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CELLS AND CELLULARITY IN INFINITE-DIMENSIONAL NORMED LINEAR SPACES

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I. INTRODUCTION

There are certain topological properties in Euclidean n-space which can be conveniently studied as properties in normed linear spaces. In this dissertation concepts such as open and closed cells, cellular sets, point-like sets, and decomposition spaces are studied and related. Many, but not all, of the relationships between these concepts in infinite-dimensional normed linear spaces resemble the corresponding relationships in finite-dimensional spaces.

The concept of cellularity in Euclidean n-space was introduced by Morton Brown in (8). Relationships between cellular sets, point-like sets, and certain decomposition spaces in finite-dimensional spaces were established in (6). In particular, in Euclidean n-space, a set is cellular if and only if it is point-like. Properties such as these have been used in studying topological n-manifolds. Perhaps generalizing these concepts to infinite-dimensional spaces will lead to the study of "infinite-dimensional manifolds." Such manifolds which are locally Hilbert space are currently being investigated, see for example (19).

Throughout this dissertation E will denote an arbitrary normed linear space, and θ will represent the zero element of E. For any positive real number r and any $x \in E$, let $B_r(x) = \{y \in E : \|x-y\| \leq r\}$ and $S_r(x) = \{y \in E : \|x-y\| = r\}$.

For convenience let $B_r = B_r(\theta)$ and $S_r = S_r(\theta)$. R^n and l_2 will represent Euclidean n-space and Hilbert space, respectively, and X will denote an arbitrary topological space. Finally for a subset Y of X, Int Y, Cl Y, and Bd Y will be used to indicate the interior, closure, and boundary of Y in X, respectively.

Many techniques used in proofs of the theorems in this dissertation owe their existence to Theorem 1.1 which is due to V. L. Klee.

Theorem 1.1. Let E be an infinite-dimensional normed linear space. Let B be a bounded, closed, convex subset of E, and let A be a compact subset of E contained in Int B. Then there exists a homeomorphism from B onto B - A which is the identity on Bd B.

Klee first stated this theorem for nonreflexive normed linear spaces (13). Later, in (14), he indicated that the theorem could be generalized to arbitrary infinite-dimensional normed linear spaces by applying the following lemma. By a linearly bounded set is meant a set such that its intersection with any infinite ray is bounded.

Lemma 1.1. Every infinite-dimensional normed linear space contains a decreasing sequence of unbounded but linearly bounded, closed, convex sets whose intersection is empty.

A number of useful corollaries follow from Theorem 1.1.

Corollary 1.1. Let E be infinite-dimensional, and let A be a compact subset of E. Then $E - A$ is homeomorphic to E.

Corollary 1.2. Let E be infinite-dimensional. Then there exists a homeomorphism of $E - \operatorname{Int} B_1$ onto B_1 which is the identity on S_1.

Corollary 1.3. Let E be infinite-dimensional. Let H be a hyperplane in E, and let Q be the closed half-space bounded by H. Then there exists a homeomorphism from Q onto B_1 which takes H onto S_1.

If $x \in E$ and r is a positive real number, then by Corollary 1.2, there is a homeomorphism from $E - \operatorname{Int} B_r(x)$ onto $B_r(x)$ which is the identity on $S_r(x)$. The inverse of this homeomorphism takes $B_r(x)$ onto $E - \operatorname{Int} B_r(x)$ and is fixed on $S_r(x)$. Together these define a homeomorphism, which throughout the rest of this dissertation will be called $i\langle x;r\rangle$, from E onto itself such that

$$i\langle x;r\rangle\left[B_r(x)\right] = E - \operatorname{Int} B_r(x),$$

$$i\langle x;r\rangle \mid S_r(x) = \text{identity},$$

$$(i\langle x;r\rangle)^{-1} = i\langle x;r\rangle.$$

This class of homeomorphisms is a primary tool used in proofs of many of the theorems which follow.

For counterexamples and special cases, a number of results in Hilbert space are of interest. Since most results in this dissertation will be topological, results stated for Hilbert space will also hold true for any separable infinite-dimensional Banach space. This of course is because all such spaces are homeomorphic (12). A theorem due to R. D. Anderson (1) is the following.

Theorem 1.2. l_2 is homeomorphic to $\prod_{i=1}^{\infty} R_i$, where each $R_i = R^1$.

This in turn follows from the next theorem, also due to Anderson (2).

Theorem 1.3. The complement of a σ-compact set in l_2 is homeomorphic to l_2.

Klee established that l_2 is homogeneous with respect to compact subsets (15).

Theorem 1.4. Any homeomorphism from a compact subset of l_2 into l_2 can be extended to a homeomorphism from l_2 onto itself.

In fact, Douglas Curtis has further shown homogeneity in l_2 with respect to closed σ-compact subsets (9).

In (13), Klee showed that not only is the unit sphere in l_2 homeomorphic to l_2 (see Corollary 1.3), but also the closed unit ball in l_2 is homeomorphic to l_2. Later he extended this result to a large class of Banach spaces (16).

II. OPEN CELLS AND THE MONOTONE UNION THEOREM

Morton Brown proved the following theorem in (7).

Theorem 2.1. The union of an increasing sequence of open n-cells is an n-cell.

The idea of an open cell can easily be extended to arbitrary normed linear spaces. It turns out that an analog to Theorem 2.1 can be established for open cells in a normed linear space.

Definition 2.1. An open E-cell in X is an open subset of X which is homeomorphic to E. If a subset Q of E is an open E-cell in E, then Q will be said to be an open cell in E.

The proof of the following lemma uses a technique somewhat similar to the proof of Theorem 2.1.

Lemma 2.1. Let $\{Q_i\}_{i=1}^{\infty}$ be an increasing sequence of open E-cells in X. If h_i is a homeomorphism of E onto Q_i such that $h_{i+1}^{-1}h_i$ is bounded for each i, then $\bigcup_{i=1}^{\infty} Q_i$ is an open E-cell in X.

Proof. Define $H_1 = h_1$. Since $h_2^{-1}h_1(B_2)$ is bounded in E, choose N_1 such that $h_2^{-1}h_1(B_2) \subset B_{N_1}(h_2^{-1}h_1(\theta))$.

Suppose the numbers N_i and the homeomorphisms H_i from E into Q_i have been defined for $1 \leq i \leq k$ such that $h_{i+1}^{-1}H_i$ is bounded, $H_i | B_{i-1} = H_{i-1} | B_{i-1}$, and h_{i+1}^{-1} $(H_i(B_{i+1}) \cup$

$\left[\bigcup_{j=1}^{i-1} h_j(B_{i-j})\right]$ $) \subset B_{N_i}(h_{i+1}^{-1}H_i(\theta))$. Then inductively define N_{k+1} and H_{k+1} in the following manner. Set $c = h_{k+1}^{-1}H_k(\theta)$. Since Q_k is open in Q_{k+1}, $h_{k+1}^{-1}H_k(\text{Int } B_k)$ is open in E and contains c. Hence there exists an $\varepsilon > 0$ such that $B_\varepsilon(c) \subset h_{k+1}^{-1}H_k(\text{Int } B_k)$. Set $D = \text{Int } B_{\frac{\varepsilon}{2}}(c)$. Since $h_{k+1}(D) \subset H_k(E)$, then $H_k^{-1}h_{k+1}(D)$ is open in E and contains θ. Therefore there exists a $\delta > 0$ such that $B_\delta \subset H_k^{-1}h_{k+1}(D)$. Define f and g, homeomorphisms of E onto itself, so that they have the following properties:

$$f(B_k) = B_\delta ,$$

$$f \mid (E - B_{k+1}) = \text{identity},$$

$$g(B_\varepsilon(c)) = B_{N_k}(c),$$

$$g \mid [B_{\frac{\varepsilon}{2}}(c) \cup (E - B_{N_{k+1}}(c))] = \text{identity}.$$

Let F be the homeomorphism of E onto itself defined by $F(x) = h_{k+1}^{-1}H_k f H_k^{-1}h_{k+1}(x)$ if $x \in h_{k+1}^{-1}H_k(B_{k+1})$, and $F(x) = x$ otherwise. Then let H_{k+1} be the homeomorphism of E into Q_{k+1} defined by $H_{k+1} = h_{k+1}F^{-1}gFh_{k+1}^{-1}H_k$. From their definitions, it is clear that f and g are bounded. Since $h_{k+1}^{-1}H_k(B_{k+1})$ is bounded in E, then F is bounded. Thus since $h_{k+1}^{-1}H_k$, F, g, F^{-1}, and $h_{k+2}^{-1}h_{k+1}$ are all bounded, their composition $h_{k+2}^{-1} H_{k+1}$ is bounded. Hence choose N_{k+1} such that $h_{k+2}^{-1} (H_{k+1}(B_{k+2}) \cup [\cup_{j=1}^{k} h_j(B_{k-j+1})]) \subset B_{N_{k+1}} (h_{k+2}^{-1}H_{k+1}(\theta))$. Since $Fh_{k+1}^{-1}H_k(B_k) = h_{k+1}^{-1}H_k f(B_k) = h_{k+1}^{-1}H_k(B_\delta) \subset B_{\frac{\varepsilon}{2}}(c)$ and since g is the identity on $B_{\frac{\varepsilon}{2}}(c)$, then $H_{k+1} \mid B_k = h_{k+1}F^{-1}gFh_{k+1}^{-1}H_k \mid B_k = H_k \mid B_k$.

