{1}{w}$, $v = y + cw$, the Jacobian $J = \frac{1}{w^2}$, then, upon performing the necessary simplification, we obtain

$$P_2 = \int_{\frac{1}{a}}^{\infty} \int_b^{\infty} \frac{(y+cw)^{\frac{n_3}{2}-1} w^{\frac{n_1}{2}-1}}{[1+y+(1+c)w]^{\frac{n_1+n_2+n_3}{2}}} dy dw \quad (44)$$

We now integrate by parts with respect to w . Let

$$u = w^{\frac{n_1}{2}-1} (y+cw)^{\frac{n_3}{2}-1}, \quad dv = \frac{(1+c)}{[1+y+(1+c)w]^{\frac{n_1+n_2+n_3}{2}}} dw$$

It follows that

$$du = \left[\left(\frac{n_3}{2} - 1 \right) cw^{\frac{n_1}{2} - 1} (y+cw)^{\frac{n_3}{2} - 2} + \left(\frac{n_1}{2} - 2 \right) w^{\frac{n_1}{2} - 2} (y+cw)^{\frac{n_3}{2} - 1} \right] dw,$$

$$v = - \frac{\left[1+y+(1+c)w \right]^{-\left(\frac{n_1+n_2+n_3}{2} - 1 \right)}}{\frac{n_1+n_2+n_3}{2} - 1}.$$

$$P_2(n_1, n_3) = -c_6 \int_b^\infty \frac{w^{\frac{n_1}{2} - 1} (y+cw)^{\frac{n_3}{2} - 1}}{\left[1+y+(1+c)w \right]^{\frac{n_1+n_2+n_3}{2} - 1}} dy \quad (45)$$

$w = \frac{1}{a}$

$$+ c_6 \int_{\frac{1}{a}}^\infty \int_b^\infty \frac{\left(\frac{n_3}{2} - 1 \right) cw^{\frac{n_1}{2} - 1} (y+cw)^{\frac{n_3}{2} - 2}}{\left[1+y+(1+c)w \right]^{\frac{n_1+n_2+n_3}{2} - 1}} dy dw \quad (46)$$

$$+ c_6 \int_{\frac{1}{a}}^\infty \int_b^\infty \frac{\left(\frac{n_1}{2} - 1 \right) w^{\frac{n_1}{2} - 2} (y+cw)^{\frac{n_3}{2} - 1}}{\left[1+y+(1+c)w \right]^{\frac{n_1+n_2+n_3}{2} - 1}} dy dw, \quad (47)$$

where

$$c_6 = \frac{k}{\left(\frac{n_1+n_2+n_3}{2} - 1 \right) (1+c)}. \quad (48)$$

Let us designate the three terms of P_2 given by (45), (46) and (47) as P_{21} , P_{22} and P_{23} , respectively. It follows that

$$P_{21}(n_1, n_3) = c_7 \int_b^{\infty} \frac{\left(y + \frac{b}{u_1}\right)^{\frac{n_3}{2} - 1}}{\left(1 + y + \frac{1+c}{a}\right)^{\frac{n_1+n_2+n_3}{2} - 1}} dy, \quad (49)$$

$$\text{where } c_7 = \frac{c_6}{\frac{n_1}{2} - 1} \quad (50)$$

Now let

$$z = \frac{y + \frac{b}{u_1}}{1 + \frac{1}{a} + y + \frac{b}{u_1}}$$

It follows that

$$y + \frac{b}{u_1} = \frac{z}{1-z} \left(1 + \frac{1}{a}\right), \quad dy = \frac{1 + \frac{1}{a}}{(1-z)^2} dz.$$

It is easily verified now that

$$P_{21}(n_1, n_3) = c_8 \int_{x'_1}^1 z^{\frac{n_3}{2} - 1} (1-z)^{\frac{n_1+n_2}{2} - 2} dz,$$

where
$$c_8 = \frac{c_7}{\left(1 + \frac{1}{a}\right)^{\frac{n_1+n_2}{2}-1}} \quad (51)$$

and
$$x'_1 = \frac{b\left(1 + \frac{1}{u_1^0}\right)}{1 + \frac{1}{a} + b\left(1 + \frac{1}{u_1^0}\right)} .$$

Therefore,
$$P_{21}(n_1, n_3) = c_9 \left[1 - I_{x'_1} \left(\frac{n_3}{2}, \frac{n_1+n_2}{2}-1\right)\right] , \quad (52)$$

where
$$c_9 = B\left(\frac{n_3}{2}, \frac{n_1+n_2}{2}-1\right) c_8 . \quad (53)$$

Equation (52) may also be written as

$$P_{21}(n_1, n_3) = c_9 I_{x_1} \left(\frac{n_1+n_2}{2}-1, \frac{n_3}{2}\right) , \quad (54)$$

where
$$x_1 = 1 - x'_1 = \frac{1 + \frac{1}{a}}{1 + \frac{1}{a} + b\left(1 + \frac{1}{u_1^0}\right)} . \quad (55)$$

Substituting for c_9 , c_8 , c_7 , c_6 and k from (53), (51), (50), (48) and (14), respectively, and simplifying, we obtain

$$P_{21}(n_1, n_3) = \frac{\left(\frac{1}{a}\right)^{\frac{n_1}{2}-1} I_{x_1} \left(\frac{n_1+n_2}{2}-1, \frac{n_3}{2}\right)}{(1+c)\left(\frac{n_1+n_2}{2}-1\right) B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1 + \frac{1}{a}\right)^{\frac{n_1+n_2}{2}-1}} . \quad (56)$$

Now let us consider the component

$$P_{22}(n_1, n_3) = \left(\frac{n_3}{2} - 1\right) c_6 c \int_{\frac{1}{a}}^{\infty} \int_b^{\infty} \frac{w^{\frac{n_1}{2}-1} (y+cw)^{\frac{n_3}{2}-2}}{\left[1+y+(1+c)w\right]^{\frac{n_1+n_2+n_3}{2}-1}} dy dw. \quad (57)$$

Comparison of (57) and (43) shows us that

$$P_{22}(n_1, n_3) = \frac{\left(\frac{n_3}{2} - 1\right) c_6 c}{k} P_2(n_1, n_3 - 2).$$

Upon substitution for c_6 and k from (48) and (14), respectively, we obtain

$$P_{22}(n_1, n_3) = \frac{c}{1+c} P_2(n_1, n_3 - 2). \quad (58)$$

Similarly,

$$\begin{aligned} P_{23}(n_1, n_3) &= \left(\frac{n_1}{2} - 1\right) c_6 \int_{\frac{1}{a}}^{\infty} \int_b^{\infty} \frac{w^{\frac{n_1}{2}-2} (y+cw)^{\frac{n_3}{2}-1}}{\left[1+y+(1+c)w\right]^{\frac{n_1+n_2+n_3}{2}-1}} dy dw \\ &= \left(\frac{n_1}{2} - 1\right) \frac{c_6}{k} P_2(n_1 - 2, n_3). \end{aligned} \quad (59)$$

Upon substitution for c_6 and k as done above, we obtain

$$P_{23}(n_1, n_3) = \frac{1}{1+c} P_2(n_1 - 2, n_3). \quad (60)$$

Combining the results obtained in (56), (58) and (60), we find that

$$P_2(n_1, n_3) = \frac{\left(\frac{1}{a}\right)^{\frac{n_1}{2}-1} I_{x_1}\left(\frac{n_1+n_2}{2}-1, \frac{n_3}{2}\right)}{(1+c)\left(\frac{n_1+n_2}{2}-1\right) B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{1}{a}\right)^{\frac{n_1+n_2}{2}-1}} \quad (61)$$

$$+ \frac{c}{1+c} P_2(n_1, n_3 - 2) + \frac{1}{1+c} P_2(n_1 - 2, n_3) .$$

Formula (61) requires two sets of initial values, a set for $n_3 = 2$ $[P_2(n_1, 2)]$ and a set for $n_1 = 2$ $[P_2(2, n_3)]$. It is readily seen from (59) that $P_{23}(2, n_3) = 0$, and hence that

$$P_2(2, n_3) = \frac{I_{x_1}\left(\frac{n_2}{2}, \frac{n_3}{2}\right)}{(1+c)\left(1+\frac{1}{a}\right)^{\frac{n_2}{2}}} + \frac{c}{1+c} P_2(2, n_3-2) , \quad (62)$$

from simplification of (56) and (58).

For the case $n_3 = 2$,

$$P_2(n_1, 2) = k_1 \int_{\frac{1}{a}}^{\infty} \int_b^{\infty} \frac{w^{\frac{n_1}{2}-1}}{[1+y+(1+c)w]^{\frac{n_1+n_2}{2}+1}} dy dw,$$

where $k_1 = \frac{n_1 + n_2}{2B(\frac{n_1}{2}, \frac{n_2}{2})}$.

Integration with respect to y gives us

$$P_2(n_1, 2) = \frac{k_1}{\frac{n_1 + n_2}{2}} \int_{\frac{1}{a}}^{\infty} \frac{w^{\frac{n_1}{2} - 1}}{[1+b+(1+c)w]^{\frac{n_1+n_2}{2}}} dw.$$

The transformation

$$z = \frac{1+b}{1+b+(1+c)w}, \quad w = \left(\frac{1-z}{z} \right) \left(\frac{1+b}{1+c} \right),$$

$$dw = \left(-\frac{1}{z^2} \right) \left(\frac{1+b}{1+c} \right) dz,$$

gives us

$$P_2(n_1, 2) = c_{11} \int_0^{x_2} z^{\frac{n_2}{2} - 1} (1-z)^{\frac{n_1}{2} - 1} dz, \quad (63)$$

$$\text{where } c_{11} = \frac{k_1}{\left(\frac{n_1+n_2}{2} \right) (1+b)^{\frac{n_2}{2}} (1+c)^{\frac{n_1}{2}}} \quad (64)$$

$$\text{and } x_2 = \frac{a(1+b)}{1+c+a(1+b)} \quad (65)$$

Evaluation of (26) and substitution for k_1 gives us

$$P_2(n_1, 2) = \frac{I_{x_2}\left(\frac{n_2}{2}, \frac{n_1}{2}\right)}{\frac{n_2}{(1+b)2} \frac{n_1}{(1+c)2}} \quad (66)$$

D. Approximate Formulas for Large n_1

We now derive simpler approximate formulas. We first consider P_2 . Writing $F_1 = F_{n_2, n_1}(a_1)$, $F_2 = F_{n_3, n_2}(a_2)$, $F_3 = F_{n_3, n_1+n_2}(a_3)$,

we have

$$P_2 = \text{Pr.} \left\{ V_2/V_1 \leq F_1 \text{ and } V_3/V \geq F_3 \right\}.$$

As $n_1 \rightarrow \infty$ both $V_1 \rightarrow \sigma_1^2$ and $V \rightarrow \sigma_1^2$ and, in the limit, the two ratios V_2/V_1 and V_3/V are independently distributed. It is therefore suggested that for large n_1 we use the approximation

$$\begin{aligned} P_2 &\doteq \text{Pr.} \left\{ V_2/V_1 \leq F_1 \right\} \text{Pr.} \left\{ V_3/V \geq F_3 \right\} \\ &\doteq (1 - I_{x_{10}}\left(\frac{1}{2}n_1, \frac{1}{2}n_2\right) I_{x_{11}}\left(\frac{1}{2}(n_1+n_2), \frac{1}{2}n_3\right), \end{aligned} \quad (67)$$

where

$$x_{10} = 1 / (1 + \frac{1 - x(a_1)}{\theta_{21} x(a_1)}) , \quad (68)$$

and

$$x_{11} = (n_1 + n_2) / (n_1 + n_2 + \frac{(n_2 \theta_{21} + n_1)(1 - x(a_3))}{\theta_{21} \theta_{32} x(a_3)}) .$$

Next we turn to

$$P_1 = \Pr. \left\{ V_2/V_1 \geq F_1 \text{ and } V_3/V_2 \geq F_2 \right\}.$$

Here we could use a similar argument if we were to let $n_2 \rightarrow \infty$. This limit would, however, not yield useful results. The important situation in pooling procedures is one in which n_2 is moderate or small, for otherwise nothing is gained by pooling V_2 with another mean square V_1 .

Instead we use the well known normal approximation to $\log V_i$. M. S. Bartlett and D. G. Kendall (1946) have shown that $\log V_i$ is approximately $N(\log \sigma_i^2, \frac{2}{n_i - 1})$ provided that n_i is not too small. We decided to limit the use of this approximation to cases in which all three n_i are ≥ 12 . Writing

$$u = \log V_2 - \log V_1 \text{ and } z = \log V_3 - \log V_2,$$

it follows that the joint distribution of u and z is approximately bivariate normal with correlation coefficient

$$\rho = -1 / \left\{ \left(1 + \frac{n_2 - 1}{n_3 - 1} \right) \left(1 + \frac{n_2 - 1}{n_1 - 1} \right) \right\}^{\frac{1}{2}} . \quad (69)$$

We may therefore employ the tables of the double probability integral of a bivariate normal surface of K. Pearson (1936), Tables VIII and IX. If x and y follow a bivariate normal distribution with both means equal to 0, correlation coefficient ρ , and both standard deviations equal to unity, then these tables give the probabilities

$$P \rho(h, k) \text{ for } x \geq h \text{ and } y \geq k.$$

In our case ρ is given by (69) and h and k by

$$h = \frac{2z_{n_2, n_1}(\alpha_1) - \log \theta_{21}}{\sqrt{\frac{2}{n_1-1} + \frac{2}{n_2-1}}}, \quad k = \frac{2z_{n_3, n_2}(\alpha_2) - \log \theta_{32}}{\sqrt{\frac{2}{n_2-1} + \frac{2}{n_3-1}}}, \quad (70)$$

where $z_{n_i, n_j}(\alpha)$ is the upper 100α % point of Fishers' z distribution with numerator degrees of freedom n_i and denominator degrees of freedom n_j . If the normal approximation is used for the distribution of $\log V_i$, this same approximation may of course be employed to simplify h and k as follows: replace

$$\frac{2z_{n_i, n_j}(\alpha)}{\sqrt{\frac{2}{n_i-1} + \frac{2}{n_j-1}}}$$

by the α % point of the normal distribution.

E. Theory of Reduction of Mixed Model to Random Model

Certain mixed models of analysis of variance were described in Section I. In this subsection we develop the distribution theory for these situations. No new formulas are required, as we shall show that the joint distribution of the three mean squares is, at least approximately, equal to that of the component of variance model. The exact specifications of the distribution for the mixed model being considered are as follows. (Primed parameters will be used to specify the parameters for the mixed model.)

(a) The error mean square V_2 and the doubtful error mean square V_1 are distributed as $\chi^2_i \sigma_i^2 / n'_i$ ($i = 1, 2$), where χ^2_i is the central χ^2 statistic with n'_i degrees of freedom. On the other hand, the treatment mean square V_3 is distributed as $\chi^2_{n'_3} \sigma_2^2 / n'_3$, where $\chi^2_{n'_3}$ is the non-central χ^2 statistic with n'_3 degrees of freedom and non-centrality parameter

$$\lambda = \frac{n'_3 \sigma_3^2 - n'_3 \sigma_2^2}{2\sigma_2^2} = \frac{n'_3}{2} (\theta_{32} - 1) . \quad (71)$$

where $\theta_{32} = \sigma_3^2 / \sigma_2^2$. V_1 , V_2 and V_3 are independent.

(b) The main purpose of the analysis is to test the hypothesis $H_0: \sigma_3^2 = \sigma_2^2$ against the alternative $H_1: \sigma_3^2 > \sigma_2^2$.

(c) The true error mean square, V_2 , has an expectation σ_2^2 which is greater than or equal to the expectation, σ_1^2 , of the doubtful error mean square.

The probability P of rejecting H_0 is obtained as the sum of the two components,

$$P_1 = \text{Pr.} \left\{ V_2/V_1 \geq F_{n_2, n_1}^1(a_1') \text{ and } V_3/V_2 \geq F_{n_3, n_2}^1(a_2') \right\} \quad (72)$$

and

$$P_2 = \text{Pr.} \left\{ V_2/V_1 < F_{n_2, n_1}^1(a_1') \text{ and } V_3/V_2 \geq F_{n_3, n_1+n_2}^1(a_3') \right\}. \quad (73)$$

In evaluating these probabilities we use the approximation first used by Patnaik (1949). We replace $\chi_{n_1}^2$ by $C\chi_{\nu_3}^2$, $\chi_{\nu_3}^2$ being the central χ^2 statistic based upon ν_3 degrees of freedom, where

$$\nu_3 = n_3' + \frac{4\lambda^2}{n_3' + 4\lambda} \quad \text{d.f.}$$

and

$$C = 1 + \frac{2\lambda}{n_3' + 2\lambda}.$$

Since the use of this approximation reduces the non-central χ^2 statistic to a central χ^2 statistic, all three statistics are now central. This suggests the possibility of applying the random model results to the present mixed model.

The power for the random model is defined as the sum of the two probabilities given by (10) and (11) in Subsection B, and may be regarded as an 8 parameter function, the 8 parameters being $n_1, n_2, n_3, \theta_{32}, \theta_{21}, a_1, a_2$ and a_3 . This power has been evaluated for various combinations of these parameters (these evaluations being restricted to cases corresponding to $a_2 = a_3 = .05$).

We now return to the power for the mixed model as defined by (72) and (73), and compare this with the corresponding formulas (10) and (11) for modified values of the 8 parameters as indicated in Table 5.

