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I. INTRODUCTION

The first calculations of the elastic electron-deuteron cross section were performed by Schiff[1] and Jankus.[2] These calculations considered relativistic electrons and nonrelativistic deuterons, while treating the nucleons as point particles. The calculation was extended by Glendenning and Kramer,[3] and Gourdin[4] who included relativistic kinematics for the recoil deuteron, and nucleon form factors. Both of these calculations employ the impulse approximation, using the nonrelativistic deuteron wavefunction to describe the deuteron structure, ignoring meson exchange effects, and treating the nucleons as though they were on their mass shell. The method of Gourdin is reviewed below.

In the impulse approximation, the nonrelativistic deuteron current is given by

$$\langle d_f | j_d^\mu | d_i \rangle = \int d^3r\, \Psi^+_{d_f}(r) \left\{ \left[j_p^\mu \right]_{NR} e^{i/2\, \bar{q}\cdot\bar{r}} + \left[j_n^\mu \right]_{NR} e^{-i/2\, \bar{q}\cdot\bar{r}} \right\} \Psi_{d_i}(r) \tag{1.1}$$

where the momentum transfer, in terms of the deuteron momenta, is

$$\bar{q} = \bar{d}_f - \bar{d}_i \tag{1.2}$$

and the nonrelativistic deuteron wavefunction is

$$\Psi_d(r) = \frac{1}{\sqrt{4\pi}\, r}\left[u(r) + \frac{1}{\sqrt{8}}\left(\frac{3}{r^2}\,\bar{\sigma}_p\cdot\bar{r}\;\bar{\sigma}_n\cdot\bar{r} - \bar{\sigma}_p\cdot\bar{\sigma}_n\right) w(r)\right]\chi_d \,. \tag{1.3}$$

The nonrelativistic reduction of the nucleon currents is performed in the Breit frame, using the Foldy-Wouthuysen transformation scheme. In terms of the Dirac and Pauli form factors, the results are

$$[j_N^o]_{NR} = \left(F_{1N} + \frac{q\cdot q\;\kappa_N}{4M_N^2}F_{2N}\right)\chi^+_{N'}\,\chi_N \tag{1.4a}$$

$$[\vec{j}_N]_{NR} = -\frac{1}{2M_N}\left(F_{1N} + \frac{q\cdot q\;\kappa_N}{4M_N^2}F_{2N}\right) 2\bar{p}\,\chi^+_{N'}\,\chi_N$$

$$+\frac{i}{2M_N}\left(F_{1N} + \kappa_N F_{2N}\right)\chi^+_{N'}(\bar{\sigma}_N \times \bar{q})\,\chi_N \tag{1.4b}$$

where $\bar{p}$ is the momentum of the nucleon spectator. So, insertion of Eqs. (1.4) and (1.3) into Eq. (1.1) yields the deuteron current.

A particle of spin 1 has, in general, three form factors.[5] Comparison of the above current with the general form of the deuteron current yields the electromagnetic form factors of the deuteron. They are scalar functions of q·q, given by:

$$G_0 = (G_{Ep} + G_{En})\int_0^\infty [u^2(r) + w^2(r)]\; j_0\left(\frac{qr}{2}\right) dr \,, \tag{1.5a}$$

$$G_1 = 2\,(G_{Mp} + G_{Mn}) \int_0^\infty \Bigg[\left(u^2(r) - \frac{1}{2} w^2(r)\right) j_0 \left(\frac{qr}{2}\right)$$

$$+ \frac{w(r)}{\sqrt{2}} \left(u(r) + \frac{w(r)}{\sqrt{2}}\right) j_2 \left(\frac{qr}{2}\right)\Bigg] dr \;,$$

$$+ \frac{3}{2}\,(G_{Ep} + G_{En}) \int_0^\infty w^2(r) \left[j_0 \left(\frac{qr}{2}\right) + j_2 \left(\frac{qr}{2}\right)\right] dr \qquad (1.5b)$$

$$G_2 = 2\,(G_{Ep} + G_{En}) \int_0^\infty w(r) \left[u(r) - \frac{w(r)}{2\sqrt{2}}\right] j_2 \left(\frac{qr}{2}\right) dr \;, \qquad (1.5c)$$

where the cross section is

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{NS} \Bigg\{\left[G_0^2 + G_2^2 + \frac{2}{3}\eta\, G_1^2\right]$$

$$+ \left[\frac{4}{3}\eta\,(1+\eta)\, G_1^2\right] \tan^2 \frac{\theta}{2}\Bigg\} \qquad (1.6)$$

with

$$\left(\frac{d\sigma}{d\Omega}\right)_{NS} = \frac{\alpha^2}{(2E)^2} \frac{\cos^2 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2}} \frac{1}{\left[1 + 2\frac{E}{M} \sin^2 \frac{\theta}{2}\right]} \qquad (1.7a)$$

and

$$\eta = -\frac{\underline{q}\cdot\underline{q}}{4M^2} \;. \qquad (1.7b)$$

The electric and magnetic form factors of the nucleons are

$$G_{EN} = F_{1N} + \frac{\kappa_N\, q\cdot q}{4M_N^2} F_{2N} \qquad (1.8a)$$

$$G_{MN} = F_{1N} + \kappa_N F_{2N} \tag{1.8b}$$

and in the static limit

$$G_0(0) = 1 \tag{1.9a}$$

$$G_1(0) = 2\left[\mu_p + \mu_n - \frac{3}{2} P_D \left(\mu_p + \mu_n - \frac{1}{2}\right)\right] \tag{1.9b}$$

$$G_2(0) = M^2 Q \tag{1.9c}$$

where Q is the quadrupole moment of the deuteron and P_D is its D-state probability.

The most complete calculations of relativistic effects in the elastic e-d cross section have been made by Gross,[6] and Coester and Ostebee.[7] Coester derives the relativistic corrections using a group theoretical treatment of the two-body problem based upon that of Bakamjian and Thomas.[8] In Gross' work they are derived using the impulse approximation and a relativistic Bethe-Salpeter type wavefunction for the deuteron. For local potentials the results are identical. In Gross' work the corrections are of two types, a kinematic correction arising from the expansion of the current to order M_N^{-2} and a correction arising from the treatment of the deuteron wavefunction in a relativistic manner so as to retain all terms of order M_N^{-2}. Gross' work is briefly reviewed.

The equations for the electromagnetic form factors of the deuteron are calculated to order M_N^{-2} using the impulse approximation and a relativistic Bethe-Salpeter type wavefunction. The wavefunction is

$$\Psi(p_1,p_2) = S_F(p_1)\ \Lambda^{\alpha}(p_1,p_2)\ C\ S_F^T(p_2)\ \mathcal{E}_{\alpha} \tag{1.10}$$

where $S_F(p)$ is the Feynman propagator for a nucleon of momentum p, $\Lambda(p_1,p_2)C$ is the $d \rightarrow n+p$ vertex function with both nucleons off mass shell, and $\mathcal{E}_{\alpha}$ is the deuteron polarization four vector. The nucleon spectator is placed on its mass shell, and the wavefunction for a deuteron of total momentum $\bar{d}$ and internal momentum $\bar{p}$ becomes

$$\phi(\bar{d},\bar{p}) = \phi^{++}_{\alpha\beta}(\bar{d},\bar{p})\ u_{\alpha}\left(\frac{1}{2}\,\bar{d}+\bar{p}\right)\ u^T_{\beta}\left(\frac{1}{2}\,\bar{d}-\bar{p}\right)$$

$$+\ \phi^{-+}_{\alpha\beta}(\bar{d},\bar{p})\ v_{\alpha}\left(-\frac{1}{2}\,\bar{d}-\bar{p}\right)\ u^T_{\beta}\left(\frac{1}{2}\,\bar{d}-\bar{p}\right)\quad . \tag{1.11}$$

The functions ϕ^{++} and ϕ^{-+} contain the vertex function with one nucleon on mass shell and have the physical interpretation that ϕ^{++} is the part in which both nucleons have positive energy and ϕ^{-+} is the part in which one has a negative energy. The corresponding functions ϕ^{+-} and ϕ^{--} were eliminated when one of the nucleons was restricted to its mass shell. The functions ϕ^{++} and ϕ^{-+} are solved by first expressing them in two-component Pauli form, and then expanding the terms to order M_N^{-2} while making the identification

$$\phi^{++}(\bar{o},\bar{p}) = \phi^{NR}(\bar{p}) \tag{1.12a}$$

$$\phi^{-+}(\bar{o},\bar{p}) = 0\quad . \tag{1.12b}$$

Here, $\phi^{NR}(\bar{p})$ is the Fourier transform of the wavefunction defined in Eq. (1.3).

The form factors are defined in terms of the relativistic deuteron current using the impulse approximation and dropping the ϕ^{-+} contribution. In the Breit frame they are

$$G_D^{\mu}(q) = \frac{e}{(2\pi)^3} \int d^3p \, \frac{M_N}{E_N} \, \mathrm{Tr}\left[\bar{\phi}\left(\frac{\bar{q}}{2}, \bar{p} + \frac{\bar{q}}{4}\right) \Gamma^{\mu}(q) \, \phi\left(-\frac{\bar{q}}{2}, \bar{p} - \frac{\bar{q}}{4}\right)\right] \tag{1.13}$$

where

$$\bar{\phi}(\bar{d},\bar{p}) = - C \gamma^5 \phi^T(\bar{d},\bar{p}) \; (C \gamma^5)^{-1} \tag{1.14a}$$

and

$$\Gamma^{\mu}(q) = \left(F_{1p} + F_{1n}\right) \gamma^{\mu} + \frac{i\sigma^{\mu\nu} q_{\nu}}{2M_N} \left(\kappa_p F_{2p} + \kappa_n F_{2n}\right) \quad . \tag{1.14b}$$

A Pauli reduction is performed on the current, keeping terms of the appropriate order, and then the trace is taken. The electromagnetic form factors of the deuteron are then obtained from the appropriate combinations of the form factors in Eq. (1.13). They are

$$G_0 = \frac{1}{\sqrt{1+\eta}} \left\{ G_{ES} \, C_E + G_{ES} \, I_C + (2 \, G_{MS} - G_{ES}) \, J_C \right\} \tag{1.15a}$$

$$\begin{aligned} G_1 = \frac{1}{\sqrt{1+\eta}} \Bigg\{ & \left(1 + \frac{1}{2}\eta\right) G_{ES} \, C_L - 3\eta \, (G_{ES} - G_{MS}) \, C_L \\ & + 2 \left(1 + \frac{1}{2}\eta\right) G_{MS} \, C_S + 2 \, G_{ES} \, I_{m1} + 2 \, G_{MS} \, (I_{m2} + J_{m1}) \\ & + 2 \, (G_{ES} - G_{MS}) \, J_{m2} \Bigg\} \end{aligned} \tag{1.15b}$$

$$G_2 = \frac{1}{\sqrt{1+\eta}} \left\{ G_{ES}\, C_Q + G_{ES}\, I_Q + (2\, G_{MS} - G_{ES})\, J_Q \right\} \tag{1.15c}$$

where

$$G_{ES} = \frac{1}{2}\,(G_{Ep} + G_{En}) \tag{1.16a}$$

$$G_{MS} = \frac{1}{2}\,(G_{Mp} + G_{Mn}) \tag{1.16b}$$

and the functions C, I, and J are given in Ref. (6).

Other important considerations are meson exchange currents and dispersion relation calculations of the deuteron structure. The contributions to the deuteron current from meson exchange currents have been discussed by several authors,[9-14] and should be significant at high momentum transfer. However, they have not been observed experimentally,[15,16] even in the ranges where they should dominate.[17] Attempts have also been made to describe the deuteron structure using dispersion relations.[18-20] However, at their present stage of development, there is no general agreement regarding their value as a quantitative tool.