Define the function H from E into X by $H(x) = H_i(x)$, where $x \in B_i - \text{Int } B_{i-1}$, $i = 1,2,3,\ldots$ $(B_0 = \emptyset)$. H is indeed a function since $H_i \mid B_{i-1} = H_{i-1} \mid B_{i-1}$ for each i. Since each H_i is a homeomorphism, H is clearly continuous, open, and one-to-one. To see that H is onto $\cup_{i=1}^{\infty} Q_i$, let $x \in \cup_{i=1}^{\infty} Q_i$. Then there is some positive integer n such that $x \in Q_n$. Also there is some positive integer m such that $x \in h_n(B_m)$. Then by the way in which N_{m+n} was chosen, $h_{m+n+1}^{-1}(x) \in B_{N_{m+n}}[h_{m+n+1}^{-1}H_{m+n}(\theta)]$. But by the construction of H_{m+n+1}, $B_{N_{m+n}}[h_{m+n+1}^{-1}H_{m+n}(\theta)] \subset h_{m+n+1}^{-1}H_{m+n+1}(B_{m+n+1})$. Hence $x \in H_{m+n+1}(B_{m+n+1})$. Therefore H is a homeomorphism of E onto $\cup_{i=1}^{\infty} Q_i$, which completes the proof.

If E is finite-dimensional, all homeomorphisms of E into itself are bounded, so that Theorem 2.1 is an immediate consequence of Lemma 2.1.

Difficulty arises when E is infinite-dimensional, since a homeomorphism of E into itself may not be bounded. However, there is a way of getting around the difficulty by using the homeomorphisms $i\langle x;r\rangle$. These homeomorphisms can be used in the following way. If A is a subset of E which is not dense in E, then an $x \in E$ and an $r > 0$ can be found so that $B_r(x) \subset E - A$. Therefore, whether A is bounded in E or not, $i\langle x;r\rangle(A)$ will be bounded in E. This technique is applied to the following lemma. It replaces the boundedness condition in the hypotheses of Lemma 2.1 with a non-denseness condition which turns out to be easier to deal with.

Lemma 2.2. Let E be infinite-dimensional, and let $\{Q_i\}_{i=1}^{\infty}$ be an increasing sequence of open E-cells in X. If each Q_i is not dense in $\bigcup_{i=1}^{\infty} Q_i$, then $\bigcup_{i=1}^{\infty} Q_i$ is an open E-cell in X.

Proof. For each i, let h_i be a homeomorphism of E onto Q_i. Set $n_1 = 1$ and $g_1 = h_1$. Suppose the integers n_i and the homeomorphisms g_{n_i} from E onto Q_{n_i}, such that $g_{n_i}^{-1} g_{n_{i-1}}$ is bounded (for $i>1$), have all been defined for $1 \leq i \leq k$. Then inductively define n_{k+1} and $g_{n_{k+1}}$ as follows. Since Q_{n_k} is not dense in $\bigcup_{i=1}^{\infty} Q_i$, there exists a point

$x \in \bigcup_{i=1}^{\infty} Q_i$ such that x is contained in some open subset of $\bigcup_{i=1}^{\infty} Q_i$ which does not intersect Q_{n_k}. Also there is a positive integer n_{k+1} such that $x \in Q_{n_{k+1}}$. Choose an $\varepsilon > 0$ such that $B_\varepsilon[h_{n_{k+1}}^{-1}(x)] \cap h_{n_{k+1}}^{-1}(Q_{n_k}) = \emptyset$, and set $g_{n_{k+1}} = h_{n_{k+1}} i<h_{n_{k+1}}^{-1}(x);\varepsilon>$. Since $g_{n_{k+1}}^{-1}(Q_{n_k}) \subset B_\varepsilon[h_{n_{k+1}}^{-1}(x)]$, $g_{n_{k+1}}^{-1} g_{n_k}$ is bounded. Also since $\bigcup_{i=1}^{\infty} Q_{n_i} = \bigcup_{i=1}^{\infty} Q_i$, and since $\{Q_{n_i}\}_{i=1}^{\infty}$ and $\{g_{n_i}\}_{i=1}^{\infty}$ satisfy the hypotheses of Lemma 2.1, then $\bigcup_{i=1}^{\infty} Q_i$ is an open E-cell in X.

The next lemma then "comes to within a point" of the desired result.

<u>Lemma 2.3</u>. Let E be infinite-dimensional, and let $\{Q_i\}_{i=1}^{\infty}$ be an increasing sequence of open E-cells in X. Then for any $x \in \bigcup_{i=1}^{\infty} Q_i$, $\bigcup_{i=1}^{\infty} Q_i - \{x\}$ is an open E-cell in X.

<u>Proof</u>. Let $x \in \bigcup_{i=1}^{\infty} Q_i$. Without loss of generality assume that $x \in Q_1$. For each i, let h_i be a homeomorphism of E onto Q_i. Let g_1 be the homeomorphism which radially shrinks E onto Int $B_1(h_1^{-1}(x))$. Set $\varepsilon_1 = 1$, $f_1 = h_1 i<h_1^{-1}(x); 1>g_1$, and $P_1 = f_1(E)$.

Suppose ε_i, f_i, and P_i have been defined for $1 \leq i \leq k$ such that f_i is a homeomorphism from E into Q_i, $P_{i-1} \subset P_i = f_i(E) = Q_i - h_i[\bar{B}_{\varepsilon_i}(h_i^{-1}(x))]$, and $Q_i - h_i[\bar{B}_{\frac{1}{i}}(h_1^{-1}(x))] \subset P_i$. Then inductively define ε_{k+1}, f_{k+1}, and P_{k+1} in the following manner. Since $h_k[\text{Int } B_{\varepsilon_k}(h_k^{-1}(x))]$ and $h_1[\text{Int } B_{\frac{1}{k}}(h_1^{-1}(x))]$ are open in Q_{k+1}, there exists an $\varepsilon_{k+1} > 0$ such that $B_{\varepsilon_{k+1}}(h_{k+1}^{-1}(x)) \subset h_{k+1}^{-1}(h_k[\text{Int } B_{\varepsilon_k}(h_k^{-1}(x))] \cap h_1[\text{Int } B_{\frac{1}{k+1}}(h_1^{-1}(x))])$. Let g_{k+1} be the homeomorphism which radially shrinks E onto Int $B_{\varepsilon_{k+1}}(h_{k+1}(x))$. Finally, set $f_{k+1} = h_{k+1} i(<h_{k+1}^{-1}(x);\varepsilon_{k+1}>g_{k+1}$ and $P_{k+1} = f_{k+1}(E)$

Each P_i is an open E-cell in X. Also $P_i \subset P_{i+1}$ for all i, $\bigcup_{i=1}^{\infty} Q_i - \{x\} = \bigcup_{i=1}^{\infty} P_i$, and P_i is not dense in $\bigcup_{i=1}^{\infty} P_i$ for all i. Thus by Lemma 2.2, $\bigcup_{i=1}^{\infty} Q_i - \{x\}$ is an open E-cell in X.

To complete the proof of the analog to Theorem 2.1, the next theorem "fills in" the point left out in the conclusion of Lemma 2.3.

