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Deductive systems and finite axiomatization properties

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Deductive systems and finite axiomatization properties

by

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Dedicated to Marek

TABLE OF CONTENTS

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GENERAL INTRODUCTION

One of the central research areas in mathematical logic is the study of deductive systems. Traditionally, the name "deductive system" refers only to what we here call a 1-deductive system, for example the deductive systems of classical, intuitionistic, modal and many-valued logics as well as predicate logic. To determine such a system one specifies a set of *formulas* and a set of *rules*, which can be used to deduce theorems of the system. A set of formulas, in turn, is specified by a set of propositional *variables* and a set Λ of *connectives*. The variables are in a sense inessential for the deductive system: a systematic replacement of one variable with another does not change the meaning of the rule. Thus a deductive system can be identified with a set Λ of connectives and a set of deduction rules.

From the observation that the connectives of Λ are nothing but algebraic operations on the set of formulas, originated the area of *algebraic logic*. The set of formulas is treated as the algebra of terms and the set of theorems forms a *filter*, i.e., a subset closed under the deduction rules. Some properties of a given deductive system are studied by means of its *models*, called *matrices*, which are identified with an algebra with some subset of so-called *designated elements*, representing the values of truth. In some special cases, for example in classical, intuitionistic, modal propositional deductive systems, it is possible to identify these matrix models with just algebras:

Boolean algebras for classical and pseudo-boolean algebras for intuitionistic logic are examples. In these cases one can use the whole realm of the universal algebraic methods to investigate the deductive system.

While in the 1-deductive systems we deal with the deduction of a term from a set of terms, the deductive system used in algebra is of a different character. Here the role of formulas is played by equations, i.e., pairs of terms. In [4] a deductive system in which the deductions are performed on pairs of terms is called 2-*deductive*. In general, a system is k-deductive if the role of formulas on which deductions are performed is played by k-tuples of terms.

Gentzen-style systems, introduced by Gentzen in the 1930's, are another important type of deductive systems in proof theory. Their importance has grown in proportion to that of proof theory itself, which has become a central area of research in mathematical logic as computation has become increasingly identified with formal deduction; see for instance [13].

A concept that generalizes the concepts of a k-deductive system and of a Gentzen system is that of a K-deductive system, where K is some (finite or infinite) predicate language, i.e., a set of predicate symbols with an arity function. A K-deductive system becomes a k-deductive system when K has exactly one predicate and this predicate has the arity k. This unique k-ary predicate is interpreted as the "truth" predicate; while for a general K, we deal with many different "truth" predicates. A Gentzen system can be viewed as a K-deductive system where for every nonzero n, K contains exactly one predicate symbol of arity n. The concept of a K-deductive system is equivalent to that of a universal Horn theory and [11] studies the models of universal Horn theories from the same perspective as it is done in our Part I, Chapter 2 or Part II, Chapter 2. We have chosen the name "K-deductive systems" to stress the connection with the k-deductive system and Gentzen systems. Also, these systems are studied under the name "generalized logical systems" in [43].

For a 1-deductive system the connection with the universal Horn theory is as follows. Let $\mathcal{S}$ be a 1-deductive system and let D be a unary predicate symbol, which we interpret as a predicate of truth. Then a rule of $\mathcal{S}$ that allows to deduce from some terms $t_1, \ldots, t_n$ another term t, is now interpreted as a first-order formula

$$D(t_1), \ldots, D(t_n) \rightarrow D(t).$$

A formula of this form is called a Horn formula. For a general K, a typical rule of a K-deductive system allows to deduce a statement $D(t_1, \ldots, t_n)$, where D is one of the truth predicates of K and $t_1, \ldots, t_n$ are terms, from some statements

$$D_1(t_1^1, \ldots, t_{n_1}^1), \ldots, D_m(t_1^m, \ldots, t_{n_m}^m),$$

where $D_1, \ldots, D_m$ are truth predicates of K and t_i^j are terms. Such a rule corresponds to a Horn formula

$$D_1(t_1^1, \ldots, t_{n_1}^1) \wedge \cdots \wedge D_m(t_1^m, \ldots, t_{n_m}^m) \rightarrow D(t_1, \ldots, t_n).$$

Let φ denote the above Horn formula and let $\vec{x} = \langle x_1, \ldots, x_p \rangle$ be the list of all variables occurring in terms t_i^j, t_i. Then the expression $\forall_{\vec{x}}\varphi$ is called a universal closure of φ or a universal Horn formula. Let T be a set of universal Horn formulas. If every Horn formula that can be deduced from T is again in T, then T is called a universal Horn theory. Every set X of universal Horn formulas generates a universal Horn theory: it is the set of all universal Horn formulas that can be deduced from X. Thus with every K-deductive system we can associate a universal Horn theory generated by the set of universal Horn formulas obtained from the rules of $\mathcal{S}$ in the way described above. Conversely, with every universal Horn theory we can associate a K-deductive system.

The key notion of our investigations is that of the *Leibniz congruence*. It was introduced in [3, 4] for one-deductive systems and k-deductive systems, respectively, and motivated by the proposal of G. Leibniz [22] to treat two objects a and b as identical if they cannot be distinguished by any property. In the language of contemporary logic this translates to defining an equivalence relation Ω on some set of "objects" in such a way that two objects are equivalent if and only if every sentence that is "true" for a is also "true" for b and vice-versa. More precisely, let $\mathfrak{A}$ be a model of some language $\mathcal{L}$ and let a and b be two elements of this model. We say that a and b are equivalent modulo the Leibniz relation $\Omega(\mathfrak{A})$ if, for every formula $\varphi(x)$ of $\mathcal{L}$, $\varphi(a)$ holds in $\mathfrak{A}$ if and only if $\varphi(b)$ holds in $\mathfrak{A}$. It turns out that for every model $\mathfrak{A}$ the relation $\Omega(\mathfrak{A})$ defined above is a congruence of the underlying algebra of $\mathfrak{A}$. Moreover, if $\mathfrak{A}$ is also a model of some deductive system with some sort of "equivalence" connective (like the $\leftrightarrow$ connective of the classical logic), then the Leibniz relation has a particularly simple presentation. For example, if $\mathfrak{A}$ is a model of classical logic, then a and b are equivalent modulo $\Omega(\mathfrak{A})$ iff $a \leftrightarrow b$ is true in $\mathfrak{A}$ (see, for example, [5]). In general, a set Δ of connectives is called an equivalence system if it has some properties of the classical equivalence connective and in particular, if the Leibniz relation can be retrieved from Δ in a way analogous to the way described above for the case of the classical $\leftrightarrow$.

For a logician investigating a concrete deductive system the existence of a system of equivalence connectives is particularly helpful: in this case many of the constructions and methods used in universal algebra can be applied in the study of the models of the deductive system. This was shown for k-deductive systems in [4] and for the general K-deductive systems in [11] (see also Part II, Chapter 2 of this dissertation).

A model is a pair consisting of an algebraic structure (called an algebra) and a system of relations called a *filter*. If $\mathcal{S}$ is a K-deductive system, then an $\mathcal{S}$-model is a model in which the filter is an $\mathcal{S}$*-filter* (Definition 2.23). Thus, with a single algebra $\mathbf{A}$ one can associate in general many $\mathcal{S}$-models, by pairing $\mathbf{A}$ with different $\mathcal{S}$-filters. The operator that to a given $\mathcal{S}$-filter F on $\mathbf{A}$ assigns the Leibniz relation $\Omega(\langle \mathbf{A}, F\rangle)$ is called the *Leibniz operator* on the lattice of $\mathcal{S}$-filters of $\mathbf{A}$. Let us fix a finite set Λ of finitary algebraic operation symbols. A Λ-algebra is a set with operations denoted by the symbols from Λ. For example, if Λ has one binary operation symbol, then Λ-algebras are exactly monoids. A K-deductive system such that the Leibniz operator on the lattice of $\mathcal{S}$-filters of every Λ-algebra is monotone, is called *protoalgebraic*. It turns out, [4, Theorem 13.2], that a 1-deductive system $\mathcal{S}$ over Λ has a system of equivalence connectives iff $\mathcal{S}$ is protoalgebraic. A K-deductive system is called *algebraizable* if the Leibniz operator is injective and continuous. By [5, Theorem 4.2] a 1-deductive system $\mathcal{S}$ is algebraizable iff it has equivalence connectives satisfying some strong conditions. These results, and in particular [4, Theorem 13.2.] motivated our research presented in Part I. In Chapter 3 we carry over the characterization of protoalgebraicity to the general case of K-deductive systems. In particular, we disprove the characterization claimed for k-deductive systems, $k \neq 1$ in [4]. We correct and expand this and other results of [4, Section 13]. We apply it to Gentzen systems in Chapter 4, where we also investigate natural conditions that allow simplified equivalence formulas, also called equivalence sequents and associate protoalgebraicity with certain weak notion of the (CUT)-rule. In Chapter 5 we extend the main result of [5] to K-deductive systems.

Another important series of results concerns properly defining and characterizing the notion of a system of implication connectives, in a manner parallel to the characterization of a system of equivalence connectives. Chapter 6 contains several partial results which set the direction for future research. A key theorem is proved in Chapter 5 on the equivalence of two systems, one a K_1-deductive system and the other a K_2-deductive system, is used.

In the last two parts of the dissertation we turn to the questions of being *finitely axiomatizable* and *finitely based.* A deductive system $\mathcal{S}$ is finitely axiomatizable if there is a finite set of rules such that every tautology of $\mathcal{S}$ can be derived from the empty set of premisses using the rules of this set. If in addition all the inference rules of the system follow from some finite set of rules, we say that $\mathcal{S}$ is finitely based. A matrix is finitely axiomatizable or finitely based, if the deductive system determined by this matrix is finitely axiomatizable or finitely based. In Part II we look at the question of finite base from the general perspective of arbitrary K-deductive systems. The main results of this part, Theorems 3.1 and 3.2, state that if the language has only finitely many symbols, then every protoalgebraic and filter-distributive deductive system that is determined by some finite set of finite matrices is finitely based. This simultaneously extends the results of [42] and [3] and consequently a famous result of Baker ([1]) stating that every finitely generated congruence-distributive variety is finitely based. In Part III, we consider the question of finite axiomatizability of deductive systems determined by a single finite matrix or algebra. In Chapter 3 we answer a question of [46, 61, 10] of finding a nonfinitely axiomatizable matrix of the smallest possible size which would also be the "simplest possible". It was proved in [46] that every two-element matrix must be finitely axiomatizable. Since a five-element, and later a four-element example was found ([61, 10]), the above question translates to the question of whether a three-element non-finitely axiomatizable matrices exist and also if there are such matrices of a particular kind. In Chapter 3 we present two such simple examples.

For the deductive systems of equational logic the finite axiomatizability and finite basis questions translate respectively to the questions of whether or not for a given finite algebra **A** there is a finite set of quasi-identities of **A** such that all identities, and all quasi-identities of **A** are consequences of this finite set of quasi-identities. Although examples of nonfinitely based algebras have been known for some time, the first example of a nonfinitely axiomatizable algebra was found only recently ([21]). The underlying algebra of a finite, nonfinitely axiomatizable matrix may be finitely axiomatizable and even finitely based, as shown in Chapter 4. We also consider the finite axiomatization problem at the second-order level of equational logic. A finite algebra **A** is second-order finitely axiomatizable if there is a finite set of second-order rules admissible (or valid, in a stronger version of the concept) which can be used to derive all first-order rules, i.e., all quasi-identities of **A**. A *second-order rule* r for equational logic is a pair consisting of a finite set X of quasi-identities and a quasi-identity φ. In Chapter 5 we present, among other things, a proof that a finite algebra that does not have homomorphic images and subalgebras is second-order finitely axiomatizable.

In the preliminary part (Preliminaries and notation) we revise some classical concepts and theorems of universal algebra and logic. For a reader unfamiliar with universal algebra or logic, this part is a prerequisite to the entire dissertation. In Part I the prerequisites are as follows: Chapter 1 contains basic definitions and should be read before any other chapter. For Chapter 4 one needs to read Chapters 2 and 3 first. Chapter 6 depends on Chapter 5 and for Chapter 5 one needs only Chapter 1. For Part III, the only prerequisite is Part I, Chapter 2; for Part II, both Chapter 1 and Chapter 3 of Part I are needed.

PRELIMINARIES AND NOTATION

The set of natural numbers will be denoted by $\mathbf{N}$ and its cardinality by ω. We sometimes omit parentheses when applying functions to elements or sets or when speaking of the image or inverse image of a set under a function.

0.1 Set-theoretic preliminaries

The power set of a set A, i.e., the set consisting of all subsets of A will be denoted by $\mathcal{P}(A)$.

Definition 0.1 *Let $X = \{X_i : i \in I\}$ be a family of sets indexed by a set I. Then the coproduct of X is the set*

$$\coprod X = \coprod_{i \in I} X_i := \{\langle i, x \rangle : x \in X_i\}.$$

We also call the coproduct of X the **disjointed union of** X.

In particular, if for every $i, j \in I$, $i \neq j \Rightarrow X_i \cap X_j = \emptyset$, then there is a bijection from $\coprod X$ onto the union $\bigcup_{i \in I} X_i$ of all X_i and in this case we can identify $\coprod X$ with $\bigcup_{i \in I} X_i$.

For a natural number $n \geq 1$, an *n-ary relation on a set A* is a subset R of the cartesian power A^n of A. The number n is called the *arity* of R. Thus if $n = 1$ then R is a subset of A. Relations of arity one are called *unary*, of arity two–*binary* and those of arity three–*ternary*. An important example of a binary relation on A is the equality relation $\mathrm{id}_A = \{\langle x, y\rangle \in A^2 : x = y\}$.

Sequences of elements will often be identified with strings and the notation $\vec{a}$ is used to denote the sequence $\langle a_1, \dots, a_n\rangle$. We often write $\langle a_1, \dots, a_n\rangle$ simply as $a_1 \dots a_n$. We write $\vec{a} \subseteq A$ to express that $\vec{a}$ is a string of elements of A.

Thus if R is an n-ary relation on a set A, then the expression $\vec{a} \in R$ stands for " $\vec{a}$ is an n-element sequence of elements of A,i.e., $\vec{a} = \langle a_1, \dots, a_n\rangle$ and $\langle a_1, \dots, a_n\rangle \in R$."

0.2 Universal algebraic preliminaries

0.2.1 Operations on elements

An *n-ary operation on A* is a function $f : A^n \longrightarrow A$. The number n is also called *the arity* of f.

Definition 0.2 *An* **algebra** *is a pair* $\mathbf{A} = \langle A, F\rangle$ *consisting of a set A and a set F of operations on A.*

We will write $\langle A, f_1, \dots, f_n\rangle$ for $\langle A, \{f_1, \dots, f_n\}\rangle$. The set A is called the *underlying set* of the algebra $\mathbf{A}$. We always use *boldface* capital letters (or groups of letters) to denote algebras and *roman* capital letters (or groups of letters) to denote sets. Moreover, unless we say otherwise, if some boldface letter is used to denote an algebra, then the corresponding Roman letter is used to denote its underlying set.

Definition 0.3 (i) *An* **algebraic language** *is a pair* $\Lambda = \langle \Lambda', \rho\rangle$ *consisting of some set* $\Lambda' = \{\lambda_1, \dots, \lambda_n\}$ *of n symbols and a function* $\rho : \Lambda \longrightarrow \mathbf{N}$*, called* **arity**. *The elements of Λ are called* **operation symbols**. *We identify the language with its set of operations, i.e.,* $\Lambda = \Lambda'$.

(ii) *Let* $\Lambda = \langle\{\lambda_\kappa : \kappa < \alpha\}, \rho\rangle$ *be an algebraic language, where* α *is some (finite or infinite) cardinal number. A* Λ-**algebra** *is a pair* $\mathbf{A} = \langle A, \{\lambda_\kappa^{\mathbf{A}} : \kappa < \alpha\}\rangle$, *where* A *is a set and for every* $\kappa < \alpha$, $\lambda_\kappa^{\mathbf{A}}$ *is a* $\rho(\lambda_\kappa)$*-ary operation on* A. *If* Λ *is finite, we say that* $\mathbf{A}$ *is an algebra of a* **finite type**.

When the algebra is clear from the context, we omit the superscript $\mathbf{A}$ and write simply λ_i for $\lambda_i^{\mathbf{A}}$. We say that two algebras $\mathbf{A}$ and $\mathbf{B}$ are of the same *type* if there is an algebraic language Λ such that $\mathbf{A}$ and $\mathbf{B}$ are Λ-algebras. Notice that every algebra is a Λ-algebra for some Λ.

Definition 0.4 *Let* Λ *be an algebraic language and let* X *be some set disjoint with* Λ. *We define the notion of a* Λ-**term in variables** X, *or simply a* **term,** *inductively as follows. Every element of* X *is a term. If* $t_1, \ldots, t_n$ *are terms and* $\lambda \in \Lambda$ *is an* n*-ary operation symbol, then the expression* $\lambda(t_1, \ldots, t_n)$ *is also a term. Nothing else is a term.*

The set of all terms in variables X will be denoted by $\mathrm{Te}_\Lambda(X)$. If we omit X and write Te_Λ, we understand that X is some set of cardinality ω. Also, if Λ is clear from the context, we omit the subscript Λ. For any X, the set $\mathrm{Te}_\Lambda(X)$ equipped with operations $\lambda^{\mathbf{Te}} = \lambda \in \Lambda$, which, to a given n-tuple $t_1, \ldots, t_n$ of terms assigns the term $\lambda(t_1, \ldots, t_n)$, forms a Λ-algebra. We denote this algebra by $\mathbf{Te}_\Lambda(X)$. The conventions above on omitting subscripts Λ or X apply also to the term algebras.

Terms will usually be denoted by small roman letters $t, s, \ldots$, possibly with subscripts.

Definition 0.5 *Let* $t \in \mathrm{Te}_\Lambda(X)$. *The set* $\mathrm{Var}(t)$ *of* **variables of** t *is defined recursively as follows:*

1. *If* $t \in X$, *then* $\mathrm{Var}(t) = \{t\}$.

2. $\mathrm{Var}(\lambda(t_1, \ldots, t_n)) = \bigcup_{i=1}^{n} \mathrm{Var}(t_i)$.

0.2.2 Universal algebraic constructions

Homomorphisms and subalgebras. Let $\mathbf{A}$ and $\mathbf{B}$ be two Λ-algebras.

Definition 0.6 *A function* $h : A \to B$ *is called an* **algebra homomorphism** *from* $\mathbf{A}$ *to* $\mathbf{B}$ *if for every* $i < n$ *and for every sequence of elements* $a_1, \ldots, a_{\rho(i)} \in A$, *we have*

$$h(f(a_1, \ldots, a_{\rho(i)})) = f(h(a_1), \ldots, h(a_{\rho(i)})).$$

An **isomorphism** *is a homomorphism* h *that is a bijection. In this case the inverse of this bijection is also a homomorphism, called the* **inverse of** h. *We say that two* Λ*-algebras* $\mathbf{A}$ *and* $\mathbf{B}$ *are* **isomorphic** *and write* $\mathbf{A} \cong \mathbf{B}$ *if there is an isomorphism* $h : \mathbf{A} \to \mathbf{B}$.

Definition 0.7 *A* **subalgebra** *of a* Λ*-algebra* $\mathbf{A}$ *is a* Λ*-algebra* $\mathbf{B}$ *such that* $B \subseteq A$ *and for every* $\lambda \in \Lambda$ *and all strings* $\vec{b}$ *of length* $\rho(\lambda)$ *of elements of* B, $\lambda^{\mathbf{A}}(\vec{b}) = \lambda^{\mathbf{B}}(\vec{b})$. *We write* $\mathbf{B} \subseteq \mathbf{A}$ *to say that* $\mathbf{B}$ *is a subalgebra of* $\mathbf{A}$.

Proposition 0.8 *A subset* B *of* A *is the underlying set of a subalgebra of* $\mathbf{A}$ *if the image of the restriction to* B *of every operation* $f \in F$ *is contained in* B.

Definition 0.9 *Let* $X \subseteq A$. *We say that* X **generates** *a subalgebra* $\mathbf{B}$ *of the algebra* $\mathbf{A}$ *if* $\mathbf{B}$ *is the smallest subalgebra of* $\mathbf{A}$ *containing* X. *For a cardinal number* α *the algebra* $\mathbf{A}$ *is* α*-generated if it is generated by some subset of cardinality less than or equal to* α.

Definition 0.10 *Let* $\mathbf{A}$ *be an algebra,* $X \subseteq A$*, and let* $\mathcal{K}$ *be a class of* Λ*-algebras. We say that* $\mathbf{A}$ *has the* **universal mapping property** *over* X *with respect to the class* $\mathcal{K}$ *if for every* $\mathbf{B} \in \mathcal{K}$ *and for every function* $f : X \to \mathbf{B}$ *there is exactly one homomorphism* $h : \mathbf{A} \to \mathbf{B}$ *such that the restriction* $h|X$ *of* h *to* X *is equal to* f*, i.e.,* $h|X = f$.

Theorem 0.11 ([7, 10.6, 10.7]) *Let* X *and* Y *be two sets of the same cardinality. If* $\mathbf{A}$ *has the universal mapping property over* X *with respect to* $\mathcal{K}$*,* $\mathbf{B}$ *has the universal mapping property over* Y *with respect to* $\mathcal{K}$*, and if* $\mathbf{A}, \mathbf{B} \in \mathcal{K}$*, then* $\mathbf{A} \cong \mathbf{B}$ *and in fact any bijection* $f : X \to Y$ *extends uniquely to an isomorphism* $f^* : \mathbf{A} \to \mathbf{B}$.

Theorem 0.12 ([7, Theorem 10.8]) *Let* X *be some set of variables and let* $\mathbf{A}$ *be a* Λ*-algebra. Then for every function* $f : X \to A$ *there is exactly one homomorphism* $h : \mathbf{Te}(X) \to \mathbf{A}$ *such that the restriction of* h *to* X *coincides with* f*, i.e., the term algebra* $\mathbf{Te}(X)$ *has the universal mapping property with respect to the class of all* Λ*-algebras.*

0.2.3 Congruences

Definition 0.13 *Let* $\mathbf{A} = \langle A, F\rangle$ *be an algebra. An equivalence relation* θ *is called a* **congruence** *if it satisfies the following* substitution property

$$\langle a_i, b_i\rangle \in \theta \text{ for all } i = 1, \ldots, n \Rightarrow \langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)\rangle \in \theta, \qquad (0.1)$$

for every n*-ary operation in* F.

The set of all congruences of an algebra $\mathbf{A}$ *will be denoted by the symbol* $\mathrm{Co}(\mathbf{A})$*. The smallest congruence on* $\mathbf{A}$*, with respect to inclusion, that includes some given* *set X of pairs of elements of A is called* **the congruence generated by** X. *It is denoted by* $\Theta(X)$.

Theorem 0.14 *(A.Malcev) Let* $\mathbf{A}$ *be an algebra and let* X *be some set of pairs of elements of* A. *Then* $\langle a,b\rangle \in \Theta(X)$ *iff there exists a sequence* $a = a_1, a_2, \ldots, a_n = b$ *of elements of* A *such that for every* $i = 1,\ldots,n$ *there is a term* $t(x_1,\ldots,x_m)$ *and elements* $c_1,\ldots,c_{m-1}$ *such that* $a_i = t(c, c_1,\ldots,c_{m-1})$ *and* $a_{i+1} = t(d, c_1, \ldots, c_{m-1})$ *for some* a, b *such that* $\langle c,d\rangle \in X$ *or* $\langle d,c\rangle \in X$.

If $\mathbf{A} = \langle A, F\rangle$ is an algebra and $h : \mathbf{A} \to \mathbf{B}$ a homomorphism, then the *kernel of* h is the inverse image $\ker(h) = h^{-1}(\mathrm{id}_B)$ of the identity relation on B. Observe that $\ker(h)$ is a congruence.

0.2.4 Equations, quasi-equations and related classes

Definition 0.15 *Let* $\mathbf{A}$ *be a* Λ*-algebra. Every homomorphism* $h : \mathbf{Te}_\Lambda \to \mathbf{A}$ *is called a* **valuation**. *A valuation* $\sigma : \mathbf{Te}_\Lambda \to \mathbf{Te}_\Lambda$ *is called a* **substitution**.

In view of the universal mapping property (0.12), every valuation is uniquely determined by its values on the set of variables. We will therefore identify valuations with their restrictions to the set of variables. Similarly, we will identify a substitution with its values on variables. Let v be a valuation into $\mathbf{A}$ and let $t = t(x_1,\ldots,x_n)$ be a term. The value $v(t)$ is denoted by $t^{\mathbf{A}}(v)$ or by $t^{\mathbf{A}}(a_1,\ldots,a_n)$, where for $i = 1,\ldots,n$, $a_i = v(x_i)$.

Every valuation v can be extended to the set of all atomic formulas in the following way. Let $\phi = R(t_1,\ldots,t_n)$. Then $v(\phi) := R((t_1(v)),\ldots,(t_n(v)))$. Also, if $\vec{t}$ is the sequence $t_1,\ldots,t_n$ of terms, then $(\vec{t}(v)) := (t_1(v)),\ldots,(t_n(v))$. In particular, for a substitution σ, $\sigma(\phi) := R(\sigma(t_1), \dots, \sigma(t_n))$ and if $\vec{t}$ is the sequence $t_1, \dots, t_n$ of terms, then $\sigma(\vec{t}) := \sigma(t_1), \dots, \sigma(t_n)$.

Definition 0.16 (i) *An* (Λ-)**equation** *is a pair of terms* $\langle t_1, t_2 \rangle$*, usually written as* $t_1 \approx t_2$*. A valuation* v **satisfies** *an equation* $t_1 \approx t_2$ *if* $t_1^{\mathbf{A}}(v) = t_2^{\mathbf{A}}(v)$*. If every valuation* v *satisfies an equation* ε *then* ε *is called an* **identity of A** *and we write* $\mathbf{A} \models \varepsilon$*. The set of all identities of* $\mathbf{A}$ *is denoted by* $\mathrm{Id}(\mathbf{A})$.

(ii) *An expression* q *of the form*

$$\bigwedge_{i<m} \epsilon_i \to \epsilon,$$

where $m \in \mathbf{N}$ *and* $\epsilon_1, \dots, \epsilon_m, \epsilon$ *are equations, is called a* **quasi-equation***. A valuation* v **satisfies** *a quasi-equation* q *above if it satisfies* ϵ *whenever it satisfies all* ϵ_i*, for* $i = 1, \dots, m$*. If every valuation* v *satisfies* q*, then we say that* $\mathbf{A}$ *satisfies* q*, write* $\mathbf{A} \models q$*, and call* q *a* **quasi-identity of A***. The set of all quasi-identities of* $\mathbf{A}$ *will be denoted by* $\mathrm{QId}(\mathbf{A})$.

Note that an equation is a special case of a quasi-equation, namely for $m = 0$. Also an equation ϵ is an identity of $\mathbf{A}$ iff it is a quasi-identity of $\mathbf{A}$.

Let $\mathcal{K}$ be a class of algebras. A quasi-equation q is called a *quasi-identity of* $\mathcal{K}$ if it is a quasi-identity of every algebra $\mathbf{A} \in \mathcal{K}$. A quasi-identity of $\mathcal{K}$ that is an equation is called an *identity of* $\mathcal{K}$. The sets of all identities and of all quasi-identities of $\mathcal{K}$ are denoted by $\mathrm{QId}(\mathcal{K})$ and $\mathrm{Id}(\mathcal{K})$, respectively.

Let Σ be some set of quasi-equations. An algebra $\mathbf{A}$ is a *model of* Σ iff $\Sigma \subseteq \mathrm{QId}(\mathbf{A})$. The class of all models of Σ is denoted by $\mathrm{Mod}(\Sigma)$. A class $\mathcal{K}$ such that $\mathcal{K} = \mathrm{Mod}(\Sigma)$ for some set of quasi-equations Σ, is called a *quasi-equational class.* If all elements of Σ are equations, then $\mathcal{K}$ is called an *equational* class.

Operators H, I, S, P, P_U. Definitions of a product and an ultraproduct of algebras can be found in [7].[1] We define the following operators on classes of algebras.

Definition 0.17 *Let $\mathcal{K}$ be a class of Λ-algebras.*

$$I(\mathcal{K}) := \{\mathbf{B}\text{: } \mathbf{B} \cong \mathbf{A}, \textit{ for some } \mathbf{A} \in \mathcal{K}\};$$

$$H(\mathcal{K}) := \{\mathbf{B} : \textit{there exists } \mathbf{A} \in \mathcal{K} \textit{ and a surjective homomorphism } f : \mathbf{A} \to \mathbf{B}\};$$

$$S(\mathcal{K}) := \{\mathbf{B}\text{:}\mathbf{B} \subseteq \mathbf{A}, \textit{ for some } \mathbf{A} \in \mathcal{K}\};$$

$$P(\mathcal{K}) := \{\textstyle\prod_{i\in I} \mathbf{A}_i\text{: } \{\mathbf{A}_i : i \in I\} \subseteq \mathcal{K}\}.$$

$$P_U(\mathcal{K}) := \{\textstyle\prod_{i\in I} \mathbf{A}_i/U\text{: } \{\mathbf{A}_i : i \in I\} \subseteq \mathcal{K} \textit{ and } U \textit{ is an ultrafilter on } I\};$$

Varieties and quasivarieties. We say that a class $\mathcal{K}$ is closed under an operator O on classes of algebras if the result of applying O to algebras in $\mathcal{K}$ is also in $\mathcal{K}$. For example, $\mathcal{K}$ is closed under the formation of direct products if the direct product of any family of algebras from $\mathcal{K}$ is also in $\mathcal{K}$.

Definition 0.18 *Let Λ be some algebraic language.*

(i) *A* **variety** *is a class of Λ-algebras closed under the operations H, S and P.*

(ii) *A* **quasivariety** *is a class of Λ-algebras closed under I, S, P and P_U.*

The following two theorems characterize equational and quasi-equational classes in terms of their closures under certain operators.

[1]On page 70 we will give definitions of a product and ultraproduct of so-called *matrices*, of which the product and ultraproduct of algebras are special cases.

Theorem 0.19 (G. Birkhoff [2]) *Let $\mathcal{K}$ be a class of Λ-algebras, for some fixed language Λ. Then the following are equivalent:*

(i) *$\mathcal{K}$ is equational.*

(ii) *$\mathcal{K}$ is a variety.*

(iii) *$HSP(\mathcal{K}) \subseteq \mathcal{K}$, i.e., $\mathcal{K}$ is closed under the operator HSP.*

(iv) *$HSP(\mathcal{K}) = \mathcal{K}$.*

Theorem 0.20 (A. Malcev [29]) *Let $\mathcal{K}$ be a class of Λ-algebras, for some fixed algebraic language Λ. Then the following are equivalent.*

(i) *$\mathcal{K}$ is quasi-equational.*

(ii) *$\mathcal{K}$ is a quasivariety.*

(iii) *$ISPP_U(\mathcal{K}) \subseteq \mathcal{K}$, i.e., $\mathcal{K}$ is closed under the operator $ISPP_U$.*

(iv) *$\mathcal{K} = ISPP_U(\mathcal{K})$.*

Definition 0.21 *Let $\mathcal{K}$ be some class of Λ-algebras. The* **variety (quasivariety) generated by** *$\mathcal{K}$ is the smallest (with respect to inclusion) variety (resp. quasivariety) including $\mathcal{K}$. If $\mathcal{K}$ is a finite set of finite algebras then the variety (quasivariety) generated by $\mathcal{K}$ is called* **finitely generated.**

Proposition 0.22 *A variety $\mathcal{V}$ is finitely generated iff it is generated by a single finite algebra.*

Proof. A variety generated by one finite algebra is finitely generated by definition. Conversely, if a finite set $\mathcal{K}$ of finite algebras generates $\mathcal{V}$, then also the product $\prod \mathcal{K}$ of all algebras of $\mathcal{K}$ generates $\mathcal{V}$. For by definition, this product is in $\mathcal{V}$ and conversely, each algebra of $\mathcal{K}$ is a homomorphic image of $\prod \mathcal{K}$ and therefore is in every variety containing $\mathcal{K}$. Now the product of a finite number of finite algebras is finite, hence $\mathcal{V}$ is generated by a single finite algebra. $\Box$ A class $\mathcal{K}$ of algebras is *locally finite* if every finitely generated member of $\mathcal{K}$ is finite. It follows from the universal mapping property that a quasivariety $\mathcal{Q}$ is locally finite iff every free matrix on finitely many generators is finite.

Theorem 0.23 ([7, Theorem 10.16.]) *A finitely generated variety is locally finite.*

Corollary 0.24 *A finitely generated quasivariety is locally finite.*

Proof. Let $\mathcal{K}$ be a finite set of finite algebras. The quasivariety generated by $\mathcal{K}$ is included in the variety generated by $\mathcal{K}$. The latter is locally finite by the previous theorem. Hence the former is also locally finite, as local finitness is preserved by subclasses. $\Box$

The following corollary follows directly from Theorems 0.19 and 0.20.

Corollary 0.25 *Let Λ be an algebraic language. Let $\mathcal{K}$ be some class of Λ-algebras. The variety and quasivariety generated by $\mathcal{K}$ are, respectively, $HSP(\mathcal{K})$ and $ISPP_U(\mathcal{K})$.*

Definition 0.26 *An algebra* $\mathbf{A}$ *is* **subdirectly irreducible** *if the set* $\mathrm{Co}(\mathbf{A}) \setminus \{\mathrm{id}_A\}$ *ordered by inclusion contains a smallest element. The class of all subdirectly irreducible members of $\mathcal{K}$ is denoted by $\mathcal{K}_{SI}$.*

Theorem 0.27 (G. Birkhoff) *Every algebra* $\mathbf{A}$ *is isomorphic to a subdirect product of subdirectly irreducible algebras that are homomorphic images of* $\mathbf{A}$.

For a class $\mathcal{K}$ of algebras of the same type let

$$P_{SD}(\mathcal{K}) := \{\mathbf{A} : \mathbf{A} \text{ is a subdirect product of a family } \mathcal{A} \subseteq \mathcal{K}\}.$$

Thus Theorem 0.27 says that $\mathbf{A} \in P_{SD}H(\mathbf{A})$.

Corollary 0.28 *Let* $\mathcal{K}$ *be a class of algebras of the same type. Then* $HSP(\mathcal{K}) = HSP(\mathcal{K}_{SI})$.

Corollary 0.29 *A quasivariety is determined by its subdirectly irreducible members, i.e., if* $\mathcal{K}$ *and* $\mathcal{L}$ *are quasivarieties, then*

$$(\mathcal{K})_{SI} = (\mathcal{L})_{SI} \Rightarrow \mathcal{K} = \mathcal{L}.$$

We also have versions of the above theorems for quasi-varieties.

Definition 0.30 *Let* $\mathcal{Q}$ *be a quasivariety and let* $\mathbf{A} \in \mathcal{Q}$. *A congruence* Θ *on* $\mathbf{A}$ *is called* **relative to** Q *if* $\mathbf{A}/\Theta \in Q$. *The set of all congruences on* $\mathbf{A}$ *that are relative to* Q *is denoted by* $\mathrm{Co}(Q)\mathbf{A}$.

Proposition 0.31 *For every quasivariety* Q *and* $\mathbf{A} \in Q$, *the set* $\mathrm{Co}(Q)\mathbf{A}$ *ordered by inclusion forms a lattice.*

The lattice of congruences of $\mathbf{A}$ relative to Q will be denoted by $\mathbf{Co}Q\mathbf{A}$.

Definition 0.32 *The algebra* $\mathbf{A}$ *is* **subdirectly irreducible relatively to** Q *if the lattice* $\mathbf{Co}_Q\mathbf{A}$ *has the smallest nonempty element. The set of all algebras in* Q *that are subdirectly irreducible relative to* Q *is denoted by* Q_{QSI}. *Given a class* $\mathcal{K}$ *of algebras, an algebra* $\mathbf{A}$ *is called subdirectly irreducible relative to* $\mathcal{K}$ *if it is subdirectly irreducible relative to* $ISPP_U(\mathcal{K})$. *By* $\mathcal{K}_{RSI}$ *we denote the class of all algebras that are subdirectly irreducible relative to* $\mathcal{K}$.

Notice that $\mathcal{K}_{RSI} \subseteq \mathcal{Q}(\mathcal{K})$.

Theorem 0.33 *Let* Q *be a quasivariety. Then every algebra* $\mathbf{A} \in Q$ *is isomorphic to a subdirect product of subdirectly irreducible algebras relative to* Q *that are homomorphic images of* $\mathbf{A}$.

Corollary 0.34 *Let* $\mathcal{K}$ *be a class of algebras of the same type. Then*

$$ISPP_U(\mathcal{K}) = ISPP_U(\mathcal{K}_{RSI}).$$

Corollary 0.35 *A quasivariety is determined by its relatively subdirectly irreducible members, i.e., if* $\mathcal{K}$ *and* $\mathcal{L}$ *are quasivarieties, then*

$$(\mathcal{K})_{RSI} = (\mathcal{L})_{RSI} \Rightarrow \mathcal{K} = \mathcal{L},$$

Definition 0.36 *Let* $\mathcal{K}$ *be a class of* Λ*-algebras and let* $\mathbf{A}$ *be a* Λ*-algebra generated by some set* X *of cardinality* α. *Then* $\mathbf{A}$ *is called a* **free algebra over** $\mathcal{K}$ **in** α **generators** *if* $\mathbf{A} \in \mathcal{K}$ *and it has the universal mapping property over* X *with respect to* $\mathcal{K}$ *(see Definition 0.10).*

By Theorem 0.11, if $\mathbf{A}$ and $\mathbf{B}$ are both free algebras with α generators for a class $\mathcal{K}$, then they must be isomorphic. The symbol $\mathbf{F}_{\mathcal{K}}(\alpha)$ denotes a free algebra over $\mathcal{K}$ in

α generators–it is unique up to isomorphism. For a class closed under isomorphisms the statement "$\mathbf{F}_{\mathcal{K}}(\alpha) \in \mathcal{K}$" means that any one, and therefore all, free algebras in α generators for $\mathcal{K}$ are in $\mathcal{K}$.

Theorem 0.37 (Birkhoff [2]) *If $\mathcal{K}$ is a quasivariety, then for every cardinal number $\alpha > 0$, $\mathbf{F}_{\mathcal{K}}(\alpha) \in \mathcal{K}$. Moreover, for any class $\mathcal{K}$ of algebras, $\mathcal{K} \in IS_{SD}(\mathcal{K})$.*

Theorem 0.38 *If $\mathcal{K}$ is a variety then the free algebra over $\mathcal{K}$ on ω generators generates $\mathcal{K}$, i.e.,*

$$\mathcal{K} = HSP(\mathbf{F}_{\mathcal{K}}(\omega)).$$

Let Λ be an algebraic language and consider the class of all Λ-algebras. This class is equational, for it is the class of models of the empty set of equalities. Thus it has a free algebra and it is easy to see that the free algebras over this class are exactly the term algebras.

Corollary 0.39 (Birkhoff [2]) *Let $\mathcal{K}$ be a class of algebras, let $\mathcal{V}$ be the variety generated by $\mathcal{K}$, and let $\mathbf{F}$ be the free algebra in $\mathcal{V}$ on ω generators. Then*

$$\mathrm{Id}(\mathcal{K}) = \mathrm{Id}(V) = \mathrm{Id}(\mathbf{F}).$$

0.3 First-order languages and structures

Definition 0.40 *A* **relational language** *is a pair $K = \langle K', \rho\rangle$ consisting of a set K' of* **relation** *or* **predicate symbols** *and a function ρ assigning to every relation symbol $R \in K$ a natural number $\rho(R)$ called its arity.*

We identify the set K of the relation symbols of K′ with the language K.

For a large part of the paper we will be concerned with the structures which are both models of some relational language K and of some algebraic language Λ. We will use the convention, that Λ always denotes an algebraic language, so whenever we say "language Λ" it should be clear that we speak about an algebraic language.

Definition 0.41 *A* **first-order language** *is a pair $\langle \Lambda, K\rangle$, where K is a relational language and Λ is an algebraic language. Alternatively, a first-order language can be viewed as a triple $\langle \Lambda,\ K, \rho\rangle$, where Λ and K are disjoint sets of symbols and $\rho : \Lambda \cup\ K \to \mathbf{N}$. The elements of Λ are called* **operation symbols,** *the elements of K are called* **predicate symbols** *and ρ is called* **arity**.

Notice, that an algebraic language Λ can be considered as a first-order language with $\mathrm{K} = \emptyset$ and a relational language K is a first-order language with $\Lambda = \emptyset$.

Definition 0.42 *An* **atomic formula** *of a first-order language $\langle \Lambda, K\rangle$ is an expression of the form $R(t_1, \ldots, t_n)$, where $R \in\ K$ is n-ary and $t_1, \ldots, t_n$ are terms. A* **conjunction of atomic formulas** *is an expression of the form $\bigwedge_{i\in I} \varphi_i$, where for each $i \in I$, φ_i is an atomic formula. A* **Horn formula** *is a first-order formula of the form*

$$\bigwedge_{i<m} \varphi_i \to \varphi \tag{0.2}$$

where m is some natural number and φ_i and φ are atomic formulas. The atomic formulas φ_i are called **premisses** *and the atomic formula φ is called the* **conclusion** *of the Horn formula* (0.2)*. A Horn formula preceded with a universal quantifier is called a* **universal Horn formula**.

By analogy with quasi-equations we will also refer to the universal Horn formulas as quasi-formulas (for example see Definition 2.7 (ii)). If $X = \{\varphi_i : i \in I\}$, then the conjunction $\bigwedge_{i\in I}\varphi_i$ can also be written as $\bigwedge X$. Let φ be an atomic formula. Then according to the above definition, there is an $R \in$ K and a sequence of terms $t_1, \ldots, t_{\rho(R)}$, such that $\varphi = R(t_1, \ldots, t_n)$. The set $\bigcup_{i\leq n} \mathrm{Var}(t_i)$ is called the *set of variables of* φ and is denoted by $\mathrm{Var}(\varphi)$. The notation $\varphi(x_1, \ldots, x_n)$ assumes that $\mathrm{Var}(\varphi) \subseteq \{x_1, \ldots, x_n\}$. For a conjunction $\psi = \bigwedge_{i\in I}\varphi_i$ of atomic formulas, $\mathrm{Var}(\psi) = \bigcup_{i\in I}\mathrm{Var}(\varphi_i)$. For a Horn formula $\psi = \bigwedge_{i\in I}\varphi_i \to \varphi$, $\mathrm{Var}(\psi) = \bigcup_{i\in I}\mathrm{Var}(\varphi_i) \cup \mathrm{Var}(\varphi)$.

Definition 0.43 *A* **structure** *of the first order language* $\langle\Lambda, K\rangle$ *is a triple* $\mathfrak{A} = \langle A, P, F\rangle$, *such that* $\langle A, F\rangle$ *is a* Λ*-algebra and*

$$P \subseteq \coprod_{R\in K} A^{\rho(R)}.$$

A K-**matrix** *is a structure for* $\langle\Lambda, K\rangle$, *for some* Λ.

We will always use capital gothic letters to denote models of first-order languages. The structure $\mathfrak{A}$ from Definition 0.43 can be identified with $\langle \mathbf{A}, R^{\mathfrak{A}} : R \in \mathrm{K}\rangle$, where $\mathbf{A}$ is the algebra $\langle A, F\rangle$. This algebra is called the *underlying algebra* of $\mathfrak{A}$. The reason for calling it a $\langle\Lambda, K\rangle$-matrix will be made clear in Part I, Chapter 2. We will always use the convention that if a gothic letter denotes a structure of a first order language, then the corresponding Roman boldface capital letter denotes its underlying algebra and, as we already said, the ordinary Roman capital letter denotes the underlying set.

Definition 0.44 *Let* $\langle\Lambda, K\rangle$ *be a first-order language and let* $\mathfrak{A}$ *be a structure for* $\langle\Lambda, K\rangle$. *Let* $\varphi(x_1, \ldots, x_n) = R(\vec{t}(x_1, \ldots, x_n))$ *be an atomic formula; let* $\varphi_1, \ldots, \varphi_n$ *be* *atomic formulas in variables* $x_1, \ldots, x_n$. *Let* $a_1, \ldots, a_n \in A$ *and let* v *be a valuation such that* $v(x_i) = a_i$, *for* $i = 1, \ldots, n$.

(i) *The valuation* v **satisfies** φ *if* $t^{\mathbf{A}}(a_1, \ldots, a_n) \in R^{\mathfrak{A}}$.

(ii) *We say that* $\mathfrak{A}$ **satisfies** φ, *and write* $\mathfrak{A} \models \varphi$, *if for every choice* $\vec{a} = \langle a_1, \ldots, a_n \rangle$ *of elements of* A, $t^{\mathbf{A}}(\vec{a}) \in R^{\mathfrak{A}}$.

(iii) *The valuation* v **satisfies** $\bigwedge_{i \leq m} \varphi_i$, *in symbols* $\mathfrak{A} \models \bigwedge_{i \leq m} \varphi_i(v)$, *if for every* $i \leq m$, v *satisfies* φ_i.

(iv) $\mathfrak{A}$ **satisfies** $\bigwedge_{i \leq m} \varphi_i$ *if for every valuation* v, $\mathfrak{A} \models \bigwedge_{i \leq m} \varphi_i(v)$.

(v) *The valuation* v **satisfies the Horn formula** $\bigwedge_{i \leq m} \varphi_i \rightarrow \varphi$ *if either* $\mathfrak{A} \models \varphi(a_1, \ldots, a_n)$ *or, for some* $i \leq m$, $\mathfrak{A} \not\models \varphi_i(a_1, \ldots, a_n)$. *We write* $\mathfrak{A} \models (\bigwedge_{i \leq m} \varphi_i \rightarrow \varphi)(v)$ *in this case.*

(vi) $\mathfrak{A}$ **satisfies** $\bigwedge_{i \leq m} \varphi_i \rightarrow \varphi$, $\mathfrak{A} \models \bigwedge_{i \leq m} \varphi_i \rightarrow \varphi$, *if for every valuation* v, $\mathfrak{A} \models (\bigwedge_{i \leq m} \varphi_i \rightarrow \varphi)(v)$. *In this case we write* $\mathfrak{A} \models (\bigwedge_{i \leq m} \varphi_i \rightarrow \varphi)$.

Atomic formulas will be denoted by small greek letters. A *universal Horn theory* is a set of universal Horn formulas that is closed under ordinary rules of the first order classical logic.

Definition 0.45 *Let* $\mathcal{K}$ *be a class of structures of a first-order language and let* Φ *be a set of formulas which are built from the set of atomic formulas by means of the standard connectives* $\vee, \wedge, \rightarrow, \neg$ *and quantifiers* $\forall, \exists$. *We say that the class* $\mathcal{K}$ **satisfies** Φ *and write* $\mathcal{K} \models \Phi$ *if every element* $\mathfrak{A} \in \mathcal{K}$ *satisfies every formula* $\varphi \in \Phi$.

The class $\mathcal{K}$ is **axiomatized by** Φ *if* $\mathcal{K} \models \Phi$ *and for every formula* φ *such that* $\mathcal{K} \models \varphi$, φ *can be derived from* Φ *by means of ordinary first order classical logic.*

In this subsection we assume the reader's familiarity with basic first-order logic; in particular with the notion of a first-order formula and satisfaction. Roughly speaking, a first-order formula is a formula constructed from atomic formulas by means of the connectives $\wedge, \vee, \neg, \rightarrow$ and quantifiers in certain standard way.

Definition 0.46 *Let* $\langle \Lambda, K \rangle$ *be a language and let* $\mathcal{K}$ *be a class of structures for this language. We say that* $\mathcal{K}$ *is* **elementary** *if there is a set of first-order formulas* Φ *such that* $\mathcal{K} = \{\mathfrak{A} : \mathfrak{A} \models \Phi\}$. $\mathcal{K}$ *is* **strictly elementary** *if the set* Φ *can be found finite.*

0.4 Lattice theoretical preliminaries

Here we review some basic facts and definitions concerning lattices. Proof of these facts can be found, for example, in [7].

A partial ordering on a set L, such that every pair a, b of elements of L has an infimum $a \wedge b$ and a supremum $a \vee b$ in L with respect to this ordering, is called *a lattice ordering* on L. Equivalently, a partial ordering $\leq$ is a lattice ordering iff every finite set of elements $a_1, \ldots, a_n$ of elements of L has an infimum $\bigwedge_{i=1}^{n} a_i$ and supremum $\bigvee_{i=1}^{n} a_i$ with respect to $\leq$. If there is a lattice ordering on a set L, then the algebra $\langle L, \vee, \wedge \rangle$ is called *a lattice.* A lattice is *complete* if the suprema and infima exist for sets of arbitrary cardinality. An element a of a complete lattice L is *compact* if, for every set $\{a_i : i \in I\} \subseteq L$, $a = \bigvee_{i \in I} a_i \Rightarrow a = \bigvee_{i \in J} a_i$ for some finite subset J of I. A lattice L is algebraic, if every element of L is a supremum (possibly infinite) of some family of compact elements.

Definition 0.47 *A subset $\{a_i : i \in I\}$ of a lattice L, where I is some set, is called* **directed** *if for all $i, j \in I$ there is $k \in I$ such that $a_i \vee a_j \leq a_k$.*

Definition 0.48 *Let $\mathbf{L}_1, \mathbf{L}_2$ be two lattices and let $f : L_1 \to L_2$ be a function (not necessary a homomorphism). Then we say that f is* **continuous** *if for every set I of indices and every directed subset $\{a_i : i \in I\}$ of L_1,*

$$f(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} a_i.$$

Let $\mathbf{A}$ be an algebra and let $\mathrm{Co}(\mathbf{A})$ be the set of all congruences on $\mathbf{A}$. Then the relation $\subseteq$ on $\mathrm{Co}(\mathbf{A})$ is a lattice ordering. With respect to inclusion, the infimum of two congruences is their intersection and their supremum is the congruence generated by $\theta \cup \psi$. Thus $\mathrm{Co}(\mathbf{A})$ with these operations forms a lattice which we denote by $\mathbf{CoA}$. It is well known that $\mathbf{CoA}$ is a complete algebraic lattice, whose compact elements are exactly the finitely generated congruences on $\mathbf{A}$.

Let $\mathbf{L}$ be a lattice and let $a \in L$. Then the set $\{b \in L : a \leq b\}$ is denoted by $[a)$.

Proposition 0.49 *Let A be a set. Let L be a lattice of subsets of A ordered by inclusion and let $\{a_i : i \in I\}$ be a directed subset of L. Then $\bigvee_{i\in I} a_i = \bigcup_{i\in I} a_i$.*

Definition 0.50 *A lattice L is* **distributive** *if it satisfies the following condition*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{0.3}$$

or, equivalently,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z). \tag{0.4}$$

Lemma 0.51 *A lattice* $\langle L, \wedge, \vee\rangle$ *is* distributive *iff for every finite family* $a_1, \ldots, a_n$, $b_1, \ldots, b_m$ *of elements of* L

$$\bigvee_{i=1}^{n} a_i \cap \bigvee_{j=1}^{m} b_j \leq \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} a_i \cap b_j.$$

A class $\mathcal{K}$ of Λ-algebras such that for every $\mathbf{A} \in \mathcal{K}$, $\mathrm{Co}(\mathbf{A})$ is distributive is called *congruence distributive.*

PART I.

DEDUCTIVE SYSTEMS

CHAPTER 1. INTRODUCTION

Mathematical logic can be viewed as a study of deductive systems. Originally, only one deductive system was considered, namely the deductive system of the classical logic. In the first part of this century, the interest of logicians was captured also by the new, so-called, non-classical deductive systems. These deductive systems were at first studied only at the level of propositions (propositional deductive systems) and in fact the investigations focused at first on the propositional *theories* rather than on deductive systems. A *theory* is a set of propositions (that are "true" in some interpretation), while the deductive system can be identified with a set of *rules* that we are allowed to use in order to make "correct" deductions. Gradually, attention turned to the study of the "deductive apparatus" used to obtain the theorems of a given deductive system; this deductive apparatus can be formalized as a set of rules, or as a *deductive system*[1].

Many deductive systems have been studied separately and many of them have interesting applications. They have, however, many common properties that can be formalized and studied jointly. The first to realize this were J. Łukasiewicz and A. Tarski and their Lwów-Warsaw School of Logic. The most recent summary of results in the area of deductive systems can be found in [59].

[1]In A. Tarski's early work these notions of deductive system and theory are reversed.

We put the words "true" and "correct" above in quotation marks, for according to the key observation of Tarski, the truth of a proposition is always relativized to a class of (intended) *models*, or equivalently (intended) *interpretations*. A model of a deductive system is an *algebra* with a special subset of *designated elements*. Elements of the designated subset are called *designated elements* and are interpreted as "true". For example, the two-element Boolean algebra $\langle\{0,1\},\vee,\wedge,\rightarrow,\neg\rangle$ with $\{1\}$ as the designated subset is a model of the classical deductive system. We can see in this example why a model is an algebra and not a set—we must be able to interpret logical operations. Thus with every deductive system we can associate a class of models, called the *semantics* of the deductive system. Properties of a deductive system are reflected in its semantics and vice versa and hence the semantical methods have been useful in investigation of the deductive systems.

A deductive system and its corresponding semantics of a somewhat different type have been used in algebra. The role of formulas is played here by equations and the role of inference rules by quasi-equations. Equations do not correspond to elements, but rather to pairs of elements of an algebra and, therefore, we have a designated relation rather than a set of designated elements. In the special system that we have in mind, such a relation is called a congruence relation. Thus a model of the deductive system of algebra consists of an algebra $\mathbf{A}$ and a congruence relation on $\mathbf{A}$.

The first systematic investigation of these two systems in a common framework can be found in Blok and Pigozzi [4]. They introduced the concept of *k-deductive system*, in which the notions of formula and equation are simultaneously generalized to that of k-formulas. It is convenient in this context to identify terms and formulas. A

k-formula is a k-sequence $\langle t_1, \ldots, t_k \rangle$ of terms. An example of a k-formula, for $k = 2$ is an equation, and for $k = 1$, a formula. The historical notion of a deductive system can be identified with that of a 1-deductive system in this context. Blok and Pigozzi also remark that the formalism of a k-deductive system is equivalent to the formalism of universal Horn logic with one predicate symbol and they mention the possibility of generalizing their theory to universal Horn theories with an arbitrary number of predicate symbols– what we call here *K-deductive systems.* If u universal Horn theory has only finitely many predicate symbols, then the K-deductive system is called also a $\vec{k}$-deductive system.) Blok and Pigozzi obtain, among others, many interesting results on the semantics of the k-deductive systems and remark that many of these results can be carried over to the more general case of K-deductive systems. They also characterize the k-deductive systems for which the semantical results parallel the results of universal algebra.

Gentzen style deductive systems have also played an important role in the study of logical systems and can be put in the same framework. Here the role of formulas is played by *sequents*, where a sequent can be identified with a finite sequence of terms (formulas). In the terminology of [3], a sequent of length k can be viewed as a k-term, and in the same way that a k-term $\langle t_1, \ldots, t_k \rangle$ corresponds to an atomic formula $D(t_1, \ldots, t_k)$ of universal Horn logic with one predicate, a sequent can be viewed as an atomic formula of the universal Horn logic with *infinitely* many predicate symbols, D_k, one for each natural number k.

Thus we see that k-deductive systems, universal Horn logic and Gentzen systems can all be treated as special cases of the same general concept, that of K-deductive systems which we introduce in Chapter 2. The work presented in Part I was inspired by the results of [3, 5, 4, 6]on algebraizable and protoalgebraic k-deductive systems.

We start by redefining the concepts of [4] in the more general context of K-deductive system. Most of the facts can be carried over from [4] without difficulties but we present some of the proofs for completeness.

We then turn in Chapter 2 to the generalization of the concept of the *protoalgebraic logic.* Roughly speaking, a logic, or more properly a deductive system, is protoalgebraic if the so-called Leibniz operator (see General introduction or Definition 2.40) defined for models of this system is monotone. The algebraic methods can be modified to apply to the semantics of protoalgebraic deductive systems. Here, we first observe that for K-deductive systems the concept can be relativized in two different ways and then we prove a theorem characterizing these relativized concepts along the lines of the characterization of protoalgebraic 1-deductive systems in [4, Theorem 13.2]. In fact, [4, Theorem 13.2] claims a characterization of protoalgebraic k-deductive systems, but the proof for $k \geq 1$ contains a gap and the theorem is false (see Chapter 3 for a counterexample). We also relativize this characterization for a particular relation symbol R and a "place" k. As other results of [4, Sec. 13] depend on Theorem 13.2 of [4], we reformulate this theorem and also consider its relativized version in Chapter 3.

The results of Chapter 3 apply to Gentzen systems. However, for all Gentzen systems studied in the literature, a much better characterization of protoalgebraicity can be given than the one that follows from Chapter 3. Namely, Theorem 3.10 implies that a Gentzen system is protoalgebraic if there is a family of finite sets of sequents, called equivalence sequents, with some special properties. However, all the systems arising naturally from different calculi are protoalgebraic and the system of equivalence sequents is much simpler: it is not only a family of finite sets but in fact just one finite set. In chapter 4 we study some syntactical conditions on the Gentzen systems that allow us to conclude that if a system is protoalgebraic then the system of equivalence connectives is finite.

Chapter 5 is devoted to the concept and properties of equivalent deductive systems over different predicate languages. Roughly speaking, two systems are equivalent if there are mutual translations that translate formulas of one language into another in such a way that an inference is valid in one system iff its tanslation is valid in the other. For example, consider the 1-deductive system of classical sentential logic and the 2-deductive equational system whose models are exactly Boolean algebras. Then it can be easily seen (see for example page 143) that these two systems are equivalent. The class of Boolean algebras is called the *algebraic semantics* of the deductive system of classical logic. In general, a K-deductive system is called *algebraizable* if there is some *quasivariety* of algebras such that $\mathcal{S}$ is equivalent to the 2-deductive system whose models are exactly the algebras of this quasivariety. The notion of algebraizable 1-deductive system was introduced in [5] and the more general concept of equivalence between a k- and an l-deductive system in [6]. It was proved in [6] that two systems are equivalent iff there is an isomorphism between their lattices of theories that commutes with substitutions. This theorem has as its special case Theorem 3.7. of [5]. On the other hand, a deep Theorem 4.2 of [5] that characterizes algebraizable 1-deductive systems in terms of injectivity and continuity of the Leibniz operator, does not have an analogue for the equivalence of k-deductive systems. We show, however, that this theorem can also be generalized, although its generalization is more limited than the generalization of [5, Theorem 3.7.]. To this end we introduce the notion of a "Birkhoff-like K-deductive system" and the notion of "compatibility relations". We then prove that if $\mathcal{S}_1$ is an arbitrary K_1-deductive system and $\mathcal{S}_2$ is a Birkhoff-like K_2-deductive system, then these systems are equivalent iff a certain operator associated with the systems and the compatibility relations C is injective and continuous (Theorem 5.19). This single theorem allows one to prove not only the characterizations of algebraizable k-deductive systems and Gentzen systems, but also certain deductive systems with the connectives that behave like implication. In the next chapter, Chapter 6, we make use of this criterion to discuss the problem of properly defining the notion of a system of implication connectives in a 1-deductive system. In the future this work may be carried over to k-deductive systems and Gentzen systems.

CHAPTER 2. PRELIMINARIES ON K-DEDUCTIVE SYSTEMS

2.1 Introduction

The key concept of this thesis, that of a K-deductive system and also the concept of a K-consequence, generalize the standard notions of consequence operator and deductive system defined, for example, in [59, page 22]. The standard notions correspond in our formalization to 1-consequence and 1-deductive system. The first step of this generalization was done in [4], where the k-deductive systems were introduced and studied. It was anticipated there that the formalism and a large part of the theory can be extended to universal Horn logic. The formalism of ω-deductive system was proposed by Pigozzi [41] and that of more general Gentzen systems of so-called type (α, β) by Rebagliato and Verdú [50]. The notions of a k-deductive system, ω-deductive system and a Gentzen system of type (α, β) are special cases of the notion of a K-deductive system, which we define in Section 2.2. This notion is equivalent to the notion of universal Horn logic (possibly without equality). We have chosen the name "K-deductive system" to stress the origin and applications of this work for deductive systems of various kinds. The K-deductive system have been also studied in [11], where some results of this chapter (and also of Chapter 2, Part II) were independently obtained. We discuss examples of K-deductive systems in Section 2.3 and define second-order-deductive systems in Section 2.4. Basic semantics of a K-deductive system is introduced in Section 2.5. The Leibniz operator, defined in Section 2.6, serves to define reduced matrix semantics (Section 2.7) and protoalgebraicity (Section 2.8). A few first results illustrating the claim that the reduced semantics of protoalgebraic K-deductive systems parallels the algebraic semantics of quasi-equational logic are given in Section 2.8. In a later parts of the dissertation, (Part I, Chapter 3 and especially Part II, Chapter II) the semantics of protoalgebraic K-deductive systems is discussed in more detail.

Leibniz operator, reduced semantics and protoalgebraic deductive systems were introduced in [3] and developed in [4] for 1- and k- deductive systems, respectively (1- and k-deductive systems are defined in Definitions 2.18 and 2.17). Almost all material presented in this Chapter is a straightforward modification of the content of [4, Sections 1–8] and much of it has already appeared in the independent work of R. Elgueta in [11].

2.2 Basic definitions

Let Λ be an arbitrary but fixed algebraic language.

Definition 2.1 *Let $K := \langle\ K, \rho\rangle$ be some relational language. Recall that* Te *and* **Te** *denote respectively the set and the algebra of Λ-terms. Let* $\mathrm{Fm}_{\mathrm{K}} := \coprod_{R\in\ K} \mathrm{Te}^{\rho(R)}$. *Thus a typical K-term is of the form $\langle R, \vec{t}\rangle = \langle R, \langle t_1, \ldots, t_{\rho(R)}\rangle\rangle$, or simply $\langle R, \vec{t}\rangle$, where $R \in\ K$ and $\vec{t} = \langle t_1, \ldots, t_{\rho(R)}\rangle \in \mathrm{Te}^{\rho(R)}$. The elements of* Fm_{K} *are called* K-**terms** *or* K-**formulas**. *We will identify a K-term $\langle R, \vec{t}\rangle$ with the (Λ, K)-atomic formula $R(\vec{t})$ and write this as $\langle R, \langle t_1, \ldots, t_{\rho(R)}\rangle\rangle$ or more simply, $R(\vec{t})$.*

We will usually denote the K-terms by small Greek letters $\varphi, \psi, \ldots$. If K consists of only one predicate symbol, D, with $\rho(D) = 1$, then the elements of Fm_{K} are of the form $D(t)$, where t is a term. Thus Fm_{K} can be identified with Te in this case, which motivates the name K-term. The alternative name "K-formula" is partly motivated by those cases where the language Λ is a set of connectives of some concrete logic, say $\Lambda = \{\vee, \wedge, \neg, \rightarrow\}$. The terms of this language are traditionally called formulas of Λ, or, in the algebraic setting, terms of Λ. Both words will be used here.

If K consists of one binary predicate symbol, then the elements of Fm_{K} are identified with pairs of terms. If this unique predicate symbol is the equality symbol $\approx$, then the elements of Fm_{K} are called *equations.*

If K has one predicate symbol R which is, say, k-ary for some integer k, Fm_{K} can be identified with Te^k, i.e., sequences $\langle t_1, \ldots, t_k\rangle$. Such sequences are called *k-formulas* ([4]). Thus terms are *1-formulas* and equations can be viewed as *2-formulas.*

If K is finite, the elements of Fm_{K} are called *$\vec{k}$-formulas*, where $\vec{k} = \langle k_R : R \in \mathrm{K}\rangle$ is the finite sequence of arities of the symbols of R, i.e., $k_R = \rho(R)$. Hence a typical $\vec{k}$-formula is of the form $R(t_1, \ldots, t_{\rho(R)})$, where $\langle t_1, \ldots, t_{\rho(R)}\rangle$ is an k_R-formula; it may be identified with a universally quantified atomic formula of some first-order language with finitely many predicate symbols.

Finally, in the special case that for every $n = 1, 2, \ldots$ there is exactly one n-ary predicate symbol, say R_n in K, the coproduct $\mathrm{Fm}_{\mathrm{K}} = \coprod_{R_n \in \mathrm{K}} \mathrm{Te}^{\rho(R_n)}$ becomes the disjointed union $\coprod_{n>0} \mathrm{Te}^n$, which we identify with $\bigcup_{n>0} \mathrm{Te}^n$. The elements of this union are called ω-formulas. Thus a typical ω-formula is a non-empty sequence $\langle t_1, \ldots, t_n\rangle$ of terms. If this sequence is written in the form $t_1, \ldots, t_{n-1} \rightarrow t_n$ then we call it a (Λ-)*sequent.*

Definition 2.2 *Let X be a set, $\leq$ a partial ordering on X and let $f : X \to X$. Then f is*

- **idempotent** *if for all $a \in X$, $f(f(a)) = f(a)$*
- **monotone** *if for all $a, b \in X$,*

$$a \leq b \Rightarrow f(a) \leq f(b), \text{ and}$$

- **increasing with respect to** $\leq$ *if for all $a \in X$, $a \leq f(a)$.*

Definition 2.3 *Let A be a set. A function $C : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ that is idempotent, monotone and increasing with respect to inclusion is called a* **closure operator**.

Topological closure or the function assigning to a subset X of some algebra $\mathbf{A}$ the universe of the subalgebra of $\mathbf{A}$ generated by X are closure operators. Definition 2.5, page 38, provides another example.

Definition 2.4 *A closure operator C on A is*

1. **algebraic** *or* **finitary** *if for all $X \subseteq A$,*

$$C(X) \subseteq \bigcup_{Y \subseteq_{\mathcal{P}_\omega} X} \mathrm{Cn}(Y). \tag{2.1}$$

2. *A closure operator C on Fm_{K} is called* **structural** *if for every substitution σ and every $X \subseteq \mathrm{Fm}_{\mathrm{K}}$,*

$$\sigma(C(X)) \subseteq C(\sigma(X)). \tag{2.2}$$

Definition 2.5 *Let $\langle \Lambda, K \rangle$ be a first-order language. An algebraic and structural closure operator Cn on Fm_{K} is called a* **(Λ, K)-consequence operator**.

Usually, when it does not lead to a misunderstanding, we will drop one or both prefixes Λ and K. Let us mention that other authors often mean by "consequence operator" just the closure operator on the set of formulas of the appropriate deductive system and then consider "finitary and structural consequence operators".

Thus in addition to (2.1) and (2.2) a consequence operator has the following properties, for all $X, Y \subseteq \mathrm{Fm_K}$:

$$X \subseteq Y \Leftrightarrow \mathrm{Cn}(X) \subseteq \mathrm{Cn}(Y), \tag{2.3}$$

$$X \subseteq \mathrm{Cn}(X) \quad \text{and} \tag{2.4}$$

$$\mathrm{CnCn}(X) \subseteq \mathrm{Cn}(X). \tag{2.5}$$

Definition 2.6 *A triple $\langle \Lambda, K, \mathrm{Cn}_{\mathcal{S}} \rangle$, where $\langle \Lambda, K \rangle$ is a first-order language and* $\mathrm{Cn}_{\mathcal{S}}$ *is a (Λ, K)-closure operator is called a (Λ, K)-***deductive system** *or simply a* **K-deductive system** *when Λ is known from the context or is left unspecified. The subscript $\mathcal{S}$ on* $\mathrm{Cn}_{\mathcal{S}}$ *will be omitted when $\mathcal{S}$ is clear from the context.*

For a given K-deductive system $\mathcal{S}$ as above, the consequence operator determines the *consequence relation,* i.e., the relation $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\mathrm{Fm_K}) \times \mathrm{Fm_K}$ defined by $\langle X, \varphi \rangle \in \vdash_{\mathcal{S}}$ iff $\varphi \in \mathrm{Cn}(X)$. We write $X \vdash_{\mathcal{S}} \varphi$ for $\langle X, \varphi \rangle \in \vdash_{\mathcal{S}}$. Thus the conditions (2.3)-(2.5) translate into the following conditions in terms of $\vdash_{\mathcal{S}}$:

$$X \vdash_{\mathcal{S}} \varphi \text{ implies } X, \psi \vdash_{\mathcal{S}} \varphi \tag{2.6}$$

$$X \vdash_{\mathcal{S}} \varphi \text{ for every } \varphi \in X \tag{2.7}$$

$$\text{If } X \vdash_{\mathcal{S}} \varphi \text{ and } Y \vdash_{\mathcal{S}} \psi, \text{ for all } \psi \in X, \text{ then } Y \vdash_{\mathcal{S}} \varphi. \tag{2.8}$$

and the conditions (2.1)–(2.2)

$$X \vdash_{\mathcal{S}} \varphi \text{ implies } Y \vdash_{\mathcal{S}} \varphi, \text{ for some finite } Y \subseteq X \tag{2.9}$$

$$X \vdash_{\mathcal{S}} \varphi \text{ implies } \sigma X \vdash_{\mathcal{S}} \sigma\varphi \quad \text{for every substitution } \sigma. \tag{2.10}$$

The consequence operator $\mathrm{Cn}_{\mathcal{S}}$ and the consequence relation $\vdash_{\mathcal{S}}$ are interderivable and can be used interchangeably. Thus $\mathcal{S}$ is often expressed in the form $\langle \Lambda, K, \vdash_{\mathcal{S}} \rangle$. If $\mathcal{S}$ is defined by Cn we write $\vdash_{\mathrm{Cn}}$ for $\mathcal{S}$.

Definition 2.7 *Let $\langle \Lambda, K \rangle$ be a first-order language.*

(i) *A pair $r = \langle X, \varphi \rangle \in \mathcal{P}(\mathrm{Fm_K}) \times \mathrm{Fm_K}$, with X finite, will be called a (Λ, K)-***rule***. The elements of the set X are called the* **premisses** *of r and φ its* **conclusion***. A rule r can also be written in the form $\dfrac{X}{\varphi}$.*

(ii) *A (Λ, K)-***sequent** *is an expression of the form $\varphi_1, \ldots, \varphi_n \rightarrow \varphi$ where $\varphi_1, \ldots, \varphi_n$ and φ are K-formulas.*

(iii) *A (Λ, K)-***quasi-formula** *is an expression of the form $\varphi_1 \wedge \ldots \wedge \varphi_n \rightarrow \varphi$, where $n \in \mathbf{N}$ and $\varphi_1, \ldots, \varphi_n, \varphi$ are K-formulas.*

In any of the expressions (ii) – (iii), *the formulas φ_i are called* **premisses** *and the formula φ is called the* **conclusion***. If Λ is known from the context or is unspecified we simply say K-***rule**, *K-***sequent**, *K-***quasi-formula**.

The three terms defined above have been used in the literature in different contexts: the rules are traditionally associated with deductive system, the sequents with Gentzen systems and the quasi-formulas with the Horn logic. A sequent differs from a rule only in the fact that the premisses are ordered and the difference between a sequent and quasi-formula lies only in the presence or absence of the symbol $\wedge$ between the premisses. In many (although not all) situations these differences are inessential and the three notions can be identified in these cases. One of our goals is to study all the three areas in the common framework and the above definition is a good illustration how this can be done.

Definition 2.8 *A rule of the form $\langle \emptyset, \varphi \rangle$ is called* **axiomatic** *or simply an* **axiom**. *We will identify an axiom $\langle \emptyset, \varphi \rangle$ with its conclusion φ.*

Definition 2.9 *A (Λ, K)-rule $r = \langle X, \varphi \rangle$ is called an* **inference rule** *or a* **derived rule** *of a (Λ, K)-deductive system $\mathcal{S}$, if $X \vdash_{\mathcal{S}} \varphi$. The set of all inference rules of $\mathcal{S}$ will be denoted by $\Gamma_{\mathcal{S}}$ or by Γ_{Cn} if $\mathrm{Cn} = \mathrm{Cn}_{\mathcal{S}}$.*

Every set Γ of rules determines a consequence operator Cn_{Γ}, and therefore a K-deductive system $\mathcal{S}_{\Gamma} = \langle \Lambda, K, \mathrm{Cn}_{\Gamma} \rangle$, in such a way that for any deductive system $\mathcal{S}$, $\mathrm{Cn}_{\Gamma_{\mathcal{S}}} = \mathrm{Cn}_{\mathcal{S}}$. To see this let Γ be a set of rules and let the relation $\vdash_{\Gamma}$ be recursively defined as follows. For a set X of K-formulas and a K-formula φ let $X \vdash_{\Gamma} \varphi$ if and only if

1. $\varphi \in X$ or
2. there is a set $Z \subseteq \mathrm{Fm}_{\mathrm{K}}$, a substitution σ and a K-formula ψ such that
   - $\sigma(\psi) = \varphi$,
   - $\langle Z, \psi \rangle \in \Gamma$ and
   - for every $\xi \in Z$, $X \vdash_{\Gamma} \sigma(\xi)$.

It is easy to show that this relation is the smallest relation $\vdash \subseteq \mathcal{P}(\mathrm{Fm_K}) \times \mathrm{Fm_K}$ satisfying (2.6), (2.7), (2.8) and (2.10) and such that $\Gamma \subseteq \vdash$. Also, since the set of premisses of every rule is finite by definition, it is not difficult to see that $\vdash_\Gamma$ must also satisfy (2.9). Thus it is a consequence relation.