Table 5. Modified parameters for random model corresponding to specified parameters for mixed model

Specified parameters for mixed model	Modified parameters for random model
n'_1	$n_1 = n'_1$
n'_2	$n_2 = n'_2$
n'_3	$n_3 = \check{v}_3 = n'_3 + \frac{4\lambda^2}{n'_3 + 4\lambda}$
a'_1	$a_1 = a'_1$
a'_2	$a_2 = \text{Root of } F_{n'_3, n'_2}(a'_2) = F_{\check{v}_3, n'_2}(a_2)$
a'_3	$a_3 = \text{Root of } F_{n'_3, n'_1+n'_2}(a'_3) = F_{\check{v}_3, n'_1+n'_2}(a_3)$
$\theta'_{21} = \sigma_2^2 / \sigma_1^2$	$\theta_{21} = \theta'_{21}$
	$\theta_{32} = \frac{2\lambda + n'_3}{n'_3}$

Entering the random model tables with these altered parameters we obtain the mixed model powers. It will be seen that when we deal with the size for the mixed model we have $\lambda = 0$ and hence $\check{v}_3 = n_3$, so that all primed parameters agree with those without primes. Thus our entire

size discussion to follow is directly applicable to the mixed model. On the other hand, the power evaluations, which refer to $\alpha_2 = \alpha_3 = .05$, will in general provide answers for larger values of α'_2 and α'_3 , and these levels α'_2 and α'_3 will vary with λ . For a proper evaluation of power corresponding to a given pair of significance levels α'_2 and α'_3 , say $\alpha'_2 = \alpha'_3 = .05$, a more extensive tabulation of (10) and (11) would be required.

F. Application of Derived Formulas

The recurrence formulas derived in Part D were used to construct master tables of P_1 and P_2 . These master tables were constructed for

$$\frac{n_2}{2} = 5, \quad \frac{n_1}{2} = 3(1)10 \text{ and } \frac{n_3}{2} = 1(1)6.$$

Also, tables were constructed for $\frac{1}{2} n_2 = 3$, in order that the effect of small error degrees of freedom could be better studied. However, the latter tables were confined to the values

$$\frac{1}{2} n_3 = 1 \text{ and } \frac{1}{2} n_1 = 8, 14 \text{ and } 20,$$

in order to save on computational expenses. The P_1 values were obtained by starting with the set of initial values from (42) and then using the recurrence relation (39), for grids of suitably selected values of x_0 and d . Similarly, tables of P_2 values were obtained by first computing the two sets of initial values from (62) and (66), and then using (61), for selected values of a , x_1 and x_2 . The values of the arguments used in the master

tables were selected so as to cover the ranges of variation of the values which would arise in the power function problems to be investigated.

For examples of master tables see Tables 8 and 9. *

To compute the power component P_1 in a given problem from these master tables, for specified degrees of freedom n_1 , n_2 and n_3 and levels of significance α_1 , α_2 , the values of the parameters u_1^0 and u_2^0 are computed from (16). From these values and those specified for θ_{21} and θ_{32} the corresponding values of a and d from (19) and hence the value of x from (37) are computed. P_1 is then obtained by interpolation in the appropriate master table.

The procedure used to compute the component P_2 is similar. For specified degrees of freedom n_1 , n_2 and n_3 and levels of significance α_1 and α_3 , the values of the parameters u_1^0 and u_3^0 are computed from (16). From these values and those specified for θ_{21} and θ_{32} the corresponding values of a , b , and c from (19) and (26) and hence the values of x_1 from (55) and x_2 from (65) are computed; P_2 is then obtained by interpolation in the appropriate master table. For the power computations that were actually made, interpolation with respect to a was avoided by choosing values of θ_{21} which would result in tabular values of a . This accounts for the decimal values of θ_{21} found in Tables 13 - 29.

The computational procedures outlined above were performed for various combinations of α_1 , n_1 , n_2 , and n_3 , and for several values

* Tables 8-31 have been assembled in the Appendix.

of θ_{32} , for the case $a_2 = a_3 = .05$, primarily. In computing the P_1 components, three point Lagrangian interpolation was used with respect to d ; and linear interpolation with regard to x , followed by a second difference correction. (Lagrangian interpolation coefficients were obtained from tables prepared by the National Bureau of Standards (1944)). For the P_2 evaluations, linear interpolation was used on the logarithms of the tabular values, followed by application of a second difference correction.

The P_1 interpolation was performed first on d , then on x . For the interpolation on d , Lagrangian interpolation coefficients were preferred to the adjusted linear interpolation, because of the fact that, for the problems studied, the same d value (and hence the same interpolation coefficient) was used for a number of x values. On the other hand, for the remainder of the interpolation for P_1 and for all of the P_2 interpolation, the adjusted linear interpolation was less time consuming. It is believed that these interpolations gave results accurate to at least two decimals, and in most cases to three decimals.

The series formulas derived in Subsection (1) of Section C were used to compute special cases not covered by the computed master tables. These formulas also proved useful as independent checks for the recurrence computations.

The approximate formula (67) for P_2 derived in Subsection D is exact for $n_1 = \infty$ and was found to be very effective for large n_1 , large θ_{32} and small θ_{21} , n_2 and n_3 . Since the size refers to $\theta_{32} = 1$

and arbitrary values of θ_{21} , it was necessary to correct the values computed for $n_1 = 60$ by an adjustment. This adjustment was obtained by evaluating the difference between the approximate value and the corresponding exact value of P_2 for $n_1 = 20$, and interpolating harmonically between this difference and that for $n_1 = \infty$ (where the difference is zero) to find the corresponding adjustment for $n_1 = 60$.

IV. DISCUSSION OF POWER AND SIZE CURVES AND COMPARISON OF TEST PROCEDURES

A. Type of Recommendations Attempted

We have seen that the power of our test procedures depends upon the following 8 parameters: The degrees of freedom n_1 , n_2 and n_3 ; the variance ratios $\theta_{21} = \sigma_2^2/\sigma_1^2$, $\theta_{32} = \sigma_3^2/\sigma_2^2$; and the levels of significance α_1 , α_2 , α_3 . Of these, the degrees of freedom n_1 , n_2 and n_3 are completely determined by the analysis of variance table, while the variance ratios are generally unknown (except in the case of the size of the procedure, when $\theta_{32} = 1$). Any recommendations that are to be made must therefore be confined to the levels of significance, α_1 , α_2 , and α_3 . We shall here be primarily concerned with the type one error being in the vicinity of .05. Most of the discussion to follow will therefore be confined to test procedures in which $\alpha_2 = \alpha_3 = .05$, that is, to procedures in which the significance levels of both final tests are .05. However, the remaining parameter, α_1 , the level of significance for the preliminary test, is entirely at our disposal. In attempting recommendations, therefore, we shall be concerned with the choice of the level of α_1 ; should α_1 be, say, .05, .25, .50, or should we use what Paull (1948, p. 4) has called the borderline test, where α_1 will be near .70 to .80? In choosing the level of α_1 , we shall consider

- (i) the variation in the size of our test procedure as a function of the parameter θ_{21} ; and
- (ii) a comparison of the power of our test procedure with that of the never pool test of the same size.

Both of these considerations were studied by Paull for his special cases. It will be seen, however, that when evaluating, as we do, the size and power for wider and more representative sets of degrees of freedom, much larger size disturbances (than those reported by Paull) occur for many of the sets considered; and that consequently we must modify his recommendations in certain respects in order to achieve size control, for an acceptable power.

B. Size Curves

The size of our test procedure does not equal the nominal level of .05, but varies about this level as α_1 and θ_{21} vary. Figures 1 to 15* give us examples of size curves, illustrating the variations in type one error with variation in θ_{21} for fixed values of the remaining parameters.

Note that as θ_{21} becomes large the size approaches .05; for, as $\theta_{21} \rightarrow \infty$ the preliminary test will almost certainly be significant, pooling will almost certainly not occur, and hence the final test will almost certainly be that of V_3/V_2 , having a size of .05.

At the lower extreme, that is, at $\theta_{21} = 1$, the size is less than .05. The reason for this is more complex. The probabilities of pooling or not pooling, in accordance with the result of the preliminary test, are now

* All figures have been assembled in the Appendix.

$1 - \alpha_1$ and α_1 , respectively. But on the final test the probability of obtaining a significant result when not pooling is much smaller than $(\alpha_1)(.05)$ because in these situations V_2 is comparatively large. On the other hand the probability of V_3/V being significant when pooling is still about $(1 - \alpha_1)(.05)$. Thus, when $\theta_{21} = 1$, the total size is below $[(\alpha_1)(.05) + (1 - \alpha_1)(.05)] = .05$.

Of particular interest are the maximum size disturbance (the size peak) and the minimum value of the size; the latter occurs at $\theta_{21} = 1$.

We first consider the size peak. Referring to the size curves for a preliminary test carried out at the 5 per cent level, we note that the peak is usually very high. Clearly, a preliminary test carried out at this level will in many cases admit an unacceptable size disturbance. This is due to the fact that at this level the preliminary test will frequently admit pooling V_2 and V_1 when σ_1^2 is smaller than the true error mean square σ_2^2 , and thereby increase the probability of type one error. We therefore seek a preliminary test in which pooling is admitted less readily; we next investigate the level $\alpha_1 = .25$. At this level (see Figures 5-13) size control is considerably better, and in many cases the peaks do not go beyond .08. In fact, for $n_3 = 2$, the peak is usually below .07. It should be noted that, in general, the size peak increases as n_1 or n_3 increases or as n_2 decreases.*

It is of course quite arbitrary to specify any rules for maintaining an acceptable upper tolerance for the size peak, since what is considered acceptable is a matter of opinion. In using a nominal size of .05, if we stipulate that our size peak should not go much beyond 10 per cent,

*The nature of the dependence of the peak on n_1 , n_2 and n_3 was obtained from observation of the curves, not from any analytic considerations.

then we find that even with the 25 per cent preliminary level, there are situations in which this upper limit is exceeded. Generally speaking, these unacceptable size peaks occur when

$$n_3 \geq n_2$$

and

$$n_1 \geq 5n_2 \quad (75)$$

(It should be noted that the occurrence of $n_3 > n_2$ is clearly rare.) This means that when the treatment degrees of freedom are greater than or equal to the error degrees of freedom we must be careful if at the same time the doubtful error degrees of freedom are greater than or equal to five times the true error degrees of freedom; or, briefly, we must be careful when pooling promises a large gain in the precision of the error estimate. In these situations a more conservative level of α_1 would be appropriate. From a study of a number of size curves it appears that a preliminary test at the 50 per cent level will ensure adequate control of the size peak in these cases (see

Figure 12).

Not only the size peak, but also the size minimum is affected by the level of the preliminary test. From theorems proved by Paull, (1948, Chapter 4) we know that the size of our test procedures is a minimum with respect to θ_{21} at θ_{21} equal to one, and that the lower bound for the size for this value of θ_{21} is $(1-\alpha_1)(.05)$. These lower bounds are .0475, .0375 and .025 for $\alpha_1 = .05$, .25 and .50, respectively. For some of our curves the plotted minimum sizes are

situated very close to these lower bounds. For the borderline test, where the size is always less than .05, this lower bound lies approximately between .01 and .015. We have computed actual minimum size values for the borderline test for selected values of n_1 , n_2 and n_3 (see Table 12). For small n_2 and n_3 these are very close to their lower bounds, irrespective of n_1 . A person using this test should therefore remember that he may be using a test which has a considerably lower size than .05. The actual disturbance is of course small, but the proportional disturbance is considerable. However, since the borderline test size disturbance is a reduction rather than a size increase and is therefore on the conservative side, we are not attempting to make any definite rules as to when the experimenter should avoid the use of this test but merely remind him that large size disturbances occur when n_2 and n_3 are both small (≤ 6).

Summarizing our considerations of size control, therefore, we have narrowed down our recommendable range of α_1 to $\alpha_1 \geq .25$, with the reservations that in certain cases characterized by inequalities (75), $\alpha_1 = .25$ would not be desirable, as it would admit too large a peak in the size curve; and that for very small values of n_2 and n_3 the experimenter may not wish to use the borderline test, as this would admit too low a size minimum.

The discussion thus far has been concerned with test procedures in which $\alpha_2 = \alpha_3 = .05$. A few special cases for $\alpha_2 = \alpha_3 = .01$ and $\alpha_1 = .25$ have also been investigated. In all these situations, larger proportional

size disturbances than those found for $\alpha_2 = \alpha_3 = .05$ were experienced, even for cases which our rule would accept (see Figure 14).

C. Frequency of Pooling

We have been discussing the effect of increasing α_1 in order to achieve size control. It is obvious that for $\alpha_1 = 1$, our preliminary F per cent point would be zero, and pooling would never occur. The question arises as to the relative frequency of pooling for the intermediate values of α_1 that we have been considering. When $\theta_{21} = 1$, the probability that V_2/V_1 exceeds $F_{\alpha_1}(n_2, n_1)$ is α_1 , so that pooling occurs with relative frequency $1 - \alpha_1$. As θ_{21} increases this frequency rapidly decreases, approaching the limit zero as θ_{21} becomes infinite. Evaluations of these frequencies of pooling, which are summarized in Tables 10 and 11, show that, while for $\alpha_1 = .25$ and small values of θ_{21} pooling will occur in the majority of experiments, when $\alpha_1 = .50$ the frequency is usually well below .50. This frequency of pooling is of course even smaller for the borderline test, where α_1 usually takes on values in the neighborhood of .7 to .8; when such large values of α_1 are employed, pooling occurs in only about twenty five per cent of all situations for which $\theta_{21} = 1$, and this pooling percentage rapidly decreases as θ_{21} increases. While this property by itself cannot be regarded as a disadvantage of the borderline test, it is clear that, if this test were the only recommended one to the experimenter, he would hardly ever pool.

D. Power Curves

We now attempt a comparison of the power of our sometimes pool procedure with that of the never pool test. As is well known, any comparison of power of any two test procedures is a fair comparison if the two test procedures have the same size. We have seen that the size of our sometimes pool procedures is not at the constant level of .05, but varies about this, depending upon the parameter θ_{21} . The method of power comparison we have therefore adopted is as follows:

- (i) Assume a fixed value of the parameter θ_{21} .
- (ii) For this value of θ_{21} evaluate the size of the sometimes pool test.
- (iii) For this level of size evaluate the power curve of the never pool test; this power curve is then directly comparable with that of the sometimes pool test corresponding to the chosen value of θ_{21} .

For an illustration of this comparison see Figure 16. Here we have $n_1 = 20$, $n_2 = 10$, $n_3 = 12$, $\alpha_1 = \alpha_2 = \alpha_3 = .05$, and a fixed value of $\theta_{21} = 1.174$ ($\sigma_2^2 = 1.174 \sigma_1^2$). For this value of θ_{21} the size of the sometimes pool test (see Figure 4) is .072; for this size we have evaluated the power curve of the never pool test, which is plotted against θ_{32} , using log scales in both cases, in Figure 16. In the same figure is shown the power curve for our sometimes pool procedure, which, in the present case, is always above that of the never pool test. This means that in the present example the power of the sometimes pool procedure is greater than that of the never pool test even though the latter is per-

mitted to have the same size (.072) as the former. Figures 17 and 18 give two similar comparisons of power curves corresponding to the same set of degrees of freedom, but to $\theta_{21} = 1.789$ and 2.866, respectively. The first of these shows intersecting power curves; the second shows a case in which the never pool test is more powerful. The three figures illustrate the fact that the sometimes pool procedure is more powerful for small θ_{21} and less powerful for large θ_{21} . (This is a general property shared by all those sometimes pool procedures for which α_1 is smaller than the α_1 level for the borderline test, as was proved by Paull (1948, p. 55)). For various other combinations of $n_1, n_2, n_3, \alpha_1 (\alpha_2 = \alpha_3 = .05), \theta_{21}$ and θ_{32} , corresponding power comparisons are tabulated in Tables 13 to 31. In order to show more clearly the dependence of these power comparisons on θ_{21} , we have plotted in Figures 19 to 24 the difference between two corresponding power points against θ_{21} . Here each curve corresponds to a fixed value of θ_{32} , that is, that value of θ_{32} at which the difference between the power ordinates of the power curves was taken. It will be seen, again, that for small θ_{21} the differences are positive (the sometimes pool procedure is more powerful than the never pool test), while for larger θ_{21} the position is reversed. As $\theta_{21} \rightarrow \infty$ the difference tends to 0, since both procedures tend to the never pool test at the .05 level of significance. The transition from favorable to unfavorable power conditions generally occurs between $\theta_{21} = 1.5$ and $\theta_{21} = 2.0$. The magnitude of these power gains and losses increase with increasing n_3 , or decreasing n_2 or increasing n_1 .

The three particular Figures 19, 20 and 21 illustrate the effect of decreasing the per cent point of F for the preliminary test. In these figures corresponding power comparisons are given, respectively, for $\alpha_1 = .05$, $F_1 = 2F_{.50}(n_2, n_1)$ and $\alpha_1 = .25$. There is a general tendency for both power gains and power losses to diminish as the per cent point of F decreases, that is, as α_1 increases from values such as .05 through intermediate values such as .25 to the level of the borderline test (approximately $\alpha = .70$ to .80). Here the gain in power has diminished further (see Tables 30 and 31) but the power losses have completely disappeared. In fact, a theorem by Paull (1948, p. 61) proves that the borderline test is always more powerful than the corresponding never pool test of the same size, although the power gain is small for large θ_{21} . However, as we have seen (see Subsection B), this size is below the nominal level of .05. If we compare the borderline test power with that of the never pool test at the nominal level of .05, the former is always less powerful. It is likewise less powerful than our sometimes pool procedures for $\alpha_1 = .50$ and $\alpha_1 = .25$, which have, of course, a larger size.

We now attempt recommendations, considering the relative merits of the procedures at $\alpha_1 = .25$, $\alpha_1 = .50$ and $\alpha_1 = .7$ to .8 (the borderline level). Unfortunately, these recommendations are somewhat subjective, since they are contingent upon what the experimenter may regard as a reasonable assumption concerning the parameter θ_{21} .

(1) If the experimenter is reasonably certain that only small values of θ_{21} can be envisaged as a possibility, he is advised to use $\alpha_1 = .25$

except in the cases (75), when he should use $\alpha_1 = .50$, in order to ensure size control. Our Figures 19 to 24 show that the range of small values of θ_{21} , when the sometimes pool procedure gives a gain in power, is approximately between 1 and 1.5 to 2. An experimenter about to adopt this recommendation but not quite certain about his assumptions may wish to know the consequences which result from his adopting this procedure when, in fact, unknown to him, θ_{21} is large. It is seen from Figures 1 to 14 that in such a situation he will still have control of the size of his test; in fact the size will be near .05 for large θ_{21} . All he loses (as is shown by Figures 17 to 24 and Tables 13 to 29) is the power of his test; this is a risk that he may well be prepared to take.