The deuteron cross section has been the object of extensive experimental study. A good review of the results for values of $q\cdot q$ less than 1 $(\text{GeV}/c)^2$ may be found in Refs. (15) and (16). Also, some data now exists for values of $q\cdot q$ up to 4 $(\text{GeV}/c)^2$.[17]

In this work, a technique for calculating the electromagnetic form factors of the deuteron is developed, that employs the standard rules for Feynman graphs. A model of the vertex for $d \to n+p$ is constructed and the scattering amplitude is calculated in the one photon exchange approximation. Finally, the deuteron form factors are calculated for the S-wave part of the deuteron wavefunction and are found to agree with Eqs. (1.5) in the appropriate limit.

The advantage of this technique is that it does not rely upon an expansion in terms of $q{\cdot}q/M^2$, an important property, since the experimental range is now to the point where that parameter is one. Also, if an energy range is found where our model for the vertex function proves inadequate, it should be possible to generalize it. Finally, the technique has the advantage of employing the standard Feynman diagram techniques with their prescriptions for including more complicated graphs or calculating similar problems.

II. THE DEUTERON WAVEFUNCTION

In this chapter the free deuteron wavefunction is constructed and some of its properties are explored. The first section of this chapter reviews a technique for describing the wavefunction of a free massive particle with any spin. This technique, developed by Weaver, Hammer, and Good,[21] is the best of the several techniques for our purposes. In the other section, the manner in which this technique may be applied is demonstrated by constructing the standard column matrix form of the wavefunctions of spin $\frac{1}{2}$ and spin 1 particles. Finally, this method is used to construct the 4×4 symmetric matrix form of the deuteron wavefunction. It is this form which is compatible with the techniques employed in Chapter III to determine the scattering amplitude for electron deuteron scattering.

A. General Remarks

In this section a few general remarks are made concerning the construction of wavefunctions for free massive particles with any spin. First, the manner in which these wavefunctions transform is reviewed. Then, the particle's wavefunction is constructed in its rest frame. Finally, the general form of the wavefunction is given and a few specific remarks are made about Lorentz scalars and normalization.

For Lorentz transformations continuous with the identity

$$x'^{\mu} = a^{\mu}{}_{\nu}(\bar{\tau})\ x^{\nu} \quad , \tag{2.1}$$

the wavefunction transformation rule is

$$\Psi'(x') = \Lambda(\bar{\tau})\ \Psi(x) \quad , \tag{2.2}$$

where

$$\Lambda(\bar{\tau}) = \begin{pmatrix} e^{i\bar{\tau}\cdot\bar{s}} & 0 \\ 0 & e^{i\bar{\tau}^{*}\cdot\bar{s}} \end{pmatrix} \quad . \tag{2.3}$$

Here the matrices $\bar{s}$ are the standard spin matrices and $\bar{\tau}$ is a three-vector with complex components parameterizing the Lorentz transformation.

To relate this transformation rule to spinor analysis, note that a symmetric spinor with 2s upper undotted indices provides a basis for an irreducible representation of the Lorentz group. The spinors transform according to the rule

$$\phi'(x')^{\alpha\cdots} = \left(e^{\frac{1}{2}\ \bar{\tau}^{*}\cdot\bar{\sigma}}\right)^{\alpha\beta} \cdots \phi(x)^{\beta\cdots} \tag{2.4}$$

for each of these 2s indices. The spinors have 2s+1 independent components, which when organized into column matrix form transform according to

$$\phi'(x') = e^{i\bar{\tau}^{*}\cdot\bar{s}}\ \phi(x) \quad . \tag{2.5}$$

The conjugate representation transforms like

$$\chi'(x') = e^{i\bar{\tau}\cdot\bar{s}}\ \chi(x), \tag{2.6}$$

where χ is generated from the symmetric spinor with 2s lower dotted indices. In these terms, the wavefunction is

$$\Psi(x) = \begin{pmatrix} \chi(x) \\ \\ \phi(x) \end{pmatrix} . \tag{2.7}$$

To relate the parameters $\bar{\tau}$ to the elements of the transformation coefficients $a^{\mu}{}_{\nu}$, consider the 2 × 2 representation of Lorentz transformations continuous with the identity, where

$$\left(e^{\frac{i}{2}\bar{\tau}\cdot\bar{\sigma}}\right)^{+} \sigma^{\mu} e^{\frac{i}{2}\bar{\tau}\cdot\bar{\sigma}} = a^{\mu}{}_{\nu}\sigma^{\nu} . \tag{2.8}$$

For a pure Lorentz transformation between an unprimed frame moving with velocity $\bar{v}$ in the primed frame, the transformation coefficients are

$$a^{i}{}_{j} = \delta^{ij} + (\gamma-1)\ v^{i}v^{j}/v^{2} , \tag{2.9a}$$

$$a^{i}{}_{o} = a^{o}{}_{i} = \gamma v^{i} , \tag{2.9b}$$

$$a^{o}{}_{o} = \gamma . \tag{2.9c}$$

So, if $\bar{\tau}$ is pure imaginary, substitution of the identity

$$e^{\frac{i}{2}\bar{\tau}\cdot\bar{\sigma}} = \cos\frac{1}{2}\tau + i\ \frac{\bar{\tau}\cdot\bar{\sigma}}{\tau}\ \sin\frac{1}{2}\tau \tag{2.10}$$

into Eq. (2.8), for σ^{o}, yields

$$\cos\tau = \gamma \tag{2.11a}$$

$$i\ \sin\tau = v\gamma \tag{2.11b}$$

and

$$\cos \frac{1}{2} \tau = \sqrt{\frac{1}{2} (\gamma+1)} \tag{2.12a}$$

$$i \sin \frac{1}{2} \tau = \sqrt{\frac{1}{2} (\gamma-1)} \quad . \tag{2.12b}$$

By a similar argument, performed for the σ^i, it can be demonstrated that if $\bar{\tau}$ is real the transformation is a space rotation in the right-hand sense about the direction $\hat{\bar{\tau}}$ through the angle τ.

Consider a particle or antiparticle with mass M. If the wave equation in the particle's rest system is

$$M \beta \Psi_R = i \frac{\partial \Psi_R}{\partial t} \quad , \tag{2.13}$$

then the solutions can be written in the form

$$\Psi_{R\varepsilon}(t_R, m) = v_{R\varepsilon}(m) \; e^{-i\varepsilon M t_R} \quad , \tag{2.14}$$

where the $v_{R\varepsilon}(m)$ are eigenfunctions of the Hamiltonian

$$\beta M \; v_{R\varepsilon}(m) = \varepsilon M v_{R\varepsilon}(m) \quad ; \tag{2.15}$$

the particle/antiparticle operator

$$\beta \; v_{R\varepsilon}(m) = \varepsilon v_{R\varepsilon}(m) \quad ; \tag{2.16}$$

and the polarization operator

$$\beta \; s_3 \; v_{R\varepsilon}(m) = m \; v_{R\varepsilon}(m) \quad . \tag{2.17}$$

The R indicates rest system quantities, ε is ± 1 for the particle/antiparticle, and m ranges from $-s$ to s.

To obtain the solution to these equations, note that if the normalization is

$$v^{+}_{R\varepsilon}(m)\ v_{R\varepsilon'}(m') = \delta_{\varepsilon\varepsilon'}\ \delta_{mm'} \quad , \tag{2.18}$$

then for a particle

$$v_{R+1}(m) = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_R(m) \\ \\ \chi_R(m) \end{pmatrix} \quad , \tag{2.19}$$

where

$$s_3\ \chi_R(m) = m\ \chi_R(m) \quad , \tag{2.20}$$

$$(s_1 \pm i s_2)\ \chi_R(m) = [s(s+1) - m(m\pm1)]^{\frac{1}{2}}\ \chi_R(m\pm1) \quad , \tag{2.21}$$

and

$$\chi^{+}_R(m)\ \chi_R(m') = \delta_{mm'} \quad . \tag{2.22}$$

And finally, if the normalization is chosen such that

$$\int d^3x\ \Psi^{+}_{R+1}\ (t_R, m)\ \Psi_{R+1}\ (t_R, m) = 1 \quad , \tag{2.23}$$

then the rest frame wavefunction is

$$\Psi_{R+1}\ (t_R, m) = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2}} \begin{pmatrix} \chi_R(m) \\ \\ \chi_R(m) \end{pmatrix} e^{-iMt_R} \quad . \tag{2.24}$$

So the particle's wavefunction, in any frame, is

$$\Psi(x) = \sqrt{\frac{M}{EV}}\ u(\bar{\tau},m)\ e^{-i\ p\cdot x}\quad , \tag{2.25}$$

where τ is given by Eqs. (2.11) and

$$u(\bar{\tau},m) = \Lambda(\bar{\tau})\ v_{R+1}(m) \tag{2.26}$$

and

$$p\cdot x = Et - \bar{p}\cdot\bar{x}\quad . \tag{2.27}$$

This choice yields the scalar

$$\begin{aligned} \bar{u}(\bar{\tau},m)\ u(\bar{\tau},m) &= (\Lambda(\bar{\tau})\ v_{R+1}(m))^{+} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\Lambda(\bar{\tau}) v_{R+1}(m)) \\ &= \chi_R^{+}(m)\ \chi_R(m) \\ &= 1\quad ; \end{aligned} \tag{2.28}$$

and the invariant integral

$$-\frac{i}{2M}\int d^3x\ \left(\frac{\partial\bar{\Psi}}{\partial t}\Psi - \bar{\Psi}\frac{\partial\Psi}{\partial t}\right) = 1\quad . \tag{2.29}$$

So, since the form of the wavefunction of a particle in its rest frame is known, its wavefunction in any frame can be found by using the above techniques.

B. The Deuteron Wavefunction

In this section the free deuteron wavefunction is constructed. In order to gain insight into the techniques necessary to construct this wavefunction, the case of a spin $\frac{1}{2}$ particle is considered first. Then the case of a spin 1 particle is considered and two separate forms of the deuteron wavefunction are constructed. Finally, a few remarks are made about the nonrelativistic deuteron wavefunction, including both S- and D-state wavefunctions.

Throughout the remainder of this chapter, a convention is employed. An index denoted by one of the letters from a to n is an integer ranging from 1 to 3, while an integer ranging from 1 to 6 is denoted by one of the letters ranging from o to z. If it is denoted by a letter from the Greek alphabet, then the symbols α to δ imply 1 or 2 and ε to ω imply 1 to 4.