<u>Theorem 2.2</u>. Let E be infinite-dimensional, and let X be a Hausdorff space with the property that there exist two points $x, y \in X$ such that $X - \{x\}$ and $X - \{y\}$ are open E-cells in X. Then X is an open E-cell.

Proof. Let h_x and h_y be homeomorphisms from E onto $X - \{x\}$ and $X - \{y\}$ respectively. Without loss of generality assume that $h_y^{-1}(x) = \theta$. Since X is Hausdorff, choose U to be an open set in X containing x such that $y \notin$ Cl U. $h_y^{-1}(U)$ is open in E, so that there exists an $\varepsilon > 0$ such that $B_\varepsilon \subset h_y^{-1}(U)$.

By Theorem 1.1, there exists a homeomorphism f of B onto $B_\varepsilon - \{\theta\}$ such that $f \mid S_\varepsilon$ = identity. Define h, a homeomorphism of X onto $X - \{x\}$, by $h(z) = h_y f h_y^{-1}(z)$ if $z \in h_y(B_\varepsilon)$, and $h(z) = z$ otherwise. Then $h^{-1}h_x$ is a homeomorphism from E onto X.

The desired analog to Theorem 2.1 is then a consequence of Theorem 2.1, Lemma 2.3, and Theorem 2.2.

Theorem 2.3. (Monotone Union Theorem) The union of an increasing sequence of open E-cells in X is an open E-cell in X.

In the Monotone Union Theorem, the condition that each cell be open in X is necessary, as shown by the following examples.

Example 2.1. For each positive integer i, let Y_i = $\{ (x,y) \in R^2 : -2 \leq y < 1,\ x = 0;$ or $y = -2,\ 0 \leq x \leq \frac{1}{2\pi};$ or $-2 \leq y \leq 0,\ x = \frac{1}{2\pi};$ or $y = \sin\frac{1}{x},\ \frac{1}{2(i+1)\pi} < x \leq \frac{1}{2\pi} \}$.

Set $Y = \bigcup_{i=1}^{\infty} Y_i$, and let Y and each Y_i have the relative topology of R^2. Then each Y_i is homeomorphic to R^1, but Y is not homeomorphic to R^1.

Example 2.2. For an infinite-dimensional example, let $Q_i = \prod_{j=1}^{\infty} Y_j$, where each $Y_j = Y_i$, and $X = \bigcup_{i=1}^{\infty} Q_i$ in the relative topology of $\prod_1^{\infty} Y$ (Y and Y_i are defined as in Example 2.1). By Theorem 1.2, each Q_i is homeomorphic to Hilbert space. However, X is not locally connected, so that it is not homeomorphic to Hilbert space.

Consider next an example on which Theorem 2.2 can be applied. Let E be an arbitrary normed linear space. Let $\tilde{E} = E \cup \{w\}$, where w is some point not in E. Define the topology on $\tilde{E}$ by letting U be open in $\tilde{E}$ if and only if either

1) $U \subset E$ and U is open in E, or

2) $w \in U$ and $E - U$ is closed and bounded in E.

Call $\tilde{E}$ the one-point co-bounded extension of E. If E is finite-dimensional, then $\tilde{E}$ is just the one-point compactification of E. If E is infinite-dimensional, then the topology of $\tilde{E}$ is finer than the topology of the one-point compactification of E.

Lemma 2.4. $\tilde{E} - \{\theta\}$ is an open E-cell.

Proof. Define the function φ from E into $\tilde{E} - \{\theta\}$ by $\varphi(x) = \frac{1}{\|x\|^2} x$ if $x \neq \theta$, and $\varphi(x) = w$ if $x = \theta$. φ is onto since if $y \in E - \{\theta\}$, then $\varphi(\frac{1}{\|y\|^2} y) = y$. To see that φ is one-to-one, let $\frac{1}{\|x\|^2} x = \frac{1}{\|y\|^2} y$. Taking the norms of both sides gives $\|x\| = \|y\|$, so that $x = y$. Because of continuity of the norm and of scalar multiplication, φ is continuous except possibly at θ. Then let U be an open set in $\tilde{E} - \{\theta\}$ containing w, so that $\tilde{E} - U$ is closed and bounded in E. Then $\tilde{E} - U \subset B_N$ for some number N. If $y \in \text{Int } B_{\frac{1}{N}} - \{\theta\}$, then $\|\varphi(y)\| > N$. So that $\varphi(\text{Int } B_{\frac{1}{N}}) \subset E - B_N \subset U$, which shows continuity at θ. A similar type of argument shows continuity of φ^{-1}, so that φ is the desired homeomorphism.

The next theorem then follows from Lemma 2.4 and Theorem 2.2.

Theorem 2.4. If E is infinite-dimensional, the one-point co-bounded extension of E is an open E-cell.

By Theorem 2.4, since the unit sphere in l_2 is homeomorphic to l_2, then $\tilde{l}_2$ is homeomorphic to the unit sphere in l_2. Also since the unit sphere in each finite-dimensional space is homeomorphic to the one-point compactification of the space, it is natural to ask the following question.

Question 2.1. If E is infinite-dimensional, is the one-point co-bounded extension of E homeomorphic to the unit sphere in E?

If this were true then E itself would be homeomorphic to its unit sphere, which seems to be an open question for arbitrary infinite-dimensional normed linear spaces.

The following theorem is a consequence of the Monotone Union Theorem.

Theorem 2.5. Let X be a second countable space which satisfies:

1) Every point in X is contained in some open E-cell in X.

2) If Q_1 and Q_2 are two open E-cells in X, then $Q_1 \cup Q_2$ is contained in some open E-cell in X.

Then X is an open E-cell.

Proof. Since by 1) all the open E-cells in X cover X and since X is second countable, there exists a sequence $\{ Q_i \}_{i=1}^{\infty}$ of open E-cells in X such that $\bigcup_{i=1}^{\infty} Q_i = X$. Define $\overline{Q}_1 = Q_1$. Suppose the open E-cells $\overline{Q}_i$ have been defined for $1 \leq i \leq k$ such that $\overline{Q}_{i-1} \cup (\bigcup_{j=1}^{i} Q_j) \subset \overline{Q}_i$ $(\overline{Q}_0 = \emptyset)$. Then inductively define $\overline{Q}_{k+1}$ as follows. By 2) there exists an open E-cell $\overline{Q}_{k+1}$ such that $\overline{Q}_k \cup Q_{k+1} \subset \overline{Q}_{k+1}$. So that $\overline{Q}_k \cup (\bigcup_{j=1}^{k+1} Q_j) \subset \overline{Q}_{k+1}$. Then by the Monotone Union Theorem, $\bigcup_{i=1}^{\infty} \overline{Q}_i$ is an open E-cell. Since $X = \bigcup_{i=1}^{\infty} \overline{Q}_i$, X is an open E-cell.

III. CLOSED CELLS AND CELLULAR SETS

Definition 3.1. A closed subset C of X is a closed E-cell in X if there exists a homeomorphism from the pair (B_1, S_1) in E onto the pair (C, Bd C) in X. If a subset C of E is a closed E-cell in E, then C will be said to be a closed cell in E.

Definition 3.2. If A is a subset of E, a cellular sequence for A is a decreasing sequence, $\{C_i\}_{i=1}^{\infty}$, of closed cells in E such that $\bigcap_{i=1}^{\infty} C_i = A$ and $C_{i+1} \subset \operatorname{Int} C_i$ for each i. A will be called cellular in E if there exists a cellular sequence for A.

Definition 3.3. A subset A of E is said to be strongly cellular in E if there exists a cellular sequence, $\{C_i\}_{i=1}^{\infty}$, for A such that for each open set U containing A there exists an integer n such that $C_n \subset U$. Such a cellular sequence will be called a strongly cellular sequence for A.

In finite-dimensional spaces, cellularity and strong cellularity are equivalent and agree with the usual definition of cellularity there. However, in general they are not equivalent, as indicated by some of the following examples.