(ii) If, however, the experimenter can make no such assumption about θ_{21} , and wishes to guard against the possibility of power losses, he may then use the borderline test, which would ensure a power gain, although he must realize

- (a) that for large θ_{21} this gain would be very small;
- (b) that for small θ_{21} he would use a test procedure of a very much smaller size than $\alpha_1 = .05$ (particularly when n_2 and n_3 are ≤ 6) and accordingly a test which is much less powerful than the never pool test of size .05. In fact, he may in these circumstances prefer not to pool at all.

It may be correctly argued that, in order to control the size peak, to advocate $\alpha_1 = .50$ in the cases characterized by (75), and $\alpha_1 = .25$ otherwise, introduces an artificial discontinuity in our recommenda-

tions. It would be quite feasible (although it would require a considerable effort in computation) to evaluate for any given triplet n_1 , n_2 and n_3 that value of α_1 which results in a size peak of 0.10 exactly. Since this level of α_1 would depend on the degrees of freedom n_1 , n_2 and n_3 , it would be necessary to evaluate the associated per cent points of F . For such recommendations to be useful this table of $F_{\alpha_1}(n_1, n_2)$ (which would be a large 3 parametric table with n_1 , n_2 and n_3 as arguments) would have to be published. To encumber the experimenter with special tables for the preliminary F -test in addition to the standard F tables for the final F -tests appeared to us to be unnecessary, and the use of the published Merrington and Thompson (1943) 25% and 50% points of F preferable.

We should note here that a rule favored by Paull (1944, Chapter 6) advocating testing the ratio V_2/V_1 against $2F_{.50}(n_2, n_1)$ will not ensure adequate control of the size peak since $2F_{.50} > F_{.25}$ in general, and we have just seen that $F_{.25}$ is sometimes too large and hence not always acceptable as a significance level for the preliminary test. Also, it would appear to us that no rule of the form $V_2/V_1 > \text{constant}$ is very satisfactory, for with such a rule the frequency with which pooling occurs as well as the size vary considerably with the degrees of freedom n_1 and n_2 .

Concerning recommendation 2, the experimenter would require knowledge of the precise level of α_1 for the borderline test, or, better still, the value of F associated with it. Paull (1944, p. 20) gives a

simple formula from which the following is derived:

F point for borderline test

$$= \frac{n_1 F_{n_3, n_1+n_2}(a_3)}{(n_1+n_2)(F_{n_3, n_2}(a_2) - n_2 F_{n_3, n_1+n_2}(a_3))}$$

where $F_{n_3, n_2}(a_2)$ represents the 100 a_2 per cent point of F with numerator d.f. n_3 and denominator d.f. n_2 , and similar statements can be made for the other symbols.

V. ILLUSTRATION OF RECOMMENDED PROCEDURES WITH PRACTICAL EXAMPLES

We now show, with the help of two selected examples, how the recommended test procedures work in practical cases. Both of these examples deal with data taken from actual industrial problems and discussed in textbooks. In both cases there is some discussion as to how to choose the valid error for the main test of significance. Moreover, the examples are situations for which, in our opinion, a suitable model is provided either by a component of variance model or mixed model.

A. Tests on Samples of Portland Cement

O. L. Davies (1947, p. 90) gives data on the comparative breaking strength of Portland cement cubes. Each of three gaugers (mixers) mixed the cement from which 12 cubes were made, and each of three breakers (operators of the testing machine) tested the strength of four cubes from each of the mixers. We have therefore a two-way classification with cell repetition. The analysis of variance of the breaking strength values in 10 lb./sq. in. is set out below.

If we regard the breakers and gaugers as respective samples of size three from larger populations of breakers and gaugers, we may represent the above data by a component of variance model. Alternatively, the three gaugers may be regarded as permanent key-personnel of

Table 6. Breaking strength of Portland cement.
Analysis of variance

Source of variation	d.f.	Mean square	Exp. mean square
Between breakers	2	12,530	
Between gaugers	$n_3 = 2$	$V_3 = 4,482$	σ_3^2
Breakers x gaugers	$n_2 = 4$	$V_2 = 1,659$	σ_2^2
Within groups of 4 cubes	$n_1 = 27$	$V_1 = 2,746$	σ_1^2

the plant, and if the tests are concerned with the comparative performance of these particular three men, a mixed model would be appropriate. Either case is covered by our recommendation made in Section IV, which would be applied as follows: It is a priori unlikely that θ_{21} is large (that σ_2^2 is much larger than σ_1^2), as this would mean that there is a large breaker x gauger interaction component adding to the cube to cube variation. This is very unlikely because the breaker's function was merely to operate the testing machine and read off the answers; further, as Davies (1947, p. 92) states, the breakers did not know whose gaugers' cubes they were testing. We may therefore decide (in accordance with our recommendation) to use either the 25 per cent or the 50 per cent F point for our preliminary test, and, as $n_2 > n_3$, we should use the 25 per cent point. The observed F-ratio $V_2/V_1 = .604$ is clearly not significant. (In fact it would have been

not significant at the 50 per cent point of F as well.) We therefore pool V_2 and V_1 , compute

$$V = \frac{n_1 V_1 + n_2 V_2}{n_1 + n_2} = 2606,$$

and test $V_3/V = 1.72$ at the 5 per cent point of F for 2, 31 degrees of freedom, for which the 5 per cent point is 3.31, so that our observed F ratio is not significant. In the case of a completely random model the same procedure may be used for testing differences between breakers. Using the breaker mean square as V_3 and testing $V_3/V = 4.81$, we obtain a result which is significant. Differences between the testing personnel are therefore indicated, suggesting that a better standardization of the test procedure would be desirable. Indeed, Davies states that for these tests an old machine was in use, which would explain the variation of results with the tester operating it.

B. Porosity of Condenser Paper

H. A. Freeman (1942, p. 70) discusses data on the porosity of condenser paper. Three readings were made on each of 9 rolls from each of 3 lots. We now have an example of a hierarchical classification. The analysis of variance table is given below. Here it might be appropriate to regard the three lot means as fixed parameters, but the roll effects and measurement effects as respective random samples from larger populations of roll effect variates and measurement effect variates. If these assumptions are accepted, we have a mixed

Table 7. Porosity readings. Analysis of variance

Source of variation	d.f.	Mean square	Exp. mean square
Between lots	$n_3 = 2$	$V_3 = 3.95$	σ_3^2
Between rolls within lots	$n_2 = 24$	$V_2 = 3.87$	σ_2^2
Between measurements within rolls	$n_1 = 54$	$V_1 = 0.78$	σ_1^2

hierarchical model, to which our recommendations apply. Without making any assumptions concerning the magnitude of possible roll effects, that is, irrespective of our a priori assumptions concerning the magnitude of θ_{21} , the preliminary test at the 25 per cent level is significant. In fact, it is significant at the 1 per cent level. Therefore we are not allowed to pool the within roll component. For the final test, then, we compute $V_3/V_2 = 1.021$ which is not significant at the 5 per cent level. The data indicate that variation in porosity is due largely to differences among rolls within lots, and, as the author suggests, attempts should be made to eliminate these differences.

VI. LITERATURE CITED

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VII. ACKNOWLEDGMENTS

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VIII. APPENDIX

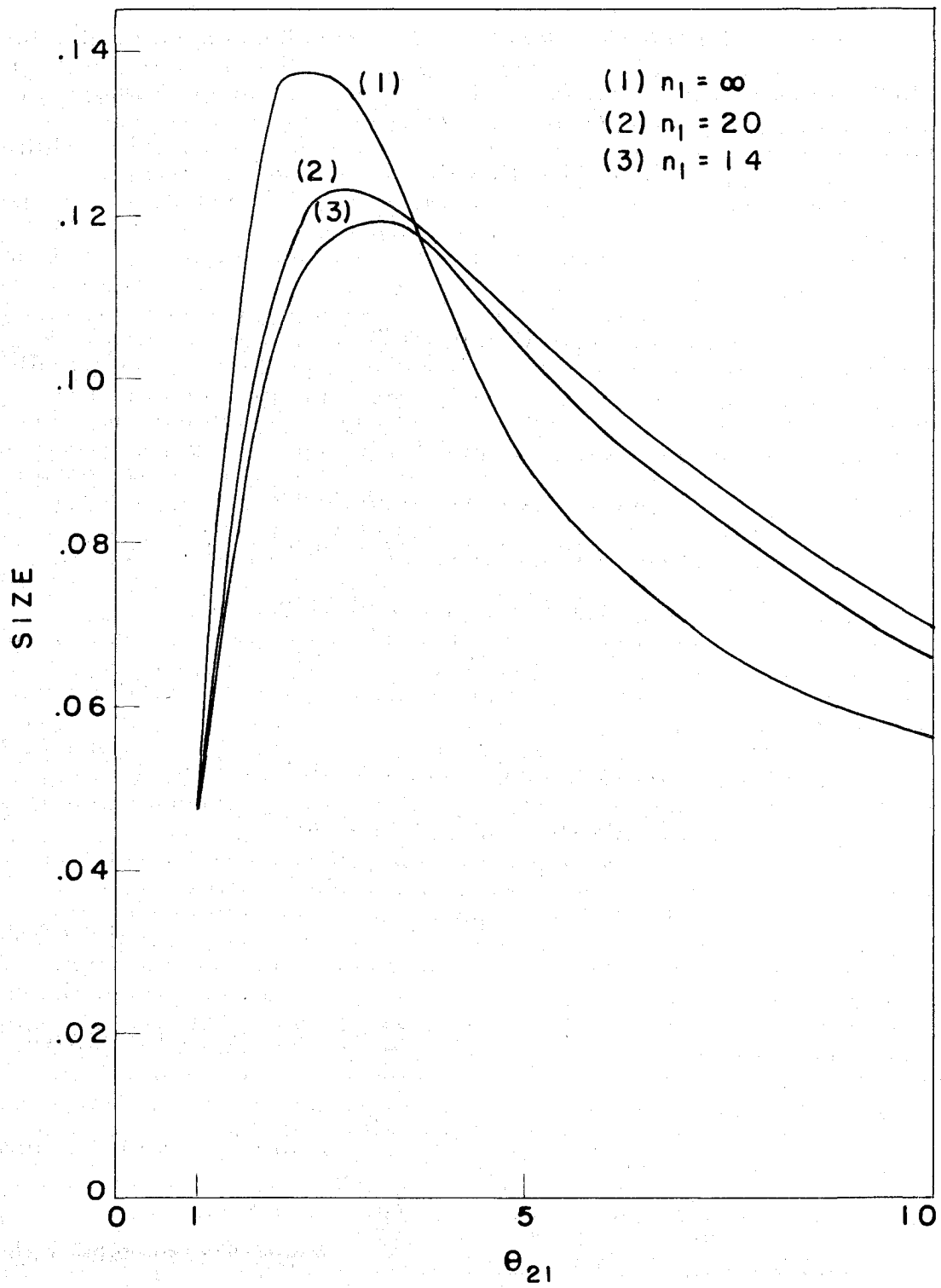


Figure 1. Size curves for $n_3 = 2$, $n_2 = 6$, $a_1 = a_2 = a_3 = .05$

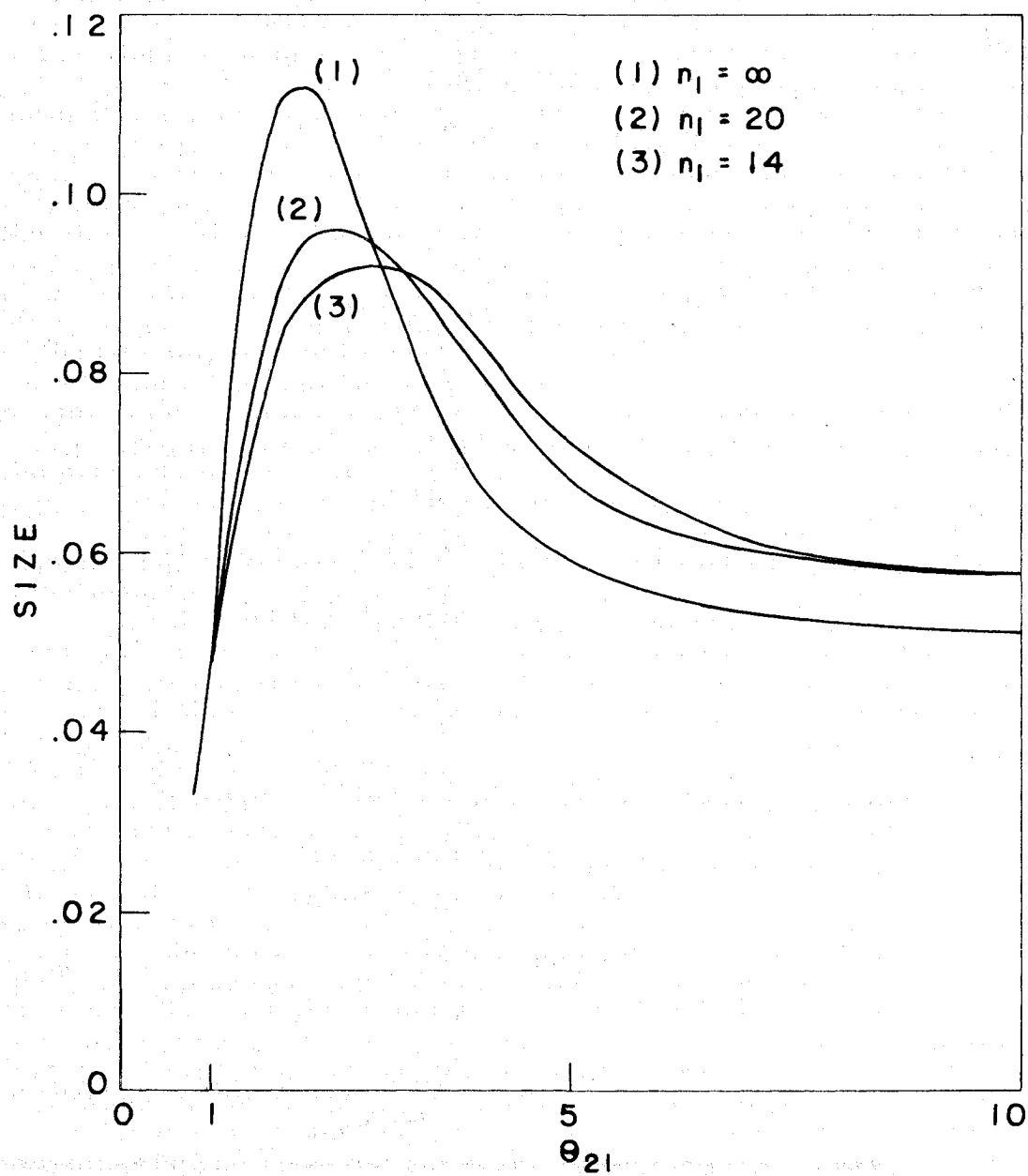


Figure 2. Size curves for $n_3 = 2$, $n_2 = 10$, $a_1 = a_2 = a_3 = .05$

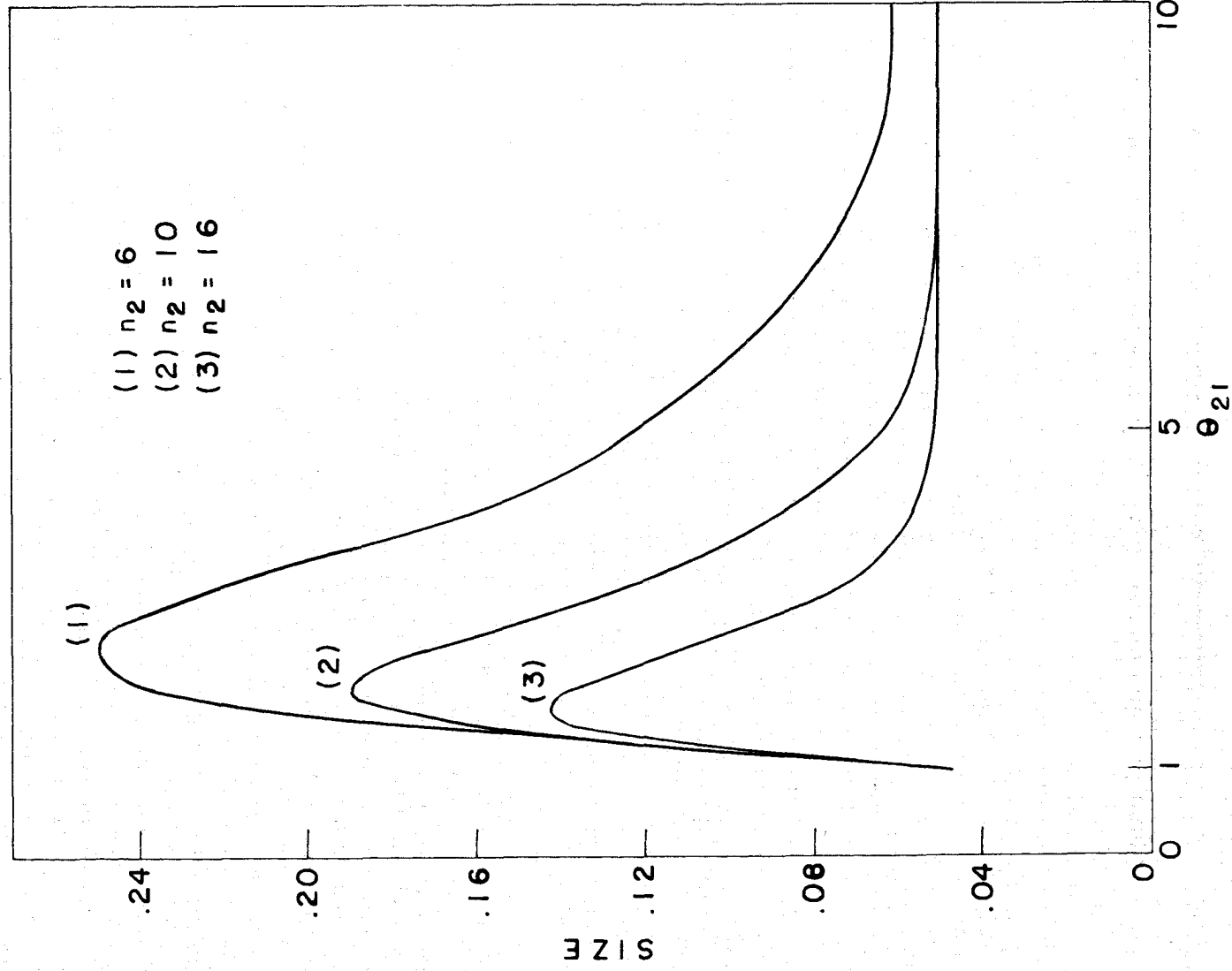


Figure 3. Size curves for $n_1 = \infty$, $n_3 = 6$, $\alpha_1 = \alpha_2 = \alpha_3 = .05$