First, consider the case of a massive particle with spin $\frac{1}{2}$. If a particle of mass M is moving at the velocity $\bar{p}/E$, then the wavefunction is

$$\Psi(x) = \sqrt{\frac{M}{EV}}\, u(\bar{p},m)\, e^{-i\, p\cdot x} \quad , \tag{2.30}$$

where

$$u_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi \\ \phi \end{pmatrix}_\mu \tag{2.31}$$

and

$$p\cdot x = Et - \bar{p}\cdot\bar{x} \quad . \tag{2.32}$$

The spinors are the standard two component spinors with the transformation rules

$$\chi' = e^{\frac{i}{2}\bar{\tau}\cdot\bar{\sigma}}\,\chi \qquad (2.33a)$$

$$\phi' = e^{\frac{i}{2}\bar{\tau}^{*}\cdot\bar{\sigma}}\,\phi\ . \qquad (2.33b)$$

So, since $\bar{v}$ is $\bar{p}/E$ and γ is E/M, the substitution of Eqs. (2.12a) and (2.12b) into Eq. (2.10) yields the explicit connection between these spinors and their form in the rest frame

$$\chi(m) = \sqrt{\frac{E+M}{2M}}\left(1 + \frac{\bar{\sigma}\cdot\bar{p}}{E+M}\right)\chi_R(m) \qquad (2.34a)$$

$$\phi(m) = \sqrt{\frac{E+M}{2M}}\left(1 - \frac{\bar{\sigma}\cdot\bar{p}}{E+M}\right)\chi_R(m)\ . \qquad (2.34b)$$

In the standard representation where σ_3 is diagonal

$$\chi_R\left(\frac{1}{2}\right) = \begin{pmatrix}1\\ \\ 0\end{pmatrix} \qquad (2.35a)$$

$$\chi_R\left(-\frac{1}{2}\right) = \begin{pmatrix}0\\ \\ 1\end{pmatrix}\ . \qquad (2.35b)$$

The normalization of the wavefunction also yields several interesting properties. The scalar is

$$\bar{u}(\bar{p},m)\ u(\bar{p},m) = 1 \qquad (2.36)$$

with

$$u^{+}(\bar{p},m)\ u(\bar{p},m) = E/M \quad . \tag{2.37}$$

The invariant integral is

$$\int d^3\mathbf{x}\ \Psi^{+}(\mathbf{x})\ \Psi(\mathbf{x}) = 1 \quad . \tag{2.38}$$

And finally, the nonrelativistic limit of the wavefunction is

$$\Psi(\mathbf{x}) = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2}}\begin{pmatrix}\chi_R \\ \\ \chi_R\end{pmatrix} e^{-i\ p\cdot x} \quad . \tag{2.39}$$

Next, consider the standard wavefunction of a massive spin 1 particle. For a particle of mass M moving at the velocity $\bar{p}/E$ the six component wavefunction is

$$\Psi(\mathbf{x}) = \sqrt{\frac{M}{EV}}\ u(\bar{p},m)\ e^{-i\ p\cdot x} \quad , \tag{2.40}$$

where

$$u_r = \frac{1}{\sqrt{2}}\begin{pmatrix}\chi \\ \\ \phi\end{pmatrix}_r \quad . \tag{2.41}$$

The spinors are the standard 3 component spinors and their transformation rules are

$$\chi' = e^{i\ \bar{\tau}\cdot\bar{s}}\ \chi \tag{2.42a}$$

$$\phi' = e^{i\ \bar{\tau}^{*}\cdot\bar{s}}\ \phi \quad . \tag{2.42b}$$

For spin 1 matrices

$$e^{i\ \bar{\tau}\cdot\bar{s}} = 1 + i\ \bar{s}\cdot\hat{\bar{\tau}}\ \sin\tau + (\bar{s}\cdot\hat{\bar{\tau}})^2\ (\cos\tau - 1). \tag{2.43}$$

So, substitution of Eqs. (2.11a) and (2.11b) into Eq. (2.43) yields the explicit connection between these spinors and their form in the rest frame

$$\chi(m) = \left[1 + \frac{\bar{s}\cdot\bar{p}}{M} + \frac{(\bar{s}\cdot\bar{p})^2}{M(E+M)}\right] \chi_R(m) \tag{2.44a}$$

$$\phi(m) = \left[1 - \frac{\bar{s}\cdot\bar{p}}{M} + \frac{(\bar{s}\cdot\bar{p})^2}{M(E+M)}\right] \chi_R(m) \quad . \tag{2.44b}$$

In the standard representation where s_3 is diagonal

$$\chi_R(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{2.45a}$$

$$\chi_R(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{2.45b}$$

$$\chi_R(-1) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad . \tag{2.45c}$$

The normalization also yields several interesting properties. The scalar is

$$\bar{u}(\bar{p},m)\; u(\bar{p},m) = 1 \quad . \tag{2.46}$$

The invariant integral is

$$-\frac{1}{2}\frac{i}{M}\int d^3x \left(\frac{\partial\bar{\Psi}}{\partial t}\Psi - \bar{\Psi}\frac{\partial\Psi}{\partial t}\right) = 1 \quad . \tag{2.47}$$

And finally, the nonrelativistic limit of the wavefunction is

$$\Psi(x) = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2}}\begin{pmatrix}\chi_R\\ \\ \chi_R\end{pmatrix} e^{-i\,p\cdot x} \quad . \tag{2.48}$$

Finally, the form of the deuteron wavefunction employed in subsequent sections is constructed. For a particle of mass M and spin 1 moving at the velocity $\bar{p}/E$, the 4×4 symmetric matrix form of the wavefunction is

$$\Psi(x) = \sqrt{\frac{M}{EV}}\, u(\bar{p},m)\; e^{-i\,p\cdot x} \quad , \tag{2.49}$$

where

$$u_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix}\chi_{\dot{1}\dot{1}} & \chi_{\dot{1}\dot{2}} & 0 & 0\\ \chi_{\dot{2}\dot{1}} & \chi_{\dot{2}\dot{2}} & 0 & 0\\ 0 & 0 & \phi^{11} & \phi^{12}\\ 0 & 0 & \phi^{21} & \phi^{22}\end{pmatrix}_{\mu\nu} \quad . \tag{2.50}$$

The transformation rules for the spinors are

$$\chi'_{\dot{\alpha}\dot{\beta}} = \left[e^{\frac{i}{2}\bar{\tau}\cdot\bar{\sigma}}\;\chi\;\left(e^{\frac{i}{2}\bar{\tau}\cdot\bar{\sigma}}\right)^T\right]_{\dot{\alpha}\dot{\beta}} \tag{2.51a}$$

$$\phi'^{\alpha\beta} = \left[e^{\frac{i}{2}\bar{\tau}^*\cdot\bar{\sigma}}\;\phi\;\left(e^{\frac{i}{2}\bar{\tau}^*\cdot\bar{\sigma}}\right)^T\right]^{\alpha\beta} \quad . \tag{2.51b}$$

So, the explicit connection between these spinors and their form in the rest frame is

$$\chi_{\dot{\alpha}\dot{\beta}}(m) = \frac{E+M}{2M}\left[\left(1 + \frac{\bar{\sigma}\cdot\bar{p}}{E+M}\right)\chi_R(m)\left(1 + \frac{\bar{\sigma}^T\cdot\bar{p}}{E+M}\right)\right]_{\dot{\alpha}\dot{\beta}} \tag{2.52a}$$

$$\phi^{\alpha\beta}(m) = \frac{E+M}{2M}\left[\left(1 - \frac{\bar{\sigma}\cdot\bar{p}}{E+M}\right)\chi_R(m)\left(1 - \frac{\bar{\sigma}^T\cdot\bar{p}}{E+M}\right)\right]^{\alpha\beta} \quad . \tag{2.52b}$$

If the explicit form of the rest frame spinor is

$$\chi_R(+1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.53a}$$

$$\chi_R(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.53b}$$

$$\chi_R(-1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad , \tag{2.53c}$$

then an examination of Eqs. (2.45) and (2.53), and Eqs. (2.44) and (2.52) shows that the relationship between the symmetric spinors and the standard 3 component spinors is

$$\begin{pmatrix} \chi_{\dot{1}\dot{1}} & \chi_{\dot{1}\dot{2}} \\ \chi_{\dot{2}\dot{1}} & \chi_{\dot{2}\dot{2}} \end{pmatrix} = \begin{pmatrix} \chi_1 & \frac{1}{\sqrt{2}}\chi_2 \\ \frac{1}{\sqrt{2}}\chi_2 & \chi_3 \end{pmatrix} \tag{2.54a}$$

$$\begin{pmatrix} \phi^{11} & \phi^{12} \\ \phi^{21} & \phi^{22} \end{pmatrix} = \begin{pmatrix} \phi_1 & \frac{1}{\sqrt{2}}\phi_2 \\ \frac{1}{\sqrt{2}}\phi_2 & \phi_3 \end{pmatrix} \quad . \tag{2.54b}$$

This is as expected, since in the bracket notation

$$|1\ M\rangle = \sum_{m_1 m_2} \langle \tfrac{1}{2}\,\tfrac{1}{2}\,m_1 m_2|1\ M\rangle |\tfrac{1}{2}\,m_1\rangle\ |\tfrac{1}{2}\,m_2\rangle \tag{2.55}$$

$$\bar{s}|1\ M\rangle = \sum_{m_1 m_2} \langle \tfrac{1}{2}\,\tfrac{1}{2}\,m_1 m_2|1\ M\rangle\ \frac{1}{2}\ (\bar{\sigma}_1+\bar{\sigma}_2)|\tfrac{1}{2}\,m_1\rangle\ |\tfrac{1}{2}\,m_2\rangle, \tag{2.56}$$

so that the boosts are

$$\left(1 \pm \frac{\bar{s}\cdot\bar{p}}{M} + \frac{(\bar{s}\cdot\bar{p})^2}{M(E+M)}\right) |1\ M>$$

$$= \sum_{m_1 m_2} <\frac{1}{2}\,\frac{1}{2}\,m_1 m_2|1\ M> \left(\frac{E+M}{2M}\right)\left(1 \pm \frac{\bar{\sigma}_1\cdot\bar{p}}{E+M}\right)|\frac{1}{2}\,m_1> \left(1 \pm \frac{\bar{\sigma}_2\cdot\bar{p}}{E+M}\right) |\frac{1}{2}\,m_2> . \tag{2.57}$$

Finally, this connection between the different representations leads to

$$(\phi)^{\dot{\alpha}\dot{\beta}}(\chi)_{\dot{\beta}\dot{\alpha}} = (\phi^+)_a\ (\chi)_a = 1 \tag{2.58a}$$

$$(\chi)_{\alpha\beta}(\phi)^{\beta\alpha} = (\chi^+)_a\ (\phi)_a = 1\quad . \tag{2.58b}$$

The normalization of the wavefunction also yields several interesting properties. If

$$\bar{u} = \beta\ u^+\ \beta\quad , \tag{2.59}$$

then

$$(\bar{u})_{\mu\nu}\ u_{\nu\mu} = \frac{1}{2}\,\mathrm{Tr}\left[\begin{pmatrix} \phi^{\dot{1}\dot{1}} & \phi^{\dot{1}\dot{2}} & 0 & 0 \\ \phi^{\dot{2}\dot{1}} & \phi^{\dot{2}\dot{2}} & 0 & 0 \\ 0 & 0 & \chi_{11} & \chi_{12} \\ 0 & 0 & \chi_{21} & \chi_{22} \end{pmatrix}\begin{pmatrix} \chi_{\dot{1}\dot{1}} & \chi_{\dot{1}\dot{2}} & 0 & 0 \\ \chi_{\dot{2}\dot{1}} & \chi_{\dot{2}\dot{2}} & 0 & 0 \\ 0 & 0 & \phi^{11} & \phi^{12} \\ 0 & 0 & \phi^{21} & \phi^{22} \end{pmatrix}\right]$$

$$= \frac{1}{2}\left[\phi^{\dot{\alpha}\dot{\beta}}\ \chi_{\dot{\beta}\dot{\alpha}} + \chi_{\alpha\beta}\ \phi^{\beta\alpha}\right]$$

$$= (\bar{u})_r\ u_r = 1\quad . \tag{2.60}$$

Also, there is a scalar produced from two-spin $\frac{1}{2}$ wavefunctions and a spin-1 wavefunction

$$\bar{u}_\mu \, \bar{u}_\nu \, u_{\mu\nu} = \frac{1}{2\sqrt{2}} \left(\phi^{\dot{\alpha}} \chi_{\dot{\alpha}\dot{\beta}} \phi^{\dot{\beta}} + \chi_\alpha \phi^{\alpha\beta} \chi_\beta \right) . \qquad (2.61)$$

The invariant integral is the same as in the six-component case and the nonrelativistic limit of the wavefunction is

$$\Psi(x) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_1 & \frac{1}{\sqrt{2}}\chi_2 & 0 & 0 \\ \frac{1}{\sqrt{2}}\chi_2 & \chi_3 & 0 & 0 \\ 0 & 0 & \chi_1 & \frac{1}{\sqrt{2}}\chi_2 \\ 0 & 0 & \frac{1}{\sqrt{2}}\chi_2 & \chi_3 \end{pmatrix}_R e^{-i\, p\cdot x} . \qquad (2.62)$$