Example 3.1. For an example of a strongly cellular set consider any compact convex subset A of E. For each i, let

$C_i = \{ x \in E : \| x-y \| \leq \frac{1}{2^i}, y \in A \}$. Since A is compact and convex, each C_i is bounded, closed, and convex. Therefore each C_i is a closed cell in E. Since A is compact, any open set containing A contains some C_i. Hence $\{ C_i \}_{i=1}^{\infty}$ is a strongly cellular sequence for A, so that A is strongly cellular in E. If A is only a bounded, closed, convex subset of E, $\{ C_i \}_{i=1}^{\infty}$ is a cellular sequence for A, but not necessarily a strongly cellular sequence for A.

The next theorem then follows from Theorem 1.4 and the preservation of strong cellularity by space homeomorphisms.

<u>Theorem 3.1</u>. Any subset of l_2 which is homeomorphic to a compact convex subset of l_2 is strongly cellular in l_2.

<u>Corollary 3.1</u>. Any subset of l_2 which is homeomorphic to either I^n or I^{∞} is strongly cellular in l_2.

The converse to Theorem 3.1 is not true. That is, sets in l_2 which are homeomorphic to compact convex subsets of l_2 do not characterize strongly cellular sets in l_2. The following example demonstrates this.

<u>Example 3.2</u>. Let M be a cellular set in R^3 which does not have the fixed-point property. Such an example has been given by Borsuk and others (see for example (4)). Let M' be a subset of l_2 which is homeomorphic to M. Then by a later theorem in this dissertation, Theorem 4.2, M' is strongly cellular in l_2. But by the Tychonoff Fixed-point

Theorem (18), M' is not homeomorphic to a convex subset of l_2 since it does not have the fixed-point property. This example then also shows that a strongly cellular set in E does not necessarily have the fixed-point property, and therefore is also not necessarily an absolute retract.

<u>Theorem 3.2</u>. Any compact subset of an infinite-dimensional normed linear space E is cellular in E.[1]

<u>Proof</u>. Let A be a compact subset of E. For each integer i, let $N_i = \{ y \in E : \inf_{x \in A} \| y - x \| < \frac{1}{i} \}$. Let $\{K_i\}_{i=1}^{\infty}$ be a decreasing sequence of linearly bounded, closed, convex sets whose intersection is empty (see Lemma 1.1). For each integer i, let $M_i = \{ y \in E : \inf_{x \in K_i} \| y - x \| < \frac{1}{i} \}$ and $C_i = \mathrm{Cl}\,(\bigcup \{ [x:y] : x \in M_i \text{ and } y \in N_i \})$. Klee has shown that each C_i is starshaped from M_i, each $C_{i+1} \subset \mathrm{Int}\, C_i$, and $\bigcap_{i=1}^{\infty} C_i = A$ (13). It can also be seen that each C_i is linearly bounded, so that each C_i is actually a closed cell in E.

<u>Definition 3.4</u>. A closed cell in E is tame if there exists a homeomorphism of E onto itself which takes B_1 onto the cell.

<u>Example 3.3</u>. A tame closed cell in E, where E is infinite-dimensional, is an example of a cellular set which is not compact and therefore not strongly cellular

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[1]The author would like to thank V. L. Klee and R. D. Anderson for pointing out this result to him.

because of the next theorem.

Theorem 3.3. Any strongly cellular set in E is compact and connected.

Proof. It is clear that the theorem is true when E is finite-dimensional, so assume that E is infinite-dimensional.

First suppose that A is strongly cellular in E but that Int A $\neq \emptyset$. Assume without loss of generality that A is bounded in E and that $\theta \in$ Int A. Let $\{C_i\}_{i=1}^{\infty}$ be a strongly cellular sequence for A, and let $\{x_i\}_{i=1}^{\infty}$ be a sequence of points of S_1 in E which has no limit point. For each integer i, there exists $y_i \in (C_i \cap T_i) - A$, where T_i is the half-infinite ray starting at θ and passing through x_i. Then $E - \{y_i\}_{i=1}^{\infty}$ is open, contains A, and contains no C_i; which is a contradiction. Hence if A is strongly cellular in E, Int A $= \emptyset$.

Next suppose that A is strongly cellular in E but is not compact. Let $\{C_i\}_{i=1}^{\infty}$ be a strongly cellular sequence for A. Let $\{z_i\}_{i=1}^{\infty}$ be a sequence of points of A which has no limit point, and let $\{U_i\}_{i=1}^{\infty}$ be a mutually disjoint sequence of open subsets of E such that $z_i \in U_i$ and $U_i \subset B_{\frac{1}{2^i}}(z_i)$ for each i. Since U_i is not contained in A because Int A $= \emptyset$, for each integer i there exists a

$w_i \in (C_i \cap U_i) - A$. Then $E - \{w_i\}_{i=1}^{\infty}$ is open, contains A, and contains no C_i; which is again a contradiction. Therefore if A is strongly cellular in E, A is compact.

Assume that A is not connected. Then A can be covered by two disjoint open subsets U and V of E. Since A is strongly cellular, there is some integer n such that $C_n \subset U \cup V$. But since $C_n \cap U \neq \emptyset$ and $C_n \cap V \neq \emptyset$, this contradicts the connectivity of C_n. Hence A is connected.

An example of a cellular set in E which is not compact or connected, and hence not strongly cellular in E, is the following.

Example 3.4. Let E be infinite-dimensional and let $x, y \in E$. Define A to be the line segment from x to y, which is cellular in E. Let $\{C_i\}_{i=1}^{\infty}$ be a cellular sequence for A, with h_i a homeomorphism of (B_1, S_1) onto $(C_i, \text{Bd } C_i)$ for each i. Also for each i, by Theorem 1.1, let g_i be a homeomorphism of B_1 onto $B_1 - \{h_i^{-1}(z)\}$ (where $z = \frac{1}{2}x + \frac{1}{2}y$) such that $g_i | S_1 =$ identity. Let f be a homeomorphism of $E - \{z\}$ onto E, by Corollary 1.1. Define $C = f(A - \{z\})$. Then C is not connected, but $\{fh_ig_i(B_1)\}_{i=1}^{\infty}$ is a cellular sequence for C.

The following theorem gives simple examples of compact subsets of an arbitrary normed linear space E which are not strongly cellular in E. Its proof is based on theory which can be found in chapter ten of (4).

Theorem 3.4. Any strongly cellular subset of E has trivial Čech homology groups.

Proof. Let $\{C_i\}_{i=1}^{\infty}$ be a strongly cellular sequence for the subset A of E. Let $\hat{X}$ denote the Stone-Čech compactification of a space X. For each i, $\widehat{C_i}$ can be considered as the closure of C_i in the space $\hat{E}$. Since A is compact, it is closed in $\hat{E}$.

Let $x \in \hat{E} - A$. Since $\hat{E}$ is a regular space, let U be an open subset of $\hat{E}$ such that $A \subset U$ and $x \notin \mathrm{Cl}\ U$ in $\hat{E}$. Since $U \cap E$ is open in E and contains A, there exists an integer n such that $C_n \subset U \cap E$. But then $\widehat{C_n} = \mathrm{Cl}\ C_n \subset \mathrm{Cl}\ U$ in $\hat{E}$. Since $x \notin \mathrm{Cl}\ U$ in $\hat{E}$, $x \notin \widehat{C_n}$. Hence $x \notin \bigcap_{i=1}^{\infty} \widehat{C_i}$, so that $\bigcap_{i=1}^{\infty} \widehat{C_i} = A$.

By continuity properties of the Čech homology theory, $\check{H}_q(\bigcap_{i=1}^{\infty} \widehat{C_i})$ is isomorphic to the inverse limit of $\{\check{H}_q(\widehat{C_i})\}$. Since each C_i is a normal space, $\check{H}_q(\widehat{C_i}) \cong \check{H}_q^f(C_i)$ for each q and i ($\check{H}_q^f$ denotes the qth Čech homology group using finite coverings). The closed unit ball in E is contractible, so that each $\check{H}_q^f(C_i)$ is the qth homology group of a point. Therefore each $\check{H}_q(\widehat{C_i})$ is the qth homology group of a point, so that so also is $\check{H}_q(A) = \check{H}_q(\bigcap_{i=1}^{\infty} \widehat{C_i})$.

Example 3.5. For later use in this dissertation, let Σ be a subset of E which is the homeomorphic image of an n-sphere, so that Σ is not strongly cellular in E.

Theorem 3.5. Any strongly cellular ANR in E is homotopically trivial.