Definition 2.10 *Let Γ be a set of K-rules. The smallest consequence relation such that $\Gamma \subseteq \vdash$ is called the* **consequence relation determined by Γ**. *The associated consequence operator is called the* **consequence operator determined by Γ** *and is denoted by* Cn_Γ.

Notice that for a given consequence operator Cn, $\vdash_{\mathrm{Cn}}$ satisfies (2.6), (2.7), (2.8) and (2.10) and therefore the smallest consequence relation containing Γ_{Cn} is equal to Γ_{Cn}, i.e., $\vdash_{\Gamma_{\mathrm{Cn}}} = \Gamma_{\mathrm{Cn}} = \vdash_{\mathcal{S}}$, where $\mathcal{S} = \langle \Lambda, K, \mathrm{Cn} \rangle$. Therefore $\mathrm{Cn}_{\Gamma_{\mathrm{Cn}}} = \mathrm{Cn}$.

Definition 2.11 *A rule $\langle X, \varphi \rangle \in \vdash_\Gamma$ is called a* **secondary** *or* **derived rule** *of* Cn_Γ.

When $X \vdash_\Gamma \varphi$, we also say that φ is derivable from X by means of rules in Γ.

Definition 2.12 *A* **proof** *or a* **derivation** *of φ from the set of* **premisses** X **by means of** *the rules of Γ is a sequence $\tau_1, \ldots, \tau_n = \varphi$ of K-terms such that for every $i = 1, \ldots n-1$, either $\tau_i \in X$ or there is a substitution σ and a rule $\langle Y, \psi \rangle \in \Gamma$, such that $\tau_i = \sigma\psi$ and, for every $\gamma \in Y$, there is a $j = 1, \ldots, i-1$ such that $\sigma\gamma = \tau_j$.*

Note that τ_1 must either be an element of X or a substitution instance of the conclusion of an axiomatic rule from Γ.

Note also, that $X \vdash_\Gamma \varphi$ if there is a proof of φ from some finite subset Y of X by means of Γ.

Given a deductive system $\mathcal{S}$ and a set of rules Γ we can consider a consequence operator $\mathrm{Cn}_{\mathcal{S},\Gamma}$ determined by $\Gamma \cup \Gamma_{\mathcal{S}}$. The deductive system $\mathcal{S}_\Gamma = \langle \Lambda, \mathrm{Cn}_{\mathcal{S},\Gamma}\rangle$ is called an *extension* of $\mathcal{S}$ by Γ. An extension is called *axiomatic* if all the rules in Γ are axiomatic.

Definition 2.13 (compare with, e.g., [8, 63]) *Let $\mathcal{S} = \langle \Lambda, \mathrm{Cn}_{\mathcal{S}}\rangle$ and $\mathcal{R} = \langle \Lambda, \mathrm{Cn}_{\mathcal{R}}\rangle$ be two K-deductive systems.*

1. *If for some set Γ of rules $\mathrm{Cn}_{\mathcal{R}} = \mathrm{Cn}_{\mathcal{S},\Gamma}$, then Γ is called a* **basis of $\mathcal{R}$ over $\mathcal{S}$**, *or* **relative to $\mathcal{S}$**, *and we say that $\mathcal{R}$ is* **based over** *$\mathcal{S}$ by Γ . If $\mathrm{Cn} = \mathrm{Cn}_{\mathcal{S},\Gamma}$ then we say that Γ is a* basis *of* Cn **over** *or* **relative to the K-deductive system $\mathcal{R}$**.

2. *If $\mathrm{Cn}_{\mathcal{R}} = \mathrm{Cn}_\Gamma$ then Γ is called a* **basis of $\mathcal{R}$** *and $\mathcal{R}$ is* **based** *by Γ. A* **basis** *of a consequence operator $\mathrm{Cn}_{\mathcal{S}}$ is the basis of the system $\mathcal{S}$.*

3. *A K-deductive system $\mathcal{R}$ and the consequence operator $\mathrm{Cn}_{\mathcal{R}}$ are called* **finitely based** *(possibly* **over $\mathcal{S}$***) if there is a finite set Γ of rules such that Γ is a basis of $\mathcal{R}$ (over $\mathcal{S}$).*

Observe, that a set Γ of rules is a basis of $\mathcal{R}$ iff it is a basis of $\mathcal{R}$ relative to the system $\mathcal{S}$ based by the empty set of rules.

Definition 2.14 *Given a K-deductive system $\mathcal{S}$, an $\mathcal{S}$-***theory** *is a set of K-terms closed under $\mathrm{Cn}_{\mathcal{S}}$. The set of all $\mathcal{S}$-theories is denoted by $\mathrm{Th}_{\mathcal{S}}$ or Th_Γ, where Γ is a set of rules such that $\mathrm{Cn}_{\mathcal{S}} = \mathrm{Cn}_\Gamma$.*

A set X such that $T = \mathrm{Cn}_{\mathcal{S}}(X)$ is called the set of *generators of T* and we say that T is *generated* by X. If X is finite, we say that T is *finitely generated.* For a given deductive system $\mathcal{S}$, the set of all $\mathcal{S}$-theories ordered by inclusion forms an algebraic lattice, the compact elements of which are the finitely generated theories. The smallest element of the lattice $\mathrm{Th}_{\mathcal{S}}$ is the theory $\mathrm{Cn}_{\mathcal{S}}(\emptyset)$ generated by the empty set. The elements of $\mathrm{Cn}_{\mathcal{S}}(\emptyset)$ are called *theorems of* $\mathcal{S}$.

One of the most often asked questions about a logical theory or a deductive system is the problem of finitely axiomatizing or finding a finite basis. As Wojtylak points out in [61] the word "axiomatize" had different meaning for different authors. Part (iii) of the definition below was proposed essentially by him ([61]).

Definition 2.15 *Let $\mathcal{S} = \langle \Lambda, \mathrm{Cn}_{\mathcal{S}} \rangle$ be a K-deductive system.*

(i) *Let T be an $\mathcal{S}$-theory. A* **basis of** T **over** $\mathcal{S}$ *is a set E of K-terms such that $T = \mathrm{Cn}_{\mathcal{S}}(E)$. If E is a basis of T over $\mathcal{S}$ then we say that T is* **based by** E **over** $\mathcal{S}$. *If E is finite, we say that T is* **finitely based by** E **over** $\mathcal{S}$.

(ii) *Let T be an $\mathcal{S}$-theory. An* **axiomatization of** T **over** $\mathcal{S}$ *is a set Γ of (Λ, K)-rules such that $T = \mathrm{Cn}_{\mathcal{S},\Gamma}(\emptyset)$. In this case we also say that T is* **axiomatized by** Γ **over** $\mathcal{S}$ *and if Γ can is finite, that T is* **finitely axiomatized by** Γ **over** $\mathcal{S}$. *T is* **finitely axiomatizable over** $\mathcal{S}$ *if it is finitely axiomatized by some finite Γ over $\mathcal{S}$.*

(iii) *Let T be an $\mathcal{S}$-theory. An* **axiomatization of** T *is the axiomatization of T relative to the deductive system determined by the empty set of rules. In this case we also say that T is* **axiomatized by** Γ *and if Γ can be chosen finite, that T is* **finitely axiomatized by** Γ. *T is* **finitely axiomatizable** *if it is axiomatized by some finite Γ.*

Notice that a "basis of a deductive system $\mathcal{S}$" (Definition 2.13) is different from a "basis of the theorems of $\mathcal{S}$ over $\mathcal{S}$". The first is a set of *rules*, the second is a set of *K-formulas.* The first allows to derive all *rules* of $\mathcal{S}$, while the second allows to derive all *theorems* of $\mathcal{S}$, by means of the rules of $\mathcal{S}$.

Every basis Γ for the consequence operator Cn is also an axiomatization for the set $\mathrm{Cn}_{\mathcal{S}}(\emptyset)$ of all theorems of $\mathcal{S}$. Thus if $\mathrm{Cn}_{\mathcal{S}}$ is finitely based, then $\mathrm{Cn}_{\mathcal{S}}(\emptyset)$ is finitely axiomatized. Also, if T is an $\mathcal{S}$-theory and $\mathcal{S}$ is finitely based, then if T is finitely based over $\mathcal{S}$, then also T is finitely axiomatized. Clearly, every consequence operator has some basis and every $\mathcal{S}$-theory has some basis over $\mathcal{S}$ and some axiomatization. But these bases and axiomatization don't need to be finite.

2.3 Examples

A K-deductive system $\mathcal{S}$ corresponds to a universal Horn theory as follows. With each K-formula $R(\vec{t}(\mathbf{x}))$ of $\mathcal{S}$, where $\mathbf{x}$ is a sequence of variables, we associate the universal sentence $\forall_{\mathbf{x}} R(\vec{t}(\mathbf{x}))$; and with every rule $\langle X, \varphi\rangle$, where $X = \{R_i(\vec{t^i}(\mathbf{x})) : i = 1, \dots, n\}$ and $\varphi = R(\vec{t}(\mathbf{x}))$ for some sequences of terms $\vec{t}, \vec{t^i}$, $i = 1, \dots, n$, we associate the universal Horn formula $\forall_{\mathbf{x}} \bigwedge_{i\leq n} R_i(\vec{t^i}(\mathbf{x})) \to R(\vec{t}(\mathbf{x}))$. Let $\mathcal{F}$ be the universal Horn theory axiomatized by the sentences associated with all the logical axioms and rules of $\mathcal{S}$. Then a rule $r = \langle Y, \psi\rangle$ is a derived rule of $\mathcal{S}$ iff the universal Horn formula associated with r is a theorem of $\mathcal{F}$.

Definition 2.16 *Let K be a finite set. Let $\vec{k} := \langle k_{\rho(R)} : R \in K\}$. Then a K-deductive system is called a* **$\vec{k}$-deductive system.**

A $\vec{k}$-deductive system $\mathcal{S}$ corresponds to a universal Horn theory with finitely many predicates. The term "$\vec{k}$-deductive" has been chosen for its connection with the systems considered in the next definition.

Definition 2.17 ([4]) *Let k be a natural number and let K consist of only one, k-ary predicate symbol. Then a K-deductive system is called* k-**deductive.**

The concept of a k-deductive system was motivated on one hand by 1-deductive systems (Definition 2.18) and by equational logic, which is 2-deductive, on the other.

Definition 2.18 *A* **1-deductive system** *is K-deductive system, where K consists of a single, unary predicate. A* **2-deductive system** *is a K-deductive system, where K consists of a single, binary predicate.*

Until recently, only 1-deductive systems were called "deductive systems". For example the deductive systems of classical, intuitionistic, modal, relevance, BCK, multi-valued and other non-classical logics, are 1-deductive. We will be particularly interested in propositional fragments of these logic and the symbols CPC, IPC will be used to denote the 1-deductive systems of classical and intuitionistic propositional logics, respectively. These systems are determined by the sets of these rules r that the set of all classical, and respectively intuitionistic, tautologies is closed under r. The symbol $BCK+\wedge$ will be used to denote the so-called BCK-logic with conjunction. It has two algebraic operations: conjunction $\wedge$ and implication, $\rightarrow$ and is axiomatized by the modus ponens rule and the following axioms.

B $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z))$

C $(x \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow (x \rightarrow z))$

K $x \to (y \to x)$

$\wedge_1$ $(x \to (y \to z)) \to ((x \wedge y) \to z)$

$\wedge_2$ $((x \wedge y) \to z) \to (x \to (y \to z))$.

Its algebraic semantics is formed by the class of all so-called BCK-algebras with the operation (S), see for example [38] for definition, or, equivalently, by the class of all algebras dual to the class of all ordered groupoids with residuation, [64]. If we reverse the order such a groupoid duallyx, then the conjunction corresponds to the groupoid operation and implication to the residuation.

We adopt the convention that the formulas of a 1-deductive system (1-formulas) are written as terms, i.e., instead of $D(t(\mathbf{x}))$ we write $t(\mathbf{x})$. The formulas of a 2-deductive systems (2-formulas) are often, but not always, written as pairs of terms. If K has one binary symbol $\approx$, then we write $t \approx s$ for the K-formula $\langle t, s\rangle$ and call it an *equation.* If the only symbol of K is a binary symbol $\leq$, then the K-formulas are written as $t \leq s$ and called *inequalities.*

In Chapter 5 we consider some 2-deductive systems. The most important of them is the following system of equational logic.

Definition 2.19 *Let Λ be an algebraic language. Let K have one binary predicate $\approx$. The* **(restricted) Birkhoff system over** Λ *is the* 2*-deductive system* $\mathcal{B}$ *axiomatized as follows. It has one axiom*

(I) $x \approx x$,

and the following rules of inference:

(S) $\dfrac{x \approx y}{y \approx x}$.

(T) $\dfrac{x \approx y, y \approx z}{x \approx z}$,

(R) $\dfrac{x_1 \approx y_1, \ldots, x_l \approx y_l}{\lambda(x_1, \ldots, x_l) \approx \lambda(y_1, \ldots, y_l)}$, *for each l-ary operation symbol $\lambda \in \Lambda$.*

The above system is called the "restricted" Birkhoff system, because it differs from the system introduced by Birkhoff in that that it does not have the following *substitution rule.*

$$\frac{t \approx s}{\sigma(t) \approx \sigma(s)},$$

where σ ranges over arbitrary substitutions. In Part I we will call the restricted Birkhoff system just Birkhoff system. In Part III, however, we will use the full Birkhoff system, with the substitution rule. It is easy to see that the above set of rules is finite if and only if the language Λ has finitely many operation symbols. This system, used in every algebraic reasoning, was formalized in [2] and since then this formalization played an important role in universal algebra, for example in solving problems of finite basis.

Using our convention that the 2-formulas can be written as pairs of terms, Birkhoff's system $\boldsymbol{B}$ can be defined as a 2-deductive system axiomatized by the following axiom and rules.

(I) $\langle x, x \rangle$,

(S) $\dfrac{\langle x, y \rangle}{\langle y, x \rangle}$,

(T) $\dfrac{\langle x, y \rangle, \langle y, z \rangle}{\langle x, z \rangle}$,

(R) $\dfrac{\langle x_1, y_1 \rangle, \ldots, \langle x_l, y_l \rangle}{\langle \lambda(x_1, \ldots, x_l), \lambda(y_1, \ldots, y_l) \rangle}$, for each l-ary operation symbol λ.

The axiom (I) is called *identity* and the rules (S), (T), and (R) are respectively called *symmetry*, *transitivity*, and *replacement.*

Another important type of systems are Gentzen systems. For some deductive systems a Gentzen system can be viewed as a so-called *second-order deductive system.*

Definition 2.20 *Suppose that for every natural number $n \geq 1$ there is exactly one n-ary predicate, R_n, in K. Then a K-deductive system is called a* **Gentzen system**.

As mentioned in section 2.2, we identify a K-formula $R_n(t_1, \ldots, t_n)$ with the sequent

$$t_1, \ldots, t_{n-1} \rightarrow t_n.$$

If $\mathcal{S}$ is a 1-deductive system then, as we mentioned above, its rules can be identified with sequents, i.e., formulas of some Gentzen system $\mathcal{G}$.

2.4 Second-order deductive systems

Let us observe that the operator associating with each set of rules Γ the consequence relation $\vdash_\Gamma$ has the following properties.

1. $\Gamma \subseteq \Delta \Rightarrow \vdash_\Gamma \subseteq \vdash_\Delta$.

2. $\vdash_{\vdash_\Gamma} = \vdash_\Gamma$.

3. $\Gamma \subseteq \vdash_\Gamma$.

4. If $\langle X, \varphi \rangle \in \vdash_\Gamma$, then there is a finite subset $\Delta \subseteq \Gamma$ such that $\langle X, \varphi \rangle \in \vdash_\Delta$.

5. If $\langle X, \varphi \rangle \in \vdash_\Gamma$, then for every substitution σ, $\langle \sigma(X), \sigma(\varphi) \rangle \in \vdash_{\sigma(\Gamma)}$.

Thus the operator C such that $C(\Gamma) = \vdash_\Gamma$ is itself idempotent, monotone, increasing, finitary and structural and hence is a consequence operator, except that it is defined on sets of K-sequents rather than sets of K-formulas. We call the operator C *a second order consequence operator.* By contrast, a K-deductive system in the sense of our original definition (Definition 2.1 is called a *first-order deductive system.*

Definition 2.21 *For a fixed* Λ *and* K *let* $\Gamma_{(\Lambda,K)}$ *denote the set of all* (Λ, K)*-rules. An operator* $\mathrm{Cn} : \mathcal{P}(\Gamma_{(\Lambda,K)}) \to \mathcal{P}(\Gamma_{(\Lambda,K)})$ *that is monotone, increasing, idempotent, finitary and structural, i.e., satisfies the conditions* (2.1– 2.5) *is called a* **second-order consequence operator.**

Consequence operators of orders higher than two can be defined in a natural way. But we will not do it here.

Note that the ω-deductive systems are the second-order deductive system for K with only one unary predicate.

A basic second order non-structural deductive system is the system $\mathcal{S}_0$ determined by the requirements (2.6)-(2.8) for $\vdash$. (page 39). Its rules are:

$$\frac{X\vdash\varphi}{X,\psi\vdash\varphi}, \qquad \frac{\emptyset}{\varphi\vdash\varphi}, \qquad \frac{X,\varphi\vdash\psi \text{ and } Y\vdash\varphi}{X,Y\vdash\psi}.$$

Here X and Y represent finite sets of K-formulas and we use the identification of a rule $X\vdash\varphi$ with the sequent $\varphi_1,\ldots,\varphi_n \to \varphi$. Thus the first of the above rules is really an infinite set of rules: one rule for each n. Also, the third rule represents an infinite set of rules. Another important second-order deductive system, which we denote by $\mathcal{S}_f$, is the one obtained from $\mathcal{S}_0$ by adding the following rules, expressing the structurality condition (2.10):

$$\frac{X\vdash\varphi}{\sigma X\vdash\sigma\varphi}$$

for every substitution σ.

Of course, every first-order non-structural K-deductive system $\mathcal{S}$ is some (second-order) axiomatic extension of $\mathcal{S}_0$, and every first-order structural K-deductive system is an axiomatic extension of $\mathcal{S}_1$, where every second-order axiom is a first-order rule.

Given a K-deductive system $\mathcal{S}$, we will be particularly interested in the second-order axiomatic extension $\mathcal{S}_2$ of $\mathcal{S}_1$ on $\mathcal{P}(\mathrm{Fm_K}) \times \mathrm{Fm_K}$ that satisfies all the inference rules of $\mathcal{S}$. Note that the derived rules of this second-order system are exactly those pairs $\langle X, s\rangle$ with $X \subseteq \Gamma_{(\Lambda,K)}, s \in \Gamma_{(\Lambda,K)}x$ such that whenever $\sigma X \subseteq\ \vdash_{\mathcal{S}}$ then also $\sigma s \in \vdash_{\mathcal{S}}$, for every substitution σ.

Similarly as for first-order deductive systems we can define the axiomatization and relative axiomatization of a second-order deductive system $\mathcal{S}$. A second-order axiomatization of a system $\mathcal{S}_2$ above relative to $\mathcal{S}_1$ deserves a special mention. It is a set Γ of first-order rules such that $\mathrm{Cn}_\Gamma = \mathrm{Cn}_{\mathcal{S}_1}$. Thus Γ is an axiomatization of $\mathcal{S}_2$ iff it is a *basis of* $\mathcal{S}$. By analogy with the first-order case, a theory of a second-order deductive system $\mathcal{S}_2$ is a set of first-order rules closed under all second-order rules of $\mathcal{S}_2$. Let us also remark that every second-order formula is a Horn formula. Thus second-order theories coincide with the first-order systems and coincide with the universal Horn theories over the language $\langle\Lambda,\ \mathrm{K}\rangle$.

When K has just one unary predicate symbol, then K-rules can be identified with ω-formulas, so in this case C is a consequence operator in the sense of our original definition. Thus a second-order consequence operator can be associated with a Gentzen system, see Definition 2.20 in this case. In Part III, Chapter 5, we formalize the second order deductive system of equational logic as a Gentzen system, for $K = 2$.

2.5 Basic semantics of K-deductive systems

For a fixed algebraic language Λ, the semantics of a K-deductive system $\mathcal{S}$ is a class of (Λ, K)-structures, called also ($\mathcal{S}$-)matrices, that satisfy all the rules of $\mathcal{S}$. In this section we define matrices (subsection 2.5.2 and discuss the consequence operator determined by a matrix (subsection 2.5.3). If $\mathcal{S}$ is an extension of the Birkhoff's system $\mathcal{B}$, then the $\mathcal{S}$-matrices are pairs consisting of an algebra $\mathbf{A}$ and a congruence relation Θ on $\mathbf{A}$.

2.5.1 K-elements and K-subsets

Definition 2.22 *A* **K-element** *of a set A is a pair $\langle R, \vec{a}\rangle$, where $R \in K$ and $\vec{a} \in A^{\rho(R)}$. The set of all K-elements of a set A is denoted by $E_K(A)$, notice that $E_K(A) = \coprod_{R\in K} A^{\rho(R)}$. A K-element $\langle R, \vec{a}\rangle$ of A is also written in the form $R\vec{a}$. If $X \subseteq E_K(A)$, then X is called a K-subset of A.*

Let us stress that the expression $R\vec{a}$ is *not* used here as an assertion. On the other hand the expression $R\vec{a} \in E_K(A)$ is an assertion, equivalent to "$\vec{a}$ is a sequence of length $\rho(R)$ of elements of A". K-elements of $\mathbf{A}$ will be denoted by Greek letters α, β. Note that in case that $\mathbf{A} = \mathbf{Te}$, the K-elements coincide with K-formulas introduced earlier. Recall that K-formulas are denoted by the Greek letters φ, ψ.

Also observe, that 1-subsets are just subsets and 2- subsets are binary relations. Every K-subset X is of the form $X = \coprod_{R\in \mathrm{K}} X_R$, where each X_R is a subset of $A^{\rho(R)}$ in the usual sense and is called the *R-component of X*. We write $R(\vec{a}) \in X$ for $\langle R, \vec{a}\rangle \in X$. Notice that $R(\vec{a}) \in X$ iff $\vec{a} \in X_R$. Let X, Y be two K-subsets of A with components $X_R, Y_R, R \in \mathcal{N}$, respectively. We say that $X \subseteq Y$ if, for every $R \in$ K,

$X_R \subseteq Y_R$. The intersection $X \cap Y$ and the union $X \cup Y$ are defined coordinatewise:

$$X \cap Y = \coprod_{R \in \mathrm{K}} X_R \cap Y_R, X \cup Y = \coprod_{R \in \mathrm{K}} X_R \cup Y_R.$$

2.5.2 Filters and matrices

Notice that Fm_{K} is a K-subset of Te. Also every $\mathcal{S}$-theory T is a K-subset of Te. In fact, $\mathcal{S}$-theories are exactly those K-subsets of Te that are *closed* under $\mathrm{Cn}_{\mathcal{S}}$, see the next definition. Let K and Λ be fixed and let $\mathfrak{A} = \langle \mathbf{A}, R^{\mathfrak{A}} : R \in \mathrm{K}\rangle$ be a model of the first-order language $\langle \Lambda, K\rangle$. Notice that $\coprod_{R \in \mathrm{K}} R^{\mathfrak{A}}$ is a K-subset of A.

Definition 2.23 (i) *A K-subset F of a Λ-algebra* $\mathbf{A}$ *is* **closed under a rule** $r = \langle X, \alpha\rangle$ *if for every substitution σ, if $\sigma(X) \subseteq F$ then $\sigma\alpha \in F$. It is* **closed under a set of rules** Γ, *if for every rule $r \in \Gamma$, it is closed under r. It is* **closed under a consequence operator** Cn, *if it is closed under every rule of* Cn.

(ii) *Let $\mathcal{S} = \langle \Lambda, \mathrm{Cn}\rangle$ be a K-deductive system and* $\mathbf{A}$ *a Λ-algebra. A K-subset F of A, which is closed under* Cn *is called an* $\mathcal{S}$**-filter** *of* $\mathbf{A}$. *If* $\mathrm{Cn} = \mathrm{Cn}_{\Gamma}$ *for some set of rules* Γ, *we call F also an* Γ**-filter on** $\mathbf{A}$. *The set of all $\mathcal{S}$- (respectively Γ-) filters on* $\mathbf{A}$ *is denoted by* $Fi_{\mathcal{S}}(\mathbf{A})$ *(by* $Fi_{\Gamma}(\mathbf{A})$*, respectively).*

(iii) *A* (Λ, K)**-matrix** *is a pair* $\mathfrak{A} = \langle \mathbf{A}, F\rangle$, *where* $\mathbf{A}$ *is a Λ-algebra and F is a K-subset of A. If F is an $\mathcal{S}$-filter, for some (Λ, K)-deductive system $\mathcal{S}$, then $\mathfrak{A}$ is called an* $\mathcal{S}$**-matrix**. *If* $\mathrm{Cn}_{\mathcal{S}} = \mathrm{Cn}_{\Gamma}$, *for some set Γ of rules, then an $\mathcal{S}$-matrix is also called a* Γ**-matrix**.

(iv) *Let $\mathfrak{A} = \langle \mathbf{A}, D_{\mathfrak{A}}\rangle$ be an $\mathcal{S}$-matrix. An $\mathcal{S}$-filter F on* $\mathbf{A}$ *such that $D_{\mathfrak{A}} \subseteq F$ is called an* $\mathcal{S}$**-filter on the matrix** $\mathfrak{A}$. *A* Γ**-filter on** $\mathfrak{A}$ *is a Γ-filter on* $\mathbf{A}$ *such* *that $D_{\mathfrak{A}} \subseteq F$. The sets of $\mathcal{S}$-filters on $\mathbf{A}$ and on $\mathfrak{A}$ are denoted $Fi_{\mathcal{S}}(\mathbf{A}), Fi_{\mathcal{S}}(\mathfrak{A})$, respectively. Similarly, $Fi_{\Gamma}(\mathbf{A})$ and $Fi_{\Gamma}(\mathfrak{A})$ denote the sets of Γ-filters on algebra $\mathbf{A}$ and on the matrix $\mathfrak{A}$, respectively.*

Notice that $\langle \mathbf{A}, X \rangle$ is a (Λ, K)-matrix iff $\langle \mathbf{A}, X_R : R \in \mathrm{K} \rangle$ is a model of the first-order language $\langle \Lambda, K \rangle$. Conversely, if $\langle \mathbf{A}, R^{\mathfrak{A}} : R \in \mathrm{K} \rangle$ is a model of $\langle \Lambda, K \rangle$ then $\langle \mathbf{A}, \coprod_{R \in \mathrm{K}} R^{\mathfrak{A}} \rangle$ is a (Λ, K)-matrix. Hence the (Λ, K)-matrices can be identified with the models of $\langle \Lambda, K \rangle$. Similarly, if $\mathcal{S}$ a K-deductive system and $\mathcal{F}$ is the universal Horn theory associated with $\mathcal{S}$ (see page 45), then an $\mathcal{S}$-matrix is a model of $\mathcal{F}$ in the usual terminology of the first-order logic.

The filter F of the matrix $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ is most often denoted by $D_{\mathfrak{A}}$. Its elements are called *designated elements of the matrix* $\mathfrak{A}$. If $D_{\mathfrak{A}} = \coprod_{R \in \mathrm{K}} A^{\rho(R)}$, i.e., $D_{\mathfrak{A}} = E_K(A)$ or if $D_{\mathfrak{A}}$ is empty, the matrix $\langle \mathbf{A}, D_{\mathfrak{A}} \rangle$ is called *trivial.* We will denote matrices by capital gothic letters and their underlying algebras by the corresponding boldface capital letters. For a K-deductive system $\mathcal{S}$, an algebra $\mathbf{A}$ (matrix $\mathfrak{A}$, resp.) and a K-subset X of A, an *$\mathcal{S}$-filter* on $\mathbf{A}$ (on $\mathfrak{A}$, resp.) *generated by* X is the intersection of all $\mathcal{S}$-filters on $\mathbf{A}$ ($\mathfrak{A}$, resp.) that contain X. This intersection is always an $\mathcal{S}$-filter.

For two $\mathcal{S}$-filters F, G on $\mathbf{A}$, $F \vee G$ is the $\mathcal{S}$-filter on $\mathbf{A}$ generated by $F \cup G$. The sets $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A}), \mathrm{Fi}_{\Gamma}(\mathbf{A}), \mathrm{Fi}_{\mathcal{S}}(\mathfrak{A}), \mathrm{Fi}_{\Gamma}(\mathfrak{A})$ together with the operations $\cap, \vee$ are algebraic lattices. Let $h : A \to B$ be a function and D and E some K-subsets of A and B, respectively. If for some $R \in \mathrm{K}$, some $\vec{a} = \langle a_1, \ldots, a_{\rho(R)} \rangle \in A$, $\langle b_1, \ldots, b_{\rho(R)} \rangle \in B$, we have $ha_i = b_i$ for $i = 1, \ldots, \rho(R)$, then we define $h(R\vec{a}) = R(h\vec{a})$. We define the image hD of D under h as

$$h(D) = \{R(h\vec{a}) : R\vec{a} \in D\}$$

and the inverse image h^{-1} of E under h as

$$h^{-1}(E) = \{R\vec{a} : R(h\vec{a}) \in E\}.$$

An image and an inverse image of a K-subset is also a K-subset.

Definition 2.24 *An algebra homomorphism* $h : \mathbf{A} \rightarrow \mathbf{B}$ *is* **a matrix homomorphism** *between the matrices* $\mathfrak{A}$ *and* $\mathfrak{B}$ *if* $h(D_{\mathfrak{A}}) \subseteq D_{\mathfrak{B}}$. *This is equivalent to* $D_{\mathfrak{A}} \subseteq h^{-1}(D_{\mathfrak{B}})$. *If* h *is surjective and* $D_{\mathfrak{A}} = h^{-1}(D_{\mathfrak{B}})$, *then* h *is called* reductive, $\mathfrak{B}$ *is called a* **reduction** *of* $\mathfrak{A}$.

For matrices $\mathfrak{M} = \langle \mathbf{M}, D\rangle$ and $\mathfrak{N} = \langle \mathbf{N}, D'\rangle$, a one-one algebra homomorphism $f : \mathbf{M} \longrightarrow \mathbf{N}$ is called a *matrix embedding* or simply an *embedding* if $f^{-1}(D') = D$. In this case we also say that $\mathfrak{M}$ is embeddable into $\mathfrak{N}$. An embedding which is onto is called a matrix *isomorphism.* Two matrices are *isomorphic* if there is an isomorphism between them. The matrix $\mathfrak{M}$ is a *submatrix* of a matrix $\mathfrak{N}$ if $M \subseteq N$ and the identity homomorphism is an embedding.

2.5.3 Tautologies and a consequence of a matrix

A homomorphism h from the term algebra into a Λ-algebra is called a *valuation.* Let $\mathfrak{A}$ be a matrix, $R \in$ K and $\vec{t} \in \mathrm{Te}^{\rho(R)}$. If for every valuation $h : \mathbf{Te} \rightarrow \mathbf{A}$, $h(R\vec{t}) \in D_{\mathfrak{A}}$, then $R\vec{t}$ is a *tautology of the matrix* $\mathfrak{A}$. The set of all tautologies of $\mathfrak{A}$ is denoted by $E(\mathfrak{A})$. It is also called the *content* of $\mathfrak{A}$.

Recall that a rule is a pair $\langle X, t\rangle$, such that X is a finite set of K-terms and t is a K-term.

Definition 2.25 *A rule* $r = \langle X, \varphi\rangle$ *is* **valid in** $\mathfrak{M}$ *if, for every valuation* $f : \mathbf{Te} \longrightarrow$

$\mathbf{M}$, *$f(X) \subseteq D$ implies $f(\varphi) \in D$. A rule r is* **admissible for** $\mathfrak{M}$ *if it is valid in the matrix* $\langle \mathbf{Te}, E(\mathfrak{M}) \rangle$.

Thus $r = \langle X, \varphi \rangle$is admissible for $\mathfrak{M}$ iff, for every substitution $\sigma : \mathbf{Te} \longrightarrow \mathbf{Te}$, whenever $\sigma(X) \subseteq E(\mathfrak{M})$, then also $\sigma(t) \in E(\mathfrak{M})$.

Every K-matrix determines a K-deductive system $\mathcal{S}_{\mathfrak{A}} = \langle \Lambda, \mathrm{Cn}_{\mathfrak{A}} \rangle$ in the following way.

Definition 2.26 *The consequence* $\mathrm{Cn}_{\mathfrak{A}}$ *is the consequence operator determined by the rules valid in* $\mathfrak{A}$.

(The consequence operator determined by a set of rules was defined on page 41.) Similarly, every class of matrices determines a set of theorems $E(\mathcal{K})$ and a consequence operator: a K-formula φ is a *theorem of* $\mathcal{K}$ if it is a theorem of every matrix $\mathfrak{A} \in \mathcal{K}$. A rule r is *valid in* $\mathcal{K}$ if it is valid in every matrix $\mathfrak{A} \in \mathcal{K}$. The consequence operation $\mathrm{Cn}_c K$. The consequence operation $\mathrm{Cn}_{\mathcal{K}}$ is the consequence operator determined by the set of the valid rules of $\mathcal{K}$. Let $\mathbf{F_M}$ be the free denumerably generated algebra in HSP($\mathbf{M}$) and $E'(\mathfrak{M})$ the set of all elements $t \in F$ such that $f(t) \in D$ for every homomorphism $f : \mathbf{F_M} \longrightarrow \mathbf{M}$. (Notice that if θ is a congruence on $\mathbf{Te}$ such that $\mathbf{F_M} \cong \mathbf{Te}/\theta$, then $E'(\mathfrak{M}) \cong E(\mathfrak{M})/\theta$.) The matrix $\mathfrak{F}_{\mathfrak{M}} = \langle \mathbf{F_M}, E'(\mathfrak{M}) \rangle$ will be called a *free* matrix over $\mathfrak{M}$. It is easy to show that a rule r is admissible for $\mathfrak{M}$ iff r is valid in the free matrix $\mathfrak{F}_{\mathfrak{M}}$. The following connection between valid and admissible rules is well-known (e.g. [45]).

Proposition 2.27 *Every rule valid in* $\mathfrak{M}$ *is admissible for* $\mathfrak{M}$.

In general, the converse of Proposition 2.27 does not hold (see, e.g., [45], page 110). The matrices in which all admissible rules are valid are called *structurally* *complete.* It is easy to see that a sufficient condition for a matrix $\mathfrak{M}$ to be structurally complete is that $\mathfrak{M}$ be embeddable in the free matrix over $\mathfrak{M}$, i.e., we have

Lemma 2.28 *If $\mathfrak{M}$ is embeddable $\mathfrak{F}_{\mathfrak{M}}$, then every rule admissible for $\mathfrak{M}$ is also valid in $\mathfrak{M}$.*

Definition 2.29 *Let S be K-deductive system, for some K and let $\mathfrak{A}$ be an S-matrix. An* **axiomatization** *of $\mathfrak{A}$ (possibly over S) is an axiomatization of $E(\mathfrak{A})$ (over S). A* **basis** *of $\mathfrak{A}$ (possibly over S) is a basis of the consequence operation $\mathrm{Cn}_{\mathfrak{A}}$ (over S). The matrix $\mathfrak{A}$ is* **finitely based** *(***finitely axiomatizable***) if there exists a finite basis (axiomatization) of $\mathfrak{A}$. It has* **finitely based theorems over** S *if there is a finite basis of theorems of $\mathfrak{A}$ over S.*

(Finite) axiomatization and basis for a class $\mathcal{K}$ of matrices (possibly over S) are defined similarly.

Let us stress that, according to the above definition (which we borrow from [60, 61, 10]), to axiomatize a matrix means to axiomatize its tautologies, possibly with rules that are only admissible for the matrix. As P. Wojtylak pointed out in [61] this notion of axiomatizability is the weakest of all notions of axiomatizability considered in the literature. Thus if a matrix cannot be finitely axiomatized in the above sense, then it cannot be axiomatized in any other sense existing in the literature. In particular, the consequence operation of such a matrix cannot be finitely based.

Notice that if Γ axiomatizes $\mathfrak{A}$, then every rule in Γ is admissible for $\mathfrak{A}$. However, these rules do not need to be valid in $\mathfrak{A}$.

Definition 2.30 *A matrix $\mathfrak{A}$ is* **axiomatized by valid rules** *if there exists an axiomatization Γ of $\mathfrak{A}$ such that all the rules in Γ are valid.*

Proposition 2.31 *If $\mathfrak{A}$ is finitely axiomatized by valid rules, then $\mathfrak{A}$ is finitely axiomatizable.*

Proposition 2.32 *A set Γ of rules axiomatizes $\mathfrak{A}$ iff all rules in Γ are admissible for $\mathfrak{A}$ and $E(\mathfrak{A}) \subseteq \mathrm{Cn}(\Gamma, \emptyset)$.*

2.5.4 Examples

We apply the notion of a matrix to the special deductive systems considered earlier.

1-matrices.

The notion of a matrix as a model of a 1-deductive system was defined by A. Tarski and J. Lukasiewicz in [24], although the idea itself can be traced back to Ch. Peirce and E. Schröder (see [58, section 31.5]). The theory of 1-matrices was developed in papers of J. Loś [23], J. Kalicki [16, 17, 18, 19], M. Wajsberg [56], S. Jaśkowski [14], A. Tarski [54] and others. The theory of logical matrices has been used in the papers of many authors, especially in Poland.

It follows from Definition 2.23 that for a given algebraic language Λ, a 1-matrix, or a matrix of a 1-deductive system, is a pair $\langle \mathbf{A}, D \rangle$ such that $\mathbf{A}$ is a Λ-algebra and D is a subset of A.

For example, if $\mathcal{S}$ is the classical deductive system, then a pair $\langle \mathbf{A}, \{1\} \rangle$, where $\mathbf{A}$ is a Boolean algebra and 1 is the largest element of $\mathbf{A}$ is a $\mathcal{S}$-matrix. Similarly, a 1-matrix $\langle \mathbf{A}, \{1\} \rangle$ is a

(i) matrix of the intuitionistic deductive system if **A** is a pseudo-Boolean algebra and 1 is its greatest element

(ii) matrix of the BCK-logic, if **A** is a BCK-algebra and 1 is its greatest element.

(iii) matrix of the implicative logic of Rasiowa [44], if **A** is an implicative algebra and 1 is the largest element of **A**.

The third example is more general then the first two. In all these examples, it is sufficient to have one designated value. The first to consider two values was J. Lukasiewicz. His matrices $\langle\{0,\frac{1}{2},1\},\rightarrow,\{1\}\rangle$ and $\langle\{0,\frac{1}{2},1\},\rightarrow,\{\frac{1}{2},1\}\rangle$ are models for the so-called Lukasiewicz 3-valued logics ([24]). The tautologies of the first of these matrices are all the terms which are theorems in certain 1-deductive system, and those of the second are the formulas which are true or possible in this system. At first, the interest of logicians was focused on the tautologies of a matrix. Thus the first of the axiomatizability/ basis notions defined above was the one of axiomatizing a matrix (i.e., its set of tautologies) relatively to a given set of rules. For example in Wajsberg's paper (reference), this set of rules consists just of the modus ponens rule. Later, the matrices were considered as models of the deductive systems, and the other notions developed. The standard notion of a deductive system used now is that of for example [59] and it coincides with our 1-deductive system.

k- and $\mathcal{B}$-matrices

For a set A, let id_A be the identity relation on A. As we already said in 0.43, a K-matrix can be identified with an (Λ, K)-structure. It follows from Definition 2.23 that for a given algebraic language Λ, a k-matrix, or a matrix of a k-deductive system, is a pair $\langle \mathbf{A}, D\rangle$ such that $\mathbf{A}$ is a Λ-algebra and D is a subset of A^k. For example, a 2-matrix is a algebra with a binary relation. Consider Birkhoff's 2-deductive system $\mathcal{B}$ introduced in Definition 2.19. It follows from the axioms $\boldsymbol{\mathcal{B}}$-matrix is a pair $\mathfrak{A} = \langle \mathbf{A}, \Theta \rangle$, where Θ is a congruence relation on $\mathbf{A}$. A valuation v satisfies an equation $t_1 \approx t_2$ if $t_1(v)\Theta t_2(v)$ and $t_1 \approx t_2$ is a tautology of $\mathfrak{A}$ iff for every valuation v, $t_1(v)\Theta t_2(v)$, i.e., if $t_1 \approx t_2$ is an identity of the quotient algebra $\mathbf{A}/\Theta$.

Thus we have the first part of the following proposition ([4])

Proposition 2.33 *Let* $\mathfrak{A} = \langle \mathbf{A}, \Theta \rangle$ *be a* $\boldsymbol{\mathcal{B}}$*-matrix. Then*

(i) $E(\mathfrak{A}) = \mathrm{Id}(\mathbf{A}/\Theta)$.

(ii) *A rule* $\langle X, \varepsilon \rangle$ *is valid in* $\mathfrak{A}$ *iff* $\bigwedge_{\varphi \in X} \varphi \to \varepsilon \in \mathrm{QId}(\mathbf{A}/\Theta)$.

(iii) *If* $\Theta = \mathrm{id}_A$*, then an equation* ε *is a tautology of* $\mathfrak{A}$ *iff* ε *is an identity of* $\mathbf{A}$*; a quasi-equation* $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$ *is a valid rule of* $\mathfrak{A}$ *iff* $\varepsilon_1, \ldots, \varepsilon_n \to \varepsilon$ *is a quasi-identity of* $\mathbf{A}$.

Proof. Straightforward. □

The proposition above says that a quasi-equation is a *valid* rule of $\langle bA, \mathrm{id}_A \rangle$ iff it corresponds to a quasi-identity of $\mathfrak{A}$. If it corresponds to a rule that is only sound (or admissible) then it is called a sound quasi-identity.

Definition 2.34 *A quasi-equation* $\bigwedge_{\varphi \in X} \varphi \to \varepsilon$ *is a* **sound quasi-identity** *of an algebra* $\mathbf{A}$ *if* $\langle X, \varepsilon \rangle$ *is a sound rule of the matrix* $\langle \mathbf{A}, \mathrm{id}_A \rangle$.

Thus a quasi-identity $\bigwedge_{\varphi \in X} \varphi \to \varepsilon$ is sound if "it does not lead outside the set of all tautologies" of $\mathfrak{A}$, i.e., if for some substitution σ, $\sigma(\varphi) \in \mathrm{Id}(\mathbf{A})$, for all $\varphi \in X$, then also $\sigma(\varepsilon) \in \mathrm{Id}(\mathbf{A})$. Substitutions are the valuations into the term algebra **Te**; it follows that a quasi-equation is a sound quasi-identity of $\mathbf{A}$, iff it is a quasi-identity of the free algebra in the variety generated by $\mathbf{A}$.

2.6 Leibniz operator

The definitions and results of this section are routine modification of the material of [4, Section 5] concerning reductive homomorphism, congruence compatible with a filter, Leibniz congruence associated with a filter and Leibniz operation.

2.6.1 Reductive homomorphisms

For the rest of this section let $\langle \Lambda, K \rangle$ be a first-order language and let $\mathfrak{A} = \langle \mathbf{A}, D_{\mathfrak{A}} \rangle, \mathfrak{B} = \langle \mathbf{B}, D_{\mathfrak{B}} \rangle$ be K matrices.

Definition 2.35 *A homomorphism $h : \mathbf{A} \to \mathbf{B}$ is said to be* **reductive** *from $\mathfrak{A}$ to $\mathfrak{B}$ if h is onto B and $h^{-1}(D_{\mathfrak{B}}) = D_{\mathfrak{A}}$. $\mathfrak{B}$ is a* **reduction** *of $\mathfrak{A}$ and $\mathfrak{B}$ is an* **expansion** *of $\mathfrak{A}$ if there exists a reductive homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. A K matrix $\mathfrak{A}$ is* **reduced** *if for every $\mathfrak{B}$, if $\mathfrak{B}$ is a reduction of $\mathfrak{A}$, then $\mathfrak{A} \cong\ fB$.*

Proposition 2.36 (compare with [4, Proposition 5.1]) *Let $h : \mathfrak{A} \to \mathfrak{B}$ be a reductive homomorphism. Let $R \in K$ with $\rho(R) = n$ and let $\vec{t} = \vec{t}(x_1, \ldots, x_n)$ be a sequence of terms. Let $\varphi = R(\vec{t}) \in \mathrm{Fm_K}, \Gamma \subseteq \mathrm{Fm_K}$ and $a_1, \ldots, a_n \in A$. Then*

(i) $\vec{t}(a_1, \ldots, a_n) \in R^{\mathfrak{A}}$ *iff* $\vec{t}(ha_1, \ldots, ha_n) \in R^{\mathfrak{B}}$.

(ii) $\Gamma \models_{\mathfrak{A}} \varphi$ *iff* $\Gamma \models_{\mathfrak{B}} \varphi$.

Proof. The first claim follows immediately from the assumption that $h^{-1}(D_{\mathfrak{B}}) = D_{\mathfrak{A}}$ and that h is a homomorphism. The second claim follows from the first claim and definition of satisfaction, def. 0.44.

For example, let $\mathfrak{A}, \mathfrak{B}$ be $\mathcal{B}$-matrices, where $\mathcal{B}$ is the Birkhoff deductive system of equational logic, page 48. Then $\mathfrak{A} = \langle \mathbf{A}, \Theta \rangle$ and $\mathfrak{B} = \langle \mathbf{B}, \Psi \rangle$, where Θ, Ψ

are congruences on $\mathbf{A}, \mathbf{B}$, respectively. Let $h : \mathfrak{A} \to \mathfrak{B}$ be a matrix homomorphism. It follows from the proposition 2.36 that if h is reductive then for all terms $t(x_1, \ldots, x_n), s(x_1, \ldots, x_n)$ and for all $a_1, \ldots, a_n \in A$, $t(a_1, \ldots, a_n)\theta^{\mathfrak{A}} s(a_1, \ldots, a_n)$ iff $t(ha_1, \ldots, ha_n)\theta^{\mathfrak{A}} s(ha_1, \ldots, ha_n)$. Thus $\mathbf{A}/\Theta$ and $\mathbf{B}/\Psi$ are isomorphic.

2.6.2 Leibniz operator

Definition 2.37 *Let $\Theta \in \mathrm{Co}(\mathbf{A})$ and let X be a K-subset of A. We say that Θ is* **compatible with** *X if for every $R \in K$ and for all sequences $\vec{a}, \vec{b} \in A^{\rho(R)}$, if $\vec{a} \in X_R$ and $a_j \Theta b_j$ for all $j = 1, \ldots, \rho(R)$ then $\vec{b} \in X$.*

If Θ is a relation on a set A, we write $\vec{a}\Theta^n\vec{b}$ for the conjunction of statements: $\vec{a} = \langle a_1, \ldots, a_n\rangle$; $\vec{b} = \langle b_1, \ldots, b_n\rangle \in A^n$ and for every $i = 1, \ldots, n$, $\langle a_i, b_i\rangle \in \Theta$.

Proposition 2.38 *Let $\Theta \in \mathrm{Co}(\mathbf{A})$ and let X be a K-subset of A. Then Θ is compatible with X iff for every $R \in K$, for every sequence $\vec{a} = \langle a_1, \ldots, a_{\rho(R)}\rangle \in X_R$,*

$$a_1/\Theta \times a_2/\Theta \times \ldots \times a_{\rho(R)}/\Theta \subseteq X_R.$$

Therefore,

$$X_R = \bigcup_{\vec{a} \in X} a_1/\Theta \times \ldots a_{\rho(R)}/\Theta.$$

Proof. Immediate from the definition. □

Lemma 2.39 *Let $\mathcal{F}$ be a family of congruences on an algebra $\mathbf{A}$ compatible with a K-subset X of A. Then $\bigvee \mathcal{F}$ is also compatible with X.*

Proof. We need to check that for every $R \in K$, if $\mathbf{a}(\bigvee \mathcal{F})^{\rho(R)}\mathbf{b}$ and $R\mathbf{a} \in X$ (i.e., $\vec{a} \in X_R$), then $R\vec{b} \in X$. But it is sufficient to prove that for every $k \leq$

$\rho(R)$, for all $a_1, \ldots, a_n, b_k \in A$, if $\langle a_k, b_k\rangle \in \bigvee \mathcal{F}$, and $R(a_1, \ldots, a_n) \in \bigvee(\mathcal{F})$, then $R(a_1, \ldots, a_{k-1}, b_k, a_{k+1}, \ldots, a_n) \in \bigvee(\mathcal{F})$. But $\bigvee(\mathcal{F})$ is generated by $\bigcup(\mathcal{F})$. By Theorem 0.14 it suffices to prove that if $\langle c, d\rangle \in \bigcup \mathcal{F}$ and

$$R(a_1, \ldots, a_{k-1}, t(c, \vec{e}), a_{k+1}, \ldots, a_n) \in X,$$

for some term t and some sequence $\vec{e}$ of elements of A, then also

$$R(a_1, \ldots, a_{k-1}, t(d, \vec{e}), a_{k+1}, \ldots, a_n) \in X.$$

But $\langle c, d\rangle \in \Theta$, for some $\Theta \in \mathcal{F}$ and therefore $\langle t(c, \vec{e}), t(d, \vec{e})\rangle \in \Theta$. Since Θ is compatible with X, it follows that $R(a_1, \ldots, a_{k-1}, t(d, \vec{e}), a_{k+1}, \ldots, a_n) \in X$. □
Thus the largest congruence compatible with a K-subset X always exists.

Definition 2.40 *The largest congruence on* $\mathbf{A}$ *compatible with* X *is called the* **Leibniz congruence** *of* X *and is denoted by* $\Omega_{\mathcal{S}}^{\mathbf{A}}(X)$. *We omit the subscript* $\mathcal{S}$ *or superscript* $\mathbf{A}$ *when* $\mathcal{S}$ *or, respectively,* $\mathbf{A}$ *is clear from the context. The operator* $\Omega_{\mathcal{S}}^{\mathbf{A}}$ *is called the* **Leibniz operator** *on* $\mathbf{A}$ **associated with the** K**-deductive system** $\mathcal{S}$.

Proposition 2.41 (compare with [4, Prop. 5.3]) *Two elements* a *and* b *of* A *are identified by* $\Omega^{\mathbf{A}}(X)$ *iff for every* K*-formula* $R(\vec{t}(x, x_1, \ldots, x_n))$ *and every choice of elements* $\vec{c} \in A^{\rho(R)}$, $\vec{t}(a, \vec{c}) \in X_R$ *iff* $\vec{t}(b, \vec{c}) \in X_R$.

Proof. A routine modification of Proposition 5.3. in [4]. □

Lemma 2.42 (compare with Lemma 5.4 in [4]) *Let* $\mathcal{S}$ *be a* K*-deductive system. Let* $\mathbf{A}, \mathbf{B}$ *be* Λ*-algebras and* $h : \mathbf{A} \to \mathbf{B}$ *a surjective homomorphism. Then for every* $F \in Fi_{\mathcal{S}}(\mathbf{B})$, $\Omega^{\mathbf{A}}(h^{-1}F) = h^{-1}\Omega^{\mathbf{B}}(F)$. *In particular, if* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are matrices and* $h : \mathfrak{A} \to \mathfrak{B}$ *is a reduction, then* $\Omega^{\mathbf{A}}(D_{\mathfrak{A}}) = h^{-1}\Omega^{\mathbf{B}}(D_{\mathfrak{B}})$.

Proof. Same as the proof of Lemma 5.4. in [4]. □

For any matrix $\mathfrak{A} = \langle \mathbf{A}, D_{\mathfrak{A}} \rangle$ we define

$$\mathfrak{A}^* := \langle A/\Omega(D_{\mathfrak{A}}), D_{\mathfrak{A}}/\Omega(D_{\mathfrak{A}}) \rangle,$$

where $D_{\mathfrak{A}}/\Omega(D_{\mathfrak{A}}) := \{R(\vec{a}/\Omega(D_{\mathfrak{A}})) : R\vec{a}\rangle \in D_{\mathfrak{A}}\}$. The matrix $\mathfrak{A}^*$ is a reduction of $\mathfrak{A}$. In fact, it is a minimal reduction of $\mathfrak{A}$ in the sense that for any other reduction $\mathfrak{B}$ of $\mathfrak{A}$, $\mathfrak{B}^*$ is isomorphic to $\mathfrak{A}^*$. We shall identify $\mathfrak{A}$ with $\mathfrak{A}^*$ if $\mathfrak{A}$ is reduced. For any class of K-matrices $\mathbf{K}$ we denote by $\mathbf{K}^*$ the class of all minimal reductions of the matrices in $\mathbf{K}$.

Lemma 2.43 (compare with [4, Lemma 5.5.]) *If Θ is a congruence compatible with $D_{\mathfrak{A}}$, then $\mathfrak{A}^*$ is isomorphic to $\langle \mathbf{A}/\Theta, D_{\mathfrak{A}}/\Theta \rangle^*$.*

□

By Proposition 2.36 (ii), $\mathcal{S}_{\mathbf{K}} = \mathcal{S}_{\mathbf{K}^*}$. The following theorem is an immediate consequence of this fact.

Theorem 2.44 ([4, Thm. 5.6.]) *For any K-deductive system $\mathcal{S}$, any set of K-formulas $\Gamma \cup \{\varphi\}$ we have $\Gamma \vdash_{\mathcal{S}} \varphi$ iff $\Gamma \models_{\mathrm{Mod}^*\mathcal{S}} \varphi$.*

□

Also, the Proposition 5.7. of [4] says, among others, that for a congruence Θ on the algebra $\mathbf{A}$, $\Omega(\Theta) = \Theta$. Hence the reduction of a 2-matrix $\langle \mathbf{A}, \Theta \rangle$ is $\langle \mathbf{A}/\Theta, \Delta_{A/\Theta} \rangle$. So the reduced models of the Birkhoff's system $\mathcal{B}$ are the matrices $\langle \mathbf{A}, \Delta_A \rangle$, which can be identified with the algebra $\mathbf{A}$. Therefore the reduced semantics for an extension $\mathcal{S}$ of $\mathcal{B}$ is, after this identification, exactly the quasivariety defined by all the quasi-equations $\varepsilon_1 \wedge \cdots \wedge \varepsilon_n \to \varepsilon$ such that the rule $\langle \{\varepsilon_1, \ldots, \varepsilon_n\}, \{\varepsilon\} \in \vdash_{\mathcal{S}}$.

Definition 2.45 *Let $\mathfrak{A}$ be a matrix and F a filter on $\mathfrak{A}$. Then the reduction of $\mathfrak{A}$ by the filter F is the matrix $\mathfrak{A}/F := \langle \mathbf{A}/\Omega(F), F/\Omega(F)\rangle$.*

Hence $\mathfrak{A}^*$ is exactly the reduction of $\mathfrak{A}$ by the filter $D_{\mathfrak{A}}$. It also follows that the reduction of a $\mathcal{B}$-matrix $\langle \mathbf{A}, D_{\mathfrak{A}}\rangle$ by a $\mathcal{B}$-filter θ can be identified with the quotient $\mathbf{A}/\theta$.

2.6.3 Leibniz operator relativized to a predicate

When K has more than one element, then for every $R \in$ K, we can define a relativized compatibility in the following way.

Definition 2.46 *A congruence Θ on $\mathbf{A}$ is R-**compatible** with a K-subset X of A if for every sequence of pairs $\langle a_i, b_i\rangle \in \Theta$, $i = 1, \ldots, \rho(R)$,*

$$R(a_1, \ldots, a_{\rho(R)}) \in X \Rightarrow R(b_1, \ldots, b_{\rho(R)}) \in X.$$

Definition 2.47 *For $R \in K$ and $k \in \{1, \ldots, \rho(R)\}$, we say that a congruence Θ on $\mathbf{A}$ is (R, k)-compatible with a K-subset X of A, if for every pair $\langle a, b\rangle \in \Theta$, for every sequence of $\rho(R) - 1$ elements $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_{\rho(R)} \in A$, we have*

$$\langle a_1, \ldots, a_{\rho(R)}\rangle \in X_R \Rightarrow \langle b_1, \ldots, b_{k_1}\rangle \in X_R.$$

Observe, that Θ is R-compatible with X iff for all pairs $\langle a, b\rangle \in \Theta$, for every $k = 1, \ldots, \rho(R)$ and for every sequence $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n$,

$$R(c_1, \ldots, c_{k-1}, a, c_{k+1}, \ldots, c_n) \in X \text{ iff } R(c_1, \ldots, c_{k-1}, a, c_{k+1}, \ldots, c_n) \in X.$$

Proposition 2.48 (i) $\Omega(X) = \bigcap_{R\in K} \Omega_R(F)$.

(ii) $\Omega_R(X) = \bigcap_{k=1}^{\rho(R)} \Omega_{(R,k)}(F)$.

Proof. Clearly, a congruence Θ is compatible with X iff for every $R \in$ K it is R-compatible with X. So for every $R \in$ K, $\Omega(X)$ is R-compatible with X and therefore $\Omega(X) \subseteq \Omega_R(X)$, for every R. Hence $\Omega(X) \subseteq \bigcap_{R\in \mathrm{K}} \Omega_R(X)$. Also, $\bigcap_{R\in \mathrm{K}} \Omega_R(X)$ is a congruence R-compatible with X, for every R. So it is compatible with X and (i) follows. Part (ii) can be proved similarly. □

Proposition 2.49 *Let $\mathfrak{A}$ be a matrix. Then the join of a family of congruences R-compatible with $D_{\mathfrak{A}}$ is also R-compatible with $D_{\mathfrak{A}}$. Similarly, the join of a family of congruences Rk-compatible with $D_{\mathfrak{A}}$ is Rk-compatible with $D_{\mathfrak{A}}$.*

Proof. Similar to the proof of Lemma 2.39 □

By the above proposition, for every K-subset of A, for every $R \in$ K there is the largest congruence $\Omega_R(X)$, which is R-compatible with X; and also the largest congruence $\Omega_{Rk}(X)$, which is (R,k)-compatible with X.

Definition 2.50 *The largest congruence on an algebra $\mathbf{A}$, which is R-compatible with X is called the* **Leibniz congruence relative to** *R and denoted by $\Omega_R^{\mathbf{A}}(X)$). The largest congruence Rk-compatible with X is called the* **Leibniz congruence relative to** *(R,k) and denoted by $\Omega_{(R,k)}^{\mathbf{A}}(X)$. We omit the superscript $\mathbf{A}$ if $\mathbf{A}$ is known from the context.*

Proposition 2.51 *Let $\mathbf{A}$ be an algebra and let X be a K-subset of A. Then a pair $\langle a,b\rangle$ of elements of A is in $\Omega_R(X)$ iff for every sequence of terms $\vec{t}(x,x_1,\ldots,x_m) \in$ $\mathrm{Te}^{\rho(R)}$ and for every choice of elements $\vec{c} = c_1,\ldots,c_m \in A$,*

$$R\vec{t}(a,\vec{c}) \in X \textit{ iff } R\vec{t}(b,\vec{c}) \in X.$$

A pair $\langle a, b\rangle$ of elements of A is in $\Omega_{(R,k)}(X)$ iff for every m and a sequence of variables $x, x_1, \ldots, x_m$, for every sequence of terms $\langle t_1, \ldots, t_{k-1}, t, t_{k+1}, \ldots, t_{\rho(R)}\rangle \in$ $\mathrm{Te}^{\rho(R)}$ such that for $i \neq k$, $\mathrm{Var}(t_i) \subseteq \{x_1, \ldots, x_m\}$ and $\mathrm{Var}(t) \subseteq \{x_1, \ldots, x_m, x\}$ and for every choice of elements $\vec{c} \in A^{|\vec{x}|}$,

$$R(t_1(\vec{c}), \ldots, t_{k-1}(\vec{c}), t_k(a, \vec{c}), t_{k+1}(\vec{c}), \ldots, t_n(\vec{c})) \in X \tag{2.11}$$

$$\textit{iff} \quad R(t_1(\vec{c}), \ldots, t_{k-1}(\vec{c}), t_k(b, \vec{c}), t_{k+1}(\vec{c}), \ldots, t_n(\vec{c})) \in X.$$

Proof. We prove the second statement of the proposition. The first follows by induction on $n - k$, where $k + 1, \ldots, n$ from the first and Proposition 2.48 (ii). Since $\Omega_{(R,k)}$ is symmetric, it suffices to prove the second statement of the proposition with "iff" in 2.11 replaced by "implies". Let $\vec{t}$ be a sequence of terms as in the statement of the proposition. Since $\Theta := \Omega_{(R,k)}(X)$ is a congruence, $\langle a, b\rangle \in \Theta$ implies that $\langle t_k(a, \vec{c}), t_k(b, \vec{c})\rangle \in \Theta$. The necessity of the condition follows immediately from the fact that Θ is (R, k)-compatible with X. For the proof of the sufficiency we define a relation Ψ by $\langle a, b\rangle \in \Psi$ iff for every sequence of terms $\langle t_1, \ldots, t_{k-1}, t, t_{k+1}, \ldots, t_{\rho(R)}\rangle \in$ $\mathrm{Te}^{\rho(R)}$ such that for $i \neq k$, $\mathrm{Var}(t_i) \subseteq \{x_1, \ldots, x_m\}$ and $\mathrm{Var}(t) \subseteq \{x_1, \ldots, x_m, x\}$ and for every choice of elements $\vec{c} \in A^{|\vec{x}|}$,

$$R(t_1(\vec{c}), \ldots, t_{k-1}(\vec{c}), t_k(a, \vec{c}), t_{k+1}(\vec{c}), \ldots, t_n(\vec{c})) \in X \text{ iff}$$

$$R(t_1(\vec{c}), \ldots, t_{k-1}(\vec{c}), t_k(b, \vec{c}), t_{k+1}(\vec{c}), \ldots, t_n(\vec{c})) \in X.$$

Notice that Ψ is a congruence on $\mathbf{A}$ and that Ψ is compatible with X. Therefore $\Psi \subseteq \Omega_R(X)$. The second statement of the proposition follows. □

2.7 Semantics and reduced matrix semantics of K-deductive systems.

From the point of view of its deductive power, a class $\mathcal{C}$ of K-matrices is equivalent to the class $\mathcal{C}^*$ of all reduced matrices of $\mathcal{C}$. In [3] and [4] it was shown that the reduced matrices under many respects behave similarly as the algebraic models of the quasi-equational logic. This is particularly true for the protoalgebraic deductive systems defined in section 2.8.

In this section we extend, to arbitrary K-deductive systems, the results of [4, Section 6] on reduced matrix semantics of k-deductive systems. These results are stated here without proof, because either exactly the same proof or a straightforward modification of a proof in [4] applies. When the modification is not completely obvious, we indicate it.

Let the algebraic language Λ be arbitrary and fixed and let K be a fixed relational language.

Definition 2.52 *A class $\mathcal{H}$ of reduced matrices of the form* $\mathrm{Mod}^*\mathcal{S}$ *for some structural and finitary K-deductive system $\mathcal{S}$ is called a* reduced universal Horn K-class. $\mathcal{H}$ *is* generated *by an arbitrary class $\mathcal{C}$ of reduced K-matrices if it is the smallest reduced universal Horn K-class including $\mathcal{C}$, equivalently, if $\mathcal{S} = \langle \Lambda, \models^f_{\mathcal{C}} \rangle$.*

Recall that for a set A, by $E_K(A)$ we denote the set of all K-elements of A, i.e., $E_K(A) = \coprod_{R \in \mathrm{K}} A^{\rho(R)}$. For a filter F on an algebra $\mathbf{B}$ and a subalgebra $\mathbf{A}$ of $\mathbf{B}$, let $F|\mathbf{B} := F \cap E_K(A)$.

Definition 2.53 *Let $\mathfrak{A} = \langle \mathbf{A}, D_{\mathfrak{A}} \rangle, \mathfrak{B} = \langle \mathbf{B}, D_{\mathfrak{B}} \rangle$ be K-matrices. $\mathfrak{A}$ is a* **submatrix** *of $\mathfrak{B}$, in symbols $\mathfrak{A} \leq \mathfrak{B}$, if $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $D_{\mathfrak{A}} = D_{\mathfrak{B}}|\mathbf{A} = D_{\mathfrak{B}} \cap E_K(A)$.*

Let $\mathfrak{A}_i, i \in I$, be a system of K-matrices. The **direct product** *of* $\{\mathfrak{A}_i : i \in I\}$ *is*

$$\prod_{i\in I} \mathfrak{A}_i; = \langle \prod_{i\in I} \mathbf{A}_i, \prod_{i\in I} D_{\mathfrak{A}_i} \rangle.$$

We identify $\prod_{i\in I} D_{\mathfrak{A}_i}$ with $\coprod_{R\in \mathrm{K}} D_R$, where D_R is the set of $\rho(R)$ tuples $\langle\langle a_{i1} : i \in I\rangle, \dots, \langle a_{i\rho(R)} : i \in I\rangle\rangle \in A^{\rho(R)}$ such that for every $i \in I$, $\langle a_{i1}, \dots, a_{i\rho(R)}\rangle \in R^{\mathfrak{A}_i}$. The set I may be empty, in which case $\prod_{i\in I} \mathbf{A}_i$ is the trivial, one-element algebra and $\prod_{i\in I} D_{\mathfrak{A}_i} = \prod_{i\in I} \mathbf{A}_i$. In [4, sec. 6] the examples are given that the reduced Horn classes are in general not closed under subalgebras or direct products.

Proposition 2.54 ([4, Proposition 6.1]) *Let $\mathfrak{A}, \mathfrak{B}$ be K-matrices such that $\mathfrak{A} \leq \mathfrak{B}$. If $\mathfrak{A}$ is reduced, then $\mathfrak{A}$ is isomorphic to a submatrix of $\mathfrak{B}^*$.*

Let $\mathcal{F} \subseteq \mathcal{P}I$ be a lattice filter on $\langle \mathcal{P}I, \cap, \cup\rangle$. We identify $(\prod_{i\in I} A_i)^k$ with $(\prod_{i\in I} A_i^k)$ under the natural mapping

$$\langle\langle a_{i1} : i \in I\rangle, \dots, \langle a_{ik} : i \in I\rangle\rangle \to \langle\langle a_{i1}, \dots, a_{ik}\rangle : i \in I\rangle.$$

We define

$$D^{\mathcal{F}}_{\prod \mathfrak{A}_i} := \{R(a_1, \dots, a_{\rho(R)}) \in E_K(A) : \{i \in I : R(a_{i1}, \dots, a_{i\rho(R)} \in D_{\mathfrak{A}_i}\} \in \mathcal{F}\}.$$

Equivalently,

$$D^{\mathcal{F}}_{\prod \mathfrak{A}_i} := \coprod_{R\in \mathrm{K}} (R^{\prod \mathfrak{A}_i}_{\mathcal{F}}) \text{ where for each } R$$

$$(R^{\prod \mathfrak{A}_i}_{\mathcal{F}} := \{\{\vec{\mathbf{a}} \in (\prod_{i\in I} A_i)^{\rho(R)} : \{i \in I : \langle a_{i1}, \dots, a_{i\rho(R)}\rangle \in R^{\mathfrak{A}_i}\} \in \mathcal{F}\}.$$

Then we let

$$(\prod_{i\in I} \mathfrak{A}_i)^{\mathcal{F}} := \langle \prod_{i\in I} \mathbf{A}_i, D^{\mathcal{F}}_{\prod \mathfrak{A}_i}\rangle.$$

We also define

$$\Theta(\mathcal{F}) := \{\langle \vec{a}, \vec{b}\rangle \in \prod_{i \in I} \mathbf{A}_i : \{i : a_i = b_i\} \in \mathcal{F}\},$$

$$\prod_{i \in I} \mathbf{A}_i/\mathcal{F} := \langle \prod_{i \in I} \mathbf{A}_i/\Theta(\mathcal{F}), D_{\prod \mathfrak{A}_i}/\mathcal{F}\rangle \text{ where}$$

$$D_{\prod \mathfrak{A}_i}/\mathcal{F} := D^{\mathcal{F}}_{\prod \mathfrak{A}_i}/\Theta(\mathcal{F}) = \coprod_{R \in \mathrm{K}} (D^{\mathcal{F}}_{\prod \mathfrak{A}_i})_R/\Theta(\mathcal{F})$$

Note that $\prod_{i \in I} \mathbf{A}_i/\mathcal{F}$ is the usual filter product of algebras. Finally define

$$\prod_{i \in I} \mathfrak{A}_i/\mathcal{F} := \langle \prod_{i \in I} \mathbf{A}_i/\mathcal{F}, D_{\prod \mathfrak{A}_i}/\mathcal{F}.$$

$\prod_{i \in I} \mathbf{A}_i/\mathcal{F}$ is called the *matrix filtered product* of $\{\mathfrak{A}_i : i \in I\}$ by $\mathcal{F}$. $D^{\mathcal{F}}_{\prod \mathfrak{A}_i}$ is an $\mathcal{S}$-filter on the direct product and $\Theta(\mathcal{F})$ is obviously compatible with it. In general $\Theta(\mathcal{F})$ is smaller that $\Omega(D^{\mathcal{F}}_{\prod \mathfrak{A}_i})$ and whence the matrix filtered product is not usually reduced. By lemma 2.43 we have that $(\prod_{i \in I} \mathfrak{A}_i/\mathcal{F})^*$ is isomorphic to $\prod^{\mathcal{F}}_{i \in I} \mathfrak{A}_i/D^{\mathcal{F}}_{\prod \mathfrak{A}_i}$.

An arbitrary reduced universal Horn K-class need not be closed under matrix ultraproducts. This is shown in [4]. For any class $\mathcal{C}$ of K-matrices we define

$$\mathbf{I}\mathcal{C} := \{\mathfrak{A} : \mathfrak{A} \text{ isomorphic to some } \mathfrak{B} \in \mathcal{C}\},$$

$$\mathbf{S}\mathcal{C} := \{\mathfrak{A} : \mathfrak{A} \leq \mathfrak{B} \text{ for some } \mathfrak{B} \in \mathcal{C}\},$$

$$\mathbf{P}\mathcal{C} := \{\textstyle\prod_{i \in I} \mathfrak{A}_i : \mathfrak{A}_i \in \mathcal{C} \text{ all } i \in I\}$$

$$\mathbf{P}_\omega\mathcal{C} := \{\mathfrak{A}_1 \times \cdots \times \mathfrak{A}_n : \mathfrak{A}_1, \ldots, \mathfrak{A}_n \in \mathcal{C}, n < \omega\},$$

$$\mathbf{P}_F\mathcal{C} := \{\textstyle\prod_{i \in I} \mathfrak{A}_i/\mathcal{F} : \mathfrak{A} \in \mathcal{C} \text{ for all } i \in I, \text{ all lattice filters } \mathcal{F} \text{ on } \mathcal{P}I\},$$

$$\mathbf{P}_U\mathcal{C} := \{\textstyle\prod_{i \in I} \mathfrak{A}_i/\mathcal{F} : \mathfrak{A} \in \mathcal{C} \text{ for all } i \in I, \text{ all lattice ultrafilters } \mathcal{F} \text{ on } \mathcal{P}I\}.$$

Also, for each operator $\mathbf{Q} \in \{\mathbf{I}, \mathbf{S}, \mathbf{P}, \mathbf{P}_\omega, \mathbf{P}_F, \mathbf{P}_U\}$ define $\mathbf{Q}^*(\mathcal{C}) := \{\mathfrak{A}^* : \mathfrak{A} \in \mathbf{Q}(\mathcal{C}\})$. We will often omit parentheses and write $\mathbf{Q}\mathcal{C}$ for $\mathbf{Q}(\mathcal{C})$, for any of the reduced or not operators defined above.

Theorem 2.55 (see [4, Thm. 6.2.]) *Let $\mathcal{C}$ be any class of reduced K-matrices and let $\mathcal{S} := \langle \Lambda, \models^{f}_{\mathcal{C}} \rangle$. Then* $\mathrm{Mod}^{*}\mathcal{S} = \mathbf{IS}^{*}\mathbf{P}^{*}_{U}\mathbf{P}^{*}_{\omega}\mathcal{C}$.

Proof. The inclusion from right to left is straightforward.

For the inclusion from left to right, the proof of [4, Theorem 6.2.] applies with the following modification at the beginning. Let $\mathfrak{A} \in \mathrm{Mod}^{*}\mathcal{S}$. Let $\{a_{\kappa} : \kappa < \alpha\}$ be a fixed system of generators of A. Let $\{x_{\kappa} : \kappa < \alpha\}$ be a corresponding system of variable symbols, and let Te_{α} be the set of all terms in these variables. According to a remark in section Universal Horn Logic we identify Fm_{K} with the set of all universal Horn formulas $R(\vec{t})$, where $R \in$ K and $\vec{t} \in \mathrm{Te}_{\alpha}^{\rho(R)}$.