In conclusion, consider the nonrelativistic wavefunction of the deuteron. It is a 2 × 2 matrix

$$\Psi_d(\bar{x}_p, \bar{x}_n) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{4\pi}} \frac{1}{r} \left[u(r) + \frac{1}{\sqrt{8}} S_{pn} \, w(r) \right] \chi_d \ ; \qquad (2.63)$$

where

$$r = |\bar{x}_p - \bar{x}_n| \qquad (2.64)$$

and

$$S_{pn} = \frac{3}{r^2} \bar{\sigma}_p \cdot \bar{r} \, \bar{\sigma}_n \cdot \bar{r} - \bar{\sigma}_p \cdot \bar{\sigma}_n \ . \qquad (2.65)$$

Its normalization is

$$\int d^3x_p \int d^3x_n \, \Psi_d^+(\bar{x}_p,\bar{x}_n) \, \Psi_d(\bar{x}_p,\bar{x}_n)$$

$$= \int_0^\infty dr \, [|u(r)|^2 + |w(r)|^2] \quad , \tag{2.66}$$

and its transform

$$\phi(\bar{p}) = \int d^3y \, e^{-i \, \bar{p}\cdot\bar{y}} \, \Psi_d(y) \tag{2.67}$$

is

$$\phi(\bar{p}) = \left[\phi_u(p) + \frac{1}{\sqrt{8}} \left(3 \, \bar{\sigma}_p\cdot\hat{\bar{p}} \, \bar{\sigma}_n\cdot\hat{\bar{p}} - \bar{\sigma}_p\cdot\bar{\sigma}_n\right) \phi_w(p)\right] \chi_d \quad , \tag{2.68}$$

where

$$\phi_u(p) = \sqrt{4\pi} \int_0^\infty ydy \; u(y) \; j_0(py) \tag{2.69a}$$

$$\phi_w(p) = - \sqrt{4\pi} \int_0^\infty ydy \; w(y) \; j_2(py) \quad . \tag{2.69b}$$

III. THE SCATTERING MATRIX

In this chapter the scattering amplitude for electron-deuteron scattering is calculated. The calculation employs the standard rules for Feynman graphs and is carried out in the one photon exchange approximation. In order to achieve the calculation, the vertex for $d \to n + p$ must be constructed. This is done to an overall factor. Then, the Feynman rules are applied to the pair of graphs shown in Figure 1, and the scattering amplitude is calculated. Finally, a special limit of the scattering amplitude is examined in order to determine the overall factor multiplying the vertex function.

A. The Vertex

In this section the matrix structure of the vertex V, for $d \to n + p$, is determined to an overall factor. To achieve this, it is assumed that $\bar{\Psi}_p \bar{\Psi}_n V \Psi_d$ is a Lorentz scalar, and the non-relativistic amplitude for the deuteron consisting of a neutron and a proton is used as a guide in making this determination. Once the relative matrix structure of V is found, the scattering amplitude can be calculated.

In the nonrelativistic limit, the amplitude for the deuteron consisting of a neutron and a proton is

$$A = \int d^3x_p \int d^3x_n \, \Psi_p^+(\bar{x}_p) \, \Psi_n^+(\bar{x}_n) \, \Psi_d(\bar{x}_p,\bar{x}_n) \quad , \tag{3.1}$$

where

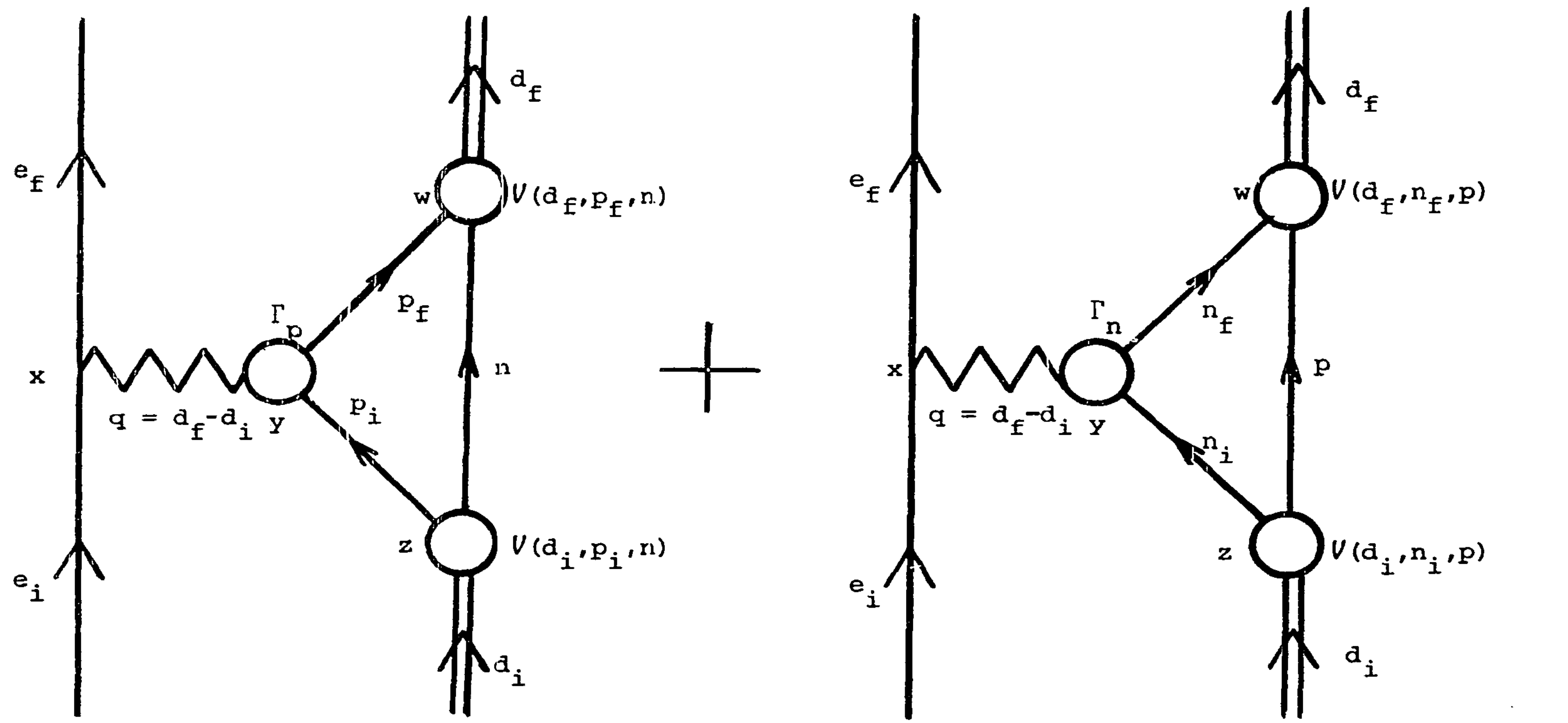

Figure 1. Electron deuteron elastic scattering in the one photon exchange approximation.

$$\Psi_p(\bar{x}_p) = \frac{1}{\sqrt{V}} e^{i\,\bar{p}\cdot\bar{x}_p}\,\chi_p \tag{3.2a}$$

$$\Psi_n(\bar{x}_n) = \frac{1}{\sqrt{V}} e^{i\,\bar{n}\cdot\bar{x}_n}\,\chi_n \tag{3.2b}$$

and the wavefunction of the deuteron is given by Eq. (2.63). Substitution of these wavefunctions into Eq. (3.1) yields

$$A = \frac{1}{V^{\frac{3}{2}}} \int d^3x_p \int d^3x_n\; e^{-i\,(\bar{p}\cdot\bar{x}_p+\bar{n}\cdot\bar{x}_n)}\,\chi_p^+\,\chi_n^+$$

$$\frac{1}{\sqrt{4\pi}}\,\frac{1}{r}\left[u(r) + \frac{1}{\sqrt{8}}\,S_{pn}\;w(r)\right]\chi_d \quad . \tag{3.3}$$

Then making the change of variables

$$\bar{R} = \frac{m_p\bar{x}_p + m_n\bar{x}_n}{m_p + m_n} \tag{3.4a}$$

$$\bar{r} = \bar{x}_p - \bar{x}_n \quad , \tag{3.4b}$$

which has a Jacobian that is one, yields the amplitude

$$A = \frac{(2\pi)^3\;\delta^3(\bar{p}+\bar{n})}{V^{\frac{3}{2}}}\,\chi_p^+\,\chi_n^+\left[\phi_u(p) + \frac{1}{\sqrt{8}}\,(3\;\bar{\sigma}_p\cdot\hat{\bar{p}}\;\bar{\sigma}_n\cdot\hat{\bar{p}} - \bar{\sigma}_p\cdot\bar{\sigma}_n)\phi_w(p)\right]\chi_d \;, \tag{3.5}$$

where ϕ_u and ϕ_w are given by Eqs. (2.69).

The form of V which this amplitude suggests is

$$V(p,n,d) = \zeta(p,n,d)\left\{\sqrt{2}\;\phi_u(\rho) + \left[\frac{3}{2M^2}\,\frac{1}{\rho^2}\,\sigma^p_{\mu\nu}\;d^\mu p^\nu \sigma^n_{\alpha\beta}\;d^\alpha n^\beta\right.\right.$$

$$\left.\left. - \frac{1}{8}\,\sigma^p_{\mu\nu}\;\sigma_n^{\mu\nu}\right]\phi_w(\rho)\right\} \quad , \tag{3.6}$$

where

$$\rho^2 = \frac{1}{2}\left[\left(\frac{n\cdot d}{M}\right)^2 - n\cdot n\right] + \frac{1}{2}\left[\left(\frac{p\cdot d}{M}\right)^2 - p\cdot p\right] \tag{3.7}$$

and ζ is a scalar function to be determined. To see that this is a reasonable guess, first note that in the deuteron's rest frame

$$\rho^2 = \bar{p}\cdot\bar{p} \quad . \tag{3.8}$$

Then, observe that if the neutron and proton are nonrelativistic in this rest frame, the amplitude A' defined by

$$A' = \int d^4x \; \bar{\Psi}_p(x) \; \bar{\Psi}_n(x) \; V \, \Psi_d(x) \tag{3.9}$$

becomes

$$A' = \frac{(2\pi)^4 \; \delta^3(\bar{p}+\bar{n}) \; \delta(E_p+E_n-M)}{V^{\frac{3}{2}}} \chi_p^+ \chi_n^+$$

$$\zeta \left[\phi_u(p) + \frac{1}{\sqrt{8}} (3 \; \bar{\sigma}_p\cdot\hat{\bar{p}} \; \bar{\sigma}_n\cdot\hat{\bar{p}} - \bar{\sigma}_p\cdot\bar{\sigma}_n) \; \phi_w(p)\right] \chi_d \quad . \tag{3.10}$$

So, this has the same matrix form as the nonrelativistic amplitude for the deuteron consisting of a neutron and a proton, Eq. (3.5).

B. The Scattering Amplitude

In this section the vertex suggested above is combined with the Feynman rules, to calculate the scattering amplitude. The calculation will be carried out in the one photon exchange approximation using the graphs shown in Figure 1.