Proof. Let $\{C_i\}_{i=1}^{\infty}$ be a strongly cellular sequence for the subset A of E. Let U be an open subset of E such that A is a retract of U. Since A is strongly cellular, there is some integer n such that $C_n \subset U$. Hence A is a retract of C_n, which is homotopically trivial. Therefore A is homotopically trivial.

The next theorem demonstrates that the definition for cellularity is actually a little stronger than it appears at first glance.

Definition 3.5. If C is a closed cell in E, a closed subset K of E - Int C is a collar of C if there exists a homeomorphism h of the pair $(B_1, B_{\frac{1}{2}})$ onto the pair $(K \cup C, C)$ such that $h(S_1) = \text{Bd}\,(K \cup C)$.

Lemma 3.1. Let C be a closed cell in E contained in Int B_1. Then there exists a closed cell C' in E contained in B_1 such that $C \subset \text{Int } C'$ and $B_1 - \text{Int } C'$ is a collar of C'.

Proof. Assume without loss of generality that $\theta \in C$. Let $d(x, C) = \inf_{z \in C} \{ \|x - z\| \}$. Define the function f from $S_{\frac{1}{2}}$ into Int $B_1 - C$ by $f(x) = [2 - d(2x, C)]x$. Since $d(\cdot, C)$ is a continuous function from E onto the non-negative real numbers, f is continuous. Also since f maps each point $x \in S_{\frac{1}{2}}$ into the line segment $[\theta : 2x]$, then f is one-to-one, and f^{-1} is continuous. Define the homeomorphism h of B_1 onto itself so that, for each $x \in S_{\frac{1}{2}}$, h maps the line segment $[\theta : x]$ linearly onto $[\theta : f(x)]$ and maps the line segment $[x : 2x]$ linearly onto $[f(x) : 2x]$. Then $h(B_{\frac{1}{2}})$ is the desired closed cell.

Lemma 3.2. If A is cellular in E, there exists a cellular sequence $\{C_i\}_{i=1}^{\infty}$ for A such that C_i - Int C_{i+1} contains a collar of C_{i+1} for each i.

Proof. Let $\{A_i\}_{i=1}^{\infty}$ be a cellular sequence for A. For each i, by Lemma 3.1, define C_i to be a closed cell in E such that $C_i \subset$ Int A_i, $A_{i+1} \subset$ Int C_i, and A_i - Int C_i is a collar of C_i. $\{C_i\}_{i=1}^{\infty}$ is then the desired cellular sequence for A.

Lemma 3.3. Let C and C' be two closed cells in E such that $C' \subset C$ and C - Int C' contains a collar K of C'. Then there exists a homeomorphism H of (B_1, S_1) onto (C, Bd C) such that $C' \subset H(B_{\frac{1}{2}}) \subset K \cup C'$.

Proof. Let g be a homeomorphism of (B_1, S_1) onto (C, Bd C), and let h be a homeomorphism of $(B_1, B_{\frac{1}{2}})$ onto $(K \cup C', C')$ such that $h(S_1) =$ Bd $(K \cup C')$. Choose $x \in B_1$ and $\varepsilon > 0$ so that $B_\varepsilon(x) \subset g^{-1}(\text{Int } C')$. Let f be a homeomorphism of B_1 onto itself so that $f(B_{\frac{1}{2}}) = B_\varepsilon(x)$ and $f \mid S_1 =$ identity. Choose $y \in B_1$ and $\delta > 0$ so that

$B_\delta(y) \subset h^{-1}g(B_\varepsilon(x))$. Let F be a homeomorphism of B_1 onto itself so that $F(B_\delta(y)) = B_{\frac{1}{2}}$ and $F \mid S_1$ = identity. Define G, a homeomorphism of C onto itself, by $G(x) = hFh^{-1}(x)$ if $x \in K \cup C'$, and $G(x) = x$ otherwise. Then set $H = Ggf$, so that H is the desired homeomorphism.

Definition 3.6. If C and C' are two closed subsets of E such that $C' \subset \text{Int } C$, then C and C' will be said to have annular difference if there exists a homeomorphism h of $B_1 - \text{Int } B_{\frac{1}{2}}$ onto $C - \text{Int } C'$ such that $h(S_1) = \text{Bd } C$ and $h(S_{\frac{1}{2}}) = \text{Bd } C'$.

Theorem 3.6. If A is cellular in E, there exists a cellular sequence for A such that each element of the sequence is tame and each two elements of the sequence have annular difference.

Proof. By Lemma 3.2, there exists a cellular sequence, $\{C_i\}_{i=1}^{\infty}$, for A such that $C_i - \text{Int } C_{i+1}$ contains a collar of C_{i+1} for each i. By Lemma 3.3, for each i, there exists a homeomorphism h_i of (B_1, S_1) onto $(C_i, \text{Bd } C_i)$ such that $C_{i+1} \subset h_i(B_{\frac{1}{2}})$. If E is infinite-dimensional, choose $\sigma > 0$ and $w \in E$ such that $B_\sigma(w) \subset E - C_1$, and let $A_1 = E - \text{Int } B_\sigma(w)$. If E is finite-dimensional, let A_1 be some closed ball containing C_1. In either case there exists a homeomorphism f_1 of E onto itself such that $f_1(B_1) = A_1$.

Suppose the subsets A_i of E and the homeomorphisms f_i from E onto itself have been defined for $1 \leq i \leq k$ such that $f_i(B_1) = A_i$, $f_i(B_2) = A_{i-1}$, and $C_i \subset \text{Int } A_i \subset \text{Int } C_{i-1}$ ($A_o = f_1(B_2)$ and $C_o = E$). Then inductively define A_{k+1} and f_{k+1} as follows. Let $\varepsilon > 0$ and $x \in E$ be chosen so that $B_\varepsilon(x) \subset f_k^{-1}(C_{k+1})$. Also choose $\delta > 0$ and $y \in E$ so that $B_\delta(y) \subset h_k^{-1} f_k(B\ (x))$. Since $B_\delta(y) \subset B_{\frac{1}{2}}$, define the homeomorphism g of B_1 onto itself so that $g(B_\delta(y)) = B_{\frac{1}{2}}$ and $g \mid S_1 =$ identity. Define G, a homeomorphism of E onto itself, by $G(x) = f_k^{-1} h_k g\, h_k^{-1} f_k(x)$ if $x \in f_k^{-1}(C_k)$, and $G(x) = x$ otherwise. Let H be a homeomorphism of E onto itself such that $H(B_1) = B_\varepsilon(x)$ and $H(B_2) = B_1$. Then set $f_{k+1} = f_k GH$ and $A_{k+1} = f_{k+1}(B_1)$. Then $\{A_i\}_{i=1}^{\infty}$ is a cellular sequence for A satisfying the conditions of the theorem.

IV. POINT-LIKE SETS AND DECOMPOSITION SPACES

Definition 4.1. A subset of a homogeneous space X is point-like in X if it is closed and its complement in X is homeomorphic to the complement of a point in X.

When E is an infinite-dimensional normed linear space, the complement of a point in E is homeomorphic to E, by Corollary 1.1. Therefore if E is infinite-dimensional, a subset of E is point-like in E if and only if its complement is an open cell in E.

It is known that cellular sets are equivalent to connected point-like sets for finite-dimensional spaces (6). This is not true for infinite-dimensional spaces as the following example shows.

Example 4.1. Let s be the countable infinite product of open intervals (-1,1), which by Theorem 1.2 is homeomorphic to l_2. For each i, let $K_i = \{(x_1, x_2, \dots) \in s :$ $x_j \in [-\frac{1}{2}, \frac{1}{2}]$ for $j \leq i$, and $x_j = 0$ for $j > i\}$. Let $K = \bigcup_{i=1}^{\infty} K_i$. Then K is a connected σ-compact subset of s which is not closed in s. Therefore by Theorem 1.3, K is point-like in s. But since K is not closed, it is not cellular.

Theorem 4.1. A cellular set A in E is point-like in E.

Proof. (For E infinite-dimensional) By Theorem 3.6 there exists a cellular sequence $\{C_i\}_{i=1}^{\infty}$ for A such that each C_i is tame. Then each $E - C_i$ is open in E and homeomorphic to $E - B_1$. But $E - B_1$ is homeomorphic to E when E is infinite-dimensional, so that $E - A = E - \bigcap_{i=1}^{\infty} C_i = \bigcup_{i=1}^{\infty}(E - C_i)$ which is an open cell in E by the Monotone Union Theorem.