Let

$$\mathrm{Diag} := \{R^{\mathfrak{A}}(\vec{t}) : R \in \mathrm{K}, \vec{t} \in \mathrm{Te}_{\alpha}^{\rho(R)}, \vec{t}(\vec{a}) \in D_{\mathfrak{A}}\}.$$

Diag is called the *diagram* of $\mathfrak{A}$. Let

$$\Delta := \mathcal{P}_{\omega}(\mathrm{Diag}) \times \mathcal{P}_{\omega}(\mathrm{Fm}_{\mathrm{K}} \setminus \mathrm{Diag})$$

If $\mathfrak{B}$ is any matrix and $b_1, \ldots, b_n \in B$, then for every $S = \langle S^{+}, S^{-} \rangle \in \Delta$ we write $\models_{\mathfrak{B}} S[b_1, \ldots, b_n]$ if

$$\vec{t}^{\mathbf{B}}(b_1, \ldots, b_n) \in R^{\mathfrak{B}} \text{ for all } R\vec{t} \in S^{+} \quad \text{and} \quad \vec{t}^{\mathbf{B}}(b_1, \ldots, b_n) \notin R^{\mathfrak{B}} \text{ for all } R^{\mathfrak{A}}\vec{t} \in S^{-}.$$

The proof now is continued exactly as the proof of [4, Theorem 6.2], except that we consider K-matrices rather than k-matrices, and K-elements, rather than sequences of k elements.

Corollary 2.56 (see [4, Corollary 6.3.]) *Let $\mathcal{C}$ be any set of reduced K-matrices and let $\mathcal{H}$ be the reduced universal Horn K-class generated by $\mathcal{C}$. Then* $\mathcal{H} = \mathbf{IS}^{*}\mathbf{P}^{*}\mathbf{P}^{*}_{U}\mathcal{C}$, *and, if* $\mathbf{K}$ *is a finite class of finite matrices,* $\mathcal{H} = \mathbf{IS}^{*}\mathbf{P}^{*}\mathbf{K}$.

Corollary 2.57 (see [4, Corollary 6.4.]) *A class $\mathcal{C}$ of reduced K-matrices is a reduced universal Horn K-class iff it is closed under reduced ultraproducts, reduced direct products, and reduced subalgebras.*

Another useful construction is that of a subdirect product.

Definition 2.58 *A submatrix $\mathfrak{B} \subseteq \prod_{i\in I}\mathfrak{A}_i$ is called a* **subdirect product** *of the system $\{\mathfrak{A}_i : i \in I\}$, in symbols $\mathfrak{B}\subseteq_{\mathrm{SD}}\prod_{i\in I}\mathfrak{A}_i$, if the projection $\pi_i : B \to A_i$ is surjective for every $i \in I$. The class of all subdirect products of matrices from $\mathcal{C}$ is denoted by $\mathbf{P}_{SD}(\mathcal{C})$.*

We will use this definition in Chapter 3 and in particular in Part II. A useful characterization of subdirect products is contained in Part II, Proposition 2.9.

Let $\mathcal{S}$ be an extension of the 2-deductive system $\mathcal{B}$. Let us observe, that an $\mathcal{S}$-matrix $\mathfrak{A} = \langle \mathbf{A}, \Theta\rangle$ is reduced iff Θ is the identity relation id_A on $\mathbf{A}$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two such reduced matrices. Then a matrix homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ is reductive iff it is an isomorphism. A product of a family of reduced $\mathcal{S}$-matrices $\mathfrak{A}_i = \langle \mathbf{A}_i, \mathrm{id}_{\mathbf{A}_i}\rangle, i \in I$ is the reduced matrix $\mathfrak{A} = \langle \prod_{i\in I}\mathbf{A}_i, \mathrm{id}_{\mathbf{A}}\rangle$ and hence can be identified with the product of algebras $\mathbf{A}_i$. Similarly, filtered products and subdirect products of reduced $\mathcal{S}$-matrices can be identified with the algebraic filtered products and subdirect products of the underlying algebras.

2.8 Protoalgebraic K-deductive systems

The concept of a protoalgebraic 1-deductive system was introduced in [3], generalized to k-deductive systems in [4], to ω-deductive systems in [41] and to the universal Horn classes in [11]. The authors of [3, 4] realized that all what is needed for certain important universal algebraic results to have their analogues for a k-deductive system $\mathcal{S}$ is the monotonicity of the Leibniz operator for every $\mathcal{S}$-model. They called those k-deductive systems protoalgebraic.

Similarly as in the previous section, the results of this section parallel the results of Section 7 of [4] and can be proved by a straightforward modification of the methods of [4]. Another presentation of these results, can be found in the independent work [11].

Definition 2.59 *A K-deductive system S is protoalgebraic if for each Λ-algebra $\mathbf{A}$, the Leibniz operator $\Omega_{\mathcal{S}}^{\mathbf{A}}$ is monotone, i.e., for all $F, G \in Fi_{\mathcal{S}}(\mathbf{A}), F \subseteq G$ implies $\Omega(F) \subseteq \Omega(G)$.*

Corollary 2.60 (Corollary 7.2 in [4]) *The Birkhoff system $\mathcal{B}$ is protoalgebraic.*

Definition 2.61 *A K-deductive system has the* **compatibility property** *if for every Λ-algebra and every $\Theta \in \mathrm{Co}(A)$ if Θ is compatible with an $\mathcal{S}$-filter F on $\mathbf{A}$, then it is also compatible with every filter that includes F.*

Let $\mathbf{A}, \mathbf{B}$ be Λ-algebras and let $h : \mathbf{A} \to \mathbf{B}$ be a surjective homomorphism. If $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$, then $h(F)$ need not be an $\mathcal{S}$-filter on $\mathbf{B}$. Let $\hat{h}_{\mathcal{S}}F$ be the $\mathcal{S}$-filter generated by hF, i.e.,

$$\hat{h}_{\mathcal{S}}F := \bigcap\{G \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{B}) : hF \subseteq G\}$$

It is easy to see that if $\mathcal{S} = \mathcal{B}$ and thus the $\mathcal{S}$-filters are the congruences, then $\hat{h}_{\mathcal{S}}\Theta$ is the transitive closure of $h\Theta$.

Lemma 2.62 (see [4, lemma 7.4.]) *Let $\mathbf{A}, \mathbf{B}$ be algebras and $h : \mathbf{A} \to \mathbf{B}$ a surjective homomorphism. Let $F \in Fi_{\mathcal{S}}(\mathbf{A})$ such that F is compatible with $h^{-1}\Delta_B$, i.e., with the relation kernel of h. Then $\hat{h}_{\mathcal{S}}F = hF$.*

Proof. By an easy modification of the proof of Lemma 7.4 of [4]. □

Definition 2.63 ([4, defin. 7.5]) *A K-deductive system $\mathcal{S}$ has the* **filter correspondence property** *if, for every surjective homomorphism $h : \mathbf{A} \to \mathbf{B}$, and for every $F \in Fi_{\mathcal{S}}(\mathbf{A})$ and $G \in Fi_{\mathcal{S}}(\mathbf{B})$,*

$$h^{-1}(\hat{h}_{\mathcal{S}}F \vee G) = F \vee h^{-1}G,$$

where the joins are taken in $Fi_{\mathcal{S}}(\mathbf{B})$ and $Fi_{\mathcal{S}}(\mathbf{A})$, respectively.

The following theorem was formulated in [4] for k-deductive system, but its proof applies without changes to arbitrary K-deductive systems.

Theorem 2.64 (see Theorem 7.6. of [4]) *Let $\mathcal{S}$ be a K-deductive system. The following are equivalent:*

1. *$\mathcal{S}$ is protoalgebraic;*

2. *$\mathcal{S}$ has the compatibility property;*

3. *$\mathcal{S}$ has the filter correspondence property.*

Corollary 2.65 (see [4, Corollary 7.7]) (The Correspondence Theorem) *Let $\mathbf{A}$ and $\mathbf{B}$ be algebras and $h : A \to B$ a surjective homomorphism. Let $\mathcal{S}$ be a K-deductive protoalgebraic system. Then for any $F \in Fi_{\mathcal{S}}(\mathbf{B})$ the mapping $G \to h^{-1}G$ is an isomorphism between $[F)$ in $Fi_{\mathcal{S}}(\mathbf{B})$ and $[h^{-1}F)$ in $Fi_{\mathcal{S}}(\mathbf{A})$.*

It also follows from the correspondence property, that if $\mathfrak{A}$ is a model of a protoalgebraic K-deductive system $\mathcal{S}$ and $F \in \mathrm{Fi}_{\mathcal{S}}(\mathfrak{A})$, then the lattice of $\mathcal{S}$ filters on $\mathfrak{A}/F$ is isomorphic to the sublattice of $\mathrm{Fi}_{\mathcal{S}}(\mathfrak{A})$ generated by F, i.e., to the lattice interval

$$[F) = \{G \in \mathrm{Fi}_{\mathcal{S}}(\mathfrak{A}) : F \subseteq G\}$$

in $\mathrm{Fi}_{\mathcal{S}}(\mathfrak{A})$.

Theorem 2.66 (see Theorem 8.1 of [4]) *Let $\mathfrak{A}$ and $\mathfrak{B}$ be models of protoalgebraic K-deductive system $\mathcal{S}$. Then every surjective matrix homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ induces a matrix homomorphism $h^* : \mathfrak{A}^* \to \mathfrak{B}^*$, of the respective reductions, defined by $h(a/\Omega(D_{\mathfrak{A}})) = ha/\Omega(D_{\mathfrak{B}})$.*

Proof. (by a modification of the proof in [4]). □

Suppose that the system $\mathcal{S}$, in the statement of the above theorem, is a extension of the system $\mathcal{B}$. Let $\mathfrak{A}\langle \mathbf{A}, \Theta\rangle, \mathfrak{B} = \langle \mathbf{B}, \mathbf{\Psi}\rangle$ be two $\mathcal{S}$-matrices. A matrix homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ is an algebra homomorphism such that $f^{-1}(\Psi) \subseteq \Theta$. The reductions of $\mathfrak{A}$ and $\mathfrak{B}$ are the quotient matrices $\langle \mathbf{A}/\Theta, \mathrm{id}\rangle$ and $\langle \mathbf{B}/\mathbf{\Psi}, \mathrm{id}\rangle$, respectively. Thus Theorem 2.66 says that if $h^{-1}(\Psi) \subseteq \Theta$ for some algebra homomorphism $h : \mathbf{A} \to \mathbf{B}$, then h induces algebra homomorphism $h^* : \mathbf{A}/\Theta \to \mathbf{B}/\mathbf{\Psi}$. This is a corollary to the homomorphism theorem in universal algebra.

An immediate consequence of the filter correspondence property is the following

Theorem 2.67 (compare with [4, Theorem 8.3.]) *If $\mathcal{S}$ is protoalgebraic, then $\mathbf{Fi}_{\mathcal{S}}(\mathfrak{A})$ is isomorphic to $\mathbf{Fi}_{\mathcal{S}}(\mathfrak{A}^*)$.*

□

Definition 2.68 *Let $\mathfrak{A} = \langle \mathbf{A}, D_{\mathfrak{A}}\rangle$ be a K-matrix and F any subset of $\coprod_{R\in K} A^R$ that includes $D_{\mathfrak{A}}$. We define*

$$\mathfrak{A}/F := \langle \mathbf{A}, F\rangle^*.$$

$\mathfrak{A}/F$ is a K-matrix and is called the **quotient matrix of $\mathfrak{A}$ by F**.

Note, that if $K = 2$ and $F = \Theta$ is a congruence, then the quotient matrix of $\mathfrak{A}$ by F is equivalent to the quotient of $\mathbf{A}$ by Θ. We call the natural algebra homomorphism $n : \mathbf{A} \to \mathbf{A}/\Omega(F)$, the *natural map of* F. Also, by analogy with universal algebra, $h^{-1}D_{\mathfrak{A}}$ the inverse image of the filter $D_{\mathfrak{A}}$ under a homomorphism $h : \mathfrak{A} \to \mathfrak{B}$, is called the *filter kernel of* h.

Theorem 2.69 (compare with [4, Theorem 8.4]) *Assume that* $\mathfrak{A}, \mathfrak{B}$ *are models of a protoalgebraic* K*-deductive system* $\mathcal{S}$*, and let* $h : \mathfrak{A} \to \mathfrak{B}$ *be surjective matrix homomorphism. Then*

1. *If* $\mathfrak{B}$ *is reduced,* $\mathfrak{A}/h^{-1}D_{\mathfrak{A}}$ *is isomorphic to* $\mathfrak{B}$.

2. *Assume that* $F \in Fi_{\mathcal{S}}(\mathfrak{A})$ *such that* $F \subseteq h^{-1}D_{\mathfrak{B}}$*. Then there is a surjective matrix homomorphism* $g : \mathfrak{A}/F \to \mathfrak{B}$ *such that* $h = g \circ n$ *where* n *is the natural map of* F.

Notice that when applied to the models of the Birkhoff's system $\boldsymbol{\mathcal{B}}$, the above theorem becomes the first isomorphism and the homomorphism theorems of universal algebra ([7, Theorem 6.12.])

Definition 2.70 *A* K*-deductive system* $\mathcal{S}$ *is R-protoalgebraic, if for every* $\mathcal{S}$*-matrix* $\mathfrak{A}$*, the operator* Ω_R *on* $Fi_{\mathcal{S}}(\mathfrak{A})$ *is monotone.*

Similarly, $\mathcal{S}$ *is Rk-protoalgebraic if* Ω_{Rk} *is monotone on* $Fi_{\mathcal{S}}(\mathfrak{A})$*, for every* $\mathcal{S}$*-matrix* $\mathfrak{A}$.

By Proposition 2.48, if for every $R \in$ K, $\mathcal{S}$ is R-protoalgebraic then it is protoalgebraic. In the next chapter we will give an example that the converse need not be true.

CHAPTER 3. REPRESENTATION OF EQUIVALENCE AND EQUALITY FOR K-DEDUCTIVE SYSTEMS

3.1 Introduction

It was proved in [4] that a 1-deductive system is protoalgebraic if and only if it has a so-called system *equivalence formulas* (Definition 3.3), or, equivalently, a (not necessary finite) system of *congruence formulas with parameters* (Definition 3.14). Similar result is claimed there for arbitrary k-deductive system, but the proof contains a gap and the result. as stated in [4, Theorem 13.2], is not true (see Example 3.1). Consequently, several results of [4, Section 13] are either incorrect or require a different argument.

The main idea of the incorrect proof of Theorem 13,2, of [4], can, however, be used in the proof of a different characterization of protoalgebraicicity of not only k-, but in general K-deductive systems; and also in the proof of a characterization of the protoalgebraic relativized to a predicate (Theorems 3.10, 3.11,3.12). This is one of the main goals of this chapter. We define *system of equivalence formulas with parameters* $\mathbf{z}$, Definition 3.3 and prove that a K-deductive system is protoalgebraic iff it has an equivalence system with parameters $\mathbf{z}$ (Theorem 3.10). A similar characterization (Theorem 3.12) of R-protoalgebraic K-deductive system, where $R \in$ K and a partial characterization of Rk-protoalgebraic K-deductive systems, where $k \leq \rho(R)$ is also presented (Theorem 3.12). Theorem 3.10 allows to correct, and extend, the content of [4, section 13]. In particular, the concepts of congruential and weakly congruential deductive systems, introduced in [4], now acquire relatives: congruential and weakly congruential systems with parameters $\mathbf{z}$ (Definition 3.24). It turns out, that the notions with and without parameters $\mathbf{z}$ coincide exactly for those K-deductive systems $\mathcal{S}$ whose classes of reduced models are closed under the operator S, i.e, systems $\mathcal{S}$ such that a submatrix of a reduced $\mathcal{S}$-matrix is also reduced.

In section 3.3, Theorem 3.22 we characterize protoalgebraic and R-protoalgebraic K-deductive system s as those having systems of so-called congruence formulas with parameters $\mathbf{z}$, where $\mathbf{z}$ is a sequence of variables of length closely associated with the arities of the predicate symbols of K (Definition 3.14).

Let the first-order language $\langle \Lambda, K \rangle$ be fixed. Let $\mathbf{z}$ denote a fixed sequence of variables defined as follows: If $\max\{\rho(R) : R \in \mathrm{K}\}$ exists, and in particular when K is finite, then $\mathbf{z} = \langle z_1, \ldots, z_n \rangle$, where $n = \max\{\rho(R) : R \in \mathrm{K}\} - 1$. (Thus if K has one unary predicate then $\mathbf{z}$ is the empty sequence.) Otherwise, $\mathbf{z}$ is the infinite sequence $\mathbf{z} = \langle z_1, z_2, \ldots \rangle$. If $\mathbf{z} = \langle z_1, \ldots, z_m \rangle$, $k \leq m$ (or $\mathbf{z} = \langle z_1, z_2, \ldots \rangle$) and $t \in Te$, then $\mathbf{z}[t/k]$ denotes the sequence $\langle z_1, \ldots, z_{k-1}, t, z_k, \ldots, z_m \rangle$ (or the sequence $\langle z_1, \ldots, z_{k-1}, t, z_k, \ldots, z_m \rangle$, respectively). We will write $R(\mathbf{z}[t/k])$ for

$$R\langle z_1, \ldots, z_{k-1}, t, z_k, \ldots, z_{\rho(R)-1} \rangle;$$

thus the notation $R(\mathbf{z}[t/k])$ assumes that not all, but only first $\rho(R) - 1$ variables of $\mathbf{z}$ are involved in $R(\mathbf{z}[x/k])$. By $\{x, y, \mathbf{z}\}$ we mean the union of $\{x, y\}$ and the set containing all variables z_i listed in $\mathbf{z}$.

3.2 Equivalence formulas in protoalgebraic K-deductive systems

Definition 3.1 *A K-formula of the form $R(t_1, \ldots, t_{\rho(R)})$, where $R \in K$ and $t_1, \ldots, t_n$ are terms, is called an* R-**formula.**

Definition 3.2 *Let $\mathcal{S}$ be a K-deductive system, let $R \in K$ and let $t_1, \ldots, t_{\rho(R)}$ be terms, with $\mathrm{Var}(t_i) \subseteq \{x, y, \mathbf{z}\}$.*

1. *The K-formula $\varphi(x, y, \mathbf{z}) = R(t_1, \ldots, t_n)$ is called an* $\mathcal{S}$-**reflexive** *formula with parameters* $\mathbf{z}$ *if*

$$\vdash_{\mathcal{S}} R(t_1, \ldots, t_n)(x, x, \mathbf{z}). \tag{3.1}$$

Let I is be a set and let $\Delta = \Delta(x, y, \mathbf{z}) = \{\Delta_i(x, y, \mathbf{z}) : i \in I\}$ be a set of K-formulas. Then

1. *The set Δ has the* **modus ponens property relative to** (R, k) *if*

$$\Delta(x, y, \mathbf{z}), R(\mathbf{z}[x/k]) \vdash_{\mathcal{S}} R(\mathbf{z}[y/k]). \tag{3.2}$$

2. *If for every $k \leq \rho(R)$, Δ has the modus ponens property relative to (R, k) then we say that Δ has the* **modus ponens property relative to** R.

3. *If for every $R \in K$, Δ has the modus ponens property relative to R then we say that Δ has the* **modus ponens property** *or that it is a* **system of modus ponens K-formulas with parameters z for $\mathcal{S}$**.

The key concepts of this section, which we now introduce, are partly motivated by the concept of a system of equivalence k-formulas in [4, Definition 13.1]. When the relational language K has more than just one predicate symbol, then in addition to the notion of a system of equivalence K-formulas we also can define the relativized notions of equivalence system.

Definition 3.3 *Let $\mathcal{S}$ be a K-deductive system and let $R \in K$.*

1. *A system $\Delta(x, y, \mathbf{z})$ of modus ponens formulas with parameters $\mathbf{z}$ for $\mathcal{S}$ such that each $\varphi \in \Delta$ is reflexive is called an* $\mathcal{S}$**-equivalence system with parameters** $\mathbf{z}$ *or just* **equivalence system for $\mathcal{S}$ with parameters.**

2. *A set of reflexive K-formulas with parameters $\mathbf{z}$ that has the modus ponens property relative to R is called an* $\mathcal{S}$**-equivalence system relative to R with parameters** $\mathbf{z}$ *or just* $\mathcal{S}$**-equivalence system relative to R with parameters.**

3. *A set of reflexive R-formulas with parameters $\mathbf{z}$ that has the modus ponens property relative to R is called an* $(R, \mathcal{S})$**-equivalence system with parameters** $\mathbf{z}$ *or* R**-equivalence system with parameters $\mathbf{z}$ for $\mathcal{S}$**, *or* $(R, \mathcal{S})$**-equivalence system with parameters.**

4. *If the parameters $\mathbf{z}$ do not occur in Δ, i.e., $\mathrm{Var}(\Delta_i) \subseteq \{x, y\}$ for each $\Delta_i \in \Delta$, then Δ is called, respectively, an* $\mathcal{S}$**-equivalence system,** $\mathcal{S}$**-equivalence system relative to R and** $(R, \mathcal{S})$**-equivalence system** *if it is a $\mathcal{S}$-equivalence system with parameters, $\mathcal{S}$-equivalence system relative to R with parameters, $(R, \mathcal{S})$-equivalence system with parameters, respectively.*

When $\mathcal{S}$ is known from the context, we just say "equivalence system with parameters $\mathbf{z}$, equivalence system relative to R with parameters $\mathbf{z}$, R-equivalence system *with parameters* $\mathbf{z}$*," respectively, for "*$\mathcal{S}$*-equivalence system with parameters* $\mathbf{z}$*,* $\mathcal{S}$*-equivalence system relative to* R *with parameters* $\mathbf{z}$*,* $(R, \mathcal{S})$*-equivalence system with parameters* $\mathbf{z}$*." We call an (R)equivalence system(with parameters* $\mathbf{z}$*) also a* **system of (*R*)-equivalence formulas (with parameters z)**. *We say that* $\mathcal{S}$ *has an equivalence system, an equivalence system relative to* R*, and an* R*-equivalence system, possibly with parameters* $\mathbf{z}$ *(or with parameters) when the respective systems for* $\mathcal{S}$ *exist.*

Notice that the notions defined in parts 2 and 3 of the above definition are not equivalent: Although every $(R, \mathcal{S})$-equivalence system is also an $\mathcal{S}$-equivalence system relative to R, the converse is not true. For example, consider $\mathrm{K} = \{R, T\}$, where both R and T are binary and let $\mathcal{S}$ be the K-deductive system (we leave Λ unspecified here) determined by the axiom $R(x, x)$ and the following rules:

$$R(x, y), R(x, z) \vdash_{\mathcal{S}} R(y, z)$$

$$R(x, y), R(z, x) \vdash_{\mathcal{S}} R(z, y)$$

$$R(x, y), T(x, z) \vdash_{\mathcal{S}} T(y, z)$$

$$R(x, y), T(z, x) \vdash_{\mathcal{S}} T(z, y).$$

By definition, the set $\{R(x, y)\}$ is an equivalence system relative to T, but it is not a T-equivalence system.

Proposition 3.4 *Let* $\mathcal{S}$ *be a* K*-deductive system. Let* $\Delta(x, y, \mathbf{z})$ *a set of* K*-formulas and let* $R \in K$*.*

1. Δ *is an* $\mathcal{S}$*-equivalence system iff it is an* $\mathcal{S}$*-equivalence system relative to every* $R \in K$*.*

2. *A set* Δ *of* K*-formulas with parameters is an* $\mathcal{S}$*-equivalence system if and only if* $\Delta = \bigcup_{R\in K} \Delta^R$*, where for every* $R \in K$*,* Δ^R *is an* $\mathcal{S}$*-equivalence system with parameters for* R*.*

3. *If* Δ *is an* $(R, \mathcal{S})$*-equivalence system then it also is an* $\mathcal{S}$*-equivalence system relative to* R*.*

4. *If for every* $R \in K$ *there is an* $(R, \mathcal{S})$*-equivalence system then there also is an* $\mathcal{S}$*-equivalence system.*

Proof. Immediate from definition. □

In view of the above lemma, part 2, an $\mathcal{S}$-equivalence system is a union of sets Δ^R, where each Δ^R is a $\mathcal{S}$-equivalence system with parameters for R. Of special interest is the case when all Δ^R can be chosen finite.

Definition 3.5 *An* $\mathcal{S}$*-equivalence system is called* **finitary** *if* $\Delta = \bigcup_{R\in K} \Delta^R$*, where for every* $R \in K$*,* Δ^R *is a finite* $\mathcal{S}$*-equivalence system with parameters for* R*.*

Proposition 3.6 1. *If a* K*-deductive system has an* R*-equivalence system with parameters* $\mathbf{z}$*, then it also has a finite* R*-equivalence system with parameters* $\mathbf{z}$*.*

2. *If a* K*-deductive system has an equivalence system with parameters* $\mathbf{z}$*, then it also has a finitary equivalence system with parameters* $\mathbf{z}$*.*

Proof. It suffices to show that if Δ is a system of modus ponens formulas with parameters $\mathbf{z}$ relative to R, then there is a finite subset Δ^R that is also a system of modus ponens formulas with parameters $\mathbf{z}$ relative to R. For if every formula in Δ is reflexive, then Δ^R is a system of modus ponens formulas with parameters $\mathbf{z}$ relative to R and if in addition each formula in Δ is reflexive, then so is every formula in Δ^R. So if Δ is an R-equivalence system with parameters $\mathbf{z}$ then so is Δ^R and 1. is proved. If Δ is an $\mathcal{S}$-equivalence system with parameters $\mathbf{z}$, then for every $R \in$ K we have a finite equivalence system with parameters $\mathbf{z}$ Δ^R for R and 2. is proved.

So suppose that Δ has the modus ponens property relatively to an $R \in$ K. Then

$$\Delta(x, y, \mathbf{z}), R(x, \mathbf{z}) \vdash_{\mathcal{S}} R(y, \mathbf{z}).$$

Since $\mathcal{S}$ is finitary, it follows that there is a finite subset $\Delta^R \subseteq \Delta$ such that

$$\Delta^R(x, y, \mathbf{z}), R(x, \mathbf{z}) \vdash_{\mathcal{S}} R(y, \mathbf{z}).$$

□

Proposition 3.7 *If K is finite, then a K-deductive system has an equivalence system with parameters* $\mathbf{z}$ *iff it has a finite equivalence system with parameters* $\mathbf{z}$.

Proof. Immediate from Proposition 3.6 and Definition 3.5. □

A deductive system S which has a system of equivalence formulas with parameters $\mathbf{z}$ is called *an equivalence theory.* ([4]).

The above notion of an $\mathcal{S}$-equivalence system for R and also the notion of the $(R, \mathcal{S})$-equivalence system can be relativized to $k \leq \rho(R)$.

Definition 3.8 *Let $R \in K$ and $k \in \{1, \ldots, \rho(R)\}$.*

1. *A set Δ of reflexive K-formulas that has the modus ponens property relative to (R, k) is called a* $\mathcal{S}$**-equivalence system of K-formulas with parameters $\mathbf{z}$ relative to** (R, k).

2. *A set Δ of reflexive R-formulas that has the modus ponens property relative to (R, k) is called a* **system of (R,S)-equivalence formulas with parameters z relative to** (R, k) *or a* R**-equivalence system with parameters z relative to** (R, k).

Of course, if K is finite, then an $\mathcal{S}$-equivalence system is finitary if and only if it is finite. Theorem 13.2 in [4] reads:

A k-deductive system $\mathcal{S}$ is protoalgebraic iff it has a finite system of equivalence formulas.

The proof of the necessity of this condition is, however, incorrect. It proves only that for every k-deductive system $\mathcal{S}$ there exists a finite system of equivalence formulas with parameters, Theorem 3.9 below. The idea of this argument can also be used to prove the relativized version of Theorem 3.9 (Theorem 3.11 below). Example 3.1 shows that Theorem 13.2. of [4] is false.

Theorem 3.9 *Let $\mathcal{S}$ be a protoalgebraic K-deductive system. Then there exists a finitary $\mathcal{S}$-equivalence system. If K is finite, then there exists a finite $\mathcal{S}$-equivalence system.*

Proof.

Assume that $\mathcal{S}$ is protoalgebraic. Let

$$T := \{R(\vec{t}(x, y, \mathbf{z})) : \vdash_{\mathcal{S}} R(\vec{t}(x, x, \mathbf{z})) : R \in \mathrm{K}, \vec{t} \in \mathrm{Te}^{\rho(R)}\}.$$

First observe, that T is an $\mathcal{S}$-filter on $\mathbf{Te}(x, y, \mathbf{z})$. For if $T \vdash_{\mathcal{S}} S(\vec{t}(x, y, \mathbf{z}))$, then, by structurality of $\mathcal{S}$ and by the definition of T, $\vdash_{\mathcal{S}} S(\vec{t}(x, x, \mathbf{z}))$. By definition of T, the K-term $S(\vec{t}(x, y, \mathbf{z}))$ in T.

We now claim, that $\langle x, y\rangle \in \Omega^{\mathbf{Te}(x,y,\mathbf{z})}(T)$. To show this we use Proposition 2.51. For let $R(\vec{t}(u,\vec{v}))$ be an arbitrary K-term, where $\vec{v} = v_1, \dots, v_m$ is some sequence of variables. Suppose that for some choice of elements $v_j = v_j(x, y, \mathbf{z})$ of $\mathrm{Te}(x, y, \mathbf{z})$, where $j = 1, \dots m$ we have that $R\vec{s}(x, v_1(x, y, \mathbf{z}), \dots, v_m(x, y, \mathbf{z})) \in T$. To show our claim we need to prove that $R(\vec{s}(y, v_1, \dots, v_m)) \in T$, where v_j stands for $v_j(x, y, \mathbf{z})$. Let $\vec{t}(x, y, \mathbf{z}) := \vec{s}(y, v_1, \dots, v_m)$. By assumption that

$$R(\vec{s}(x, v_1(x, y, \mathbf{z}), \dots, v_m(x, y, \mathbf{z}))) \in T,$$

we have $\vdash_{\mathcal{S}} R(\vec{t}(x, x, \mathbf{z}))$, which implies that $R(\vec{t}(x, y, \mathbf{z})) \in T$ and finishes the proof that $\langle x, y\rangle \in \Omega^{\mathbf{Te}(x,y,\mathbf{z})}(T)$.

Now since $\mathcal{S}$ is protoalgebraic, for every $k = 1, \dots, \rho(R)$, the pair $\langle x, y\rangle$ is also in $\Omega^{\mathbf{Te}(x,y,\mathbf{z})}(T \cup \{R(\mathbf{z}[x/k])\})$. But $R(\mathbf{z}[x/k]) \in (T \cup \{R(\mathbf{z}[x/k])\})$, hence $R(\mathbf{z}[y/k]) \in (T \cup \{R(\mathbf{z}[x/k])\})$. So T is an $\mathcal{S}$-equivalence system with parameters $\mathbf{z}$. The theorem now follows from Propositions 3.6 and 3.7. □

Modifying an argument used in the proof of [4, Theorem 13.2.], we get the following corollary.

Theorem 3.10 Representation Theorem for protoalgebraic K-deductive systems

1. *A K-deductive system $\mathcal{S}$ is protoalgebraic iff there is a finitary $\mathcal{S}$-equivalence system with parameters.*

2. *A $\vec{k}$-deductive system $\mathcal{S}$ is protoalgebraic iff there is a finite $\mathcal{S}$-equivalence system with parameters.*

Proof. Let Δ be a system of $\mathcal{S}$-equivalence formulas with parameters $\mathbf{z}$. Let $\mathbf{A}$ be a $\Lambda-$algebra and let $F, G \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ such that $F \subseteq G$. Suppose $R(\vec{a}) \in G$ and

$\vec{a} \equiv \vec{b}(\Omega(F)^{\rho(R)})$ (recall that this means that sequences $\vec{a}, \vec{b}$ have length $\rho(R)$ and for every $i \leq \rho(R)$, $a_i \equiv b_i(\Omega(F))$). Let us fix $\varphi \in \Delta$ and assume that φ is of the form $S(\vec{t}(x, y, \mathbf{z}))$, for some $S \in$ K and some terms $t_1, \ldots, t_{\rho(S)}$. For every $i = 1, \ldots, \rho(R)$, for every $k = 1, \ldots, \rho(S)$ and for all $\vec{c} \subseteq A$, we have

$$t_k^{\mathbf{A}}(a_i, a_i, \vec{c}) \equiv_{\Omega(F)} t_k^{\mathbf{A}}(a_i, b_i, \vec{c}).$$

By the reflexivity condition 3.1, $S(t_1^{\mathbf{A}}(a_i, a_i, \vec{c}), \ldots, t_{\rho(S)}^{\mathbf{A}}(a_i, a_i, \vec{c})) \in F$. It follows that also $S(t_1^{\mathbf{A}}(a_i, b_i, \vec{c}), \ldots, t_{\rho(S)}^{\mathbf{A}}(a_i, b_i, \vec{c})) \in F \subseteq G$, by compatibility of $\Omega(F)$ with F. Thus $\Delta(a_i, b_i, \vec{c}) \subseteq G$. Now, Δ is a set of modus ponens formulas with parameters $\mathbf{z}$ for $\mathcal{S}$. Hence for every $i = 1, \ldots, \rho(R)$, if

$$R(a_1, \ldots, a_i, b_{i+1}, \ldots, b_{\rho(R)}) \in G,$$

then also

$$R(a_1, \ldots, b_i, b_{i+1}, \ldots, b_{\rho(R)}) \in G.$$

Since $R(\vec{a}) \in G$, it follows that $R(\vec{b}) \in G$. We have shown that $\Omega(F)$ is compatible with G. Therefore $\Omega(F) \subseteq \Omega(G)$. This shows that $\mathcal{S}$ is protoalgebraic. The reverse implication follows from Theorem 3.9. $\square$

We now turn to the protoalgebraicity relativized to R.

Theorem 3.11 *Let $\mathcal{S}$ be a K-deductive system and let $R \in$ K. Then $\mathcal{S}$ is R-protoalgebraic iff it has a finite system of R-equivalence formulas.*

Proof. Let $R \in$ K and assume that $\mathcal{S}$ is R-protoalgebraic. Let

$$T := \{R(\vec{t}(x, y, \mathbf{z})) \in \mathrm{Fm}_{\mathrm{K}} : \vdash_{\mathcal{S}} R(\vec{t}(x, x, \mathbf{z})), \vec{t} \in \mathrm{Te}^{\rho(R)}\}.$$

(This set T differs from the one we used in the proof of Thm.3.9 in this that here R is fixed and there it ranged over the set K.) First observe, that if $T \vdash_{\mathcal{S}} R(\vec{t}(x, y, \mathbf{z}))$, then, by structurality of $\mathcal{S}$ and by the definition of T, $R(\vec{t}(x, y, \mathbf{z})) \in T$.

Similarly as in the proof of Theorem 3.9, we now claim, that

$$\langle x, y \rangle \in (\Omega_R^{\mathbf{Te}(x,y,\mathbf{z})})(T).$$

For let $R(\vec{t}(u, \vec{v})$ be an arbitrary K-term, where $\vec{v} = v_1, \ldots, v_m$ is some sequence of variables. Suppose that for some choice of elements $v_j(x, y, \mathbf{z})$ of $\mathrm{Te}(x, y, \mathbf{z})$, where $j = 1, \ldots m$ we have that $R(\vec{t}(x, v_1(x, y, \mathbf{z}), \ldots, v_m(x, y, \mathbf{z})) \in T$. Let $\vec{s}(x, y, \mathbf{z}) := \vec{t}(y, v_1(x, y, \mathbf{z}), \ldots, v_m(x, y, \mathbf{z}))$. To prove that $\langle x, y \rangle \in \Omega^{\mathbf{Te}(x,y,\mathbf{z})}(T)$, we need to show that also $R(\vec{s}(x, y, \mathbf{z})\rangle \in T$. But by structurality and the assumption that

$$R(\vec{t}(x, v_1(x, y, \mathbf{z}), \ldots, v_m(x, y, \mathbf{z})) \in T$$

we have $\vdash_{\mathcal{S}} R(\vec{s}(x, x, \mathbf{z})\rangle$, which implies that $R(\vec{s}(x, y, \mathbf{z})\rangle \in T$ and finishes the proof that $\langle x, y \rangle \in \Omega_R^{\mathbf{Te}(x,y,\mathbf{z})}(T)$.

Now since $\mathcal{S}$ is R-protoalgebraic, for every $k = 1, \ldots, \rho(R)$, the pair $\langle x, y \rangle$ is also in $\Omega_R^{\mathbf{Te}(x,y,\mathbf{z})}(T \cup \{R(\mathbf{z}[x/k]\})$. But $R(\mathbf{z}[x/k] \in (T \cup \{R(\mathbf{z}[x/k]\})$, hence $R(\mathbf{z}[y/k] \in (T \cup \{R(\mathbf{z}[x/k]\})$. Therefore, there is some $n = n(i, k)$ and some R-formulas $\varphi_j^{R,k} = \varphi_j^{R,k}(x, y, \mathbf{z})$ for $j = 1, \ldots, n$ such that

$$\varphi_1(x, y, \mathbf{z}), \ldots, \varphi_j(x, y, \mathbf{z}), R(\mathbf{z}[x/k]) \vdash_{\mathcal{S}} R(\mathbf{z}[y/k]). \tag{3.3}$$

Also, by the definition of T,

$$\vdash_{\mathcal{S}} \varphi_j(x, x, \mathbf{z}). \tag{3.4}$$

Now the number $\rho(R)$ is finite, so the union $\Delta^R = \bigcup_{k \leq \rho(R)} \{\varphi_j^{R,k} : j = 1, \ldots m_k\}$ is finite. It follows that Δ is a finite $(R, \mathcal{S})$-equivalence system.

To prove that the existence of an $(R, \mathcal{S})$-equivalence system Δ^R implies that $\mathcal{S}$ is R-protoalgebraic, let $F \subseteq G$ be two $\mathcal{S}$-filters on an algebra $\mathbf{A}$. We show that $\Omega_R(F)$ is -compatible with G. For let $R(\vec{a}) \in G$ and assume that $\vec{a} \equiv \vec{b}(\Omega_R(F))$. Let $\varphi \in \Delta^R$ be arbitrary but fixed. Then φ is of the form $R(\vec{t}(x, y, \mathbf{z})$, for some sequence of terms t of length $\rho(R)$. For every $i = 1, \ldots, \rho(R)$, for every $k = 1, \ldots, \rho(R)$ and for every sequence $\vec{c}$ of elements of A, $t_k(a_i, a_i, \vec{c}) \equiv t_k(a_i, b_i, \vec{c})(\Omega_R(F))$, since $\Omega_R(F)$ is a congruence. By reflexivity of Ω_R, $R(t_k(a_i, a_i, \vec{c})) \in F$ and by the compatibility of $\Omega_R(F)$ with F, $R(t_k(a_i, b_i, \vec{c})) \in F$, for every $k \leq \rho(R)$. Therefore $\Delta^R(a_i, b_i, \vec{c}) \subseteq F \subseteq G$. Since $R(\vec{a}) \in G$ and Δ^R has the modus ponens property 3.2, we can prove by induction that also $R(\vec{b}) \in G$. This shows that $\Omega_R(F)$ is compatible with G. Hence $\Omega_R(F) \subseteq \Omega_R(G)$, which finishes the proof that $\mathcal{S}$ is R-protoalgebraic. □

Theorem 3.11 above gives necessary and sufficient condition for a K-deductive system $\mathcal{S}$ to be protoalgebraic. We are not aware of any similar condition characterizing the (R, k)-protoalgebraicity: a sufficient condition on (R, k)-protoalgebraicity which can be obtained using the method of proof of Theorem 3.11 is strictly stronger than the necessary condition obtained this way.

Theorem 3.12 *Let $\mathcal{S}$ be a K-deductive system, let $R \in K$ and let $k \leq \rho(R)$.*

1. *If $\mathcal{S}$ is (R, k)-protoalgebraic, then there is a finite system of $(R,\mathcal{S})$-equivalence formulas for (R, k).*

2. *If there is a finite system $\Delta^{(R,k)}$ of $(R,\mathcal{S})$-equivalence formulas for (R, k) such that every $\varphi \in \Delta^{(R,k)}$ is of the form*

$$R(t_1(x, \mathbf{z}), \ldots, t_{k-1}(x, \mathbf{z}), t_k(x, y, \mathbf{z}), t_{k+1}(x, \mathbf{z}), \ldots, t_{\rho(R)}(x, \mathbf{z}), \tag{3.5}$$

   then $\mathcal{S}$ is (R, k)-protoalgebraic.

Proof. To show that the existence of a finite (R,S)-equivalence system of the form 3.5. above implies that $\mathcal{S}$ is (R,k)-protoalgebraic, let $\mathfrak{A}$ be a model of $\mathcal{S}$, let $F \subseteq G$ be two $\mathcal{S}$-filters on $\mathfrak{A}$. In order to show that $\mathcal{S}$ is (R,k)-protoalgebraic, it suffices to show that $\Omega_{(R,k)}(F)$ is (R,k)-compatible with G . So let $\langle a,b\rangle \in \Omega_{(R,k)})$ and suppose that $\langle c_1,\ldots,c_{k-1},a,c_{k+1},\ldots,c_{\rho(R)}\rangle \in G$. Suppose that we have a finite (R,k)-equivalence system $\Delta^{(R,k)}$ and that $\langle a,b\rangle \in \Omega_{(R,k)}$. Hence also, for every $\varphi(x,y,\mathbf{z}) \in \Delta^{(R,k)}$, $\langle t_k(a,a,\vec{c}),t_k(a,b,\vec{c})\rangle \in \Omega_{(R,k)}(F)$. By (3.1),

$$R(t_1(a,\vec{c}),\ldots,t_{k-1}(a,\vec{c}),t_k(a,a,\vec{c}),t_{k+1}(a,\vec{c}),\ldots,t_{\rho(R)}(a,\vec{c})) \in F$$

and therefore also, by (R,k)-compatibility of $\Omega_{(R,k)}$ with F,

$$R(t_1(a,\vec{c}),\ldots,t_{k-1}(a,\vec{c}),t_k(a,a,\vec{c}),t_{k+1}(a,\vec{c}),\ldots,t_{\rho(R)}(a,\vec{c})) \in F \subseteq G,$$

using the fact that the formulas in $\Delta^{(R,k)}$ are of the special form. But then, by (3.2), $\langle c_1,\ldots,c_{k-1},b,c_{k+1},\ldots,c_{\rho(R)}\rangle \in G_R$, as desired.

For the proof of 1. let us fix $R \in$ K and $k = 1,\ldots,\rho(R)$ and assume that $\mathcal{S}$ is (R,k)-protoalgebraic. Let

$$T_k := \{R(\vec{t}(x,y,\mathbf{z})) \in \mathrm{Fm_K} : \vdash_{\mathcal{S}} R(\vec{t}(x,x,\mathbf{z})), \vec{t} \in \mathrm{Te}^{\rho(R)}\}.$$

Similarly as in the proof of Theorems 3.9 and 3.11 we can show that the pair $\langle x,y\rangle \in \Omega_{(R,k)}(T)$ and by monotonicity of $\Omega_{(R,k)}$ it is also in $(\Omega_{(R,k)}{}^{\mathbf{Te}(x,y,\mathbf{z})}(T \cup \{R(\mathbf{z}[x/k])\}))$. But $\mathbf{z}[x/k] \in (T \cup \{R(\mathbf{z}[x/k])\})$, hence

$$R(\mathbf{z}[y/k]) \in (T \cup \{R(\mathbf{z}[x/k])\}).$$

Therefore, there are some $n = n(i,k)$ and some K-formulas $\varphi_j(x,y,\mathbf{z}) = \varphi_j^{(R,k)}(x,y,\mathbf{z})$ for $j = 1,\ldots,m$, such that

$$\varphi_1(x,y,\mathbf{z}),\ldots,\varphi_j(x,y,\mathbf{z}), R(\mathbf{z}[x/k]) \vdash_{\mathcal{S}} R(\mathbf{z}[y/k]). \tag{3.6}$$

Also, by the definition of T,

$$\vdash_{\mathcal{S}} \varphi_j(x, x, \mathbf{z}) \tag{3.7}$$

and the formulas $\varphi_j(x, y, \mathbf{z})$ are all of the form $R(\vec{t})$ where $\vec{t}$ are sequences of terms of length $\rho(R)$. These formulas form therefore an (R,S)-equivalence system for (R, k) with parameters $\mathbf{z}$. As we mentioned above, a system which is (R, k)-protoalgebraic for every $k \leq \rho(R)$ must be k-protoalgebraic, but we do not expect the converse to be, in general, true. We would like to be able to characterize the (R,k)-protoalgebraic K-deductive system in a manner similar to the characterization of R-protoalgebraic K-deductive system above. So we ask the following

Open questions:

1. Is there a characterization of (R, k)-protoalgebraic K-deductive system similar to the one given in Theorems 3.10, 3.11 for protoalgebraic and R-protoalgebraic K-deductive systems?

2. If $\mathcal{S}$ is (R, k)-protoalgebraic for every $k \leq \rho(R)$, does it follow that $\mathcal{S}$ is R-protoalgebraic?

Notice that when K has one predicate symbol which is unary, then the answer to both questions are obviously positive. For in this case Theorems 3.10, 3.11 and 3.12 coincide. Notice also, that in all these theorems we may omit the restriction on the equivalence systems that they involve the parameters, i.e., for example a 1-deductive system $\mathcal{S}$ is protoalgebraic iff it has a finite system of equivalence formulas (without parameters). This is the content of [4, Thm.13.2] for the special case of 1-deductive systems.

Corollary 3.13 ([4, Thm.13.2] for 1-deductive systems) *A* 1*-deductive system* $\mathcal{S}$ *is protoalgebraic iff it has a finite system of equivalence* 1*-formulas.*

Proof. It follows from Theorem 3.10 that $\mathcal{S}$ is protoalgebraic iff it has a finitary equivalence system with parameters $\mathbf{z}$, where $\mathbf{z}$ is the sequence of variables z_i of length $1 - 1$, i.e., $\mathbf{z}$ is the empty sequence of parameters. Hence there is a finitary $\mathcal{S}$-equivalence system without parameters. But as we already observed, in case that K is finite, the existence of a finitary system of equivalence formulas is equivalent to the existence of a finite system of equivalence formulas. □

The question remains, whether also for $K \neq 1$, protoalgebraic K-deductive system must have a finite system of equivalence formulas without parameters. This question is equivalent to the question whether the existence of a finite system of equivalence formulas with parameters implies the existence of a finite system of equivalence formulas without parameters. A relativized question asks if the existence of a finite system of R -equivalence formulas with parameters implies the existence of such formulas without parameters. (We do not consider here this question relativized to (R,k).) The following example shows that the answer to this last question is negative, even in the simplest case that K has only one predicate, which is binary. Since there is only one predicate in K, the same example proves that the answer to the first question is negative.

Example 3.1 Let Λ be the similarity type of one binary operation and let K be the relational language consisting of one binary predicate R. Since R is the only predicate symbol of K, the notions of R-protoalgebraic and protoalgebraic coincide as also do the notions of R-equivalence system for $\mathcal{S}$ (with parameters) and of equivalence system for $\mathcal{S}$ (with parameters). The result of the operation of Λ on terms t and

s will be written as juxtaposition. Let $\mathcal{S}$ be the 2-deductive system given by the following axiom and rules:

$$\vdash R(x,x) \tag{3.8}$$

$$R(xz,yz), R(x,z) \vdash R(y,z) \tag{3.9}$$

$$R(xz,yz), R(z,x) \vdash R(y,z) \tag{3.10}$$

The system consisting of one 2-formula $R(xz,yz)$ forms an R-equivalence system with parameter z for $\mathcal{S}$. Thus by Thm. 3.11, $\mathcal{S}$ is R-protoalgebraic.

We claim, however, that $\mathcal{S}$ does not have an R- equivalence system without parameter, i.e., that no set of 2-formulas in variables x and y forms an equivalence system.

To see this, let $\Delta = \Delta(x,y)$ be some set of pairs of terms in variables x, y and let $\tilde{\Delta} := (\mathrm{Cn}_{\mathcal{S}}\Delta) \cap \mathrm{Te}(x,y)$. Note, that $\tilde{\Delta}$ is closed under rules 3.9 and 3.10.

We now claim that

$$\mathrm{Cn}_{\mathcal{S}}(\Delta, R(x,z)) = \tilde{\Delta} \cup \{R(x,z)\} \cup \{\langle t,t\rangle : t \in \mathrm{Te}\}. \tag{3.11}$$

Let RHS denote the right hand side of the above equation. It is clear that RHS is included in the left hand side and that $\Delta \cup \{R(x,z)\}$ is included in RHS.

It suffices to show that RHS is an $\mathcal{S}$ theory. It is clearly closed under the rule 3.8. We now show that it is also closed under rules 3.9 and 3.10. For the rule 3.9 suppose that $R(tu,su), R(t,u)$ is in R. We want to show that then also $R(s,u)$ is in this set. This is obvious, if $t = s$, which is the case when $tu = su$ as terms. Note that $\langle tu, su\rangle \neq \langle x,z\rangle$. So we may assume that $R(tu,su) \in \tilde{\Delta}$. In particular, the only variables occurring in t, u, s are x, y. Therefore, if $R(t, u)$ is contained in the third component of RHS then it also is contained in $\tilde{\Delta}$. It follows that $R(t, u) \in \tilde{\Delta}$ and therefore $R(s, u) \in \tilde{\Delta}$, since $\tilde{\Delta}$ is closed under rule 3.3. This verifies our claim for 3.9. The proof for 3.10 is similar: suppose that $R(tu, su), R(u, t)$ is in RHS. If $t = s$, $R(u, s) \in$ RHS by assumption. So assume that $t \neq s$. Then $R(tu, su) \in \tilde{\Delta}$ and therefore $\mathrm{Var}(t, u, s) \subseteq \{x, y\}$. In particular, $t \neq z$ and therefore $R(u, t) \neq R(x, z)$. So $R(t, u) \in \tilde{\Delta}$ or $t = u$, so in any case $R(t, u) \in \tilde{\Delta}$. It follows that $R(u, s) \in \tilde{\Delta} \subseteq$ RHS. This verifies that $\mathcal{S}$ is an $\mathcal{S}$-theory and therefore the equation 3.11.

Now $R(y, z)$ is not contained in RHS and therefore is not in $\mathrm{Cn}_{\mathcal{S}}(\Delta, R(x, z))$.

This shows that Δ is not a modus ponens system. Since Δ was an arbitrary set of 2-formulas in variables x, y, it follows that $\mathcal{S}$ does not have an equivalence system without parameter z. □

3.3 Systems of congruence K- and R-formulas

Definition 3.14 *Let $\mathcal{S}$ be a K-deductive system. Let* $\mathbf{z}$ *be as defined at the beginning of section 3.2. Let* $\mathbf{w}$ *be some sequence of variables, possibly infinite.*

1. *We say that a set $\Delta(x, y, \mathbf{z}, \mathbf{w})$ of K-formulas has the* $\mathcal{S}$**-replacement property** *(relative to x and y), or just* $\mathcal{S}$**-replacement property** *if for every term $t(x, \mathbf{v})$, where* $\mathbf{v}$ *is a sequence of variables,*

$$\Delta(x, y, \mathbf{z}, \mathbf{w}) \vdash_{\mathcal{S}} \Delta(t(x, \mathbf{v}), t(y, \mathbf{v}), \mathbf{z}, \mathbf{w}). \tag{3.12}$$

2. *An $\mathcal{S}$-equivalence system with parameters* $\mathbf{z}$ *Δ is called an* $\mathcal{S}$**-congruence system with parameters z and w** *if it has the replacement property 3.12.*

If such a system exists, then we also say that $\mathcal{S}$ **has a congruence system** *with parameters* $\mathbf{z}$.

3. *An* $(R, \mathcal{S})$*-equivalence system with parameters* $\mathbf{z}$ *and* $\mathbf{w}$ *is called a* **(R,S)-congruence system with parameters z and w** *or a* R**-congruence system with parameters z and w for** $\mathcal{S}$ *if it has the replacement property 3.12.*

4. *If for any of the above systems the sequence* $\mathbf{w}$ *is empty, we say that it is a congruence system (*$\mathcal{S}$*-congruence system,*$(R, \mathcal{S})$*-congruence system) with parameters* $\mathbf{z}$.

Theorem 3.15 *Let* $\mathcal{S}$ *be a* K*-deductive system and let* $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ *be an* $\mathcal{S}$*-matrix.*

1. *Let* $R \in K$*. If* Δ^R *is an* R*-congruence system for* $\mathcal{S}$ *with parameters* $\mathbf{z}$ *and* $\mathbf{w}$*, then*

$$a \equiv b\,(\Omega_R(F)) \text{ iff } (\Delta^R)^{\mathbf{A}}(a, b, \vec{d}) \subseteq F$$

for all sequences $\vec{d}$ *of elements of* A *of the length equal to the sum of lengths of* $\mathbf{z}$ *and* $\mathbf{w}$.

2. *If* Δ *is a congruence system for* $\mathcal{S}$ *with parameters* $\mathbf{z}$ *and* $\mathbf{w}$*, then*

$$a \equiv b\,(\Omega(F)) \text{ iff } \Delta^{\mathbf{A}}(a, b, \vec{d}) \subseteq F$$

for all sequences $\vec{d}$ *of elements of* A *of length equal to the sum of lengths of* $\mathbf{z}$ *and* $\mathbf{w}$.

Proof. Let $\mathcal{S}$ and $\mathfrak{A}$ be as in the statement of the theorem. First observe, that if $\langle a, b \rangle \in \Omega(F)$ then since $\Omega(F)$ is a congruence, also $\langle t(a, a, \vec{d}), t(a, b, \vec{d}) \rangle \in \Omega(F)$.

Similarly, for $R \in$ K, $\langle a, b\rangle \in \Omega_R(F)$ implies $\langle t(a, a, \vec{d}), t(a, b, \vec{d})\rangle \in \Omega_R(F)$ for every term $t(x, y, \mathbf{z})$.

Now suppose that $X(x, y, \mathbf{z})$ is a system of reflexive formulas with parameters $\mathbf{z}$. Then in particular, for every $\varphi \in X$, $\varphi(a, a, \vec{d}) \in F$. If in addition X has the replacement property relative to an $R \in$ K, then for every $\varphi \in X$ of the form $\varphi = R(\vec{t}(x, y, \mathbf{z})$, we have that $\varphi(a, b, \vec{d}) \in F$, too. Now an R-congruence system Δ^R with parameters $\mathbf{z}$ has the replacement property relative to R and also every $\varphi \in \Delta^R$ is an R-formula, so by the above argument, $\Delta^R(a, b, \vec{d}) \subseteq F$, for every sequence of elements $\vec{d}$ of the same length as $\mathbf{z}$. An $\mathcal{S}$-congruence system Δ with parameters $\mathbf{z}$ has the replacement property with respect to every $R \in$ K. Also, for every $\varphi \in \Delta$, φ is an R-formula, for some $R \in K$ and $\Omega(F) = \bigcap_{R\in \text{ K}} \Omega_R(F)$. Hence if $\langle a, b\rangle \in \Omega(F)$ and $\varphi \in \Delta$, then for some $R \in$ K, φ is an R-formula and $\langle a, b\rangle \in \Omega_R(F)$. The above argument implies that $\varphi(b, a, \vec{d}) \in F$, and whence $\Delta(a, b, \vec{d}) \subseteq F$, for all $\vec{d}$.

For a set X of K-formulas define the following relation on A:

$$\langle a, b\rangle \in \Theta_X \text{ iff } \Delta^{\mathbf{A}}(a, b, \vec{d}) \subseteq F$$

for all sequences $\vec{d}$ of elements of A of the same length as $\mathbf{z}$.

To prove the theorem it remains to show that if Δ^R is an R-congruence system with parameters $\mathbf{z}$ and $\mathbf{w}$, then $\Theta_{(\Delta^R)} \subseteq \Omega_R(F)$ and that if Δ is a $\mathcal{S}$-congruence system with parameters $\mathbf{z}$ and $\mathbf{w}$, then $\Theta_\Delta \subseteq \Omega(F)$. In the series of lemmas 3.16–3.20 we will show that $\Theta_{(\Delta^R)}$ and Θ_Δ are congruences on $\mathbf{A}$ that are, respectively, R-compatible and compatible with F.

Lemma 3.16 *Let $\mathcal{S}$ be a K-deductive system and let $\Delta(x, y, \mathbf{z})$ be some set of K-formulas with parameters $\mathbf{z}$. Let $\mathfrak{A} = \langle \mathbf{A}, F\rangle$ be an $\mathcal{S}$-matrix. Then Θ_Δ is reflexive* *iff every $\varphi(x, y, \mathbf{z}) \in \Delta$ is a reflexive K-formula with parameters $\mathbf{z}$;*

Proof. Immediate by definition. $\square$

Lemma 3.17 *Let $\mathcal{S}$ be a K-deductive system and let $\mathfrak{A} = \langle \mathbf{A}, F\rangle$ be an $\mathcal{S}$-matrix.*

1. *Suppose that $R \in K$ and let $\Delta^R(x, y, \mathbf{z})$ be some set of R-formulas that has the replacement property with respect to R. Then the relation $\Theta_{(\Delta^R)}$ is symmetric.*

2. *If a set of K-formulas has the replacement property for $\mathcal{S}$ (for every $R \in K$), then Θ_Δ is symmetric.*

Proof. Let $\langle a, b\rangle \in \Theta_{(\Delta^R)}$. To show that $\langle b, a\rangle \in \Theta_{(\Delta^R)}$, we need to show that $\Delta^R(b, a, \vec{d}) \subseteq F$, for all sequences of elements of A $\vec{d}$ of length of $\mathbf{z}$. Let $R(\vec{t}(x, y, \mathbf{z}) \in \Delta^R$. We need to show that

$$R(\vec{t}(b, a, \vec{d})) \in F. \tag{3.13}$$

Since Δ^R is reflexive, we know that $R(\vec{t}(a, a, \vec{d})) \in F$. Notice that 3.13 follows by induction from the following claim.

Claim 1 *If*

$$R(t_1(a, a, \vec{d}), \ldots, t_k(a, a, \vec{d}), t_{k+1}(b, a, \vec{d}), \ldots, t_{\rho(R)}(b, a, \vec{d})) \in F \textit{ then}$$

$$R(t_1(a, a, \vec{d}), \ldots, t_{k-1}(a, a, \vec{d}), t_k(b, a, \vec{d}), \ldots, t_{\rho(R)}(b, a, \vec{d})) \in F.$$

To prove the claim, we use the assumption that $\langle a, b\rangle \in \Theta_{(\Delta^R)}$ and therefore also

$$\langle t_k(a, a, \vec{d}), t_k(b, a, \vec{d})\rangle \in \Theta_{(\Delta^R)}.$$

By definition of $\Theta_{(\Delta^R)}$ we know that for every sequence $\vec{e}$ of elements of A,

$$\Delta(t_k(a, a, \vec{d}), t_k(b, a, \vec{d}), \vec{e}) \subseteq F.$$

Let $\vec{e}$ be the sequence $\langle t_1(a,a,\vec{d}),\ldots,t_{k-1}(a,a,\vec{d}),t_{k+1}(,a,\vec{d}),\ldots,t_{\rho(R)}(b,a,\vec{d})\rangle$. The claim follows from the modus ponens property with respect to R. This proves the first statement of the lemma.

For the second statement, observe that if Δ is a $\mathcal{S}$-congruence system with parameters $\mathbf{z}$ and $\mathbf{w}$ for $\mathcal{S}$, then it has the replacement property with respect to every $R \in$ K and therefore the above proof can also be applied to show that Θ_Δ is symmetric.

Lemma 3.18 *Let $\mathcal{S}$ be a K-deductive system and let $\mathfrak{A} = \langle \mathbf{A}, F\rangle$ be an $\mathcal{S}$-matrix.*

1. *Suppose that $R \in K$ and let $\Delta^R(x,y,\mathbf{z})$ be some set of R-formulas that has the replacement property with respect to R. Then the relation $\Theta_{(\Delta^R)}$ is transitive.*
2. *If a set of K-formulas has the replacement property for $\mathcal{S}$ (for every $R \in K$), then Θ_Δ is transitive.*

Proof. Fix $R \in$ K and let X be either Δ^R or Δ. Then let Θ be Θ_X , i.e., Θ is either $\Theta_{(\Delta^R)}$ or Θ_Δ. Let $\langle a,b\rangle,\langle b,f\rangle \in \Theta$. Consider an R-formula $\varphi = R(t_1,\ldots,t_{\rho(R)}) \in \Theta_{(\Delta^R)}$, where for $i = 1,\ldots,\rho(R)$, $t_i = t_i(x,y,\mathbf{z})$. Let $s(x,\vec{v}) \in$ Te. Fix two sequences of elements of A: $\vec{c}$ of the same length as $\vec{v}$ and $\vec{d}$ of the same length as $\mathbf{z}$.lengths. We claim that $R(\vec{t}(s(a,\vec{c}),\vec{t}(f,\vec{c}),\vec{d})\rangle \in F$. For $k = 1,\ldots,\rho(R)$ let Let $t'_k := t_k(\vec{t}(a,\vec{c}),\vec{t}(b,\vec{c}),\vec{d})$ and $t''_k := t_k(\vec{t}(a,\vec{c}),\vec{t}(f,\vec{c}),\vec{d})$. Since $\langle b,f\rangle \in \Theta_{(\Delta^R)}$, we have, for every $k \leq \rho(R)$: $\langle t'_k,t''_k\rangle \in \Theta$ and therefore, by definition of Θ,

$$\Delta(t_k(\vec{t}(a,\vec{c}),\vec{t}(b,\vec{c}),\vec{d}),(t_k(\vec{t}(a,\vec{c}),\vec{t}(f,\vec{c}),\vec{d}) \subseteq F. \tag{3.14}$$

To complete proof of the claim we need to show that

$$R(t''_1,\ldots,t''_{\rho(R)}) \in F \tag{3.15}$$

Notice that $R(t'_1, \ldots, t'_{\rho(R)}) \in F$. Let us fix $k \leq \rho(R)$ and suppose that

$$R(t'_1, \ldots, t'_k, t''_{k+1}, \ldots, t''_{\rho(R)}) \in F. \tag{3.16}$$

Since X is R-compatible with F, it follows that

$$R(t'_1, \ldots, t'_{k-1}, t''_k, t''_{k_1}) \in F.$$

Proceeding by induction we get that $R(t''_1, \ldots, t''_{\rho(R)}) \in F$, i.e., the claim is true. Now if $X = \Delta^R$, it follows that $\Delta^R(a, f, \vec{d}) \subseteq F$, i.e., $\langle a, f \rangle \in \Theta_{(\Delta^R)}$ and the proof of 1. is finished.

If $X = \Delta$, notice the above argument is independent of R and therefore independent of the choice of $\varphi \in X$, i.e., for every $\varphi \in \Delta$, $\langle a, b \rangle, \langle b, f \rangle \in \Theta$ implies $\varphi(a, f, \vec{d}) \in F$, all $\vec{d}$. Hence $\langle a, f \rangle \in \Theta$.

$$R(\vec{t}(s(a, \vec{c}), \vec{t}(b, \vec{c}), \vec{d}) R(\vec{t}(s(a, \vec{c}), \vec{t}(f, \vec{c}), \vec{d}) \rangle.$$

Lemma 3.19 *Let $\mathcal{S}$ be a K-deductive system and let $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ be an $\mathcal{S}$-matrix.*

1. *Suppose that $R \in K$ and let $\Delta^R(x, y, \mathbf{z})$ be some set of R-formulas that has the modus ponens property with parameters with respect to R. Then the relation $\Theta_{(\Delta^R)}$ is compatible with F.*

2. *If a set of K-formulas has the replacement property for $\mathcal{S}$ (for every $R \in K$), then Θ_Δ is compatible with F.*

Proof. Immediate from definition of the modus ponens and compatibility properties.
□

Lemma 3.20 *Let $\mathcal{S}$ be a K-deductive system and let $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ be an $\mathcal{S}$-matrix.*

1. *Suppose that $R \in K$ and let $\Delta^R(x, y, \mathbf{z})$ be some set of reflexive R-formulas that has the replacement property with respect to R. Then the relation $\Theta_{(\Delta^R)}$ is a congruence.*

2. *If a set of reflexive K-formulas has the replacement property for $\mathcal{S}$ (for every $R \in K$), then Θ_Δ is a congruence.*

Proof. From the previous lemmas it follows that Θ_Δ and $\Theta_{(\Delta^R)}$ are equivalence relations. The fact that they are congruences now follows immediately from the replacement property. □

The theorem follows from lemmas 3.19 and 3.20. □

We will now apply Theorem 3.15 to protoalgebraic K-deductive systems.

Lemma 3.21 *Let $\mathcal{S}$ be a K-deductive system and let $R \in K$. Assume that $\Delta(x, y, \mathbf{z})$ is a set of K-formulas. Define*

$$\widehat{\Delta}(x, y, \mathbf{z}, \mathbf{w}) := \bigcup_{\mathbf{v}} \bigcup_{t(x,\mathbf{v}) \in \mathrm{Te}} \Delta(t(x, \mathbf{v}), t(y, \mathbf{v}), \mathbf{z}),$$

where the first union is indexed by the finite sequences of variables $\mathbf{v}$ and $\mathbf{w}$ is the infinite sequence of these variables. Then

1. *If Δ is an $\mathcal{S}$-equivalence system, then $\widehat{\Delta}$ is an $\mathcal{S}$-congruence system;*

2. *If Δ is an R-equivalence system for $\mathcal{S}$, then $\widehat{\Delta}$ is an R-congruence system for $\mathcal{S}$.*

Proof. By definition, $\widehat{\Delta}$ has the replacement property. If Δ is a set of reflexive formulas, then so is $\widehat{\Delta}$ and if Δ has the modus ponens property (relative to R), then since $\Delta \subsetneq \hat{\Delta}$, it follows that

$$\hat{\Delta}(x, y, \mathbf{z}), R(\mathbf{z}[x/k]) \vdash_{\mathcal{S}} R(\mathbf{z}[y/k],$$

for every $k \leq \rho(R)$. Therefore $\hat{\Delta}$ has the modus ponens property in this case. Thus if Δ is an $\mathcal{S}$-equivalence system with parameters $\mathbf{z}$, then so is $\hat{\Delta}$, and if Δ is an $(R, \mathcal{S})$-equivalence system with parameters $\mathbf{z}$, then so is $\hat{\Delta}$. The lemma follows. □

Theorem 3.22 (see Thm. 13.5 in[4]) *Let $\mathcal{S}$ be a protoalgebraic K-deductive system with a system $\Delta(x, y, \mathbf{z})$ of equivalence formulas with parameters $\mathbf{z}$. Then for every Λ-algebra $\mathbf{A}$, all $a, b \in A$, and every $F \in Fi_{\mathcal{S}}(\mathbf{A})$*

$$a \equiv b(\Omega(F)) \quad \textit{iff} \quad \Delta^{\mathbf{A}}(s^{\mathbf{A}}(a, \vec{c}), s^{\mathbf{A}}(b, \vec{c}), \vec{d}) \subseteq F \tag{3.17}$$

for all $s \in \bigcup_{\vec{y}} Te(x, \vec{y}), \vec{c} \subseteq A$, and all sequences $\vec{d}$ of elements of A of the same length as $\mathbf{z}$.

Proof. By Lemma 3.21 and Theorem 3.15. □

Theorem 3.9 and Lemma 3.21 imply that every protoalgebraic K-deductive system has a $\mathcal{S}$-congruence system with parameters $\mathbf{z}$ and $\mathbf{w}$. On the other hand, if $\mathcal{S}$ has a congruence system with parameters $\mathbf{z}$, then $\mathcal{S}$ is protoalgebraic. The following theorem should replace [4, Theorem 13.10.].

Theorem 3.23 *1. A K-deductive system is protoalgebraic iff it has a system of congruence formulas with parameters $\mathbf{z}$ and $\mathbf{w}$.*

2. A K-deductive system is R-protoalgebraic iff it has an R-congruence system with parameters $\mathbf{z}$ and $\mathbf{w}$.

Proof. That the condition is necessary follows from Theorem 3.9 and lemma 3.21.

By Proposition 3.6 and Theorem 3.10 we need to show that the existence of a $\mathcal{S}$-congruence system with parameters $\mathbf{z}$ and $\mathbf{w}$ implies the existence of an equivalence system with parameters $\mathbf{z}$ (but without parameters $\mathbf{w}$) and that the existence of an R-equivalence system with parameters $\mathbf{z}$ and $\mathbf{w}$ implies the existence of an R-equivalence system with parameters $\mathbf{z}$.

Assume that $\Delta(x, y, \mathbf{z}, \vec{v})$ is a congruence system with parameters $\mathbf{z}$ and $\mathbf{w}$. Take $\mathbf{A}$ in 3.15 to be the term algebra over denumerably many generators, including $x, y, \mathbf{z}$. Let T be a theory generated by $R((z_1, \ldots, z_{k-1}, x, z_{k+1}, \ldots, z_{\rho(R)})$ together with all substitution instances of of congruence formulas of the form $\varphi(x, y, \mathbf{z}, \vec{t})$, with $\varphi \in \Delta$ $\vec{t} \in \mathrm{Te}^{|\vec{v}|}$, i.e.,

$$T = \mathrm{Cn}(\{R(\mathbf{z}[x/k])\} \cup \{\varphi(x, y, \mathbf{z}, \vec{t}) : \varphi \in \Delta, \vec{t} \in \mathrm{Te}^{|\vec{v}|}\})$$

The variables z_i may occur in the terms t_k of $\vec{t}$. Since $\Omega(T)$ is compatible with T and $\langle x, y\rangle \in \Omega(T)$, we have that $R(\mathbf{z}[y/k]) \in T$. This means that

$$R(\mathbf{z}[x/k]), \Delta(x, y, \mathbf{z}, \vec{t}) \vdash_{\mathcal{S}} R(\mathbf{z}[y/k]).$$

Furthermore, since $\mathcal{S}$ is finitary, there exists a finite set $\Gamma \subseteq \Delta(x, y, \mathbf{z}, \vec{t})$ such that $R(\mathbf{z}[x/k])\Gamma \vdash_{\mathcal{S}} R(\mathbf{z}[y/k])$. Let $\Delta'(x, y, \mathbf{z})$ be the result of substituting x in Γ for every variable different from $x, y, \mathbf{z}$. Then, since $\mathcal{S}$ is structural, we have

$$R(\mathbf{z}[x/k])\Delta'(x, y, \mathbf{z}) \vdash_{\mathcal{S}} R(\mathbf{z}[y/k])$$

Clearly, $\vdash_{\mathcal{S}} \Delta'(x, x, \mathbf{z})$. Thus $\Delta'(x, y, \mathbf{z})$ is a finite equivalence system with parameter $\mathbf{z}$ for $\mathcal{S}$, hence $\mathcal{S}$ is protoalgebraic. □

The proof of the second statement is similar. □

In section 3.3 we have characterized the protoalgebraic K-deductive systems by the existence of a $\mathcal{S}$-congruence system with parameters $\mathbf{z}$ and $\mathbf{w}$. This system in general is infinite. In the next section we turn to the special cases when the congruence system can be found finite and also to the cases where one of the two sets of parameters may be omitted.

3.4 Congruential and related K-deductive systems

In addition to the concepts of a congruential and weakly congruential K-deductive systems introduced (for k-deductive system) in [4, Definition], we consider (weakly) congruential systems with parameters $\mathbf{z}$. It turns out that the concepts with and without parameters $\mathbf{z}$ coincide exactly for these K-deductive systems for which the class of reduced models is closed under the operator S. We verify, or if necessary correct, [4, 13.6–13.13] for K-deductive systems. Some of these results can be relativized to a predicate symbol R. This research is still in progress.

A *system of congruence formulas with parameters* $\mathbf{z}$ is a system Δ of congruence formulas with parameters $\mathbf{z}$ and $\mathbf{w}$, where $\mathbf{w}$ is the empty string. A *system of congruence formulas* is a system Δ of congruence formulas with parameters $\mathbf{z}$, where $\mathbf{z}$ is the empty string. We also say that Δ is a system of congruence formulas without *any* parameters when Δ is a system of congruence formulas.

Definition 3.24 *Let $\mathcal{S}$ be a K-deductive system.*

1. *$\mathcal{S}$ is* **weakly congruential with parameters** $\mathbf{z}$ *if it has, possibly infinite, system of congruence formulas with parameters* $\mathbf{z}$.

2. $\mathcal{S}$ *is* **congruential with parameters z** *if it has a finite system of congruence formulas with parameters* $\mathbf{z}$.

3. $\mathcal{S}$ *is* **weakly congruential** *if it has, possibly infinite, system of congruence formulas.*

4. $\mathcal{S}$ *is* **congruential** *if it has a finite system of congruence formulas.*

Corollary 3.25 *Let* $\mathcal{S}$ *be a* K*-deductive system and* $\mathfrak{A} \in \mathrm{Mod}\mathcal{S}$.

1. *If* $\mathcal{S}$ *is weakly congruential with parameters* $\mathbf{z}$, *whith an* $\mathcal{S}$*-congruence system with parameters* $\mathbf{z}$ $\Delta(x, y, \mathbf{z})$, *then* $\mathfrak{A}$ *is reduced iff* $\mathfrak{A}$ *satisfies*

$$[\forall_{\mathbf{z}}(\bigwedge \Delta(x, y, \mathbf{z}))] \rightarrow x \approx y. \tag{3.18}$$

Notice that the conjunction in the antecedent is infinite.