Considering the diagrams in Figure 1, the scattering amplitude is

$$S_{fi} = (-i) \int d^4x \left[e\, \bar{\Psi}_{e_f}(x) \gamma_\mu \Psi_{e_i}(x) \right] \int d^4y\, D_F\, (y-x)\ e_p$$

$$\int d^4w \int d^4z\ \bar{\Psi}_{d_f}(w)\ \bar{V}_f S_{F_p}(w-y)\ \Gamma^\mu_p\ S_{F_p}(y-z)\quad S_{F_n}(w-z)$$

$$V_i\ \Psi_{d_i}(z) + [n \leftrightarrow p] \quad , \tag{3.11}$$

where

$$\bar{V} = \gamma^o\ V^+\ \gamma^o \quad . \tag{3.12}$$

Here $e < 0$ is the electron charge and $e_p > 0$ is the proton charge. Substituting into Eq. (3.11)

$$\Psi(x) = \sqrt{\frac{M}{EV}}\ u(\bar{p},s)\ e^{-i\ p\cdot x} \quad , \tag{3.13}$$

$$D_F(y-x) = \int \frac{d^4q}{(2\pi)^4}\ e^{-i\ q\cdot(y-x)} \left(\frac{-1}{q\cdot q+i\varepsilon}\right) \quad , \tag{3.14}$$

$$S_F(y-z) = \int \frac{d^4p}{(2\pi)^4}\ e^{-i\ p\cdot(y-z)}\ \frac{\not{p}+m}{p\cdot p-m^2+i\varepsilon} \quad ; \tag{3.15}$$

and

$$\Gamma^\mu = F_1\gamma^\mu + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\kappa\ F_2 \quad ; \tag{3.16}$$

and then computing the integrals, the scattering amplitude is obtained. It is

$$S_{fi} = i\,\frac{ee_p}{V^2}\,(2\pi)^4\,\delta^4\,[(d_f-d_i)-(e_i-e_f)]\sqrt{\frac{m_e^2}{E_f E_i}\,\frac{M^2}{\tilde{E}_f\tilde{E}_i}}$$

$$[\bar{u}(\bar{e}_f,s_f)\gamma^\mu\,u(\bar{e}_i,s_i)]\,\frac{1}{q\cdot q}\,[\bar{u}(\bar{d}_f,s_f')A_d^\mu\,u(\bar{d}_i,s_i')]$$

$$+\,[n\leftrightarrow p]\quad, \tag{3.17}$$

where the symbols denote:

$$A_d^\mu = \int \frac{d^4n}{(2\pi)^4}\,\bar{\mathcal{V}}_f\,\frac{\not{d}_f-\not{n}+m_p}{(d_f-n)^2-m_p^2+i\varepsilon}\,\Gamma_p^\mu\,\frac{\not{d}_i-\not{n}+m_p}{(d_i-n)^2-m_p^2+i\varepsilon}\,\frac{\not{n}+m_n}{n\cdot n-m_n^2+i\varepsilon}\,\mathcal{V}_i\;, \tag{3.18}$$

$$\mathcal{V} = \zeta\left\{\sqrt{2}\,\phi_u(\rho)-\left[\frac{3}{2M^2}\,\frac{1}{\rho^2}\,\sigma^p_{\mu\nu}d^\mu n^\nu\sigma^n_{\alpha\beta}d^\alpha n^\beta+\frac{1}{8}\,\sigma^p_{\mu\nu}\,\sigma^{\mu\nu}_n\right]\phi_w(\rho)\right\}, \tag{3.19}$$

$$\rho = \left[\left(\frac{n\cdot d}{M}\right)^2 - n\cdot n\right]^{\frac{1}{2}}\quad, \tag{3.20}$$

$$q^\mu = (d_f - d_i)^\mu\quad, \tag{3.21}$$

$$\not{d} = \tilde{E}\gamma^0 - \bar{d}\cdot\bar{\gamma}\;, \tag{3.22}$$

and

$$d^\mu = (\tilde{E},\,\bar{d})\quad. \tag{3.23}$$

C. The Overall Factor

In this section the overall factor multiplying the matrices of the vertex function is determined. To achieve this, the internal propagators are expressed in an interesting way. This expression displays the propagators as a sum of particle and antiparticle energy projection operators, and reduces the number of poles in each term of that sum. Then, considering only the particle aspects of the internal fermion lines in the scattering amplitude, the overall factor is determined.

To express the propagators as sums of energy projection operators, notice the following expression. Since

$$\sum_s u(\bar{p},s)\ \bar{u}(\bar{p},s) = \frac{E\,\gamma^0 - \bar{p}\cdot\bar{\gamma} + m}{2m} \tag{3.24a}$$

and

$$-\sum_s v(-\bar{p},s)\ \bar{v}(-\bar{p},s) = \frac{-E\,\gamma^0 + \bar{p}\cdot\bar{\gamma} + m}{2m} \tag{3.24b}$$

where

$$E = [\bar{p}\cdot\bar{p} + m^2]^{\frac{1}{2}} \quad , \tag{3.24c}$$

the internal propagators can be expressed in terms of energy projection operators. The expression is

$$\frac{p^0\gamma^0 - \bar{p}\cdot\bar{\gamma} + m}{p\cdot p - m^2 + i\varepsilon} = \frac{m}{p\cdot p - m^2+i\varepsilon}\left\{\left(1 + \frac{p^0}{E}\right) \sum_s u(\bar{p},s)\ \bar{u}(\bar{p},s) - \left(1 - \frac{p^0}{E}\right) \sum_s v(-\bar{p},s)\bar{v}(-\bar{p},s)\right\}. \tag{3.25}$$

This expression, together with Eq. (3.18), yields the following result

$$\bar{u}(\bar{d}_f,s_f')A_d^\mu\, u(\bar{d}_i,s_i')$$

$$= \bar{u}(\bar{d}_f,s_f') \int \frac{d^4n}{(2\pi)^4}\, \bar{V}\,(d_f-n,n,d_f) \sum_{\substack{s_p's_p\\ s_n}} \frac{m_p}{(d_f-n)^2-m_p^2+i\varepsilon} \left[\left(1+\frac{\tilde{E}_f-n^o}{E_f^p}\right)\right.$$

$$\left. u(\bar{d}_f-\bar{n},s_p')\,\bar{u}(\bar{d}_f-\bar{n},s_p') - \left(1-\frac{\tilde{E}_f-n^o}{E_f^p}\right) v(\bar{n}-\bar{d}_f,s_p')\,\bar{v}(\bar{n}-\bar{d}_f,s_p')\right]$$

$$\Gamma_p^\mu \frac{m_p}{(d_i-n)^2 - m_p^2 + i\varepsilon} \left[\left(1+\frac{\tilde{E}_i-n^o}{E_i^p}\right) u(\bar{d}_i-\bar{n},s_p)\,\bar{u}(\bar{d}_i-\bar{n},s_p)\right.$$

$$\left. - \left(1-\frac{\tilde{E}_i-n^o}{E_i^p}\right) v(\bar{n}-\bar{d}_i,s_p)\,\bar{v}(\bar{n}-\bar{d}_i,s_p)\right] \frac{m_n}{n\cdot n - m_n^2 + i\varepsilon} \left[\left(1+\frac{n^o}{E^n}\right)\right.$$

$$\left. u(\bar{n},s_n)\,\bar{u}(\bar{n},s_n) - \left(1-\frac{n^o}{E^n}\right) v(\bar{n},s_n)\,\bar{v}(\bar{n},s_n)\right] \bar{V}(d_i-n,n,d_i)\, u(\bar{d}_i,s_i') \tag{3.26}$$

where

$$E^p = [(\bar{d}-\bar{n})^2 + m_p^2]^{\frac{1}{2}} \tag{3.27a}$$

$$E^n = [\bar{n}\cdot\bar{n} + m_n^2]^{\frac{1}{2}} \quad . \tag{3.27b}$$

Supposing that the particle aspects of the internal fermion lines dominate the antiparticle aspects, this result reduces to

$$\bar{u}(\bar{d}_f,s_f')A_d^{\mu}\,u(\bar{d}_i,s_i')$$

$$= \bar{u}(\bar{d}_f,s_f') \int \frac{d^4n}{(2\pi)^4}\,\frac{m_p^2}{E_f^pE_i^p}\,\frac{m_n}{E^n}\,\frac{(n^o - \tilde{E}_f - E_f^p)(n^o - \tilde{E}_i - E_i^p)}{(n^o - \tilde{E}_f + E_f^p - i\varepsilon)(n^o - \tilde{E}_f - E_f^p + i\varepsilon)}$$

$$\frac{(n^o + E^n)}{(n^o - \tilde{E}_i + E_i^p - i\varepsilon)(n^o - \tilde{E}_i - E_i^p + i\varepsilon)(n^o + E^n - i\varepsilon)(n^o - E^n + i\varepsilon)}$$

$$\bar{V}(d_f-n,n,d_f) \sum_{\substack{s_p's_p\\ s_n}} u(\bar{d}_f-\bar{n},s_p')\,\bar{u}(\bar{d}_f-\bar{n},s_p')\Gamma_p^{\mu}$$

$$u(\bar{d}_i-\bar{n},s_p)\,\bar{u}(\bar{d}_i-\bar{n},s_p)\,u(\bar{n},s_n)\,\bar{u}(\bar{n},s_n)$$

$$V(d_i-n,n,d_i)\,u(\bar{d}_i,s_i') \quad . \tag{3.28}$$

Notice that in the n^o plane, there is now only one nonvanishing pole below the real axis, the one which places the spectator particle on the mass shell.

Finally, to determine the overall factor, perform the n^o integration closing the contour below the real axis. This yields

$$\bar{u}(\bar{d}_f,s_f')A_d^{\mu}\,u(\bar{d}_i,s_i') = -\,i \int \frac{d^3n}{(2\pi)^3}\,\frac{1}{[E_f^p + E^n - \tilde{E}_f][E_i^p + E^n - \tilde{E}_i]}$$

$$\frac{m_p^2}{E_f^pE_i^p}\,\frac{m_n}{E^n} \sum_{\substack{s_p's_p\\ s_n}} \bar{u}(\bar{d}_f,s_f')\,\bar{V}_f\,u(\bar{d}_f-\bar{n},s_p')\,u(\bar{n},s_n)\,\bar{u}(\bar{d}_f-\bar{n},s_p')$$

$$\Gamma_p^{\mu}\,u(\bar{d}_i-\bar{n},s_p)\,\bar{u}(\bar{d}_n-\bar{n},s_p)\,\bar{u}(\bar{n},s_n)\,V_i\,u(\bar{d}_i,s_i') \quad , \tag{3.29}$$

which suggests that the overall factor is

$$\zeta_{\substack{f\\i}} = \left[\rho^2_{\substack{f\\i}} + m^2_\rho\right]^{\frac{1}{2}} + \left[\rho^2_{\substack{f\\i}} + m^2_n\right]^{\frac{1}{2}} - M \quad . \tag{3.30}$$

To see that this is a reasonable choice, notice that after the n^o integration

$$\zeta = [\bar{n}\cdot\bar{n} + m^2_p]^{\frac{1}{2}} + [\bar{n}\cdot\bar{n} + m^2_n]^{\frac{1}{2}} - M \tag{3.31}$$

in the frame where $|\bar{d}| = 0$. Thus terms in the denominator of the order of the binding energy are cancelled.

IV. THE DEUTERON FORM FACTORS

In this chapter the electromagnetic form factors of the deuteron are calculated. The discussion begins with a summary of the conventions, the matrix representations, and the properties of the Lorentz frame in which the work is conducted. Then, in the second section, the actual calculation is performed. First, the deuteron current is related to the invariant amplitude and the calculation of the current is reduced to a trace and a four-dimensional integral. Then, restricting the discussion to the S-wave part of the deuteron wavefunction, the trace is evaluated and the results are organized into an expansion in terms of the complete set of spin-1 matrices. This expansion of the current defines the deuteron form factors. Finally, these results are compared with those of previous researchers.