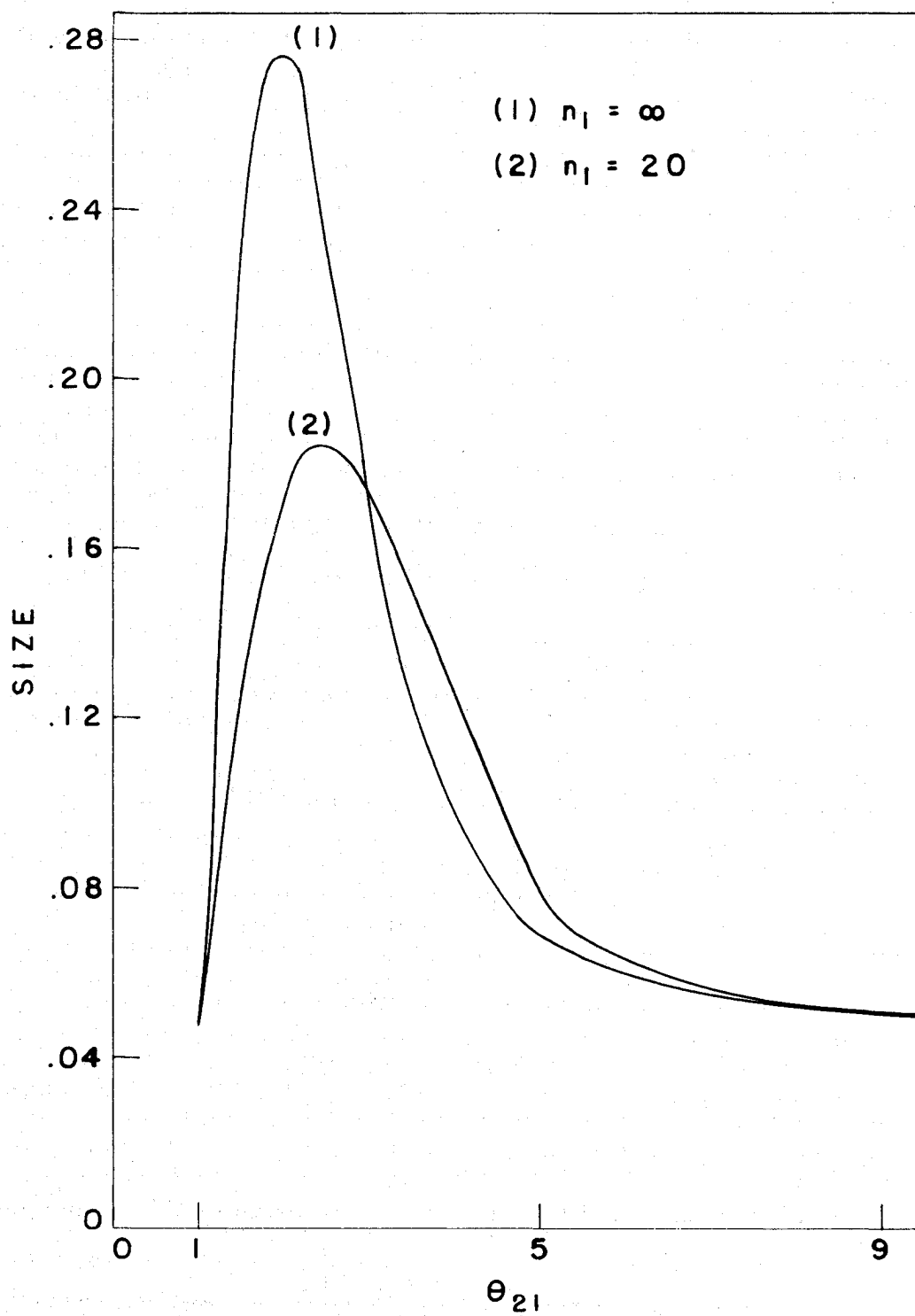


Figure 4. Size curves for $n_3 = 12$, $n_2 = 10$, $a_1 = a_2 = a_3 = .05$

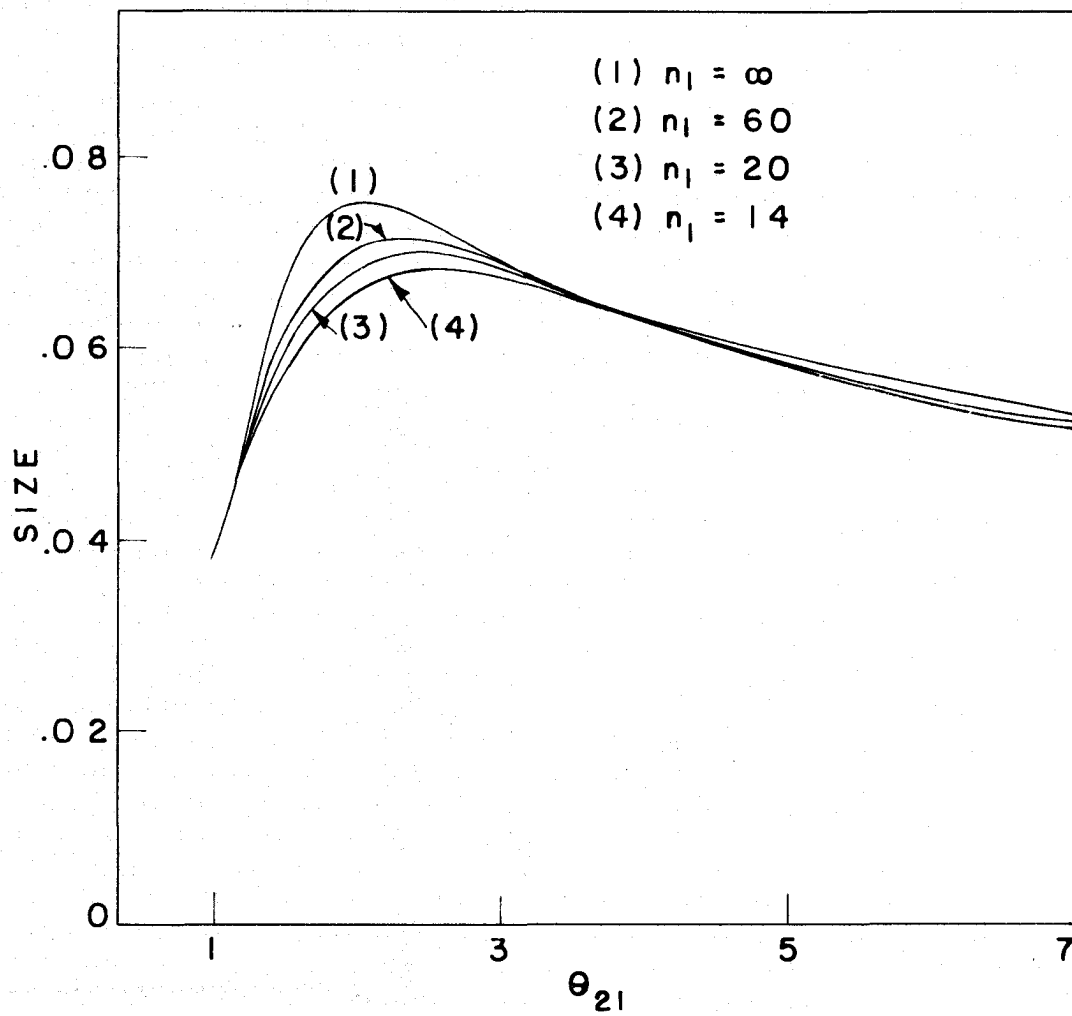


Figure 5. Size curves for $n_3 = 2$, $n_2 = 6$, $a_1 = .25$, $a_2 = a_3 = .05$

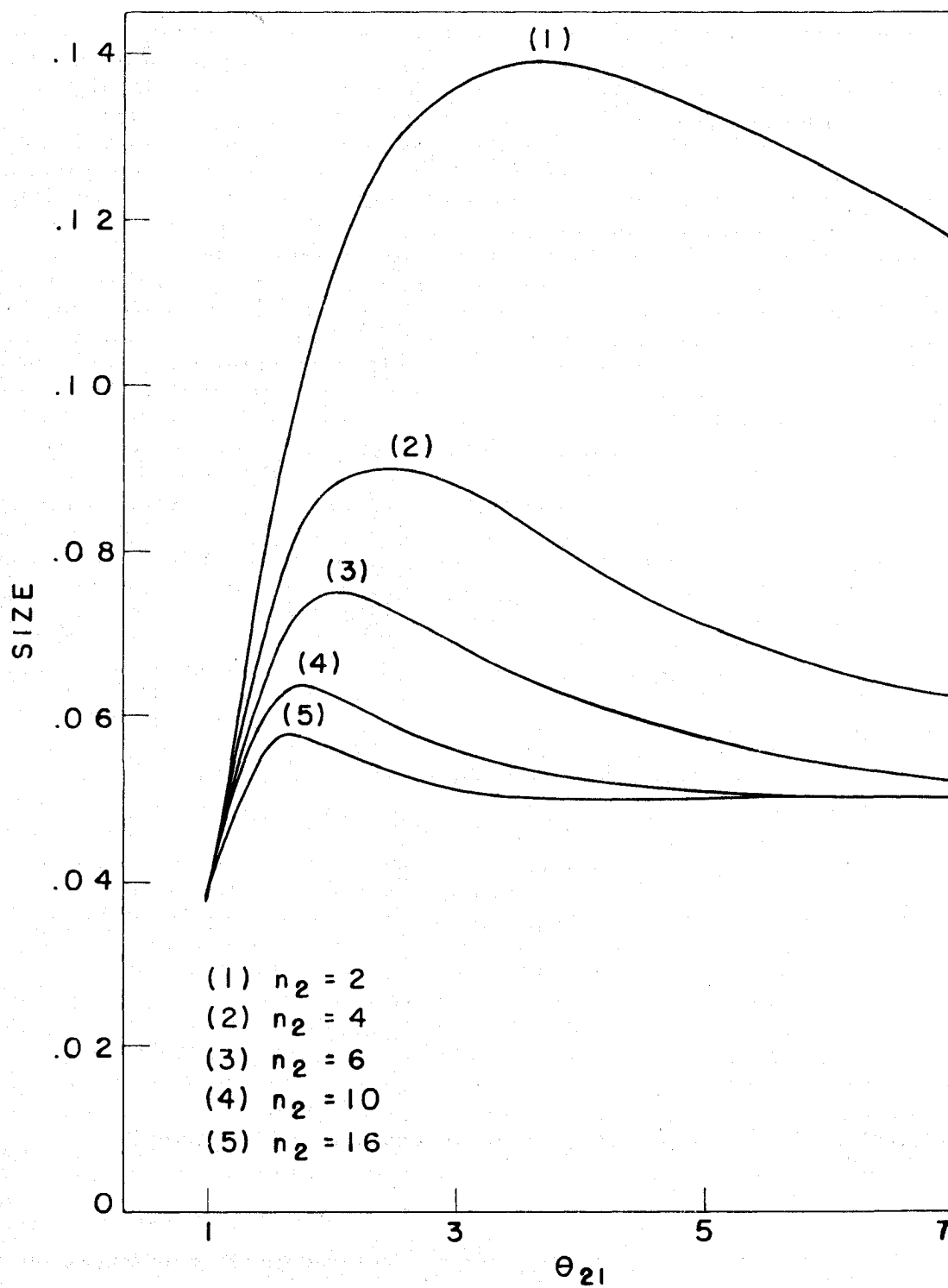


Figure 6. Size curves for $n_3 = 2$, $n_1 = \infty$, $a_1 = .25$, $a_2 = a_3 = .05$