Question 4.1. Which point-like sets in an infinite-dimensional normed linear space are cellular in the space?

The following theorem is a partial answer to this question for Hilbert space, since under the hypotheses of the theorem, the subset of l_2 is necessarily point-like in l_2.

Theorem 4.2. Any subset of l_2 which is homeomorphic to some point-like set in R^n is strongly cellular in l_2.

Proof. Let A be a subset of l_2 and let h be a homeomorphism of A into $K = \{(x_1, x_2, \ldots) \in l_2 : x_i = 0 \text{ for } i > n\}$ such that h(A) is point-like in K. Since A is compact, by Theorem 1.4, the homeomorphism h between A and h(A) can be extended to H of l_2 onto itself. Since K is finite-dimensional and since H(A) is point-like in K, H(A) is strongly cellular in K. So there exists a decreasing sequence $\{C_i\}_{i=1}^{\infty}$ of closed sets contained in K such that each C_i is homeomorphic to I^n, C_i and C_{i+1} have annular difference, $\bigcap_{i=1}^{\infty} C_i = H(A)$, and for each open set U containing H(A) there exists an n such that $C_n \subset U$. Then for each i, by Corollary 3.1, $H^{-1}(C_i)$ is strongly cellular in l_2. For each

i, let $\{C_{ij}\}_{j=1}^{\infty}$ be a strongly cellular sequence for $H^{-1}(C_i)$. Let $n_1 = 1$, and for $i > 1$, let n_i be an integer such that $C_{in_i} \subset \text{Int } H^{-1}(C_{i-1})$. Then set $A_i = C_{in_i}$. It can be seen that $\{A_i\}_{i=1}^{\infty}$ is a strongly cellular sequence for A.

Question 4.2. Is there a type of converse to Theorem 4.2 such as: "Strongly cellular sets in l_2 are homeomorphic either to some point-like set in R^n or to I^{∞}."?

Let D be a decomposition of E, and let $H[D]$ be the set of non-degenerate elements of D. E/D denotes the decomposition space of E defined by D. If $H[D]$ consists of the single element A, then E/A will be used to denote E/D.

When E is finite-dimensional, it is known that A is cellular in E if and only if E/A is homeomorphic to E. Theorem 4.3 is a corresponding relationship for strongly cellular sets in infinite-dimensional normed linear spaces. A good summary of results and problems for finite-dimensional decomposition spaces (in particular for R^3) can be found in (3).

Theorem 4.3. A is strongly cellular in E if and only if E/A is homeomorphic to E.

Proof. (For E infinite-dimensional) For $i = 0,1,\ldots,$ let D_i and T_i denote $B_{\frac{1}{2^i}}$ and $S_{\frac{1}{2^i}}$, respectively.

Necessity. Let $\{C_i\}_{i=1}^{\infty}$ be a strongly cellular sequence for A. By the proof of Theorem 3.6, there exists a cellular sequence $\{A_i\}_{i=1}^{\infty}$ for A such that $A_{i+1} \subset C_i$ for each i, and there exist homeomorphisms f_i of E onto itself such that $f_i(B_1) = A_i$ for $i \geq 1$, and for $i > 1$, $f_i(B_2) = A_{i-1}$. Also, for $i > 1$, let h_i be a homeomorphism of E onto itself such that $h_i(B_2) = D_{i-2}$ and $h_i(B_1) = D_{i-1}$, and let $g_i = f_i h_i^{-1}$.

Let H_1 be the homeomorphism from E - Int B_1 onto E - Int A_1 defined by $H_1 = f_1 \mid (E - \text{Int } B_1)$. Suppose H_i, homeomorphisms of D_{i-2} - Int D_{i-1} onto A_{i-1} - Int A_i ($D_{-1} = A_0 = E$), have been defined for $1 \leq i \leq k$ such that $H_i \mid T_{i-2} = H_{i-1} \mid T_{i-2}$ for $i > 1$. Then define the homeomorphism H_{k+1} from D_{k-1} - Int D_k onto A_k - Int A_{k+1} in the following manner. $g_{k+1}^{-1} H_k \mid T_{k-1}$ is a homeomorphism of T_{k-1} onto itself. This induces a homeomorphism F_{k+1} of D_{k-1} - Int D_k onto itself by mapping radial segments passing through points in T_{k-1} linearly onto radial segments passing through the images of these points in T_{k-1} under $g_{k+1}^{-1} H_k \mid T_{k-1}$. Then set $H_{k+1} = g_{k+1} F_{k+1}$.

Piecewise, the H_i define a homeomorphism H of E - $\{\theta\}$ onto E - A. Extend H to a function $\overline{H}$ from E to E/A by letting $\overline{H}(\theta) = \{A\}$. Then $\overline{H}$ is the desired homeomorphism.

Sufficiency: Let h be a homeomorphism of E/A onto E, let f be the canonical map of E onto E/A, and let F = hf. Without loss of generality assume that $F(A) = \theta$. For each integer i, define the closed cell A_i in the following manner. Let $\varepsilon > 0$ and $x \in E$ be such that $B_\varepsilon(x) \subset F^{-1}(D_{i-1} - D_i)$. Also define $\delta > 0$ and $y \in E$ so that $B_\delta(y) \subset F(B_\varepsilon(x))$. Choose $\sigma > 0$ such that $B_\sigma(i\langle y;\delta\rangle(\theta)) \subset i\langle y;\delta\rangle(D_i)$. Let g be a homeomorphism of E onto itself such that $g(B_\delta(y)) = B_\sigma(i\langle y;\delta\rangle(\theta))$ and $g \mid B_{\frac{\sigma}{2}}(i\langle y;\delta\rangle(\theta)) =$ identity. Define $\tau > 0$ so that $B_\tau \subset i\langle y;\delta\rangle\,[B_{\frac{\sigma}{2}}(i\langle y;\delta\rangle(\theta))]$.

Let G be a homeomorphism of E onto itself such that $G(B_\tau) = D_i$ and $G \mid [B_{\frac{\tau}{2}} \cup (E - D_{i-1})] =$ identity. Define H, a homeomorphism of E onto itself, by $H(x) = F^{-1}Gi\langle y;\delta\rangle gi\langle y;\delta\rangle F(x)$ if $x \in E - A$, and $H(x) =$ otherwise. Let φ be a homeomorphism of E onto itself such that $\varphi(B_1) = E - \operatorname{Int} B_\varepsilon(x)$, and set $A_i = H\varphi(B_1)$. With each A_i defined as above, $\{A_i\}_{i=1}^{\infty}$ is the desired strongly cellular sequence for A.

It has been asked whether the decomposition space defined from any compact upper semi-continuous decomposition of Hilbert space is homeomorphic to Hilbert space. More generally, consider any normed linear space E. Since Σ, defined in Example 3.5, is not strongly cellular in E, and since Σ plus the singleton sets in $E - \Sigma$ make an upper semi-continuous decomposition of E, then by Theorem 4.3 E/Σ is an example of a decomposition space defined from a compact upper semi-continuous decomposition of E which is not homeomorphic to E.

Definition 4.2. A collection D of mutually disjoint closed subsets of a topological space X is discrete if the union of the elements of any subcollection of D is closed in X.

Definition 4.3. A topological space X is collectionwise normal if for each discrete collection D of subsets of X there exists a collection U of mutually disjoint open subsets of X which cover the union of the elements of D such that each element of U intersects at most one element of D.

For a discussion of the concepts of discrete collections and collectionwise normal spaces see (5). In particular, a normed linear space is collectionwise normal since it is metrizable.

Theorem 4.3 can be generalized slightly. A decomposition of E is said to be strongly cellular if every element of it is strongly cellular.

Theorem 4.4. If D is a strongly cellular decomposition of E such that $H[D]$ is finite, then E/D is homeomorphic to E.