A. The Conventions

In this section the conventions and representations that will be used throughout the remainder of the chapter are given. Except for the representation of the γ-matrices, they are those of Bjorken and Drell.[22]

The basic conventions are quite familiar. The index notation is

$$A^{\mu} = (A^{0},\bar{A}) \quad , \tag{4.1}$$

$$A_{\mu} = (A^{0},-\bar{A}) = g_{\mu\nu}A^{\nu} \quad , \tag{4.2}$$

with

$$\not{A} = A_\mu \gamma^\mu \quad , \tag{4.3}$$

and the matrix representation is

$$\gamma^o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.4}$$

$$\gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad . \tag{4.5}$$

Frequently appearing combinations are

$$\gamma^5 = i\ \gamma^0\gamma^1\gamma^2\gamma^3 \tag{4.6}$$

and

$$\sigma^{\mu\nu} = \frac{i}{2} (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \quad . \tag{4.7}$$

In this representation

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.8}$$

and the components of $\sigma^{\mu\nu}$ are

$$\sigma^{oo} = 0 \quad , \tag{4.9a}$$

$$\sigma^{oi} = i\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \quad , \tag{4.9b}$$

and

$$\sigma^{ij} = \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \quad . \tag{4.9c}$$

Finally, the Pauli spin matrices are

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.10a}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.10b}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{4.10c}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.10d}$$

and the spin-1 matrices are

$$S^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.11a}$$

$$S^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{4.11b}$$

$$S^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad . \tag{4.11c}$$

So, defining

$$S^{ij} = S^i S^j + S^j S^i \tag{4.12}$$

gives

$$S^{11} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} , \tag{4.13a}$$

$$S^{12} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \tag{4.13b}$$

$$S^{13} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} , \tag{4.13c}$$

$$S^{22} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} , \tag{4.13d}$$

$$S^{23} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix} , \tag{4.13e}$$

$$S^{33} = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.13f}$$

In the next section it will be convenient to employ an unusual technique when taking traces. To prepare this technique, consider two sets of Pauli spin matrices, τ^μ and σ^μ, operating in different spaces. Then, the γ-matrices can be written as a direct product of τ and σ matrices. Specifically,

$$\gamma^o = \tau^1 \times \sigma^o \tag{4.14a}$$

$$\gamma^i = (-i\tau^2) \times \sigma^i \tag{4.14b}$$

$$\sigma^{oi} = (i\tau^3) \times \sigma^i \tag{4.14c}$$

$$\sigma^{ij} = \tau^o \times (\varepsilon^{ijk} \sigma^k) \tag{4.14d}$$

$$\gamma^5 = \tau^3 \times \sigma^o \tag{4.14e}$$

and

$$\not{A} = (\tau^1) A^o\sigma^o + (i\tau^2) \bar{A} \cdot \bar{\sigma} \tag{4.14f}$$

$$\not{A}^T = (\tau^1) A^o\sigma^o - (i\tau^2) \bar{A}\cdot\bar{\sigma}^T \quad . \tag{4.14g}$$

So, traces of γ-matrices may be taken in two steps, since

$$\mathrm{Tr}(A \times B) = \mathrm{Tr}A \cdot \mathrm{Tr}B \quad . \tag{4.15}$$

Finally, the calculation will be carried out in the Breit frame. The Breit frame is the Lorentz frame where

$$\bar{d}_f = \bar{q}/2 \quad , \tag{4.16}$$

$$\bar{d}_i = - \bar{q}/2 \quad , \tag{4.17}$$

and

$$\tilde{E}_f = \tilde{E}_i = \tilde{E}$$

$$= \left[\frac{\bar{q}\cdot\bar{q}}{4} + M^2\right]^{\frac{1}{2}} \quad . \tag{4.18}$$

Also the z-direction is frequently chosen so that

$$\bar{q} = (0,\ 0,\ q). \tag{4.19}$$

B. The Deuteron Current

In this section the electromagnetic form factors of the deuteron are calculated. They are derived from the deuteron current, which in turn is deduced from the scattering amplitude. In a suitable limit, these form factors agree with the accepted results.

To relate the deuteron current to the scattering amplitude, note that Eq. (3.17) gives

$$S_{fi} = \frac{i}{V^2} (2\pi)^4 \; \delta^4 \; [(d_f - d_i) - (e_i - e_f)] \sqrt{\frac{m_e^2}{E_f E_i}} \sqrt{\frac{M^2}{\tilde{E}_f \tilde{E}_i}} \, M_{fi} \quad , \qquad (4.20)$$

where the invariant amplitude is

$$M_{fi} = \frac{ee_p}{q \cdot q} \bar{u}(\bar{e}_f, s_f) \gamma_\mu \, u(\bar{e}_i, s_i) \; \bar{u}(\bar{d}_f, s_f') A_d^\mu \, u(\bar{d}_i, s_i') \; + [p \leftrightarrow n]$$

$$= \frac{ee_p}{q \cdot q} \langle e_f | j_\mu^e | e_i \rangle \langle d_f | j_d^\mu | d_i \rangle \quad . \qquad (4.21)$$

So, if only the S-wave part of the deuteron wavefunction is considered, the deuteron current in the Breit frame is

$$\langle d_f | j_d^\mu | d_i \rangle = \int \frac{d^4 n}{(2\pi)^4} \; \frac{2 \, \phi_u^*(\rho_f) \; \phi_u(\rho_i) \; \zeta_f^* \, \zeta_i}{[p_f \cdot p_f - m_p^2 + i\varepsilon] \, [p_i \cdot p_i - m_p^2 + i\varepsilon] \, [n \cdot n - m_n^2 + i\varepsilon]}$$

$$\times \langle f | \theta^\mu | i \rangle + [p \leftrightarrow n] \qquad (4.22)$$

where

$$\langle f | \theta_p^\mu | i \rangle = \bar{u} \left(\frac{\bar{q}}{2}, s_f' \right) \; [(\not{p}_f + m_p) \Gamma_p^\mu (\not{p}_i + m_p)] \, [(\not{n} + m_n)] \; u \left(-\frac{\bar{q}}{2}, s_i' \right) \; , \quad (4.23)$$

$$\Gamma_p^\mu = F_{1p}\,\gamma^\mu - \frac{\kappa_p}{2m_p} F_{2p}\, i\,\sigma^{\mu j}\, q^j \quad , \tag{4.24}$$

$$p_f^\mu = \left(\tilde{E} - n^o,\ \frac{\bar{q}}{2} - \bar{n}\right) \quad , \tag{4.25}$$

$$p_i^\mu = \left(\tilde{E} - n^o,\ -\frac{\bar{q}}{2} - \bar{n}\right) \quad , \tag{4.26}$$

$$\rho_i^f = \left[\left(\frac{n^o\tilde{E} \mp \bar{n}\cdot\bar{q}/2}{M}\right)^2 - n\cdot n\right]^{\frac{1}{2}} \quad , \tag{4.27}$$

and

$$\zeta = [\rho^2 + m_p^2]^{\frac{1}{2}} + [\rho^2 + m_n^2]^{\frac{1}{2}} - M \quad . \tag{4.28}$$

Examination of this current yields two interesting characteristics. First, since $\phi(\rho)$ is real, performance of the substitution $\bar{n} \rightarrow -\bar{n}$ yields

$$p_f\cdot p_f = (\tilde{E} - n^o)^2 - \left(\frac{\bar{q}}{2} - \bar{n}\right)^2$$

$$\leftrightarrow (E - n^o) \quad - \left(\frac{\bar{q}}{2} + \bar{n}\right)^2 = p_i\cdot p_i \quad , \tag{4.29}$$

$$n\cdot n = (n^o)^2 - \bar{n}\cdot\bar{n}$$

$$\leftrightarrow (n^o)^2 - \bar{n}\cdot\bar{n} = n\cdot n \quad , \tag{4.30}$$

$$\rho_f = \left[\left(\frac{n^o\tilde{E} - \bar{n}\cdot\bar{q}/2}{M}\right)^2 - n\cdot n\right]^{\frac{1}{2}}$$

$$\leftrightarrow \left[\left(\frac{n^o\tilde{E} + \bar{n}\cdot\bar{q}/2}{M}\right)^2 - n\cdot n\right]^{\frac{1}{2}} = \rho_i \quad , \tag{4.31}$$

and

$$\zeta_f \leftrightarrow \zeta_i \quad . \tag{4.32}$$

So, the part of the integral multiplying $<f|\theta^\mu|i>$ is even in $\bar{n}$, restricting the form of $<f|\theta^\mu|i>$ to terms also even in $\bar{n}$. Second, the matrix multiplication in Eq. (4.23) is

$$\begin{aligned}<f|\theta^\mu_p|i> &= (\bar{u}_f)_{\eta\theta}[(\not{p}_f+m_p)\Gamma^\mu_p(\not{p}_i+m_p)]_{\eta\kappa}\,[\not{n}+m_n]_{\theta\lambda}(u_i)_{\kappa\lambda} \\ &= [\not{n}^T+m_n]_{\lambda\theta}(\bar{u}_f)_{\theta\eta}[(\not{p}_f+m_p)\Gamma^\mu_p(\not{p}_i+m_p]_{\eta\kappa}(u_i)_{\kappa\lambda} \\ &= \mathrm{Tr}[(\not{n}^T+m_n)\bar{u}_f(\not{p}_f+m_p)\Gamma^\mu_p(\not{p}_i+m_p)u_i] \quad .\end{aligned} \tag{4.33}$$

So the deuteron current is expressed in terms of a trace and a four-dimensional integral.

The deuteron wavefunction is

$$\begin{aligned}u\left(-\frac{\bar{q}}{2}, m\right) &= \begin{pmatrix} u_-(m) & 0 \\ 0 & u_+(m)\end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix} u_-(m) + \begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix} u_+(m)\end{aligned} \tag{4.34}$$

where an examination of Eqs. (2.52) and (2.53) shows

$$u_-(m) = \frac{1}{\sqrt{2}}\begin{pmatrix} \left(\frac{\tilde{E}-q/2}{M}\right)\chi_{R1}(m) & \frac{1}{\sqrt{2}}\chi_{R2}(m) \\ \frac{1}{\sqrt{2}}\chi_{R2}(m) & \left(\frac{\tilde{E}+q/2}{M}\right)\chi_{R3}(m)\end{pmatrix} \tag{4.35}$$

and

$$u_+(m) = \frac{1}{\sqrt{2}} \begin{pmatrix} \left(\frac{\tilde{E}+q/2}{M}\right) \chi_{R1}(m) & \frac{1}{\sqrt{2}} \chi_{R2}(m) \\ \frac{1}{\sqrt{2}} \chi_{R2}(m) & \left(\frac{\tilde{E}-q/2}{M}\right) \chi_{R3}(m) \end{pmatrix} . \tag{4.36}$$

Similarly

$$\bar{u}(+\bar{q}/2,\ m') = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u_-^+(m') + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} u_+^+(m') . \tag{4.37}$$

This is the form of the wavefunction that will be inserted into the trace calculation to find the deuteron current.