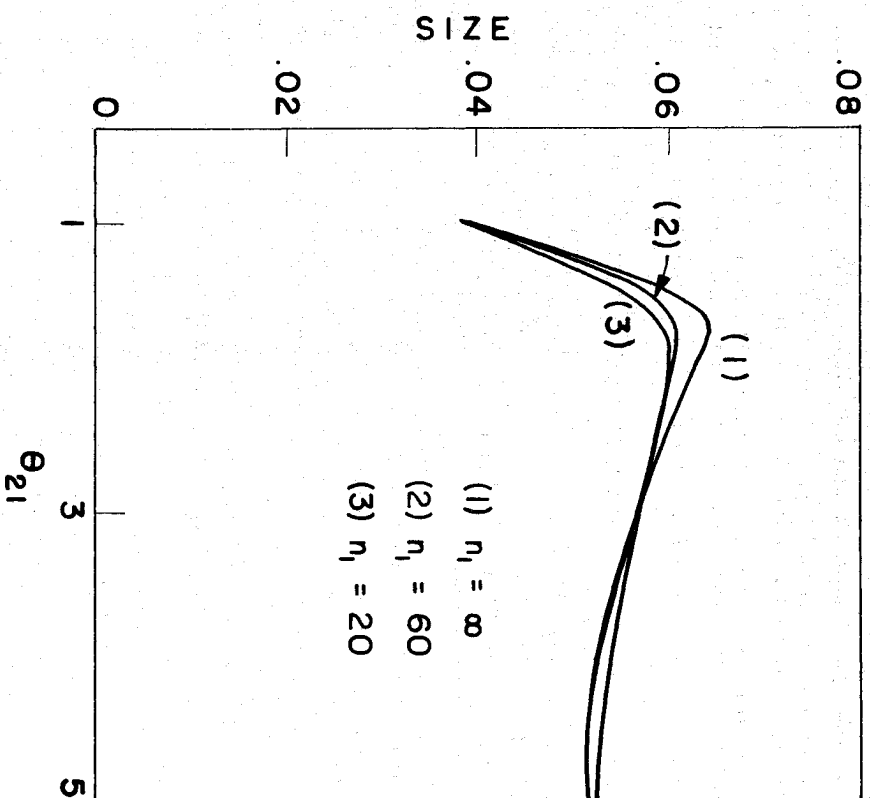


Figure 7. Size curves for $n_3 = 2$, $n_2 = 10$, $a_1 = .25$,
 $a_2 = a_3 = .05$

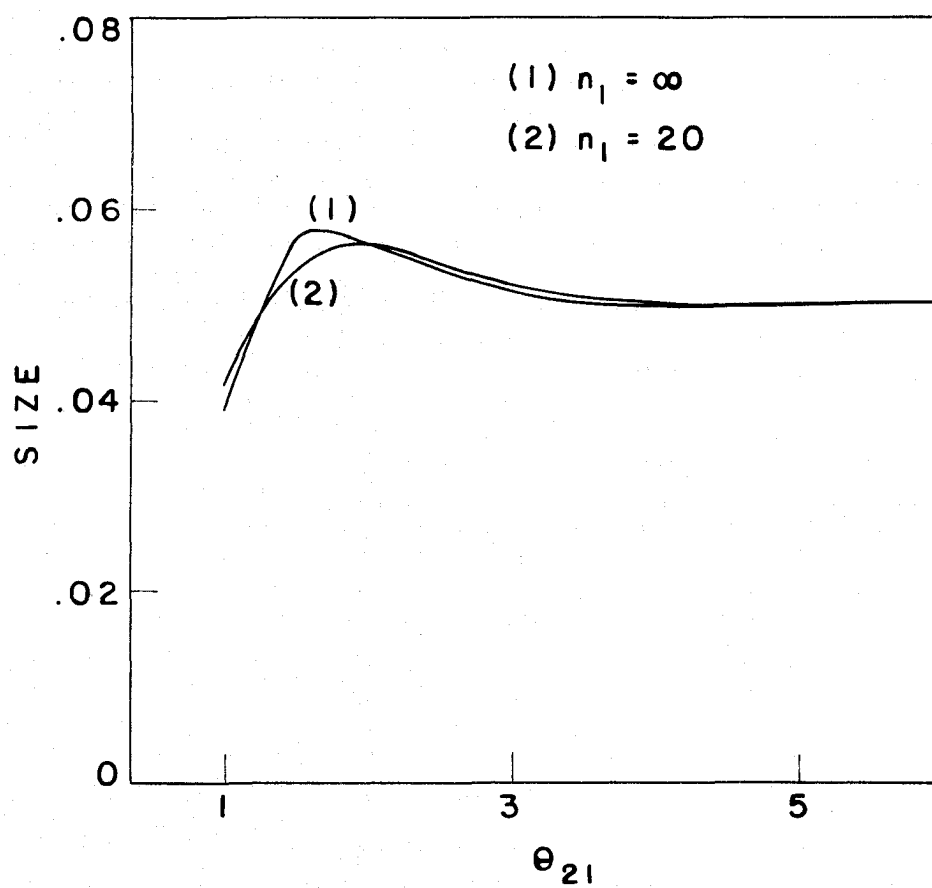


Figure 8. Size curves for $n_3 = 2$, $n_2 = 16$, $a_1 = .25$,

$$a_2 = a_3 = .05$$

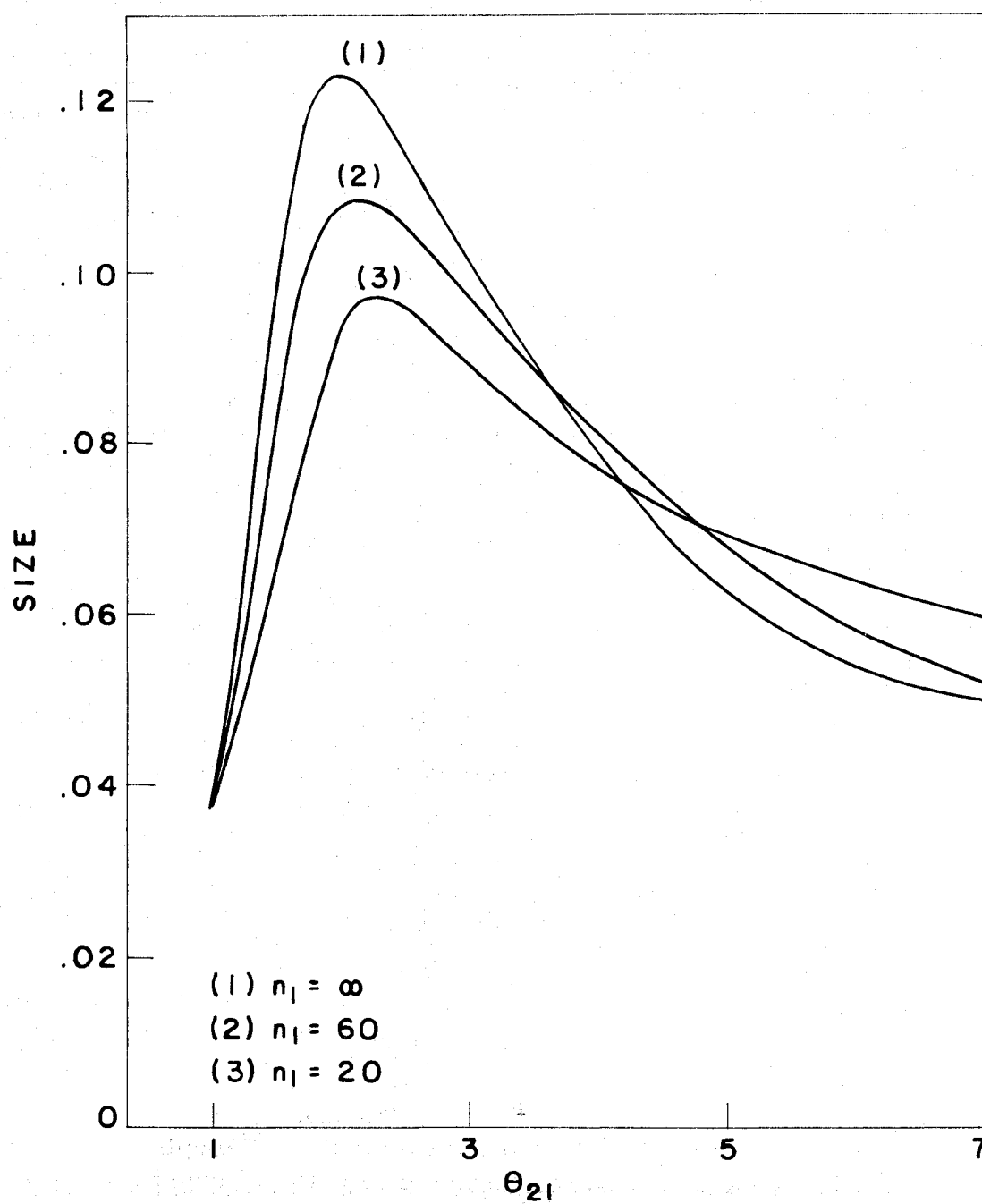


Figure 9. Size curves for $n_3 = 6$, $n_2 = 6$, $a_1 = .25$, $a_2 = a_3 = .05$

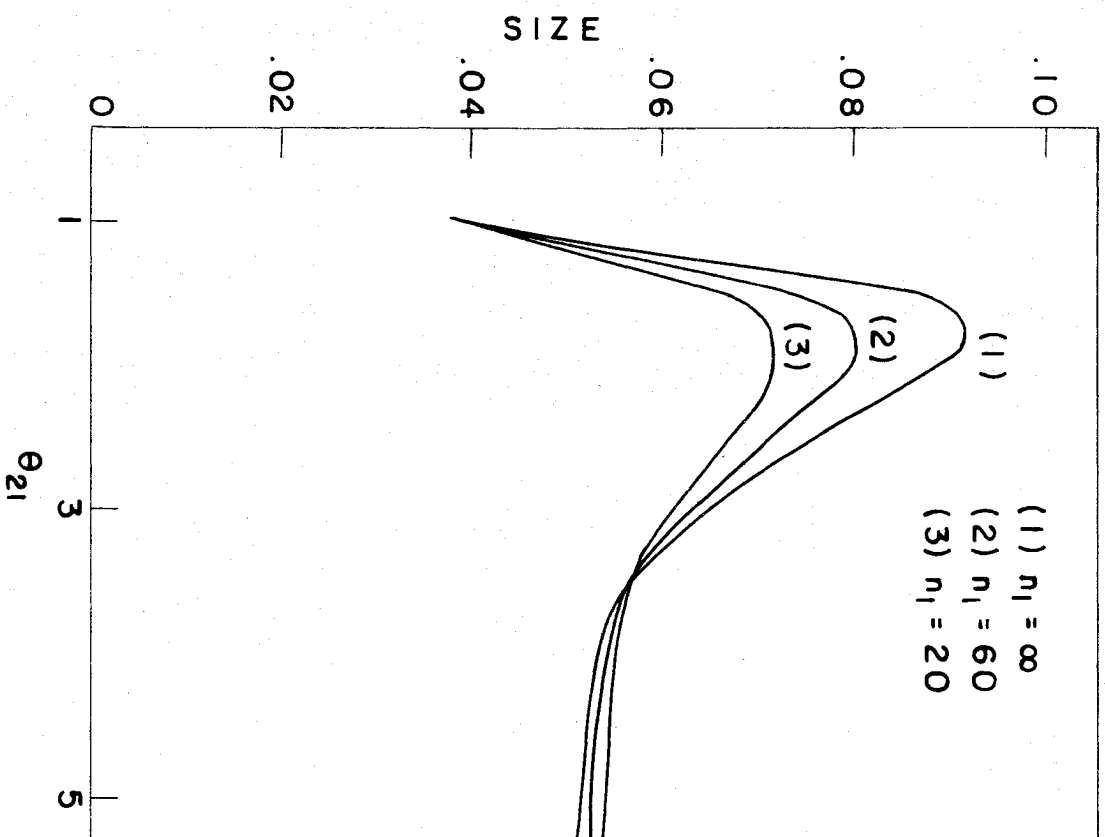


Figure 10. Size curves for $n_3 = 6$, $n_2 = 10$,
 $a_1 = .25$, $a_2 = a_3 = .05$

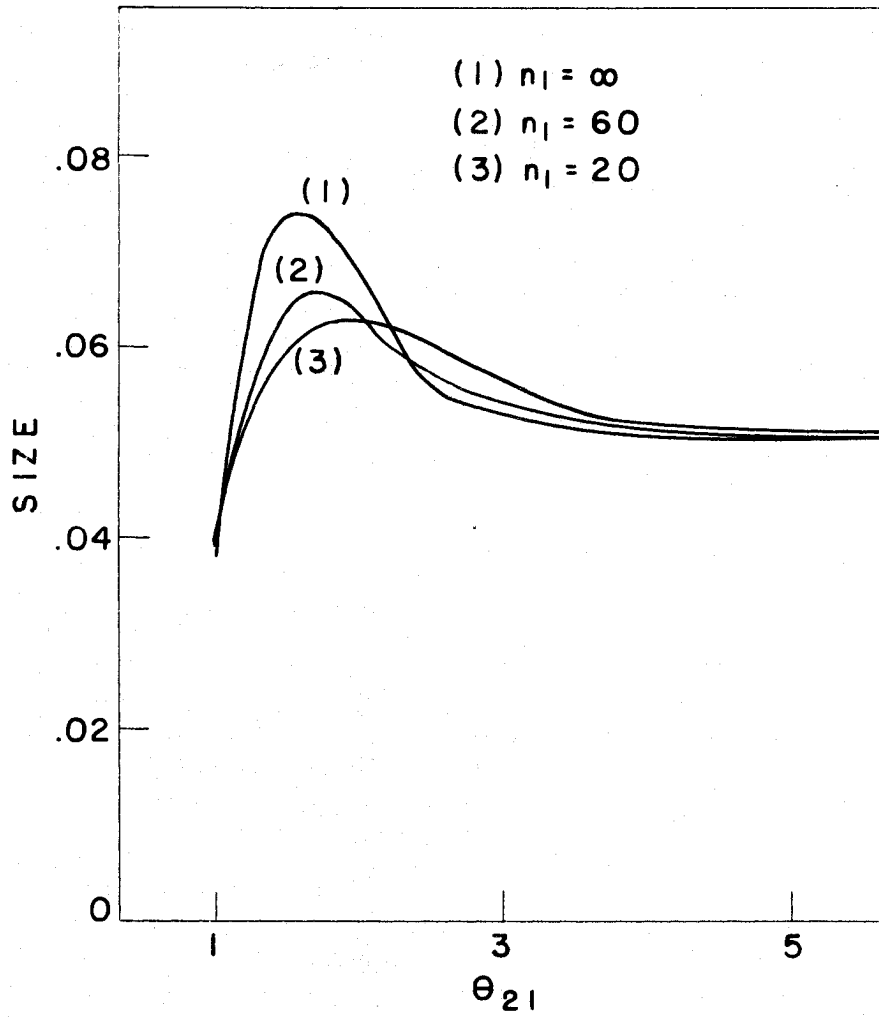


Figure 11. Size curves for $n_3 = 6$, $n_2 = 16$,
 $a_1 = .25$, $a_2 = a_3 = .05$

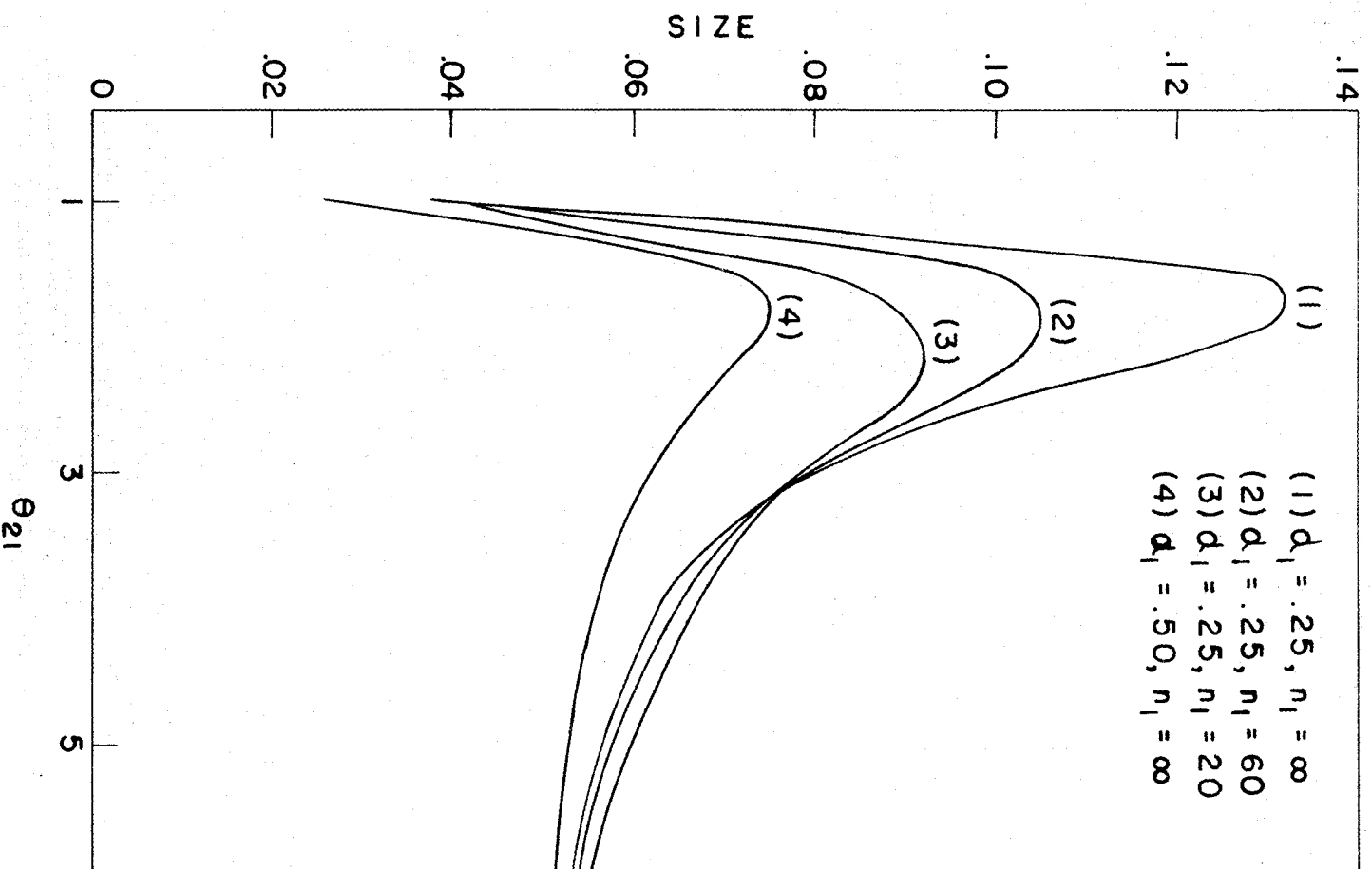


Figure 12. Size curves for $n_3 = 12$, $n_2 = 10$,
 $\alpha_2 = \alpha_3 = .05$

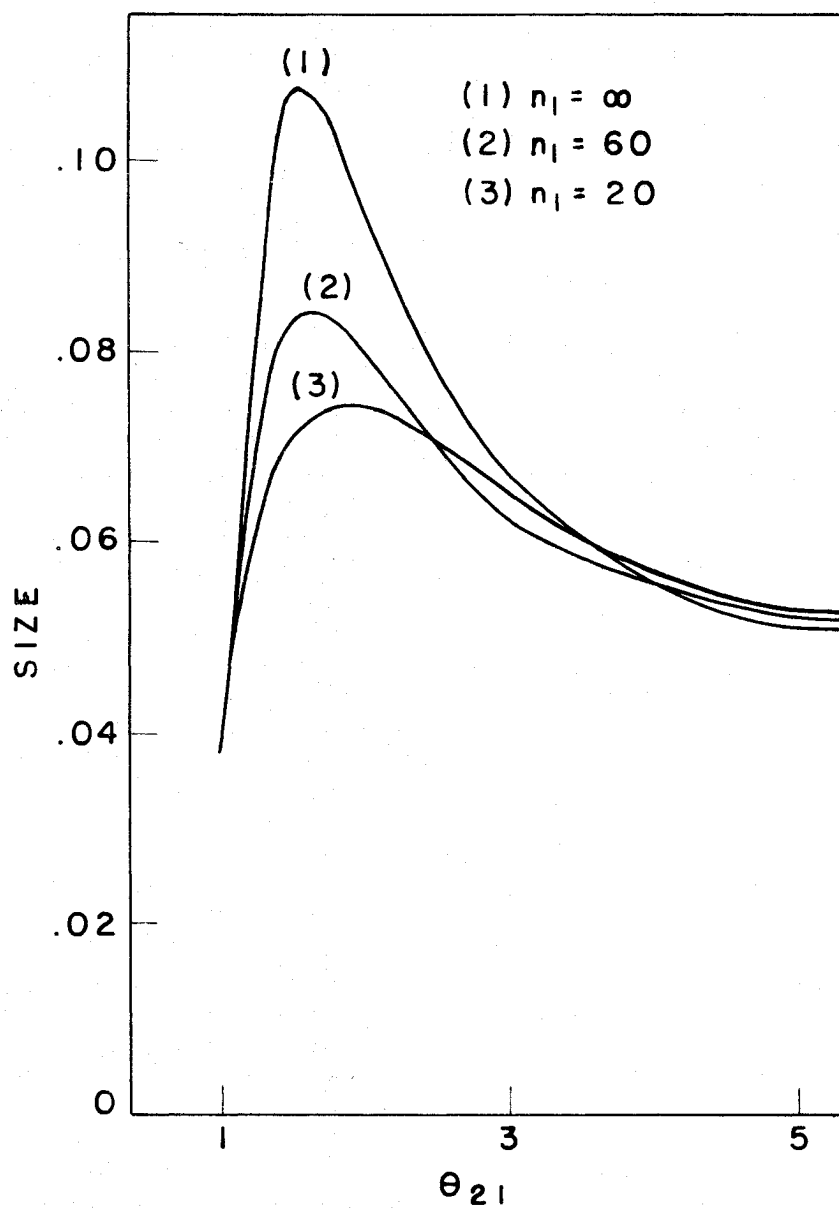


Figure 13. Size curves for $n_3 = 12$, $n_2 = 16$,
 $a_1 = .25$, $a_2 = a_3 = .05$