2. *If* $\mathcal{S}$ *is weakly congruential, where* $\Delta(x, y)$ *is a congruence system, then* $\mathfrak{A}$ *is reduced iff* $\mathfrak{A}$ *satisfies*

$$(\bigwedge \Delta(x, y, \mathbf{z})) \rightarrow x \approx y. \tag{3.19}$$

Again, the conjunction in the antecedent is infinite.

3. *If* $\mathcal{S}$ *is congruential with parameters* $\mathbf{z}$, *where*

$$\Delta(x, y, \mathbf{z}) = \{\Delta_1(x, y, \mathbf{z}), \ldots, \Delta_n(x, y, \mathbf{z})\}$$

is a finite $\mathcal{S}$*-congruence system with parameters* $\mathbf{z}$, *then* $\mathfrak{A}$ *is reduced iff* $\mathfrak{A}$ *satisfies*

$$[\forall_{\mathbf{z}}(\Delta_1(x, y, \mathbf{z}) \wedge \cdots \wedge \Delta_n(x, y, \mathbf{z}))] \rightarrow x \approx y. \tag{3.20}$$

4. *If $\mathcal{S}$ is congruential, where $\Delta(x,y) = \{\Delta_1(x,y),\ldots,\Delta_n(x,y)\}$ is a finite congruence system, then $\mathfrak{A}$ is reduced iff it satisfies the universal Horn sentence*

$$\Delta_1(x,y) \wedge \cdots \wedge \Delta_n(x,y) \rightarrow x \approx y.$$

Proof. By Theorem 3.15. □

Part 4. of Corollary 3.25 has first been stated in [4, Corollary 13.6 (i)] for k-deductive systems.

Corollary 3.26 *If $\mathcal{S}$ is a K-deductive system then*

1. *If $\mathcal{S}$ is weakly congruential then* $\mathrm{Mod}^*\mathcal{S}$ *is closed under the operator S, of forming submatrices.*

2. *If $\mathcal{S}$ is congruential with parameters $\mathbf{z}$, then* $\mathrm{Mod}^*\mathcal{S}$ *is closed filtered products.*

3. *If $\mathcal{S}$ is congruential then* $\mathrm{Mod}^*\mathcal{S}$ *is closed under the formation of submatrices and filtered products.*

Proof. By Corollary 3.25. □

Parts 1 and 3 of Corollary 3.26 have first been stated in [4, Corollary 13.6 (i)] for k-deductive systems.. An example of a K-deductive system $\mathcal{S}$ which has congruence formulas with $\mathbf{z}$ but $\mathrm{Mod}^*\mathcal{S}$ is not closed under $\mathbf{S}$ will be given below (Example 3.2).

Theorem 3.27 ([4, Theorem 13.7]) *Let $\mathcal{S}$ be a K-deductive system with a finite system of congruence formulas without parameters. Then for all $\mathbf{K} \subseteq \mathrm{Mod}^*\mathcal{S}$, the reduced universal Horn class generated by $\mathbf{K}$ is $\mathbf{ISPP}_U\mathbf{K}$.*

Proof. By Corollaries 2.56 and 3.26. □

Corollary 3.28 ([4, Corollary 13.8]) *If S is a K-deductive system with a finite system of congruence formulas without parameters, then the class of underlying algebras of* Mod^*S *forms a quasivariety. If C is a class of reduced models then the class of the underlying algebras of the reduced universal Horn class generated by C coincides with the quasivariety generated by the underlying algebras of the elements of C.*

Proof. By Theorem 3.27. □

The following theorem characterizes semantically models of protoalgebraic as well as weakly congruential and congruential K-deductive systems (i.e., without $\mathbf{z}$). The theorem is due to Blok and Pigozzi, [4, Theorem 13.2.]. They state this theorem for k-deductive systems, but the same proof works for arbitrary K-deductive systems. It is an open question, how to characterize the classes corresponding to (ii) and (iii) below, but with $\mathbf{z}$. It will be shown below (Example 3.2), that the class of all models of a congruential with $\mathbf{z}$ system does not need to be closed under $\mathbf{S}$.

Theorem 3.29 *Let $\mathcal{H} = \mathrm{Mod}^*S$, for some K-deductive system S.*

- *S is protoalgebraic iff it is closed under $\mathbf{P}_{SD}$.*
- *S is a weakly congruential iff $\mathcal{H}$ is closed under $\mathbf{S}$ and $\mathbf{P}$.*
- *S is a congruential iff $\mathcal{H}$ is closed under $\mathbf{S}, \mathbf{P}$ and $\mathbf{P}_U$.*

Proof. [4, Proof of Theorem 13.12]. □

Theorem 3.30 *1. If S has a system of congruence formulas with parameters $\mathbf{z}$ and Mod^*S is closed under $\mathbf{S}$, then S has a system of congruence formulas without $\mathbf{z}$.*

2. If $\mathcal{S}$ has a finite system of congruence formulas with parameters $\mathbf{z}$ *and* $\mathrm{Mod}^*\mathcal{S}$ *is closed under* $\mathbf{S}$*, then $\mathcal{S}$ has a finite system of congruence formulas without* $\mathbf{z}$.

Proof. Let $\mathcal{S}$ be a K-deductive system and let $\Delta(x, y, \mathbf{z})$ be a system of congruence formulas for $\mathcal{S}$. Let

$$X := \{\varphi(x, y, \vec{t}) : \varphi \in \Delta, \vec{t} \in \mathrm{Te}(x, y)\}$$

and let T be the $\mathcal{S}$-filter generated by X in $\mathbf{Te}(x, y, \mathbf{z})$. Let $\mathfrak{B}$ be the reduced matrix $\langle \mathrm{Te}(x, y, \mathbf{z}), T\rangle^* \in \mathrm{Mod}^*\mathcal{S}$. Thus

$$\mathfrak{B} = \langle \mathbf{Te}(x, y, \mathbf{z})/\Omega F, F/\Omega F\rangle,$$

where Ω denotes $\Omega^{\mathrm{Te}(x,y,z)}$. Let $A := \{t/\Omega(T) : t \in \mathrm{Te}(x, y)\}$ and let $T' := \{t/\Omega(T) : t \in T \cap \mathrm{Te}(x, y)\}$, i.e., $T' = (T \cap \mathrm{Te}(x, y))/\Omega(T)$. Note that $T' = (T \cap \mathrm{Te}(x, y))/\Omega(T)$. Since $\Omega(T)$ is compatible with T, we can conclude that $T' = T/\Omega(T) \cap Te(x, y)/\Omega(T)$. The inclusion from left to right is obvious and for the inclusion from right to the left let $t \in \mathrm{Te}(x, y)$ and $s \in T$ be such that $t/\Omega(T) = s/\Omega(T)$. Hence $\langle t, s\rangle \in \Omega(T)$ and since $\Omega(T)$ is compatible with T, we have that $t \in T$. Hence $t \in T \cap \mathrm{Te}(x, y)$ and $t/\Omega(T) \in (T \cap \mathrm{Te}(x, y))/\Omega(T) = T'$. It follows that the matrix $\mathfrak{A} := \langle \mathbf{A}, T'\rangle$ is a submatrix of $\mathfrak{B}$. By assumption that $\mathrm{Mod}^*\mathcal{S}$ is closed under S we conclude that $\mathfrak{A}$ is reduced. Let $x^* = x/\Omega(T), y^* = y/\Omega(T)$. We claim that $\langle x^*, y^*\rangle \in \Omega^A(T')$. In view of Thm. 3.15, it suffices to show that for every sequence $\vec{t}/\Omega(T)$ of elements of $\mathrm{Te}(x, y)/\Omega(T)$ we have $\Delta(x, y, \vec{t})/\Omega(T) \subseteq T'$, i.e., for every sequence $\mathbf{t}$ of terms in x, y, $\Delta(x, y, t) \in T$. But this is true by definition of T. So $\langle x^*, y^*\rangle \in \Omega(T')$ and since $\mathfrak{A}$ is reduced, $x^* = y^*$ in $\mathfrak{A}$. Therefore also $x^* = y^*$ in $\mathfrak{B}$. In other words, $\langle x, y\rangle \in \Omega(T)$ and hence $\Delta(x, y, z) \subseteq T$ in $\mathrm{Te}(x, y, \mathbf{z})$, by Thm. 3.15. Hence $X \vdash_{\mathcal{S}} \Delta(x, y, \mathbf{z})$. By assumption, $X \subseteq \mathrm{Te}(x,y)$ and also X is reflexive, has modus ponens and replacement property. Thus it is a $\mathcal{S}$-congruence system without parameters $\mathbf{z}$. If $\Delta(x,y,\mathbf{z})$ is finite, then by the finitary character of $\mathcal{S}$, there is a finite subset $\Delta'(x,y) \subseteq X$ such that $\Delta'(x,y) \vdash_{\mathcal{S}} \Delta(x,y,\mathbf{z})$. This subset is a finite congruence system for $\mathcal{S}$. □

Corollary 3.31 to the proof of Theorem 3.30.
Let $\mathcal{S}$ be weakly congruential with parameters $\mathbf{z}$. Let $\mathbf{A},\mathbf{B}$,T and T' be as in the above proof. If $\Omega^{\mathfrak{B}}(T) \cap A^2 = \Omega^{\mathbf{A}}(T')$ then $\mathcal{S}$ is weakly congruential.

Proof. As in the proof of Theorem 3.30 we prove that $\langle x^*, y^* \rangle \in \Omega(T')$. By assumption, $\langle x^*, y^* \rangle \in \Omega^{\mathfrak{B}}(T) \cap A^2$. Since $\mathfrak{B}$ is reduced, $\Omega^{\mathfrak{B}}(T)$ is the identity relation on $\mathbf{B}$. Hence $x^* = y^*$ and by the argument used in the proof of Theorem 3.30, we prove that $\mathcal{S}$ has a congruence system (without $\mathbf{z}$). □

Corollary 3.32 *If $\mathrm{Mod}^*\mathcal{S}$ is closed under S, then*

1. *If $\mathcal{S}$ is weakly congruential with $\mathbf{z}$, then $\mathcal{S}$ is weakly congruential.*

2. *If $\mathcal{S}$ is congruential with $\mathbf{z}$, then $\mathcal{S}$ is congruential.*

Proof. By Theorem 3.30.

Corollary 3.33 *Let $\mathcal{S}$ be a K-deductive system.*

1. *If $\mathcal{S}$ is weakly congruential with parameters $\mathbf{z}$, then the following are equivalent:*

   (a) $\mathcal{S}$ is weakly congruential

   (b) $\mathrm{Mod}^*\mathcal{S}$ is closed under S

(c) *Let* $\mathfrak{B} \in \mathrm{Mod}\mathcal{S}$ *and let* $\mathfrak{A} \subseteq \mathfrak{B}$*. Then*

$$\Omega^{\mathfrak{B}}(F) \cap A^2 = \Omega^{\mathfrak{A}}(F \cap A) \tag{3.21}$$

2. *If* $\mathcal{S}$ *is congruential with parameters* $\mathbf{z}$*, then the following are equivalent:*

   (a) $\mathcal{S}$ *is congruential*

   (b) $\mathrm{Mod}^*\mathcal{S}$ *is closed under* S

   (c) *Let* $\mathfrak{B} \in \mathrm{Mod}\mathcal{S}$ *and let* $\mathfrak{A} \subseteq \mathfrak{B}$*. Then*

$$\Omega^{\mathfrak{B}}(F) \cap A^2 = \Omega^{\mathfrak{A}}(F \cap A) \tag{3.22}$$

Proof. We will prove 1 and 2 simultaneously. The implication from (a) to (b) follows from Corollary 3.26 and the implication from (b) to (a) from Theorem 3.30. From the corollary 3.31 to the proof of 3.30 the implication (c) to (a) follows. Finally, for the proof of (a) implies (c) suppose that $\mathcal{S}$ is weakly congruential or congruential with the congruence system $\Delta(x, y)$, and $\mathbf{A} \leq \mathbf{B}$, $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{B})$. Let $a, b \in A$. Then $\langle a, b\rangle \in \Omega^{\mathbf{A}}(F \cap A)$ iff $\Delta(a, b) \subseteq F \cap A$. Since $a, b \in A$, this is equivalent to $\Delta(a, b) \subseteq F$ and therefore to $\langle a, b\rangle \in \Omega^{\mathbf{B}}(F) \cap F$. □

Recall that an operator O between two lattices is continuous, if it preserves the unions of directed sets (Definition 0.48).

Theorem 3.34 (compare with [4, Theorem 13.13 (i)]) *Let* $\mathcal{S}$ *be* K*-deductive system. If for every algebra* $\mathbf{A}$*, the Leibniz operator* $\Omega : Fi_{\mathcal{S}}(\mathbf{A}) \to \mathrm{Co}(\mathbf{A})$ *is continuous, then* $\mathcal{S}$ *is congruential with parameters* $\mathbf{z}$*.*

Proof. Since Ω is continuous, it also is monotone. Thus $\mathcal{S}$ is protoalgebraic and by Theorem refpral implies delta we have a finite equivalence system $\Delta(x, y, \mathbf{z})$. We use the convention here that $\vec{s}$ denotes a sequence of terms of the same length as $\mathbf{z}$. Let here, exceptionally, $\mathbf{Te} := \mathbf{Te}(x, y, \mathbf{z})$ and let $\mathbf{Te}' := \mathrm{Te}(x, y, v, \mathbf{z})$. For all $S \subseteq \mathrm{Te}, U \subseteq \mathrm{Te}'$ define

$$X_{(S,U)} := \{\varphi(t(x/v), t(y/v), \vec{s}) : \varphi \in \Delta(x, y, \mathbf{z}), t(v, x, y, \mathbf{z}) \in U, \vec{s} \subseteq S\}.$$

Notice that $X_{(S,U)} \subseteq \mathrm{Te}$. Let, for every pair S, U,

$$T_{(S,U)} := \mathrm{Cn}_{\mathcal{S}} X_{(S,U)}.$$

Then $T_{(\mathrm{Te},\mathrm{Te}')} = \bigcup\{T_{(S,U)} : S \subseteq \mathrm{Te}, U \subseteq \mathrm{Te}', S, U \text{ finite}\}$. By continuity, $\Omega(T_{(\mathrm{Te},T e')}) = \bigcup\{\Omega T_{(S,U)} : S, U \text{ finite}\}$. Since $\Delta(t(x/v), t(y/v), \vec{s}) \subseteq T_{(\mathrm{Te},\mathrm{Te}')}$, for all $t \in \mathrm{Te}(v, x, y, \mathbf{z})$, all $\vec{s} \subseteq \mathrm{Te}$, we conclude, by Theorem 3.22, that $\langle x, y\rangle \in \Omega T_{(\mathrm{Te}',\mathrm{Te})}$. So $\langle x, y\rangle \in \Omega(T_{(S,U)})$, for some finite S, U. Therefore for all $t \in \mathrm{Te}'$ and all $\vec{s} \subseteq \mathrm{Te}$, $\langle t(x/v), t(y/v)\rangle \in \Omega T_{(S,U)}$ and $\Delta(t(x/z), t(y/z)), \vec{s}) \subseteq T_{(S,U)}$. Hence $X_{\mathrm{Te}',\mathrm{Te}} \subseteq T_{(S,U)} \subseteq T_{\mathrm{Te}',\mathrm{Te}}$. Therefore $T_{(S,U)} = T_{(\mathrm{Te}',\mathrm{Te})}$ and $X_{(S,U)} \vdash_{\mathcal{S}} X_{(\mathrm{Te}',\mathrm{Te})}$. In particular, $X_{(S,U)}$ has the replacement property. Let $\Xi(x, y, \mathbf{z}) : X_{(S,U)}$. Then Ξ is finite, reflexive and $\Xi \vdash_{\mathcal{S}} \Delta(x, y, \mathbf{z})$, and therefore Ξ has the modus ponens property. Hence Ξ is a congruence system. □

The converse to Thm. 3.34 is, in general, false (Example 3.2 and Theorem 3.43 below). If, however, the system $\mathcal{S}$ is congruential without any parameters, then the converse, proved for k-deductive systems in [4, Theorem 13.13 (i)], is true.

Theorem 3.35 compare with [4, Theorem 13.13(i), necessity] *If a K-deductive system is congruential then the Leibniz operator $\Omega : Fi_{\mathcal{S}}(\mathbf{A}) \to \mathrm{Co}(\mathbf{A})$ is continuous for every algebra $\mathbf{A}$.*

It is not hard to see that proofs very similar to those of Theorems 3.34 and 3.35 can be used to prove a relativized version of this theorem:

Theorem 3.36 *If for every* $\mathbf{A}$ *the operator* $\Omega_R : Fi_{\mathcal{S}}(\mathbf{A}) \to \mathrm{Co}(\mathbf{A})$ *is continuous, then* $\mathcal{S}$ *is R-congruential without parameter. If* $\mathcal{S}$ *is R-congruential without parameters, then the operator* Ω_R *is continuous.*

Theorem 3.37 *If for every algebra* $\mathbf{A}$ *the Leibniz operator* $\Omega : Fi_{\mathcal{S}}(\mathbf{A}) \to \mathrm{Co}(\mathbf{A})$ *is monotone and for every filter* F *on* $\mathbf{A}$ *and a subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$,

$$\Omega(F|B) = \Omega(F) \cap B^2,$$

then $\mathcal{S}$ *is weakly congruential without any parameters.*

Proof. Since Ω is monotonic, $\mathcal{S}$ is protoalgebraic and therefore there exists an infinite set of congruence formulas with arbitrary parameters. Let $\Delta(x, y, \mathbf{z}, \mathbf{v})$ be such a set. Let $\mathbf{A}$ be the term algebra Te and let $\mathbf{B}$ be its subalgebra generated by x and y. Let F be the $\mathcal{S}$-filter on $\mathbf{A}$ generated by the union of all sets $\Delta(x, y, \mathbf{t}, \mathrm{s})$, where $\mathbf{t}$ and s range over all sequences of terms in variables x, y, of the same length that $\mathbf{z}$ and $\mathbf{v}$ respectively. Let $\mathbf{z}$ be among the variables generating $\mathbf{A}$ and let G be the restriction of F to B. Then $\Omega(G) = \Omega(F) \cap B^2$, by assumption. By definition of F, the pair $\langle x, y \rangle$ is in $\Omega^{\mathbf{B}}(G)$. So $\langle x, y \rangle \in \Omega^{\mathbf{A}}(F)$. Therefore $\Delta(x, y, \mathbf{z}) \subseteq F$, which means that $\bigcup_{\mathbf{t},\mathrm{s}} \bigcup \Delta(x, y, \mathbf{t}, \mathrm{s}) \vdash_{\mathcal{S}} \delta(x, y, \mathbf{z})$. But this means that $\bigcup_{\mathbf{t},\mathrm{s}} \bigcup \Delta(x, y, \mathbf{t}, \mathrm{s})$ is a congruence system without parameters. This finishes the proof. □

Corollary 3.38 *A K-deductive system* $\mathcal{S}$ *is weakly congruential iff it is protoalgebraic and for every algebra* $\mathbf{A}$, *every filter* F *on* $\mathbf{A}$ *and a subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$,

$$\Omega^{\mathbf{B}}(F|B) = \Omega^{\mathbf{A}}(F) \cap B^2.$$

Proof. If $\mathcal{S}$ is weakly congruential, then it has a congruence system $\Delta(x, y)$. This is also an equivalence system, so Ω is monotonic. Let $\mathbf{B} \leq \mathbf{A}$ and let F be an $\mathcal{S}$-filter on $\mathbf{A}$. Then for any elements a and b of B, we have that $\langle a, b\rangle \in \Omega^{\mathbf{B}}(F|B)$ iff $\Delta(a, b) \subseteq F|B$ iff $\Delta(a, b) \subseteq F$ iff $\langle a, b\rangle \in \Omega^{\mathbf{A}}(F)$ iff $\langle a, b\rangle \in \Omega^{\mathbf{A}}(F) \cap B^2$. Thus $\Omega^{\mathbf{B}}(F|B) = \Omega^{\mathbf{A}}(F) \cap B^2$. The other direction is the content of Thm. 3.37. □

In the previous subsection we gave an example of a protoalgebraic 2-deductive system $\mathcal{S}$ such that every equivalence systems for $\mathcal{S}$ must depend on parameter z. Below we present an extension $\mathcal{T}$ of this system by rules which has a finite congruence system with parameter z but does not have a congruence system, finite nor infinite, without parameter. In fact, it does not even have an equivalence system without parameter. Thus this single example supercedes Example 3.1.

Example 3.2 *Let $\mathcal{T}$ be the 2-deductive system over Λ determined by the rules 3.8-3.10 and the following additional rules:*

$$R(xz, yz) \vdash R((xu)z, (yu)z) \tag{3.23}$$

$$R(xz, yz) \vdash R((ux)z, (uy)z) \tag{3.24}$$

Theorem 3.39 *$\mathcal{T}$ is congruential with parameters* $\mathbf{z}$ *but does not have an equivalence system without parameters* $\mathbf{z}$. *In particular, $\mathcal{T}$ is not congruential.*

Proof. The system consisting of one R-formula $R(xz, yz)$ forms a congruence system with parameter z for $\mathcal{T}$. We now show, that no set of R-formulas in variables x, y

only, can form an equivalence system. This will prove the theorem. To see this, let $\Delta = \Delta(x, y)$ be some set of R-formulas all of whose variables are among x, y. Let $\widehat{\Delta} = \widehat{\Delta}(x, y) := \mathrm{Cn}_{\mathcal{T}}(\Delta) \cap (\mathrm{Te}(x, y))^2$. For the proof by contradiction, suppose that

$$\Delta(x, y), R(x, z) \vdash_{\mathcal{T}} R(y, z).$$

Then also

$$\widehat{\Delta}(x, y), R(x, z) \vdash_{\mathcal{T}} R(y, z).$$

Let $\mathcal{T}'$ be the system based by the axiom 3.8 and the rules 3.23 and 3.24. Notice that the rules 3.23 and 3.24 "preserve" variables, i.e., if $X \vdash_{\mathcal{T}'} R(t, s)$, for some set X of K-formulas and some terms t, s, then $\mathrm{Var}(X) \subseteq \mathrm{Var}(t, s)$. Therefore $R(y, z)$ cannot be derived from $\widehat{\Delta}(x, y) \cup R(x, z)$ just by means of $\mathcal{T}'$, i.e., every proof of $R(y, z)$ from $\widehat{\Delta}(x, y) \cup R(x, z)$ in the system $\mathcal{T}$ must contain an application of at least one of the rules 3.9 and 3.10. Let Γ be the shortest proof of $R(y, z)$ from $\widehat{\Delta}(x, y), R(x, z)$. Consider the first application of one of the rules 3.9, 3.10. One of the derivations below is such an application

$$\frac{R(tr, sr), R(t, r)}{R(s, r)} \tag{3.25}$$

or the derivation

$$\frac{R(tr, sr), R(r, t)}{R(r, s)}, \tag{3.26}$$

for some terms t, r, s such that $\widehat{\Delta}(x, y), R(x, z) \vdash_{\mathcal{T}'} R(tr, sr), R(t, s)$.

If $\mathrm{Var}(t, r, s) \subseteq \{x, y\}$, then $R(s, r), R(r, s) \in \widehat{\Delta}$ and the conclusion of the rule $R(s, r)$, or $R(r, s)$ is in $\widehat{\Delta}$. But this contradicts the assumption that the proof Γ is the shortest proof of $R(y, z)$ from $\widehat{\Delta}$ and $R(x, z)$. So we assume that $\mathrm{Var}(t, r, s) \not\subseteq \{x, y\}$. We now use the following two lemmas.

Lemma 3.40 *For every set E of K-formulas,*

$$\mathrm{Cn}_{T'}(E \cup \{R(x,z)\}) = \mathrm{Cn}_{T'}(E) \cup \{R(x,z)\}$$

Lemma 3.41 *For every set E of K-formulas and a K-formula φ, $\varphi \in \mathrm{Cn}_{T'}(E)$ iff*

$$\varphi = R(T(t_1, \ldots, t_n, t)r, T(t_1, \ldots, t_n, s)r)$$

for some $T, t_1, \ldots, t_n, r, t, s \in \mathrm{Te}$ and $R(tr, sr) \in E\}$.

Assume Lemmas 3.40 and 3.41. By Lemma 3.40 $\widehat{\Delta}(x,y) \vdash_{T'} R(tv, sv)$. By Lemma 3.41

$$\mathrm{Var}(v) \subseteq \{x, y\} \tag{3.27}$$

and $\mathrm{Var}(t) \cup \{x,y\} = \mathrm{Var}(s) \cup \{x,y\}$. Therefore, by assumption that $\mathrm{Var}(t,r,s) \not\subseteq \{x,y\}$, we have that

$$\mathrm{Var}(t) \not\subseteq \{x, y\}. \tag{3.28}$$

Therefore, $R(t,r)$ is not the premiss $R(x,z)$. Also, $R(r,t)$ cannot be $R(z,x)$, for if $t = z$ and $r = x$, then we have that $\widehat{\Delta} \vdash_{cT} R(zx, sx)$, which by lemma 3.41 implies that $R(z,s) \in \widehat{\Delta}$, a contradiction. Hence by Lemma 3.41, if 3.25 was applied, then $R(t,r)$ is derivable from $\widehat{\Delta}$, and if 3.26 was applied, then $R(r,t)$ is derivable from $\widehat{\Delta}$. By lemma 3.41, $\mathrm{Var}(t) \subseteq \mathrm{Var}(r) \cup \{x,y\}$. But by 3.27, this last set is just $\{x,y\}$, which contradicts 3.28. This finishes the proof that Δ does not have the modus ponens property relative to R and the proof of the theorem.

It remains to prove the lemmas. For Lemma 3.40, notice that $R(x,z)$ cannot be a premiss of a substitution instance of any of the rules that are basis for $\mathcal{S}$.

For lemma 3.41, it is obvious, that the left-hand side of the equality is included in the right-hand side. Clearly, the right-hand-side is closed under $\mathrm{Cn}_{\mathcal{T}'}$. Thus the two sides must be equal. □

Corollary 3.42 *The class* $\mathrm{Mod}^*\mathcal{T}$ *of reduced models of the system* $\mathcal{T}$ *defined in Example 3.2 is not closed under* S.

Proof. By Theorem 3.30.

Theorem 3.43 *The operator* $\Omega_{\mathcal{T}}$ *is not continuous.*

Proof.

For every finite subset S of $\mathrm{Te}(x, y)$ define $X_{(S,U)} := \{\langle xt, yt\rangle : t \in S\}$. Let F_S be the $\mathcal{Q}$-filter on $\mathbf{Te}(x, y)$ generated by X_S. We claim that

$$F_S = \{\langle s, s\rangle : s \in \mathrm{Te}(x, y)\} \cup$$

$$\{\langle t[x/z]s, t[y/z]s\rangle : s \in S, t \in \mathrm{Te}(x, y, z) \text{ where } z \text{ occurs in } t \text{ only once }\}.$$

We give now the proof of the inclusion from left to right. The other inclusion is immediate.

Claim *Suppose that* $\langle r, tq\rangle \in F_S$. *Then either* $r = tq$ *or* $|q| \leq max\{|s| : s \in S\}$.

Proof of Claim. If $\langle r, tq\rangle \in F_S$, then there must be a $\mathcal{T}$-derivation of $\langle r, tq\rangle$ using axioms X_S and rules of $\mathcal{T}$. If the last rule used in this derivation was (3.8), then $r = tq$ and OK. If the last rule was (3.23) or (3.24) then also OK, by the induction hypothesis that for all the R-formulas with shorter derivations the claim holds. So suppose that the last rule used was (3.9). Then there are some R-terms $R(u(tq), r(tq))$, $R(u, tq)$ for which the claim holds. Thus either $u(tq) = r(tq)$, in which case $u = r$ and we are done since $\langle r, tq\rangle = \langle u, tq\rangle$ satisfies the claim by the induction hypothesis, or else $|tq| \leq max\{|s| : s \in S\}$, which implies that also $|q| \leq max\{|s| : s \in S\}$. The case that the lat rule used was (3.10) is similar. The claim is proved.

It follows from the claim, that if s is the member of S of maximal length, then

$$\langle x(sx), y(sx)\rangle \notin F_S.$$

Since $\langle xz, yz\rangle$ is a congruence system with z for $\mathcal{T}$, by Thm 3.15 we have that $\langle x, y\rangle \notin \Omega(F_S)$, for any finite S. But by the same theorem, $\langle x, y\rangle \in \Omega(\bigcup\{F_S : S \subseteq \mathrm{Te}(x,y) \text{ finite}\})$, hence $\Omega(\bigcup\{F_S : S \subseteq \mathrm{Te}(x,y) \text{ finite}\}) \neq \bigcup\{\Omega(F_S) : S \subseteq \mathrm{Te}(x,y) \text{ finite }\}$. But the family of all F_S is directed, hence Ω is not continuous. □

3.5 Matrix homomorphisms and quotient matrices of protoalgebraic K-deductive systems

Some of the results of Chapter 2 concerning protoalgebraic K-deductive systems, can now be proved as corollaries to the representation theorem, Theorem 3.10. For example, let us reprove the following theorem.

Theorem 3.44 (Theorem 2.66) *Let $\mathfrak{A}$ and $\mathfrak{B}$ be models of protoalgebraic K-deductive system $\mathcal{S}$. Then every surjective matrix homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ induces a matrix homomorphism $h^* : \mathfrak{A}^* \to \mathfrak{B}^*$, of the respective reductions, defined by $h^*(a/\Omega(D_{\mathfrak{A}})) = ha/\Omega(D_{\mathfrak{B}})$.*

Proof. One way to prove the theorem is by a modification of the proof in [4]. We can also use Theorems 3.15 and 3.23 as follows. We need to show that h^* is well defined.

Let $\Delta(x, y, \mathbf{z}, \mathbf{w})$ be an $\mathcal{S}$-congruence system with parameters $\mathbf{z}$ and $\mathbf{w}$ that exists by Theorem 3.23. By theorem 3.15 If $a/\Omega(D_{\mathfrak{A}}) = b/\Omega(D_{\mathfrak{A}})$, then $\Delta(a, b, \vdash) \subseteq D_{\mathfrak{A}}$, for all sequences $\vdash$ of elements of A of length same as the sum of lengths of $\mathbf{z}$ and $\mathbf{w}$. Let $\vec{c}$ be a sequence of elements of B of length equal to the sum of lengths of $\mathbf{w}$ and $\mathbf{z}$. Since h is surjective, $\vec{c} = h(\vdash)$, for some $\vdash \subseteq A$. By assumption, $\Delta(a, b, \vdash) \subseteq D_{\mathfrak{A}}$, hence $\Delta(ha, hb, \vec{c}) \subseteq (D_{\mathfrak{A}}) \subseteq D_{\mathfrak{B}}$. Since $\vec{c}$ was arbitrary, this proves that $ha/\Omega(D_{\mathfrak{A}}) = hb/\Omega(D_{\mathfrak{B}})$ and h^* is well defined. □

3.6 Summary

In spite of the gap in the proof of Theorem 13.2, most of the results of [4, Section 13] remain true. By definition, Corollary 13.6 of [4] is true (Corollary 3.25 part 4 and Corollary 3.26 part 3, here) and as a consequence, Theorem 13.7 and Corollary 13.8 are also true. Although the proof of Theorem 13.10 contains a gap, the result is true and demonstrated here as Theorem 3.23(1.). Theorems 13.12 and 13.13 (ii) are also true. Theorem 13.15 will be proved in Chapter 5.

We do not know if Theorem 13.13 (i) is true. Partial results are our Theorems 3.34 and 3.35. The two results of [4, Section 13] require modification: 13.2, and 13.5. However, due to the new characterization of protoalgebraicity involving parameters $\mathbf{z}$, the discussion of [4, section 13] should be exteded to cover also the classes defined by the congruence systems with parameters $\mathbf{z}$. Some results are presented here, but we still do not have a full semantical characterization of systems that are congruential or weakly congruential with parametrs $\mathbf{z}$. We only know that a sufficient and necessary condition for such systems to be congruential (weakly congruential) is the closure of the class of reduced models under the operator S. Similarly, we do not know if the characterization of congruential systems as those for which the Leibniz operator is continous, Theorem 13.13 (i), is correct. We only know that the continuity is a necessary condition, and that it is sufficient for a system to be congruential with parameters $\mathbf{z}$.

We have also relativized some results of [4, section 13] to the predicate R and partially to (R, k). It will be interesting to relativize other results of this chapter.

The results presented in Chapter 3 will be used in Chapter 4 to protoalgebraic Gentzen systems and in Chapter 2 to study the semantics of protoalgebraic K-deductive systems. Besides protoalgebraic and congruential K-deductive systems, another interesting class is the class of so-called algebraizable K-deductive system. A K-deductive system is *algebraizable* iff the Leibniz operator Ω is one-one and continuous. We discuss the algebraizable K-deductive systems in Chapter 5.

CHAPTER 4. PROTOALGEBRAIC GENTZEN SYSTEMS

4.1 Introduction

We discuss the consequences of the representation theorem, Theorem 3.10, for protoalgebraic Gentzen systems. Following [50, 49, 41], Gentzen systems are formalized as ω-deductive systems. Theorem 3.10 implies that a Gentzen system $\mathcal{G}$ is protoalgebraic iff there is a finitary (but possibly infinite) $\mathcal{G}$-equivalence system with parameters $\mathbf{z}$. It is an easy observation that for all Gentzen systems that have a so-called (CUT) *rule* (page 121), a *finite* equivalence system can be found, but the converse is not true. We discuss the connection between the (CUT) rule and protoalgebraicity. In section 4.5 we consider a mild condition guaranteeing that a protoalgebraic Gentzen system has a *finite* equivalence system with parameters $\mathbf{z}$. Let us mention that another way of explaining why the equivalence systems are finite for the Gentzen system known from the literature, leads through a different formalization of a Gentzen system that will be explained in a separate work.

4.2 Gentzen-systems as ω-deductive systems

We formalize here a Gentzen system as an ω-deductive system. The basic notion is that of a sequent, which here is identified with an ω-term (Chapter 2, page 37). Recall that for a given algebraic language Λ, ω-formulas can be identified with (Λ-

)*sequents*, i.e., expressions of the form $t_1, \ldots, t_n \to t$, where $t_1, \ldots, t_n, t$ are terms of a given language Λ. A more general definition of a sequent was given in 2.7. Although it is possible to carry over the discussion presented here to this more general context, as well as to the so-called sequents of type (α, β) considered in [50], we will restrict ourselves here to the sequents understood as ω-formulas.

For an algebraic language Λ let $\mathrm{Seq}_\Lambda = \mathrm{Seq}$ be the set off all Λ-sequents. An ω-deductive system will be called here also a Gentzen system. We now restate Definition 2.20.

Definition 4.1 *A* **Gentzen system** *is a pair* $\langle \Lambda, \mathrm{Cn} \rangle$, *where* Λ *is an algebraic language and* $\mathrm{Cn} : \mathcal{P}_\omega(\mathrm{Seq}) \to \mathrm{Seq}$ *is an algebraic and structural closure operator, i.e.,* Cn *satisfies conditions (2.1)-(2.5) for all subsets* X, Y *of the set* Seq *and all substitutions* σ.

The rules of a Gentzen system take the form:

$$\frac{S_1, \ldots, S_n}{S},$$

where $S_1, \ldots, S_n, S$ are sequents.

We say that a sequent S is *derivable* in a Gentzen system $\mathcal{G}$ if $S \in \mathrm{Cn}_{\mathcal{G}}(\emptyset)$.

It is often convenient, see Example 4.1 below, to present rules of a Gentzen system by so-called schemata of rules, called also rule-schemata. These schemata contain meta-variables Γ, Φ ranging over finite sequences of terms. An ω-rule is a special case of a rule-schema, namely a schema with no meta-variables. If, however, there is at least one meta-variable in a rule-schema, then this rule represents an infinite number of ω-rules. The notion of a schema of rules can be formalized as follows. Suppose that in addition to the set of (first-order) variables $\mathrm{Var} = \{x, y, z, x_1, y_1, z_1, x_2, \ldots\}$, which we here also denote by Var_1, we have a set $\mathrm{Var}_2 := \{\Gamma, \Delta, \Sigma, \ldots\}$ of *second-order variables.* A *schema of sequents* or *sequent-schema* is an expression of the form $X_1, \ldots, X_n \rightarrow X$, where $X_1, \ldots, X_n, X \in \mathrm{Te}(Var_1) \cup \mathrm{Var}_2$. Let $S_1, \ldots, S_n, S$ be schemata of sequents. Then the expression $\dfrac{S_1, \ldots, S_n}{S}$ is called a *schema of rules* or a *rule-schema.* A *second-order substitution* is a pair of functions $f = \langle f_1, f_2 \rangle$ such that f_1 is a substitution (i.e., a homomorphism on $\mathbf{Te}(\mathrm{Var}_1)$) and $f_2 : \mathrm{Var}_2 \rightarrow \mathcal{P}_\omega(\mathrm{Te})$. For $X \in \mathrm{Te}(Var_1) \cup \mathrm{Var}_2$, $f(X) = f_1(X)$, if $X \in \mathrm{Te}(\mathrm{Var}_1)$ and $f(X) = f_2(X)$, if $X \in \mathrm{Var}_2$. For a sequent-schema $S = X_1, \ldots, X_n \rightarrow X$, $f(S) = f(X_1), \ldots, f(X_n) \rightarrow f(X)$, and for a rule-schema $r = \dfrac{S_1, \ldots, S_n}{S}$, $f(r) = \dfrac{f(S_1), \ldots, f(S_n)}{f(S)}$. Now for a Gentzen system $\mathcal{G}$ we say that a rule-schema r is *valid in* $\mathcal{G}$ or is *a rule-schema of* $\mathcal{G}$ iff for every second-order substitution $f = \langle f_1, f_2 \rangle$, the ω-rule $f(r)$ is a rule of $\mathcal{G}$. An *instance* of a rule-schema r is a rule $f(r)$, where f is a second-order substitution. A Gentzen system is *based* by a set B of rule-schemata if it is based by the set of all second-order substitutions of rules from B.

Example 4.1 (based on [53, pages 9–11]) Consider the the following schema of rules.

Weakening	$\dfrac{\Gamma \rightarrow z}{\Gamma x \rightarrow z}$ (W)	$\dfrac{\Gamma \rightarrow z}{\Gamma \rightarrow z \vee x}$ (Wr)
Contraction	$\dfrac{\Gamma, z, z \rightarrow x}{\Gamma z \rightarrow x}$ (C)	$\dfrac{\Gamma z \rightarrow (x \vee y) \vee y}{\Gamma z \rightarrow x \vee y}$ (Cr)
Exchange	$\dfrac{\Gamma, x, y, \Phi \rightarrow z}{\Gamma, y, x, \Phi \rightarrow z}$ (EX)	$\dfrac{\Phi \rightarrow x \vee (y \vee z) \vee u}{\Phi \rightarrow x \vee (z \vee y) \vee u}$ (EXr)
(CUT)	$\dfrac{\Gamma, x, \Sigma \rightarrow y; \ \Phi \rightarrow x}{\Gamma, \Phi, \Sigma \rightarrow y}$	
$\Rightarrow$ introduction	$\dfrac{x\Gamma \rightarrow z}{\Gamma \rightarrow x \Rightarrow z} \Rightarrow \mathrm{r}$	$\dfrac{\Gamma \rightarrow z, \quad \Delta x \rightarrow D}{\Gamma, \Delta, z \Rightarrow x \rightarrow D} \Rightarrow \mathrm{l}$
$\wedge$ introduction	$\dfrac{\Gamma, x \rightarrow z}{\Gamma x \wedge y \rightarrow z}, \dfrac{\Gamma, x \rightarrow z}{\Gamma y \wedge x \rightarrow z}$ ($\wedge$ l)	$\dfrac{\Gamma \rightarrow z \quad \Phi \rightarrow u}{\Gamma, \Phi \rightarrow z \wedge u}$ ($\wedge$r)
$\vee$ introduction	$\dfrac{\Gamma, x \rightarrow z \quad \Phi y \rightarrow z}{\Gamma, \Phi, x \vee y \rightarrow z}$ ($\vee$l)	$\dfrac{\Gamma \rightarrow z}{\Gamma \rightarrow z \vee u}, \dfrac{\Gamma \rightarrow z}{\Gamma \rightarrow u \vee z}$ ($\vee$ r)
$\neg$ introduction	$\dfrac{\Gamma \rightarrow u \vee z}{\Gamma \neg u \rightarrow z}$ ($\neg$l)	$\dfrac{\Gamma, x \rightarrow z}{\Gamma \rightarrow \neg x \vee z}$ ($\neg$r)

The rules (W), (C), (EX), (CUT) will be called *structural* and the remaining rules are called *logical.* Because each of the rules in the above example is a *schema* of rules in which the second-order variables actually appear, each of them represents an infinite set of ω-rules. For example, (W) represents the set of rules of the form

$$\frac{t_1, \ldots, t_n \to t}{s, t_1, \ldots, t_n \to t},$$

with t, t_i ranging over terms and n ranging over natural numbers. Since we are assuming structurality, the meta-variables ranging over terms can be replaced by elements of the set Var of all first-order variables, so (W) can be replaced by the infinite number of rules

$$\frac{x_1, \ldots, x_n \to x}{y, x_1, \ldots, x_n \to x},$$

one for each $n \in \mathbf{N}$.

Example 4.2 We list some Gentzen systems, known from the literature, that can be based by sets of rules listed in the previous example.

For the purpose of this example we will say that a Gentzen system $\mathcal{G}$ *axiomatizes* a deductive system $\mathcal{S}$ if for every $t \in \mathrm{Te}$, t is a tautology of $\mathcal{S}$ iff the sequent $\to t$ is derivable in $\mathcal{G}$. Thus for example we say that a Gentzen system $\mathcal{G}$ axiomatizes the deductive system CPC of classical propositional logic iff, for every term t in the language $\{\Rightarrow, \wedge, \vee\}$, t is a tautology of CPC iff the sequent $\to t$ is derivable in $\mathcal{G}$.

(LKp) The propositional fragment of the system (LK) ([53]) that axiomatizes CPC is based by all the rules listed in Example 4.1. We also denote this system here, exceptionally, by (LK).

(LJ) If we add to the language the constant $\perp$, remove the rules (Cr) and (EXr), and replace the rules (Wr), ($\neg$l), ($\neg$r) by the following rules

(Wr') $\dfrac{\Gamma \to \perp}{\Gamma \to x}$

($\neg$l') $\dfrac{\Gamma \to u}{\Gamma \neg u \to \perp}$

($\neg$r') $\dfrac{\Gamma, x \to \perp}{\Gamma \to \neg x}$

respectively, then the Gentzen system based by the resulting rules is called (LJ) and it axiomatizes the deductive system IPC of the propositional fragment of intuitionistic logic ([53]).

If S is some subset of connectives $\{\vee, \wedge, \Rightarrow, \neg\}$, and we consider a Gentzen system based by all the structural rules listed in Example 4.1 and exactly these of the logical rules in Example 4.1 that involve the connectives of S, then this Gentzen system axiomatizes the so-called S-fragment of the classical logic. Similarly, we get a Gentzen system axiomatizing the S-fragment of intuitionistic logic by considering all structural rules of (LJ) and all and only these logical rules of (LJ) that involve the connectives from S. For example, if we consider only $S = \{\Rightarrow\}$, then we get the following Gentzen system that axiomatizes the implicational fragment of IPC.

(INT) The Gentzen system (INT) is based by the following rules: (W), (Wr'), (EX), (C), (CUT), ($\Rightarrow$ r) and ($\Rightarrow$ l); and equivalently, by (W), (EX), (C), (CUT), ($\Rightarrow$ r) and ($\Rightarrow$ l). It corresponds to the implicational fragment of the intuitionistic logic.

(BCK) Removing from (INT) the rule (C), we get the Gentzen system called (BCK). It axiomatizes the so-called (BCK)-logic, defined for example in [64]. The system itself is due to Y. Komori, see [39].

(BCK+$\wedge$) is the Gentzen system obtained from (BCK) by addition of ($\wedge$r) and ($\wedge$), ([38]).

We also consider Gentzen systems resulting from the systems above by removing the (CUT) rule. These are denoted by (LK\CUT), (LJ\CUT), (BCK\CUT) and ((BCK+$\wedge$)\CUT), respectively.

Two Gentzen systems $\mathcal{G}_1$ and $\mathcal{G}_2$ are *sequent-derivability equivalent* if a sequent S is derivable in $\mathcal{G}_1$ iff it is derivable in $\mathcal{G}_2$, i.e., if $\mathcal{G}_1$ and $\mathcal{G}_2$ axiomatize the same deductive system.

Usually, a logician working with a concrete Gentzen system $\mathcal{G}$ is interested in its power to derive sequents. Our approach here is different: we look at the *deductive* power of a Gentzen system, i.e., at the set of derived rules of $\mathcal{G}$. We say that two Gentzen systems $\mathcal{G}_1$ and $\mathcal{G}_2$ are *equivalent* iff a (Gentzen-) rule r is a derived rule of $\mathcal{G}_1$ iff it is a derived rule of $\mathcal{G}_2$. As is easy to predict, and as we will below see (Example 3.2), two Gentzen systems that are sequent-derivability equivalent, do not need be equivalent. The next Theorem, says that each of (LK), (LJ), (INT), (BCK) is sequent-derivability equivalent to a Gentzen system without (CUT). This contrasts with Corollary 4.17 below.

Theorem 4.2 ([53] for (LK) and (LJ), [37] for (BCK) and (BCK+$\wedge$)) *Let $\mathcal{G}$ be one of (LK), (LJ), (INT), (BCK), (BCK+$\wedge$) Gentzen system. Then a sequent S is derivable in $\mathcal{G}$ iff S is derivable in $\mathcal{G} \setminus$ (CUT).*

□

4.3 Semantics of Gentzen systems and the Leibniz operator. A review.

We apply here definitions from Chapter 2 to the case of an ω-deductive system. Let $\mathcal{G}$ be a Gentzen system.

Definition 4.3 *(Definition 0.40) A model of $\mathcal{G}$ is a pair $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ consisting of a Λ-algebra $\mathbf{A}$ and a system $F = \langle F_n : n \in \mathrm{N} \rangle$, where for each n, $F_n \subseteq A^n$, such that for every rule $\dfrac{S_1, \ldots, S_k}{S}$ of $\mathcal{G}$, where $S_1, \ldots, S_k, S$ are sequents of length $n_1, \ldots, n_k, n$, respectively, and for every valuation $f : \mathrm{Te} \to \mathbf{A}$, the following implication holds.*

$$f(S_1) \in F_{n_1}, \ldots, f(S_k) \in F_{n_k} \Rightarrow f(S) \in F_n.$$

The system F is called a **filter on A** *and is sometimes identified with the disjoint union of all F_i.*

Definition 4.4 *(Definition 2.23) If $\mathfrak{A} = \langle \mathbf{A}, F \rangle$ is a model of $\mathcal{G}$ then a filter on $\mathfrak{A}$ is any filter G on $\mathbf{A}$ such that $F \subseteq G$, where inclusion is defined coordinatewise, i.e., if $F = \langle F_n : n \in \mathrm{N} \rangle$ and $G = \langle G_n : n \in \mathrm{N} \rangle$, then $F \subseteq G$ iff for every $n \in \mathrm{N}$, $F_n \subseteq G_n$.*

Corollary 4.5 *(to Proposition 2.41) Let $\mathfrak{A}$ be a model of a Gentzen system $\mathcal{G}$. Then two elements a, b of A are in the relation $\Omega_{\mathcal{G}}(F)$ on $\mathbf{A}$, if for every positive integer k, for every $i < k$, for every sequence $t_0, \ldots, t_k$ of terms in $\mathrm{Te}(x, x_1, x_2, \ldots)$, for every pair of homomorphisms $f, g : \mathrm{Te}(x, x_1, x_2, \ldots) \to A$ such that $f(x) = a, g(x) = b, f(x_i) = g(x_i)$ for all i, we have:*

(i)

$$f(t_0), \ldots, f(t_{k-1}) \to f(t_k) \in F_k \Rightarrow$$

$$f(t_1), \ldots, f(t_{i-1}), g(t_i), f(t_{i+1}), \ldots, f(t_{k-1}) \to f(t_k) \in F_k \text{ and}$$

(ii)

$$f(t_0),\ldots,f(t_{k-1}) \to f(t_k) \in F \Rightarrow f(t_0),\ldots,f(t_{k-1}) \to g(t_k) \in F.$$

Equivalently, $\langle a,b\rangle \in \Omega_{\mathcal{G}}(F)$ *if for every* $k, i < k$*, for every pair of homomorphisms* f, g *as above, for every sequence of elements* $c_0,\ldots,c_k \in A$*, we have*

$$c_0,\ldots,c_{i_1}, f(t), c_{i+1},\ldots,c_{k-1} \to c_k \in F \Rightarrow$$

$$c_0,\ldots,c_{i_1}, g(t), c_{i+1},\ldots,c_{k-1} \to c_k \in F \text{ and}$$

$$c_0,\ldots,c_{k-1} \to f(t) \in F \Rightarrow c_0,\ldots,c_{k-1} \to g(t) \in F.$$

Definition 4.6 *(Definitions 2.59 and 3.5) A Gentzen system* $\mathcal{G}$ *is protoalgebraic if the operator* $\Omega_{\mathcal{G}}$ *described above is monotone for every algebra* $\mathbf{A}$*.*

Definition 4.7 *(Definition 3.3) Let* $\mathcal{G}$ *be a Gentzen system. A* **finitary system of equivalence sequents for** $\mathcal{G}$ *is a set* $\Delta(x,y,\mathbf{z})$ *of sequents, where* $\mathbf{z} = \langle z_1,\ldots,z_n,\ldots\rangle$ *is a (possibly infinite) sequence of variables, called parameters, which are different of* x, y*, such that for every natural* $k \geq 1$*, there is a finite set* $\Delta_k(x,y,\mathbf{z})$ *of sequents such that* $\Delta(x,y,\mathbf{z}) = \bigcup_{k\in\mathbf{N}} \Delta_k(x,y,\mathbf{z})$ *and for every natural* $k \geq 1$*, for every* $i < k$

$$\vdash_S \Delta_k(x,x,z_1,\ldots,z_k) \tag{4.1}$$

$$\frac{\Delta_k(x,y,z_1,\ldots,z_k), \quad z_1,\ldots,z_{i-1},x,z_i,\ldots,z_{k-1} \to z_k}{z_1,\ldots,z_{i-1},y,z_i,\ldots,z_{k-1} \to z_k} \quad \text{and} \tag{4.2}$$

$$\frac{\Delta_k(x,y,z_1,\ldots,z_{k-1}), \quad z_1,\ldots,z_{k-1} \to x}{z_1,\ldots,z_{k-1} \to y}. \tag{4.3}$$

Corollary 4.8 *(to Thm. 3.10) A Gentzen system* $\mathcal{G}$ *is protoalgebraic iff it has a finitary system of equivalence sequents with parameters* $\mathbf{z}$*, where* $\mathbf{z} = \langle z_1, z_2, \ldots\rangle$*.*

Let us stress that in general $\Delta(x, y, \mathbf{z})$ need not be finite, as shown in an example at the end of this chapter (Example 4.5). On the other hand for all well-known Gentzen system investigated in the literature, for example the systems (LK), (LJ), (BCK), (BCK+$\wedge$), (INT), the finitary equivalence system, if exists, is finite. We later discuss (sections 4.2, 4.5) some extra condition which imply that a protoalgebraic Gentzen system has a *finite* system of equivalence sequents.

In this Chapter we often will say that a system has some rule to mean that this rule is derivable. Also, for a schema of rules we say that it is derivable, if every instance of this schema is derivable. Thus for example, we will say that a Gentzen system $\mathcal{G}$ has

$$\text{(CUT)}\ \frac{\Gamma, x, \Sigma \to y;\ \Delta \to x}{\Gamma, \Delta, \Sigma \to y}$$

if for every n the rule

$$\frac{x_1, \ldots, x_n, x, y_1, \ldots, y_m \to y;\ z_1, \ldots, z_k \to x}{x_1, \ldots, x_n, z_1, \ldots, z_k, y_1, \ldots, y_m \to y}$$

is a derived rule of $\mathcal{G}$.

4.4 Protoalgebraicity versus Cut rule and preservation of subterms

It is obvious, that if a Gentzen system $\mathcal{G}$ has (CUT) and $x \to x$ is an axiom of $\mathcal{G}$, then $\{x \to y, y \to x\}$ forms an equivalence system for $\mathcal{G}$ that is finite and without parameters. This is the case with the systems (LK), (LJ), (INT), (BCK+$\wedge$). An example below (Example 4.3) shows that the converse need not be true. In this section we define *the system of weak (CUT) rules,* Definition 4.12). Every Gentzen system $\mathcal{G}$ with these rules is protoalgebraic and in the next section we show that if

$\mathcal{G}$ satisfies some mild condition then it is protoalgebraic iff it has a system of weak (CUT) rules.

4.4.1 (CUT)-rule

Proposition 4.9 *Let $\mathcal{G}$ be a Gentzen system such that $x \to x$ is a theorem of $\mathcal{G}$. If* (CUT) *is a derived rule of $\mathcal{G}$ then $\mathcal{G}$ is protoalgebraic.*

Proof. Let $\mathcal{G}$ be a Gentzen system in which (CUT) is a derived rule and $x \to x$ is a theorem. We claim that the set $\Delta(x, y) := \{x \to y, y \to x\}$ is an equivalence system for $\mathcal{G}$. By assumption, (4.1) holds. The (CUT) rule yields (4.2) and (4.3). □

On the other hand, not in every protoalgebraic Gentzen system the (CUT) rule can be derived. Below we give an example of a protoalgebraic Gentzen system which does not have (CUT). In fact, (CUT) is not even admissible for this system.

Example 4.3 *Let Λ have the following connectives: nullary connectives (i.e., constants) 1 and $\top$, and a binary connective $\equiv$. Let $\mathcal{G}$ be the following Gentzen system*

Axioms $\top \to 1, \quad 1 \to \top, \quad \to x \equiv x$

Rules

$$\frac{\Delta x \Gamma \to y, \quad \Sigma \to x \equiv z}{\Delta, \Sigma z \Gamma \to y} \tag{4.4}$$

$$\frac{\Delta \to x, \quad \Sigma \to x \equiv z}{\Delta \Sigma \to z} \tag{4.5}$$

We now show that (CUT) is not a derived rule of $\mathcal{G}$ and moreover it is even not admissible, even though $\mathcal{G}$ is protoalgebraic, lemma 4.11.

Lemma 4.10 (CUT) *is not admissible in $\mathcal{G}$ and moreover, it is not even derivable.*

Proof. We first claim that the only sequents of the form

$$S := \Sigma \to x \equiv z \tag{4.6}$$

that are derivable in $\mathcal{G}$ are those of the form $\to x \equiv x$ where $\Sigma = \emptyset$ and $x = z$. If S is an axiom, then it certainly satisfies the condition that $\Sigma = \emptyset$ and $x = z$. So consider a proof P of S and assume that the claim holds for every sequent S' with a proof shorter than P. Let us first observe that the rule (4.4) could not be the last rule used in the derivation. For otherwise, some substitution instance $t_1, \ldots, t_k, t, s_1, \ldots, s_m \to x \equiv z$ of its first premiss would have a shorter proof than P and would be of the form (4.6) with nonempty antecedent. So the last rule used in P was (4.5) and a premiss $t_1, \ldots, t_m \to t \equiv (x \equiv z)$ has a proof shorter than P. But by our induction hypothesis this is impossible. This establishes the claim.

Now suppose that one of the two rules of $\mathcal{G}$ has been applied to two sequents S_1, S_2 that are derivable from the empty set of premisses. It follows from the claim above that one of the sequents, say S_2 is of the form $\to t \equiv t$. Therefore the conclusion S must be exactly S_1. It follows that the only sequents that are derivable in $\mathcal{G}$ are the axioms of $\mathcal{G}$.

In particular, $\top \to \top$ is not derivable. But $\top \to 1$, $1 \to \top$ are axioms, so they are derivable. It follows that the (CUT) rule is not an admissible rule of $\mathcal{G}$. $\square$

Lemma 4.11 *The system $\mathcal{G}$ is protoalgebraic.*

Proof. As noticed above, to show that $\mathcal{G}$ is protoalgebraic it suffices to observe that there is a system of equivalence sequents for $\mathcal{G}$. Let $\Delta(x, y, z)$ be $\{\to x \equiv y\}$. Then the third axiom of $\mathcal{G}$, i.e., $\to x \equiv x$, together with the two rules establish the conditions for $\Delta(x, y)$ to be a system of equivalence sequents. $\square$

Thus a protoalgebraic Gentzen system does not need to have the (CUT) rule. This is not surprising, since even in the presence of (CUT) the system of equivalence sequents consists of $x \to y, y \to x$ and not just $x \to y$. Thus the axiom and rules (4.1)-(4.2) suggest the question whether every protoalgebraic Gentzen system should have the following rules:

$$(\mathrm{ECUT}_l) \quad \frac{x \to y \quad y \to x \quad \Gamma x \Delta \to z}{\Gamma y \Delta \to z}$$

$$(\mathrm{ECUT}_r) \quad \frac{x \to y \quad y \to x \quad \Gamma \to x}{\Gamma \to y}$$

The above pair of rules, jointly denoted by (ECUT) is called *a system of equivalence* (CUT) *rules.*

Again, it is easy to see that if (ECUT) is admissible in a Gentzen system $\mathcal{G}$, then the sequents $x \to y$ and $y \to x$ form an equivalence system for $\mathcal{G}$ and therefore $\mathcal{G}$ is protoalgebraic.

However a modification of the exampleabove by adding a new constant $\bot$ and a new axiom $\bot \to \top$, shows that a protoalgebraic Gentzen system need not have (

Example 4.4 Let Λ consist of the nullary connectives $\top, \bot, 1$ and binary connective $\equiv$. Let $\mathcal{G}$ be the Gentzen system determined by the following axioms and rules.

Axioms $\to x \equiv x, \quad \top \to 1, \ 1 \to \top$

Rules

$$\frac{\Gamma x \Delta \to z, \quad \to x \Leftrightarrow y}{\Gamma y \Delta \to z}$$

$$\frac{\Gamma \to x, \quad \to x \Leftrightarrow y}{\Gamma \to y}$$

It is easy to prove by induction on the length of derivation, that a sequent of the form $\rightarrow x \Leftrightarrow y$ is derivable in $\mathcal{G}$ iff $x = y$. It follows that the only sequents which can be derived in $\mathcal{G}$ are the axioms. Therefore the sequent $\bot \rightarrow 1$ is not derivable. Since $\bot \rightarrow \top, \top \rightarrow 1$ and $1 \rightarrow \top$ are derivable, this means that (ECUT) is not an admissible rule of $\mathcal{G}$. $\square$

4.4.2 Weak equivalence cut

The conditions (4.2)-(4.3) of Example 4.4 establish some rules that are similar to, although weaker than (ECUT).

We call the pair of these rules weak equivalence cut rules, (WEC).

Definition 4.12 *Let $\mathcal{G}$ be a Gentzen system.*

We say that $\mathcal{G}$ has a **weak-equivalence cut property** *or just* **weak-cut property**, (WEC)*-property for short, if there is a finite set $\Delta(x, y, z)$ of sequents such that the following rules, called* **weak-cut rules** *and denoted (WEC), are derivable in $\mathcal{G}$.*

$$(WEC_l)\ \frac{\Gamma, x, \Sigma \rightarrow z;\ \ \Delta(x, y, z)}{\Gamma, y, \Sigma \rightarrow z}$$

$$(WEC_r)\ \frac{\Gamma \rightarrow x;\ \ \Delta(x, y, z)}{\Gamma \rightarrow y}$$

Notice that the set Δ in the above definition forms a system of equivalence sequents with parameters $\mathbf{z}$, in fact just one parameter, for $\mathcal{G}$. Thus we have the following proposition.

Proposition 4.13 *If a Gentzen system $\mathcal{G}$ has a* (WEC)*-property, then $\mathcal{G}$ is protoalgebraic.*

Proof. By Definition 4.12 and Corollary 4.8. □

In the Gentzen systems defined in Example 4.2 all the rules, except (CUT), preserve subterms (Definition 4.14 below). On the other hand, Gentzen systems with (CUT) are protoalgebraic. We now observe that existence in a Gentzen system $\mathcal{G}$ of some rule that does not preserve subterms is necessary for $\mathcal{G}$ to be protoalgebraic.

4.4.3 Preservation of subterms

Definition 4.14 *A Gentzen style schema of rules* **preserves subterms** *if for every variable (of first or second order) occurring as a subterm in any of the premisses of a rule, this variable also occurs in the conclusion of this rule.*

Notice that each of the (W), (Ex), (C), (Wr), (EXr), (Cr) preserves subterms. All the logical rules defined in Example 4.1 preserve subterms, too. If a rule $\frac{S_1,\ldots,S_n}{S}$ preserves subterms, then for every second-order substitution σ, every subterm of a term occurring in $\sigma S_i, i = 1,\ldots,n$, also occurs in $\sigma(S)$. This justifies our terminology. Observe, that the (CUT) rule does not preserve subterms. For the variable x does occurs in the premiss of the first sequent while it does not occur in the conclusion.

Similarly, the rule

$$(\mathrm{WEC}_l)\ \frac{\Gamma, x, \Sigma \rightarrow z;\ \Delta(x,y,z)}{\Gamma, y, \Sigma \rightarrow z}$$

does not preserve subterms.

Thus we have the following

Lemma 4.15 *Let $\mathcal{G}$ be a Gentzen system based by some set R of rules. If each of the rules of R preserves subterms, then $\mathcal{G}$ does not have weak-equivalence cut property.*

□

Theorem 4.16 *Let $\mathcal{G}$ be one of the systems (LK\CUT), (LJ\(CUT), (BCK\(CUT), (INT\CUT). Then $\mathcal{G}$ is not protoalgebraic.*

Proof. Since each of rhe systems in the assumption has the variable preservation property, the conclusion follows by Lemma 4.15. □

Corollary 4.17 *Let $\mathcal{G}$ be one of the Gentzen systems (LK), (LJ), (INT), (BCK), (BCK+$\wedge$). Then $\mathcal{G}$ is not equivalent to $\mathcal{G} \setminus (CUT)$. In particular, (CUT) is not a derived rule of $\mathcal{G} \setminus (CUT)$.*

By the Cut elimination theorem, Theorem 4.2 (CUT) is an admissible rule of $\mathcal{G} \setminus (CUT)$, where $\mathcal{G}$ is as above. It, however, is not a derived rule, according to Corollary 4.17.

4.5 Accumulative Gentzen systems

The rules (W), (EX), (CUT), and most of the logical rules considered above are of the form $\dfrac{S_1, \ldots, S_n}{S}$, where all the second order variables occurring in the sequent-schemata $S_1, \ldots, S_n$ are pairwise distinct and all of them occur in S. Gentzen systems based by such rule-schemata have the following property.

Definition 4.18 *A Gentzen system $\mathcal{G}$ has the* **accumulation property** *if for every set T of sequents, all finite sets of terms Σ, Γ and all terms t, s, r, if*

$$\frac{T;\ \Gamma \to t}{\Sigma \to s}$$

is derivable in $\mathcal{G}$, then also

$$\frac{T;\ r, \Gamma \to t}{r, \Sigma \to s}$$

is derivable in $\mathcal{G}$.

The name "accumulation" is borrowed from [50], where Gentzen systems with left and right accumulation are considered, but our definition slightly differs from the definition of [50].

We have seen that a Gentzen system has, possibly infinite, system of equivalence formulas, with infinitely many parameters. On the other hand the systems with (CUT) and (WCUT) are examples of protoalgebraic Gentzen systems that have a finite system of equivalence formulas without parameters. This leads us to the following questions.

Questions Let $\mathcal{G}$ be a protoalgebraic Gentzen system.

1. Under which additional conditions does $\mathcal{G}$ have a finite system of equivalence formulas?

2. Under which conditions the equivalence system for $\mathcal{G}$ does depend only on two variables, i.e., is of the form $\Delta(x, y)$ for some set of sequents Δ?

We give here a partial answer to these question, by considering systems with the accumulation property.

Lemma 4.19 *If a Gentzen system* Γ *has the accumulation property, then for every set* T *of sequents, all finite sets of terms* Σ, Γ, Φ *and all terms* $t, s,$, *if*

$$\frac{T;\ \Gamma \to t}{\Sigma \to s}$$

is derivable in $\mathcal{G}$, *then also*

$$\frac{T;\ \Phi, \Gamma \to t}{\Phi, \Sigma \to s}$$

is derivable in $\mathcal{G}$.

Proof. By induction on the cardinality of Φ and using definition. □

Theorem 4.20 *If $\mathcal{G}$ is protoalgebraic Gentzen system with the accumulation property, then there is a finite system of equivalence sequents with one parameter z.*

Proof. Since $\mathcal{G}$ is protoalgebraic, for every k there is a finite set Δ_k of sequents with $k-1$ parameters such that their union forms a finitary system of equivalence sequents. In particular, for $k = 2$ we have a finite set of sequents $\Delta(x, y, z) = \Delta_2(x, y, z)$ such that

$$\vdash_S \Delta(x, x, z)$$

$$\frac{\Delta(x,y,z), \quad \to x}{\to y} \quad \text{and}$$

$$\frac{\Delta(x,y,z), \quad x \to z}{y \to z}.$$

By the corollary of the definition of a natural Gentzen system, we conclude that also

$$\frac{\Delta(x,y,z), \quad \Phi \to x}{\Phi \to y} \quad \text{and}$$

$$\frac{\Delta(x,y,z), \quad \Phi x \to z}{\Phi y \to z}.$$

Thus for every $k_i < k$, we have

$$\frac{\Delta(x,y,z), \quad x_1 \cdots x_k \to x}{x_1, \ldots, x_k \to y} \quad \text{and}$$

$$\frac{\Delta(x,y,z), \quad x_1 \cdots x_{i-1} x x_{i+1} \cdots x_k \to z}{x_1 \cdots x_{i-1} y x_{i+1} \cdots x_k \to z},$$

which means that $\Delta(x, y, z)$ is an equivalence system for $\mathcal{G}$. □

It follows from the above theorem that if a $\mathcal{G}$ has accumulation property, then $\mathcal{G}$ is protoalgebraic iff it has weak-cut property.

4.5.1 Examples

Lemma 4.21 *Let $\mathcal{G}$ be one of the following Gentzen systems: (LK), (LJ), (BCK), (INT), (LK\CUT), (LJ\CUT), (BCK\CUT), (INT\CUT). Then $\mathcal{G}$ has the accumulation property.*

□

We have seen, Proposition 4.9, that the first four systems are protoalgebraic, with the (WEC) system $\{x \to y, y \to x\}$, while the remaining four are not, Theorem 4.16.

We conclude with an example of a Gentzen system that is protoalgebraic, but does not have the weak-equivalence cut property. Hence this is also an example of a system that does not have the accumulation property.

Example 4.5 *For a rational number q let $[q]$ be the largest natural number n such that $n \leq q$. Let $\mathcal{G}$ be the following Gentzen system. It has one axiom:*

(Ax) $\Gamma, x \to x$*;*

and the following rules

(EX) $\dfrac{\Gamma, x, y, \Psi \to z}{\Gamma, y, x, \Psi \to z}$

(1) $\dfrac{\Gamma, x \to z \quad \Gamma, y \to z}{\Gamma, y \to z}$

(2) $\dfrac{\to x, \quad x \to y}{\to y}$

Note that the axiom (Ax) as well as the rule (1) represent infinite families of ω-axioms and rules.

Proposition 4.22 *TheGentzen system $\mathcal{G}$ is protoalgebraic.*

Proof. In view of Corollary 4.8 it suffices to show that there is a system $\Delta(x, y, \mathbf{z})$ of equivalence sequents with parameters. Let

$$\Delta(x, y, \mathbf{z}) := \{z_1, \ldots, z_n, x \to y : n \geq 0\} \cup \{z_1, \ldots, z_n, y \to x : n \geq 0\}.$$

Then

$$\Delta = \bigcup_{n \in \mathbf{N}} \Delta_n,$$

where $\Delta_n = \{z_1, \ldots, z_n, x \to y, \quad z_1, \ldots, z_n, y \to x\}$. The set $\Delta(x, x, \mathbf{z})$ consists of axioms, so $\vdash_{\mathcal{G}} \Delta(x, x, \mathbf{z})$. The condition 4.2 clearly holds by rule (1) and 4.3 can be demonstrated as follows. If $k = 1$, then

$$\frac{\Delta_k(x, y, z_1, \ldots, z_{k-1}), \quad z_1, \ldots, z_{k-1} \to x}{z_1, \ldots, z_{k-1} \to y}$$

holds by rule (2). Let $k \geq 2$. Consider the following instance of rule (1).

$$\frac{z_1, \ldots, z_{k-2}, x \to y, \quad z_1, \ldots, z_{k-1} \to x}{z_1, \ldots, z_{k-1} \to y}.$$

Since the left premiss is in Δ, we conclude that 4.3 holds. Hence Δ is an equivalence system with parameters for $\mathcal{G}$. $\Box$

We now show that $\mathcal{G}$ does not have a finite system of equivalence sequents.

Theorem 4.23 *There is no finite set Δ of sequents that forms an equivalence system, with parameters, for $\mathcal{G}$.*

Proof. The proof will be completed in a series of lemmas.

Lemma 4.24 *Suppose that a sequent $\Gamma \to u$ is a theorem of $\mathcal{G}$, where Γ is a sequence of terms. Then u is a member of Γ.*

Proof. We show that the set $A := \{\Gamma \to u : u \in \Gamma\}$ is closed under the axiom and rules of $\mathcal{G}$. By definition, every axiom is in A. Clearly, A is closed under (EX). For (1), if $\Pi, t \to s$ and $\Pi, r \to t$ are in A, then either $s \in \Pi$, in which case $\Pi, r \to s \in A$, or $s = t$. Since $\Pi, r \to t \in A$, it follows that $s = t \in \Pi, r$ and therefore $\Pi, r \to s \in A$. Hence A is closed under (1). The proof that it is closed under (2) is straightforward. □

Let T be a set of sequents. Then let $\mathrm{Ex}(T)$ be the set of all sequents of the form

$$u_{\pi(1)}, \ldots u_{\pi(n)} \to u$$

such that $u_1, \ldots, u_n \to u \in T$ and π is a permutation of the set $\{1, \ldots, n\}$.

Lemma 4.25 *Let* $S_1 := v_1, \ldots, v_n \to v$ *and* $S_2 := w_1, \ldots, w_m \to w$, *where* $m \neq n, m, n \geq 1$. *Then* $\mathrm{Cn}(S_1, S_2) = \mathrm{Ex}(S_1, S_2) \cup A$.

Proof. Let R be the right-hand-side of the above equality. It suffices to show that R is closed under the axioms and rules of $\mathcal{G}$. It clearly is closed under axioms and under (Ex). By assumption that both m and n are greater than 1, and by the definition of A, the rule (2) can not be applied to any pair of sequents from R. We now show that R is closed under (1). Let $S, S' \in R$. Of course $\mathrm{Ex}(S_1, S_2) = \mathrm{Ex}(S_1) \cup \mathrm{Ex}(S_2)$, and since $m \neq n$, if (1) can be applied to S and S',

$$\frac{S, S'}{S''} \quad (1)$$

then either one of S, S' is an axiom or both $S, S' \in \mathrm{Ex}(S_i)$, for $i = 1$ or $i = 2$. But it is clear that in the latter case, $S'' \in \mathrm{Ex}(S_i)$, as well. If $S = \Gamma, t \to s \in A$, $S' = \Gamma, u \to t \in \mathrm{Ex}(S_i)$, i=1,2, and $S'' = \Gamma, u \to s$, then either $t = s$, in which case $S'' = S'$, or else $s \in \Gamma$, in which case $S'' \in A$. In both cases $S'' \in R$. Hence R is closed under (1) and the proof of the lemma is finished. □

Lemma 4.26 *Let $N \geq 1$ and let*

$$R := \{u_1, \ldots, u_n \to u : n \leq N\}.$$

Then $\mathrm{Cn}(R) = R \cup A$.

Proof. Obviously, $R \cup A$ is closed under axioms and rules of $\mathcal{G}$. □

Lemma 4.27 *Let N be a fixed natural number. Suppose that $\Delta = \Delta(x, y, \mathbf{z})$ is a set of sequents such that for every $u_1, \ldots, u_n \to u \in \Delta$, $n \leq N$. Suppose further that $\vdash_{\mathcal{G}} \Delta(x, x, \mathbf{z})$. Let $S := z_1, \ldots, z_{N+1}, x \to z$. Then* $\mathrm{Cn}(\Delta, S) = \mathrm{Cn}(\Delta) \cup \mathrm{Ex}(S) \cup A$.