Proof. Let $H[D] = \{A_i\}_{i=1}^{n}$, where each A_i is strongly cellular in E. For each i, let f_i be the canonical map from E onto $E/\{A_1, A_2, \ldots, A_i\}$. By Theorem 4.3, there exists a homeomorphism h_1 from E/A_1 onto E. Suppose a homeomorphism h_i from $E/\{A_1, A_2, \ldots, A_i\}$ onto E has been defined for each i, $1 \leq i \leq k < n$. Then inductively define the homeomorphism h_{k+1} from $E/\{A_1, A_2, \ldots, A_{k+1}\}$ onto E as follows. Let U be an open subset of E containing A_{k+1}, but disjoint from $\bigcup_{i=1}^{k} A_i$. There is a strongly cellular sequence for A_{k+1} contained in U, so that $h_k f_k(A_{k+1})$ is strongly cellular in E. Then again by Theorem 4.3, there exists a homeomorphism H from $E/h_k f_k(A_{k+1})$ onto E. Let g be the canonical map from E onto $E/h_k f_k(A_{k+1})$. Hence define $h_{k+1} = Hgh_k f_k f_{k+1}^{-1}$. Therefore, carrying this inductive step to n, h_n is defined and is the desired homeomorphism from E/D onto E.

Theorem 4.5. Let D be a strongly cellular decomposition of E such that $H[D]$ is discrete. If the Annulus Conjecture for E (see Chapter V.) is true, then E/D is homeomorphic to E.

Proof. Since E is collectionwise normal, there exists a collection U of mutually disjoint open subsets of X which cover the union of the elements of $H[D]$ such that each element of U intersects at most one element of $H[D]$. Let $A \in H[D]$ and let V be the union of the elements of U which intersect A. By Theorem 3.6, let $\{A_i\}_{i=1}^{\infty}$ be a strongly cellular sequence for A contained in V such that each A_i is tame and each two elements of $\{A_i\}_{i=1}^{\infty}$ have annular difference. Let h be a homeomorphism of E onto itself such that $h(B_1, S_1) = (A_1, \text{Bd } A_1)$ and $h(B_{\frac{1}{2}}, S_{\frac{1}{2}}) = (A_2, \text{Bd } A_2)$. Let $x \in A$ and $\varepsilon > 0$ be such that $B_\varepsilon(x) \subset \text{Int } A_1$. By the truth of the Annulus Conjecture, there exists a homeomorphism f of B_1 onto itself such that $f(B_{\frac{1}{2}}) = h^{-1}(B_\varepsilon(x))$ and $f \mid S_1 =$ identity. Let g be the homeomorphism of E onto itself defined by $g(x) = hfh^{-1}(x)$ if $x \in A$, and $g(x) = x$ otherwise. By an argument similar to that in Theorem 4.3, it is possible to define a map from $B_\varepsilon(x)$ onto itself which is fixed on $S_\varepsilon(x)$ and which takes $g(A_i - \text{Int } A_{i+1})$ homeomorphically onto $B_{\frac{\varepsilon}{2^{i-2}}} - \text{Int } B_{\frac{\varepsilon}{2^{i-1}}}$ for $i \geq 2$. Thus A_1/A is homeomorphic to A_1 by a homeomorphism fixed on Bd A_1. Therefore, composing such homeomorphisms, one for each element of $H[D]$, gives the desired homeomorphism from E/D onto E.

Theorem 4.6. Let D be a decomposition of E such that $H[D]$ is discrete and E/D is homeomorphic to E. Then D is strongly cellular.

Proof. Let h be a homeomorphism from E/D onto E, and let f be the canonical map from E onto E/D. Let $A \in H[D]$. Since $H[D]$ is discrete, there is an open subset U of E containing A but disjoint from the remaining elements of $H[D]$. Choose $\varepsilon > 0$ so that $B_\varepsilon[hf(A)] \subset hf(U)$. In (17), D. E. Sanderson has established a Schoenfliess theorem for infinite-dimensional normed linear spaces. Hence the closure of the complementary domain of $f^{-1}h^{-1}(S_{\frac{\varepsilon}{2}}[hf(A)])$ containing A, call it C, is a closed cell in E. $hf(\text{Int } C) = \text{Int } B_{\frac{\varepsilon}{2}}[hf(A)]$ which is homeomorphic to E. Therefore $(\text{Int } C)/A$ is homeomorphic to E. So by Theorem 4.3, since $\text{Int } C$ is homeomorphic to E, A is strongly cellular in $\text{Int } C$. Then since $\text{Int } C$ is open in E, A is strongly cellular in E.

V. HALF-OPEN ANNULUS THEOREM

A half-open annulus theorem for arbitrary normed linear spaces can be established analogous to the Half-open Annulus Conjecture proven true in finite-dimensional spaces, see for example (11).

<u>Lemma 5.1</u>. Let C be a closed cell in E, and let f be a homeomorphism of the pair (B_1, S_1) onto the pair (C, Bd C). Then there exists a homeomorphism h from E onto itself such that $h \mid B_{\frac{1}{2}} = f \mid B_{\frac{1}{2}}$.

<u>Proof</u>. Set $D_i = B_{\frac{2^i-1}{2^i}}$, for i = 1,2,... . Since there is a homeomorphism of E onto itself which takes C into B_1 and $f(\theta)$ onto θ, it can be assumed without loss of generality that $C \subset B_1$ and $f(\theta) = \theta$. Choose $\varepsilon > 0$ so that $B_\varepsilon \subset f(D_1)$, and choose $\delta > 0$ so that $B_\delta \subset f^{-1}(B_{\frac{\varepsilon}{2}})$. Define a homeomorphism F_1 of E onto itself so that $F_1(D_1) = B_\delta$ and $F_1 \mid (E - D_2) =$ identity. Let G_1 be a homeomorphism of E onto itself such that $G_1(B_\varepsilon) = B_2$ and $G_1 \mid [B_{\frac{\varepsilon}{2}} \cup (E - B_3)] =$ identity. Then define h_1, a homeomorphism of E onto itself, by $h_1(x) = fF_1^{-1}f^{-1}G_1fF_1f^{-1}(x)$ if $x \in f(D_2)$, and $h_1(x) = G_1(x)$ otherwise.

Suppose homeomorphisms h_i of E onto itself have been defined for $1 \leq i \leq k$ such that $h_i \mid h_{i-1}h_{i-2} \ldots h_1 f(D_i) =$ identity (for $i = 1$, $h_i \mid f(D_i) =$ identity), $B_{2i} \subset h_i h_{i-1} \ldots h_1 f(D_{i+1})$, and $h_i h_{i-1} \ldots h_1 f(B_1) \subset B_{2i+1}$. Then inductively define h_{k+1} in the following manner. Define a homeomorphism F of E onto itself so that $F(D_{k+1}) = D_k$ and $F \mid (E - D_{k+2}) =$ identity. Let G be a homeomorphism of E onto itself such that $G(B_{2k}) = B_{2k+2}$ and $G \mid [B_{2k-1} \cup (E - B_{2k+3})] =$ identity. Then define h_{k+1}, a homeomorphism of E onto itself, by $h_{k+1}(x) = h_k \ldots h_1 f F^{-1} f^{-1} h_1^{-1} \ldots h_k^{-1} G h_k \ldots h_1 f F f^{-1} h_1^{-1} \ldots h_k^{-1}(x)$ if $x \in h_k \ldots h_1 f(D_{k+2})$, and $h_{k+1}(x) = G(x)$ otherwise. Let g be a homeomorphism from E onto Int B_1 such that $g \mid B_{\frac{1}{2}} =$ identity. Then $h = \ldots h_3 h_2 h_1 f g$ is the desired homeomorphism.

The next theorem follows from Lemma 5.1 (also see (17)).

Theorem 5.1. A closed cell in E is tame if and only if it has a collar.

D. E. Sanderson has given an example of a closed cell in l_2 which is not tame (17).

Lemma 5.2. Let C be a closed cell in E contained in B_1 such that B_1 - Int C contains a collar of C. Then there exists a homeomorphism h of Int B_1 onto itself such that $h(B_{\frac{1}{2}}) = C$.

Proof. By Lemma 3.3, it can be assumed without loss of generality that $C \subset B_{\frac{1}{2}}$. If K is a collar of C contained in B_1 - Int C, then there exists a homeomorphism f of the pair $(B_1, B_{\frac{1}{2}})$ onto the pair $(K \cup C, C)$ such that $f(S_1) = \mathrm{Bd}$ $(K \cup C)$. By Lemma 5.1, there exists a homeomorphism F of E onto itself such that $F \mid B_{\frac{1}{2}} = f \mid F_{\frac{1}{2}}$. Let g be a homeomorphism of Int B_1 onto E so that $g \mid B_{\frac{1}{2}}$ = identity. Then define the desired homeomorphism by $h = g^{-1}Fg$.