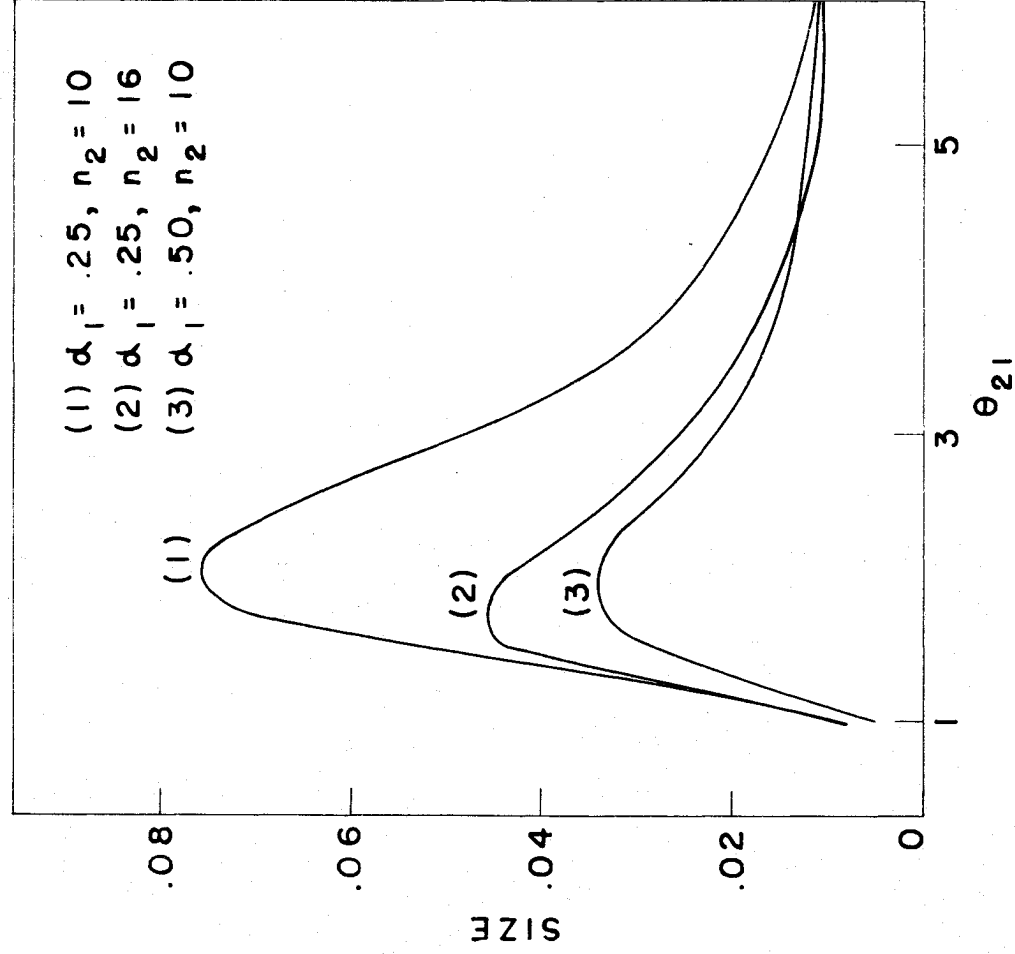


Figure 14. Size curves for $n_3 = 12, n_1 = \infty,$

$$\alpha_2 = \alpha_3 = .01$$

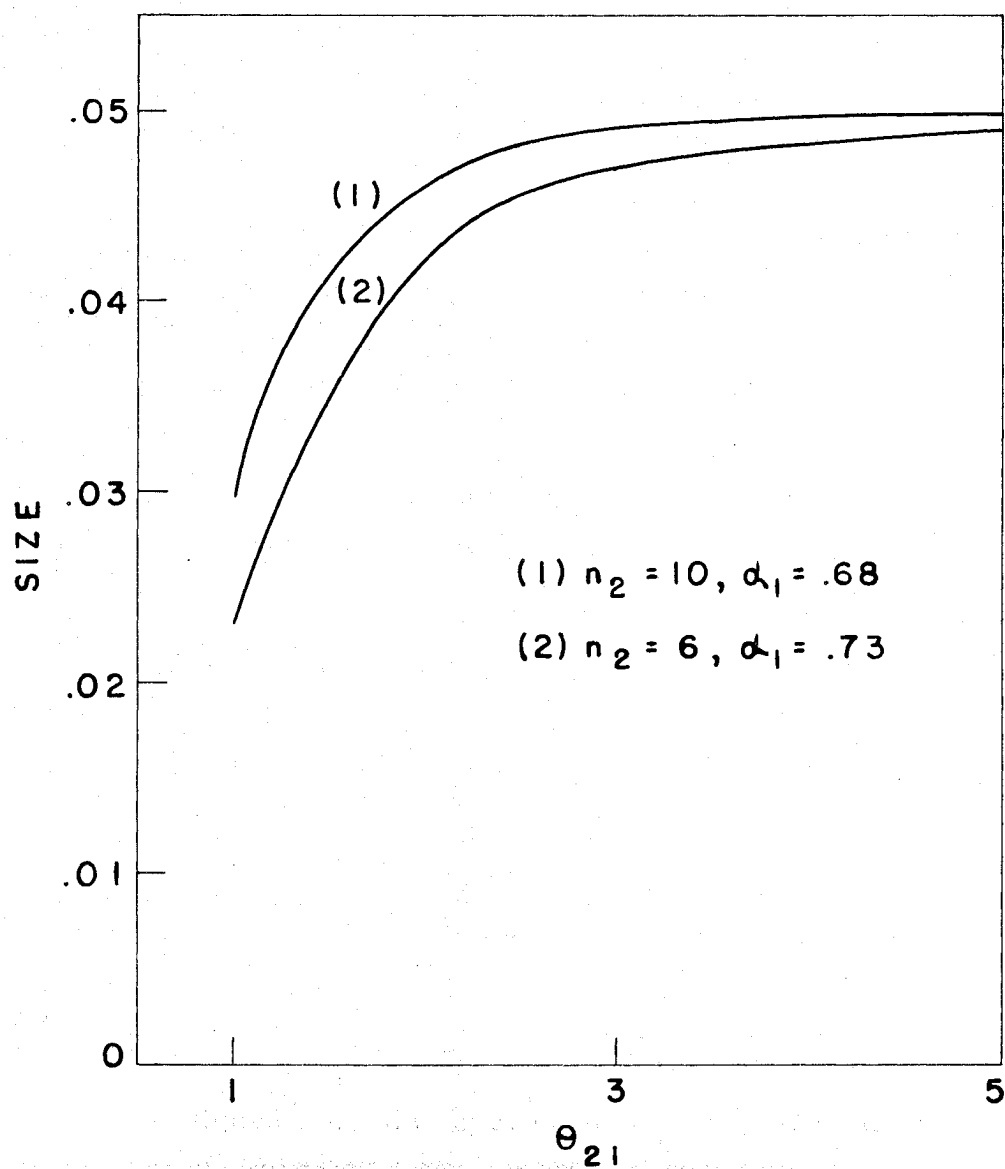


Figure 15. Size curves for $n_3 = 2, n_1 = 20$, for the borderline test, for $\alpha_2 = \alpha_3 = .05$

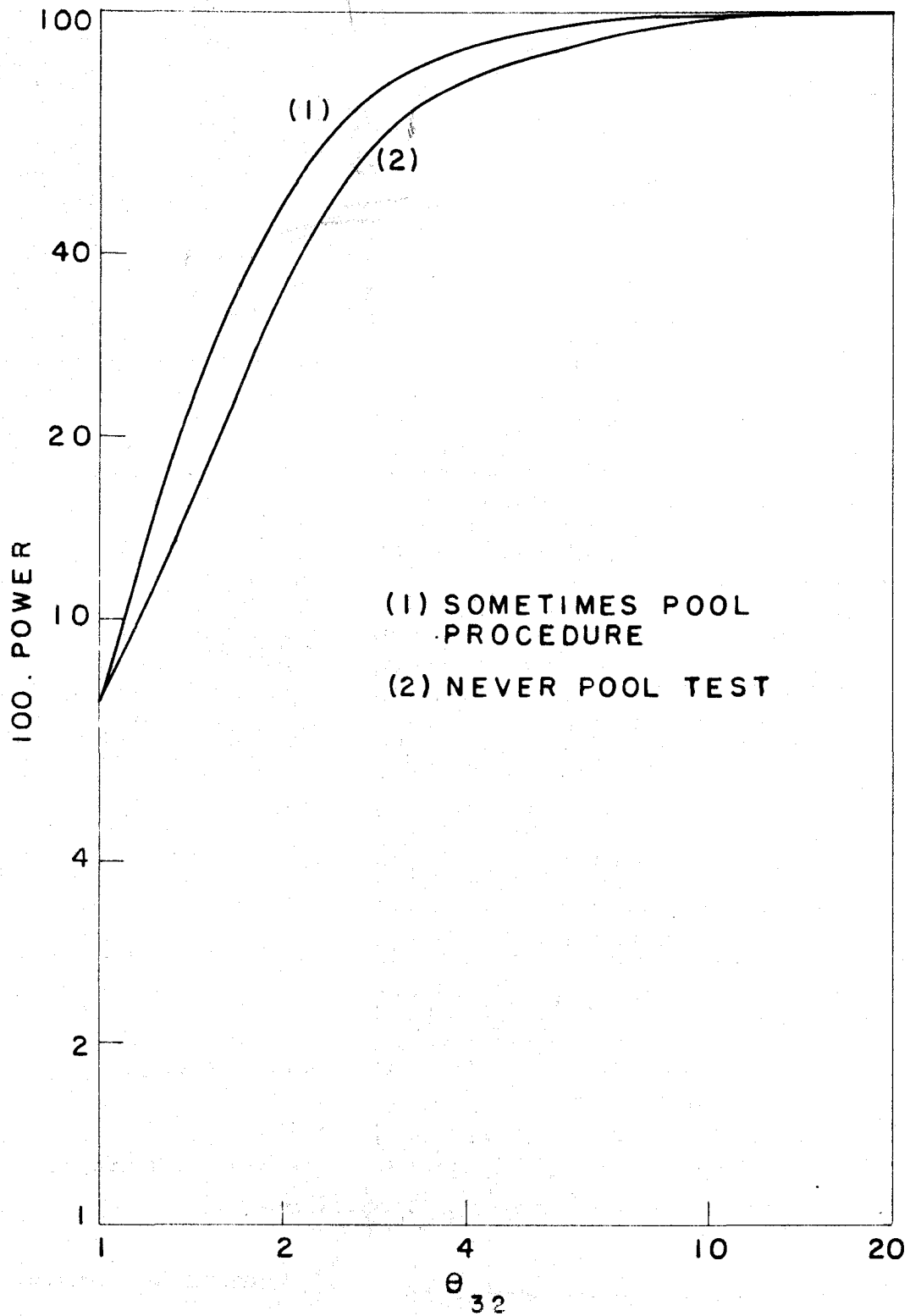


Figure 16. Power curves for $n_3 = 12$, $n_2 = 10$, $n_1 = 20$,
 $\alpha_1 = \alpha_2 = \alpha_3 = .05$, $\theta_{21} = 1.174$

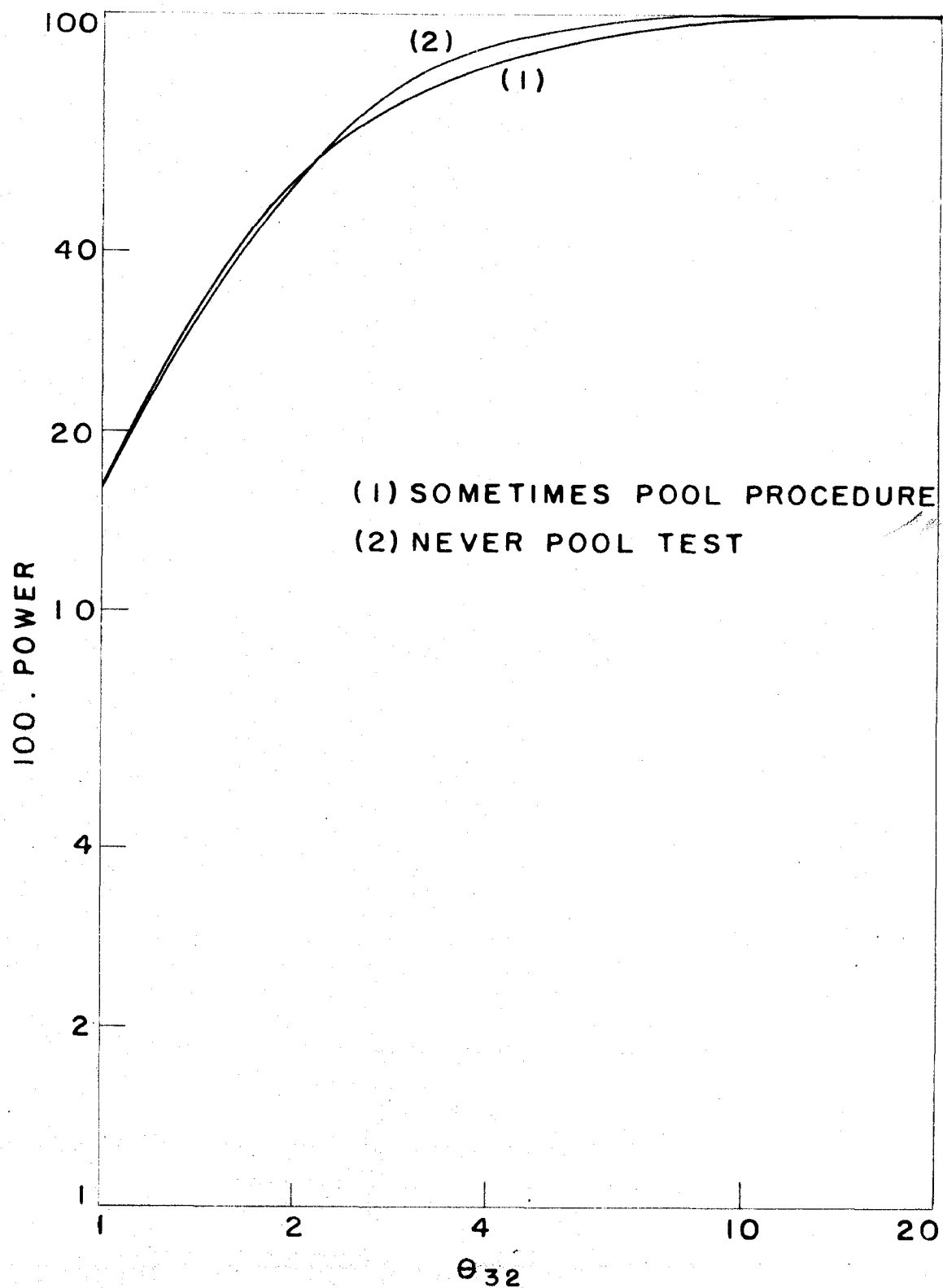


Figure 17. Power curves for $n_3 = 12$, $n_2 = 10$, $n_1 = 20$,
 $\alpha_1 = \alpha_2 = \alpha_3 = .05$, $\theta_{21} = 1.789$

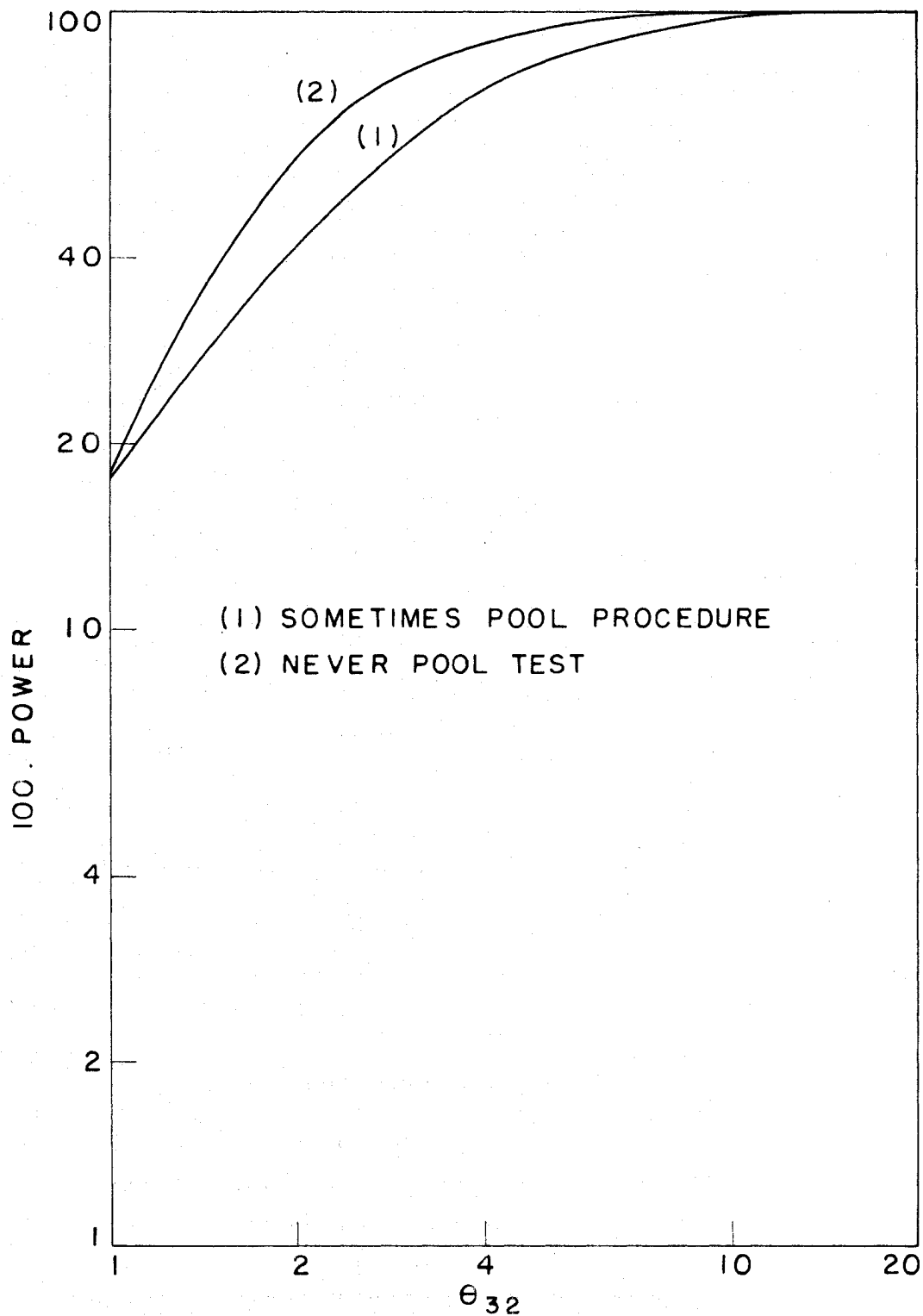


Figure 18. Power curves for $n_3 = 12$, $n_2 = 10$, $n_1 = 20$,
 $\alpha_1 = \alpha_2 = \alpha_3 = .05$, $\theta_{21} = 2.866$