In order to demonstrate how this calculation proceeds, a part of it will be done in detail. Consider the portion of θ_p^i proportional to F_{2p}, i.e.

$$\frac{2i\ m_p}{\kappa_p F_{2p}} \langle f | \theta_{2p}^i | i \rangle$$

$$= \mathrm{Tr}\ [(\not{n}^T + m_n)\bar{u}_f(\not{p}_f + m_p)\sigma^{ij}q^j(\not{p}_i + m_p)u_i] . \tag{4.38}$$

If all the terms with an odd number of γ's are discarded, and then Eqs. (4.14), (4.34), and (4.37) are substituted into Eq. (4.38) and the τ trace is taken with all terms odd in $\bar{n}$ being deleted, we arrive at an intermediate result. It is

$$\frac{2i\,m_p}{\kappa_p F_{2p}} \langle f|\theta^i_{2p}|i\rangle =$$

$$\mathrm{Tr}\,[u^+_-(f)\sigma^j u_-(i) + u^+_+(f)\sigma^j u_+(i)] \left\{\epsilon^{ikj}\, q^k/2\, m_n \left[m_p^2 + (\tilde{E}-n^o)^2 + \bar{n}\cdot\bar{n} - \frac{\bar{q}\cdot\bar{q}}{4}\right] + n^j\, m_n (\bar{n}\times\bar{q})^i\right\}$$

$$+ \mathrm{Tr}\,[u^+_-(f)\sigma^j u_+(i) + u^+_+(f)\sigma^j u_-(i)] \left\{\epsilon^{ikj}\, q^k m_p n^o (\tilde{E}-n^o)\right\}$$

$$+ \mathrm{Tr}\,[u^+_-(f)\sigma^j u_+(i) - u^+_+(f)\sigma^j u_-(i)] \left\{2i\, m_p n^o \left(\delta^{ij}\frac{\bar{q}\cdot\bar{q}}{4} - q^i q^j\right)\right\}$$

$$+ \mathrm{Tr}\,[\sigma^{Tj}(u^+_-(f)u_+(i) + u^+_+(f)u_-(i)] \left\{m_p (\bar{n}\times\bar{q})^i\, n^j\right\} . \qquad (4.39)$$

Next, the σ traces must be taken. In order to achieve this, first establish the relations

$$\mathrm{Tr}\,[u^+(f)\; u(i)] = u^+(f)\; u(i) \qquad (4.40a)$$

$$\mathrm{Tr}\,[u^+(f)\;\sigma^i\; u(i)] = u^+(f)\; s^i u(i) \qquad (4.40b)$$

$$\mathrm{Tr}\,[\sigma^{Ti}\; u^+(f)\; u(i)] = u^+(f)\; s^i u(i) \qquad (4.40c)$$

$$\mathrm{Tr}\,[\sigma^{Ti}\; u^+(f)\sigma^j\; u(i)] = u^+(f)\; s^{ij} u(i) - \delta^{ij}\; u^+(f)u(i), \qquad (4.40d)$$

by considering each value of i and j, and the u_+ and u_- cases, individually. On the left side of Eqs. (4.40) the u's are given by Eqs. (4.35) and (4.36), and on the right side they are

$$u_{\pm}(m) = \frac{1}{\sqrt{2}}\left(1 \pm \frac{(\bar{s}\cdot\bar{q})}{2M} + \frac{(\bar{s}\cdot\bar{q})^2}{4M(E+M)}\right) \begin{pmatrix}\chi_1(m)\\ \chi_2(m)\\ \chi_3(m)\end{pmatrix}_R . \qquad (4.41)$$

Application of these relations and multiplication of the resulting matrices for each value of i and j establishes the needed trace relations. The complete set of these relations, needed for the entire calculation, is

$$\mathrm{Tr}\ [u_-^+(f)u_-(i) + u_+^+(f)u_+(i)] = \chi_f^+\chi_i + \frac{q^k q^l}{4M^2}\chi_f^+ s^{kl}\chi_i \ , \tag{4.42a}$$

$$\mathrm{Tr}\ [u_-^+(f)u_-(i) - u_+^+(f)u_+(i)] = -\frac{\tilde{E}q^k}{M^2}\chi_f^+ s^k \chi_i \ , \tag{4.42b}$$

$$\mathrm{Tr}\ [u_-^+(f)u_+(i) + u_+^+(f)u_-(i)] = \chi_f^+\chi_i \ , \tag{4.42c}$$

$$\mathrm{Tr}\ [u_-^+(f)u_+(i) - u_+^+(f)u_-(i)] = 0 \ , \tag{4.42d}$$

$$\mathrm{Tr}\ [u_-^+(f)\sigma^i u_-(i) + u_+^+(f)\sigma^i u_+(i)]$$

$$= \frac{\tilde{E}}{M}\chi_f^+ s^i \chi_i + \frac{2\tilde{E}+M}{M^2(\tilde{E}+M)}\frac{q^i q^k}{4}\chi_f^+ s^k \chi_i \ , \tag{4.42e}$$

$$\mathrm{Tr}\ [u_-^+(f)\sigma^i u_-(i) - u_+^+(f)\sigma^i u_+(i)] = -\frac{q^k}{2M}\chi_f^+ s^{ik}\chi_i$$

$$-\frac{q^i q^k q^l}{8M^2(\tilde{E}+M)}\chi_f^+ s^{kl}\chi_i \ , \tag{4.42f}$$

$$\mathrm{Tr}\ [u_-^+(f)\sigma^i u_+(i) + u_+^+(f)\sigma^i u_-(i)]$$

$$= \frac{\tilde{E}}{M}\chi_f^+ s^i \chi_i - \frac{q^i q^k}{4M(\tilde{E}+M)}\chi_f^+ s^k \chi_i \ , \tag{4.42g}$$

$$\mathrm{Tr}\ [u_-^+(f)\sigma^i u_+(i) - u_+^+(f)\sigma^i u_-(i)] = -\frac{i\varepsilon^{ikl}}{2M} q^l \chi_f^+ s^k \chi_i \ , \tag{4.42h}$$

$$\mathrm{Tr}\ [\sigma^{Ti}(u_-^+(f)u_+(i) + u_+^+(f)u_-(i)]$$

$$= \frac{\tilde{E}}{M} \chi_f^+ s^i \chi_i - \frac{q^i q^k}{4M(\tilde{E}+M)} \chi_f^+ s^k \chi_i \ , \tag{4.42i}$$

$$\mathrm{Tr}\ [\sigma^{Ti}(u_-^+(f)u_+(i) - u_+^+(f)u_-(i))] = -\frac{i\varepsilon^{ikl}}{2M} q^l \chi_f^+ s^k \chi_i \ , \tag{4.42j}$$

$$\mathrm{Tr}\ [\sigma^{Ti}u_-^+(f)\sigma^j u_+(i) + \sigma^{Ti}u_+^+(f)\ \sigma^j u_-(i)]$$

$$= -\delta^{ij} \chi_f^+ \chi_i + \left[\frac{\tilde{E}^2}{M^2}\delta^{im}\delta^{jn} - \frac{q^k q^l}{4M^2}\varepsilon^{ikm}\varepsilon^{jln} - \frac{\tilde{E}\, q^n q^k}{4M^2(\tilde{E}+M)}\left(\delta^{ik}\delta^{jm} + \delta^{jk}\delta^{im}\right) + \frac{q^i q^j q^m q^n}{16M^2(\tilde{E}+M)^2}\right] \chi_f^+ s^{mn} \chi_i \ , \tag{4.42k}$$

and

$$\mathrm{Tr}\ [\sigma^{Ti}u_-^+(f)\sigma^j u_+(i) - \sigma^{Ti}u_+^+(f)\sigma^j u_-(i)]$$

$$= -\frac{i}{M}\left[\frac{\tilde{E}\, q^k}{2M}\left(\delta^{in}\varepsilon^{kjm} + \delta^{jn}\varepsilon^{kim}\right) - \frac{q^k q^i q^n}{8M(\tilde{E}+M)}\left(\delta^{kj}\varepsilon^{lim} + \delta^{ki}\varepsilon^{ljm}\right)\right] \chi_f^+ s^{mn} \chi_i \ . \tag{4.42l}$$

Substituting the needed trace relations into Eq. (4.39), and collecting the terms, yields

$$\frac{2im_p}{\kappa_p F_{2p}} \langle f|\theta^i_{2p}|i\rangle = \chi_f^+ \, s^j \, \chi_i \left\{ (\bar{n}\times\bar{q})^i \, n^j \left[\frac{(m_p+m_n)\tilde{E}}{2M}\right] \right.$$

$$\left. - \varepsilon^{ijk} q^k \left[\frac{m_n \tilde{E}}{2M}\left(m_p^2 + (\tilde{E}-n^o)^2 + \bar{n}\cdot\bar{n} - \frac{\bar{q}\cdot\bar{q}}{4}\right) + \frac{m_p n^o (M^2 - n^o E)}{M}\right]\right\} . \tag{4.43}$$

An examination of Eqs. (4.22) and (4.43) shows that the part of the deuteron current proportional to F_{2p} is of the form

$$j^i_{d2} = \int d^3n \, [(\bar{n}\times\bar{q})^i \, \bar{s}\cdot\bar{n} \, A + (\bar{s}\times\bar{q})^i \, M^2 B] \tag{4.44}$$

where A and B are functions of $\bar{n}\cdot\bar{n}$ and $(\bar{n}\cdot\bar{q})^2$. To convert this to a standard form, note

$$j^i_{d2} = \int n^2 dn \, d\,(\cos\theta)\, d\phi \, A \, \varepsilon^{ijk} \, n^j \, q^k \, n^l \, s^l + (\bar{s}\times\bar{q})^i \int d^3n \, M^2 B$$

$$= \int n^2 dn \, d\,(\cos\theta)\, d\phi \, A \, q_3 \begin{pmatrix} n\sin\theta\sin\phi \\ -n\sin\theta\cos\phi \\ 0 \end{pmatrix}^i$$

$$(n\sin\theta\cos\phi \, s_1 + n\sin\theta\sin\phi \, s_2 + n\cos\theta \, s_3)$$

$$+ (\bar{s}\times\bar{q})^i \int d^3n \, M^2 B$$

$$= \int n^2 dn \, d\,(\cos\theta)\, \pi \, A \, q_3 \begin{pmatrix} n^2(1-\cos^2\theta)s_2 \\ -n^2(1-\cos^2\theta)s_1 \\ 0 \end{pmatrix} + (\bar{s}\times\bar{q})^i \int d^3n \, M^2 B$$

$$= (\bar{s}\times\bar{q})^i \int d^3n \left[\frac{1}{2}\left(\bar{n}\cdot\bar{n} - \frac{(\bar{n}\cdot\bar{q})^2}{\bar{q}\cdot\bar{q}}\right) A + M^2 B\right] \tag{4.45}$$

So, since A and B are defined by Eq. (4.43), the part of the deuteron current density proportional to F_{2p} is given in Eq. (4.45).

Similar arguments may be made for the deuteron charge density and for the rest of the current density. The result of those arguments is

$$j_d^o = D_0 - \frac{3\, q^i q^j}{2\sqrt{2}\ \underline{q}\cdot\underline{q}}\, s_{IR}^{ij}\, D_2 \tag{4.46}$$

$$j_d^i = \frac{i}{2M}\, (\bar{s}\times\bar{q})^i\, D_1 \tag{4.47}$$

where

$$s_{IR}^{ij} = s^i s^j + s^j s^i - \frac{4}{3}\,\delta^{ij} \quad . \tag{4.48}$$

The D_0, D_1, and D_2 are the electromagnetic form factors of the deuteron. They are scalar functions of $\underline{q}\cdot\underline{q}$ and will be given in Eqs. (4.60).

In the final two parts of this section, the cross section for electron deuteron scattering is calculated in terms of the form factors, and it is shown that in a suitable limit these form factors agree with accepted results.