Proof. By lemma 4.24, $\to x$ is not a theorem of $\mathcal{G}$ and therefore $N \geq 1$, by assumption that $\vdash_{\mathcal{G}} \Delta(x, x, \mathbf{z})$. By lemma 4.26, if $v_1, \ldots, v_m \to v \in \mathrm{Cn}(\Delta)$, then $m \leq N$ and therefore, by lemma 4.25, $\mathrm{Cn}(\mathrm{Cn}(\Delta) \cup \mathrm{Ex}(S)) \subseteq \mathrm{Cn}(\Delta) \cup \mathrm{Ex}(S) \cup A$. Hence $\mathrm{Cn}(\Delta, S) \subseteq \mathrm{Cn}(\mathrm{Cn}(\Delta) \cup \mathrm{Ex}(S)) \subseteq \mathrm{Cn}(\Delta) \cup \mathrm{Ex}(S) \cup A \subseteq \mathrm{Cn}(\Delta, S)$. □

Let Δ and S be as in the assumptions of Lemma 4.27. It follows from Lemma 4.27 and Lemma 4.26 that $z_1, \ldots, z_n, y \to z \notin \mathrm{Cn}(\Delta, S)$. In particular, a finite set Δ satisfies the assumptions of Lemma 4.27 for some N. It follows that no finite $\Delta(x, y, \mathbf{z})$ satisfies condition 4.2 and therefore there is no finite equivalence system for $\mathcal{G}$. The theorem is proved. □

Since $\mathcal{G}$ is protoalgebraic but does not have a finite system of equivalence sequents, by Theorem 4.20 it can not have the accumulation property. Indeed,

$$\frac{x \to z, \quad y \to x}{y \to z}$$

is an instance of the rule (1). However, let $\Delta := \{z_1, x \to z\}$. By lemma 4.27, $\dfrac{z_1, x \to z, \quad y \to x}{z_1, y \to z}$ is not a derived rule of $\mathcal{G}$.

Let us make two more remarks. First, if we add to the rules of $\mathcal{G}$ the weakening rule, then the resulting Gentzen system has a finite equivalence system with parameters **z**, namely $\{x \to y, y \to x\}$. Hence this extended system is an example of a non-accumulative Gentzen system that has (WEC) and therefore a finite system of equivalence system. Second, although the Gentzen system $\mathcal{G}$ of example 3.9oes not have a finite system of equivalence system with parameters, it has a finite set of sequent-schemata $\Delta(x, y, \Gamma) = \{\Gamma, x \to y, \Gamma, y \to x\}$ that has the properties

$$\vdash_{\mathcal{G}} \Delta(x, x, \Gamma),$$

$$\frac{\Delta(x, y, \Gamma), \Gamma, x \to z}{\Gamma, y \to z}$$

$$\frac{\Delta(x, y, \Gamma), \Gamma \to x}{\Gamma \to y},$$

that are analogous to the properties of equivalence sequents. Gentzen systems with finite systems of sequent-schemata that have the above properties are studied in a separate work.

4.6 Summary

A Gentzen system is protoalgebraic iff it has a finitary system of equivalence sequents. For example, a Gentzen system that has (CUT) and the axiom $x \to x$ must be protoalgebraic. Thus (LK), (LJ), (INT), (BCK), (BCK+$\wedge$) are all protoalgebraic. On the other hand, (LK\CUT), (LJ\CUT), (INT\CUT), (BCK\CUT), (BCK+ $\wedge$ \CUT) are not. We also considered here the question whether a protoalgebraic Gentzen system must have a finite system of equivalence sequents. Example 4.5, Theorem 4.23, shows that no, but if we assume that the system has the accumulation property, then the answer is positive.

CHAPTER 5. ALGEBRAIZATION AND EQUIVALENCE THEOREMS

5.1 Introduction

A *deductive system* is a $(\Lambda,\ K)$-deductive system, for some first order language $\langle\Lambda, K\rangle$. In this chapter we consider a concept of equivalence of two deductive systems (Definition 5.1). We give certain sufficient condition for two systems to be equivalent. This allows for a generalization of Theorem 4.4 of [6] and Theorem 2.20 of [50] (Theorems 5.19, 5.29). Theorem 5.19 will be applied, in chapter 6, to characterize the deductive systems with implication.

5.2 Definition of equivalent systems

The following definition is a straightforward generalization of a definition that was first proposed for k and l-deductive systems in [6, page 12].

Definition 5.1 *Let $\mathcal{S}_1, \mathcal{S}_2$ be, respectively, a K_1- and a K_2-deductive systems over the same fixed algebraic language Λ. By a (K_1, K_2)-***translation** *we mean a sequence $\tau = \{\tau_R : R \in K_{K_1}\}$, such that each τ_R is a finite set of K_2-formulas in k variables, where $k = \rho(R)$. Thus*

$$\tau_R(p_0, \dots, p_{k-1}) = \{\tau_R^i(p_0, \dots, p_{k-1}) : i < m_R\}$$

for some positive integer m_R.

If φ is a K_1-formula, e.g., $\varphi = R(t_1, \ldots, t_k)$, for some $R \in K_1$ and some sequence of terms $\vec{t} = \langle t_1, \ldots, t_k \rangle$, then $\tau(\varphi) = \{\tau_R^i(\vec{t}) : i \leq k\}$. For $\Gamma \subseteq \mathrm{Fm}_{\mathrm{K}}$, we let

$$\tau(\Gamma) = \bigcup\{\tau(\varphi) : \varphi \in \Gamma\}.$$

A (K_1, K_2)-translation τ is called an **interpretation** *of $\mathcal{S}_1$ in $\mathcal{S}_2$ if, for all $\Gamma \subseteq \mathrm{Fm}_{\mathrm{K}_1}, \varphi \in \mathrm{Fm}_{\mathrm{K}_1}$, we have*

$$\Gamma \vdash_{\mathcal{S}_1} \varphi \;\; \textit{iff} \;\; \tau(\Gamma) \vdash_{\mathcal{S}_2} \tau(\varphi). \tag{5.1}$$

We say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are **equivalent** *if there is an interpretation τ of $\mathcal{S}_1$ in $\mathcal{S}_2$ and an interpretation v of $\mathcal{S}_2$ in $\mathcal{S}_1$ that are inverse to one another in the following sense*

$$\varphi \dashv\vdash_{\mathcal{S}_1} v(\tau(\varphi)) \tag{5.2}$$

for all $\varphi \in \mathrm{Fm}_{\mathrm{K}_1}$ and

$$\varphi \dashv\vdash_{\mathcal{S}_2} \tau(v(\varphi)) \tag{5.3}$$

for all $\varphi \in \mathrm{Fm}_{\mathrm{K}_2}$.

Thus $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent iff there is a (K_1, K_2)-translation τ and a (K_2, K_1)-translation v such that

$$\Gamma \vdash_{\mathcal{S}_2} \varphi \;\; \text{iff} \;\; v(\Gamma) \vdash_{\mathcal{S}_1} v(\varphi), \tag{5.4}$$

$$\Gamma \vdash_{\mathcal{S}_1} \varphi \;\; \text{iff} \;\; \tau(\Gamma) \vdash_{\mathcal{S}_2} \tau(\varphi), \tag{5.5}$$

$$\varphi \dashv\vdash_{\mathcal{S}_1} v(\tau(\varphi)) \tag{5.6}$$

for all $\varphi \in \mathrm{Fm}_{\mathrm{K}_1}$ and

$$\varphi \dashv\vdash_{\mathcal{S}_2} \tau(v(\varphi)) \tag{5.7}$$

for all $\varphi \in \mathrm{Fm}_{\mathrm{K}_2}$.

Lemma 5.2 *Let $\mathcal{S}_1, \mathcal{S}_2$ be K_1 and K_2-deductive systems, respectively and let τ and v be (K_1, K_2)- and (K_2, K_1)-translations, respectively. Then the conjunction of the conditions (5.5) and (5.7) is equivalent to the conjunction of (5.4) and (5.6).*

Proof. By (5.5), $v(\Gamma) \vdash_{\mathcal{S}_1} v\varphi$ iff $\tau v \vdash_{\mathcal{S}_2} \tau v \varphi$. By (5.7), this is equivalent to $\Gamma \vdash_{\mathcal{S}_2} \varphi$. Hence (5.4). To show (5.6), observe that by (5.5), $v\tau\varphi \vdash_{\mathcal{S}_1} \varphi$ iff $\tau v \tau \varphi \vdash_{\mathcal{S}_2} \tau\varphi$. By (5.7), this is equivalent to $\tau\varphi \vdash_{\mathcal{S}_2} \tau\varphi$, which is true. The condition $\varphi \vdash_{\mathcal{S}_1} v\tau\varphi$ is shown similarly. Hence (5.6) holds. Thus (5.5) and (5.7) imply (5.4) and (5.6). Reversing the roles of $\mathcal{S}_1$ and $\mathcal{S}_2$ and of τ and ν, one gets that (5.4) and (5.6) imply (5.5) and (5.7). □

In particular, let $\mathcal{S}_1$ be classical propositional, intuitionistic propositional or BCK-logic in some language that contains at least the binary implication connective $\rightarrow$. Let $\mathcal{S}_2$ be $\models_K$, where K is the class of Boolean, Heyting or BCK-algebras, respectively. Consider the translations $\tau(x) = \{x \approx 1\}$ and $v(x, y) = \{x \rightarrow y, y \rightarrow x\}$. It is well known that

$$\Gamma \vdash_{\mathcal{S}_1} t \text{ iff } \{s \approx 1 : s \in \Gamma\} \vdash_{\mathcal{S}_2} t \approx 1 \text{ and}$$

$$t \approx s \dashv\vdash_{\mathcal{S}_2} t \rightarrow s \approx 1, s \rightarrow t \approx 1.$$

But these are conditions (5.5) and (5.7), hence $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent and $\mathcal{S}_1$ is algebraizable. More examples of algebraizable as well as examples of non-algebraizable 1-deductive systems can be found in [5].

Definition 5.3 *Let Σ_1 and Σ_2 be a K_1 and K_2-deductive systems, respectively. Let σ : Te $\rightarrow$ Te be a substitution. Then we say that a function $\Sigma : \mathrm{Th}_{\mathcal{S}_1} \longrightarrow \mathrm{Th}_{\mathcal{S}_2}$* **commutes with** σ *if for every $\mathcal{S}_1$-theory T,*

$$\mathrm{Cn}_{\mathcal{S}_2}(\sigma(\Sigma(T))) = \Sigma(\mathrm{Cn}_{\mathcal{S}_1}(\sigma(T))) \tag{5.8}$$

The next theorem generalizes [6, theorem 4.4] and also [5, theorem 3.7]. To prove it we use the idea of the proofs in [5] and [6].

Theorem 5.4 *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be K_1- and K_2-deductive systems, respectively. Then the conditions (i)-(iii) below are equivalent.*

1. *$\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent;*

2. *There exists an isomorphism Σ from $\mathrm{Th}\mathcal{S}_1$ to $\mathrm{Th}\mathcal{S}_2$ that commutes with all substitutions;*

3. *There exists an isomorphism Σ from $\mathrm{Th}\mathcal{S}_1$ onto $\mathrm{Th}\mathcal{S}_2$ that commutes with surjective substitutions.*

If this is the case, the interpretations τ of $\mathcal{S}_1$ into $\mathcal{S}_2$ that exists by definition of equivalence, can be chosen in such a way that for every $\mathcal{S}_1$-theory T, $\Sigma(T) = \mathrm{Cn}_{\mathcal{S}_2}(\{\tau(\psi) : \psi \in T\}) = \{\varphi : v(\varphi) \subseteq T\}$.

Proof. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent with translations τ and v such that (5.5) and (5.7) hold. For a $\mathcal{S}_1$-theory T define $\Sigma(T) := \mathrm{Cn}_{\mathcal{S}_2}\tau(T)$. By definition, $\Sigma : \mathrm{Th}_{\mathcal{S}_1} \longrightarrow \mathrm{Th}_{\mathcal{S}_2}$. We claim that Σ is an isomorphism commuting with substitutions. To show that it is 1-1, let $T, S \in \mathrm{Th}\mathcal{S}_1$ and assume that $\Sigma(T) = \Sigma(S)$. This means that $\tau(T) \vdash_{\mathcal{S}_2} \tau(S)$, which, by (5.5), is equivalent to $T = S$.

To show that Σ is onto, let $\Phi \in \mathrm{Th}\mathcal{S}_2$. Define T by

$$T = \mathrm{Cn}_{\mathcal{S}_1} v(\Phi).$$

We claim that $\Phi = \Sigma(T)$. Let $\varphi \in \Phi$. By definition of T and Σ, $\tau v \varphi \in \Sigma(T)$. By (5.7), $\varphi \in \Sigma(T)$. Hence $\Phi \subseteq \Sigma(T)$. On the other hand, let $\varphi \in \Sigma(T)$. Then there are some

$\alpha_1, \ldots, \alpha_n \in T = \mathrm{Cn}_{\mathcal{S}_1}(v(\Phi))$ such that $\tau(\alpha_1), \ldots, \tau(\alpha_n) \vdash_{\mathcal{S}_2} \varphi$. But $T = \mathrm{Cn}_{\mathcal{S}_1}(v(\Phi))$, hence there are some $\varphi_1, \ldots, \varphi_m \in \Phi$ such that $v(\varphi_1), \ldots, v(\varphi_n) \vdash_{\mathcal{S}_1} \alpha_1, \ldots, \alpha_n$. Therefore, $\tau v(\varphi_1), \ldots, \tau v(\varphi_m) \vdash_{\mathcal{S}_2} \varphi$ which is equivalent to $\varphi_1, \ldots, \varphi_m \vdash_{\mathcal{S}} \varphi$ and therefore $\varphi \in \Phi$. This finishes the argument that Σ is onto.

Finally, we want to show that Σ commutes with substitutions. Let σ be a substitution and let T be an $\mathcal{S}_1$-theory.

Then $\Sigma(\sigma(T) = \mathrm{Cn}_{\mathcal{S}_2}(\{\tau(\varphi(\sigma\vec{x} : \varphi(\vec{x}) \in T\}$. But if $\varphi(\vec{x}) \in T$, then $\tau(\varphi(\vec{x})) \in \Sigma(T)$ and $\tau(\varphi(\sigma\vec{x})) \in \sigma(\Sigma(T))$. Therefore $\Sigma(\sigma(T)) \subseteq \mathrm{Cn}_{\mathcal{S}_2}\sigma(\Sigma(T))$. Also,

$$\begin{aligned}
\mathrm{Cn}_{\mathcal{S}_2}(\sigma(\Sigma(T))) &= \mathrm{Cn}_{\mathcal{S}_2}\{\alpha(\sigma\vec{x}) : \alpha(\vec{x}) \in \mathrm{Cn}_{\mathcal{S}_2}\tau(T)\} \\
&\subseteq \mathrm{Cn}_{\mathcal{S}_2}\{\sigma(\alpha(\vec{x})) : \sigma(\alpha(\vec{x})) \in \mathrm{Cn}_{\mathcal{S}_2}(\tau(\sigma(T)))\} \\
&\subseteq \mathrm{Cn}_{\mathcal{S}_2}\{\beta : \beta \in \mathrm{Cn}_{\mathcal{S}_2}(\tau(\sigma(T)))\} \\
&= \mathrm{Cn}_{\mathcal{S}_2}(\tau(\sigma(T))) \subseteq \mathrm{Cn}_{\mathcal{S}_2}\tau(\mathrm{Cn}_{\mathcal{S}_1}(\sigma(T))) \\
&= \Sigma(\sigma(T).
\end{aligned}$$

This finishes the proof of the theorem in one direction. For the other direction, suppose that we have an isomorphism Σ that commutes with substitutions. First fix an $R \in \mathrm{K}_1$ and let $T := \mathrm{Cn}_{\mathcal{S}_1}(R\vec{x})$, where $\vec{x} = \langle x_1, \ldots, x_{\rho(R)-1}\rangle$. Since Σ is an isomorphism, $\Sigma(T)$ is finitely generated and therefore there exist some set τ_R of terms $\tau_R^1(\vec{x}), \ldots, \tau_R^n(\vec{x})$ such that

$$\Sigma(T) = \mathrm{Cn}_{\mathcal{S}_2}(\tau_R(\vec{x})).$$

Since R was arbitrary, we have a sequence $\tau = \{\tau_R : R \in \mathrm{K}\}$ which is a (K_1, K_2)-translation. Now let $\varphi = R(\vec{t})$, where $\vec{t} = \langle t_1, \ldots, t_{\rho(R)}\rangle$ is a sequence of terms, be a K_2-formula and let σ be a substitution such that for every $i = 1, \ldots, \rho(R)$, we have $\sigma(x_i) = t_i$. Note, that σ can be taken surjective. Since Σ commutes with substitutions,

$$\begin{aligned}
\Sigma(\mathrm{Cn}_{\mathcal{S}_1}(\varphi)) &= \Sigma(\mathrm{Cn}_{\mathcal{S}_1}(\sigma(T_R))) \\
&= \mathrm{Cn}_{\mathcal{S}_2}(\sigma(\Sigma(T_R))) \\
&= \mathrm{Cn}_{\mathcal{S}_2}(\tau_R(\sigma(\vec{x}))) \\
&= \mathrm{Cn}_{\mathcal{S}_2}(\tau_R(t_1, \ldots, t_r)) \\
&= \mathrm{Cn}_{\mathcal{S}_2}(\tau(\varphi)).
\end{aligned}$$

Because of this and since Σ is an isomorphism, for $\Gamma \subseteq \mathrm{Fm}_{\mathrm{K}_1}$, we have the following sequence of equalities.

$$\begin{aligned}
\mathrm{Cn}_{\mathcal{S}_1}(\Gamma) &= \bigvee_{\gamma\in\Gamma} \Sigma(\mathrm{Cn}_{\mathcal{S}_1}(\gamma) \\
&= \bigvee_{\gamma\in\Gamma} \mathrm{Cn}_{\mathcal{S}_2}(\tau(\gamma)) && (5.9) \\
&= \mathrm{Cn}_{\mathcal{S}_2}(\tau(\Gamma)). && (5.10)
\end{aligned}$$

It follows from this and the fact that $\Gamma \vdash_{\mathcal{S}_1} \varphi$ is equivalent to $\Sigma(\mathrm{Cn}_{\mathcal{S}_1}\varphi) \subseteq \Sigma(\mathrm{Cn}_{\mathcal{S}_2}(\Gamma))$, that for $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathrm{K}_1}$,

$$\Gamma \vdash_{\mathcal{S}_1} \varphi \text{ iff } \tau(\Gamma) \vdash_{\mathcal{S}_2} \tau(\varphi).$$

By a symmetric argument, we prove that there is a (K_2, K_1)-translation v such that

$$\Sigma^{-1}(\mathrm{Cn}_{\mathcal{S}_2}(\varphi)) = \mathrm{Cn}_{\mathcal{S}_1}(v(\varphi))$$

and therefore also for all sets $\Phi \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathrm{K}_2}$

$$\Phi \vdash_{\mathcal{S}_2} \varphi \text{ iff } v(\Phi) \vdash_{\mathcal{S}_1} v\varphi.$$

Also, $\mathrm{Cn}_{\mathcal{S}_1}(\varphi)\mathrm{Cn}_{\mathcal{S}_1}(\varphi) = \Sigma\Sigma^{-1}(\mathrm{Cn}_{\mathcal{S}_1}(\varphi)) = \Sigma(\mathrm{Cn}_{\mathcal{S}_2}(v(\varphi))) = \mathrm{Cn}_{\mathcal{S}_1}(\tau v(\varphi)))$, and we conclude that

$$\tau(v(\vec{\phi})) \vdash_{\mathcal{S}_2} \varphi \text{ and } \varphi \vdash_{\mathcal{S}_2} \tau v\vec{\phi}.$$

This finishes the proof that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent. It follows from the proof that $\Sigma(T) = \mathrm{Cn}_{\mathcal{S}_2}(\{\tau(\psi) : \psi \in T\}) = \{\varphi : v(\varphi) \subseteq T\}$. $\square$

5.3 Birkhoff-like deductive systems

Theorem [5, Theorem 4.2] states that a 1-deductive system $\mathcal{S}$ is equivalent to some extension of the Birkhoff system $\mathcal{B}$ iff $\Omega_{\mathcal{S}}$ is injective and continuous. Similar result can be proved if $\mathcal{B}$ is replaced by a so-called Birkhoff-like deductive system (Definition 2.19). Theorem 5.8 justifies our choice of the name "Birkhoff-like".

Definition 5.5 *A set Γ of K_2-rules is called* **Birkhoff-like** *if for all K_2-formulas $\varphi_1, \ldots, \varphi_n, \varphi, \psi$ the following conditions B(i) and B(ii) hold:*

B(i) *For every surjective substitution σ such that $\dfrac{\sigma(\varphi)}{\psi}$ is an instance of a Γ-rule, there is an K_2-term ξ such that $\sigma(\xi) = \psi$ and $\dfrac{\varphi}{\xi}$ is an instance of a rule of Γ.*

B(ii) *Let $\dfrac{\varphi_1, \ldots, \varphi_n}{\varphi}$ and $\dfrac{\varphi}{\psi}$ be some instances of rules in Γ. Then there are K_2-formulas $\xi_1, \ldots, \xi_n$ such that for every $i = 1, \ldots, n$, $\dfrac{\varphi_i}{\xi_i}, \dfrac{\xi_1, \ldots, \xi_n}{\psi}$ are also instances of rules of Γ.*

If a K_2-deductive system $\mathcal{S}_2$ is based by some Birkhoff-like set Γ of rules, then $\mathcal{S}_2$ is called **Birkhoff-like**.

Remark Condition B(ii) says that every derivation of the form

$$\dfrac{\dfrac{\varphi_1, \ldots, \varphi_n}{\varphi}}{\psi} \tag{5.11}$$

can be replaced by a derivation

$$\frac{\frac{\varphi_1}{\xi_1}, \ldots, \frac{\varphi_n}{\xi_n}}{\psi}, \tag{5.12}$$

i.e., every $\mathcal{S}$-proof in which some multiple-premiss rules have been used can be replaced by one in which every single-premiss rule is used before any multiple-premiss rule. It follows from the condition B(i) that if σ is a surjective substitutionand ψ can be derived from $\sigma(\varphi)$ using only a nonempty sequence of one-premiss rules of a Birkhoff-like set of rules Γ, then there is a ξ such that $\varphi \vdash_\Gamma \xi$, also by means of only one-premiss rules, such that $\psi = \sigma(\xi)$. Notice that a condition similar to B(i) holds trivially in case that $\psi = \sigma(vph)$, namely if $\psi = \sigma\varphi$ then there is a ψ' such that $\psi = \sigma\varphi$ and $\varphi \vdash_\Gamma \psi'$ by means of at most single-premiss rules. So

Example 5.1 We will consider K_2 consisting of one binary relation and we will write a K_2- fla either as a pair $\langle t, s\rangle$ or as an inequality $t \leq s$, with $t, s \in$ Te. Suppose that with every $\lambda \in \Lambda$ there are associated two, possibly empty, sets: $P_\lambda, N_\lambda \subseteq \{1, \ldots, \rho(\lambda)\}$. The system $P : \langle\langle P_\lambda, N_\lambda\rangle : \lambda \in \Lambda\rangle$ determines a K_2-deductive (i.e., a 2-deductive) system $\mathcal{S}_P$ as follows.

$\mathcal{S}_P$ is the 2-deductive system axiomatized by the axiom (I) and the rules , (T), (Rp), (Rn) below:

(I) $\langle x, x\rangle$

(T) $\dfrac{\langle x, y\rangle, \langle y, z\rangle}{\langle x, z\rangle}$

(Rp) $\dfrac{x \leq y}{\lambda(z_1, \ldots, z_n)[x/z_k] \leq \lambda(z_1, \ldots, z_n)[y/z_k]}$ for all λ and $k \in P_\lambda$

(Rn) $\dfrac{y \leq x}{\lambda(z_1,\ldots,z_n)[x/z_k] \leq \lambda(z_1,\ldots,z_n)[y/z_k]}$ for all $\lambda \in \Lambda$ and $k \in N_\lambda$.

We will be using this system and its generalization in chapter 6. Notice that $\mathcal{S}_P$ depends on the choice of pairs of sets P_λ, N_λ.

We now show that the above axiomatization is Birkhoff-like and therefore $\mathcal{S}_P$ is Birkhoff-like.

Theorem 5.6 *For every choice of $P = \langle P_\lambda, N_\lambda \rangle$ as above, the system $\mathcal{S}_P$ is Birkhoff-like.*

Proof. For B(i), observe that all single–premiss rules are of the form (Rp) or (Rn). So suppose that for some surjective substitution σ, $\dfrac{\sigma(\varphi)}{\psi}$ is an instance of the rule (Rp). This means that there is some n-ary λ and some $k \leq n$, such that $\varphi = \langle t, s\rangle, \psi = \langle u, v\rangle$, $u = \lambda(t_1,\ldots,t_{k-1},\sigma(t),t_{k+1},\ldots,t_n)$ and $v = \lambda(t_1,\ldots,t_{k-1},\sigma(s),t_{k+1},\ldots,t_n)$. For $1 \leq i \leq n, i \neq k$ let t_i' be a term such that $\sigma(t_i') = t_i$. Let also

$$\xi := \langle \lambda(t_1',\ldots,t_{k-1}',t,t_{k+1}',\ldots,t_n'), \lambda(t_1',\ldots,t_{k-1}',s,t_{k+1}',\ldots,t_n')\rangle.$$

Clearly, $\dfrac{\varphi}{\xi}$ is an instance of the rule (Rp). The case of the rule (Rn) is handled similarly.

For B(ii), note that if φ or ψ in the statement of B(ii) is an instance of the axiom, then the conclusion is obvious. So we need to prove that every derivation (5.11) in which the first rule used was (T) and the second rule used was either (Rp) or (Rn) can be replaced by a derivation of the form (5.12). So assume that this last rule is (Rp) for some λ and k and suppose that we have a derivation

$$\dfrac{\dfrac{\langle t_1, t_2, \rangle, \langle t_2, t_3\rangle}{\langle t_1, t_3\rangle}}{\langle \lambda(t_1), \lambda(t_3)\rangle},$$

where $\lambda(t_1)$ stands really for $\lambda(s_1, \ldots, s_{k-1}, t, s_{k+1}, \ldots, s_n)$, where n is the arity of λ, k is some number between 1 and n and s_i, for $1 \leq i \leq n, i \leq k$, are some terms and $\lambda(t_3)$ stands for $\lambda(s_1, \ldots, s_{k-1}, s, s_{k+1}, \ldots, s_n)$, with the same k and terms s_i. This derivation can be replaced by applying the rule (Rp) to both $\langle t_1, t_2 \rangle$ and $\langle t_2, t_3 \rangle$ first and then applying the transitivity rule (T), i.e.,

$$\frac{\dfrac{\langle t_1, t_2 \rangle}{\langle \lambda(t_1), \lambda(t_2) \rangle} \quad \dfrac{\langle t_2, t_3 \rangle}{\langle \lambda(t_2), \lambda(t_3) \rangle}}{\langle \lambda(t_1), \lambda(t_3) \rangle}.$$

The proof for the rule (Rn) is similar. □

Notice that $\mathcal{S}_P$ matrices are pairs $\langle \mathbf{A}, \leq \rangle$, where $\mathbf{A}$ is a Λ-algebra and $\leq$ is a quasi-order on A with the property that if $k \in P_\lambda$ then the polynomial

$$p(x) = \lambda(a_1, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_n), \tag{5.13}$$

for a_i for $1 \leq i \leq n, i \neq k$, is monotone with respect to $\leq$ and if $k \in N_\lambda$, then (5.13) is anti-monotone with respect to $\leq$.

As a special case consider $\Lambda = \{\neg, \rightarrow, \vee\}$ and let $P_\neg = \emptyset = N_\vee$, $P_\rightarrow = \{2\}$, $P_\vee = \{1, 2\}$ and $N_\neg = N_\rightarrow = \{1\}$.

Then $\mathcal{S}_P$ is axiomatized by (I), (T), (S) and the following rules:

$$\frac{x \leq y}{\neg y \leq \neg x}$$

$$\frac{x \leq y}{z \rightarrow x \leq z \rightarrow y}$$

$$\frac{x \leq y}{x \rightarrow z \leq y \rightarrow z}$$

$$\frac{x \leq y}{x \vee z \leq y \vee z}$$

$$\frac{x \leq y}{z \vee x \leq z \vee y}.$$

Boolean algebras are examples of $\mathcal{S}_P$-matrices for this π.

Similarly, if for the language $\{+,-,0\}$, where $+$ is binary, $-$ unary and 0 nullary, we let $P_+ = \{1,2\}$, $N_+ = \emptyset$, $P_- = \{1\}$, $N_- = \emptyset$ and $P_0 = N_0 = \emptyset$ (the only possible choice as the arity of 0 is 0), then we get the rules that are clearly satisfied by $\langle \mathbf{Z}, +, -, 0, \leq \rangle$ and $\langle \mathbf{R}, +, -, 0, \leq \rangle$, where $\mathbf{Z}$ and $\mathbf{R}$ are the set of integers and reals, respectively, $+$ is the operation of addition of real numbers, $-$ is the operation of taking the number with the opposite sign and same absolute value, 0 is interpreted as the number 0 and $\leq$ is the standard ordering of real numbers. equipped with the ordinary addition and taking the opposite as well as 0 and the usual ordering relation on numbers.

If in Example 5.1 we let $N_\lambda = \emptyset$ for every $\lambda \in \Lambda$, then we obtain as a special case the following Birkhoff-like system.

Example 5.2 Let $\mathcal{S}_{(\leq)}$ be based by the following axiom and rules.

I $x \leq x$

T $\dfrac{x \leq y,\ y \leq z}{x \leq z}$

R′ $\dfrac{x \leq y}{\lambda(x_1, \ldots, x_n)[x/x_k] \leq \lambda(z_1, \ldots, z_n)[y/z_k]}$, for every n, every n-ary operation symbol $\lambda \in \Lambda$ and every $k < n$.

Corollary 5.7 *The system $\mathcal{S}_{(\leq)}$ considered above is Birkhoff-like.*

Proof. The corollary follows directly from lemma 5.6. □

The rule (R′) is called a replacement rule and in the presence of the transitivity rule (T), is equivalent to the rule (R) introduced in chapter 2. Let us mention, however, that (R) is a multi-premiss rule and that the axiomatization (I), (T), (R) of $\mathcal{S}_{(\leq)}$ is not Birkhoff-like. The models of system $\mathcal{S}_{(\leq)}$ considered here are called "ordered algebras", i.e., they are pairs $\langle \mathbf{A}, \leq \rangle$ consisting of an algebra $\mathbf{A}$ and a quasi-order $\leq$, with respect to which all operations are monotone. Classes of ordered algebras are studied in [57, section 4.2.1]. Results of [57, section 4.2.1] concerning ordered algebras are now also consequences of the general theory of protoalgebraic K-deductive systems.

The following theorem motivates the name "Birkhoff-like".

Theorem 5.8 *The deductive system $\mathcal{B}$ of equational logic is Birkhoff-like.*

Proof. Recall that $\mathcal{B}$ is axiomatized by (I), (T), (R′) and the symmetry rule (S). Suppose that $\dfrac{\sigma(\varphi)}{\psi}$ is an instance of the rule (S). Then φ must be of the form $\langle t, s \rangle$, for some terms t, s and $\psi = \langle \sigma(s), \sigma(t) \rangle$. Let $\xi = \langle s, t \rangle$. Then $\psi = \sigma(\xi)$ and $\dfrac{\varphi}{\xi}$ is an instance of (S). This, together with Corollary 5.7 guarantees that condition B(i) holds. To check that B(ii) holds, consider a derivation 5.11, where the first rule applied is (T) and the next is (S). Thus (5.11) is of the form

$$\dfrac{\dfrac{x \approx y \quad y \approx z}{x \approx z}}{z \approx x}.$$

This derivation can be replaced by

$$\dfrac{\dfrac{y \approx z}{z \approx y} \quad \dfrac{x \approx y}{y \approx x}}{z \approx x}.$$

This together with Corollary 5.7 guarantees that B(ii) holds for $\mathcal{S}_P$. □

Example 5.3 Let $\mathcal{S}_t$ be the 2-deductive system based by (I), (R′) and (S).

It follows from the proof of Theorem 5.8 that $\mathcal{S}_t$ is Birkhoff-like. A $\mathcal{S}_t$-matrix is a pair consisting of an algebra together with a reflexive symmetric relation that is closed under the rule (R′). Such relations are called *tolerance relations* on $\mathbf{A}$.

5.4 Compatibility relation

Definition 5.9 *Let $\mathcal{S}_1$ be a K_1-deductive system and $\mathcal{S}_2$ a Birkhoff-like K_2-deductive system over the same language Λ. Let $\mathbf{A}$ be a Λ-algebra. Recall that $E_{K_2}(A) := \coprod_{R\in\ K} A_R$ is the set of all K_2-elements of A, i.e., $E_{K_2}(A)$ is the universal $\mathcal{S}_2$-filter on $\mathbf{A}$. Recall that $Fi_{\mathcal{S}_1}(\mathbf{A})$ is the set of all $\mathcal{S}_1$-filters on $\mathbf{A}$. Let $C \subseteq E_{K_2}(A) \times Fi_{\mathcal{S}_1}(\mathbf{A})$. Then C is called a $(\mathcal{S}_1, \mathcal{S}_2)$-***compatibility relation on A** with respect to $\mathcal{S}_1$ and $\mathcal{S}_2$ if the following conditions hold.*

C(i) *For every $F \in Fi_{\mathcal{S}_1}(\mathbf{A})$ and every $R\vec{a} \in \mathrm{Fg}^{\mathbf{A}}(\emptyset)$, $\langle R\vec{a}, F\rangle \in C$.*

C(ii) *For every $F \in Fi_{\mathcal{S}_1}(\mathbf{A})$, the set $\{R\vec{a} \in E_{K_2}(A) : \langle R\vec{a}, F\rangle \in C\}$ is closed under all multiple-premiss rules of $\mathcal{S}_2$.*

C(iii) *For every $R\vec{a} \in E_{K_2}(A)$ and any system $\langle F_i : i \in I\rangle \in Fi_{\mathcal{S}_1}(\mathbf{A})^I$,*

$$\langle R\vec{a}, F_i\rangle \in C \text{ for all } i \in I \text{ implies that } \langle R\vec{a}, \textstyle\bigcap_{i\in I} F_i\rangle \in C.$$

Now let $C = \langle C_{\mathbf{A}} : \mathbf{A}$ is a Λ-algebra$\rangle$ be a system of $(\mathcal{S}_1, \mathcal{S}_2)$-compatibility relations, one for each Λ-algebra $\mathbf{A}$. Then C is **uniform** *if the following condition holds*

C(iv) *For any homomorphism $f : \mathbf{A} \longrightarrow \mathbf{A}$ and any $F \in Fi_{\mathcal{S}_1}(\mathbf{A})$, for every $R\vec{a} \in E_{K_2}(A)$, if $\langle R(f(\vec{a}), F\rangle \in C_{\mathbf{B}}$ then $\langle R\vec{a}, f^{-1}F\rangle \in C_{\mathbf{A}}$ and if f is onto, then $\langle R(f(\vec{a}), F\rangle \in C_{\mathbf{B}}$ is equivalent to $\langle R\vec{a}, f^{-1}F\rangle \in C_{\mathbf{A}}$.*

The K_2-elements of A will be often denoted by the greek letters α, β, γ, etc. We will often write $C(\alpha, F)$ for $\langle \alpha, F\rangle \in C$.

Definition 5.10 *Let $\mathcal{S}_1$ be a K_1-deductive system and let K_2 have only one, binary, relation symbol. We will write the K_2-elements as pairs of $\langle a, b\rangle$ of elements of A.*

Let $\mathbf{A}$ *be a* Λ*-algebra. Define* $C = C_{\mathbf{A}} \subseteq E_{K_2}(A) \times Fi_{\mathcal{S}_1}(\mathbf{A})$ *as follows. For each* $\langle a, b\rangle \in \mathbf{A}$ *and* $F \in Fi_{\mathcal{S}_1}(\mathbf{A})$, $\langle\langle a, b\rangle, F\rangle \in C_{\mathbf{A}}$ *iff for every* $R \in K_1$ *and for every* $\vec{c} \in A^{\rho(R)-1} = \langle c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_{\rho(R)}\rangle$ *and every* i *with* $1 \leq i \leq \rho(R)$

$$R(c_1, \ldots, c_{i-1}, a, c_{i+1}, \ldots, c_{\rho(R)}) \in F \Rightarrow$$

$$R(c_1, \ldots, c_{i-1}, b, c_{i+1}, \ldots, c_{\rho(R)}) \in F. \tag{5.14}$$

The system C *is called the* **standard system of $(\mathcal{S}_1, \mathcal{S}_2)$-compatibility relations.**

The term "$(\mathcal{S}_1, \mathcal{S}_2)$-compatibility relations" in the above definition is justified by Corollary 5.12 below.

Lemma 5.11 *Let* $\mathcal{S}_1$ *be a* K_1*-deductive systemand* $\mathcal{S}_2$ *a* 2*-deductive system. Let* $C = \langle C_{\mathbf{A}} : \mathbf{A}$ *is a* Λ*-algebra*$\rangle$ *be the standard system of* $(\mathcal{S}_1, \mathcal{S}_2)$*-compatibility relations. Then:*

1. *the system* C *satisfies the conditions C(iii)–C(iv) of Definition 5.9.;*

2. *if* $\mathcal{S}_2$ *does not have other multiple-premiss rules than possibly (T), then* C *also satisfies C(ii).*

3. *Thus under the assumption of 2., a sufficient condition for* C *to be a uniform system of* $(\mathcal{S}_1, \mathcal{S}_2)$*-compatibility relations is that for every axiom* $\langle t, s\rangle$ *of* $\mathcal{S}_2$, *for every* Λ*-algebra* $\mathbf{A}$ *and* $\mathcal{S}_1$*-filter* F *on* $\mathbf{A}$, *for every relation symbol* R *of* K_2, *for every* $1 \leq i \leq \rho(R)$ *for all elements* $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_{\rho(R)}$ *and for every valuation* $f : \mathrm{Te} \to \mathbf{A}$, *we have that* $R(c_1, \ldots, c_{i-1}, f(t), c_{i+1}, \ldots, c_{\rho(R)}) \in F \Rightarrow R(c_1, \ldots, c_{i-1}, f(s), c_{i+1}, \ldots, c_{\rho(R)}) \in F$.

Proof. Let us fix some algebras $\mathbf{A}, \mathbf{B}$, a family $\langle F_i : i \in I\rangle$ of $\mathcal{S}_1$-filters on $\mathbf{A}$ indexed by some set I, some homomorphism $f : \mathbf{A} \to \mathbf{B}$ and an $\mathcal{S}_1$-filter F on $\mathbf{B}$. Since $E_{K_2}(A)$ is identified with A^2, for the first statement of the lemma we need to show

C(iii) If for every $i \in I$, $\langle\langle a, b\rangle, F_i\rangle \in C$, then also $\langle\langle a, b\rangle, \bigcap_{i\in I} F_i\rangle \in C$ and

C(iv) If $\langle\langle fa, fb\rangle, F\rangle \in C$, then also $\langle\langle a, b\rangle, f^{-1}F\rangle \in C$; and if f is onto, then $\langle\langle fa, fb\rangle, F\rangle \in C$ is equivalent to $\langle\langle a, b\rangle, f^{-1}F\rangle \in C$.

Let $R \in \mathrm{K}_1$ and let $\vec{c}$ be a sequence $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_{\rho(R)}$ of elements of A.

The condition C(iii) then follows from the following equivalence that is obviously true for every x:

$$\forall_{i\in I} R(c_1, \ldots, c_{k-1}, x, c_{k+1}, \ldots, c_{\rho(R)}) \in F_i \Rightarrow R(c_1, \ldots, c_{k-1}, x, c_n) \in \textstyle\bigcap_{i\in I} F_i.$$

For assume that $\forall_{i\in I}\langle\langle a, b\rangle, F_i\rangle \in C$ and assume that $R(c_1, \ldots, c_{k-1}, a, c_{k+1}, c_n) \in \bigcap_{i\in I} F_i$. Then $R(c_1, \ldots, c_{k-1}, a, c_{k+1}, \ldots, c_{\rho(R)}) \in F_i$, for every $i \in I$. Hence also

$$R(c_1, \ldots, c_{k-1}, b, c_{k+1}, \ldots, c_{\rho(R)}) \in F_i,$$

for every $i \in I$. This is equivalent to the condition $R(c_1, \ldots, c_{k-1}, b, c_{k+1}, \ldots, c_{\rho(R)}) \in \bigcap_{i\in I} F_i$. This shows that $\langle\langle a, b\rangle, \bigcap_{i\in I} F_i\rangle$.

For C(iv), assume that for every sequence $\vec{c'} = \langle c'_1, \ldots, c'_{k-1}, c'_{k+1}, \ldots, c'_{\rho(R)}\rangle$ of elements of $\mathbf{B}$, we have that

$$R(c'_1, \ldots, c'_{k-1}, f(a), c'_{k+1}, \ldots, c'_{\rho(R)}) \in F \Rightarrow R(c'_1, \ldots, c'_{k-1}, f(b), c'_{k+1}, \ldots, c'_{\rho(R)}) \in F. \tag{5.15}$$

Assume that $R(c_1, \ldots, c_{k-1}, a, c_{k+1}, \ldots, c_{\rho(R)}) \in f^{-1}F$. Then

$$R(f(c_1), \ldots, fc_{k-1}), f(a), f(c_{k+1}), \ldots, f(c_{\rho(R)})) \in F$$

and therefore

$$R(f(c_1),\dots,fc_{k-1}),f(b),f(c_{k+1}),\dots,f(c_{\rho(R)}))\in F$$

by assumption (5.15). This finishes the proof that

$$R(c_1,\dots,c_{k-1},a,c_{k+1},\dots,c_{\rho(R)})\in f^{-1}F\Rightarrow R(c_1,\dots,c_{k-1},b,c_{k+1},\dots,c_{\rho(R)})\in f^{-1}F \tag{5.16}$$

Hence $R(c_1,\dots,c_{k-1},a,c_{k+1},\dots,c_{\rho(R)})\in f^{-1}F$. It is clear that if f is chosen to be onto $\mathbf{B}$, then (5.15) and (5.16) are equivalent.

This finishes the proof of C(iv) and of the first statement of the lemma.

For the second statement assume that the only multiple-premiss rule is (T). But the relation $\Rightarrow$ is transitive, i.e.,

$$[(\vec{a}\in F\Rightarrow\vec{b}\in F)\text{ and }(\vec{b}\in F\Rightarrow\vec{c}\in F)]\Rightarrow(\vec{a}\in F\Rightarrow\vec{c}\in F),$$

so C(ii) holds.

The third statement follows from the first two. □

Corollary 5.12 *Let $\mathcal{S}_2$ be a Birkhoff-like 2-deductive system, such that $\mathcal{S}_2$ does not have other multiple-premiss rules than possibly (T) and the only axiom of $\mathcal{S}_2$ is $\langle x,x\rangle$. Then for every K_1 and for every K_1-deductive system $\mathcal{S}_1$, the standard system of compatibility relations (Definition 5.10) is a uniform system of $(\mathcal{S}_1,\mathcal{S}_2)$-compatibility relations. In particular, let $\mathcal{S}_1$ be a 1-deductive system. Then the system $C=\langle C_{\mathbf{A}}:\mathbf{A}$ is a Λ-algebra$\rangle$, where $C_{\mathbf{A}}(\langle a,b\rangle\rangle,F)$ iff $a\in F\Rightarrow b\in F\rangle$, is a uniform system of $(\mathcal{S}_1,\mathcal{S}_2)$-compatiblity relations.*

Proof. By 3. of lemma 5.11.

Corollary 5.13 *Let $\mathcal{S}_2$ be one of the systems $\mathcal{S}_P, \mathcal{S}_{(\leq)}, \mathcal{S}_t, \mathcal{B}$ considered above, and let $\mathcal{S}_1$ be a K_1-deductive system, for some K_1. Then the standard system of $(\mathcal{S}_1, \mathcal{S}_2)$-compatibility relations is a uniform system of $(\mathcal{S}_1, \mathcal{S}_2)$-compatiblity relations. In particular, let $\mathcal{S}_1$ be a 1-deductive system. Then the system $C = \langle C_\mathbf{A} : \mathbf{A} \rangle$, such that $C_\mathbf{A}(\langle a, b\rangle\rangle, F)$ iff $a \in F \Rightarrow b \in F\rangle$, is a uniform system of $(\mathcal{S}_1, \mathcal{S}_2)$-compatiblity relations.*

Proof. Apply Corollary 5.12. □

Recall that the condition 5.14 was used in Chapter 2, Definition 2.37 to define the compatibility of a congruence θ on an algebra $\mathbf{A}$, i.e., an $\mathcal{B}$-filter on $\mathbf{A}$, with an $\mathcal{S}$-filter on $\mathbf{A}$. We now introduce a more general concept of compatibility of an $\mathcal{S}_2$-filter with an $\mathcal{S}_1$-filter. This concept is relativized to a uniform system of compatibility relations C and therefore also to the systems $\mathcal{S}_1$ and $\mathcal{S}_2$.

Definition 5.14 *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be a K_1 and K_2-deductive system, respectively. Let C be a uniform system of compatibility relations between K_2-elements and $\mathcal{S}_1$-filters. Let $\mathbf{A}$ be a Λ-algebra. We say that an $\mathcal{S}_2$-filter θ is C-***compatible with** *an $\mathcal{S}_1$-filter F if for every $\alpha \in \theta$ we have $C(\alpha, F)$.*

Let $\mathcal{S}_2$ be the 2-deductive system $\mathcal{S}_P$ defined in the Example 5.1 page 147, and let C be the uniform system of compatibility relations (Definition 5.10). Recall that the models of $\mathcal{S}_P$ are algebras ordered by a quasi-order relation $\leq$ with the property, that the polynomials of the form $\lambda(a_1, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_n)$ are monotone in x if $k \in P_\lambda$ and antimonotone in x if $k \in N_\lambda$. Then this quasi-order $\leq$ is C-compatible with an $\mathcal{S}_1$-filter F iff for all $a \leq b$ in A, $a \in F$ implies $b \in F$.

Notice also, that if the 2-deductive system $\mathcal{S}_2$ has the symmetry rule (S), and C is standard (Definition 5.10) then an $\mathcal{S}_2$-filter θ on $\mathbf{A}$ is compatible with an $\mathcal{S}_1$-filter

F on $\mathbf{A}$, if for every $R \in \mathrm{K}_1$ and for all $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_{\rho(R)} \in A$, we have

$$R(c_1, \ldots, c_{k-1}, a, c_{k+1}, \ldots, c_{\rho(R)}) \in F \text{ iff } R(c_1, \ldots, c_{k-1}, b, c_{k+1}, \ldots, c_{\rho(R)}) \in F, \tag{5.17}$$

i.e., the implication in definition 5.10 is replaced by the equivalence. Thus we see that in case that $\mathcal{S}_2$ is the Birkhoff's deductive system for equational logic, then the congruence θ is compatible with an $\mathcal{S}_1$-filter iff it is compatible in the sense of definition from chapter 2.

5.5 Generalized Leibniz operator

Definition 5.15 *Let $\mathcal{S}_1, \mathcal{S}_2$ be K_1- and K_2-deductive systems and let C be a uniform system of $(\mathcal{S}_1, \mathcal{S}_2)$-compatibility relations. Let $\mathbf{A}$ be a Λ-algebra. The* **generalized Leibniz operator on A** *is the function that to every $\mathcal{S}_1$-filter F of $\mathbf{A}$ assigns the following K_2-subset $\Omega^C(F)$ of A :*

$$\Omega^C(F) := \{\alpha \in E_{K_2}(A) : \textit{ for every } \beta \in \mathrm{Fg}^{\mathfrak{A}}_{\mathcal{S}_2}(\alpha),\ C(\beta, F)\}.$$

Let $\mathcal{S}_1$ be a K_1-deductive system, let $\mathcal{S}_2$ be a K_2-deductive system in some Birkhoff-like axiomatization Γ. Let Γ' be the set of all axioms and one-premiss rules of Γ. Let $\mathbf{A}$ be a Λ-algebra, $F \in \mathrm{Fi}_{\mathcal{S}_1}(\mathbf{A})$.

Lemma 5.16 *Assume that $\mathcal{S}_2$ is a Birkhoff-like K_2-deductive system and let C be a uniform system of $(\mathcal{S}_1, \mathcal{S}_2)$-compatibility relations. Let α be an K_2-element of A. Then*

$$\Omega^C(F) = \{\alpha \in E_{K_2}(A) : \textit{ for every } \beta \in \mathrm{Fg}_{\Gamma'}(\alpha),\ \ C(\beta, F)\}.$$

Proof. We need to prove that

(i) for all $\beta \in \mathrm{Fg}^{\mathbf{A}}_{\Gamma'}(\alpha)$ $C(\beta, F)$

implies

(ii) for all $\beta \in \mathrm{Fg}^{\mathbf{A}}_{\Gamma}(\alpha)$ $C(\beta, F)$.

We first prove this implication for the special case that $\mathbf{A} = \mathrm{Te}$, $\alpha = \varphi \in \mathrm{Fm}_{\mathrm{K}_2}, \beta = \psi \in \mathrm{Fm}_{\mathrm{K}_2}$ and $F = T$ is an $\mathcal{S}_1$-theory. Thus we are proving the following claim

Claim 2 *Let T be an $\mathcal{S}_1$-theory, let φ be a K_2-formula. Then*

$$[\forall_{\psi}(\varphi \vdash_{\Gamma'} \psi \Rightarrow C(\psi, T))] \Rightarrow \tag{5.18}$$

$$[\forall_{\psi}(\varphi \vdash_{\Gamma} \psi \Rightarrow C(\psi, T))] \tag{5.19}$$

Assume that 5.18 is true and suppose that $\varphi \vdash_{\Gamma} \psi$. If $\varphi \vdash_{\Gamma'} \psi$, then we are done. So assume that $\varphi \not\vdash_{\Gamma'} \psi$. Therefore some multiple-premiss rules must have been used in a derivation of ψ from φ. In view of B(ii), we can assume that for some $n \geq 1$ and for some K_2-formulas $\psi_1, \ldots, \psi_n$,

$$\varphi \vdash_{\Gamma'} \psi_1, \ldots, \psi_n \tag{5.20}$$

$$\psi_1, \ldots, \psi_n \vdash_{\Gamma \setminus \Gamma'} \psi, \tag{5.21}$$

where $\psi_i \neq \psi$, for any $i = 1, \ldots, n$ and ψ is not an axiom. By assumption, for every $i = 1, \ldots, n$, we have $C(\psi_i, F)$. By C(ii), $C(\psi, T)$. This finishes the proof of Claim 1.

Now let algebra $\mathbf{A}$ be arbitrary and let us return to the proof that (i) implies (ii). So let us assume (i) and also let $\beta \in \mathrm{Fg}^{\mathbf{A}}_{\Gamma}(\alpha)$. Then there are a valuation f, K_2-formulas φ and ψ such that $f(\varphi) = \alpha$, $f(\psi) = \beta$ and $\varphi \vdash_{\Gamma} \psi$. Without the loss of generality, we can assume that f is surjective. We now claim that the condition 5.18 holds for φ and $T := f^{-1}F$. For suppose that $\varphi \vdash_{\Gamma'} \xi$, for some $\xi \in \mathrm{Fm}_{\mathrm{K}_2}$. Then by our assumption (i), we know, that $C(f(\xi), F)$ holds, and by C(iv), also $C(\xi, T)$ holds. This verifies 5.18. Therefore 5.19 holds as well and in particular, $C(\psi, T)$. By C(iv) again, $C(\beta, F)$. This finishes the proof of the lemma. □

In the next chapter, we will apply Lemma 5.16 to a generalization of the system $\mathcal{S}_P$. On page 150 we considered a special case of $\mathcal{S}_P$, namely a system $\mathcal{S}_{(\leq)}$ axiomatized by (I), (T) and (Rp). We already mentioned that this system is Birkhoff-like and that the relations C defined in definition 5.10 form a uniform system of $(\mathcal{S}_1, \mathcal{S}_2)$-compatibility relations for every system $\mathcal{S}_1$. According to lemma 5.16, for this system $\mathcal{S}_{(\leq)}$, $\langle a, b\rangle \in \Omega^C(F)$ iff for every term $t(x, \vec{y})$ and every sequence $\vec{e}$ of elements of A of the same length that $\vec{y}$, we have that $t(a, \vec{e}) \in F \Rightarrow t(b, \vec{e}) \in F$.

For $\mathcal{S}_t$, lemma 5.16 says that $\langle a, b\rangle \in \Omega^C(F)$ iff for every term $t(x, \vec{y})$ and every sequence $\vec{e}$ of elements of A of the same length as $\vec{y}$, we have that $t(a, \vec{e}) \in F$ iff $t(b, \vec{e}) \in F$. Since Lemma 5.16 characterizes Ω^C independently of the multiple-premiss rules, and since the single-premiss rules of the Birkhoff's system $\mathcal{B}$ and of $\mathcal{S}_t$ are the same, Ω^C for $\mathcal{B}$ is characterized in exactly same way as for $\mathcal{S}_t$. Notice, that this means that when $\mathcal{S}_2 = \mathcal{B}$ and C is the standard system of $(\mathcal{S}_1, \mathcal{S}_2)$-compatibility relations, then $\Omega^C(F)$ is exactly the Leibniz congruence for F, i.e., the largest congruence compatible with F. This fact is generalized in theorem 5.18. We first prove an auxiliary lemma.

Lemma 5.17 *Let $\mathcal{S}_1, \mathcal{S}_2$ be some K_1 and K_2-deductive systems and let C be some uniform system of $(\mathcal{S}_1, \mathcal{S}_2)$-compatibility relations. Let also, for some Λ-algebras* $\mathbf{A}$ *and* $\mathbf{B}$, $f : \mathbf{A} \longrightarrow \mathbf{B}$ *and* $F \in Fi_{\mathcal{S}_1}(\mathbf{B})$. *Then* $f^{-1}(\Omega^C(F)) \subseteq \Omega^C(f^{-1}F)$.

Proof. Suppose that $\alpha \in f^{-1}(\Omega^C(F))$, i.e., $f(\alpha) \in \Omega^C(F)$. Let $\beta \in \mathrm{Fg}^{\mathfrak{A}}_{\mathcal{S}_2}(\alpha)$. We want to show that $C(\beta, f^{-1}F)$. But then $f(\beta) \in \mathrm{Fg}^{\mathfrak{B}}_{\mathcal{S}_2}(f(\alpha))$ and therefore $C(f(\beta), F)$. By C(iv), $C(\beta, f^{-1}F)$. $\square$

Theorem 5.18 *Assume that $\mathcal{S}_2$ is a Birkhoff-like K_2-deductive system. Then for every K_1, every K_1-deductive system $\mathcal{S}_1$ and for every uniform system of $(\mathcal{S}_1, \mathcal{S}_2)$-compatibility relations C, for every Λ-algebra* $\mathbf{A}$ *and an $\mathcal{S}_1$-filter F on* $\mathbf{A}$, *the set $\Omega^C(F)$ is the largest $\mathcal{S}_2$-filter C-compatible with F.*

Proof. Let Γ be a Birkhoff-like axiomatization of $\mathcal{S}_2$. To show that $\Omega^C(F)$ is an $\mathcal{S}_2$-filter, we first prove the following claim.

Claim 3 *Let T be an $\mathcal{S}_1$-theory and let $\varphi_1, \ldots, \varphi_n, \psi$ be some K_2-formulas. If φ is derivable from $\varphi_1, \ldots, \varphi_n$ by means of Γ and $\varphi_1, \ldots, \varphi_n \in \Omega^C(T)$, then $C(\varphi, T)$.*

Proof of Claim.

Assume that φ is derivable from $\varphi_1, \ldots, \varphi_n$ by means of Γ. By B(ii) there are derivations $\varphi_1, \ldots, \varphi_n \vdash_{\Gamma'} \psi_1, \ldots, \psi_m$ and $\psi_1, \ldots, \psi_m \vdash_{\Gamma \setminus \Gamma'} \varphi$. Then for every $i = 1, \ldots, m$ there exists $j = 1, \ldots, n$ such that $\psi_i \in \mathrm{Cn}_{\Gamma'}(\varphi_j)$, so $\psi_i \in \Omega^C(F)$, hence $C(\psi_i, F)$ holds. By C(ii), also $C(\varphi, F)$ holds, which finishes the proof of the claim.

Now to finish the proof that $\Omega^C(F)$ is a $\mathcal{S}_2$-filter, let $\mathbf{A}$ be an arbitrary Λ-algebra, let F be a $\mathcal{S}_1$-filter on $\mathbf{A}$. We need to show that for all $\alpha_1, \ldots, \alpha_n \in \Omega^C(F)$ and for every $\beta \in \mathrm{Fg}^{\mathfrak{A}}_{\mathcal{S}_2}(\alpha_1, \ldots, \alpha_n)$, also $\beta \in \Omega^C(F)$. For this let us take $\gamma \in \mathrm{Fg}^{\mathfrak{A}}_{\mathcal{S}_2}(\beta)$. We need to show that $C(\beta, F)$. Since $\gamma \in \mathrm{Fg}^{\mathfrak{A}}_{\mathcal{S}_2}(\beta)$, $\gamma \in \mathrm{Fg}^{\mathfrak{A}}_{\mathcal{S}_2}(\alpha_1, \ldots, \alpha_n))$, as well.

But then there exist some K_2-formulas $\varphi_1, \ldots, \varphi_n, \psi$ and a valuation f such that $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{S}_2} \psi$ and $\alpha_i = f(\varphi_i), \gamma = f(\psi)$. Moreover, by lemma 5.17, $\varphi_i \in \Omega^C(T)$, where $T = f^{-1}F$. But then, by Claim, $C(\psi, T)$ and therefore $C(\gamma, F)$, as needed. This finishes the proof that $\Omega^C(F)$ is a $\mathcal{S}_2$-filter.

It follows directly from the definition of Ω^C and the fact, that $\alpha \vdash_{\mathcal{S}} \alpha$, for every system $\mathcal{S}$, that $\Omega^C(F)$ is compatible with F.

If θ is another $\mathcal{S}_2$-filter C-compatible with F, then let $\alpha \in \theta$. Let $\beta \in \mathrm{Fg}^{\mathfrak{A}}_{\mathcal{S}_2}(\alpha)$. Since θ is a $\mathcal{S}_2$-filter, it follows that $\beta \in \theta$ and therefore $C(\beta, F)$. This shows that $\alpha \in \Omega^C(F)$, hence $\theta \subseteq \Omega^C(F)$. □

5.6 Equivalent semantics Theorem

For a predicate language L and a class $\mathcal{K}$ of L-matrices, let $\mathcal{S}_{\mathcal{K}}$ be the L-deductive system determined by all the L-rules that are valid in every matrix from $\mathcal{K}$. The consequence relation $\vdash_{\mathcal{S}_{\mathcal{K}}}$ will also be denoted by $\models_{\mathcal{K}}$.

Theorem 5.19 *(Equivalent Semantics Theorem) Assume that $\mathcal{S}_1, \mathcal{S}_2$ are K_1, K_2-deductive systems, respectively, $\mathcal{S}_2$ is Birkhoff-like and C is a uniform system of $(\mathcal{S}_1, \mathcal{S}_2)$-compatibility relations. If the generalized Leibniz operator $\Omega^C : \mathrm{Th}\mathcal{S}_1 \to \mathrm{Th}\mathcal{S}_2$ is injective and continuous, then there exists a class $\mathcal{K}$ of $\mathcal{S}_2$-matrices such that $\mathcal{S}_1$ and $\models_{\mathcal{K}}$ are equivalent.*

Moreover, in this case there are (K_1, K_2)- and (K_2, K_1)-translations τ and v, respectively, such that the following conditions hold for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathrm{K}_1}$, $\psi \in \mathrm{Fm}_{\mathrm{K}_2}$ and $\mathcal{S}_1$-theories T.

$$\Gamma \vdash_{\mathcal{S}_1} \varphi \textit{ iff } \tau(\Gamma) \models_{\mathcal{K}} \tau(\varphi)$$

$$\tau(v(\psi)) \dashv\models_{\mathcal{K}} \psi \text{ and}$$

$$T \vdash_{\mathcal{S}_1} \varphi \text{ implies } C(\tau(\varphi), T).$$

Proof. The proof breaks into a sequence of lemmas.

Lemma 5.20 *Let* **A** *be a* Λ*-algebra and* F_i *a family of filters on* **A** *indexed by some set* I*. If* Ω^C *is order-preserving on* $Fi_{\mathcal{S}_1}(\mathbf{A})$*, then*

$$\Omega^C(\bigcap_{i\in I} F_i) = \bigcap_{i\in I} \Omega^C(F_i).$$

Proof. The inclusion from left to right follows from the fact that Ω^C is order preserving. To prove the other inclusion, let α be a K_2-element of **A** such that $\alpha \in \bigcap(\Omega^C(F_i)$. Then $C(\alpha, F_i)$ holds for every i. By C(iii) $C(\alpha, \bigcap F_i)$. This shows that $\bigcap \Omega^C(F_i)$ is compatible with $\bigcap F_i$. Hence $\bigcap \Omega^C F_i \subseteq \Omega^C \bigcap F_i$, by theorem 5.18. □

Lemma 5.21 *Let* **A** *be the term algebra and let* T *be an* $\mathcal{S}_1$*-theory. Then for every surjective substitution* σ,

$$\sigma^{-1}(\Omega^C T) = \Omega^C(\sigma^{-1} T).$$

Proof. The inclusion from left to right follows from lemma 5.17. For the other inclusion suppose that $\varphi \in \Omega^C(f^{-1}(T))$. To show that $\varphi \in f^{-1}(\Omega^C(T))$, assume that $\psi \in \mathrm{Cn}_{\Gamma'} f(\varphi)$. We need to show $C(\psi, T)$. Since $\psi \in \mathrm{Cn}_{\Gamma'}\varphi$, then by B(i) (see a remark on page 147), we conclude that there is a ψ' such that $\varphi \vdash_{\Gamma'} \psi'$ and $f(\psi') = \psi$. But since $\varphi \in \Omega^C(f^{-1}(T))$, we know that $C(\psi', f^{-1}F)$ and therefore, since f is onto, we have, by C(iv), that $C(\psi, F)$. This finishes the proof of the lemma. □

Lemma 5.22 *Let* $\mathcal{K} := \{\langle \mathbf{Te}, \Omega^C T\rangle : T \in \mathrm{Th}_{\mathcal{S}_1}\}$. *If* Ω^C *is continuous on* $Fi_{\mathcal{S}_1}(\mathbf{Te})$, *then*

(i) *For every* $T \in \mathrm{Th}_{\mathcal{S}_1}$, $\Omega^C(T) \in \mathrm{Th}_{\models_{\mathcal{K}}}$.

(ii) *For every* $\Phi \in \mathrm{Th}_{\models_{\mathcal{K}}}$ *there exists a* $T \in \mathrm{Th}_{\mathcal{S}_1}$ *such that* $\Phi = \Omega^C(T)$.

Proof. (i) Suppose that $\Omega^C(T) \models_{\mathcal{K}} \varphi$. But $\langle \mathbf{Te}, \Omega^C T\rangle \in \mathcal{K}$ and $\Omega^C(T) \subseteq \Omega^C(T)$. So $\varphi \in \Omega^C(T)$. (ii) Let $\Phi \in \mathrm{Th}_{\mathcal{K}}$. Assume first that Φ is finitely generated, i.e., there is a finite set Γ of K_2-terms such that $\Phi = \mathrm{Cn}_{\mathcal{K}}\Gamma$. Let $\varphi \notin \Phi$. Then $\Gamma \not\models_{\mathcal{K}} \varphi$, hence there is $\sigma : \mathbf{Te} \to \mathbf{Te}$ and $T \in \mathrm{Th}_{\mathcal{S}_1}$ such that $\sigma\Gamma \subseteq \Omega^C(T)$ and $\sigma\varphi \notin \Omega^C(T)$. So $\Gamma \subseteq \sigma^{-1}\Omega^C(T) = \Omega^C(\sigma^{-1}T)$, by Lemma 5.21 and $\varphi \notin \sigma^{-1}\Omega^C(T) = \Omega^C(\sigma^{-1}T)$. Hence also $\Phi \subseteq \Omega^C(\sigma^{-1}T)$ and $\varphi \notin \Omega^C(\sigma^{-1}T)$. Let $S := \sigma^{-1}T$. Then S is a $\mathcal{S}_1$-theory. Thus we have shown that for every $\varphi \notin \Phi$ there is $S \in \mathrm{Th}_{\mathcal{S}_1}$ such that $\Phi \subseteq \Omega^C(S)$ and $\varphi \notin \Omega^C(S)$. So

$$\Phi = \bigcap\{\Omega^C(S) : \Phi \subseteq \Omega^C(S)\},$$

so $\Phi = \Omega^C(\bigcap\{S : \Phi \subseteq \Omega^C(S)\}$, by lemma 5.20. This finishes the proof of ii) in case that Φ is finitely generated.

Now suppose that Φ is arbitrary. Then

$$\Phi = \bigvee(\{\Phi_i : \Phi_i \subseteq \Phi,\ \Phi_i \text{ is finitely generated}\}.$$

By the first part, for every Φ as above, there is T_i such that $\Phi_i = \Omega^C(T_i)$. Also $\{\Phi_i : \Phi_i \subseteq \Phi,\ \Phi_i \text{ is finitely generated}\}$ is a directed set. By the continuity of Ω^C,

$$\begin{aligned}
\Phi &= \bigvee\{\Phi_i : \Phi_i \subseteq \Phi, \Phi_i \text{ finitely generated}\}\\
&= \bigvee\{\Omega^C(T_i) : \Omega^C(T_i) \subseteq \Phi, \Omega^C(T_i) \text{ is finitely generated}\}\\
&= \Omega^C(\bigvee\{T_i : \Omega^C(T_i) \subseteq \Phi, \Omega^C(T_i) \text{ is finitely generated}\}).
\end{aligned}$$

This proves the lemma. □

Lemma 5.23 *If Ω^C is 1-1 and continuous, then Ω^C commutes with surjective substitutions.*

Proof. Let σ be a surjective substitution. Let $T' := \mathrm{Cn}_{\mathcal{S}_1}\sigma T$. Since $\Omega^C(T')$ is an $\mathcal{K}$-theory, in order to show the inclusion $\mathrm{Cn}_{\mathcal{K}}(\sigma(\Omega^C(T))) \subseteq \Omega^C(\mathrm{Cn}_{\mathcal{S}_1}\sigma T)$, it suffices to show that $\sigma(\Omega^C(T)) \subseteq \Omega^C(T')$. Let $\varphi \in \sigma\Omega^C(T)$. Then $\varphi = \sigma\varphi'$, for some $\varphi' \in \Omega^C(T)$. To show that $\varphi \in \Omega^C(T')$, let ψ be such that $\varphi \vdash_{\Gamma'} \psi$, where Γ' is the set of all single-premiss rules in some Birkhoff-like axiomatization Γ of $\mathcal{S}_2$. Moreover, by C(i), we can assume that all the rules used in such a proof had exactly one premiss. So by B(i), there exists a ψ' such that $\varphi' \vdash_{\mathcal{S}_2} \psi'$ and $\sigma\psi' = \psi$. Now, $\psi' \in \Omega^C(T)$. Also, $\sigma(T) \subseteq \mathrm{Cn}_{\mathcal{S}_1}(\sigma(T)) = T'$, so $T \subseteq \sigma^{-1}(T')$. Since Ω^C is continuous, it also is monotone and therefore $\Omega^C(T) \subseteq \Omega^C(\sigma^{-1}T')$. Hence $\varphi \in \Omega^C(\sigma^{-1}T')$. So $C(\psi', \sigma^{-1}T')$ and by C(iv), $C(\psi, T')$ This shows that $\varphi \in \Omega^C(T')$ and completes the proof of the inclusion from right to left. Now as $\mathrm{Cn}_{\mathcal{K}}\sigma(\Omega^C(T)) \in \mathrm{Th}_{\mathcal{K}}$, by the previous lemma we have that for some $\mathcal{S}_1$-theory S, $\mathrm{Cn}_{\mathcal{K}}\sigma(\Omega^C(T)) = \Omega^C(S)$. Also,

$$\Omega^C(T) \subseteq \sigma^{-1}\mathrm{Cn}_{\mathcal{K}}(\sigma(\Omega^C(T)) = \sigma^{-1}(\Omega^C(S)) = \Omega^C(\sigma^{-1}S),$$

this last equality by Lemma 5.20. Since Ω^C is 1-1 and, by lemma 5.20 $\Omega^C(T) \cap \Omega^C(\sigma^{-1}S) = \Omega^C(T \cap \sigma^{-1}S)$, we conclude that $T \subseteq \sigma^{-1}S$. Therefore $\sigma T \subseteq S$ and $\mathrm{Cn}_{\mathcal{S}_1}\sigma T \subseteq S$. So

$$\Omega^C(\mathrm{Cn}_{\mathcal{S}_1}\sigma T) \subseteq \Omega^C(S) = \mathrm{Cn}_{\mathcal{K}}\sigma(\Omega^C(T)).$$

□

It follows that $\Omega^C : \mathrm{Th}_{\mathcal{S}_1} \to \mathrm{Th}_{\mathcal{K}}$ is an isomorphism which commutes with surjective substitutions. By Thm. 5.4 $\mathcal{S}_1$ and $\models_{\mathcal{K}}$ are equivalent. This finishes the proof of the first statement of the theorem. The first two conditions in the second statement follow immediately from theorem 5.4. For the third condition, $\Sigma(T) = \mathrm{Cn}_{\mathcal{S}_2}(\tau(\psi) : \psi \in T)$. But the role of Σ is played by Ω^C here. Since $\Omega^C(T)$ is C-compatible with T, it follows that if $\psi \in T$, then $\tau(\psi) \in \Omega^C(T)$ and therefore $C(\tau(\psi), T)$. □

5.7 Corollaries

For a K_1-deductive system $\mathcal{S}_1$, a K_2-deductive system $\mathcal{S}_2$, a (K_2, K_1)-translation v and for every Λ-algebra $\mathbf{A}$, define the operator $\Omega_v : \mathrm{Fi}_{\mathcal{S}_1}(\mathbf{A}) \longrightarrow \mathcal{P}(E_{K_2}(A))$ as follows

$$\Omega_v(F) := \{\alpha \in E_{K_2}(A) : v\alpha \in F\}.$$

Lemma 5.24 *Let $\mathcal{S}_1, \mathcal{S}_2$ be a K_1- and a K_2-deductive systems, respectively. Let v be a (K_2, K_1)-translation. Then the operator Ω_v is continuous.*

Proof. Let $\mathbf{A}$ be a Λ-algebra and let $\mathcal{F} = \{F_i : i \in I\}$ be a directed set of $\mathcal{S}_1$-filters on $\mathbf{A}$. We need to show that $\Omega_\tau(\bigvee \mathcal{F}) = \bigvee_{i \in I} \Omega_\tau(F_i)$. Let $\alpha \in E_{K_2}(A)$. Since I is directed, and $v(\alpha)$ finite, $v(\alpha) \subseteq \bigvee \mathcal{F}$ iff there is an $i \in I$ such that $v(\alpha) \subseteq F_i$ and the claim follows. □

Lemma 5.25 *Let $\mathcal{S}_1$ be a K_1-deductive system and let $\mathcal{S}_2$ be a K_2-deductive system. Let further v be a (K_2, K_1)-translation and τ a (K_1, K_2)-translation such that*

$$\phi \dashv\vdash_{\mathcal{S}_1} v\tau\varphi.$$

Then Ω_v is injective.

Proof. Suppose $\Omega_v(T) = \Omega_v(S)$ and let $\alpha \in F$. Then $v\tau\alpha \in F$ and therefore $\tau\alpha \in \Omega_v(F) = \Omega_v(G)$. Thus $v(\tau(\alpha)) \in G$ and $\alpha \in G$. It follows that $F \subseteq G$ and by a symmetric argument $F = G$. □

Corollary 5.26 (Second equivalent semantics Theorem) *Assume that $\mathcal{S}_1, \mathcal{S}_2$ are a K_1- and a K_2-deductive systems, respectively, $\mathcal{S}_2$ is Birkhoff-like and C is a uniform system of $(\mathcal{S}_1, \mathcal{S}_2)$-compatibility relations. Then the generalized Leibniz operator Ω^C :* $\mathrm{Th}\mathcal{S}_1 \to \mathrm{Th}\mathcal{S}_2$ *is injective and continuous iff there exists a class $\mathcal{K}$ of $\mathcal{S}_2$-matrices and a pair of (K_1, K_2)- and (K_2, K_1)-translations τ and v, respectively, such that the following conditions hold for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathrm{K}_1}$, $\psi \in \mathrm{Fm}_{\mathrm{K}_2}$ and $\mathcal{S}_1$-theories T.*

$$\Gamma \vdash_{\mathcal{S}_1} \varphi \textit{ iff } \tau(\Gamma) \models_{\mathcal{K}} \tau(\varphi)$$

$$\tau(v(\psi)) \dashv\models_{\mathcal{K}} \psi \textit{ and}$$

$$\Omega^C = \Omega_v.$$

Proof. If Ω^C is injective and continuous, then by Theorem 5.19 there is a class $\mathcal{K}$ and translations τ and v such that the first two conditions hold. The third condition follows from the proof of Theorem 5.19. The converse follows from the fact that Ω_τ is injective and continuous, according to Lemmas 5.25 and 5.24. □

In some cases the condition $\Omega^C = \Omega_\tau$ can be dropped from the right-hand side of Corollary 5.26

Lemma 5.27 *Let $\mathcal{S}_1$ be a K_1-deductive system and let $\mathcal{S}_2$ be a K_2-deductive system. Suppose that there is a (K_2, K_1)-interpretation v, i.e., v is a translation such that* *for all* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{K_2}$

$$\Gamma \vdash_{\mathcal{S}_2} \varphi \text{ iff } v(\Gamma) \vdash_{\mathcal{S}_1} v(\varphi)$$

Then for every Λ-algebra $\mathbf{A}$ *and for every $\mathcal{S}_1$-filter F on* $\mathbf{A}$*, $\Omega_v(F)$ is an $\mathcal{S}_2$-filter of* $\mathbf{A}$.