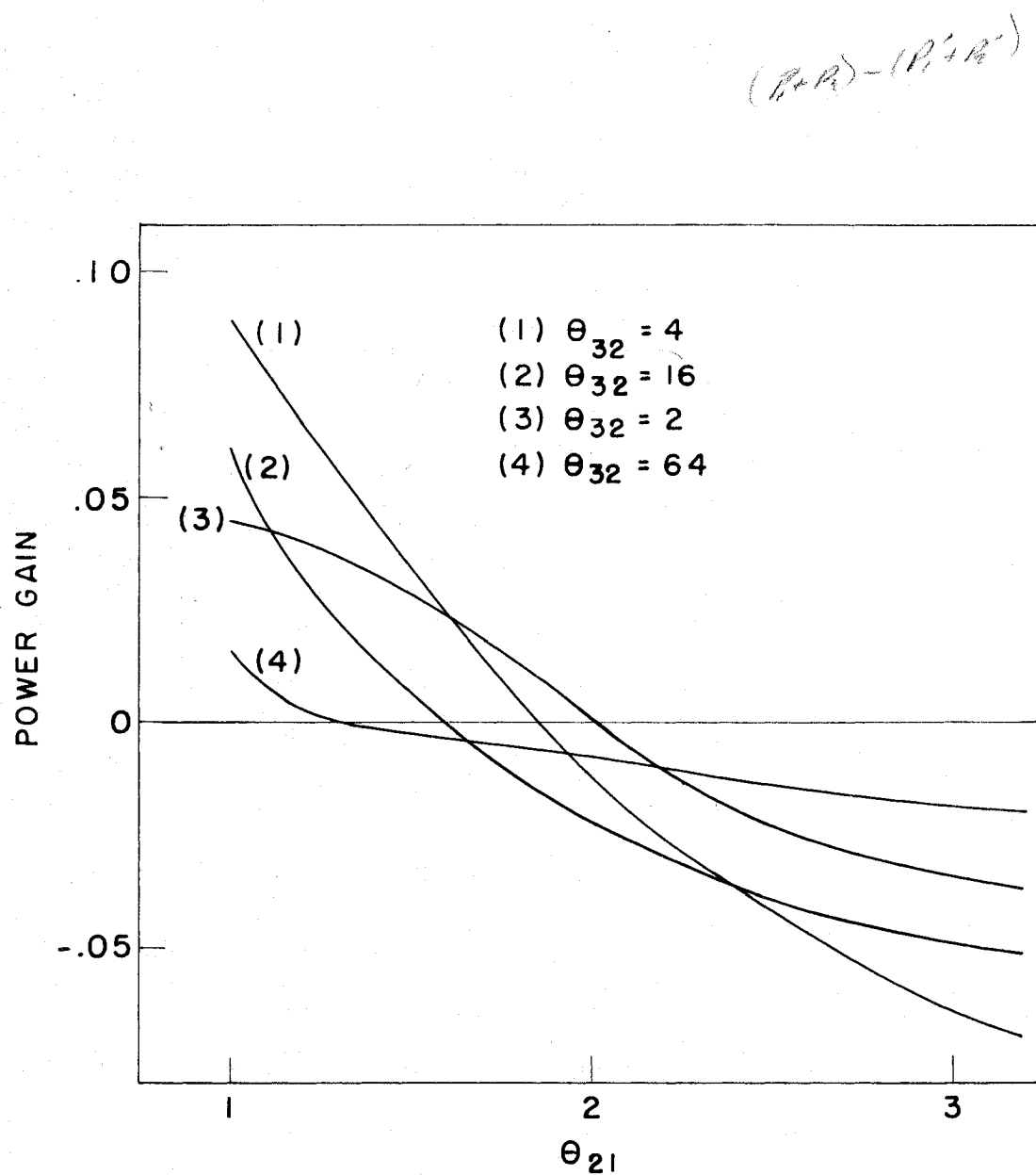


Figure 19. Power gain of the sometimes pool procedure over the never pool test of the same size for $n_1 = 20$, $n_3 = 2$, $n_2 = 6$, $\alpha_1 = \alpha_2 = \alpha_3 = .05$

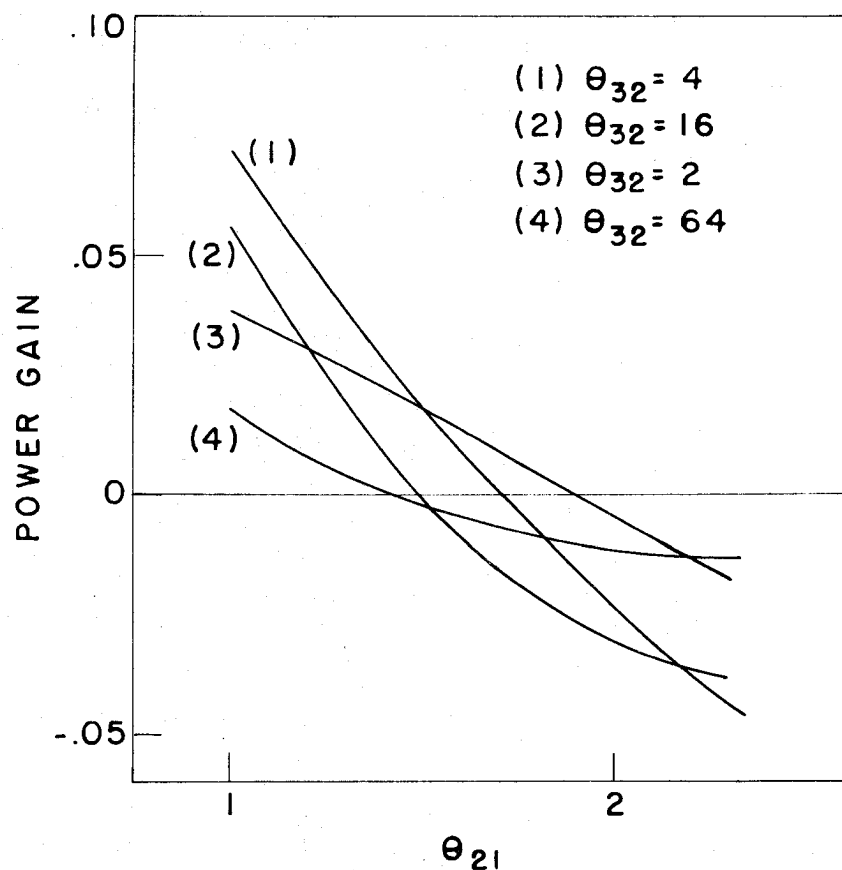


Figure 20. Power gain of the sometimes pool procedure over never pool test of the same size for $n_1 = 20$, $n_3 = 2$, $n_2 = 6$, $F_1 = 2 F_{.50}$, $\alpha_2 = \alpha_3 = .05$

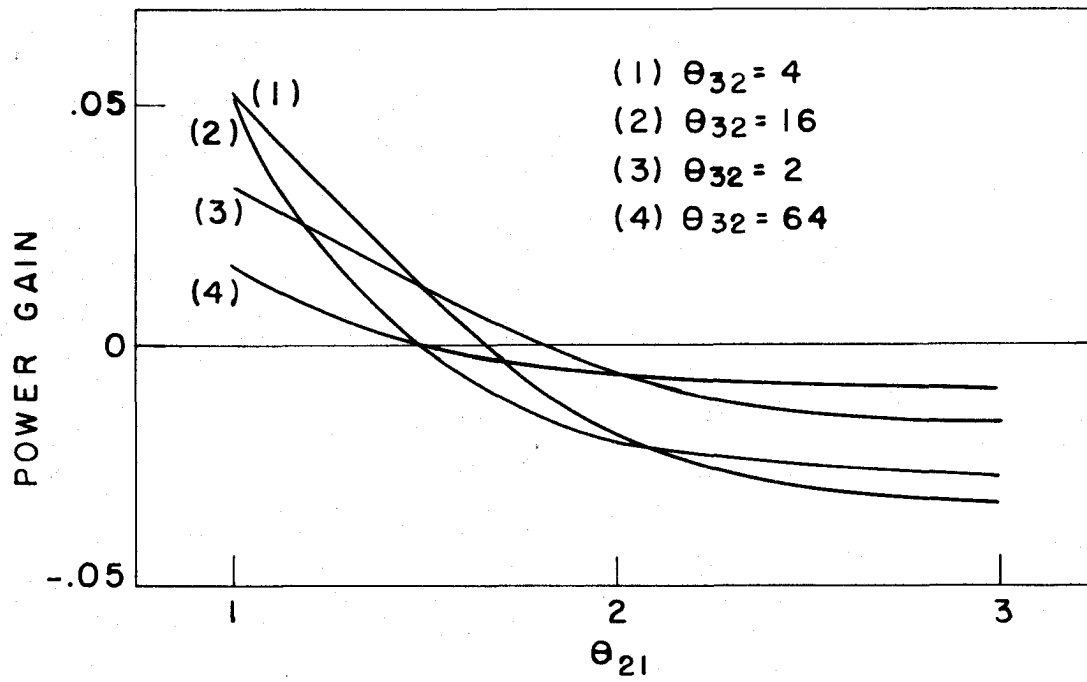


Figure 21. Power gain of the sometimes pool procedure over the never pool test of the same size for $n_1 = 20$, $n_3 = 2$, $n_2 = 6$, $\alpha_1 = .25$, $\alpha_2 = \alpha_3 = .05$

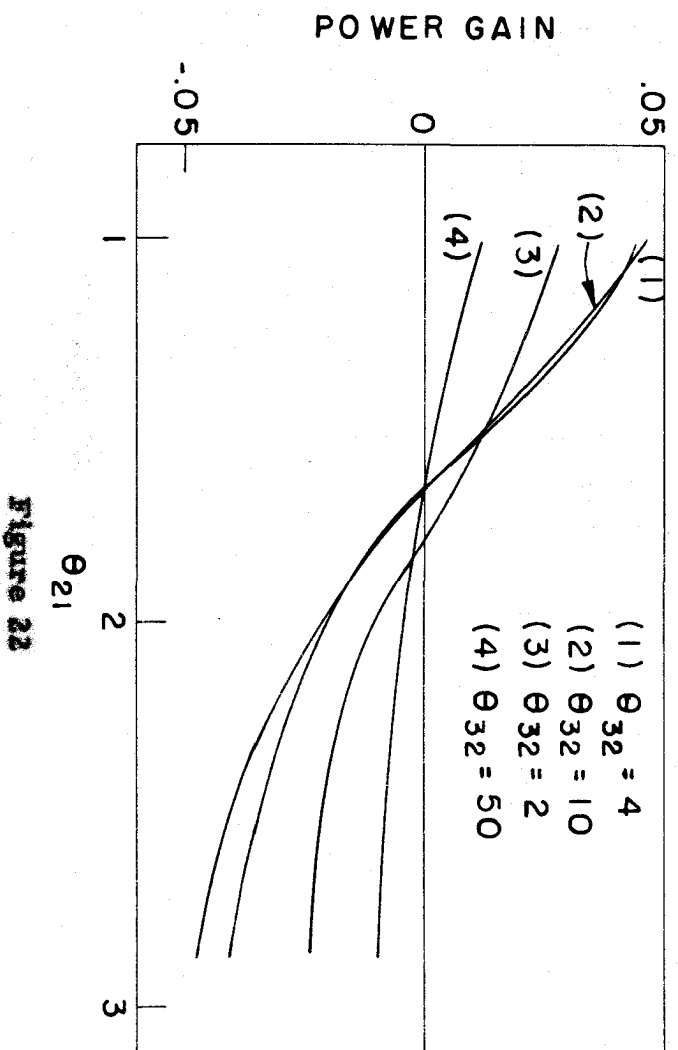


Figure 22

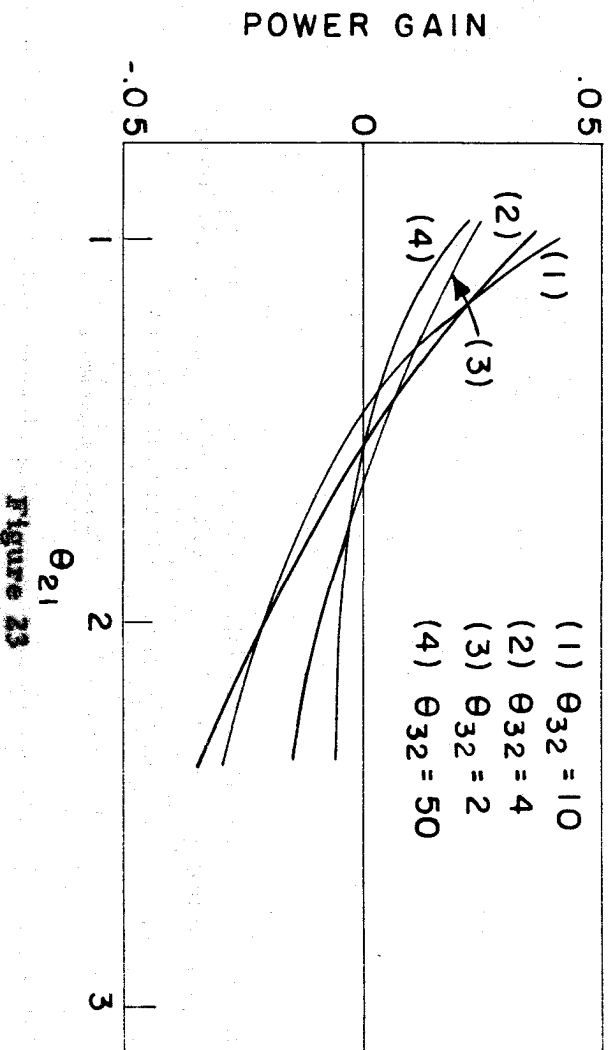


Figure 23

Figure 22. Power gain of the sometimes pool procedure over the never pool test of the same size for $n_1 = 20$, $n_3 = 2$,

$n_2 = 10$, $\alpha_1 = \alpha_2 = \alpha_3 = .05$

Figure 23. Power gain of the sometimes pool procedure over the never pool test of the same size for $n_1 = 20$, $n_3 = 2$,

$n_2 = 10$, $F_1 = 2 F_{.50}$, $\alpha_2 = \alpha_3 = .05$

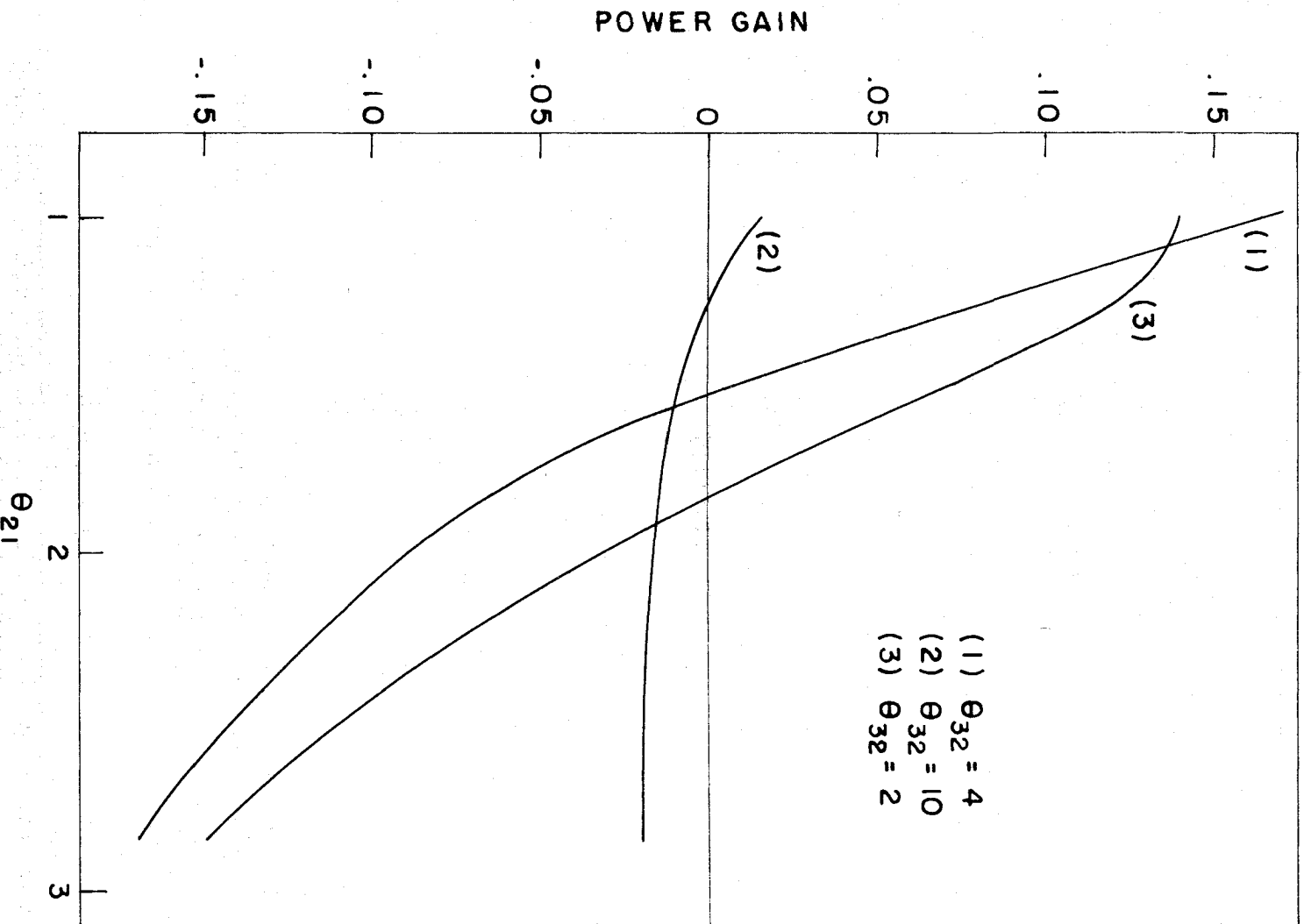


Figure 24. Power gain of the sometimes pool procedure over the never pool test of the same size for $n_1 = 20$, $n_3 = 12$, $n_2 = 10$, $\alpha_1 = \alpha_2 = \alpha_3 = .05$

Table 8. Master table for P_1 , for $n_1 = 20$, $n_2 = 10$, $n_3 = 12$

x	d				
	0	.1	.2	.3	.5
.32*				.0104	.0149
.38	.0117	.0184	.0268	.0353	.0480
.44	.0361	.0529	.0721	.0909	.1166
.50	.0898	.1224	.1572	.1893	.2290
.56	.1868	.2380	.2886	.3320	.3790
.62	.3334	.3986	.4581	.5052	.5452
.68	.5187	.5853	.6413	.6809	.6985
.74	.7116	.7643	.8045	.8280	.8148
.80	.8702	.8998	.9197	.9256	.8846
.86	.9641	.9739	.9784	.9724	.9150
	.7	1.0	1.3	2.0	3.5
.20*	.0005	.0004	.0004	.0002	.0000
.32	.0166	.0154	.0125	.0066	.0017
.38	.0519	.0470	.0378	.0196	.0110
.44	.1226	.1085	.0860	.0441	.0197
.50	.2333	.2014	.1575	.0794	.0293
.56	.3738	.3143	.2422	.1200	.0380
.62	.5207	.4264	.3237	.1577	.0443
.68	.6479	.5177	.3877	.1859	.0478
.74	.7379	.5778	.4279	.2026	.0493
.80	.7880	.6088	.4476	.2102	.0498
.86*	.8082				

* These were additional values subsequently computed to facilitate interpolation where necessary.