For an unpolarized beam, incident upon an unpolarized target, the lab. frame unpolarized cross section is[22]

$$\frac{d\sigma}{d\Omega} = \frac{m_e^2 M}{4\pi^2}\,\frac{p'}{p}\,\frac{1}{\left[M + E - \left(\frac{pE'}{p'}\right)\cos\theta\right]}\,\frac{1}{6}\,\frac{e^2}{(q\cdot q)^2}\ \mathrm{Tr}\,[j_\mu^e j_\nu^{e+}]\ \mathrm{Tr}\,[j_d^\mu j_d^{\nu+}] \tag{4.49}$$

where the lab. frame momenta are

$$d_i = (M,0) \quad , \tag{4.50a}$$

$$d_f = (M + q^o, \bar{q}) \quad , \tag{4.50b}$$

$$e_i = (E, \bar{p}) \quad , \tag{4.50c}$$

and

$$e_f = (E', \bar{p}') \quad . \tag{4.50d}$$

For the electron, in any frame, Eq. (4.21) gives

$$\mathrm{Tr}\ [j^e_\mu j^{e+}_\nu] = \frac{1}{4m_e^2}\ \mathrm{Tr}\ [(\not{e}_f + m_e)\ \gamma_\mu\ (\not{e}_i + m_e)\ \gamma_\nu]$$

$$= \frac{1}{m_e^2}\ [e_{f\mu} e_{i\nu} + e_{f\nu} e_{i\mu} - g_{\mu\nu}\ (e_f \cdot e_i - m_e^2)] \quad . \tag{4.51}$$

To obtain the corresponding result for the deuteron, note that in the Breit frame, Eqs. (4.46) and (4.47) yield

$$\mathrm{Tr}\ (j^o_d j^{o+}_d) = 3\ (D_0^2 + D_2^2) \quad , \tag{4.52}$$

$$\mathrm{Tr}\ (j^o_d j^{i+}_d) = 0 \quad , \tag{4.53}$$

and

$$\mathrm{Tr}\ (j^i_d j^{j+}_d) = \frac{D_1^2}{2M^2}\ (\delta^{ij}\ \bar{q} \cdot \bar{q} - q^i q^j) \quad . \tag{4.54}$$

So, in any frame, the most general form consistent with gauge invariance is

$$\mathrm{Tr}\ (j_d^{\mu} j_d^{\nu +}) = \left(d_f^{\mu} d_i^{\nu} + d_f^{\nu} d_i^{\mu} + \frac{q\cdot q}{2}\, g^{\mu\nu}\right) D_1^2/M^2$$

$$+ \left(d_f^{\mu} + d_i^{\mu}\right)\left(d_f^{\nu} + d_i^{\nu}\right) \left[\frac{3(D_0^2 + D_2^2) - 2D_1^2}{4M^2 - q\cdot q}\right] . \qquad (4.55)$$

Hence, since $m_e << M$ the product of the traces in the lab. frame is

$$\mathrm{Tr}\ (j_{\mu}^{e} j_{\nu}^{e+})\ \mathrm{Tr}\ (j_d^{\mu} j_d^{\nu +}) = \frac{6\ EE'\ \cos^2\theta/2}{m_e^2\ (1+\eta)} \left\{\left[D_0^2 + D_2^2 + \frac{2}{3}\,\eta\ D_1^2\right]\right.$$

$$\left. + \left[\frac{4}{3}\,\eta\ (1+\eta)\ D_1^2\right] \tan^2\theta/2\right\} \qquad (4.56)$$

where

$$\eta = -\frac{q\cdot q}{4M^2} . \qquad (4.57)$$

So, the cross section is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{NS} \left\{\left[G_0^2 + G_2^2 + \frac{2}{3}\,\eta\ G_1^2\right]\right.$$

$$\left. + \left[\frac{4}{3}\,\eta\ (1+\eta)\ G_1^2\right] \tan^2\theta/2\right\} , \qquad (4.58)$$

where

$$\left(\frac{d\sigma}{d\Omega}\right)_{NS} = \frac{\alpha^2}{(2E)^2} \frac{\cos^2\theta/2}{\sin^4\theta/2} \frac{1}{\left[1 + 2\,\frac{E}{M}\,\sin^2\theta/2\right]} \qquad (4.59)$$

is the cross section for an electron scattered from a spin o point particle. The deuteron form factors are scalar functions of $q\cdot q$, and are given by

$$G_0 = \frac{i}{[1+\eta]^{\frac{1}{2}}} D_0$$

$$= \frac{i}{[1+\eta]^{\frac{1}{2}}} \int \frac{d^4 n}{(2\pi)^4} N \left\{ F_{1p} \left[2\, m_p m_n \tilde{E} \right.\right.$$

$$+ n^o \left(m_p^2 - 2\, m_p m_n + M^2 - 2\, n^o \tilde{E} + (n^o)^2 - \frac{8}{3} m_p m_n \eta \right) + \bar{n}\cdot\bar{n} \left(\frac{5}{3} n^o - \frac{2}{3} \tilde{E} \right) \Bigg]$$

$$+ \frac{\kappa_p}{m_p} F_{2p} \left[-\frac{2}{3} m_n \eta\, (2 m_p^2 \tilde{E} + M^2 \tilde{E}) \right.$$

$$+ n^o \left(\frac{4}{3} m_n \tilde{E}\, n^o \eta - \frac{2}{3} m_n M^2 \eta - 2\, m_p M^2 \eta \right)$$

$$\left.\left. + \bar{n}\cdot\bar{n} \left(\frac{4}{3} m_n \tilde{E}\, \eta \right) - \frac{(n\cdot q)^2}{4M^2} \left(\frac{8}{3} m_n \tilde{E} \right) \right] \right\} + [p \leftrightarrow n] \quad , \qquad (4.60a)$$

$$G_1 = \frac{i}{[1+\eta]^{\frac{1}{2}}} D_1$$

$$= \frac{i}{[1+\eta]^{\frac{1}{2}}} \int \frac{d^4 n}{(2\pi)^4} N$$

$$F_{1p} \left[2\, m_p m_n \tilde{E} + n^o \; (m_p^2 + M^2 - (n^o)^2) - \bar{n}\cdot\bar{n}\; (n^o) \right.$$

$$\left. + \frac{(n\cdot q)^2}{4M^2} \left(\frac{2n^o}{\eta} \right) \right] + \frac{\kappa_p}{m_p} F_{2p} \left[m_p \tilde{E}\; (m_p^2 + M^2) \right.$$

$$\left. - n^o\; (m_p\, n^o\, \tilde{E} + 2\, m_p M^2 \eta) - \bar{n}\cdot\bar{n}\; (m_p \tilde{E}) + \frac{(n\cdot q)^2}{4M^2} \left(\frac{m_p \tilde{E} + m_n \tilde{E}}{\eta} \right) \right]$$

$$+ [p \leftrightarrow n] \quad , \qquad (4.60b)$$

$$G_2 = \frac{i}{[1+\eta]^{\frac{1}{2}}} D_2$$

$$= \frac{1}{[1+\eta]^{\frac{1}{2}}} \int \frac{d^4n}{(2\pi)^4} N \left\{ \frac{2\sqrt{2}}{3} F_{1p} \left[n^o (2\, m_p m_n\, \eta) \right.\right.$$

$$\left. + \bar{n}\cdot\bar{n}\, (n^o - \tilde{E}) + \frac{(n\cdot q)^2}{4M^2} \left(\frac{3\, (\tilde{E} - n^o)}{\eta} \right) \right]$$

$$+ \frac{2\sqrt{2}}{3} \frac{\kappa_p}{m_p} F_{2p} \left[\eta\, m_n\, (m_p^2 \tilde{E} - M^2 \tilde{E}) + n^o\, (2 m_n M^2 \eta - m_n \tilde{E}\, n^o\, \eta) \right.$$

$$\left.\left. - \bar{n}\cdot\bar{n}\, (m_n \tilde{E} \eta) + \frac{(n\cdot q)^2}{4M^2} (2\, m_n \tilde{E}) \right] \right\} + [p \leftrightarrow n] \quad , \qquad (4.60c)$$

where

$$\tilde{E} = M\, [1 + \eta]^{\frac{1}{2}} \qquad (4.60d)$$

and

$$N = \frac{2\, \phi^*_u(\rho_f)\, \phi_u(\rho_i)\, \zeta^*_f\, \zeta_i}{[p_f\cdot p_f - m_p^2 + i\varepsilon]\, [p_i\cdot p_i - m_p^2 + i\varepsilon]\, [n\cdot n - m_n^2 + i\varepsilon]} \quad . \qquad (4.60e)$$

Here p^μ, ρ, and ζ are given by Eqs. (3.22), (3.20), and (4.28).

In the limit of small momentum transfer, these form factors agree with the accepted results.[15] To show this, first do the n^o integration, conjecturing that the pole

$$n^o = [\bar{n}\cdot\bar{n} + m_n^2]^{\frac{1}{2}} \qquad (4.61)$$

gives the major contribution. Then, let

$$m_p = m_n = \frac{1}{2} M \tag{4.62}$$

and keep only the terms to first order in $|\bar{n}|$. The result of this approximation is that

$$\rho_{\substack{f\\i}} = |\bar{n} \pm \frac{\bar{q}}{4}| \quad , \tag{4.63a}$$

$$\zeta_{\substack{f\\i}} = - M + \tilde{E} \mp \frac{\bar{n}\cdot\bar{q}}{M} \quad , \tag{4.63b}$$

and

$$(d_{\substack{f\\i}} - n)^2 - \frac{M^2}{4} = M^2 - M\tilde{E} \pm \bar{n}\cdot\bar{q} \quad . \tag{4.64}$$

Finally, making the substitution

$$G_E = F_1 + \kappa \frac{q\cdot q}{M^2} F_2 \tag{4.65}$$

$$G_M = F_1 + \kappa F_2 \quad , \tag{4.66}$$

and keeping only the terms to first order in $|\bar{q}|/2M$, Eqs. (4.60) reduce to

$$G_0 = (G_E^p + G_E^n) \int \frac{d^3n}{(2\pi)^3} \phi_u^* (|\bar{n}-\bar{q}/4|)\ \phi_u(|\bar{n}+\bar{q}/4|) \tag{4.67a}$$

$$G_1 = 2(G_M^p + G_M^n) \int \frac{d^3n}{(2\pi)^3} \phi_u^* (|\bar{n}-\bar{q}/4|)\ \varphi_u(|\bar{n}+\bar{q}/4|) \quad , \tag{4.67b}$$

and

$$G_2 = 0 \quad . \tag{4.67c}$$

Notice, that

$$\int \frac{d^3n}{(2\pi)^3} \phi_u^* (|\bar{n} - \bar{q}/4|) \ \phi_u(|\bar{n} + \bar{q}/4|)$$

$$= \int \frac{d^3n}{(2\pi)^3} \int d^3y \ e^{i(\bar{n}-\bar{q}/4)\cdot\bar{y}} \ \Psi_u^*(y) \int d^3z \ e^{-i(\bar{n}+\bar{q}/4)\cdot\bar{z}} \ \Psi_u(z)$$

$$= \int d^3y \ e^{-i/2 \ \bar{q}\cdot\bar{y}} \ \Psi_u^*(y) \ \Psi_u(y)$$

$$= \frac{8\pi}{q} \int_0^\infty y \ dy \ \Psi_u^*(y) \ \Psi_u(y) \ \sin\left(\frac{qy}{2}\right)$$

$$= 4\pi \int_0^\infty y^2 \ dy \ \Psi_u^*(y) \ \Psi_u(y) \ j_0\left(\frac{qy}{2}\right) \quad , \tag{4.68}$$

which upon consideration of Eq. (2.63) yields

$$\int \frac{d^3n}{(2\pi)^3} \phi_u^* (|\bar{n} - \bar{q}/4|) \ \phi_u(|\bar{n} + \bar{q}/4|)$$

$$= \int_0^\infty dr \ u^2(r) \ j_0\left(\frac{1}{2} qr\right) \quad . \tag{4.69}$$

Here, $j_0(z)$ is a spherical Bessel function of the first kind, and is $\sqrt{\frac{\pi}{2z}} J_0(z)$.[23]

In conclusion, a technique for calculating the electromagnetic form factors of the deuteron, which does not employ expansions, has been developed. In the nonrelativistic limit, these results agree with accepted nonrelativistic results; Eqs. (4.67) and (4.58) agreeing with Eqs. (1.5) and (1.6) respectively.

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