Lemma 5.3. Let a and b be two numbers such that $0 < a < b$, and let C be a closed cell in E containing B_a. Then there exists a homeomorphism h of E onto itself such that $h(C) \subset B_b$ and $h \mid B_a$ = identity.

Proof. This clearly holds for E finite-dimensional, so assume that E is infinite-dimensional. Let $x \in E - C$ and $\varepsilon > 0$ be such that $x \notin B_{a+\varepsilon}$. Let F be a homeomorphism of E onto itself so that $F(B_{a+\varepsilon}) = B_b$ and $F \mid B_a$ = identity. Choose $\delta > 0$ such that $B_\delta[F(x)] \subset E - [B_b \cup F(C)]$. Let G be a homeomorphism of B_b - Int B_a onto itself such that $G(S_a) = S_b$, $G(S_{\frac{1}{2}(a+b)}) = S_{\frac{1}{2}(a+b)}$, and $G(S_b) = S_a$. Define the homeomorphism f of B_b onto $i<\theta;b>(E - \text{Int } B_a)$ by $f \mid (B_b - \text{Int } B_a) = i<\theta;b>G \mid (B_b - \text{Int } B_a)$ and extend radially to all of B_b. Let $\sigma > 0$ and $y \in E$ be such that $B_\sigma(y) \subset f^{-1}i<\theta;b>(B[F(x)])$. Let g be a homeomorphism of B_b onto itself such that $g[B_\sigma(y)] = B_{\frac{1}{2}(a+b)}$ and $g \mid S_b =$ identity. Define the homeomorphism H of E onto itself by $H(x) = i<\theta;b>fgf^{-1}i<\theta;b>(x)$ if $x \in E - \text{Int } B_a$, and $H(x) = x$ otherwise. Then the desired homeomorphism is $h = HF$.

Theorem 5.2. (Half-open Annulus Theorem) If C is a tame closed cell in E contained in Int B_1, then there exists a homeomorphism h of Int B_1 onto itself such that $h(B_{\frac{1}{2}}) = C$.

Proof. By Lemma 3.1, there exists a closed cell C' in E such that $C' \subset \text{Int } B_1$, $C \subset \text{Int } C'$, and $B - \text{Int } C'$ contains a collar of C'. Then by Lemma 5.2, there exists a homeomorphism f of Int B_1 onto itself such that $f(B_{\frac{1}{2}}) = C'$. Therefore it can be assumed without loss of generality that $C \subset B_{\frac{1}{2}}$. Let g be a homeomorphism of E onto itself such that $g(B_1) = C$. Then by Lemma 5.3, there exists a homeomorphism G of E onto itself such that $G_g(B_2) \subset B_1$ and $G \mid B_{\frac{1}{2}} =$ identity.

Hence $Gg(B_2 - \text{Int } B_1)$ is a collar of C contained in $B_1 -$ Int C. Therefore Lemma 5.2 gives the desired homeomorphism h.

Corollary 5.1. Let C and C' be two tame closed cells in E such that $C' \subset \text{Int } C$. Then there exists a homeomorphism from the pair $(B_1 - B_{\frac{1}{2}}, S_1)$ onto the pair $(C - C', \text{Bd } C)$.

In general it is not known whether the half-open annulus, $\text{Int } B_1 - \text{Int } C$, in Theorem 5.2 can be strengthened to an annulus, $B_1 - \text{Int } C$. This question is the Annulus Conjecture.

Annulus Conjecture for E. Let C be a tame closed cell in E contained in $\text{Int } B_1$. Then there exists a homeomorphism h of B_1 onto itself such that $h(B_{\frac{1}{2}}) = C$ and $h \mid S_1 = \text{identity}$.

If C is a tame closed cell in E contained in $B_{\frac{1}{2}}$, then $B_1 - \text{Int } C$ is an approximation to an annulus in the following sense.

Theorem 5.3. Let C be a tame closed cell in E contained in $B_{\frac{1}{2}}$. Then for any $0 < \varepsilon \leq \frac{1}{2}$, there exists a homeomorphism h of E onto itself such that $h(B_{\frac{1}{2}}) = C$ and $B_{1-\varepsilon} \subset h(B_{1-\varepsilon}) \subset B_1 \subset h(B_1) \subset B_{1+\varepsilon}$.

Proof. Let f be a homeomorphism of E onto itself such that $f(B_{\frac{1}{2}}) = C$. By Lemma 5.3 there exists a homeomorphism g_1 of E onto itself such that $g_1 f^{-1}(B_{1-\varepsilon}) \subset B_{1-\varepsilon}$ and $g_1 \mid B_{\frac{1}{2}}$ = identity. Set $F_1 = fg_1^{-1}f^{-1}$. Again by Lemma 5.3 there exists a homeomorphism G_1 of E onto itself such that $G_1F_1f(B_{1-\varepsilon}) \subset B_1$ and $G_1 \mid B_{1-\varepsilon}$ = identity.

By Lemma 5.3, there exists a homeomorphism g_2 of E onto itself such that $g_2f^{-1}F_1^{-1}G_1^{-1}(B_1) \subset B_1$ and $g_2 \mid B_{1-\varepsilon}$ = identity. Set $F_2 = G_1F_1fg_2^{-1}f^{-1}F_1^{-1}G_1^{-1}$. Also by Lemma 5.3 there exists a homeomorphism G_2 of E onto itself such that $G_2F_2G_1F_1f(B_1) \subset B_{1+\varepsilon}$ and $G_2 \mid B_1$ = identity. Then h = $G_2F_2G_1F_1$ is the desired homeomorphism.

By techniques similar to those used in this section, D. E. Sanderson has also established in (17) an annulus theorem (slightly different than the Annulus Conjecture) for a certain class of infinite-dimensional spaces, which includes Hilbert space.

VI. SUMMARY

In generalizing topological concepts such as cells and cellular sets to arbitrary normed linear spaces, the biggest difficulty seems to come from the fact that the infinite-dimensional spaces are not locally compact. Thus results which come about because of the compactness of closed n-cells in Euclidean n-space do not carry over in general. However, results in Euclidean n-space which are proven using arguments depending only on the boundedness of closed cells, can be handled by a technique inspired by one of V. L. Klee's results. So even though a closed cell is not compact in an infinite-dimensional space, it can be made bounded by a space homeomorphism. Therefore a great many results in Euclidean n-space do actually carry over to infinite-dimensional normed linear spaces.

By examining the results of this dissertation, it appears that the author's concept of strong cellularity imitates the properties of the usual concept of cellularity in Euclidean n-space better than the author's concept of cellularity in arbitrary normed linear spaces. In this sense, perhaps the names of cellular and weakly cellular would be more descriptive than strongly cellular and cellular, respectively. One reason for these concepts being named as they were, is that the author has also considered an even weaker definition of cellularity than the two above. This is the same as the definition for cellularity except "weak closed cells" are used in the definition instead of closed cells. By a weak closed cell is meant a closed homeomorph of the closed unit ball.

The following include some of the major problems and questions raised by this dissertation.

Characterize strong cellularity (cellularity) in normed linear spaces (in l_2) using other properties -- for example Theorem 4.3; also for example of partial results, Theorems 3.3, 3.4, and 4.2.

Find which point-like sets in an infinite-dimensional normed linear space (in l_2) are strongly cellular (cellular) in the space.

If D is a decomposition of E, find properties which must be put on $H[D]$ to insure either that D strongly cellular implies E/D homeomorphic to E or that E/D homeomorphic to E implies D strongly cellular.

If D is a decomposition of l_2 and if $H[D]$ consists of arcs, is E/D homeomorphic to l_2?

Is the Annulus Conjecture true in general or in any particular infinite-dimensional normed linear space?

Can the techniques and results in this dissertation promote and facilitate the study of "infinite-dimensional manifolds" -- spaces which are locally like infinite-dimensional normed linear spaces?

Can the techniques and results in this dissertation be further generalized to arbitrary locally convex linear topological spaces or to locally convex linear metric spaces? It appears that the results of this dissertation do generalize to locally convex linear metric spaces.

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