Table 9. Master table for $P'_2 = \log_{10} (P_2 \cdot 10^7)$, for
 $n_1 = 20$, $n_2 = 10$, $n_3 = 2$ and $a = 1$

x_1	x_2	P'_2	x_2	P'_2	x_2	P'_2
.50	.38	1.9685	.53	2.6010	.68	3.7538
.60	.40	3.2227	.52	3.7339	.64	4.5595
.70	.40	4.2269	.50	4.6358	.60	5.2420
.80	.43	5.2036	.50	5.5054	.57	5.9042
.90	.46	6.0903	.50	6.2728	.54	6.4864
.95	.475	6.5072	.50	6.6250	.52	6.7275
.99	.495	6.9038	.503	6.9085		
.9999	.49995	6.9582	.50004	6.9587		

Table 10. Probability of pooling, $\alpha_1 = .25$ and $\alpha_1 = .50$

		θ_{21}			
n_2	n_1	1	1.5	2	3
<hr/>					
$\alpha_1 = .25$					
2	∞	.750	.603	.500	.426
4	∞	.750	.536	.390	.227
6	20	.750	.522	.360	.184
6	∞	.750	.485	.312	.144
10	20	.750	.475	.286	.108
10	∞	.750	.407	.208	.061
16	20	.750	.432	.224	.058
16	∞	.750	.321	.117	.018
<hr/>					
$\alpha_1 = .50$					
10	∞	.50	.204	.088	.021

Table 11. Probability of pooling for the borderline test

n_3	n_2	n_1	θ_{21}			
			1	1.5	2	2.5
2	2	20	.152	.105	.080	.064
2	4	20	.225	.124	.078	.054
2	6	8	.277	.142	.081	.050
2	6	20	.268	.127	.069	.041
2	10	14	.320	.132	.060	.030
2	10	20	.317	.124	.054	.026
12	10	14	.238	.088	.037	.018
12	10	20	.228	.080	.032	.015

Table 12. Minimum size values for the borderline test

n_3	n_2	n_1	Minimum size
2	2	20	.010
2	2	60	.009
2	4	20	.018
2	6	14	.024
2	6	20	.023
2	10	14	.030
2	10	20	.030
6	6	20	.017
6	10	20	.023
6	16	20	.029
12	16	20	.030

Table 13. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 8$, $n_2 = 6$, $n_3 = 2$, $\alpha_1 = \alpha_2 = \alpha_3 = .05$

θ_{21}	Test	θ_{32}				
		1	2	4	16	64
0.940*	s.p.	.044	.175	.393	.782	.938
	n.p.	.044	.143	.324	.723	.919
1.295	s.p.	.066	.217	.439	.799	.943
	n.p.	.066	.191	.391	.768	.934
1.834	s.p.	.089	.249	.463	.803	.945
	n.p.	.089	.235	.444	.799	.944
4.093	s.p.	.109	.248	.427	.773	.934
	n.p.	.109	.270	.484	.820	.950
11.185	s.p.	.070	.175	.356	.741	.925
	n.p.	.070	.200	.402	.775	.936
	n.p.	.050	.156	.343	.737	.924

*In Tables 13-31, values of $\theta_{21} < 1$ have been inserted for interpolation purposes. These values do not of course arise in the analysis of variance models considered. For the same tables, s.p. represents the sometimes pool procedure and n.p. the never pool test.

Table 14. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 14$, $n_2 = 6$, $n_3 = 2$, $\alpha_1 = \alpha_2 = \alpha_3 = .05$

θ_{21}	Test	θ_{32}				
		1	2	4	16	64
0.834	s.p.	.034	.159	.378	.774	.937
	n.p.	.034	.118	.285	.693	.908
1.220	s.p.	.065	.225	.449	.803	.945
	n.p.	.065	.189	.387	.766	.933
1.860	s.p.	.102	.269	.475	.805	.945
	n.p.	.102	.258	.470	.813	.948
5.083	s.p.	.102	.215	.388	.753	.929
	n.p.	.102	.259	.472	.814	.949
	n.p.	.050	.156	.343	.737	.924

Table 15. The power of the sometimes pool procedure
and the never pool test of the same size,
for $n_1=14$, $n_2=10$, $n_3=2$, $\alpha_1 = \alpha_2 = \alpha_3 = .05$

θ_{21}	Test	θ_{32}				
		1	2	4	10	50
0.896	s.p.	.041	.179	.414	.695	.928
	n.p.	.041	.158	.365	.652	.915
1.269	s.p.	.065	.229	.460	.723	.937
	n.p.	.065	.212	.434	.704	.931
1.859	s.p.	.086	.257	.477	.729	.938
	n.p.	.086	.253	.480	.736	.939
4.538	s.p.	.077	.213	.418	.690	.928
	n.p.	.077	.236	.461	.723	.936
	n.p.	.050	.179	.393	.674	.922

Table 16. The power of the sometimes pool procedure
and the never pool test of the same size,
for $n_1=20$, $n_2=6$, $n_3=2$, $\alpha_1=\alpha_2=\alpha_3=.05$

θ_{21}	Test	θ_{32}				
		1	2	4	16	64
0.780	s.p.	.0278	.147	.366	.768	.935
	n.p.	.0278	.101	.256	.668	.900
1.188	s.p.	.064	.227	.453	.806	.947
	n.p.	.064	.187	.386	.765	.933
1.904	s.p.	.109	.278	.480	.804	.944
	n.p.	.109	.271	.485	.821	.951
3.247	s.p.	.122	.255	.429	.771	.933
	n.p.	.122	.292	.508	.832	.954
6.016	s.p.	.087	.192	.367	.744	.928
	n.p.	.087	.233	.442	.780	.944
	n.p.	.050	.156	.343	.737	.924

Table 17. The power of the sometimes pool procedure
and the never pool test of the same size,
for $n_1=20$, $n_2=10$, $n_3=2$, $\alpha_1=\alpha_2=\alpha_3=.05$

θ_{21}	Test	θ_{32}				
		1	2	4	10	50
1.174	s.p.	.061	.228	.459	.733	.946
	n.p.	.061	.205	.426	.698	.929
1.789	s.p.	.091	.261	.480	.731	.938
	n.p.	.091	.262	.489	.742	.941
2.866	s.p.	.094	.243	.448	.706	.932
	n.p.	.094	.267	.495	.746	.942
9.058	s.p.	.057	.180	.394	.675	.924
	n.p.	.057	.197	.416	.691	.927
	n.p.	.050	.179	.393	.674	.922

Table 18. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 14$, $n_2 = 10$, $n_3 = 12$, $\alpha_1 = \alpha_2 = \alpha_3 = .05$

θ_{21}	Test	θ_{32}				
		1	2	4	10	50
0.896	s. p.	.037	.347	.816	.989	1.000
	n. p.	.037	.229	.643	.966	1.000
1.269	s. p.	.078	.469	.850	.987	1.000
	n. p.	.078	.368	.783	.987	1.000
1.859	s. p.	.135	.506	.826	.982	1.000
	n. p.	.135	.496	.867	.994	1.000
4.538	s. p.	.124	.335	.708	.976	1.000
	n. p.	.124	.476	.855	.993	1.000
	n. p.	.050	.280	.702	.976	1.000

Table 19. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 20$, $n_2 = 10$, $n_3 = 12$, $\alpha_1 = \alpha_2 = \alpha_3 = .05$

θ_{21}	Test	θ_{32}				
		1	2	4	10	50
0.802	s.p.	.023	.314	.817	.985	1.000
	n.p.	.023	.168	.555	.946	1.000
1.174	s.p.	.072	.479	.870	.987	1.000
	n.p.	.072	.350	.769	.985	1.000
1.789	s.p.	.152	.535	.824	.982	1.000
	n.p.	.152	.529	.883	.995	1.000
2.866	s.p.	.178	.421	.734	.976	1.000
	n.p.	.178	.572	.903	.996	1.000
	n.p.	.050	.280	.702	.976	1.000

Table 20. The power of the sometimes pool procedure
and the never pool test of the same size,
for $n_1 = 8$, $n_2 = 6$, $n_3 = 2$, $F_1 = 2F_{.50}$, $\alpha_2 = \alpha_3 = .05$

θ_{21}	Test	θ_{32}				
		1	2	4	6	64
0.995	s.p.	.041	.162	.369	.759	.932
	n.p.	.041	.137	.315	.717	.917
1.457	s.p.	.059	.191	.394	.766	.933
	n.p.	.059	.176	.371	.756	.930
2.220	s.p.	.072	.200	.392	.759	.930
	n.p.	.072	.202	.405	.776	.937
3.556	s.p.	.062	.169	.352	.740	.924
	n.p.	.062	.183	.380	.761	.932
	n.p.	.050	.156	.343	.737	.924

Table 21. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 14$, $n_2 = 6$, $n_3 = 2$, $F_1 = 2F_{.50}$, $\alpha_2 = \alpha_3 = .05$

θ_{21}	Test	θ_{32}				
		1	2	4	16	64
0.802	s.p.	.030	.142	.352	.755	.930
	n.p.	.030	.108	.278	.679	.903
1.222	s.p.	.056	.195	.403	.775	.949
	n.p.	.056	.170	.363	.750	.928
1.958	s.p.	.080	.216	.407	.767	.932
	n.p.	.080	.218	.424	.788	.940
6.189	s.p.	.064	.166	.350	.739	.924
	n.p.	.064	.186	.384	.764	.933
	n.p.	.050	.156	.343	.737	.924

Table 22. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 14$, $n_2 = 10$, $n_3 = 2$, $F_1 = 2F_{.50}$, $\alpha_2 = \alpha_3 = .05$

θ_{21}	Test	θ_{32}				
		1	2	4	10	50
0.957	s.p.	.043	.176	.407	.689	.927
	n.p.	.043	.163	.372	.657	.917
1.401	s.p.	.063	.218	.441	.707	.931
	n.p.	.063	.208	.430	.701	.930
2.135	s.p.	.075	.224	.436	.702	.931
	n.p.	.075	.232	.457	.720	.935
3.420	s.p.	.068	.205	.412	.687	.928
	n.p.	.068	.220	.443	.710	.932
	n.p.	.050	.179	.393	.674	.922

Table 23. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 20$, $n_2 = 6$, $n_3 = 2$, $F_1 = 2F_{.50}$, $\alpha_2 = \alpha_3 = .05$

θ_{21}	Test	θ_{32}				
		1	2	4	16	64
0.843	s.p.	.031	.152	.367	.764	.934
	n.p.	.031	.110	.272	.683	.905
1.350	s.p.	.065	.215	.422	.780	.937
	n.p.	.065	.190	.389	.767	.934
2.304	s.p.	.089	.219	.402	.762	.931
	n.p.	.089	.236	.446	.800	.944
4.269	s.p.	.075	.182	.361	.742	.924
	n.p.	.075	.209	.414	.782	.939
	n.p.	.050	.156	.343	.737	.924

Table 24. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 20$, $n_2 = 10$, $n_3 = 2$, $F_1 = 2F_{.50}$, $\alpha_2 = \alpha_3 = .05$

θ_{21}	Test	θ_{32}				
		1	2	4	10	50
0.966	s.p.	.044	.190	.416	.710	.941
	n.p.	.044	.165	.375	.660	.918
1.473	s.p.	.071	.230	.453	.715	.934
	n.p.	.071	.225	.449	.715	.934
2.359	s.p.	.080	.228	.435	.699	.931
	n.p.	.080	.243	.469	.729	.937
7.456	s.p.	.057	.179	.393	.675	.922
	n.p.	.057	.196	.415	.690	.927
	n.p.	.050	.179	.393	.674	.922

Table 25. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 14$, $n_2 = 10$, $n_3 = 12$, $F_1 = 2F$, $\alpha_2 = \alpha_3 = .05$

θ_{21}	Test	θ_{32}				
		1	2	4	10	50
0.957	s.p.	.039	.354	.784	.986	1.000
	n.p.	.039	.240	.657	.969	1.000
1.401	s.p.	.083	.434	.805	.981	1.000
	n.p.	.083	.379	.791	.988	1.000
2.135	s.p.	.117	.412	.752	.977	1.000
	n.p.	.117	.460	.846	.993	1.000
3.420	s.p.	.106	.329	.707	.976	1.000
	n.p.	.106	.435	.831	.991	1.000
	n.p.	.050	.280	.702	.976	1.000

Table 26. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 20$, $n_2 = 10$, $n_3 = 12$, $F_1 = 2F_{.50}$, $\alpha_2 = \alpha_3 = .05$

θ_{21}	Test	θ_{32}				
		1	2	4	10	50
0.966	s.p.	.040	.383	.827	.985	1.000
	n.p.	.040	.244	.662	.970	1.000
1.473	s.p.	.103	.479	.811	.981	1.000
	n.p.	.103	.431	.828	.991	1.000
2.359	s.p.	.142	.406	.736	.976	1.000
	n.p.	.142	.512	.874	.995	1.000
	n.p.	.050	.280	.702	.976	1.000

Table 27. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 20$, $n_2 = 6$, $n_3 = 2$, $\alpha_1 = .25$, $\alpha_2 = \alpha_3 = .05$

θ_{21}	Test	θ_{32}				
		1	2	4	16	64
1	s.p.	.038	.161	.368	.757	.930
	n.p.	.038	.127	.314	.705	.913
1.5	s.p.	.060	.189	.385	.756	.930
	n.p.	.060	.178	.373	.757	.930
2	s.p.	.068	.190	.377	.751	.928
	n.p.	.068	.195	.396	.771	.935
3	s.p.	.068	.178	.361	.743	.926
	n.p.	.068	.194	.394	.770	.935
5	s.p.	.058	.164	.348	.738	.924
	n.p.	.058	.175	.369	.754	.930
	n.p.	.050	.156	.343	.737	.924

Table 28. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 14$, $n_2 = 10$, $n_3 = 12$, $\alpha_1 = .25$, $\alpha_2 = \alpha_3 = .05$

θ_{21}	Test	θ_{32}				
		1	2	4	10	50
0.714	s.p.	.013	.232	.708	.971	1.000
	n.p.	.013	.111	.448	.911	1.000
1.045	s.p.	.039	.294	.757	.980	1.000
	n.p.	.039	.240	.657	.969	1.000
1.593	s.p.	.075	.360	.727	.977	1.000
	n.p.	.075	.358	.774	.986	1.000
2.552	s.p.	.082	.311	.704	.975	1.000
	n.p.	.082	.377	.790	.988	1.000
8.066	s.p.	.050	.280	.699	.975	1.000
	n.p.	.050	.280	.702	.976	1.000

Table 29. The power of the sometimes pool procedure and the never pool test of the same size, for $n_1 = 20$, $n_2 = 10$, $n_3 = 12$, $\alpha_1 = .25$, $\alpha_2 = \alpha_3 = .05$

θ_{21}	Test	1	2	4	10	50
1.066	s.p.	.044	.354	.741	.979	1.000
	n.p.	.044	.259	.679	.973	1.000
1.708	s.p.	.086	.360	.714	.976	1.000
	n.p.	.086	.389	.799	.988	1.000
2.914	s.p.	.079	.292	.701	.975	1.000
	n.p.	.079	.369	.784	.987	1.000
5.399	s.p.	.050	.280	.697	.975	1.000
	n.p.	.050	.280	.702	.976	1.000

Table 30. The power of the borderline sometimes pool procedure and the never pool test of the same size for $n_1 = 20$, $n_2 = 6$, $n_3 = 2$

θ_{21}	Test	θ_{32}				
		1	2	4	16	64
1	s.p.	.023	.117	.307	.722	.920
	n.p.	.023	.087	.232	.646	.891
1.5	s.p.	.035	.140	.330	.732	.923
	n.p.	.035	.121	.290	.697	.910
2	s.p.	.042	.152	.338	.735	.923
	n.p.	.042	.137	.314	.716	.917
3	s.p.	.047	.156	.342	.736	.924
	n.p.	.047	.149	.333	.730	.922
	n.p.	.050	.156	.343	.737	.924

Table 31. The power of the borderline sometimes pool procedure and the never pool test of the same size for $n_1 = 20$, $n_2 = 10$, $n_3 = 2$

θ_{21}	Test	θ_{32}				
		1	2	4	10	50
1.0	s.p.	.030	.146	.362	.655	.917
	n.p.	.030	.127	.321	.615	.904
1.5	s.p.	.041	.168	.384	.669	.921
	n.p.	.041	.158	.366	.652	.916
2.0	s.p.	.046	.175	.390	.673	.922
	n.p.	.046	.170	.382	.665	.919
3.0	s.p.	.049	.178	.393	.674	.922
	n.p.	.049	.177	.391	.672	.921
	n.p.	.05	.179	.393	.674	.922