Proof. The lemma follows immediately from the assumption and definition of Ω_v. □

Lemma 5.28 *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be K_1 and K_2-deductive systems, respectively. Let C be a uniform system of $(\mathcal{S}_1, \mathcal{S}_2)$-compatibility relations. If $\mathcal{K}$ is a class of $\mathcal{S}_2$-matrices such that $\mathcal{S}_1$ and $\models_{\mathcal{K}}$ are equivalent with some translations τ, v as in Definition 5.1, then for every algebra* $\mathbf{A}$ *and every $\mathcal{S}_1$-filter F, the set $\Omega_v(F)$ is a subset of $\Omega^C(F)$. If moreover, $\alpha \in \Omega^C(F)$ implies $v(\alpha) \in F$, then the two sets are equal.*

Proof. By the previous lemma, for every algebra $\mathbf{A}$ and for every $\mathcal{S}_1$-filter F, $\Omega_v(F)$ is a $\mathcal{S}_2$-filter. The condition that $\tau\varphi \in T$ implies $C(\varphi, T)$, together with C(iv), guarantees that $\Omega_\tau(F)$ is compatible with F. □

Theorem 5.29 *Let $\mathcal{S}_2$ be one of the following 2-deductive systems: $\mathcal{B}$, $\mathcal{S}_t$ or the system $\mathcal{S}_{(\leq)}$. Let $\mathcal{S} = \mathcal{S}_1$ be a K_1-deductive system, for some K_1. Let C be the standard system of compatibility relations (Definition 5.10). Then the operator Ω^C is 1-1 and continuous on $Fi_{\mathcal{S}_1}(\mathrm{Th}_{\mathcal{S}_1})$ iff there is a class $\mathcal{K}$ of $\mathcal{S}_2$ matrices such that $\models_{\mathcal{K}}$ and $\mathcal{S}_1$ are equivalent.*

Proof. As all these systems are Birkhoff-like and C is uniform system of $(\mathcal{S}_1, \mathcal{S}_2)$-compatibility relations, for any $\mathcal{S}_1$, the direction "only if" follows immediately from the Semantic Equivalence Theorem 5.19. For the other direction, let v and τ be the interpretations, such that (5.4)-(5.7) hold. In particular, v is a (K_2, K_1)-translation, where K_2 has only one predicate and this predicate is binary. The following conditions follow from (5.4) and the fact that (I) and (Rp) are in the axiomatization of $\mathcal{S}_2$.

$$\vdash_{\mathcal{S}} v(x,x) \tag{5.22}$$

$$v(x,y) \vdash_{\mathcal{S}} v(\lambda(x/k), \lambda(y/k)) \tag{5.23}$$

where $\lambda \in \Lambda, n = \rho(\lambda)$ and $1 \leq k \leq n$ and $\lambda(x/k) := \lambda(z_1, \ldots, z_{k-1}, x, z_{k+1}, \ldots, z_n)$, $\lambda(y/k) := \lambda(z_1, \ldots, z_{k-1}, y, z_{k+1}, \ldots, z_n)$. The first condition implies that for every $\mathcal{S}_1$-filter F, $\tau(a,a) \in F$ and the second guarantees that if $\langle a, b\rangle \in \Omega^C(F)$, for some $\mathcal{S}_1$-filter F, then also the pairs $\langle \tau_1(a,a), \tau_n(a,b)\rangle$ are in $\Omega^C(F)$, where $\tau = \langle \tau_1, \ldots, \tau_n\rangle$. Since $\Omega^C(F)$ is C-compatible with F, it follows that $v(a,b) \subseteq F$. By Lemma 5.28, the operators Ω^C and Ω_τ are equal. The latter is continuous and 1-1 by lemmas 5.24 and 5.25. The theorem now follows from this and from Theorem 5.19. □.

Definition 5.30 *Let for every n, $\mathbf{x^n}$ denote the sequence of n variables $x_1, \ldots, x_n$. A K-deductive system $\mathcal{S}$ is* **algebraizable** *if it has a finite system Δ of congruence formulas and, for, every $R \in K$ with $\rho(R) = n$, a finite system of equations $\langle \varepsilon_1(\mathbf{x}^n) \approx \delta_1(\mathbf{x}^n), \ldots, \varepsilon_{m_R}(\mathbf{x}^n) \approx \delta_{m_R}(\mathbf{x}^n)$ such that*

$$\Delta(\delta_i(\mathbf{x}^n), \varepsilon(\mathbf{x}^n)) \vdash_{\mathcal{S}} R(\mathbf{x}^n).$$

A system Δ of equivalence sequents for a Gentzen system $\mathcal{G}$ is called a system of *congruence sequents* if it is a system of congruence ω-formulas in the sense of Definition 3.14. A special case of Definition 5.30 is the following

Definition 5.31 (i) *([5, Definition 13.14.]) A k-deductive system $\mathcal{S}$ is* **algebraizable** *if it has a finite system Δ of congruence formulas $\Delta_1, \ldots, \Delta_n$, with-*

out parameters, together with a pair of finite systems of k-ary terms $\delta_1, \ldots, \delta_m$ *and* $\varepsilon_1, \ldots, \varepsilon_m$ *such that*

$$\{\Delta(\delta_i(\mathbf{x}), \varepsilon_i(\mathbf{x})) : i \leq m\} \dashv\vdash_{\mathcal{S}} \mathbf{x}.$$

(ii) *([50]) A Gentzen system* $\mathcal{G}$ *is algebraizable if it has a finite system* Δ *of congruence sequents, without parameters, and for every n a finite set* τ_n *of equations* $\tau_n = \{\delta_1 \approx \varepsilon_1, \ldots, \delta_m \approx \varepsilon_m\}$*, where for* $i = 1, \ldots, m$*,* δ_i, ε_i *are terms in variables* $\mathbf{x}^n$ *such that*

$$\{\Delta(\delta_i(\mathbf{x}), \varepsilon_i(\mathbf{x})) : i \leq m\} \dashv\vdash_{\mathcal{G}} \mathbf{x}.$$

Corollary 5.32 (Algebraization Theorem) *A K-deductive system* $\mathcal{S}$ *is algebraizable iff the Leibniz operator on* Th$\mathcal{S}$ *is injective and continuous.*

Proof. By theorem 5.29. □

We list two special cases of Corollary 5.32 that have been considered in the literature.

Corollary 5.33 *([5, Theorem 4.2.]) A* 1*-deductive system* $\mathcal{S}$ *is algebraizable iff the Leibniz operator on* Th$\mathcal{S}$ *is injective and continuous.*

Corollary 5.34 *A Gentzen system* $\mathcal{G}$ *is algebraizable iff the Leibniz operator on* Th$\mathcal{G}$ *is 1-1 and continuous.*

Algebraizable Gentzen systems have been considered in [50]. In fact [50] considers so called "Gentzen systems of type (α, β)", which we do not define here, and of which the Gentzen systems in our sense are special cases. Gentzen systems of type (α, β) of [50] can be formalized as K-deductive systems, for some special K, and therefore our Corollary 5.32 applies also to these systems.

CHAPTER 6. THEORY OF ALGEBRAIC IMPLICATION

6.1 Introduction

For many well-known 1-deductive systems $\mathcal{S}$ the connective $\rightarrow$ of implication is strictly associated with some partial ordering in the models of $\mathcal{S}$. Also, the results of [4, 5] reviewed in Chapters 3 and 5 demonstrate the connections between the existence of a system of congruence formulas for a 1-deductive system $\mathcal{S}$, interpretability of equality in $\mathcal{S}$, properties of the operator Ω and the equivalence of $\mathcal{S}$ with some extension of the Birkhoff-system $\mathcal{B}$. In this chapter we turn to the question of interpretability of a partial ordering in a 1-deductive system and how this interpretability is associated, firstly, with the existence of some set of formulas, called later a *system of implication connectives*, secondly, with the equivalence of $\mathcal{S}$ with some other system, called here $\mathcal{S}_\pi$, and, thirdly, the properties of certain operator, denoted by Ω^π, that plays the role analogous to the operator Ω in the case of interpretation of equality. We would like to define a system of implication connectives, a 2-deductive system system $\mathcal{S}_\pi$ and the operator Ω^π in such a way that the theorems obtained by replacing equivalence formulas by implication formulas, equality by partial ordering, $\mathcal{B}$ by $\mathcal{S}_\pi$ and Ω by Ω^π, in the characterizations theorems of [5], hold. We are able to obtain some partial results of the desired form. Full analogy is, we believe, impossible, due to an intrinsic difference between equality in an algebra and partial ordering, or any other predicate, for this matter. We hope, however, that the study of 1-deductive systems with systems of implication formulas, initiated in this chapter, will be continued, possibly by other authors.

In this chapter, "deductive system" means 1-deductive system.

6.2 Occurrence of a variable in a term

Let the algebraic language Λ be fixed. Let $\mathbf{N}^*$ be the set of all finite (possibly empty) strings of elements of $\mathbf{N}$. The empty string is denoted by ϵ and the concatenation of two strings by their juxtaposition.

Definition 6.1 *A* **tree** *is a subset T of the set $\mathbf{N}^*$ of all finite strings of natural numbers that has the following properties.*

(i) $\epsilon \in T$.

(ii) *Let $\alpha \in \mathbf{N}^*$ and let $k \in \mathbf{N}$. If a string $\alpha k \in T$ then $\alpha \in T$ and $\alpha i \in T$, for every $i \leq k$. The string α is called a* **parent** *of αk and αk is called a* **child of** *α in T.*

The empty string ϵ is also called the **root of** *T. The elements of T are called the* **nodes** *of T. A node that has no children is called a* **leaf** *of T.*

Definition 6.2 *A Λ-tree is a finite tree whose internal nodes are labeled by elements of the set Λ and the leaves are left unlabeled. Also, every node labeled by a basic operation $\lambda \in \Lambda$, has exactly $\rho(\lambda)$ children.*

Since the language Λ is fixed, we will omit the prefix Λ and say "tree" instead of "Λ-tree".

Let T and S be trees and let N be a set of leaves of T. A *substitution of S for leaves in N in T* is the tree $T[S/N] := T \setminus N \cup \{\nu_1\nu_2 : \nu_1 \in N, \nu_2 \in S\}$.

Let N and M be two sets of strings. Then

$$NM := \{\nu\mu : \nu \in N, \mu \in M\}.$$

Definition 6.3 *A* **Λ-term** *is a Λ-tree in which every leaf is labeled by some variable. The* **parse tree of** *t is the tree $T_\Lambda(t)$ in which all nodes other than leaves are labeled the same as they are labeled in t; the leaves are left unlabeled. If Λ is known, $T_\Lambda(t)$ is also denoted by $T(t)$. A* **parse tree** *is a parse tree of some term t. In the special case that $t = \lambda(x_1, \ldots, x_n)$, $T(t)$ is denoted by λ. An* **occurrence** *of a variable x in t is a leaf ν that is labeled by x. We say that x* **occurs** *at ν in t, in this case. The set of all occurrences of a given variable x is denoted by $O_{t,x}$. If x occurs in t at ν, we write $x = o(t, \nu)$. Thus*

$$O_{t,x} := \{\nu : o(t, \nu) = x\}.$$

The set of all leaves of the parse tree of t is denoted by $\mathrm{Occ}_\Lambda(t) = \mathrm{Occ}(t)$, this is also the set of all possible occurrences of variables in t. So

$$\nu \in \mathrm{Occ}(t) \textit{ iff for some } x, o(t, \nu) = x.$$

We will not consider here trees other than parse trees. So whenever we say "tree" we mean a parse tree of some term. Note that the above definition of a Λ-term is equivalent to the Definition 0.4. We will often omit the prefix and subscripts Λ. To illustrate the above concepts by an example, let $+$ and p be a binary and ternary operations, respectively. Let $t = p(x, y, z)$ and $s = (x+y)+z$. Then $\mathrm{Occ}(t) = \{1, 2, 3\}$

and $o(t,1) = x, o(t,2) = y$. To find the elements of $\mathrm{Occ}(s)$ we note that $s = r[x+y/u]$, where $r = u + z$. Now $\mathrm{Occ}(r) = \{1,2\}$, with $u = o(r,1)$. Thus $\mathrm{Occ}(s) = \{11,12,2\}$ and $o(s,11) = x, o(s,12) = y$ and $o(s,2) = z$.

Let t and s be two terms, $N \subseteq \mathrm{Occ}(t)$. Let $T = T(t), S = T(s)$. Then by $t[s/N]$ we mean the tree $T[S/N]$ labeled so that if $\nu \in \mathrm{Occ}(t) \setminus N$ then ν has the same label as in t and if $\nu = \nu_1\nu_2, \nu_1 \in N, \nu_2 \in \mathrm{Occ}(s)$, then ν has the same label as ν_2 in s.

Intuitively, $t[s/N]$ results from t by replacing all variables occurring at leaves labeled by elements of N, by the term s. In particular, if $N = O_{t,x}$, then $t[s/N] = t[s/x]$, the result of substitution of s into t for x. In other words, a substitution $t[s/x]$ replaces all and only occurrences of a given variables x in t by s, whereas to get $t[s/N]$ we replace maybe not all occurrences of x and possibly also some occurrences of other variables in t.

Definition 6.4 *Two terms t and s are* **similar** *if their parse trees, $T(t)$ and $T(s)$ are the same.*

Note that if t and t' are similar terms, then $\mathrm{Occ}(t) = \mathrm{Occ}(t')$.

6.3 Polarity

Intuitively, a polarity is a function that to a leaf n in a parse tree T assigns either $\{+1\}$ or $\{-1\}$ or the empty set. Occurrences of variables in a term inherit polarity from the polarities on the parse tree of the term.

We also consider a set-polarity (which we also call polarity on sets), which differs from polarity in that that it assigns $\{+1\}$ or $\{-1\}$ or the empty set to some nonempty *set* of occurrences of a variable in a term, rather than to just one occurrence. Thus a polarity is in particular a set-polarity. However, for a large part of our considerations the concept of polarity will be sufficient. We first consider polarity defined for (parse) trees.

Definition 6.5 *A* **set-polarity on** Λ, *is a function* π *that to a pair* $\langle T, N\rangle$, *where* T *is a tree and* N *is some set of leaves of* T, *assigns a set* $\pi(T,N) \subseteq \{+1,-1\}$ *that has at most one element and has the following properties:*

(ROOT) *If* T *consists of only one node (and therefore this node is the root of the tree), then* $+1 \in \pi(T,\{\epsilon\})$.

(SUBST) *If* $\alpha \in \pi(T,N)$ *and* $\beta \in \pi(S,M)$, *then* $\alpha \cdot \beta \in \pi(T[S/N], NM)$, *where* $\alpha \cdot \beta$ *is the product of the two numbers* α *and* β.

The property (SUBST) is called the *substitution property* of set-polarity. We often will say "set-polarity" rather than "set-polarity on Λ". Every tree-polarity on sets determines certain function on pairs consisting of a term t and some sets $N \subseteq \mathrm{Occ}(t)$. If t is a term and ν a leaf in the parse tree $T(t)$, then we write $\pi(t,\nu)$ for $\pi(t,\{\nu\})$.

Definition 6.6 *Let* π *be a set-polarity on* Λ, *let* t *be a term and let* $N \subseteq \mathrm{Occ}(t)$. *Then we put*

$$\pi(t,N) := \pi(T(t),N) \text{ and}$$

$$\pi(t,x) := \pi(t,O_{t,x}),$$

where x *is some variable occurring in* t.

Note that the condition (ROOT) implies that $\pi(x,\epsilon) = \{+1\}$ and the condition (SUBST) implies the following property of π on terms:

If t and s are terms, $N \subseteq \mathrm{Occ}(t), M \subseteq \mathrm{Occ}(s)$, $\tau \in N, \sigma \in M$ and if $\alpha \in \pi(t, \tau)$ and $\beta \in \pi(s, \sigma)$. then $\alpha \cdot \beta \in \pi(t[s/N], NM)$.

In particular, if N is the set of all occurrences of certain variable x in t and M is the set of all occurrences of some variable y in s such that y does not occur in t, then the property (SUBST) says that, if $\alpha \in \pi(t, x), \beta \in \pi(s, y)$, then $\alpha \cdot \beta \in \pi(t[s/x], y)$.

For example, consider the language of one binary operation $+$ and one unary operation $-$. Suppose that $\pi(x+y, \{1\}) = \pi(x+y, \{2\}) = \{+1\}$ and $\pi(-x, 1) = \{-1\}$. Then it follows from (SUBST) that if π is a set-polarity, then $\pi(x + (-y), \{21\}) = \{-1\}$. In this case we can also write that $\pi(x + y, x) = \pi(x + y, y) = \{+1\}$ and $\pi(-x, x) = \{-1\}$ and therefore $\pi(x + (-y), y) = \{-1\}$, if π is a set-polarity.

Recall that in Example 5.1, we considered some system of pairs of sets $P = \langle\langle P_\lambda, N_\lambda\rangle : \lambda \in \Lambda\rangle$, where for every $\lambda \in \Lambda$ $P_\lambda, N_\lambda \subseteq \{1, \ldots, \rho(\lambda)\}$. Such a system P determines a set-polarity $\pi = \pi_P$ by the conditions $+1 \in \pi(\lambda, \{k\})$ iff $k \in P_\lambda$, $-1 \in \pi(\lambda, \{k\})$ iff $k \in N_\lambda$ and (SUBST). Conversely, every set-polarity π defines a unique system $P = \langle P_\lambda, N_\lambda : \lambda \in \Lambda\rangle$ by $+1 \in \pi(\lambda, \{k\})$ iff $k \in P_\lambda$, $-1 \in \pi(\lambda, \{k\})$ iff $k \in N_\lambda$. However, there may be many different set-polarities that define the same system P in this way. This follows from the fact, that, for any term t and $N \subseteq \mathrm{Occ}(t)$, the set-polarity $\pi_P(t, N)$ determined by P as above, is non-empty only if N is a singleton.

Definition 6.7 *1. By a* **polarity** *we mean a set-polarity π such that for every tree T and for every set N of leaves of T, if N has more than* 1 *element, then $\pi(T, N) = \emptyset$.*

2. A polarity π is **total** *if for every pair $\langle T, \nu\rangle$, such that ν is a leaf of T. $\pi(T, \nu) \neq \emptyset$.*

3. *A polarity is* **strict** *if for every tree T and for every leaf ν of T, $\pi(T,\nu)$ has at most one element.*

Consequently, if π is a total polarity, then for every term t and $N \subseteq \mathrm{Occ}(t)$, $\pi(t,N) \neq \emptyset$ iff N is singleton. For a total and strict polarity π, $\pi(t,\nu)$ is always a singleton. If x occurs more than once in t and π is a polarity, then $\pi(t,x) = \emptyset$.

Remarks

(1) A strict polarity π can be identified with a partial function on pairs $\langle T,\nu\rangle$ into $\{+1,-1\}$, which is defined only if $\pi(T,\nu) \neq \emptyset$; in this case its value on $\langle T,\nu\rangle$ is the unique element of this set. This function is total exactly when the polarity is total.

(2) If the polarity π is strict and total, then it is entirely determined by its values on the parse tree of the terms $\lambda(x_1,\ldots,x_n)$, more precisely, its values on pairs $\langle\lambda,k\rangle$ (i.e., $\langle T(\lambda(x_1,\ldots,x_n))),\ k\rangle$, where $k = 1,\ldots,n$. Namely, if there is only one value for $\pi(t,\nu)$ allowed, then this value is uniquely determined by (SUBST) and the values of $\pi(\lambda,k)$, for all λ occurring in t and all $k \leq \rho(\lambda)$.

(3) Consider again a system $P = \langle P_\lambda, N_\lambda : \lambda \in \Lambda\rangle$, as in Example 5.1. If for every $\lambda \in \Lambda$, $P_\lambda \cap N_\lambda = \emptyset$, then π_P defined previously is a strict polarity. If for every $\lambda \in \Lambda$ $P_\lambda \cup N_\lambda = \{1,\ldots,\rho(\lambda)\}$, then π_P is total. By remark (2), there is a one-one onto correspondence between the set of all strict total polarities on Λ and the set of all systems $P = \langle P_\lambda, N_\lambda : \lambda \in \Lambda\rangle$, such that for all $\lambda \in \Lambda$, $P_\lambda \cap N_\lambda = \emptyset$ and $P_\lambda \cup N_\lambda = \{1,\ldots,\rho(\lambda)\}$.

Definition 6.8 *A polarity π is called* **positive** *if $-1 \notin \pi(T,\nu)$, for all $\langle T,\nu\rangle$. In* *other words, π is positive if $\pi(T,\nu) \neq \emptyset$ implies $+1 \in \pi(T,\nu)$. A polarity π is* **negative** *if $+1 \notin \pi(T,\nu)$, for all $\langle T,\nu\rangle$.*

Given a polarity π on sets, we say that a term t is **positive** *(***negative***) in a set of occurrences N, if $+1 \in \pi(t,N)$ (resp. $-1 \in \pi(T,N)$). We say that t is* **positive (negative) in** *x if t is positive (negative) in $O_{t,x}$.*

If $\vec{z}$ is a sequence of variables indexed by the strings enumerating the leaves of T, then $T[\vec{z}]$ denotes the term resulting by labeling the leaves of T by $\vec{z}$, where the leaf numbered by ν is labeled by z_ν. Let t be a Λ-term and N set of leaves in the parse tree $T(t)$ of t. We say that a variable x *occurs in t outside of N*, if there is a leaf ν of the parse tree $T(t)$ of t such that ν is labeled by x in t and $\nu \notin N$. Let $\mathbf{A}$ be a Λ-algebra, let t be a term and let $N \subseteq \mathrm{Occ}(t)$. Let $a \in \mathbf{A}$ and let also $\vec{c}$ be a sequence of elements of A indexed by the variables z occurring in t outside of N. Then we let $t(a/N)(\vec{c}) := h(t')$, where $t' = t[x/N]$, x is a variable that does not occur in t and h is a valuation that sends x to a and a variable z that occurs in t (outside of N) to c_z. If a sequence $\vec{c}$ is indexed by some set of variables including all the variables that occur in t outside of N, then $t(a/N)(\vec{c}) := t(a/N)(\vec{c}')$, where $\vec{c}'$ is the subsequence of $\vec{c}$ indexed by the variables occurring in t outside of N. In the future, whenever we write some expression of the form $t(a/N)(\vec{c})$ it will be automatically assumed that $\vec{c}$ is a sequence of elements of the algebra $\mathbf{A}$, known from the context, that is indexed by a superset of the set of all variables occurring in t outside of N.

Definition 6.9 *Let $\mathbf{A}$ be a Λ-algebra and consider a binary relation $\leq$ on $\mathbf{A}$. Then we say that $\leq$* agrees *with a set-polarity π if for every term t and $N \subseteq \mathrm{Occ}(t)$, we have*

$$a \leq b \Rightarrow t(a/N)(\vec{c}) \leq t(b/N)(\vec{c}),$$

if $+1 \in \pi(t, N)$ *and*

$$a \leq b \Rightarrow t(b/N)(\vec{c}) \leq t(a/N)(\vec{c}),$$

if $-1 \in \pi(t, N)$.

Example 6.1 Let Λ be the language consisting of a binary operations $+$ and a unary operation $-$. Let π be the polarity determined by letting $x + y$ be positive in both x and y and $-y$ negative in y. It is clear that this determines a unique Λ-polarity. Consider now the set of all real numbers with the standard operations $+$ and $-$ and let $\leq$ be the standard ordering of the real numbers. Then it is easy to see that $\leq$ agrees with the polarity π.

Example 6.2 Let Λ have the following connectives: binary connectives $\rightarrow, \vee, \wedge$ and unary connective $\neg$. Let Λ' be a subset of Λ. The *standard polarity* on Λ' is defined as follows: $\pi(\rightarrow, 1) = \pi(\neg, 1) = \{-1\}$ and $\pi(\rightarrow, 2) = \pi(\vee, 1) = \pi(\vee, 2) = \pi(\wedge, 1) = \pi(\wedge, 2) = \{+1\}$. In other words, $x \rightarrow y$ is negative in x and positive in y, $\neg x$ is negative in x and all the remaining polarities are positive.

Let $\langle \mathbf{A}, \leq \rangle$ be an ordered Λ'-algebra, where $\Lambda' \subseteq \{\vee, \wedge\}$ such that if $\vee \in \Lambda'$ then $\vee$ is the operation supremum on A, if $\wedge \in \Lambda'$, then $\wedge$ is the operation infimum. Then the order $\leq$ agrees with the standard polarity. Also, if $\{\wedge, \rightarrow\} \subseteq \Lambda' \subseteq \{\wedge, \vee, \rightarrow\}$ and $\langle \mathbf{A}, \leq \rangle$ is a Λ'-algebra, where $\wedge$ and $\vee$ are as above and for all $a, b \in A$,

$$a \rightarrow b = \max\{z : z \wedge a \leq b\}, \tag{6.1}$$

then it is well known that $\leq$ agrees with the standard polarity. More generally, if $\rightarrow \in \Lambda' \subseteq \{\wedge, \vee, \rightarrow\}$ $\mathbf{A}, \leq$ are as above, A has the largest element 1 relatively to $\leq$ and for all $a, b, c \in A$,

$$(a \rightarrow b) \rightarrow [(c \rightarrow a) \rightarrow (c \rightarrow b)] = 1,$$

$$(a \rightarrow b) \rightarrow [(b \rightarrow c) \rightarrow (a \rightarrow c)] = 1,$$

then $\leq$ agrees with the standard polarity. Finally, if in addition Λ' contains $\neg$, and A has the least element 0 such that $\neg x = x \rightarrow 0$, then $\leq$ agrees with the standard polarity.

A BCK-*algebra* is a $\{\rightarrow\}$-subalgebra of a $\{\rightarrow\}$-reduct of a $\{\wedge, \rightarrow\}$-algebra such that 6.1 holds ([64]). Thus for every BCK-algebra, as well as for every semilattice or lattice, every pseudocomplemented lattice, every Heyting, Brouwerian or Boolean algebra, the partial order defined by $a \leq b$ iff $a \rightarrow b = 1$ agrees with the standard polarity.

6.4 A 2-deductive system $\mathcal{S}_\pi$ and the operator Ω^π

Recall from chapter 3 that a 1-deductive system $\mathcal{S}$ has a system of equivalence connectives iff $\Omega_{\mathcal{S}}$ is monotone and that this system of equivalence connectives has some strong properties iff in addition $\Omega_{\mathcal{S}}$ is continuous and 1-1. In the next sections we will be concerned with the question whether the existence and properties of a system of *implication* connectives for $\mathcal{S}$ can be characterized in a similar way by means of some operator analogous to the Leibniz operator. This new operator turns out to depend not only on the deductive system $\mathcal{S}$ itself, but also is relativized to a pre-established polarity π and therefore we call it $\Omega^\pi_{\mathcal{S}}$ or Ω^π.

If π is a strict total polarity determined by its values on Λ, or, equivalently, by sets P_λ, N_λ for $\lambda \in \Lambda$, where $P_\lambda \cap N_\lambda = \emptyset$, $P_\lambda \cup N_\lambda = \{1, \ldots, \rho(\lambda)\}$, then the definition of Ω^π has been already given in Chapter 5. We generalize this definition of Ω^π to the case that π is an arbitrary set-polarity. We first generalize the definition of the 2-deductive system $\mathcal{S}_P$ (Example 5.1).

Definition 6.10 *Let $\mathcal{S}$ be a 1-deductive system. Let π be a set-polarity. The 2-deductive system $\mathcal{S}_\pi$ over Λ is based by the following axiom and rules of inference:*

$$\frac{\emptyset}{\langle x, x\rangle} \tag{6.2}$$

$$\frac{\langle x, y\rangle, \langle y, z\rangle}{\langle x, z\rangle} \tag{6.3}$$

$$\frac{\langle x, y\rangle}{\langle t[x/N], t[y/N]\rangle} \tag{6.4}$$

for every term t and $N \subseteq \mathrm{Occ}(t)$ such that $+1 \in \pi(t, N)$; and

$$\frac{\langle x, y\rangle}{\langle t[y/N], t[x/N]\rangle} \tag{6.5}$$

for every term t and $N \subseteq \mathrm{Occ}(t)$ such that $-1 \in \pi(t, N)$.

The next theorem has already been proved for the special case that the set-polarity is determined by a system $P = \langle\langle P_\lambda, N_\lambda\rangle : \lambda \in \Lambda\rangle$ (Theorem 5.6).

Theorem 6.11 *For every set-polarity π, the 2-deductive system $\mathcal{S}_\pi$ is Birkhoff-like.*

Proof. The proof below is a modification of the proof of Theorem 5.6. For B(i), observe that all single–premiss rules are of the form (6.4) or (6.5). So suppose that for some surjective substitution σ, $\dfrac{\sigma(\varphi)}{\psi}$ is an instance of the rule (6.4). This means that there is some term τ and some $N \subseteq \mathrm{Occ}(\tau)$ such that $\varphi = \langle t, s\rangle, \psi = \langle u, v\rangle$,

$$u = \tau(\sigma(t)/N)(\vec{t}) \text{ and}$$

$$v = \tau(\sigma(s)/N)(\vec{t}),$$

for some sequence of terms $\vec{t} = \langle t_z : z \text{ occurs in } \tau \text{ outside of } N\rangle$. For each such z, let t'_z be a term such that $\sigma(t'_z) = t_z$ and let $\vec{t'} = \langle t'_z : z \text{ occurs in } \tau \text{ outside of } N\rangle$. Let also

$$\xi := \langle \tau(t/N)(\vec{t'}), \tau(s/N)(\vec{t'})\rangle.$$

Clearly, $\dfrac{\varphi}{\xi}$ is an instance of the rule (6.4). The case of the rule (6.5) is handled similarly.

For B(ii), note that if φ or ψ in the statement of B(ii) is an instance of the axiom, then the conclusion is obvious. So we need to prove that every derivation (5.11) in which the first rule used was (T) and the second rule used was either (6.4) or (6.5) can be replaced by a derivation of the form (5.12). So assume that the second rule in (5.12) is (6.4) for some τ and N and suppose that we have a derivation

$$\frac{\dfrac{\langle t_1, t_2, \rangle, \langle t_2, t_3\rangle}{\langle t_1, t_3\rangle}}{\langle \tau(t_1/N)(\vec{s}), \tau(t_3/N)(\vec{s})\rangle},$$

where $\vec{s}$ is some sequence of terms. This derivation can be replaced by applying the rule (6.4) to both $\langle t_1, t_2\rangle$ and $\langle t_2, t_3\rangle$ first and then applying the transitivity rule (T), i.e.,

$$\frac{\dfrac{\langle t_1, t_2\rangle}{\langle \tau(t_1), \tau(t_2)\rangle} \quad \dfrac{\langle t_2, t_3\rangle}{\langle \tau(t_2/N)(\vec{s}), \tau(t_3/N)(\vec{s})\rangle}}{\langle \tau(t_1/N(\vec{s})), \tau(t_3/N)(\vec{s})\rangle}.$$

The proof for the rule (6.5) is similar. □

If some total polarity π is used in the role of the set-polarity in Definition 6.10, then we get a simpler basis for $\mathcal{S}$.

Proposition 6.12 *Assume that π is a total strict polarity. Then $\mathcal{S}_\pi$ is based by the following axiom and inference rules.*

$$\vdash \langle x, x\rangle \tag{6.6}$$

$$\langle x, y\rangle, \langle y, z\rangle \vdash \langle x, z\rangle \tag{6.7}$$

$$\langle x, y\rangle \vdash \langle \lambda(x/x_i), \lambda(y/x_i)\rangle \tag{6.8}$$

for every $\lambda \in \Lambda$ such that $+1 \in \pi(\lambda, i)$ and

$$\langle x, y\rangle \vdash \langle \lambda(y/x_i), \lambda(x/x_i)\rangle \tag{6.9}$$

for every $\lambda \in \Lambda$ such that $-1 \in \pi(\lambda, i)$.

Proof. All of the above rules are included in the basis given in Definition 6.10. So it suffices to show that the rules (6.4) and (6.5) are derivable from the rules given in the proposition. Let t be a term and $N \subseteq \mathrm{Occ}(t)$. Since π is a polarity, to verify (6.4) and (6.5) we may assume that $N = \{\nu\}$ for some $\nu \in \mathrm{Occ}(t)$. If $t = \lambda$, for some $\lambda \in \Lambda$, then (6.4) and (6.5) are (6.8) and (6.9), respectively and there is nothing to prove. Suppose that $t = \lambda(t_1, \ldots, t_n)$ for some n ary $\lambda \in \Lambda$ and some terms $t_1, \ldots, t_n$, such that (6.4) and (6.5) are derivable for t_i, $i = 1, \ldots, n$ in the role of t. Suppose that $\nu = k\mu$, for some $k \leq n$ and $\mu \in \mathrm{Occ}(t_i)$, some $i = 1, \ldots, n$. By (SUBST), $\pi(t, \nu) = \{+1\}$ iff $\pi(t, \nu) = \pi(\lambda, k)$ and $\pi(t, \nu) = \{-1\}$ iff $\pi(t, \nu) \neq \pi(\lambda, k)$. By the induction hypothesis, $\dfrac{\langle x, y\rangle}{\langle t_i[x/\mu], t_i[y/\mu]\rangle}$ if $\pi(t_i, \mu) = \{+1\}$; and $\dfrac{\langle x, y\rangle}{\langle t_i[y/\mu], t_i[x/\mu]\rangle}$ if

$\pi(t_i, \mu) = \{-1\}$. Let $u = t_i[x/\mu], v = t_i[y/\mu]$, if $\pi(t_i, \mu) = \{+1\}$ and $u = t_i[y/\mu], v = t_i[x/\mu]$, if $\pi(t_i, \mu) = \{-1\}$. Now applying the rule (6.4), if $\pi(\lambda, k) = \{+1\}$, or (6.5) if $\pi(\lambda, k) = \{-1\}$, with u in the role of x and v in the role of y, we get

$$\frac{\langle x, y\rangle}{\langle t[x/\nu], t[y/\nu]\rangle}$$

if $\pi(t_i, \nu) = \{+1\}$; and

$$\frac{\langle x, y\rangle}{\langle t[y/\nu], t[x/\nu]\rangle}$$

if $\pi(t, \nu) = \{-1\}$. □

Definition 6.13 *Let $\mathcal{S}$ be a 2-deductive system. Then $\mathcal{S}$ is called a* **quasi-ordering system** *if $\vdash_{\mathcal{S}} \langle x, x\rangle$ and $\langle x, y\rangle, \langle y, z\rangle \vdash_{\mathcal{S}} \langle x, z\rangle$. Let π be a set-polarity. A quasi-ordering system $\mathcal{S}$ that satisfies 6.8 and 6.9 is called a* **π-quasi-ordering**.

Note that $\mathcal{S}$ is a π-quasi-ordering iff $\mathcal{S}$ is an extension of $\mathcal{S}_\pi$. In particular, $\mathcal{S}_\pi$ is the smallest π-quasi-ordering system. According to Definition 2.23, an $\mathcal{S}_\pi$-matrix is a Λ-algebra with a reflexive and transitive relation $\leq$ such that for all elements $a, b \in A$, and sequences $\vec{c}$ of elements of A

$$\langle a, b\rangle \in \leq \Rightarrow \langle t(a/N)(\vec{c}), t(b/N)(\vec{c})\rangle \in \leq \tag{6.10}$$

for every Λ-term t such that $+1 \in \pi(t, N)$; and

$$\langle a, b\rangle \in \leq \Rightarrow \langle t(b/N)(\vec{c}), t(a/N)(\vec{c})\rangle \in \leq \tag{6.11}$$

for every Λ-term t such that $-1 \in \pi(t, N)$.

In other words, if $\leq$ is a $\mathcal{S}_\pi$-filter on a Λ-algebra $\mathbf{A}$, then $\leq$ is a quasi-ordering on $\mathbf{A}$ that agrees with π.

Definition 6.14 *For any* 1*-deductive system* $\mathcal{S}$*, let* $C = \langle C_{\mathbf{A}} : \Lambda\text{-}algebras\mathbf{A}\rangle$ *be the standard system of* $(\mathcal{S}, \mathcal{S}_\pi)$*-compatibility relations (Definition 5.10), i.e., for each* Λ*-algebra* $\mathbf{A}$*,* $a, b \in A$*,* $F \in Fi_{\mathcal{S}}(\mathbf{A})$*,* $\langle\langle a, b\rangle, F\rangle \in C_{\mathbf{A}}$ *iff* $a \in F \Rightarrow b \in F$*. Let* $\mathbf{A}$ *be a* Λ*-algebra, let* F *be a* $\mathcal{S}$*-filter on* $\mathbf{A}$*, and let* Θ *be a quasi-ordering on* A *that agrees with the set polarity* π*. We say that* Θ *is* **compatible with** F *if* Θ *is* C*-compatible with* F*, i.e., when for all elements* a, b *of* A*, if* $\langle a, b\rangle \in \Theta$*, then* $a \in F \Rightarrow b \in F$*.*

It follows from Definition 5.15 and Theorem 5.18 that for every Λ-algebra $\mathbf{A}$ and every $\mathcal{S}$-filter F on $\mathbf{A}$, the largest quasi-ordering on $\mathbf{A}$ that agrees with π and is compatible with F, exists.

Definition 6.15 *Let* $\mathbf{A}$ *be a* Λ*-algebra and let* F *be a* $\mathcal{S}_\pi$*-filter on* $\mathbf{A}$*. Then* $\Omega^\pi(F) = \Omega^\pi_{\mathbf{A}}(F)$ *is the largest* $\mathcal{S}_\pi$*-filter on* $\mathbf{A}$ *that is compatible with* F*. Thus* $\Omega^\pi(F)$ *is the largest quasi-ordering on* A *that agrees with polarity* π *and is compatible with* F*. Also, for every* Λ*-algebra* $\mathbf{A}$ *this defines an operator* $\Omega^\pi_{\mathbf{A}} : Fi_{\mathcal{S}}(\mathbf{A}) \to Fi_{\mathcal{S}_\pi}(\mathbf{A})$*.*

Proposition 6.16 *For every* Λ*-algebra* $\mathbf{A}$ *and every* $\mathcal{S}$*-filter* F *on* $\mathbf{A}$*, we have* $\langle a, b\rangle \in \Omega^\pi(F)$ *iff for every* Λ*-term* t*,* $N \subseteq \mathrm{Occ}(t)$*, for every sequence* $\vec{c}$ *of elements of* A *indexed by the variables occurring in* t *outside of* N*, we have*

$$\textit{If } +1 \in \pi(t, N) \textit{ then } t(a/N)(\vec{c}) \in F \Rightarrow\ t(b/N)(\vec{c}) \in F \textit{ and} \tag{6.12}$$

$$-1 \in \pi(t, N) \textit{ then } t(b/N)(\vec{c}) \in F \Rightarrow\ t(a/N)(\vec{c}) \in F. \tag{6.13}$$

Proof. By Lemma 5.16 and Theorem 5.18.

Recall (Definition 6.8) that a total polarity π is positive iff for every $\lambda \in \Lambda$, for every $k \leq \rho(\lambda)$, we have $+1 \in \pi(\lambda, k)$ and negative iff for every $\lambda \in \Lambda$, for every $k \leq \rho(\lambda)$, we have $-1 \in \pi(\lambda, k)$.

If π is a positive polarity, then the condition (6.8) becomes the condition (R′) of equational deductive system $\mathcal{B}$ (Definition 2.19) and $\mathcal{S}_\pi$ differs from $\mathcal{B}$ in that $\mathcal{S}_\pi$ does not have the symmetry axiom (S). In particular,

Proposition 6.17 *Let π be a total polarity and let $\mathbf{A}$ be a Λ-algebra.*

1. *If π is positive, then every equivalence relation on $\mathbf{A}$ which agrees with π is a congruence.*

2. *If π is negative, then every equivalence relation on $\mathbf{A}$ which agrees with π is a congruence.*

Proof. For the proof of 1., suppose that Θ agrees with π. Since π is total, for every $\lambda \in \Lambda$ and for every $k \leq \rho(\lambda)$, $+1 \in \pi(\lambda, k)$. By (6.4), $a\Theta b$ implies that

$$\lambda(c_1, \ldots, c_{k-1}, a, c_k, \ldots, c_{\rho(\lambda)-1})\Theta\lambda(c_1, \ldots, c_{k-1}, b, c_k, \ldots, c_{\rho(\lambda)-1}),$$

for all $\lambda \in \Lambda$, all $a, b, c_1, \ldots, c_{\rho(\lambda)-1} \in A$. Since λ and k are arbitrary, it follows that Θ is a congruence. 2. is proved similarly. □

Lemma 6.18 *If π is a total polarity, then for every Λ-algebra $\mathbf{A}$ and for every $\mathcal{S}$-filter F on $\mathbf{A}$, the Leibniz operator $\Omega_{\mathbf{A}}(F) = \Omega^{\pi}_{\mathbf{A}}(F) \cap (\Omega^{\pi}_{\mathbf{A}}(F))^{-1}$.*

Proof. We first verify that $\Omega(F) \subseteq \Omega^{\pi}(F)$. If $\langle a, b\rangle \in \Omega(F)$, then for every term t and an occurrence $\nu \in \mathrm{Occ}(t)$, we have $t(a/\nu)(\vec{c}) \in F$ iff $t(b/\nu)(\vec{c}) \in F$. Thus in particular, if $+1 \in \pi(t, \nu)$, then $t(a/\nu)(\vec{c}) \in F$ implies $t(b/\nu)(\vec{c}) \in F$; and if

$-1 \in \pi(t, \nu)$, then $t(b/\nu)(\vec{c}) \in F$ implies $t(a/\nu)(\vec{c}) \in F$. Hence $\Omega(F) \subseteq \Omega^{\pi}(F)$. Therefore also $\Omega(F) = (\Omega(F))^{-1} \subseteq (\Omega^{\pi}(F))^{-1}$. So $\Omega(F) \subseteq \Omega^{\pi}(F) \cap (\Omega^{\pi}(F))^{-1}$.

On the other hand, if π is total, for every term t and occurrence $\nu \in \mathrm{Occ}(t)$, t is either positive or negative in ν. Suppose that $\langle a, b\rangle \in \Omega^{\pi}(F) \cap (\Omega^{\pi}(F))^{-1}$, i.e., $\langle a, b\rangle \in \Omega^{\pi}(F), \langle b, a\rangle \in \Omega^{\pi}(F)$ and assume that $t(a/\nu)(\vec{c}) \in F$. For $+1 \in (t, N)$ this implies $t(b/\nu)(\vec{c}) \in F$, by $\langle a, b\rangle \in \Omega^{\pi}(F)$ while for $-1 \in \pi(t, N)$, $t(b/\nu)(\vec{c}) \in F$ follows from $\langle b, a\rangle \in \Omega^{\pi}(F)$. We have shown that $t(a/\nu)(\vec{c}) \in F$ and $\langle a, b\rangle \in \Omega^{\pi}(F) \cap (\Omega^{\pi}(F))^{-1}$ implies $t(b/\nu)(\vec{c}) \in F$. By symmetry of $\Omega^{\pi}(F) \cap (\Omega^{\pi}(F))^{-1}$, also $t(b/\nu)(\vec{c}) \in F$ and $\langle a, b\rangle \in \Omega^{\pi}(F) \cap (\Omega^{\pi}(F))^{-1}$ implies $t(a/\nu)(\vec{c}) \in F$. This shows that $\Omega^{\pi}(F) \cap (\Omega^{\pi}(F))^{-1} \subseteq \Omega(F)$. □

Let Π be one of the operators $\Omega^{\pi}, (\Omega^{\pi})^{-1}, \Omega^{\pi} \cap (\Omega^{\pi})^{-1}, \Omega$. We say that Π is respectively *monotone, continuous or injective* iff it is monotone, continuous or injective, respectively, on the $\mathcal{S}$-filter–lattice of every Λ-algebra.

Lemma 6.19 *Let $\mathcal{S}$ be a 1-deductive system and let π be a set-polarity. Then*

1. *If Ω^{π} is monotone then $(\Omega^{\pi})^{-1}$ and $\Omega^{\pi} \cap (\Omega^{\pi})^{-1}$ are monotone.*

2. *If Ω^{π} is continuous then $(\Omega^{\pi})^{-1}$ and $\Omega^{\pi} \cap (\Omega^{\pi})^{-1}$ are continuous.*

In particular, if π is a total polarity then

3. *If Ω^{π} is monotone then Ω is monotone.*

4. *if Ω^{π} is continuous, then Ω is continuous.*

Proof. Clearly, $\Omega^{\pi}(X) \subseteq \Omega^{\pi}(Y)$ implies $(\Omega^{\pi}(X))^{-1} \subseteq (\Omega^{\pi}(Y))^{-1}$ and therefore implies $\Omega^{\pi}(X) \cap (\Omega^{\pi}(X))^{-1} \subseteq \Omega^{\pi}(X) \cap (\Omega^{\pi}(X))^{-1}$. The first claim follows.

For the second claim, let I be a directed set and let $\mathcal{F} = \{F_i : i \in I\}$ be a family of $\mathcal{S}$-filters on some algebra $\mathbf{A}$. Observe that the following statements are equivalent.

(i) $\langle a, b\rangle \in (\Omega^{\pi}(\bigvee \mathcal{F}))^{-1}$

(ii) $\langle b, a\rangle \in \Omega^{\pi}(\bigvee \mathcal{F})$

(iii) $\langle b, a\rangle \in \bigvee_{i\in I} \Omega^{\pi}(F_i) = \bigcup_{i\in I} \Omega^{\pi}(F_i)$, by continuity of Ω^{π}

(iv) $\langle a, b\rangle \in \bigcup_{i\in I}(\Omega^{\pi}(F_i))^{-1}$.

This shows that $(\Omega^{\pi})^{-1}$ is continuous, if Ω^{π} is. Therefore

$$\begin{aligned}
(\Omega^{\pi} \cap (\Omega^{\pi})^{-1})(\bigvee_{i\in I} \Omega^{\pi}(F_i)) &= \Omega^{\pi}(\bigvee_{i\in I} (F_i)) \cap (\Omega^{\pi}(\bigvee_{i\in I} (F_i)))^{-1} \\
&= \bigvee_{i\in I} \Omega^{\pi}(F_i) \cap \bigvee_{i\in I} (\Omega^{\pi}(F_i))^{-1} \\
&= \bigcup_{i\in I} \Omega^{\pi}(F_i) \cap \bigcup_{i\in I} (\Omega^{\pi}(F_i))^{-1} \\
&= \bigcup_{i\in I} \Omega^{\pi} \cap (\Omega^{\pi})^{-1}(\mathcal{F}).
\end{aligned}$$

This shows the second claim. Claims 3. and 4. follow from 1. and 2. and from lemma 6.18. $\square$

We conclude this section with two questions to which we would like to know the answer.

Questions Let π be a total polarity.

1. Does the injectivity of Ω^{π} imply the injectivity of Ω?

2. Do the continuity and injectivity of Ω^{π} jointly imply the injectivity of Ω?

6.5 Operators $\Omega_{\vdash}$ and Ω_I

Let $\mathcal{S}$ be an arbitrary but fixed 1-deductive system.

Definition 6.20 *Let* $\mathbf{A}$ *be a* Λ*-algebra and let* F *be an* $\mathcal{S}$*-filter on* $\mathbf{A}$*. Then*

$$\Omega_{\vdash}(F) := \{\langle\varphi,\psi\rangle : \psi \in \mathrm{Fg}^{\mathbf{A}}(F \cup \{\varphi\})\}$$

In particular, if $\mathbf{A} = \mathrm{Te}$ and T is an $\mathcal{S}$-theory, then

$$\Omega_{\vdash}(T) = \{\langle x,y\rangle : T, x \vdash_{\mathcal{S}} y\}$$

Lemma 6.21 *For every deductive system* $\mathcal{S}$

1. *The operator* $\Omega_{\vdash}$ *is continuous.*

2. *If* $\mathcal{S}$ *has theorems, i.e.,* $\mathrm{Cn}_{\mathcal{S}}(\emptyset) \neq \emptyset$*, then* $\Omega_{\vdash}$ *is injective.*

Proof. Let $\mathbf{A}$ be a Λ-algebra and let $\{F_i : i \in I\}$ be a directed family of $\mathcal{S}$-filters on $\mathbf{A}$. Let $\langle a,b\rangle$ be a pair of elements of A. For 1. we need to show that

$$\langle a,b\rangle \in \Omega_{\vdash}(\bigcup_{i\in I} F_i \text{ iff } \langle a,b\rangle \in \bigcup_{i\in I}(\Omega_{\vdash}(F_i)).$$

But $\langle a,b\rangle \in \Omega_{\vdash}(\bigcup_{i\in I} F_i$ iff $b \in \mathrm{Fg}(\bigcup_{i\in I} F_i \cup \{a\})$. Since $\mathcal{S}$ is finitary (Definitions 2.4, 2.5 and 2.6), this last statement is equivalent to $b \in \mathrm{Fg}(\bigvee_{i\in J} F_i \cup \{a\})$, for some finite $J \subseteq I$. Since the family $\{F_i : i \in I\}$ is directed, this is equivalent to $b \in \mathrm{Fg}(F_j \cup \{a\})$, for some $j \in I$, which in turn is equivalent to $\langle a,b\rangle \in \bigcup_{i\in I}(\Omega_{\vdash}(F_i))$, i.e., 1. holds. For 2., let a be an element of $\mathbf{A}$ that is contained in every $\mathcal{S}$-filter on $\mathbf{A}$. Such an a exists, by assumption that $\mathrm{Cn}_{\mathcal{S}}(\emptyset) \neq \emptyset$. Assume that for some $\mathcal{S}$-filters F and G on $\mathbf{A}$, $\Omega_{\vdash}(F) = \Omega_{\vdash}(G)$. Then for every element $b \in A$, $b \in F$ iff $b \in \mathrm{Fg}(F \cup \{a\})$ iff $\langle a,b\rangle \in \Omega_{\vdash}(F)$ iff $\langle a,b\rangle \in \Omega_{\vdash}(G)$ iff $b \in \mathrm{Fg}(G \cup \{a\})$ iff $b \in G$. Hence $F = G$. $\square$

Lemma 6.22 *Let π be a set-polarity. If Ω^π is monotone then $\Omega^\pi \subseteq \Omega_\vdash$.*

Proof. Let $\mathbf{A}$ be a Λ-algebra and let $F \in Fi_{\mathcal{S}}(A)$. Suppose that $\langle a, b\rangle \in \Omega^\pi(F)$. By monotonicity of Ω^π, also $\langle a, b\rangle \in \Omega^\pi(\mathrm{Fg}^{\mathbf{A}}(F \cup \{a\}))$. By compatibility of $\Omega^\pi(\mathrm{Fg}^{\mathbf{A}}(F \cup \{a\}))$ with $\mathrm{Fg}^{\mathbf{A}}(F \cup \{a\})$, $b \in \mathrm{Fg}^{\mathbf{A}}(F \cup \{a\})$. Hence $\langle a, b\rangle \in \Omega_\vdash(F)$. □

Definition 6.23 *For a given set I of binary formulas, an algebra $\mathbf{A}$ and an $\mathcal{S}$-filter$F$ on $\mathbf{A}$, define*

$$\Omega_I(F) := \{\langle a, b\rangle : I(a, b) \subseteq F\}.$$

For example, if $E(x, y)$ is a system of congruence formulas for a protoalgebraic system $\mathcal{S}$, then $\Omega_E = \Omega$ (Theorem 3.15).

Lemma 6.24 *For every finite set of binary formulas I, the operator Ω_I on the lattice of $\mathcal{S}$-filters of $\mathbf{A}$ is continuous.*

Proof. Let $\mathcal{F} = \{F_i : i \in J\}$ be a directed set of $\mathcal{S}$-filters on $\mathbf{A}$. We need to show that $\Omega_I(\bigvee \mathcal{F}) = \bigvee_{i \in J} \Omega_I(F_i)$.

But since $\{F_i : i \in J\}$ is directed, and $I(a, b)$ finite, $I(a, b) \subseteq \bigvee \mathcal{F}$ iff there is $i \in J$ such that $I(a, b) \subseteq F_i$ and the claim follows. □

Definition 6.25 *Let $\mathcal{S}$ be a fixed 1-deductive system and let $I(x, y)$ be a set of binary formulas.*

1. *I is* **reflexive over** *$\mathcal{S}$ if*

$$\vdash_{\mathcal{S}} I(x, x) \tag{6.14}$$

2. I is **transitive over** $\mathcal{S}$ *if*

$$I(x,y), I(y,z) \vdash_{\mathcal{S}} I(x,z) \tag{6.15}$$

3. I **is a modus ponens (MP) system over** $\mathcal{S}$ *or* **has detachment property over** $\mathcal{S}$ *if*

$$I(x,y), x \vdash_{\mathcal{S}} y. \tag{6.16}$$

The rule (6.16) is called **modus ponens** *or* **detachment** *rule.*

4. I **is a Deduction Theorem (DT) system over** $\mathcal{S}$ *if, for every set* $\mathcal{G} \cup \{\varphi, \psi\}$ *of formulas,*

$$\Gamma, \varphi \vdash_{\mathcal{S}} \psi \textit{ implies } \Gamma \vdash_{\mathcal{S}} I(\varphi, \psi). \tag{6.17}$$

5. I is a **Deduction–Detachment Theorem (DDT) system over** $\mathcal{S}$ *if it has both (MP) and (DT), i.e., for every set* $\Gamma \cup \{\varphi, \psi\}$ *of formulas*

$$\Gamma, \varphi \vdash_{\mathcal{S}} \psi \textit{ iff } \Gamma \vdash_{\mathcal{S}} I(\varphi, \psi). \tag{6.18}$$

We say that $\mathcal{S}$ **has the (MP) rule, (DT) or (DDT), with** I, *if* I *is a set of binary formulas that is a (MP), (DT), (DDT) system over* $\mathcal{S}$.

Note also that any set of equivalence formulas is a reflexive (MP) system, but in general is not a (DT) system.

Lemma 6.26 *Let* I *be a set of binary terms. Then* $\Omega_{\vdash} = \Omega_I$ *iff* I *is a DDT system for* $\mathcal{S}$.

Proof. Immediate by definitions. □

Definition 6.27 *Let $\mathcal{S}$ be a* 1*-deductive system, π a set-polarity, and $I(x,y)$ a set of binary formulas. We say that I is* **monotone over π and $\mathcal{S}$** *if for all $t \in \mathrm{Te}$, $N \subseteq \mathrm{Occ}(t)$, if $+1 \in \pi(t,N)$, then*

$$I(x,y) \vdash_{\mathcal{S}} I(t(x,\vec{z}),t(y,\vec{z})) \tag{6.19}$$

and for every term t and $N \subseteq \mathrm{Occ}(t)$ such that $-1 \in \pi(t,N)$

$$I(x,y) \vdash_{\mathcal{S}} I(t(y,\vec{z}),t(x,\vec{z})). \tag{6.20}$$

Definition 6.28 *Let π be a set-polarity and $I(x,y)$ a set of binary formulas. Then I* **has polarity over π** *if for each $t \in I$ either*

$$-1 \in \pi(t,x) \text{ or } +1 \in \pi(t,y). \tag{6.21}$$

Proposition 6.29 *Let π be a set-polarity and let $I(x,y)$ be a set of binary formulas.*

1. *If I is reflexive over $\mathcal{S}$ and has polarity over π and $\mathcal{S}$, then $\Omega^\pi \subseteq \Omega_I$.*
2. *If I is a reflexive, transitive, monotone (MP) system over $\mathcal{S}$ then $\Omega_I \subseteq \Omega^\pi$.*
3. *Hence if I is a reflexive, transitive, monotone (MP) system over $\mathcal{S}$ and has polarity over $\mathcal{S}$ then $\Omega_I = \Omega^\pi$.*

Proof. For the first statement, let $\mathbf{A}$ be a Λ-algebra and let F be an $\mathcal{S}$-filter. Suppose that $\langle a,b\rangle \in \Omega^\pi(F)$. Since I is reflexive, $I(a,a) \cup I(b,b) \subseteq F$. Let $t(x,y) \in I(x,y)$.

Since I has polarity over π and $\mathcal{S}$, either $+1 \in \pi(t,y)$ or $-1 \in \pi(t,x)$. In the first case, $t(a,a) \in F$ implies $t(a,b) \in F$, and in the second case, $t(b,b) \in F$ implies $t(a,b) \in F$. So $I(a,b) \subseteq F$ and $\langle a,b\rangle \in \Omega_I(F)$.

For the second statement, let $\mathbf{A}$ be a Λ-algebra and let F be an $\mathcal{S}$-filter. Ω_I is reflexive and transitive, by reflexivity and transitivity of I. Since I is a (MP) system, $\langle a,b\rangle \in \Omega_I(F), a \in F$ imply $b \in F$, i.e., $\Omega_I(F)$ is compatible with F. Suppose that $\langle a,b\rangle \in \Omega_I(F)$. Then $I(a,b) \subseteq F$. Since I is monotone over π and $\mathcal{S}$, then for every term t, and $N \subseteq \mathrm{Occ}(t)$ such that $+1 \in \pi(t,N)$, we have $I(t(a/N,\vec{c}), t(b/N,\vec{c})) \subseteq F$, for all $\vec{c} \subseteq A$. Similarly, for for every term t, and $N \subseteq \mathrm{Occ}(t)$ such that $-1 \in \pi(t,N)$, we have $I(t(b/N,\vec{c}), t(a/N,\vec{c})) \subseteq F$, for all $\vec{c} \subseteq A$. Hence Ω_I agrees with π. Since $\Omega^\pi(F)$ is the largest quasi-ordering compatible with F that agrees with π, $\Omega_I(F) \subseteq \Omega^\pi(F)$. □

By Proposition 6.29, if a set of binary formulas $I(x,y)$ is reflexive, transitive, monotone and has the (MP) property, then $\Omega_I \subseteq \Omega^\pi$. The next proposition says that also conversely, this inclusion implies that $I(x,y)$ has the (MP) property.

Proposition 6.30 *Let $\mathcal{S}$ be a 1-deductive system and let $I(x,y)$ be a set of binary formulas. Then*

1. *If $\Omega_I \subseteq \Omega^\pi$, then I has the (MP) property.*

2. *If I is reflexive, monotone and transitive then $\Omega_I \subseteq \Omega^\pi$ iff I has the (MP) property.*

Proof. Let $T = \mathrm{Cn}_{\mathcal{S}}(x, I(x,y))$. If $\Omega_I \subseteq \Omega^\pi$, then $\langle x,y\rangle \in \Omega_I(T) \subseteq \Omega^\pi(T)$. Since also $x \in T$, then by compatibility of $\Omega^\pi(T)$ with T, $y \in T$ follows, i.e., $I(x,y), x \vdash_{\mathcal{S}} y$.

This proves 1. The condition 2. follows from the condition 1. and Proposition 6.29,1. □

Proposition 6.31 *Let $\mathcal{S}$ be a 1-deductive system and let π be a set-polarity. If a set of binary formulas I has polarity over π and $\mathcal{S}$ and is monotone over π and $\mathcal{S}$, then I is transitive.*

Proof. We need to show that $I(x,y), I(y,z) \vdash_{\mathcal{S}} I(x,z)$. Let $t(x,y) \in I(x,y)$. Since I has polarity over π and $\mathcal{S}$, either $+1 \in \pi(t,y)$ or $-1 \in \pi(t,x)$. Suppose that the former is the case. Since I is monotone over π and $\mathcal{S}$,

$$I(y,z), t(x,y) \vdash_{\mathcal{S}} t(x,z).$$

$$I(x,y), I(y,z) \vdash_{\mathcal{S}} t(x,z). \tag{6.22}$$

In the second case, i.e., when $-1 \in \pi(t,x)$,

$$I(x,y), t(y,z) \vdash_{\mathcal{S}} t(x,z)$$

and therefore 6.22 also holds. □

Corollary 6.32 *Let $\mathcal{S}$ be a 1-deductive system, let π be a set-polarity and let I be a set of binary formulas. If I is a reflexive, monotone (MP) system over $\mathcal{S}$ and has polarity over $\mathcal{S}$ then $\Omega_I = \Omega^{\pi}$.*

Proof. By Propositions 6.29 and 6.31

Proposition 6.33 *Let $\mathcal{S}$ be a deductive system and let π be a set-polarity. If there exists a reflexive (MP) system over $\mathcal{S}$ that has polarity over π and $\mathcal{S}$ then Ω^{π} is monotone.*

Proof. Let $\mathbf{A}$ be a Λ-algebra and let F, G be $\mathcal{S}$-filters on $\mathbf{A}$. It suffices to show that if $F \subseteq G$, then $\Omega^{\pi}(F) \subseteq \Omega^{\pi}(G)$. Assume that $F \subseteq G$ and, for some elements a, b of A, $\langle a, b\rangle \in \Omega^{\pi}(F)$ and $a \in G$. By Proposition 6.29, $\Omega^{\pi}(F) \subseteq \Omega_I(F)$, so $I(a,b) \subseteq F \subseteq G$. By (MP), $b \in G$, which finishes the proof that $\Omega^{\pi}(F)$ is compatible with G. □

6.6 Monotonicity Theorem

Recall (Theorem 3.10) that, for a deductive system $\mathcal{S}$, $\mathcal{S}$ has a system of equivalence formulas iff the Leibniz operator $\Omega_{\mathcal{S}}$ is monotone. One of our hopes was to prove a similar result characterizing deductive systems with some sort of a *system of implication formulas*, to be yet properly defined, by the monotonicity of $\Omega^{\pi}_{\mathcal{S}}$. By a method similar to that used in [4] to prove the version of Theorem 3.10 for 1-deductive systems, we will prove in this section that if $\Omega^{\pi}_{\mathcal{S}}$ is monotone, then a system of (MP) and reflexive formulas exist (Theorem 6.34). This makes a system of (MP) and reflexive formulas natural candidates to be called implication systems. In distinction to Theorem 3.10, however, the converse of Theorem 6.34 is false (Example 6.3). Moreover, every equivalence system is reflexive and (MP), but not every equivalence system is what we would like to call "implication" system (Example 6.3). In sections 6.9 and 6.10 we show that under some special condition on the system $\mathcal{S}$ and polarity π, we can prove a converse to Theorem 6.34. The question, whether if π is strict and total, then Theorem 6.34 has the converse, is open.

Theorem 6.34 *Let us fix a language Λ and a set-polarity π. Let $\mathcal{S}$ be a 1-deductive system and suppose that the operator $\Omega^{\pi} : \mathrm{Th}_{\mathcal{S}} \to \mathrm{Th}_{\mathcal{S}_{\pi}}$ is monotone. Then there is a finite system of reflexive (MP) formulas over $\mathcal{S}$.*

Proof. Let

$$T := \{\gamma(x,y) \in \mathrm{Te}_\Lambda : \vdash_{\mathcal{S}} \gamma(x,x)\}$$

First observe that T is an $\mathcal{S}$-filter on $\mathbf{Te}(x,y)$. For suppose that $T \vdash_{\mathcal{S}} \tau(x,y)$. Then, by structurality, $T(x,x) \vdash_{\mathcal{S}} \tau(x,x)$, where $T(x,x)$ is the result of substituting y for x in every term of T. But by the definition of T, every element of $T(x,x)$ is a theorem of $\mathcal{S}$. Hence also $\tau(x,x)$ is a theorem of $\mathcal{S}$. Therefore $\tau(x,y) \in T$, by the definition of T again. We have shown that T is an $\mathcal{S}$-theory. We next claim that $\langle x,y\rangle \in \Omega^{\pi}(T)$. To show this, consider a term $\tau(z,u_1,\ldots,u_m)$, where m is some number and $z,u_1,\ldots,u_m$ are some variables. Now substitute first x for z and some elements $t_i(x,y)$ of $\mathrm{Te}(x,y)$ for u_i's and assume that $\tau(x,t_1(x,y),\ldots,t_m(x,y)) \in T$. Next, substitute the same terms for u_i's but y, rather than x, for z. In order to show our claim we need to show that also $\tau(y,t_1(x,y),\ldots,t_m(x,y)) \in T$. Call this last term $\gamma(x,y)$, i.e.,

$$\gamma(x,y) := \tau(y,t_1(x,y),\ldots,t_m(x,y)).$$

From our assumption that $\tau(x,t_1(x,y),\ldots,t_m(x,y)) \in T$ it follows that also

$$\tau(x,t_1(x,x),\ldots,t_m(x,x)),$$

which is equal to $\gamma(x,x)$, is a theorem of $\mathcal{S}$. Hence $\gamma(x,y) \in T$, as desired. This finishes the proof that $\langle x,y\rangle \in \Omega^{\pi}(T)$.

By monotonicity of Ω^{π} we have that also $\langle x,y\rangle \in \Omega^{\pi}(T \vee \mathrm{Cn}_{\mathcal{S}}(x))$. But since $x \in T \vee \mathrm{Cn}_{\mathcal{S}}(x)$, it follows that $y \in (T \vee \mathrm{Cn}_{\mathcal{S}}(x)$ and therefore

$$T, x \vdash_{\mathcal{S}} y.$$

But this implies that there is some finite set $I(x,y) \subseteq T$ such that

$$I(x,y), x \vdash_{\mathcal{S}} y,$$

i.e., (6.16) holds. Also, since $I \subseteq T$, by definition of T the condition (6.14) holds. This finishes the proof of the theorem. □

Example 6.3 Let Λ consist of one binary symbol $\leftrightarrow$ and let $\mathcal{E}$ be the deductive system, called the deductive system of equivalential logic, determined by the following axiom and rule

$$(x \leftrightarrow y) \leftrightarrow ((z \leftrightarrow y) \leftrightarrow (x \leftrightarrow z))$$

$$\frac{x, x \leftrightarrow y}{y}.$$

It can be proved that $\vdash_{\mathcal{E}} x \leftrightarrow y$. Hence the set of binary formulas $\{x \leftrightarrow y\}$ is reflexive and (MP). In particular, $\Omega_{\mathcal{E}}$ is monotone. Let $\pi(t, N) = \emptyset$, for all $t \in \Lambda$ and all $N \subseteq \mathrm{Occ}(t)$. Then, by Proposition 6.16, $\langle a, b\rangle \in \Omega^{\pi}_{\mathcal{S}}(F)$ iff $a \in F$ is equivalent to $b \in F$. Therefore in the algebra $\mathbf{Te}(x, y)$, $\langle x, y \in \Omega^{\pi}_{\mathcal{S}}(\mathrm{Fg}_{\mathcal{E}}^{\mathbf{Te}(x,y)}(\emptyset))$, while $\langle x, y\rangle \notin \Omega^{\pi}(\mathrm{Fg}_{\mathcal{E}}^{\mathbf{Te}(x,y)}(x))$. Thus Ω^{π} is not monotone.

If π is a total polarity then Theorem 6.34 is a corollary of Theorem 3.10 and Lemma 6.19, (4). Let us notice that in the proof of the above theorem it really is inessential if we use Ω^{π} or Ω. Moreover, in the role of $\Omega^{\pi}(T)$ we do not need to take the *largest* $\mathcal{S}_{\pi}$-filter compatible with T- it is enough that it is compatible.

Let us also observe that the properties (6.14) and (6.16) are exactly the defining properties of equivalence formulas. Thus in particular, we have the following corollary to the monotonicity theorem 6.34 and the representation of equality Theorem 3.10. This corollary improves Lemma 6.19, (3).

Corollary 6.35 *If Ω^{π} is monotone then also Ω is monotone.*

Proof. By 6.34, if Ω^π is monotone then there is a finite set $I(x, y)$ of binary formulas satisfying (6.14) and (6.16). But then I is also a set of equivalence formulas and therefore by Theorem 3.10, we conclude that Ω is monotone. □

Example 6.3 shows that for a 1-deductive system $\mathcal{S}$, $\Omega^\pi_\mathcal{S}$ does not need to be monotone, even if $\Omega_\mathcal{S}$ is monotone.

Open question Is there a property P of sets of binary formulas, relative to a deductive system $\mathcal{S}$, such that the formula $x \to y$ satisfies this property relatively to at least some well-known deductive systems with implication and such that for a 1-deductive system $\mathcal{S}$, $\Omega^\pi_\mathcal{S}$ is monotone iff $\mathcal{S}$ has a set of binary formulas satisfying P?

In the next section we turn to stronger conditions on $\Omega^\pi_\mathcal{S}$ and, therefore, to conditions stronger than just monotonicity, on $\Omega^\pi_\mathcal{S}$.

6.7 Definition of π-implication

A main issue here is when do we want to call a system of formulas an *implication system*. We want this concept to be general enough to apply at least to all deductive system defined in the literature that have some implication connective. On the other hand the conditions on an implication system should be strong enough to distinguish it from an equivalence in those deductive systems as well as strong enough to guarantee that Ω^π is injective and continuous, or at least that it is monotone. We considered the monotonicity of $\Omega^\pi_\mathcal{S}$ in the previous section. So let us now turn to systems of formulas the existence of which is guaranteed by the continuity and injectivity of Ω^π.

Definition 6.36 *Let $\mathcal{S}$ be a 1-deductive system and let $I(x,y)$ be a set of binary formulas. Then $I(x,y)$ is called a π-***implication system** *if I is reflexive, transitive (MP) and monotone over π and $\mathcal{S}$.*

Recall that for a class $\mathcal{K}$ of $\mathcal{S}_\pi$-matrices, $\mathcal{S}_\mathcal{K}$ is the 2-deductive system determined by the rules that are valid in every matrix in $\mathcal{K}$. We write $\models_\mathcal{K}$ for $\vdash_{\mathcal{S}_\mathcal{K}}$.

Definition 6.37 *Let $\mathcal{S}$ be a 1-deductive system and let π be a set-polarity. Then a class $\mathcal{K}$ of $\mathcal{S}_\pi$-matrices is a π-***quasi-ordered algebraic semantics for** *$\mathcal{S}$ if the systems $\mathcal{S}$ and $\mathcal{S}_\mathcal{K}$ are equivalent (Definition 5.1).*

Since the $\mathcal{S}_\pi$-filters are quasi-orderings, we will use the symbol $\leq$ for the only predicate of the language of $\mathcal{S}_\pi$ and we will write an $\mathcal{S}_\pi$-formula $\langle t,s\rangle$ as the inequality $t \leq s$. If $\vec{\varepsilon}$ and $\vec{\delta}$ are finite sequences of unary terms of the same length, then $\vec{\delta}(t) \leq \vec{\varepsilon}(t)$ stands for the set $\{\delta_i(t) \leq \epsilon_i(t) : i = 1,\dots,n\}$, where $\vec{\delta} = \{\delta_i : i = 1,\dots n\}$, $\vec{\varepsilon} = \{\epsilon_i : i = 1,\dots,n\}$. If X is a set, then $(\vec{\delta}(X) \leq \vec{\varepsilon}(X))$ is the union of all sets $(\vec{\delta}(x) \leq \vec{\varepsilon}(x))$, where $x \in X$. If $I(x,y)$ is a set of binary formulas, then

$$I(\vec{\delta}(t), \vec{\varepsilon}(t)) := \{\varphi(\delta_i(t), \varepsilon_i(t)) : i = 1\dots,n, \varphi \in I\}.$$

Proposition 6.38 *Let $\mathcal{S}$ be a 1-deductive system and let π be a set-polarity. Then $\mathcal{S}$ has an π-quasi-order algebraic semantics iff there is a set $I(x,y)$ of binary formulas, and a system $\vec{\delta} \leq \vec{\varepsilon} = \langle \delta_i \leq \vec{\varepsilon}_i : i = 1,\dots,n\rangle$ of inequalities, where δ_i, ε_i are unary terms, such that I is reflexive, transitive and π-monotone and moreover*

$$x \dashv\vdash_{\mathcal{S}} I(\vec{\delta}(x) \leq \vec{\varepsilon}(x)).$$

Proof. By Definition 5.1 with $I(x,y) := v(x,y)$ and $\tau(x) := \{\delta_i(x) \leq \varepsilon_i(x) : i = 1,\dots,n\}$ $\square$

The following theorem is a corollary to Thm 5.26.

Theorem 6.39 *Let $\mathcal{S}$ be a 1-deductive system and let π be a set-polarity. Then the operator $\Omega^{\pi}_{\mathcal{S}}$ is injective and continuous iff $\mathcal{S}$ has a π-quasi-order algebraic semantics, i.e., there exists a class $\mathcal{K}$ of $\mathcal{S}_{\pi}$-matrices, a finite set I of binary terms, and two finite sequences of unary terms $\vec{\delta}, \vec{\varepsilon}$ such that*

$$x \dashv\vdash_{\mathcal{S}} I(\vec{\delta}(x), \vec{\varepsilon}(x)) \tag{6.23}$$

$$\Phi \models_{\mathcal{K}} \alpha \leq \beta \text{ iff } \{I(\gamma, \zeta) : \gamma \leq \zeta \in \Phi\} \vdash_{\mathcal{S}} I(\alpha, \beta) \tag{6.24}$$

Moreover,

$$\Omega^{\pi} = \Omega_I \tag{6.25}$$

Proof. By Corollary 5.26 with the translation τ defined by $\tau(x) := \{\delta_i(x) \leq \varepsilon_i(x) : i = 1, \ldots, n\}$ and the translation υ defined by $\upsilon(x, y) := I(x, y)$. $\square$

Corollary 6.40 *Let $\mathcal{S}$ be a 1-deductive system and let π be a set-polarity. If $\Omega^{\pi}_{\mathcal{S}}$ is injective and continuous, then $\mathcal{S}$ has π-implication.*

Definition 6.41 *Let $\mathcal{S}$ be a 1-deductive system and let π be a set-polarity. Then a set $I(x, y)$ of binary formulas is called* **algebraizable π-implication** *if $\Omega^{\pi} = \Omega_I$ and there is a system $\vec{\delta}(x) \leq \vec{\varepsilon}(x)$ of inequalities such that the conditions 6.23 and 6.24 hold.*

Corollary 6.42 *Let $\mathcal{S}$ be a 1-deductive system and let π be a set-polarity. Then Ω^{π} is injective and continuous iff $\mathcal{S}$ has an algebraizable π-implication. If a set of binary formulas $I(x, y)$ is an algebraizable π-implication for $\mathcal{S}$, then $I(x, y)$ is also a π-implication for $\mathcal{S}$.*

Proof. The first statement follows directly from the algebraization theorem. For the second statement, let $I(x,y)$ be an algebraizable π-implication. The fact that $I(x,y)$ is reflexive, transitive and monotone follows from (6.24) and the conditions (6.2)-(6.8) of the definition of $\mathcal{S}_\pi$, Definition 6.10. Since $\Omega^\pi = \Omega_I$, then $\Omega_I(\mathrm{Cn}_{\mathcal{S}_\pi}(\Gamma))$ is compatible with Cn_Γ), where $\Gamma = Cn_{\mathcal{S}}(I(x,y),x)$. But $\langle x,y\rangle \in \Omega_I(\mathrm{Cn}_{\mathcal{S}}(\Gamma))$ and $x \in \mathrm{Cn}_{\mathcal{S}}(\Gamma)$. Therefore $y \in \mathrm{Cn}_{\mathcal{S}}(I(x,y),x)$, i.e., I is MP. □

Lemma 6.43 *If $I(x,y)$ is a system of algebraizable implication formulas, then $\Omega_I \subseteq \Omega^\pi$.*

Proof. Recall that $\Omega_I(F)$ is an $\mathcal{S}_\pi$-filter compatible with F. Since $\Omega^\pi(F)$ is the largest $\mathcal{S}_\pi$-filter compatible with F, the inclusion follows. □

Lemma 6.44 *If I is a set of algebraizable implication formulas, then the operator Ω_I is injective and continuous.*

Proof. Continuity of Ω_I has been proved in lemma 6.24. Let now $\Omega_I(F) = \Omega_I(G)$, for two $\mathcal{S}$-filters on an algebra $\mathbf{A}$. Then for all $a,b \in A$, $I(a,b) \in F$ iff $I(a,b) \in G$. Let $a \in F$. By (6.23), $I(\vec{\delta}(a),\vec{\varepsilon}(a)) \subseteq F$. Therefore also $I(\vec{\delta}(a),\vec{\varepsilon}(a)) \subseteq G$ and by (6.24) again, $x \in G$. □

Theorem 6.39 fully characterizes the 1-deductive systems with algebraizable implication as those satisfying the conjunction of 6.23, 6.24 and $\Omega = \Omega_I$. The first two of these conditions refer to the existence of formulas with certain properties, while the third one is of a different nature. It is natural to ask whether a theorem similar to Theorem 6.39 would hold, if we dropped from the right-hand-side the condition that $\Omega^\pi = \Omega_I$, i.e., whether (6.23) and (6.24), possibly with the assumption that

I is MP, imply that Ω^π is injective and continuous. Unfortunately, this is not the case as witnessed by Example 6.3 above. Recall that the Ω^π considered there is not monotone. Hence it is not continuous. However the systems $I(x,y) = \{x \leftrightarrow y\}$ and $\delta(x) = \{x\}$, $\varepsilon(x) : \{x \leftrightarrow x\}$ satisfy (6.23), (6.24) and I is (MP).

The question whether there are some syntactical properties of I that imply (in addition to being implied by the fact) that Ω^π is injective and continuous, is open. In the next sections we turn to some additional conditions on polarity or system $\mathcal{S}$ that allow to characterize the continuity and injectivity of Ω^π in purely syntactical terms.

Our goal is to find a set of properties P on polarity π or on Ω^π and a set of properties Q of $I(x,y)$ such that first, the properties Q distinguish implication from equivalence, second, Ω^π is injective and continuous and P holds iff $\mathcal{S}$ has an algebraizable system of implication formulas satisfying Q and third, Ω^π is monotone and P holds iff $\mathcal{S}$ has a system I of MP reflexive formulas satisfying Q. Moreover we would like Q to be such that for every deductive systems with implication I considered in the literature, I has Q. The problem is still open. The next three sections present some partial results.

6.8 Condition $I(x,y) \not\vdash_{\mathcal{S}} I(y,x)$

Condition

$$I(x,y) \not\vdash_{\mathcal{S}} I(y,x) \tag{6.26}$$

clearly distinguishes implication from equivalence in all standard systems. We obtain a version of the algebraization theorem for I satisfying (6.26) together with its converse. However we do not have any interesting version of the monotonicity theorem, other than Theorem 6.34. Clearly, such an implication system cannot be an equivalence system, which solves one of our problems. We obtain a version of the algebraization theorem for this implication. We still do not know, however, whether the converse to this theorem holds.

Lemma 6.45 *If $\Omega^\pi = \Omega_I$, then $\Omega^\pi \neq (\Omega^\pi)^{-1}$ iff $I(x,y) \not\vdash_S I(y,x)$.*

Proof. $\Omega^\pi \neq (\Omega^\pi)^{-1}$ iff $\Omega^\pi(T) \neq (\Omega^\pi(T))^{-1}$, for some $\mathcal{S}$-theory T iff $\Omega_I(T) \neq {\Omega_I}^{-1}(T)$, for some $\mathcal{S}$-theory T iff for some terms t, s, $I(t,s) \subseteq T$ and $I(t,s) \not\subseteq T$, for some T iff $I(x,y) \not\vdash_S I(y,x)$. □

Theorem 6.46 *Let $\mathcal{S}$ be a 1-deductive system and let π be a polarity function. Then Ω^π is continuous, injective and $\Omega^\pi \neq (\Omega^\pi)^{-1}$ iff there exist sequences of unary terms $\vec{\delta}, \vec{\varepsilon}$ of the same finite length and a finite set I of binary formulas such that (6.23)-(6.24), $\Omega^\pi = \Omega_I$ and*

$$I(x,y) \not\vdash_S I(y,x). \tag{6.27}$$

Proof. Suppose that Ω^π is continuous, injective and that $\Omega^\pi \neq (\Omega^\pi)^{-1}$. The existence of $I, \vec{\delta}$ and $\vec{\varepsilon}$ such that (6.23)-(6.25) follows from theorem 6.39. (6.27) follows from lemma 6.45.

On the other hand if the conditions on the right hand side hold then in particular $\Omega^\pi = \Omega_I$ and therefore by lemma 6.44 Ω^π is continuous and injective and $\Omega^\pi \neq (\Omega^\pi)^{-1}$ by lemma 6.45. □

Thus the condition $\Omega^\pi \neq (\Omega^\pi)^{-1}$ is strong enough to imply that the set of implication formulas cannot be an equivalence system. We however do not know if the existence of a π-implication satisfying (6.27) implies that Ω^π is injective, continuous and $\Omega^\pi \neq (\Omega^\pi)^{-1}$.

Another question is the following:

Question If Ω^π is monotone and $\Omega^\pi \neq (\Omega^\pi)^{-1}$, does it follow that $\mathcal{S}$ has a Mp reflexive system satisfying (6.27)?

6.9 Condition $y \vdash_{\mathcal{S}} I(x, y)$

In this section we define some *set*-polarity π_p which depends on the system $\mathcal{S}$. We obtain versions of algebraization and monotonicity theorems, with converses. A dark side of this approach is that the applications are too limited. For example, our theorems do not apply to relevance logic.

Definition 6.47 *For every tree T and a set N of leaves of T let $+1 \in \pi_p(T, N)$ iff $x \vdash_{\mathcal{S}} T[x/N][\vec{z}]$, where $\vec{z}$ is a sequence of variables different of x, indexed by the leaves of T that are not in N. Assume further, that $-1 \notin \pi(T, N)$ for any pair $\langle T, N\rangle$.*

Let us notice that if $\mathcal{S}$ is the deductive system of classical, intuitionistic or (BCK+$\wedge$) logic, then $\pi_p(\wedge, 1) = \emptyset$, hence π_p is not the standard polarity defined before. Of course, also the polarity for the connectives of negation and implication in the first component is not stndard, by the assumption that π is positive. The implication in the second variable is positive for classical, intuitionistic and (BCK) (also with $\wedge$) logics as well as for all logics in which $x \vdash y \to x$. However for the implication of the relevance logic, π_p is not standard, since $y \not\vdash_{\mathcal{S}} x \to y$ in this logic and hence $\pi_p(\to, 2) = \emptyset$.

We will write $x \vdash_{\mathcal{S}} T[x/N]$ if $x \vdash_{\mathcal{S}} T[x/N][\vec{z}]$ for every sequence of variables $\vec{z}$ indexed by the leaves of T that are not in N.

Proposition 6.48 *The function π_p defined in definition 6.47 is a set-polarity.*

Proof. The condition (ROOT) of definition 6.5 clearly holds for π_p. To check the condition (SUBST), let T and S be two Λ-trees and let N and M be some sets of leaves of T and S, respectively. Notice, that by Definition 6.47, $\pi(T,N) \cup \pi(S,M) \subseteq \{+1\}$. Suppose that $+1 \in \pi(S,M)$ and $+1 \in \pi(T,N)$. Then $x \vdash_{\mathcal{S}} s[x/M]$ and $u \vdash_{\mathcal{S}} t[u/N]$. Taking in the role of u $S[x/M][\vec{z}]$, we get $x \vdash_{\mathcal{S}} T[S[x/M]/N]$. Hence $x \vdash_{\mathcal{S}} T[S/N][x/MN]$. Thus $+1 \in \pi(T[S/N][x/MN], NM)$. □

It follows from the definition of π_p that if $t = t(x,\vec{z})$ is a term and x is a variable occurring in t, then $\pi(t, O_{t,x}) = \{+1\}$ iff $x \vdash_{\mathcal{S}} t(x,\vec{z})$. Recall that $\pi(t,x) = \pi(t, O_{t,x})$, for a term t and a variable x occurring in t. Note also that π is not necessary total and that π is positive.

Let now $\mathcal{S}$ be a fixed 1-deductive system and let $\pi = \pi_p$ for $\mathcal{S}$.

Lemma 6.49 *Let $\mathbf{A}$ be a Λ-algebra and let a, b be elements of $\mathbf{A}$. Let F be an $\mathcal{S}$-filter on $\mathbf{A}$. Then $\langle a,b\rangle \in \Omega^{\pi}(F)$ iff for all $\vec{c} = c_1, \ldots, c_n$, every t, S such that $+1 \in \pi_p(t,S)$, we have*

$$t[a/S][\vec{c}] \in F \Rightarrow t[b/S][\vec{c}] \in F.$$

6.9.1 A version of equivalent semantics theorem for π_p

Recall that we say that a Λ-term t is *positive in a variable* x if $+1 \in \pi_{\mathcal{S}}(t,x)$.

Lemma 6.50 *Suppose that $\pi = \pi_p$ and let F be a $\mathcal{S}$-filter on a Λ-algebra $\mathbf{A}$. Then $A \times F \subseteq \Omega^{\pi}(F)$.*

Proof. By lemma 6.49, we need to show that for every term t positive in x, for all $a, b \in A$ we have :

$$\text{If } b \in F \text{ then } t(a/x, \vec{c}) \in F \Rightarrow t(b/x, \vec{c}) \in F.$$

But if t is positive in x, then $x \vdash_{\mathcal{S}} t$ and therefore $t(b/x, \vec{c}) \in F$, if $b \in F$. □

Lemma 6.51 *Let π be an arbitrary set-polarity. Suppose that $\Omega^{\pi} = \Omega_I$ for some system of binary formulas $I(x, y)$. Then $y \vdash_{\mathcal{S}} I(x, y)$ iff for every Λ-algebra $\mathbf{A}$ and every $\mathcal{S}$-filter F on $\mathbf{A}$*

$$I(A \times F) \subseteq F \textit{ and } A \times F \subseteq \Omega^{\pi}(F)$$

Proof. The "only if " part follows directly from lemma 6.50 and our assumption that $\Omega^{\pi} = \Omega_I$. To get the "if" part let $\mathbf{A}$ be the term algebra and F the $\mathcal{S}$-theory generated by y. Then, by assumption, $\langle x, y \rangle \in \Omega^{\pi}(F) = \Omega_I(F)$ which means that $I(x, y) \subseteq \mathrm{Cn}_{\mathcal{S}}(y)$. So $I(A \times F) \subseteq F$ and $A \times F \subseteq \Omega^{\pi}(F)$. □

Theorem 6.52 *Let $\mathcal{S}$ be a 1-deductive system and let $\pi = \pi_p$. Then for every Λ-algebra $\mathbf{A}$ and $\mathcal{S}$-filter F of $\mathbf{A}$ the operator Ω^{π} is injective, continuous on the lattice of $\mathcal{S}$-filters of $\mathbf{A}$ iff there exists an algebraic implication system satisfying the condition*

$$y \vdash_{\mathcal{S}} I(x, y). \tag{6.28}$$

Proof. Assume first that there is a system of algebraizable implication formulas such that (6.28) holds. For every $t \in I$ let $O_t := O_{t,y}$, i.e., the set be the set of all occurrences of y in $t(x,y)$. It follows from definition 6.47 that $\pi(t, O_t) = \{+1\}$. Therefore, by lemma 6.29 we know that $\Omega^\pi = \Omega_I$. Hence on every Λ-algebra $\mathbf{A}$, Ω^π is injective and continuous, by lemma 6.44.

If Ω^π is injective and continuous theorems 6.42 and 6.39 imply that there is a finite system $I(x,y)$ of algebraizable implication formulas and that $\Omega^\pi = \Omega_I$. The condition (6.28) follows from lemmas 6.50 and 6.51. □

6.9.2 A version of monotonicity theorem

Theorem 6.53 *Let $\mathcal{S}$ be a 1-deductive system and let $\pi = \pi_p$ be as above. Then Ω^π is monotone iff there exists a system $I(x,y)$ of reflexive MP formulas that satisfies (6.28).*

Proof. If I satisfies the conditions on the right–hand side of the equivalence, then the $I(x,y)$ is positive in y and Ω^π is monotone by lemma 6.29. On the other hand, if Ω^π is monotone, then let

$$\Gamma := \{t(x,y) \in \mathrm{Te}(x,y) : +1 \in \pi(t,y) \text{ and } \vdash_{\mathcal{S}} t(x,x)\}.$$

Let $T := \mathrm{Cn}_{\mathcal{S}}(\Gamma)$. We claim that $\langle x,y\rangle \in \Omega^\pi(T)$. For suppose that we have a term $s(u, v_1, \ldots, v_n)$ that is positive in u and let $t_1, \ldots, t_n$ be some elements of $\mathrm{Te}(x,y)$, i.e., for every $i = 1, \ldots, n$, $t_i = t_i(x,y)$ is a term. Suppose that $s(x, t_1, \ldots, t_n) \in T$ and let $t(x,y) := s(y, t_1(x,y), \ldots, t_n(x,y))$. We want to show that also $t(x,y) \in T$. But $s(x, t_1, \ldots, t_n) \in T$ implies that $\vdash_{\mathcal{S}} t(x,x)$ and s positive in u means that $u \vdash_{\mathcal{S}}$

$s(u, v_1, \ldots, v_n)$ and therefore $y \vdash_S t(x,y)$. Hence $t(x,y) \in \Gamma$, which finishes the proof of our claim.

It follows from the claim and the monotonicity of Ω^π that $\langle x,y\rangle \in \Omega^\pi(T \vee \mathrm{Cn}_S(x))$ and therefore $\Gamma, x \vdash_S y$. So there exists a finite set $I(x,y) \subseteq \Gamma$ such that $I(x,y), x \vdash_S y$. Since $I \subseteq \Gamma$, the other two properties of I follow. $\square$

6.9.3 Some remarks

Defining set-polarity the way we did in this section has the advantage that we are able to prove both the algebraization theorem and the monotonicity theorem in both directions. Also, the condition $y \vdash_S I(x,y)$ distinguishes the implication system from the equivalence systems.

On the other hand the strong condition in the definition limits the applications. Also, since π_p is not total, we can't, in general prove, that $\Omega^\pi \cap (\Omega^\pi)^{-1} = \Omega$.

We could of course "make" π_p total by saying that $\pi_p(t, N) = \{-1\}$, whenever it is not $\{+1\}$. This, however, would limit our applications yet further. For example, with this new definition, the classical connective $\wedge$ would have negative polarity in both occurrences. But then $\Omega^\pi \neq \Omega_I$, where I is the singleton set consisting of the implication connective, because $\langle x,y\rangle \in \Omega^\pi(F)$ would then imply $\langle y \wedge z, x \wedge z\rangle \in \Omega^\pi(F)$, while $\langle x,y\rangle \in \Omega_I(F)$ does not imply $\langle y \wedge z, x \wedge z\rangle \in \Omega_I(F)$. This is the main reason why we have chosen def. 6.47 as the definition of π_p.

6.10 Polarity entailment defined

One of the most tempting "corrections" to algebraizability, and also monotonicity theorem is to tie the polarity more strongly to the deductive system $\mathcal{S}$. Here we consider some strong conditions on polarity determined by the entailment relation of $\mathcal{S}$. The converses to both algebraization and monotonicity theorems are immediate under these strong assumptions. It turns out, however, that because of the strength of the assumptions, the results of this section apply only to deductive systems with Deduction–Detachment Theorem.

Lemma 6.54 *Let $\mathcal{S}$ be a 1-deductive system and assume that for every term t and a set $N \subseteq \mathrm{Occ}(t)$ at most one of the two conditions below holds for every $\mathcal{S}$-theory Θ:*

$$\frac{x, \Theta \vdash_{\mathcal{S}} y}{t(z_1, \ldots, z_n)[x/N], \Theta \vdash_{\mathcal{S}} t(z_1, \ldots, z_n)[y/N]} \tag{6.29}$$

$$\frac{x, \Theta \vdash_{\mathcal{S}} y}{t(z_1, \ldots, z_k)[y/N], \Theta \vdash_{\mathcal{S}} t(z_1, \ldots, z_n)[x/N]} \tag{6.30}$$

Let π be the function defined b for a tree T and a set N of occurrences of variables in t by

$$+1 \in \pi(T, N) \textit{ iff (6.29)}$$

holds for $t = T[x/N][\vec{z}]$ and all $\mathcal{S}$-theories Θ and

$$-1 \in \pi(T, N) \textit{ iff (6.30)}$$

holds for same t as above and all theories Θ . Then π is a total polarity.

Proof. It is straightforward to check that π satisfies conditions (ROOT) and (SUBST). □

Definition 6.55 *Let $\mathcal{S}$ be a 1-deductive system satisfying the assumption of the lemma. Then the polarity $\pi_{\vdash}$ satisfying the the conclusion of the lemma is called the* **polarity entailment defined** .

Remark If $\mathcal{S}$ is the deductive system of classical or intuitionistic logic then $\pi_{\vdash}$ coincides with the standard polarity. As we shall see in a moment, the implication of BCK logic is neither positive nor negative, with respect to $\pi_{\vdash}$, in any of its components.

Proposition 6.56 *Let $\mathcal{S}$ be 1-deductive system and let I be a finite set of binary formulas.*

1. *If I is a DDT system for $\mathcal{S}$, then for every $t \in I$, t is positive in y and negative in x, with respect to $\pi_{\vdash}$. Moreover,*

2. *If I is a system of reflexive formulas, then I is a DDT system iff for every $t \in I$, $\pi_{\vdash}(t, y) = \{+1\}$ iff for every $t \in I$, $\pi_{\vdash}(t, x) = \{-1\}$.*

Proof. directly from definition 6.55(I may include it in the next version) □

This proposition implies that $\pi_{\vdash}$ is not standard on any deductive system possessing implication but without Deduction–Detachment Theorem.

6.10.1 A version of the monotonicity theorem

Lemma 6.57 *Let $\mathcal{S}$ be a* 1*-deductive system such that for every term t and $N \subseteq$* Occ(t)*, at least one of the conditions (6.29) and (6.30) holds. Let $\pi = \pi_{\vdash}$. Then $\Omega_{\vdash} \subseteq \Omega^{\pi}$.*

Proof. It suffices to see that for every Λ-algebra $\mathbf{A}$ and $F \in \mathrm{Fi}_{\mathcal{S}}(A)$, $\Omega_{\vdash}(F)$ is a $\mathcal{S}_{\pi}$-filter compatible with F. This will imply that $\Omega_{\vdash}(F) \subseteq \Omega^{\pi}$. $\Omega_{\vdash}(F)$ is clearly reflexive, transitive and by definition it is compatible with F. Conditions (6.29) and (6.30) guarantee that this relation satisfies the condition (6.10) and (6.11). Thus it is an $\mathcal{S}_{\pi}$-filter. □

Theorem 6.58 *Let S be a 1-deductive system such that for every term t and for every $N \subseteq \mathrm{Occ}(t)$ one of the conditions (6.29) and (6.30) holds. Let $\pi = \pi_{\vdash}$. Then the following conditions are equivalent:*

1. *Ω^{π} is monotone*

2. *$\Omega^{\pi} = \Omega_{\vdash}$*

3. *Ω^{π} is injective and continuous.*

Proof. By lemma 6.22 the monotonicity of Ω^{π} implies $\Omega^{\pi} \subseteq \Omega_{\vdash}$ and lemma 6.57 gives the other inclusion. On the other hand, if $\Omega^{\pi} = \Omega_{\vdash}$, then Ω^{π} is injective and continuous by lemma 6.21. Clearly, continuity implies monotonicity. □

Theorem 6.59 *Let S be a 1-deductive system such that for every term t and for every $N \subseteq \mathrm{Occ}(t)$ one of the conditions (6.29) and (6.30) holds. Let $\pi = \pi_{\vdash}$. Then Ω^{π} is monotone iff there exists a finite system of algebraizable implication formulas satisfying Deduction–Detachment Theorem such that $\Omega^{\pi} = \Omega_I$.*

Proof. First suppose that Ω^{π} is monotone. By previous theorem, Ω^{π} is also injective and continuous and therefore there is a finite system of algebraizable implication formulas such that $\Omega_I = \Omega^{\pi}$. Also, the monotonicity of Ω^{π} implies that $\Omega^{\pi} = \Omega_{\vdash}$ and therefore $\Omega_I = \Omega_{\vdash}$, which by lemma 6.26 is equivalent to Deduction–Detachment Theorem for S with I.

The converse follows from the fact that Ω_I is monotone. □.

Theorem 6.60 *Let S be a deductive system and let π be some polarity. Suppose that Ω^{π} is injective and continuous. Then there exists a system of algebraizable implication formulas satisfying the Deduction–Detachment Theorem for S iff $\pi = \pi_{\vdash}$.*

Proof. We have already shown that if polarity is entailment defined and Ω^π injective and continuous, then there is an implication system I with Deduction–Detachment Theorem. For the converse, let t be a term positive in ν, i.e., $+1 \in \pi(t,\nu)$ and assume that $x, T \vdash_{\mathcal{S}} y$. By Deduction Theorem, $T \vdash_{\mathcal{S}} I(x,y)$. By (6.19), $T \vdash_{\mathcal{S}} I(t[x/\nu], t[y/\nu]$ and by the detachment, $t[x/\nu], T \vdash_{\mathcal{S}} t[y/\nu]$. □

6.10.2 An open problem

Let Λ be an algebraic language and let $\mathcal{G}$ be a Gentzen system over Λ. Let $\mathcal{L} := \{t :\vdash_{\mathcal{G}} \rightarrow t\}$. Let $\mathcal{S}$ be the deductive system determined by all the rules admissible for $\mathcal{L}$. Can we use $\mathcal{G}$ to define a polarity π in such a way that:

1. For standard classical and non-classical logics this polarity agrees with the standard ones

2. there are versions of equivalent semantics and monotonicity theorems, with converses, that are applicable to all the standard logics with implication would also bring some new information about other logics?

PART II.

FINITE BASIS AND RELATED PROBLEMS

CHAPTER 1. INTRODUCTION

One of the most intriguing problems in universal algebra is the so-called finite basis problem. For a finite algebra **A** the problem asks, whether all the identities satisfied by **A** can be derived from some finite set of identities of **A**. If this is the case, then the algebra **A** is called finitely based. Recall (Corollary 0.39) that the identities of **A** are the same as the identities of the variety generated by **A**, hence the finite basis problem can be restated as asking whether every variety generated by a single finite algebra is finitely based. Recall also, Definition 0.21, that a variety generated by a finite set of finite algebras, or, equivalently (Proposition 0.22), by one finite algebra, is also called finitely generated. In general, not every finite algebra and therefore not every finitely generated variety is finitely based. The first example to show this was found by R. Lyndon, [27]. But on the other hand, there are some special classes of algebras such that if a finite algebra **A** is a member of this class, then **A** is finitely based. For example, every two element algebra, every finite group and every finite ring ([26, 33, 20, 25], respectively) are finitely based.

Another positive result that we want to mention applies to a large class of varieties generated by finite algebras. This theorem, proved by K. Baker [1] in 1977, says that a finitely generated congruence-distributive variety is finitely based.

The finite basis question was also considered in the context of quasi-equational logic. We say that a finite set $\mathcal{K}$ of finite algebras is finitely *q-based* if there is a finite set Γ of quasi-identities of $\mathcal{K}$ such that every quasi-identity of $\mathcal{K}$ can be derived from Γ. By a finite q-basis question we will understand here the question whether a finite set of finite algebras is finitely q-based. Again, there are many examples of finite but nonfinitely q-based algebras. The simplest one in our opinion is the 3-element finitely based, but non-finitely q-based, semigroup found by M. Sapir, [51]. The first general positive results were proved in [42] for so-called relatively congruence-distributive finitely generated quasivarieties. A quasivariety Q is relatively congruence-distributive if for every algebra $\mathbf{A} \in Q$, the lattice of these congruences Θ of $\mathbf{A}$ that are relative to Q (definition 0.30) is distributive. Recall that a variety or quasivariety is of *finite type* if it is a variety or quasivariety of Λ-algebras, where Λ has only finitely many operation symbols and all of them are of finite arity. The following are the main theorems proved in [42].

Theorem 1.1 *Every finitely generated and relatively congruence-distributive quasivariety of finite type is finitely q-based.*

Theorem 1.2 *Let Q be a relatively congruence distributive quasivariety of finite type. Then every finitely generated relative subvariety of Q is finitely based.*

Because of the special character of congruence-distributive varieties, [15], the above theorem generalizes Baker's theorem, see [42] for the argument. We now turn to another generalization of Baker's theorem, that has been proved in [3].

Recall that an algebra $\mathbf{A}$ can be identified with a reduced model $\mathfrak{A} = \langle \mathbf{A}, \approx \rangle$ of the 2-deductive system $\mathcal{B}$ of equational logic and the equational theory of $\mathbf{A}$ then coincides with the set of theorems of $\mathfrak{A}$. Also, if S is an extension of $\mathcal{B}$ by theorems, then a 2-subset Θ of a Λ-algebra is an $\mathcal{S}$-filter on $\mathbf{A}$ exactly when Θ is a congruence on $\mathbf{A}$. Hence Baker's theorem can be restated as follows.

Theorem 1.3 [1] *Let $\mathcal{K}$ be a finite set of finite reduced $\mathcal{B}$ matrices and let $\mathcal{S}_{\mathcal{K}}$ be the extension of the Birkhoff-system $\mathcal{B}$ by the theorems of $\mathcal{K}$. If $\mathcal{S}_{\mathcal{K}}$ is filter-distributive, then $\mathcal{S}_{\mathcal{K}}$ is finitely based over $\mathcal{B}$.*

Recall (Definition 2.15 (i)) that the *theorems of $\mathcal{K}$ are finitely based relative to $\mathcal{S}$* if there is a finite number of formulas true in $\mathcal{K}$ such that every theorem of $\mathcal{K}$ can be derived from this finite set of formulas. A natural question arises, whether Theorem 1.3 remains true, if Birkhoff's system $\mathcal{B}$ in its assumption is replaced by an arbitrary K-deductive system $\mathcal{S}$ based by a finite set of rules, i.e., one may ask the following question.

Let $\mathcal{S}$ be a K-deductive system axiomatized by some finite number of rules. Does every finite $\mathcal{S}$-matrix, have finitely axiomatizable theorems relatively to $\mathcal{S}$?

These question and similar were considered for example in [60, 61, 46, 8, 3]. In general, universal algebraic results do not carry over to the logical matrices. For example Part III, Chapters 3 and 4 contains examples of matrices $\mathfrak{M}$ with theorems nonfinitely based over *any* system $\mathcal{S}$ axiomatized by a finite set of rules while their underlying algebras *are* finitely based.

The situation changes, however, when the deductive system $\mathcal{S}$ is protoalgebraic. The results of [3] and most of all [4] show that many constructions of universal algebra can be applied, after a suitable modification, to the semantics of protoalgebraic deductive system. This allowed the authors of [3] to prove the following strengthening of Baker's theorem.

Theorem 1.4 ([3, Corollary 4.7.]) *Suppose that we have only finitely many algebraic operations. Let $\mathcal{S}$ be a protoalgebraic filter-distributive* 1*-deductive system based by a finite set of rules and let $\mathfrak{A}$ be a finite $\mathcal{S}$-matrix. Then the theorems of $\mathfrak{A}$ are finitely based over $\mathcal{S}$.*

In Chapter 2 of this part we apply the results of Part I Chapters 2 and 3 to the semantics of protoalgebraic universal Horn logic with *finitely* many predicates symbols, or, equivalently, to protoalgebraic $\vec{k}$-deductive systems and in some cases even to K-deductive systems. The content of Chapter 2 is a straightforward extension of the results of [4] to $\vec{k}$-deductive systems. Some of these extensions have been independently obtained in [11].

Similarly as in the case of k-deductive systems ([4]), many of the theorems of universal algebra have their analogues for protoalgebraic $\vec{k}$-deductive systems. In particular, the classical lemma of B. Jónsson [15] that describes the subdirectly irreducible algebras in a congruence-distributive variety can be extended to reduced models of a filter-distributive protoalgebraic $\vec{k}$-deductive systems. We use this fact in the proof of the main results of Part II, Theorems 3.1, 3.2. These theorems generalize the finite basis theorem of Pigozzi to general filter-distributive $\vec{k}$-deductive systems in the same way as the theorem 1.4 generalizes Baker's theorem. In fact we go one step further and prove this generalization for $\vec{k}$-deductive systems, while in [3] only 1-deductive systems were considered. This adds to the complication of the technique and constructions, but once this is done, most of the idea of the proof of [42] can be generalized rather easily, using techniques originated in [3].

CHAPTER 2. K-PROTOQUASIVARIETIES

2.1 Introduction

The reduced matrix semantics of a protoalgebraic K-deductive system resembles under many respects the quasi-variety semantic of equational logic. Several results to support this point for 1- and k-deductive system have been proved in [3] and [4, Sections 9-12]. The material of [4, Sections 9-12] can be extended without difficulty to K-deductive systems, in fact in most cases the proofs used in [4] are independent from the relational language K chosen, although the results are stated only for k-deductive systems. In this Chapter we restate some of the content of [4, Sections 9-12] for K-deductive systems. We include sample proofs, for example of Theorem 2.12, but in general the reader is referred to [4]. Sections 2.2-2.2 of this chapter contain the results that will be used in Chapter 3. In section 2.5 a theorem about the existence of a free object in the class of reduced models of a protoalgebraic K-deductive system is proved (Theorem 2.23). The results presented in sections 2.2 and 2.2 have been independently obtained in [11].

2.2 Protoquasivarieties and relative subvarieties

Here we define protoquasivariety, subdirect products, subdirectly irreducible matrices, meet irreducible filters and prove, among others, the subdirect representation theorem for protoquasivarities, that generalizes Theorem 0.20 (Chapter "Preliminaries and notation").

If $\mathcal{S}$ is a K-deductive system with a basis Γ, then $\mathrm{Mod}\Gamma := \mathrm{Mod}\mathcal{S}$. Recall that a class of all reduced models of a K-deductive system is called a *matrix quasivariety.*

Definition 2.1 *The class* $\mathrm{Mod}^*\mathcal{S}$ *of all reduced models of a protoalgebraic K-deductive system $\mathcal{S}$ is called* K-**protoquasivariety** *or simply* **protoquasivariety.**

Definition 2.2 *Let $\mathcal{S}$ be a K-deductive system. A* **relative subvariety of a matrix quasivariety** $\mathrm{Mod}^*\mathcal{S}$ *is the class* $\mathrm{Mod}^*\mathcal{R}$, *where $\mathcal{R}$ is some axiomatic extension of $\mathcal{S}$.*

Definition 2.3 *Let $\mathcal{Q}$ be a matrix quasivariety. A matrix homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ is a $\mathcal{Q}$-homomorphism if $\mathfrak{B} \in \mathcal{Q}$. The class of all $\mathcal{Q}$-homomorphic images of matrices in $\mathcal{C}$ is denoted by $\mathbf{H}_{\mathcal{Q}}\mathcal{C}$. If $\mathcal{Q}$ is the class of all reduced K-matrices, then $\mathbf{H}_{\mathcal{Q}}\mathcal{C}$ is abbreviated as $\mathbf{H}\mathcal{C}$.*

Proposition 2.4 $\mathfrak{B} \in \mathbf{H}_{\mathcal{Q}}\mathcal{C}$ *iff there exists an $\mathfrak{A} \in \mathcal{C}$ and $F \in Fi_{\mathcal{S}}(\mathfrak{A})$ such that* $\mathfrak{B} \cong \langle \mathbf{A}, F\rangle^*$.

Proof. By Part I, Theorem 2.69.

Proposition 2.5 *Let* $\mathcal{R} = \mathrm{Mod}^*(\Gamma \cup E)$ *be a relative subvariety of* $\mathcal{Q} = \mathrm{Mod}^*\Gamma$, *where Γ is a set of K-rules and E is a set of K-formulas. Let $\mathbf{A}$ be a Λ-algebra. Then a K-subset X of A is an $\mathcal{R}$-filter iff it is an $\mathcal{S}$-filter and $E \subseteq E(\mathfrak{A})$.*

Proof. Immediate from definition. □

Corollary 2.6 *A relative subvariety of a protoquasivariety $\mathcal{Q}$ is also a protoquasivariety.*

Proof. By Proposition 2.5. □

Theorem 2.7 ([4, Thm. 11.1]) *The relative subvariety of a protoquasivariety $\mathcal{Q}$ generated by $\mathcal{C}$ is* $\mathbf{H}_{\mathcal{Q}}\mathbf{S}^*\mathbf{P}\mathcal{C}$.

Proof. See the proof of [4, Theorem 11.1]. □

Recall that a $\vec{k}$-protoquasivariety is a K-protoquasivariety, where K is finite. By a simple modification of the proofs of [3, Lemma 1.6. and Theorem 1.7.] one can prove the following

Lemma 2.8 *Let $\mathcal{C}$ be a finite set of finite $\vec{k}$-matrices. Then the class*

$$\{\mathfrak{A} : \mathfrak{A}^* \cong \mathfrak{B}^* \textit{ for some } \mathfrak{B} \in \mathcal{C}\}$$

is strictly elementary.

Proof. A modification of the proof of Theorem 1.7. of [3]. □

2.3 Subdirect products

A subdirect representation theorem below implies that every protoquasivariety is determined by its subdirectly irreducible members. Theorems 0.27 and 0.29 of the Chapter "Preliminaries and Notation" are special cases.

Recall (Definition 2.58) that a submatrix $\mathfrak{B}$ of a direct product of a family of matrices $\mathcal{A} = \{\mathfrak{A}_i : i \in I\}$ is called **a subdirect product of $\mathcal{A}$** iff for every $i \in I$, the projection of $\mathfrak{B}$ onto $\mathfrak{A}_i$ is onto. The following proposition characterizes subdirect products.

Proposition 2.9 (see [4, Prop. 9.1.]) *A reduced matrix $\mathfrak{A}$ is isomorphic to a subdirect product of reduced matrices $\mathfrak{B}_i$, with $i \in I$, iff there exists a system $F_i, i \in I$, of K-subsets of A such that*

(i) $\bigcap_{i \in I} F_i = D_{\mathfrak{A}}$

(ii) $\mathfrak{A}/F_i$ *is isomorphic to* $\mathfrak{B}_i$, *for all* $i \in I$.

Proof. See the proof of [4, Proposition 9.1.]. □

Definition 2.10 *Let* $\mathcal{S}$ *be a* K*-deductive system with a basis* Γ. *Let* $\mathcal{C} \subseteq \mathrm{Mod}^*\mathcal{S}$. *Let* $\mathfrak{A}$ *be a nontrivial* K*-matrix.*

1. $\mathfrak{A}$ *is* **subdirectly irreducible relatively to** $\mathcal{C}$ *if the fact that* $\mathfrak{A}$ *is isomorphic to some subdirect product* $\mathfrak{B}$ *of matrices* $\mathfrak{C}_i \in \mathcal{C}$, $i \in I$, *implies that for some* $i \in I$, *the projection* π_i *is an isomorphism from* $\mathfrak{B}$ *onto* $\mathfrak{C}_i$.

2. $\mathfrak{A}$ *is* **finitely subdirectly irreducible** *iff whenever* $\mathfrak{A}$ *is isomorphic to a subdirect product of a finite family* $\mathcal{A}$ *of matrices then* $\mathfrak{A}$ *is isomorphic to some algebra from* $\mathcal{A}$.

3. *The class of all relatively (finitely) subdirectly irreducible members of* $\mathcal{C}$ *is denoted by* $\mathcal{C}_{SI}$ *(*$\mathcal{C}_{SI}$*, respectively).*

4. *Let* $\mathcal{C} = \mathrm{Mod}^*\mathcal{S}$ *and let* $\mathcal{K}$ *be some class of* K*-matrices. Then the class of all members of* $\mathcal{Q}$ *that are subdirectly irreducible relatively to* $\mathcal{C}$ *(finitely subdirectly irreducible relatively to* $\mathcal{C}$*, resp.) is denoted by* $\mathcal{Q}_{\mathcal{C}SI}$ *or by* $\mathcal{Q}_{\Gamma SI}$ *(by* $\mathcal{Q}_{\mathcal{C}FSI}$ *or by* $\mathcal{Q}_{\Gamma FSI}$*, resp.).*

In particular, let $\mathcal{C}$ in the above definition be a relative subvariety of $\mathrm{Mod}^*\mathcal{S}$. Then it follows from Proposition 2.5 that a matrix $\mathfrak{A}$ is subdirectly irreducible relatively to $\mathcal{C}$ iff it is subdirectly irreducible relatively to $\mathrm{Mod}^*\mathcal{S}$. Hence

Proposition 2.11 *If $\mathcal{R}$ is a relative matrix subvariety of a matrix protoquasivariety $\mathcal{Q}$, then*

$$(\mathcal{R})_{\mathcal{SI}} = \mathcal{R} \cap \mathcal{Q}_{\mathcal{SI}} \text{ and } (\mathcal{R})_{\mathcal{FSI}} = \mathcal{R} \cap \mathcal{Q}_{\mathcal{FSI}}.$$

Proof. By Proposition 2.5. □

Theorem 2.12 (compare with Subdirect Representation Theorem, [4, Thm.9.2].) *Assume $\mathcal{Q}$ is a protoquasivariety. Every matrix in $\mathcal{Q}$ is isomorphic to a subdirect product of matrices in $\mathcal{Q}$ that are subdirectly irreducible relatively to $\mathcal{Q}$.*

Proof. Let $\mathcal{Q} = \mathrm{Mod}^*\mathcal{S}$ with $\mathcal{S}$ protoalgebraic. Let $\mathfrak{A} \in \mathrm{Mod}^*\mathcal{S}$. For each $R \in$ K and $\vec{a} \in A^{\rho(R)} \setminus R^{\mathfrak{A}}$ choose an $\mathcal{S}$-filter $F_{R(\vec{a})}$ on $\mathfrak{A}$ that is maximal with respect to the property that $\vec{a} \notin (F_{R(\vec{a})})_R$. Such a filter exists, because $\mathrm{Fi}_{\mathcal{S}}(\mathfrak{A})$ is an algebraic lattice. By Proposition 2.9, $\mathfrak{A}$ is isomorphic to a subdirect product of the $\mathfrak{A}/F_{R(\vec{a})}$. We show that $\mathfrak{A}/F_{R(\vec{a})}$ is subdirectly irreducible relatively to $\mathcal{S}$. By the correspondence property $\mathrm{Fi}_{\mathcal{S}}(\mathfrak{A}/F_R(\vec{a}))$ is isomorphic to the lattice filter $[F_{R(\vec{a})})$ in $\mathrm{Fi}_{\mathcal{S}}(\mathfrak{A})$. Every proper filter of this lattice contains the pair $R(\vec{a})$. Thus no family of proper filters of $\mathfrak{A}/F_{(i,\vec{a})}$ intersects at $D_{\mathfrak{A}/F_{R(,\vec{a})}}$. This implies by 2.9 that $\mathfrak{A}/F_{R(\vec{a})}$ is subdirectly irreducible relative to $\mathcal{S}$. □

The above theorem has also been independently proved in [11, Theorem 6.7].

Corollary 2.13 *1. Let $\mathcal{Q}$ and $\mathcal{R}$ be protoquasivarities and suppose that $\mathcal{Q}_{SI} = \mathcal{R}_{SI}$. Then $\mathcal{Q} = \mathcal{R}$.*

2. Let $\mathcal{R}$ and $\mathcal{T}$ be relative matrix subvarieties of the same protoquasivariety $\mathcal{Q}$. If $\mathcal{R} \cap \mathcal{Q}_{SI} = \mathcal{T}_{SI}$, then $\mathcal{R} = \mathcal{T}$.

Proof. The first statement follows from Theorem 2.12 . The second statement follows from the first, by Proposition2.11. □

Let $\mathfrak{A}$ be a model of a K-deductive system S. An $\mathcal{S}$-filter F on $\mathfrak{A}$ is *(completely) meet irreducible* in the lattice $\mathrm{Fi}_{\mathcal{S}}(\mathfrak{A})$ if F cannot be expressed as the meet of a finite (arbitrary) set of $\mathcal{S}$-filters. It follows from the above argument and Prop. 2.9 that $\mathfrak{A}$ is subdirectly irreducible iff $D_{\mathfrak{A}}$ is completely meet irreducible and that $\mathfrak{A}$ is finitely subdirectly irreducible iff $D_{\mathfrak{A}}$ is meet irreducible (all these irreducibilities relative to $\mathcal{S}$).

Theorem 2.14 compare with ([4, Theorem 9.3.])

(i) *Every K-protoquasivariety is closed under subdirect products and, in particular, under direct products.*

(ii) *Conversely, every reduced universal Horn K-class that is closed under subdirect products is a K-protoquasivariety.*

Recall that for a class of $\mathcal{C}$ of K-matrices

$$\mathbf{P}_{SD}\mathcal{C} := \{\mathfrak{A} : \mathfrak{A} \subseteq_{SD} \prod_{i \in I} \mathfrak{A}_i : \mathfrak{A}_i \in \mathcal{C} \text{ all } i \in I\}.$$

In this notation the subdirect representation Theorem 2.12 can be expressed as the equality

$$\mathcal{Q} = \mathbf{IP}_{SD}(\mathcal{Q}_{\Gamma_{SI}})$$

for every protoquasivariety $\mathcal{Q}$ determined by the set of rules Γ and Theorem 2.14 says that a universal Horn K-class is protoquasivariety iff it is closed under the operator P_{SD}. Thus a protoquasivariety $\mathcal{Q}$ is determined by the class $\mathcal{Q}_{\Gamma SI}$ of its subdirectly irreducible members.

Corollary 2.15 (Corollary 9.4.) *Assume that $\mathcal{C}$ generates a K-protoquasivariety $\mathcal{Q}$. Then $\mathcal{Q} = \mathbf{IS^*PP^*}_U\mathcal{C}$. If K is a finite set of finite K-matrices, then $\mathcal{Q} = \mathbf{IS^*P}\mathcal{C}$.*

□

Theorem 2.16 ([4, Theorem 9.6.]) *Let $\mathcal{C}$ be a set of reduced matrices and $\mathcal{Q}$ the K-protoquasivariety generated by $\mathcal{C}$. Suppose that $\mathcal{Q}$ is determined by the set of rules Γ. Then $\mathcal{Q}_{\Gamma\mathrm{FSI}} \subseteq \mathbf{IS^*P^*}_U\mathcal{C}$. If $\mathcal{C}$ is a finite set of finite matrices, then $\mathcal{C}_{\Gamma\mathrm{FSI}} \subseteq \mathbf{IS^*}\mathcal{C}$.*

Corollary 2.17 *Assume that $\mathcal{C}$ is a set of reduced K-matrices and $\mathcal{Q}$ the protoquasivariety generated by $\mathcal{C}$. Then $\mathcal{Q} = \mathbf{IP}_{SD}\mathbf{S^*P^*}_U\mathcal{C}$. If $\mathcal{C}$ is a finite set of finite matrices, then $\mathcal{Q} = \mathbf{IP}_{SD}\mathbf{S^*}\mathcal{C}$.*

2.4 Filter distributivity

Recall, Definition 0.50, that a lattice L is *distributive* iff for all elements a, b, c of L, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. Equivalently, L is distributive, if for all $a, b, c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. If $\mathcal{Q} = \mathrm{Mod}^*(\Gamma)$ and for every matrix $\mathfrak{A}$ the lattice $\mathrm{Fi}_{\mathcal{Q}}(\mathfrak{A})$ of $\mathcal{Q}$-filters of $\mathfrak{A}$ is distributive, then we call the matrix quasivariety $\mathcal{Q}$ *filter distributive*. The following theorem is a straightforward generalization of Theorem 12.1 of [4], which in turn generalizes the well known lemma of Jónsson [15]. A similar theorem was proved in [11, Theorem 6.16].

Theorem 2.18 (see [4, Theorem12.1]) *Let $\mathcal{Q}$ be a filter-distributive protoquasivariety and let $\mathcal{K} \subseteq \mathcal{Q}$. Let $\mathcal{V}$ be the relative matrix subvariety of $\mathcal{Q}$ generated by $\mathcal{K}$. Then $\mathcal{V}_{\mathrm{FSI}} \subseteq \mathbf{H}^*_{\mathrm{Q}}\mathbf{S^*P}^*_{\mathrm{U}}\mathcal{K}$.*

Proof. The proof of Theorem 12.1. in [4] depends only on the semantical results generalized here, among others, in Corollary 2.65 and Lemma 2.16. Hence the same proof can be used to prove Theorem 2.18. □

Corollary 2.19 *If a relative matrix subvariety $\mathcal{V}$ of a filter-distributive protoquasivariety is a finitely generated, then $\mathcal{V}_{\mathrm{FSI}}$ is a finite set of finite matrices. If $\mathcal{K}$ is a finite set of finite matrices, then, as is well-known, $\mathbf{P}^*_U K \subseteq \mathcal{K}$. Hence, by Theorem 2.18, if $\mathcal{V}$ is finitely generated by $\mathcal{K}$, then $\mathcal{V}_{\mathrm{FSI}} \subseteq \mathbf{H}^*_Q\mathbf{S}^*(\mathcal{K})$, which is a finite set of finite matrices, if $\mathcal{K}$ is finite.*

Proof.

Recall that a matrix protoquasivariety is a class $\mathrm{Mod}^*\Gamma$, where Γ determines a protoalgebraic deductive system. We say that a class $\mathrm{Mod}^*\mathcal{S}$ of first order structures is *elementary*, if $\mathcal{S}$ can be axiomatized by a set of axioms. It is *strictly elementary*, if $\mathcal{S}$ can be axiomatized by a finite set of axioms.

Corollary 2.20 *If $\mathcal{V}$ is a finitely generated relative matrix subvariety of a filter-distributive protoquasivariety, then $\mathcal{V}_{\mathrm{FSI}}$ is strictly elementary. If $\mathcal{V}$ is a finitely generated, filter-distributive protoquasivariety then $\mathcal{V}_{\mathrm{FSI}}$ is strictly elementary.*

Proof. Notice that the second statement follows from the first. Indeed, let $\mathcal{V}$ be a filter-distributive protoquasivariety finitely generated by $\mathcal{K}$. Then $\mathcal{V}$ is a relative matrix subvariety of itself and includes $\mathcal{K}$. Now if $\mathcal{W}$ is a relative matrix subvariety of $\mathcal{V}$ that includes $\mathcal{K}$, then $\mathcal{W}$ is also a protoquasivariety, hence $\mathcal{V} \subseteq \mathcal{W}$, as $\mathcal{V}$ is the protoquasivariety generated by $\mathcal{K}$. Therefore $\mathcal{V}$ is the smallest relative matrix subvariety of itself that includes $\mathcal{K}$; thus the relative matrix subvariety of $\mathcal{V}$ generated by $\mathcal{K}$ is $\mathcal{V}$ itself. Hence $\mathcal{V}$ is a finitely generated relative matrix subvariety of $\mathcal{V}$. By the first statement of the corollary, $\mathcal{V}_{\mathrm{FSI}}$ is finite. Now if $\mathcal{V}$ is a finitely generated relative matrix subvariety of a filter-distributive protoquasivariety $\mathcal{Q}$, then by Corollary 2.19, $\mathcal{V}_{\mathrm{FSI}}$ is finite. Since $\mathcal{V}_{\mathrm{FSI}}$ generates $\mathcal{V}$ as a relative subvariety of $\mathcal{Q}$, it follows from Lemma 2.8 that $\mathcal{V}_{\mathrm{FSI}}$ is strictly elementary. □

2.5 Free matrices

For every K-deductive system $\mathcal{S}$ there exists an $\mathcal{S}$-matrix that is a free object in the class of all $\mathcal{S}$-matrices. If $\mathcal{S}$ is protoalgebraic or R-protoalgebraic, then also there is a free object in the class $\mathrm{Mod}^{*}\mathcal{S}$ of reduced models of $\mathcal{S}$.

Recall that Te_α denotes the Λ-term algebra on α generators. Let $\mathrm{Fm}_{\mathrm{K}\alpha} = \coprod_{P\in\ \mathrm{K}} \mathrm{Te}_\alpha^{\rho(R)}$.

Let $\mathcal{S}$ be a K-deductive system, and let T_α be the smallest $\mathcal{S}$-theory in $\mathrm{Fm}_{\mathrm{K}\alpha}$, i.e., $T_\alpha = \mathrm{Cn}_{\mathcal{S}}(\emptyset)$. Then let T_α be the set of all theorems of $\mathcal{S}$ in variables $x_\kappa, \kappa < \alpha$. Let $\mathcal{LT}_\alpha := \langle \mathrm{Te}_\alpha, T_\alpha \rangle$. The matrix $\mathcal{LT}_\alpha$ is called the *Lindenbaum-Tarski matrix* of $\mathcal{S}$ over α variables. By convention, $x_\kappa^* := x_\kappa/\Omega(T_\alpha)$.

A K-deductive system $\mathcal{S}$ is *R-trivial,* if either all or none R-formulas are theorems of $\mathcal{S}$. The system $\mathcal{S}$ is *trivial* if it is R-trivial for all $R \in\ \mathrm{K}$. Let $\mathcal{S}$ be a K-deductive system and let α be a non-zero cardinal number. Let $\mathbf{Te}$ be the algebra of of Λ-terms over α generators $x_\kappa, \kappa < \alpha$. Let T_α be the set of $\mathcal{S}$-theorems in variables $\mathbf{z}, x_\kappa, \kappa < \alpha$. Let, for every $\kappa < \alpha$, $x_\kappa^* := x_\kappa/\Omega(T_\alpha)$. One corollary of Theorem 3.10 is that if a protoalgebraic K-deductive system $\mathcal{S}$ is not R-trivial for at least one $R \in\ \mathrm{K}$, then in the reduced Lindenbaum-Tarski matrix, all the generators x_κ^* are distinct.

Theorem 2.21 *Let $R \in\ K$ and assume that a K-deductive system $\mathcal{S}$ is not R-trivial.*

Then

1. *If $\mathcal{S}$ is protoalgebraic, then $x_\kappa \neq x_\lambda$, for $\kappa < \lambda < \alpha$.*

2. *If $\mathcal{S}$ is R-protoalgebraic, then $x_\kappa \neq x_\lambda$, for $\kappa < \lambda < \alpha$.*

Proof. Let $x := x_\kappa, y := x_\lambda$. Let $\Delta(x, y, \mathbf{z})$ be a system of equivalence K-formulas with parameters $\mathbf{z}$ for $\mathcal{S}$, that exists by Theorem 3.10.

Suppose that $x^* = y^*$, i.e., $\langle x, y\rangle \in \Omega(T_\alpha)$.

If $\mathcal{S}$ is protoalgebraic, then by monotonicity of Ω, for every $k \leq \rho(k)$ and for every choice of variables $z_1, \ldots, z_{\rho(R)-1}$

$$\langle x, y\rangle \in \Omega(\mathrm{Cn}(T_\alpha \cup \{R(z_1, \ldots, z_{k-1}, x, z_{k+1}, \ldots, z_{\rho(R)-1})\})).$$

Since

$$R(z_1, \ldots, z_{k-1}, x, z_{k+1}, \ldots, z_{\rho(R)-1}) \in \mathrm{Cn}(T_\alpha \cup \{R(z_1, \ldots, z_{k-1}, x, z_{k+1}, \ldots, z_{\rho(R)-1})\}),$$

it follows that

$$R(z_1, \ldots, z_{k-1}, y, z_{k+1}, \ldots, z_{\rho(R)-1}) \in \mathrm{Cn}(T_\alpha \cup \{R(z_1, \ldots, z_{k-1}, x, z_{k+1}, \ldots, z_{\rho(R)-1})\}),$$

and therefore

$$R(z_1, \ldots, z_{k-1}, x, z_{k+1}, \ldots, z_{\rho(R)-1}) \vdash_{\mathcal{S}} R(z_1, \ldots, z_{k-1}, x, z_{k+1}, \ldots, z_{\rho(R)-1}).$$

But this is possible only if $\mathcal{S}$ is R-trivial, a contradiction.

The same argument applies when "protoalgebraic" is replaced by R-protoalgebraic and Ω by Ω_R. $\square$

A special case of this theorem was stated in [4], although the proof was based on a false Theorem 13.2. of [4].

Theorem 2.22 ([4, Corollary 13.3]) *If $\mathcal{S}$ is a nontrivial protoalgebraic k-deductive system, then x^*_κ are pairwise disjoint.*

The argument used in the proof of Theorem 2.21 cannot be applied if the assumption that $\mathcal{S}$ is R-protoalgebraic is replaced by the assumption that it is Rk-protoalgebraic. We would like to know whether the assumption of Rk-protoalgebraicity is sufficient for the conclusion of 2.21.

Note that if a K-deductive system is R-trivial for every R, then for every $x := x_\kappa$, $y := x_\lambda$, every $R \in$ K and for every sequence of $\rho(R)$ terms $\vec{t}(u, \mathbf{z})$, where u is a new variable and $\mathbf{z}$ is some sequence of terms, elements of T_α, we have

$$R(\vec{t}(x, \mathbf{z})) \in T_\alpha \text{ iff } R(\vec{t}(y, \mathbf{z})) \in T_\alpha.$$

Thus if $\mathcal{S}$ is protoalgebraic, then $\mathcal{S}$ is trivial iff for every pair of variables x, y, $x^* = y^*$ in every Lindenbaum-Tarski matrix $\mathcal{LT}_\alpha$.

Theorem 2.23 *Let $\mathcal{Q} = \mathrm{Mod}^*\mathcal{S}$ be a nontrivial matrix protoquasivariety. Then for every $\alpha \geq 1$, $(\mathcal{LT}_\alpha)^*$ is a free, α generated, object in the category $\mathrm{Mod}^*\mathcal{S}$. If $\mathbf{K}$ generates $\mathcal{Q}$ then $(\mathcal{LT}_\alpha)^* \in \mathbf{IP}_{SD}\mathbf{S}^*\mathbf{K}$.*

Proof. The first part of this proof is the same as the first part of the proof of [4, Thm. 10.1]. It uses Thm. 2.66 and Corollary 2.21. To prove the second part, let $\mathbf{K}$ generate $\mathcal{Q}$. For each $R\vec{\tau}(x_{\kappa_1}, \ldots, x_{\kappa_m}) \in \mathrm{Fm}_{K_\alpha} \setminus T_\alpha$ there exists $\mathfrak{A} \in \mathbf{K}$ and $a_1, \ldots, a_m$ such that $\langle a_1, \ldots, a_m \rangle \notin (D_\mathfrak{A})_R$. The rest of the proof goes the same as in[4].

A K matrix $\langle \mathbf{A}, D \rangle$ is *finitely generated* if the algebra $\mathbf{A}$ is finitely generated.

Definition 2.24 *A class $\mathcal{Q}$ of K-matrices is* **locally finite** *if every finitely generated member of $\mathcal{K}$ is finite.*

Recall (Theorem 0.23) that a finitely generated variety (of algebras) is locally finite.

Lemma 2.25 *A finitely generated protoquasivariety is locally finite.*

Proof. Since a protoquasivariety $\mathcal{Q}$ generated by $\mathcal{K}$ is $\mathbf{IS^*P^*P^*_U}(\mathcal{K})$, it follows that the underlying algebra of a matrix $\mathfrak{A} \in \mathcal{Q}$ is a member of the variety generated by the underlying algebras of matrices from $\mathcal{K}$. Hence if $\mathfrak{A} \in \mathcal{Q}$ is finitely generated, then its underlying algebra $\mathbf{A}$ is finite. Hence $\mathcal{Q}$ is locally finite. $\Box$

CHAPTER 3. FINITE BASIS THEOREM FOR FILTER-DISTRIBUTIVE PROTOQUASIVARIETIES

Assume that Λ and K have only finitely many symbols. In this chapter we generalize Theorems 1.1 and 1.2. to arbitrary filter-distributive K- deductive systems. Theorems 1.1 and 1.2 then become the special cases our theorems, for the Birkhoff system in the role of the K-deductive system. Recall that when K is finite, then a K-deductive system is also called $\vec{k}$-deductive.

In [4] most of the well-known universal algebraic results which were the starting points of the proofs in [42, Theorems 1.1 and 1.2] were generalized to protoalgebraic k-deductive systems and in Chapter 2 to protoalgebraic K-deductive systems. It turns out that using these results one can relatively easily generalize the proof of Theorems 1.1 and 1.2 to the arbitrary filter distributive $\vec{k}$-deductive systems. Recall that a $\vec{k}$-deductive system $\mathcal{S}$ is finitely based if there is a finite set Γ of rules (some of which may be axiomatic) such that $\mathrm{Cn}_{\mathcal{S}} = \mathrm{Cn}_{\Gamma}$. A protoquasivariety $\mathcal{Q} = \mathrm{Mod}^*(\mathcal{S})$ is finitely based if $\mathcal{S}$ is finitely based, equivalently, if there is a finite set Γ of rules such that $\mathcal{Q} = \mathrm{Mod}^*\Gamma$. It was proved in [3, Corollary 4.7] that if a filter-distributive 1-protoquasivariety $\mathcal{Q}$ is finitely based, then every finitely generated relative matrix subvariety of $\mathcal{Q}$ is finitely based. The extension of this result to $\vec{k}$-protoquasivarities, (Lemma 3.7), will be here used to prove the following theorems:

Theorem 3.1 *Every finitely generated and filter-distributive matrix protoquasivariety is finitely based.*

Theorem 3.2 *Let $\mathcal{Q}$ be a filter-distributive matrix protoquasivariety. Then every finitely generated relative matrix subvariety of $\mathcal{Q}$ is finitely based.*

Our proof follows closely the idea and organization of the proof in [42]. In sections 3.3 and 3.4 we generalize key notions of [42]: the notion of universally parameterized quasi-equation and that of transformation of a parameterized quasi-equation by an equation. We later use them to prove some lemmas needed in the proofs of our main theorems in section 3.5.

3.1 Universally parameterized definable principal filter meets

In this section we show that if a finitely generated protoquasivariety Q is filter-distributive then there is a finite system of universally quantified atomic formulas that define the intersection of any pair of principal filters in Q.

By a *universally parameterized atomic formula* (or simply *parameterized atomic formula*), we mean any formula of the form

$$\forall \vec{u} T(\kappa_1(\vec{x}, \vec{u}), \ldots, \kappa_{\rho(T)}(\vec{x}, \vec{u})), \tag{3.1}$$

where $R \in$ K, $\vec{x} \in \mathrm{Var}^m$ for some m, $\vec{u} \in \mathrm{Var}^n$ for some n, $\kappa_1, \ldots, \kappa_{\rho(T)} \in (\mathrm{Te}(\vec{x}, \vec{u}))$. We also write $\vec{\kappa}$ for $\langle \kappa_1, \ldots, \kappa_{\rho(T)} \rangle$ and $\vec{\kappa}(\vec{x}, \vec{u})$ for $\langle \kappa_1(\vec{x}, \vec{u}), \ldots, \kappa_{\rho(T)}(\vec{x}, \vec{u}) \rangle$. In the future, whenever we write $T(\vec{\kappa}(\vec{x}, \vec{u}))$, we will assume, without even saying so, that the strings $\vec{\kappa}, \vec{x}, \vec{u}$ are of such a length that the expression (3.1) makes sense. The variables $\vec{u}$ are called *parameters*.

A finite conjunction of parameterized atomic formulas is a formula $\forall\vec{u}\varphi(\vec{x},\vec{u})$, where $\varphi(\vec{x},\vec{u}) = \bigwedge_{i\leq m} T_i(\vec{\kappa}_i(\vec{x},\vec{u}))$, for some positive integer m, some relation symbols $T_1,\ldots,T_i \in$ K, and sequences of terms $\vec{\kappa}_i$ of length $\rho(T_i)$. According to our convention, $\vec{x},\vec{u}$ are assumed to be strings of variables of such length that the expressions make sense.

A *finite conjunctions of parameterized atomic formulas indexed by a pair* $\langle R,S\rangle \in K^2$ is a formula

$$\forall\vec{u}\varphi_{RS}(\vec{x},\vec{y},\vec{u}),$$

where $\vec{x} = \langle x_1,\ldots,x_{\rho(R)}\rangle$, $y = \langle y_1,\ldots,y_{\rho(S)}\rangle$ and

$$\varphi_{RS} = \bigwedge_{i\leq m_{RS}} T_i^{RS}(\vec{\kappa}_i^{RS}(\vec{x},\vec{y},\vec{u})) \tag{3.2}$$

for some positive integer m_{RS}, some relation symbols $T_i^{RS} \in$ K, some sequences of terms $\vec{\kappa}_i$.

Whenever we write φ_{RS} we will assume without further explanation that this is a conjunction (3.2) which depends on $\rho(R)+\rho(S)$ variables $\vec{x}\vec{y}$ and some additional variables $\vec{u}$, called *parameters.*

A system Φ of finite conjunctions of parameterized atomic formulas indexed by K^2 is a system $\Phi = \{\forall\vec{u}\varphi_{RS} : R,S \in \mathrm{K}\}$, where for every pair $\langle R,S\rangle$, ϕ_{RS} is of the form 3.2.

Lemma 3.3 *Let $\mathcal{Q}$ be a matrix quasivariety and let Φ be a system of finite conjunctions of parameterized atomic formulas indexed by K^2. Then the following are equivalent:*

1. $\forall_{R,S\in K}\forall_{\mathfrak{A}\in\mathcal{Q}}\forall\vec{a}\in A^{\rho(R)},\vec{b}\in A^{\rho(S)}$

$$\mathrm{Fg}^{\mathfrak{A}}(R\vec{a})\cap\mathrm{Fg}^{\mathfrak{A}}(S\vec{b})=\bigvee_{i<m}\bigvee_{\vec{e}\in A^n}\mathrm{Fg}^{\mathfrak{A}}(T_i^{RS}(\vec{\kappa}^i(\vec{a},\vec{b},\vec{e})))$$

2. $\forall_{R,S\in K}\forall_{\mathfrak{A}\in\mathcal{Q}}\forall\vec{a}\in A^{\rho(R)},\vec{b}\in A^{\rho(S)}$

$$\mathrm{Fg}^{\mathfrak{A}}(R\vec{a})\cap\mathrm{Fg}^{\mathfrak{A}}(S\vec{b})=D_{\mathfrak{A}}\ \textit{iff}\ \mathfrak{A}\models\forall_{\vec{u}}\varphi_{RS}(\vec{a},\vec{b},\vec{u})$$

3. $\forall_{R,S\in K}$

$$\mathcal{Q}_{\mathrm{FSI}}\models\forall_{\vec{x}}\forall_{\vec{y}}[\forall_{\vec{u}}\varphi_{RS}(\vec{x},\vec{y},\vec{u})\leftrightarrow R\vec{x}\vee S\vec{y}].$$

Proof. The implications $1\Rightarrow 2\Rightarrow 3$ are immediate. For $3\Rightarrow 1$ let $F=\mathrm{Fg}^{\mathfrak{A}}(R\vec{a})$. Then $\langle\mathbf{A},F\rangle\in\mathcal{Q}$. Suppose that $F=\bigcap_{j=1}^k F_j$, where for every $j=1,\dots k$, F_j is meet irreducible. Then for all $j=1,\dots m_{RS}$, $\langle\mathbf{A},F_j\rangle\models R\vec{a}$ and therefore by 3, $\langle\mathbf{A},F_j\rangle\models\forall_{\vec{u}}\varphi_{RS}(\vec{a},\vec{b},\vec{u})$. For every $i\leq m_{RS}$, let here $T_i=T_i^{RS},\vec{\kappa}_i=\vec{\kappa}_i^{RS}$. So for all $j\leq k$, for all $i\leq m_{RS}$, for all strings $\vec{e}$ of elements of A of the same length as $\vec{u}$, we have $\langle\mathbf{A},F_j\rangle\models T_i(\vec{\kappa}_i(\vec{a},\vec{b},\vec{e}))$. Hence $\vec{\kappa}_i(\vec{a},\vec{b},\vec{e})\in\bigcap_{j=1}^k(F_j)_{T_i}=F_{T_i}$. Thus for every i, $\mathrm{Fg}^{\mathfrak{A}}(T_i\vec{\kappa}_i(\vec{a},\vec{b},\vec{e}))\subseteq F=\mathrm{Fg}^{\mathfrak{A}}(R\vec{a})$. Similarly, for every i, $\mathrm{Fg}^{\mathfrak{A}}(T_i\vec{\kappa}(\vec{a},\vec{b},\vec{e}))\subseteq F=\mathrm{Fg}^{\mathfrak{A}}(S\vec{b})$. This proves the inclusion from left to right in 1. For the other inclusion, it suffices to show that for every finitely meet irreducible filter F of $\mathfrak{A}$, $\bigvee_{i<m}\bigvee_{\vec{e}\in A^n}\mathrm{Fg}^{\mathfrak{A}}(T_i(\vec{\kappa}^i(\vec{a},\vec{b},\vec{e})))\subseteq F\Rightarrow\mathrm{Fg}^{\mathfrak{A}}(R\vec{a})\cap\mathrm{Fg}^{\mathfrak{A}}(S\vec{b})\subseteq F$. Suppose that the inclusion in the antecedent holds. Then for every $i=1,\dots,m_{RS}$, for every $\vec{e}$, $\langle\mathbf{A},F\rangle\models T_i\vec{\kappa}(\vec{a},\vec{b},\vec{e})$. So by (3), $\langle\mathbf{A},F\rangle\models R\vec{a}\vee S\vec{b}$. Therefore $\mathrm{Fg}^{\mathfrak{A}}(R\vec{a})\cap\mathrm{Fg}^{\mathfrak{A}}(S\vec{b})\subseteq F$. $\square$

Definition 3.4 *Let $\mathcal{Q}$ be a protoquasivariety and Φ a system of finite conjunctions of parameterized atomic formulas indexed by K^2 such that one, and therefore all,* *of the conditions of Lemma 3.3 holds, for all $R, S \in R$. Then we say that $\mathcal{Q}$* **has parameterized definable principal filter meets,** *or* **parameterized** DPFM, *for short.*

Theorem 3.5 *Let $\mathcal{Q}$ be a matrix protoquasivariety. Then $\mathcal{Q}$ has parametrized* DPFM *iff $\mathcal{Q}$ is filter-distributive and $\mathcal{Q}_{\mathrm{FSI}}$ is elementary. Moreover, if parameterized* DPFM *are defined by a system $\Phi = \{\forall\vec{u}\varphi_{RS} : R, S \in K\}$ of parameterized atomic formulas indexed by K^2, then*

$$\mathcal{Q}_{\mathrm{FSI}} \subseteq \mathrm{Mod}^*(\Gamma \cup \Sigma), \textit{where} \tag{3.3}$$

$$\Sigma = \{ \bigwedge_{R,S \in K} \forall_{\vec{x},\vec{y}}(\forall_{\vec{u}}\varphi_{RS}(\vec{x},\vec{y},\vec{u}) \to R\vec{x} \vee S\vec{y})\} \textit{ and} \tag{3.4}$$

$$\Gamma \textit{ is such that } \mathcal{Q} = \mathrm{Mod}^*\Gamma. \tag{3.5}$$

Proof. For the proof of the implication from left to right, assume that $\mathcal{Q}$ has parameterized DPFM by means of some system

$$\Phi = \{\forall\vec{u}\varphi_{RS} : R, S \in \mathrm{K}\}.$$

Let Γ be such that $\mathcal{Q} = \mathrm{Mod}^*\Gamma$. Then

$$\mathcal{Q}_{\mathrm{FSI}} \subseteq \mathrm{Mod}^*(\Gamma \cup \Sigma), \tag{3.6}$$

by lemma 3.3.

To see the inclusion in the other direction, let $\mathfrak{A}$ be a reduced model of $\Gamma \cup \Sigma$. Then $\mathfrak{A} \in \mathcal{Q}$ and we want to see that $\mathfrak{A}$ is finitely meet irreducible. Let F and G be Γ-filters on $\mathfrak{A}$ such that $D_{\mathfrak{A}} = F \cap G$. Suppose also that $F \neq D_{\mathfrak{A}}$ and let $R \in \mathrm{K}$ and $\vec{a} \in A^{(}\rho R)$ be such that $\vec{a} \in F_R \setminus (D_{\mathfrak{A}})_R$. We will show that for every $S \in \mathrm{K}$ and for every $\vec{b} \in A^{\rho(S)}, \vec{b} \in G_S$ implies $\vec{b} \in (D_{\mathfrak{A}})_S$. Since $F \cap G = D_{\mathfrak{A}}, \vec{a} \in F_R, \vec{b} \in G_S$ imply that $\mathrm{Fg}^{\mathfrak{A}}(R\vec{a}) \cap \mathrm{Fg}^{\mathfrak{A}}(S\vec{b}) = D_{\mathfrak{A}}$. By lemma 3.3, part 2, this implies that $\mathfrak{A} \models \forall \vec{u} \varphi_{RS}(\vec{a}, \vec{b}, \vec{u})$. Since we have assumed that $\mathfrak{A}$ is a model of Σ, we know that $\mathfrak{A} \models R\vec{a} \vee S\vec{b}$. We also know by assumption that $\mathfrak{A} \not\models R\vec{a}$. So $\mathfrak{A} \models S\vec{b}$, i.e., $\vec{b} \in (D_{\mathfrak{A}})_S$. This proves that $G = D_{\mathfrak{A}}$. Thus

$$\mathcal{Q}_{\mathrm{FSI}} = \mathrm{Mod}^*(\Gamma \cup \{ \bigvee_{R,S \in \mathrm{K}} \forall_{\vec{x},\vec{y}}(\forall_{\vec{u}} \varphi_{RS}(\vec{x}, \vec{y}, \vec{u})) \rightarrow (R\vec{x} \vee S\vec{y})\}),$$

and therefore $\mathcal{Q}_{\mathrm{FSI}}$ is an elementary class. Now to show that $\mathcal{Q}$ is filter-distributive, we want to show that $F \cap (G_1 \vee G_2) \subseteq (F \cap G_1) \vee (F \cap G_2)$ for all $\mathfrak{A} \in \mathcal{Q}$, all $F, G_1, G_2 \in \mathrm{Fi}_{\mathcal{Q}}(\mathfrak{A})$. Since $\mathrm{Fi}_{\mathcal{Q}}(\mathfrak{A})$ is algebraic, it suffices to prove that

$$\bigvee_{j<l} \mathrm{Fg}^{\mathfrak{A}}(R_j\vec{a}_j) \cap \bigvee_{i<k} \mathrm{Fg}^{\mathfrak{A}}(S_i\vec{b}_i) \subseteq \bigvee_{j<l} \bigvee_{i<k} [\mathrm{Fg}^{\mathfrak{A}}(R_j\vec{a}_j) \cap \mathrm{Fg}^{\mathfrak{A}}(S_i\vec{b}_i)]$$

We prove this inclusion by showing that every finitely meet irreducible Γ-filter F that includes the right-hand-side also includes the left-hand side. So let F be finitely meet irreducible and suppose that the right-hand side is included in F. Then $\langle \mathbf{A}, F \rangle \in \mathcal{Q}_{\mathrm{FSI}}$ and for every $i < k, j < l$ $\langle \mathbf{A}, F \rangle \models \forall_{\vec{u}} \varphi_{R_j S_i}(\vec{a}, \vec{b}_i, \vec{u})$. Therefore for all pairs $i < k, j < l$,

$$\langle \mathbf{A}, F \rangle \models R_j\vec{a}_j \vee S_i\vec{b}_i. \tag{3.7}$$

We either have that for all $j < l$ If $\langle \mathbf{A}, F \rangle \models R_j\vec{a}_j$, in which case $\bigvee_{j<l} \mathrm{Fg}^{\mathfrak{A}}(R_j\vec{a}_j) \cap \bigvee_{i<k} \mathrm{Fg}^{\mathfrak{A}}(S_i\vec{b}_i) \subseteq F$; or else by 3.7 for this j and all $i < k$, we must have that $\mathrm{Fg}^{\mathfrak{A}}(S_i\vec{b}_i) \subseteq F$ and the left-hand side is included in F.

For the implication from right to left assume that $\mathcal{Q}$ is filter-distributive and that $\mathcal{Q}_{\mathrm{FSI}}$ is elementary. Let $R, S \in \mathrm{K}, \rho(R) = n, \rho(S) = m$. Let $\mathfrak{F}$ be the free matrix over $\mathcal{Q}$ countably generated by $x_1, \ldots, x_n, y_1, \ldots y_m, u_0, u_1, \ldots$. Let $\{\vec{\tau}_i(\vec{x}, \vec{y}, \vec{u}) : i <$

$\omega\} \subseteq \mathrm{Te}^+$ and $\{T_i : i < \omega\} \subseteq$ K be such that

$$\mathrm{Fg}^{\mathfrak{F}}(R\vec{x}) \cap \mathrm{Fg}^{\mathfrak{F}}(S\vec{y}) = \bigvee_{i<\omega} \mathrm{Fg}^{\mathfrak{F}}(T_i(\vec{\tau}_i(\vec{x},\vec{y},\vec{u})))$$

Let $\mathfrak{A}$ be a countably generated matrix in $\mathcal{Q}$, such that $\vec{a} = a_1,\ldots,a_n, \vec{b} = b_1,\ldots,b_m \in A$ and let $\vec{e} = e_0, e_1,\ldots$ be any sequence of elements of A that together with $a_1,\ldots,a_n,b_1,\ldots,b_m$ generate $\mathbf{A}$. Let h be a surjective matrix homomorphism from $\mathfrak{F}$ to $\mathfrak{A}$ such that $hx_i = a_i$ for $i = 1,\ldots,n$, $hy_j = b_j$, for $j = 1,\ldots,m$, $hu_k = e_k, k = 0,1,\ldots$. Let $F := h^{-1}(D_{\mathfrak{A}})$. Then by the correspondence property we have that for every $R \in$ K, $\vec{p} \in A^{\rho(R)}, \vec{\pi} \in \mathrm{Te}^{\rho(R)}$ such that $h(\vec{\pi}) = \vec{p}$,

$$h^{-1}(\mathrm{Fg}^{\mathfrak{A}}(R\vec{p})) = \mathrm{Fg}^{\mathfrak{A}}(R\vec{\pi}) \vee F.$$

Now making use of filter-distributivity we have:

$$\begin{aligned}
h^{-1}(\mathrm{Fg}^{\mathfrak{A}}(R\vec{a}) \cap \mathrm{Fg}^{\mathfrak{A}}(S\vec{b})) &= h^{-1}(\mathrm{Fg}^{\mathfrak{A}}(R\vec{a})) \cap h^{-1}(\mathrm{Fg}^{\mathfrak{A}}(S\vec{b}) \\
&= (\mathrm{Fg}^{\mathfrak{F}}(R\vec{x}) \vee F) \cap (\mathrm{Fg}^{\mathfrak{F}}(S\vec{y}) \vee F) \\
&= (\mathrm{Fg}^{\mathfrak{F}}(R\vec{x}) \cap \mathrm{Fg}^{\mathfrak{F}}(S\vec{y})) \vee F \\
&= \bigvee_{i<\omega} (\mathrm{Fg}^{\mathfrak{F}}(T_i(\vec{\tau}_i(\vec{x},\vec{y},\vec{u}))) \vee F) \\
&= \bigvee_{i<\omega} h^{-1}(\mathrm{Fg}^{\mathfrak{A}}(T_i\vec{\tau}_i(\vec{a},\vec{b},\vec{e}))) \\
&= h^{-1}(\bigvee_{i<\omega} \mathrm{Fg}^{\mathfrak{A}}(T_i\vec{\tau}_i(\vec{a},\vec{b},\vec{e})))
\end{aligned}$$

Thus $\mathrm{Fg}^{\mathfrak{A}}(R\vec{a}) \cap \mathrm{Fg}^{\mathfrak{A}}(S\vec{b}) = \bigvee_{i<\omega} \mathrm{Fg}^{\mathfrak{A}}(T_i\vec{\tau}_i(\vec{a},\vec{b},\vec{e}))$. Therefore for every countably generated $\mathfrak{A} \in \mathcal{Q}$ and all $\vec{a}, \vec{b}$ in A, we have

$$\mathrm{Fg}^{\mathfrak{A}}(R\vec{a}) \cap \mathrm{Fg}^{\mathfrak{A}}(S\vec{b}) = D_{\mathfrak{A}} \Leftrightarrow \forall_{i<\omega}\vec{\tau}_i(\vec{a},\vec{b},\vec{e}) \in (D_{\mathfrak{A}})_{T_i} \tag{3.8}$$

for all $\vec{e}$ in A such that $\vec{a},\vec{b},e_0,e_1,\ldots$ generate $\mathbf{A}$. We will argue that the quantification "such that $\vec{a},\vec{b},e_0,e_1,\ldots$ generate $\mathbf{A}$" can be removed from (3.8) and that (3.8) holds for all matrices $\mathfrak{A} \in \mathcal{Q}$, not necessarily countably generated. Indeed, for every matrix $\mathfrak{A} \in \mathcal{Q}$

$$\mathrm{Fg}^{\mathfrak{A}}(R\vec{a}) \cap \mathrm{Fg}^{\mathfrak{A}}(S\vec{b}) = D_{\mathfrak{A}} \Leftrightarrow \text{ for every } \mathfrak{B} \leq \mathfrak{A},\ \mathrm{Fg}^{\mathfrak{B}}(R\vec{a}) \cap \mathrm{Fg}^{\mathfrak{B}}(S\vec{b}) = D_{\mathfrak{B}}. \quad (3.9)$$

So let $\mathfrak{A}$ be an arbitrary matrix from $\mathcal{Q}$ and assume $\mathrm{Fg}^{\mathfrak{A}}(R\vec{a}) \cap \mathrm{Fg}^{\mathfrak{A}}(S\vec{b}) = D_{\mathfrak{A}}$. We want to show that $\forall_{i<\omega}\vec{\tau}_i(\vec{a},\vec{b},\vec{e}) \in (D_{\mathfrak{A}})_{T_i}$, for arbitrary sequence $\vec{e}$ from A. Let $\mathfrak{B}$ be the submatrix of $\mathfrak{A}$ generated by $\vec{a},\vec{b},\vec{e}$. By 3.9, $\mathrm{Fg}^{\mathfrak{B}}(R\vec{a}) \cap \mathrm{Fg}^{\mathfrak{B}}(S\vec{b}) = D_{\mathfrak{B}}$ and by 3.8, $\forall_{i<\omega}\vec{\tau}_i(\vec{a},\vec{b},\vec{e}) \in T_i^{\mathfrak{B}}$, and therefore $\forall_{i<\omega}\vec{\tau}_i(\vec{a},\vec{b},\vec{e}) \in (D_{\mathfrak{A}})_{T_i}$. To show (3.8) for arbitrary $\mathfrak{A}$ in the other direction, assume that $\forall_{i<\omega}\vec{\tau}_i(\vec{a},\vec{b},\vec{e}) \in (D_{\mathfrak{A}})_{T_i}$, for all sequences $\vec{e}$ of elements of A. Thus by (3.8), for every countably generated submatrix $\mathfrak{B}$ of $\mathfrak{A}$, $\mathrm{Fg}^{\mathfrak{B}}(R\vec{a}) \cap \mathrm{Fg}^{\mathfrak{B}}(S\vec{b}) = D_{\mathfrak{B}}$. By 3.9, $\mathrm{Fg}^{\mathfrak{A}}(R\vec{a}) \cap \mathrm{Fg}^{\mathfrak{A}}(S\vec{b}) = D_{\mathfrak{A}}$. We have proved that for every $\mathcal{Q}$-matrix $\mathfrak{A}$,

$$\mathrm{Fg}^{\mathfrak{A}}(R\vec{a}) \cap \mathrm{Fg}^{\mathfrak{A}}(S\vec{b}) = D_{\mathfrak{A}} \Leftrightarrow \forall_{i<\omega}\vec{\tau}_i(\vec{a},\vec{b},\vec{e}) \in (D_{\mathfrak{A}})_{T_i},$$

for all infinite sequences $\vec{e}$ of elements of A. Thus $\mathcal{Q}_{\mathrm{FSI}}$ satisfies

$$\forall_{\vec{x},\vec{y}}\Big(\bigwedge_{i<\omega} \forall_{\vec{u}} T_i(\vec{\tau}_i(\vec{x},\vec{y},\vec{u}))\Big) \leftrightarrow R\vec{x} \vee S\vec{y}. \quad (3.10)$$

Since it is an elementary class, it is easy to show that the infinite conjunction can be replaced by a finite subconjunction and

$$\mathcal{Q}_{\mathrm{FSI}} \models \forall_{\vec{x},\vec{y}} \bigwedge_{i<m} \forall_{\vec{u}} T_i(\vec{\tau}(\vec{x},\vec{y},\vec{u})) \leftrightarrow R\vec{x} \vee S\vec{y}.$$

Therefore also the infinite string of variables $\vec{u}$ can be replaced by a finite one. $\square$

If K is a class of matrices, let $\mathcal{Q}K$ be the smallest matrix quasi-variety containing K. It is called the matrix quasivariety generated by K.

Corollary 3.6 (see [3, Lemma 4.4], for 1-deductive systems) *Every finitely generated and filter-distributive protoquasivariety has parameterized DPFM.*

Proof. Let $\mathcal{Q} = \mathcal{Q}\mathcal{C}$, where $\mathcal{C}$ is a finite set of finite matrices. In view of the theorem it suffices to show that $\mathcal{Q}_{\mathrm{FSI}}$ is elementary. But by the Lemma (refJonssonlemma), $\mathcal{Q}_{\mathrm{FSI}} \subseteq I\mathbf{S}^* P_U^* \mathcal{C} = I\mathbf{S}^* \mathcal{C}$ and therefore is finite. Hence it is elementary.

3.2 First main lemma

Main Lemma 3.7 *Assume that $\mathcal{S}$ is a protoalgebraic deductive system axiomatized by some set Γ of rules consisting of finitely many proper rules and possibly infinitely many axioms, i.e., $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 is finite, $\Gamma_2 \subseteq \mathrm{At}$. Assume further that $\mathcal{Q} = \mathrm{Mod}^*\mathcal{S}$ is filter-distributiv and that $\mathcal{Q}_{\mathrm{FSI}}$ is a strictly elementary class. Then there exists a finite set $\Delta \subseteq \mathrm{At}$, such that $\mathrm{Mod}^*\mathcal{S} = \mathrm{Mod}^*(\Gamma_1 \cup \Delta)$.*

Proof. Let $\Phi = \{\varphi_{R,S} : R, S \in \mathrm{K}\}$ be a system of conjunctions of parameterized atomic formulas that defines PFM in $\mathcal{Q}$. For $R, S \in \mathrm{K}$, let

$$\psi_{RS} = \forall_{\vec{x},\vec{y}}(\forall_{\vec{u}}\varphi_{R,S}(\vec{x},\vec{y},\vec{u})) \to R\vec{x} \vee S\vec{y}.$$

Then

$$\mathcal{Q}_{\mathrm{FSI}} = \mathrm{Mod}^*(\Gamma_1 \cup \Gamma_2 \cup \{\psi_{RS} : R, S \in \mathrm{K}\}),$$

where $\mathcal{Q}_{\mathrm{FSI}}$ is the class of all finitely subdirectly irreducible matrices in $\mathcal{Q}$. Since $\mathcal{Q}$ is strictly elementary, there exists a finite subset Γ_2' of Γ_2, such that

$$\mathcal{Q}_{\mathrm{FSI}} = \mathrm{Mod}^*(\Gamma_1 \cup \Gamma_2' \cup \{\psi_{RS} : R, S \in \mathrm{K}\}). \tag{3.11}$$

Let Ω be the set consisting of the formulas (3.12)–(3.15) below, for every finite m and for all $R, S, T, T_1, \ldots, T_m \in$ K.

$$R\vec{x} \to \forall_{\vec{u}}(\varphi_{RS}(\vec{x}, \vec{y}, \vec{u})) \tag{3.12}$$

$$\forall_{\vec{u}}\phi_{RS}(\vec{x}, \vec{y}, \vec{u}) \to \forall_{\vec{u}}\phi_{SR}(\vec{y}, \vec{x}, \vec{u}) \tag{3.13}$$

$$\forall_{\vec{u}}\phi_{RR}(\vec{x}, \vec{x}, \vec{u}) \to R\vec{x} \tag{3.14}$$

$$\bigwedge_{i<m} \forall_{\vec{u}}\varphi_{T_iR}(\vec{x}, \vec{y}\vec{u}) \to \forall_{\vec{u}}\varphi_{TR}(\vec{x}, \vec{y}, \vec{u}) \tag{3.15}$$

for each rule $\bigwedge_{i<m} T_i(\vec{\tau}_i(\vec{x})) \to T(\vec{\tau}(\vec{x}))$ in Γ.

Lemma 3.8 *$\mathcal{Q} \models \Omega$, i.e., $\Gamma_1 \cup \Gamma_2 \models \Omega$.*

Proof. Let $\mathfrak{A} \in \mathcal{Q}$. Since Φ defines PFM in $\mathcal{Q}$, $\mathfrak{A} \models$ (3.12) is equivalent to the condition that, for all $R, S \in$ K and for all $\vec{a}, \vec{b}$, if $\vec{a} \in R^{\mathfrak{A}}$, then $\mathrm{Fg}^{\mathfrak{A}}(R\vec{a}) \cap \mathrm{Fg}^{\mathfrak{A}}(S\vec{b}) = D_{\mathfrak{A}}$, which is obviously true.

Similarly, $\mathfrak{A} \models$ (3.13) follows from the fact that for all $\vec{a} \in A^{\rho(R)}$ and $\vec{b} \in A^{\rho(S)}$, $\mathrm{Fg}^{\mathfrak{A}}(R\vec{a}) \cap \mathrm{Fg}^{\mathfrak{A}}(S\vec{b}) = D_{\mathfrak{A}}$ implies $\mathrm{Fg}^{\mathfrak{A}}(S\vec{b}) \cap \mathrm{Fg}^{\mathfrak{A}}(R\vec{a}) = D_{\mathfrak{A}}$, and $\mathfrak{A} \models$ (3.14) follows from the fact that $\mathrm{Fg}^{\mathfrak{A}}(R\vec{a}) \cap \mathrm{Fg}^{\mathfrak{A}}(R\vec{a}) = D_{\mathfrak{A}}$ implies $\mathfrak{A} \models R\vec{a}$. Next, let

$$\bigvee_{i \leq m} T_i(\vec{\tau}(\vec{x})) \to T(\vec{\tau}(\vec{x})) \in \Gamma.$$

Then $\mathfrak{A} \models$ (3.15) iff, for all sequences $\vec{a}, \vec{b}$ of appropriate length, $\mathrm{Fg}^{\mathfrak{A}}(T_i(\vec{\tau}_i(\vec{a}))) \cap \mathrm{Fg}^{\mathfrak{A}}(R\vec{b}) = D_{\mathfrak{A}}$ implies $\mathrm{Fg}^{\mathfrak{A}}(T(\vec{\tau}(\vec{a}))) \cap \mathrm{Fg}^{\mathfrak{A}}(R\vec{b}) = D_{\mathfrak{A}}$. But

$$\mathrm{Fg}^{\mathfrak{A}}(T(\vec{\tau}(\vec{a}))) \subseteq \bigvee_{i<m} \mathrm{Fg}^{\mathfrak{A}}(T_i(\vec{\tau}_i(\vec{a})))$$

and therefore, by filter-distributivity,

$$\begin{aligned}
[\mathrm{Fg}^{\mathfrak{A}}(T(\vec{\tau}(\vec{a})))] \cap \mathrm{Fg}^{\mathfrak{A}}(R\vec{b}) &\subseteq \Big[\bigvee_{i<m} \mathrm{Fg}^{\mathfrak{A}}(T_i(\vec{\tau}_i(\vec{a})))\Big] \cap \mathrm{Fg}^{\mathfrak{A}}(R\vec{b}) \\
&= \bigvee_{i<m} \Big[\mathrm{Fg}^{\mathfrak{A}}(T_i(\vec{\tau}_i(\vec{a}))) \cap \mathrm{Fg}^{\mathfrak{A}}(R\vec{b})\Big] \\
&= D_{\mathfrak{A}},
\end{aligned}$$

by assumption. □

Let now $L := \mathrm{Mod}^*(\Gamma_1 \cup \Gamma_2' \cup \Omega)$. Note that L is finitely axiomatizable.

Lemma 3.9 $L \cap (\mathrm{Mod}^*(\Gamma_1))_{\mathrm{FSI}} \subseteq \mathcal{Q}_{\mathrm{FSI}}$.

Proof. Let $\mathfrak{A} \in L \cap (\mathrm{Mod}^*(\Gamma_1))_{\mathrm{FSI}}$. Then $\mathfrak{A} \models \Gamma_1 \cup \Gamma_2' \cup \Omega$ and $D_{\mathfrak{A}}$ is finitely meet irreducible(relatively to Γ_1). By 3.11 it suffices to show that $\mathfrak{A} \models \psi_{RS}$, for all $R, S \in \mathrm{K}$. Let $\vec{a} \in A^{\rho(R)}, \vec{b} \in A^{\rho(S)}$ and suppose that for every sequence $\vec{c}$, $\mathfrak{A} \models \varphi_{RS}(\vec{a}, \vec{b}, \vec{c})$. We want to show that $\mathfrak{A} \models R\vec{a} \vee S\vec{b}$. For arbitrary $U, V \in \mathrm{K}$ define

$$F_V := \{\vec{y} \in A^{\rho(V)} : \mathfrak{A} \models \varphi_{RV}(\vec{a}, \vec{y}, \vec{c}) \text{ for all } \vec{c}\}$$

$$G_U := \{\vec{x} \in A^{\rho(U)} : \mathfrak{A} \models \varphi_{UV}(\vec{x}, \vec{y}, \vec{c}) \text{ for all } \vec{c}, \text{ all } V \in \mathrm{K}, \text{ all } \vec{y} \in F_V\}.$$

We claim now that $F := \coprod\{F_V : V \in \mathrm{K}\}$ and $G := \coprod_{U \in \mathrm{K}} G_U$ are Γ_1-filters. For this, assume that

$$\bigwedge_{i<m} T_i(\vec{\eta}_i(\vec{x})) \to T(\vec{\eta}(\vec{x})) \tag{3.16}$$

is a rule of Γ_1 and that for some $\vec{e}$ and for all $i < m$, $\langle \mathbf{A}, F\rangle \models T_i(\vec{\eta}_i(\vec{e}))$. Then, $\vec{\eta}_i(\vec{e}) \in F_{T_i}$, i.e., $\mathfrak{A} \models \varphi_{RT_i}(\vec{a}, \vec{\eta}_i(\vec{e}), \vec{c})$, for all $\vec{c}$. Also, since $\mathfrak{A}$ satisfies (3.15) and (3.13), we have

$$\mathfrak{A} \models \bigwedge_{i<m} \forall_{\vec{u}} \varphi_{RT_i}(\vec{y}, \vec{\eta}_i(\vec{x}), \vec{u}) \to \forall_{\vec{u}} \varphi_{RT}(\vec{y}, \vec{\eta}(\vec{x}), \vec{u}).$$

Therefore

$$\mathfrak{A} \models \varphi_{RT}(\vec{a}, \vec{\eta}(\vec{e}), \vec{c}),$$

for all $\vec{c}$, i.e., $\vec{\eta}(\vec{e}) \in F_T$ and $\langle \mathbf{A}, F\rangle \models T(\vec{\eta}(\vec{e}))$. This finishes the proof that F is a Γ_1-filter. Now suppose that (3.16) is in Γ_1 and that, for all $i < m$

$$\eta_i(\vec{e}) \in G_{T_i}. \tag{3.17}$$

Then, by (3.15),

$$\mathfrak{A} \models \bigwedge_{i<m} \forall_{\vec{u}}\varphi_{T_iV}(\vec{\eta_i}(\vec{x}), \vec{y}, \vec{u}) \to \forall_{\vec{u}}\varphi_{TV}(\vec{\eta}(\vec{x}), \vec{y}, \vec{u}).$$

(3.17) says that, for all $\vec{c}$, all $V \in$ K, and all $\vec{f} \in F_V$

$$\mathfrak{A} \models \varphi_{T_iV}(\vec{\eta_i}(\vec{e}), \vec{f}, \vec{c}).$$

Therefore,

$$\mathfrak{A} \models \varphi_{TV}(\vec{\eta}(\vec{e}), \vec{f}, \vec{c}),$$

i.e., $\vec{\eta}(\vec{e}) \in G_T$.This finishes the proof that G is a filter. We claim now that $F \cap G = D_{\mathfrak{A}}$. For if for some $V \in$ K $\vec{e} \in A^{\rho(V)}$, $\vec{e} \in F_V \cap G_V$, then, by definition of G, $\mathfrak{A} \models \varphi_{VV}(\vec{e}, \vec{e}, \vec{c})$, for all $\vec{c}$. But then $\vec{e} \in D_{\mathfrak{A}}$, by (3.14), which completes the proof of our claim that $F \cap G = D_{\mathfrak{A}}$. Since $\mathfrak{A} \in \mathcal{Q}_{\mathrm{FSI}}$, it follows that $F = D_{\mathfrak{A}}$ or $G = D_{\mathfrak{A}}$. By definitions of F and G, $S\vec{b} \in F$ and $R\vec{a} \in G$. Therefore $\mathfrak{A} \models R\vec{a} \vee S\vec{b}$. We have proved that for all $R, S \in$ K, $\mathfrak{A} \models \psi_{R,S}$ and thus completed the proof of the lemma.

□

Since $\Gamma_1 \cup \Gamma_2 \models \Gamma_1 \cup \Gamma_2' \cup \Omega$ (Lemma 3.8), there is a finite set $\Gamma_2'' \subseteq \Gamma_2$ such that $\Gamma_1 \cup \Gamma_2'' \models \Gamma_1 \cup \Gamma_2' \cup \Omega$. Therefore $\mathcal{Q} \subseteq \mathrm{Mod}^*(\Gamma_1 \cup \Gamma_2'') \subseteq L \subseteq \mathrm{Mod}^*(\Gamma_1)$. Recall that $\mathcal{Q}_{\mathrm{FSI}} = \mathcal{Q} \cap (\mathrm{Mod}^*(\Gamma_1)_{\mathrm{FSI}}) = \mathcal{Q} \cap (\mathrm{Mod}^*(\Gamma_1 \cup \Gamma_2''))_{\mathrm{FSI}}$. Hence

$$\mathcal{Q}_{\mathrm{FSI}} = \mathcal{Q} \cap (\mathrm{Mod}^*(\Gamma_1)_{\mathrm{FSI}}) \subseteq (\mathrm{Mod}^*(\Gamma_1 \cup \Gamma_2''))_{\mathrm{FSI}}.$$

It follows that

$$\mathcal{Q}_{\mathrm{FSI}} \subseteq \mathrm{Mod}^*(\Gamma_1 \cup \Gamma_2'')_{\mathrm{FSI}} \subseteq L \cap (\mathrm{Mod}^*\Gamma_1)_{\mathrm{FSI}}.$$

By Lemma 3.9, $L \cap (\mathrm{Mod}^*(\Gamma_1))_{\mathrm{FSI}} \subseteq \mathcal{Q}_{\mathrm{FSI}}$. Therefore

$$\mathcal{Q}_{\mathrm{FSI}} = (\mathrm{Mod}^*(\Gamma_1 \cup \Gamma_2''))_{\mathrm{FSI}} = L \cap (\mathrm{Mod}^*(\Gamma_1)_{\mathrm{FSI}}.$$

Thus $\mathcal{Q} = \mathrm{Mod}^*(\Gamma_1 \cup \Gamma_2'')$, and $\mathcal{Q}$ is finitely axiomatizable.

3.3 Universally parameterized Horn formulas

A *universally parameterized strict Horn formula* or just *universally parameterized Horn formula* is a formula of the

$$\xi_0(\vec{x}) \wedge \ldots \wedge \xi_{k-1}(\vec{x}) \rightarrow \eta_0 \wedge \cdots \wedge \eta_{l-1}(\vec{x}), \tag{3.18}$$

where $l \geq 1$ and for $i = 0, \ldots, k-1$ and $j = 0, \ldots, l-1$ $\xi_i(\vec{x}), \eta_j(\vec{x})$ are arbitrary parameterized atomic formulas with free variables $x_0, \ldots x_{p-1}$ for some p and such that $\vec{x} = x_0, \ldots x_{p-1}$. Thus every parameterized Horn formula is logically equivalent to a formula $\psi(\vec{x})$ of the form

$$\forall_{\vec{v}} \bigwedge_{i<k} R_i(\vec{\tau}_i(\vec{x}, \vec{v})) \rightarrow \forall_{\vec{v}} \bigwedge_{i<l} S_j(\vec{\sigma}_j(\vec{x}, \vec{v})) \tag{3.19}$$

for some sequence of variables $\vec{v}$. Note that a Horn formula (3.18) corresponds to a rule. In the sequel $\psi(\vec{x})$ will always represent a parameterized Horn formulaand $\theta(\vec{x}), \xi(\vec{x}), \eta(\vec{x})$ arbitrary parameterized atomic formulas or conjunctions of such. Every ordinary Horn formula is a parameterized Horn formula with an empty list of parameters.

The formulas (3.12)- (3.14), which constitute the set Ω of the previous section are also examples of parameterized implications.

A class $\mathcal{L}$ of matrices is called a *parameterized matrix quasivariety* if $\mathcal{L} = \text{Mod}^*\Gamma$ where Γ is a set of parameterized Horn formulas.

An arbitrary filter F on an algebra $\mathbf{A}$ is *closed under parameterized implication* (3.19) if whenever $\{\tau_i(\vec{a}, \vec{e}) : \vec{e} \in A^q\} \subseteq F_{R_i}$ for all $i < k$ then also for every $j < l, \{\sigma_j(\vec{a}, \vec{e}) : \vec{e} \in A^q\} \subseteq F_{S_j}$. If F is a filter on a Λ-algebra $\mathbf{A}$, then $\langle \mathbf{A}, F \rangle \models \psi(\vec{x})$ iff F is closed under $\psi(\vec{x})$. Clearly, if each member of some system of filters is closed under a given parameterized Horn formula then so is their intersection.

Let $\mathcal{L}$ be a parameterized quasivariety and let $\mathbf{A} \in \mathcal{L}$. A filter F on $\mathbf{A}$ is called an $\mathcal{L}$- filter if $\langle \mathbf{A}, F \rangle \in \mathcal{L}$. So F is an $\mathcal{L}$- filter if it is closed under any set of parameterized Horn formulas that form a base for $\mathcal{L}$. Hence the set of all $\mathcal{L}$-filters on $\mathbf{A}$, which we again denote by $\text{Fi}_{\mathcal{L}}(\mathbf{A})$, is closed under arbitrary intersections. This gives

Lemma 3.10 *Every parameterized quasivariety is closed under the formation of subdirect products.*

The set $\text{Fi}_{\mathcal{L}}(\mathbf{A})$ need not be closed under the formation of unions of directed sets, so $\text{Fi}_{\mathcal{L}}(\mathbf{A})$ is not, in general, an algebraic lattice.

A matrix $\mathfrak{A} = \langle \mathbf{A}, F \rangle \in \mathcal{L}$ is *finitely subdirectly irreducible relative to* $\mathcal{L}$ if F is finitely meet irreducible in the lattice $\text{Fi}_{\mathcal{L}}(\mathbf{A})$. The subclass of $\mathcal{L}$ so defined is denoted by $\mathcal{L}_{\text{FSI}}$. $\text{Fi}_{\mathcal{L}}(\mathbf{A})$ need not be algebraic, so a fixed filter need not be the meet of finitely many meet irreducible $\mathcal{L}$ filters. This is always the case, of course, when $\mathbf{A}$ is finite, so we have

Lemma 3.11 *Assume $\mathcal{L}$ is is a parameterized quasivariety and let $\mathfrak{A} \in \mathcal{L}$. If $\mathfrak{A}$ is finite, then it is a subdirect product of a finite number of matrices in $\mathcal{L}_{\text{FSI}}$.*

3.4 Calculus of transformations

We introduce the notion of *transformation* of a parameterized Horn formula by a parameterized atomic formula. This always generates another parameterized Horn formula. We look at the relationship between the model-theoretic properties of an arbitrary parameterized Horn formula and those of its transform.

Let $\vec{u} = u_0, \ldots, u_{n-1}$ be some fixed string of variables. Let

$$\Phi = \{\forall \vec{u} \varphi_{RS} : R, S \in \mathrm{K}\}$$

be arbitrary but fixed system of finite conjunctions of parameterized atomic formulas. Recall that φ_{RS} is of the form (3.2). For any pair of ordinary atomic formulas $\alpha(\vec{x}) = R(\vec{\tau}(\vec{x})), \beta(\vec{x}) = S(\vec{\rho}(\vec{x}))$ define

$$\mathrm{Tr}(\alpha, \beta) := \forall_{\vec{u}} \varphi_{RS}(\vec{\tau}(\vec{x}), \vec{\rho}(\vec{x}), \vec{u})$$

Next, for any pair $\xi(\vec{x}), \eta(\vec{x})$ of conjunctions of parameterized atomic formulas, where $\xi(\vec{x}) := \forall_{\vec{v}} \bigwedge_{i<l} \alpha_i(\vec{x}, \vec{v})$ and $\eta(\vec{z}) := \forall_{\vec{v}} \bigwedge_{j<k} \beta_j(\vec{z}, \vec{v})$ define

$$\mathrm{Tr}(\xi, \eta) = \forall_{\vec{v}} \bigwedge_{i<l} \bigwedge_{j<k} \mathrm{Tr}(\alpha_i, \beta_j)$$

Finally, for any Horn formula $\psi(\vec{x}) := \xi(\vec{x}) \rightarrow \eta(\vec{x})$, where $\xi(\vec{x}), \eta(\vec{x})$ are conjunctions of parameterized atomic formulas, and any conjunction of parameterized atomic formulas $\theta(\vec{z})$ define

$$\mathrm{Tr}(\psi, \theta) = \mathrm{Tr}(\xi, \theta) \rightarrow \mathrm{Tr}(\eta, \theta)$$

$$\mathrm{Tr}(\theta, \psi) = \mathrm{Tr}(\theta, \xi) \rightarrow \mathrm{Tr}(\theta, \eta)$$

Transformations $\mathrm{Tr}(\psi(\vec{x}), \theta(\vec{z}))$ and $\mathrm{Tr}(\theta(\vec{z}), \psi(\vec{x}))$ are called respectively the *right* and *left transforms* of $\psi(\vec{x})$ by $\theta(\vec{z})$. Sometimes we write Tr_Φ instead of just Tr to stress the fact that this operator depends on the system Φ.

Note that we define the transforms of a parameterized Horn formula by a conjunction of parameterized atomic formulas, not by another parameterized Horn formula. Note also that the transforms are again parameterized Horn formulas, so it makes sense to iterate the operation. For any parameterized Horn formula $\psi(\vec{x})$ define

$$\mathrm{Lt}^R(\psi) := \mathrm{Tr}(\psi, R\vec{z})$$

$$\mathrm{Rt}^R(\psi) := \mathrm{Tr}(R\vec{z}, \psi)$$

called the *left* and *right transforms* of ψ with respect to R, respectively. Again, to stress that Lt^R and Rt^R depend on Φ we sometimes write the names of these operations with the subscript Φ. We put $\mathrm{Lt}^\Phi(\psi) := \{\mathrm{Lt}^R_\Phi(\psi) : R \in \mathrm{K}\}$ and $\mathrm{Rt}^\Phi(\psi) := \{\mathrm{Rt}^R_\Phi(\psi) : R \in \mathrm{K}\}$

Observe that the formulas (3.15) of the section 3.2 are just the left transforms of the rules Γ.

In the proof of Lemma 3.9 we were able to conclude that the filter F was closed under the rules Γ from the fact that the matrix $\mathfrak{A}$ universally satisfied the corresponding left transforms. Similarly, G was closed under the same rules because $\mathfrak{A}$ satisfied the corresponding right transforms. This relationship between closure under rules and satisfaction of their transforms extends to parameterized Horn formulas.

Lemma 3.12 *Let $\mathfrak{A}$ be any matrix and let $X = \coprod_{T\in K} X_T$ with $X_T \subseteq A^{\rho(T)}$ for each $T \in K$. Define $F_X := \coprod_{T\in K}(F_X)_T, G_X := \coprod_{S\in K}(G_X)_S$ by:*

$$(F_X)_T := \{\vec{x} \in A^{\rho(T)} : \mathfrak{A} \models \forall_{\vec{u}}\varphi_{TS}(\vec{x}, \vec{c}, \vec{u}) : \textit{ for all } S \in K \textit{ and all } \vec{c} \in X_S\}$$

$$(G_X)_S := \{\vec{y} \in A^{\rho(S)} : \mathfrak{A} \models \forall_{\vec{u}}\varphi_{TS}(\vec{c}, \vec{y}, \vec{u}) : \textit{ for all } T \in K \textit{ and all } \vec{c} \in X_T\}.$$

Then for any parameterized Horn formula ψ

$\mathfrak{A} \models \mathrm{Lt}^{\Phi}(\psi)$ *implies that* F_X *is closed under* ψ, *and*

$\mathfrak{A} \models \mathrm{Rt}_{\Phi}(\psi)$ *implies that* G_X *is closed under* ψ.

Proof. Suppose that $\mathfrak{A} \models \mathrm{Lt}^{\Phi}(\psi)$, with $\psi = \xi(\vec{x}) \rightarrow \eta(\vec{x})$, where

$$\xi(\vec{x}) = \forall_{\vec{v}} \bigwedge_{i<l} T_i(\vec{\tau}_i(\vec{x}, \vec{v})), \tag{3.20}$$

$$\eta(\vec{x}) = \forall_{\vec{v}} \bigwedge_{i<m} S_i(\vec{\rho}_i(\vec{x}, \vec{v})). \tag{3.21}$$

Then

$$\begin{aligned} \mathrm{Lt}^{R}_{\Phi}(\psi) &= \mathrm{Tr}_{\Phi}(\psi(\vec{x}), R(\vec{z})) \\ &= \mathrm{Tr}_{\Phi}\xi(\vec{x})R(\vec{z}) \rightarrow \mathrm{Tr}_{\Phi}\eta(\vec{x})R(\vec{z}) \\ &= \forall_{\vec{v}}\forall_{\vec{u}} \bigwedge_{i<l} \phi_{T_iR}(\tau_i(\vec{x}, \vec{v}), \vec{z}, \vec{u}) \rightarrow \forall_{\vec{v}}\forall_{\vec{u}} \bigwedge_{j<m} \phi_{S_jR}(\rho_j(\vec{x}, \vec{v}), \vec{z}, \vec{u}) \end{aligned} \tag{3.22}$$

Assume that $\mathfrak{A} \models \mathrm{Lt}^{\Phi}(\psi)$, i.e., for every $R \in \mathrm{K}, \mathfrak{A} \models \mathrm{Lt}^{R}_{\Phi}(\psi)$. To complete the proof we need to verify the following implication : if $\vec{a} \in A^p$ (where p is the length of the sequence $\vec{x}$) and if for any choice of $\vec{e} \in A^r$ (where r is the length of the sequence $\vec{v}$) and for every $i < l$ we have $\vec{\tau}_i(\vec{a}, \vec{e}) \in (F_X)_{T_i}$ then for any $\vec{e} \in A^r$ and for every $j < m$ we have $\vec{\rho}_j(\vec{a}, \vec{e}) \in (F_X)_{S_j}$.

So suppose that $\vec{a} \in A^p$ and that for all $\vec{e} \in A^r$, all $i < l$ $\vec{\tau}_i(\vec{a}, \vec{e}) \in (F_X)_{T_i}$. This means that for every $R \in \mathrm{K}$, for all $\vec{c} \in X_R$, we have $\mathfrak{A} \models \forall_{\vec{u}} \bigwedge_{i<l} \phi_{T_iR}(\tau_i(\vec{a}, \vec{e}), \vec{c}, \vec{u})$. Since for every $R \in \mathrm{K}, \mathfrak{A} \models \mathrm{Lt}^{R}_{\Phi}(\psi)$, this means that also for all $R \in \mathrm{K}$, $\vec{c} \in X_R$, $j < m$ and $\vec{e} \in A^r$ we have $\mathfrak{A} \models \forall_{\vec{u}} \bigwedge_{j<l} \phi_{S_jR}(\rho_j(\vec{a}, \vec{e}), \vec{c}, \vec{u})$, i.e., that for all $\vec{e}$, all $j < l$

$\vec{\rho}_j(\vec{a},\vec{e}) \in (F_X)_{S_j}$. This finishes the proof that F_X is closed under ψ. The proof that G_X is closed under ψ is similar.

Let $\mathbf{Q}$ be a protoquasivariety for which Φ defines principal filter meets. The real content of Lemma 3.8 is that, whenever an implication ψ is a rule of $\mathbf{Q}$, so are the left and right transforms of ψ.

Lemma 3.13 *Let* $\mathbf{Q}$ *be a protoquasivariety with principal filter meets defined by a system* Φ. *Then* $\mathbf{Q} \models \mathrm{Lt}_\varphi(\psi)$ *and* $\mathbf{Q} \models \mathrm{Rt}_\varphi(\psi)$, *for every rule* ψ *such that* $\mathcal{Q} \models \psi$.

Proof. It suffices to show that $\mathcal{Q}_{\mathrm{FMI}}$ satisfies the transforms of ψ. Let $\psi = \xi \to \eta$ where ξ and η are as in the previous lemma. Then $\mathrm{Lt}_\varphi(\psi) = \{\mathrm{Lt}^R_\varphi(\psi) : R \in \mathrm{K}\}$, where $\mathrm{Lt}^R_\varphi(\psi)$ is of the form 3.22. Let $\mathfrak{A} = \langle \mathbf{A}, D_\mathfrak{A}\rangle \in \mathcal{Q}_{\mathrm{FMI}}, R \in \mathrm{K}$ and $\vec{c} \in A^{\rho(R)}$.

Assume that for all $\vec{e} \in A^r$ $\mathfrak{A} \models \forall_{\vec{u}} \bigwedge_{i<l} \phi_{T_iR}(\tau_i(\vec{a},\vec{e}),\vec{c},\vec{u})$. We want to show that for all $\vec{e} \in A^r$ $\mathfrak{A} \models \forall_{\vec{u}} \bigwedge_{j<m} \phi_{S_jR}(\rho_j(\vec{a},\vec{e}),\vec{c},\vec{u})$.

Since Φ defines principal filter meets, we have

$$\mathrm{Fg}^\mathfrak{A}(T_i(\vec{\tau}_i(\vec{a},\vec{e})) \cap \mathrm{Fg}^\mathfrak{A}(R\vec{c}) = D_\mathfrak{A}$$

for all $i < l, \vec{e} \in A^r$. Since $\mathfrak{A}$ is FMI either $\vec{\tau}_i(\vec{a},\vec{e}) \in (D_\mathfrak{A})_{T_i} = T_i^\mathfrak{A}$ for all $i < l$ or $\vec{c} \in (D_\mathfrak{A})_R = R^\mathfrak{A}$. In the first case, $\xi(\vec{a})$ holds and therefore by assumption, $\eta(\vec{a})$ holds too, i.e., $\vec{\rho}_j(\vec{a},\vec{e}) \in (D_\mathfrak{A})_{S_j} = S_j^\mathfrak{A}$ for all $j < m$, all $\vec{e} \in A^r$. So in this case $\mathrm{Fg}^\mathfrak{A}(S_j(\rho_j(\vec{a},\vec{e})))) \cap \mathrm{Fg}^\mathfrak{A}(R\vec{c}) = D_\mathfrak{A}$. Of course this is also true in the second case. So $\mathfrak{A} \models \forall_{\vec{u}} \bigwedge_{j<m} \phi_{S_jR}(\rho_j(\vec{a},\vec{e}),\vec{c},\vec{u})$ for all $\vec{e} \in A^r$ and therefore $\mathfrak{A} \models \mathrm{Lt}_\Phi(\psi)$. The proof that $\mathfrak{A} \models \mathrm{Rt}_\Phi(\psi)$ is similar. □

In order to axiomatize the property of defining principal filter meets we consider the following formulas:

$$\alpha := \mathrm{Tr}_\Phi(R\vec{x}, S\vec{y}) \to \mathrm{Tr}_\Phi(S\vec{y}, R\vec{x}), \quad \text{i.e.,} \tag{3.23}$$

$$\alpha = \forall_{\vec{u}}\varphi_{RS}(\vec{x},\vec{y},\vec{u}) \to \forall_{\vec{u}}\varphi_{SR}(\vec{y},\vec{x},\vec{u});$$

$$\beta := \mathrm{Tr}_\Phi(\mathrm{Tr}_\Phi(R\vec{x},S\vec{y}),T\vec{z}) \leftrightarrow \mathrm{Tr}_\Phi(R\vec{x},\mathrm{Tr}_\Phi(S\vec{y},T\vec{z})) \text{ i.e., ;} \tag{3.24}$$

$$\begin{aligned}
\beta &= \mathrm{Tr}_\Phi(\forall_{\vec{u}}\phi_{RS}(\vec{x},\vec{y},\vec{u})),T\vec{z}) \leftrightarrow \mathrm{Tr}_\Phi(R\vec{x},\forall_{\vec{u}}\phi_{ST}(\vec{y},\vec{z},\vec{u}))) \\
&= \forall_{\vec{u}} \bigwedge_{i<m_{RS}} \mathrm{Tr}_\Phi(T_i^{RS}(\tau_i^{RS}(\vec{x},\vec{y},\vec{u})),T\vec{z}) \leftrightarrow \forall_{\vec{u}} \bigwedge_{i<m_{RS}} \mathrm{Tr}_\Phi(R\vec{x},T_i^{ST}(\tau_i^{ST}(\vec{y},\vec{z},\vec{u})) \\
&= \forall_{\vec{u}} \bigwedge_{i<l_{ST}} \varphi_{R_i^{ST}T}(\kappa_i(\vec{x},\vec{y},\vec{u}),\vec{z}) \leftrightarrow \forall_{\vec{u}} \bigwedge_{i<l_{ST}} \varphi_{RR_i^{ST}}(\vec{x},\kappa_i(\vec{y},\vec{z},\vec{u}))
\end{aligned}$$

Note that β is the conjunction of two parameterized implications.

Lemma 3.14 *Assume that $\mathcal{Q}$ is a protoquasivariety and that Φ defines principal filter meets in $\mathcal{Q}$. Then $\mathcal{Q} \models \alpha, \beta$.*

Proof. Let $\mathfrak{A} \in \mathcal{Q}$. Notice that α is the formula (3.13 and that the fact that $\mathcal{Q} \models \alpha$ has already been proved in lemma 3.8 . It suffices to show that for every finitely meet irreducible filter F on $\mathfrak{A}$, $\langle \mathbf{A}, F\rangle \models \beta$. So let F be a finitely meet irreducible filter of $\mathfrak{A}$. To demonstrate that $\langle \mathbf{A}, F\rangle \models \beta$,i.e, that

$$\langle \mathbf{A}, F\rangle \models \mathrm{Tr}_\Phi(\mathrm{Tr}_\Phi(R\vec{x},S\vec{y}),T\vec{z}) \to \mathrm{Tr}_\Phi(R\vec{x},\mathrm{Tr}_\Phi(S\vec{y},T\vec{z})),$$

we will show that the satisfaction of either antecedent or conclusion of the above formula is equivalent to the condition that

$$\vec{x} \in F_R \text{ or } \vec{y} \in F_S \text{ or } \vec{z} \in F_T. \tag{3.25}$$

Indeed,

$$\begin{aligned}
\mathrm{Tr}_\Phi(\mathrm{Tr}_\Phi(R\vec{x},S\vec{y}),T\vec{z}) &= \mathrm{Tr}_\Phi(\forall_{\vec{u}}\varphi_{RS}(\vec{x},\vec{y},\vec{u}),\vec{\tau}\vec{z}) \\
&= \mathrm{Tr}_\Phi(\forall_{\vec{u}} \bigwedge_{i<m_{RS}} R_i^{RS}(\kappa_i^{RS}(\vec{x},\vec{y},\vec{u})),T\vec{z}) \\
&= \forall_{\vec{u}} \bigwedge_{i<m_{RS}} \forall\vec{v}\varphi_{R_i^{RS}}T(\vec{\tau}_i^{RS}(\vec{x},\vec{y},\vec{u}),\vec{z},\vec{v}).
\end{aligned}$$

This is equivalent to the condition that for every $i < m_{RS}$, for all $\vec{a} \in A^{\rho(R)}, \vec{b} \in A^{\rho(S)}, \vec{c} \in A^{\rho(T)}$

$$\langle \mathbf{A}, F\rangle \models \forall \vec{u} \forall \vec{v} \varphi_{R_i^{RS} T}(\vec{\tau}_i^{RS}(\vec{a}, \vec{b}, \vec{u}), \vec{c}, \vec{v}).$$

As F is finitely meet irreducible this in turn, for a fixed i, is equivalent to the condition that either $\vec{\tau}_i^{RS}(\vec{a}, \vec{b}, \vec{u}) \in F_{R_i^{RS}}$, for every $\vec{u}$, or $\vec{c} \in F_T$. Thus either $\vec{c} \in F_T$ or for every $i < m_{RS}, \vec{\tau}_i^{RS}(\vec{a}, \vec{b}, \vec{u}) \in F_{R_i^{RS}}$, for all $\vec{u}$. But this last condition is is equivalent to

$$\langle \mathbf{A}, F\rangle \models \forall_{\vec{u}} \bigwedge_{i<m_{RS}} T_i^{RS}(\vec{\tau}_i^{RS}(\vec{a}, \vec{b}, \vec{u})), \text{ i.e., to}$$

$$\langle \mathbf{A}, F\rangle \models \forall_{\vec{u}} \varphi_{RS}(\vec{a}, \vec{b}, \vec{u}) \text{ and whence to (3.25).}$$

The fact that $\langle \mathbf{A}, F\rangle \models \mathrm{Tr}_\Phi(R\vec{x}, \mathrm{Tr}_\Phi(S\vec{y}, T\vec{z}))$ is equivalent to (3.25) is proved similarly. Hence $\mathcal{Q} \models \beta$. □

Lemma 3.15 *Let Φ be a system of parameterized conjunctions of atomic formulas indexed by K^2. Then for every parameterized Horn formula $\psi(\vec{x})$*

$$\alpha \models \mathrm{Lt}_\varphi^R(\psi) \leftrightarrow \mathrm{Rt}_\varphi^R(\psi)$$

for all $R \in K$. □

Lemma 3.16 *Let Φ be a system of parameterized conjunctions of atomic formulas indexed by K^2, let $\psi(\vec{x})$ be a parameterized Horn formula. For $R \in K$ let z^R be a string of variables of length $\rho(R)$ such that none of the variables in the sequence $\vec{z}$ occurs in ψ. Let $\vec{z}$ be a concatenation of all z^R. Then for any conjunction $\theta(\vec{w})$ of parameterized atomic formulas*

$$\models \forall_{\vec{z}}(\bigwedge_{R \in K} (\mathrm{Tr}_\Phi(\psi(\vec{x}), R\vec{z}^R))) \to \mathrm{Tr}_\Phi(\psi(\vec{x}), \theta(\vec{w}))$$

Proof. Let $\psi(\vec{x}) = \xi(\vec{x}) \to \eta(\vec{w})$, where $\xi(\vec{x}), \eta(\vec{w})$ are parameterized conjunction of atomic formulas. Let $\theta(\vec{w}) = \forall \vec{v} \bigwedge_{i<k} S_i(t_i(\vec{w}, \vec{v}))$. Then $\mathrm{Tr}_\Phi(\psi(\vec{x}), \theta(\vec{w}))$ is logically equivalent to

$$\forall_{\vec{v}}(\bigwedge_{i<k} \mathrm{Tr}_\Phi(\xi(\vec{x}), S_i(\tau_i(\vec{w}, \vec{v})) \to \forall_{\vec{v}}(\bigwedge_{i<k} \mathrm{Tr}_\Phi(\eta_i(\vec{x}), S_i(\tau_i(\vec{w}, \vec{v}))).$$

This, however, is logically implied by

$$\forall_{\vec{v}}[\bigwedge_{i<k} \mathrm{Tr}_\Phi(\xi(\vec{x}), S_i(\tau_i(\vec{w}, \vec{v})) \to \mathrm{Tr}_\Phi(\eta_i(\vec{x}), S_i(\tau_i(\vec{w}, \vec{v}))],\ \text{which is equal to}$$

$$\forall_{\vec{v}}[\bigwedge_{i<k} \mathrm{Tr}_\Phi(\psi(\vec{x}), S_i(\tau_i(\vec{w}, \vec{v})))].$$

this in turn is a substitution instance of $\forall_{\vec{z}} \bigwedge_{i<k} \mathrm{Tr}_\Phi(\psi(\vec{x}), S_i(\vec{z}))$, which is implied by

$$\forall \vec{z} \bigwedge_{R \in \mathrm{K}} \mathrm{Tr}_\Phi(\psi(\vec{x}), R\vec{z}).$$

□

Lemma 3.17 *Let Φ be a system of parameterized conjunction of atomic formulas indexed by K^2. Then for any parameterized implication ψ and all $R, S \in K$ we have:*

$$\beta, \mathrm{Lt}^R_\Phi(\psi) \models \mathrm{Lt}^S_\Phi(\mathrm{Lt}^R_\Phi(\psi))$$

Proof. Let $\psi(\vec{x}) = \xi(\vec{x}) \to \eta(\vec{x})$, where ξ and η are of the forms (3.20), (3.21), respectively.

Then $\mathrm{Lt}^S_\Phi(\mathrm{Lt}^R_\Phi(\psi))$ is:

$$\mathrm{Tr}_\Phi(\mathrm{Tr}_\Phi(\xi(\vec{x}), R\vec{z}), S\vec{w}) \to \mathrm{Tr}_\Phi(\mathrm{Tr}_\Phi(\eta(\vec{x}), R\vec{z}), S\vec{w}) =$$

$$\forall_{\vec{v}} \bigwedge_{i<l} \mathrm{Tr}_\Phi(\mathrm{Tr}_\Phi(T_i(\vec{\tau}_i(\vec{x},\vec{v})), R\vec{z}), S\vec{w}) \to \forall_{\vec{v}} \mathrm{Tr}_\Phi(\mathrm{Tr}_\Phi(S_j(\vec{\rho}_j(\vec{x},\vec{v})), R\vec{z}), S\vec{w}).$$

Under β, a conjunction on the left is equivalent to $\mathrm{Tr}_\Phi(T_i(\tau_i(\vec{x},\vec{v})), \mathrm{Tr}_\Phi(R\vec{z}, S\vec{w}))$ and the conjunct on the right to $\mathrm{Tr}_\Phi(S_j(\rho_j(\vec{x},\vec{v})), \mathrm{Tr}_\Phi(R\vec{z}, S\vec{w}))$. Thus $\mathrm{Lt}^R_\Phi(\mathrm{Lt}^S_\Phi(\psi))$ is equivalent under β to $\mathrm{Tr}_\Phi(T_i(\psi(\vec{x}), \mathrm{Tr}_\Phi(R\vec{z}, S\vec{w}))$. But by Lemma 3.16 this last formula is a consequence of $\mathrm{Lt}_\Phi(\psi)$.

3.5 The main results

The aim of this section is to prove Theorems 3.1 and 3.2. Theorem 3.2 is a consequence of Theorem 3.1, which in turn follows from the Main Lemma and the following:

Theorem 3.18 *Let* $\mathbf{Q}$ *be a finitely generated, filter-distributive protoquasivariety. Then there exists a finite set of rules* R *such that* $\mathcal{Q} = \mathrm{Mod}^*(R \cup E(\mathcal{Q}))$, *where* $E(\mathcal{Q})$ *is the set of all the theorems of* $\mathcal{Q}$.

Proof. By hypothesis, $\mathcal{Q}$ is finitely generated; so by Lemma 2.18, $\mathcal{Q}_{\mathrm{FSI}}$ is up to isomorphism a finite set of finite matrices. So it is strictly elementary. This together with filter-distributivity implies that $\mathcal{Q}$ has parameterized DPFM (Thm. 3.5). Let $\mathbf{Q} = \mathrm{Mod}^*\Gamma$, where Γ is a possibly infinite set of Horn formula s (rules). Let Φ be a system defining PDFM in $\mathcal{Q}$. Then by theorem 3.5

$$\mathcal{Q}_{\mathrm{FSI}} = \mathrm{Mod}^*(\Gamma \cup \{\forall_{\vec{x},\vec{y}}[(\forall_{\vec{u}}\varphi_{RS}(\vec{x},\vec{y},\vec{u})) \to R\vec{x} \vee S\vec{y}] : R, S \in \mathrm{K}\}).$$

As in section 3.2, we use ψ_{RS} as an abbreviation for $\forall_{\vec{x},\vec{y}}[(\forall_{\vec{u}}\varphi_{RS}(\vec{x},\vec{y},\vec{u})) \to R\vec{x} \vee S\vec{y}]$. Since $\mathcal{Q}_{\mathrm{FSI}}$ is strictly elementary, there exists a finite subset Γ' of Γ, such that

$$\mathcal{Q}_{\mathrm{FSI}} = \mathrm{Mod}^*(\Gamma' \cup \{\psi_{RS} : R, S \in \mathrm{K}\}).$$

Let $\gamma := \forall_{\vec{u}}\varphi_{RR}(\vec{x},\vec{x},\vec{u}) \to R\vec{x}$ and let Ω be $\{\alpha,\beta,\gamma\} \cup \mathrm{Lt}_\Phi(\Gamma' \cup \{\alpha,\beta,\gamma\})$.

For any set Δ of parameterized Horn formulas let $\mathrm{Lt}_\Phi(\Delta) = \{\mathrm{Lt}_\Phi(\psi) : \psi \in \Delta\}$ and similarly for $\mathrm{Rt}_\Phi(\Delta)$.

Lemma 3.19 $\Omega \models \mathrm{Lt}^\Phi(\Gamma' \cup \Omega) \cup \mathrm{Rt}^\Phi(\Gamma' \cup \Omega)$.

Proof. Let $\psi \in \Gamma' \cup \Omega = \Gamma' \cup \{\alpha,\beta,\gamma\} \cup \mathrm{Lt}^\Phi(\Gamma' \cup \{\alpha,\beta,\gamma\})$. If $\psi \in \Gamma' \cup \{\alpha,\beta,\gamma\}$, then $\mathrm{Lt}^\Phi(\psi) \subseteq \Omega$ and we are done. If $\psi \in \mathrm{Lt}^\Phi(\Gamma' \cup \{\alpha,\beta,\gamma\})$, then $\beta,\psi \models \mathrm{Lt}^\Phi(\psi)$ by Lemma 3.17. Therefore $\Omega \models \mathrm{Lt}^\Phi(\psi)$ and thus also $\Omega \models \mathrm{Rt}^\Phi(\psi)$, since $\alpha, \mathrm{Lt}^\Phi(\psi) \models \mathrm{Rt}^\Phi(\psi)$. □

Let $\mathcal{L} = \mathrm{Mod}^*(\Gamma' \cup \Omega)$. Note that $\mathcal{L}$ is a parameterized matrix quasivariety.

Lemma 3.20 $\mathcal{L}_{\mathrm{FSI}} \subseteq \mathcal{Q}_{\mathrm{FSI}}$.

Proof. Let $\mathfrak{A} \in \mathcal{L}_{\mathrm{FSI}}$. Then $\mathfrak{A} \models \Gamma'$, so in order to show that $\mathfrak{A} \in \mathcal{Q}_{\mathrm{FSI}}$ it suffices to show that for all $R, S \in \mathrm{K}$

$$\mathfrak{A} \models \psi_{RS}, \text{ i.e.,}$$

$$\mathfrak{A} \models \forall_{\vec{x},\vec{y}}(\forall_{\vec{u}}\varphi_{RS}(\vec{x},\vec{y},\vec{u}) \to R\vec{x} \vee S\vec{y}) \qquad (3.26)$$

Since $\mathfrak{A} \in \mathcal{L}_{\mathrm{FSI}}$, we have that $F \cap G = D_{\mathfrak{A}} \Leftrightarrow F = D_{\mathfrak{A}}$ or $G = D_{\mathfrak{A}}$ for all $\mathcal{L}$-filters F, G on $\mathfrak{A}$. Let $R, S \in \mathrm{K}$ have the arities n, m, respectively. Let $\vec{a} \in A^n, \vec{b} \in A^m$ be such that $\mathfrak{A} \models \forall_{\vec{u}}\varphi_{RS}(\vec{a},\vec{b},\vec{u})$. Define $F = \coprod\{F_T : T \in \mathrm{K}\}$, by $F_T := \{\vec{c} \in A^{\rho(T)} : \mathfrak{A} \models \forall_{\vec{u}}\varphi_{TS}(\vec{c},\vec{b},\vec{u})\}$. By Lemma 3.12, applied to $X_S = \{\vec{b}\}, X_T = \emptyset$, for $T \neq S$, we conclude that F is closed under every parameterized Horn formula $\psi \in \Gamma' \cup \Omega$. Thus F is an $\mathcal{L}$-filter and $\vec{a} \in F_R$. Now define $G = \coprod\{G_T : T \in \mathrm{K}\}$ by $G_T := \{\vec{d} \in A^{\rho(T)} : \mathfrak{A} \models \forall \vec{u}\varphi_{VT}(\vec{c},\vec{d},\vec{u})$ for all $V \in \mathrm{K}$, all $\vec{d} \in F_V\}$. The fact that $\Omega \models \mathrm{Rt}^\Phi(\psi)$ for every $\psi \in \Gamma' \cup \Omega$ implies that G is an $\mathcal{L}$-filter. Also, $\vec{b} \in G_S$. Let

$\vec{c} \in (F \cap G)_T$. Then $\mathfrak{A} \models \forall_{\vec{u}}\varphi_{TT}(\vec{c}, \vec{c}, \vec{u})$. Therefore $\mathfrak{A} \models T\vec{c}$, i.e., $F \cap G = D_{\mathfrak{A}}$. This implies that either $F = D_{\mathfrak{A}}$ or $G = D_{\mathfrak{A}}$. Since $\vec{a} \in F_R$ and $\vec{b} \in G_S$, either $\mathfrak{A} \models R\vec{a}$ or $\mathfrak{A} \models S\vec{b}$ and (3.26) is established. □

To prove Theorem 3.18, we first show that $\mathcal{Q} \subseteq \mathcal{L}$. Recall that $\mathcal{L} = \mathrm{Mod}^*(\Gamma' \cup \Omega)$, where $\Omega = \{\alpha, \beta, \gamma\} \cup \mathrm{Lt}^{\Phi}(\{\alpha, \beta, \gamma\} \cup \Gamma')$. Trivially, $\mathcal{Q} \models \Gamma'$. Since Φ defines meets of principal $\mathcal{Q}$-filters in $\mathcal{Q}$, we have $\mathcal{Q} \models \alpha, \beta, \gamma$. This was shown in Lemma 3.14 for α, β and for γ see the proof of Lemma 3.8. Also, we have $\mathcal{Q} \models \mathrm{Lt}^{\Phi}(\Gamma' \cup \{\alpha, \beta, \gamma\})$ (by Lemma 3.13). Thus $\Gamma \models \Gamma' \cup \Omega$ and $\mathcal{Q} \subseteq \mathcal{L}$ as desired. Since $\Gamma' \cup \Omega$ is finite, there exists a finite subset Γ'' of Γ such that $\Gamma'' \models \Gamma' \cup \Omega$. Then $\mathcal{Q} \subseteq \mathrm{Mod}^*(\Gamma'' \cup E(\mathcal{Q})) \subseteq \mathcal{L}$. To complete the proof we show that $\mathrm{Mod}^*(\Gamma'' \cup E(\mathcal{Q})) \subseteq \mathcal{Q}$. $\mathrm{Mod}^*(\Gamma'' \cup E(\mathcal{Q}))$ is a matrix subquasivariety of the matrix variety $\mathrm{Mod}^*(E(\mathcal{Q}))$ generated by $\mathcal{Q}$, which is finitely generated by hypothesis. So $\mathrm{Mod}^*(E(\mathcal{Q}))$ is finitely generated and therefore, by lemma 2.25, locally finite. We claim that it suffices to prove that every finite member $\mathfrak{A}$ of $\mathrm{Mod}^*(\Gamma'' \cup E(\mathcal{Q}))$ is in $\mathcal{Q}$. For suppose that $\mathfrak{A} \in \mathrm{Mod}^*(\Gamma'' \cup E(\mathcal{Q})) \setminus \mathcal{Q}$. Then there is a rule $r \in \Gamma$ such that $\mathfrak{A} \not\models r$. Since r contains only finitely many variables, there is a finitely generated submatrix $\mathfrak{B}$ of $\mathfrak{A}$, which does not satisfy r. Therefore also $\mathfrak{B}^*$ is finitely generated and does not satisfy r. Now since $\mathrm{Mod}^*(\Gamma'' \cup E(\mathcal{Q}))$ is closed under S^*, $\mathfrak{B}^*$ is a finite matrix and whence belongs to the matrix quasivariety $\mathcal{Q}$. But this contradicts the fact that $\mathfrak{B}^* \notin \mathcal{Q}$. This verifies the claim. Let $\mathfrak{A} \in \mathrm{Mod}^*(\Gamma'' \cup E(\mathcal{Q}))$ be finite. Then $\mathfrak{A} \in \mathcal{L}$ and therefore $\mathfrak{A} \leq_{SD} \mathfrak{B}_0 \times \cdots \times \mathfrak{B}_{n_1}$, for some n and some matrices $\mathfrak{B}_i \in \mathcal{Q}_{\mathrm{FSI}}$, for each $i < n$. By Lemma 3.20, $\mathfrak{B}_i \in \mathcal{Q}_{\mathrm{FSI}}$, all $i < n$. Thus $\mathfrak{A} \in \mathcal{Q}$. This completes the proof of Theorem 3.18. □

Now we turn to the proof of Theorem 3.1.

Proof. of Theorem 3.1.

Let $\mathcal{Q}$ be a matrix protoquasivariety satisfying the hypothesis of the theorem. By Theorem 3.18, $\mathcal{Q} = \mathrm{Mod}^*(R \cup E(\mathcal{Q}))$ for some finite set of rules $R \subseteq \Gamma$. Also, by Corollary refcor20 s2 (to generalized Jónsson's lemma, Lemma 2.18) $\mathcal{Q}_{\mathrm{FSI}}$ is strictly elementary. So we can apply our Main Lemma 3.7 to conclude that $\mathcal{Q}$ is finitely based. □

Theorem 3.2 is now a corollary to the above theorem. Recall that $\mathcal{V}$ is a matrix subvariety of a matrix quasivariety $\mathcal{Q}$ if $\mathcal{V}$ is the intersection of $\mathcal{Q}$ with some matrix variety.

Proofof Theorem 3.2.

Let $\mathcal{V}$ be a finitely generated matrix subvariety of $\mathcal{Q}$. Then $\mathcal{V}$ is also filter-distributive matrix protoquasivariety. Let K be a finite set of finite matrices generating $\mathcal{V}$. By the generalized Jónsson lemma for filter-distributive systems, Theorem Jonlemds, $\mathcal{V}_{\mathrm{FSI}} \subseteq \mathbf{H}_Q\mathbf{S}^*P_UK \subseteq \mathbf{HS}^*K$. Since $\mathcal{V}_{\mathrm{FSI}}$ generates $\mathcal{V}$ as a matrix quasivariety, it follows that $\mathcal{V}$ satisfies the assumptions of Theorem 3.1 and therefore is also finitely based. □

3.6 Discussion

A 1-deductive system $\mathcal{S}$ has *disjunction* if there is a binary connective $\vee$ in the algebraic language of $\mathcal{S}$ such that for every set $X \cup \{\varphi, \psi\}$ of formulas, $\mathrm{Cn}_{\mathcal{S}}(X \cup \{\varphi \vee \psi\} = \mathrm{Cn}_{\mathcal{S}}(X, \varphi) \cap \mathrm{Cn}_{\mathcal{S}}(X, psi)$.

Corollary 3.21 *1. Let $\mathcal{S}$ be a protoalgebraic* 1*-deductive system with disjunction. If* $\mathrm{Mod}^*\mathcal{S}$ *is finitely generated, then $\mathcal{S}$ is finitely based.*

2. Let $\mathcal{S}$ be a protoalgebraic 1*-deductive system with disjunction and let $\mathcal{V}$ be a finitely generated relative matrix subvariety of* $\mathrm{Mod}^*\mathcal{S}$*. Then $\mathcal{V}$ is finitely based.*

Proof. It follows from [9] that a 1-deductive system with disjunction is filter-distributiv. The corollary follows from Theorem 3.1. □ A special case of the above corollary is due to J. Czelakowski, [8]. A matrix $\mathfrak{M}$ is weakly adequate for a deductive system $\mathcal{S}$, if $E(\mathfrak{M}) = \mathrm{Cn}_{\mathcal{S}}(\emptyset)$.

Corollary 3.22 *([8, Corollary III.4] Let $\mathcal{S}$ be a congruential* 1*-deductive system with disjunction. If there is a finite matrix weakly adequate for $\mathcal{S}$, then $\mathcal{S}$ is finitely based.*

Proof. Let $\mathfrak{M}$ be a finite matrix weakly adequate for $\mathcal{S}$. Let $\mathcal{V}$ be the relative matrix subvariety of $\mathrm{Mod}^*\mathcal{S}$, generated by $\mathfrak{M}$. Then $\mathcal{V} = \mathrm{Mod}^*(\mathcal{S} \cup E(\mathfrak{M})) = \mathrm{Mod}^*(\mathcal{S})$, since $E(\mathfrak{M}) = \mathrm{Cn}_{\mathcal{S}}(\emptyset)$. By part 2. of Corollary 3.21, $\mathcal{S}$ is finitely based. □

It follows from our earlier observations that if $\mathcal{S}$ is the equational deductive system of G. Birkhoff, then our Theorems 3.1 and 3.2 become Theorems 1.1 and 1.2 of [42].

Recall that Theorem [3, 4.1.] was (the instance of) our Main Lemma here. Although this theorem does not follow directly from our main results, the following weakening of this theorem is also a corollary to the results presented here.

Corollary 3.23 *([3, Corollary 4.7.])Let Λ be a finitary algebraic language and let $\mathcal{S}$ be a* 1*-deductive system over Λ. Assume that $\mathcal{S}$ can be presented by finitely many inference rules and that $\mathcal{S}$ is filter-distributive. Then any finite $\mathcal{S}$-matrix has finitely axiomatizable theorems.*

Theorem 4.1. of [3] would of course immediately follow from the positive answer to any of the following two questions

Question 1 *Assume that $\mathcal{S}$ is protoalgebraic filter-distributive $\vec{k}$-deductive system and that* $(\mathrm{Mod}^{*}\mathcal{S})_{\mathrm{FSI}}$ *is strictly elementary. Does $\mathcal{S}$ need to be finitely based?*

Question 2 *If $\mathcal{R}$ is a relative matrix subvariety of a filter-distributive protoquasivariety $\mathcal{Q}$. Does $\mathcal{R}$ need to be finitely based?*

Both of these questions were first formulated in [42] for ordinary quasi-varieties and varieties ([42, Problems 9.6. and 9.7.]).

Notice that in the proof of Theorems 3.1 and 3.2 we have essentially used the fact that K has only finitely many relation symbols. Thus the theorem applies to $\vec{k}$-deductive systems but not to Gentzen systems.

We would like to ask the following question

Question 3 *Suppose that K has infinitely many finitary predicate symbols and suppose that a K-deductive system $\mathcal{S}$ is protoalgebraic and filter-distributive. Does it follow that every finite set of finite models of $\mathcal{S}$ generates a finitely based K-potoquasivariety?*

We also mentioned in the introduction to Part III that Baker's theorem has been generalized to congruence-modular varieties in [30]. It would be interesting to see if this result can in turn be generalized to $\vec{k}$-deductive systems. Thus we would like to ask

Question 4 *Suppose that $\mathcal{S}$ is a protoalgebraic filter-modular $\vec{k}$-ds. Let $\mathcal{K}$ is a finite set of finite models of $\mathcal{S}$ and suppose that there is a number n such that every subdi-*

rectly irreducible matrix in $\mathcal{Q}(\mathcal{K})$ has at most n elements. Does it follow that $\mathcal{Q}(\mathcal{K})$ is finitely based?

For quasi-varieties this question has also already been asked in [42] ([42, Problem 9.13]).

PART III.

FINITE AXIOMATIZATION

CHAPTER 1. INTRODUCTION

In Part II we considered the finite basis problem for arbitrary $\vec{k}$-matrices. In Part III we turn to a related finite axiomatizability problem, restricted to the following 3 contexts: 1-deductive systems, equational logic and second-order equational logic.

Recall (Part I, Definition 2.29) that the finite axiomatization problem for a finite matrix $\mathfrak{A}$ asks whether there is a finite set R of rules such that φ is a theorem of $\mathfrak{A}$ iff φ is derivable from the empty set of premisses using the rules of R. This implies that the rules of R are sound for $\mathfrak{A}$. One can also ask whether $\mathfrak{A}$ has a generally stronger property that such a set of *valid* rules can be found. In the case of *structurally complete* matrices, i.e., the matrices for which every sound rule is valid, the two questions coincide. Recall also that the finite axiomatization property is weaker than the property of having a finitely based consequence operation or finitely based theorems over any, finitely axiomatizable, deductive system $\mathcal{S}$.

In the context of equational logic finite axiomatization problem for a finite algebra $\mathbf{A}$ translates to the question whether the identities of $\mathbf{A}$ are logical consequences of a finite set of quasi-identities of the algebra, or, more generally, of a finite set of quasi-equations that are sound for the algebra. The question whether every finite algebra $\mathbf{A}$ is finitely axiomatizable in this sense was first proposed by W. Rautenberg in [46] and independently by A. Wroński (see [42]). A matrix $\mathfrak{A}$ is called *left-associative* iff the term $(x(yz)$ is a tautology of $\mathfrak{A}$. Some auxiliary results about left-associative matrices and their underlying algebras are proved in Chapter 2. In Chapter 3 we prove that among 3-element left-associative algebraic matrices there exist exactly two that are nonfinitely axiomatizable. This answers the open question of [46, 61, 10]. In Chapter 4 we consider the Rautenberg–Wroński problem for the underlying algebras of the nonfinitely axiomatizable matrices of [60] and Chapter 3.

The finite axiomatization problem can also be considered in the context of second-order equational logic. Here, we require that the second-order theorems of a finite matrix be derivable from a finite set of second-order rules. This is equivalent to the condition that the first-order rules be derivable from a finite set of second-order rules.

In the context of equational logic first-order rules are quasi-equations and the second-order rules take the form

$$\frac{\bigwedge \Gamma_1 \to \varepsilon_1, \ldots, \bigwedge \Gamma_n \to \varepsilon_n}{\bigwedge \Delta \to \delta},$$

where ε_i, δ are equations and Δ, Γ_i are finite sets of equations. The second order finite axiomatization problem for a finite algebra $\mathbf{A}$ then asks whether there is a finite set of second-order rules valid, or, in the more general version, sound, for $\mathbf{A}$ such that every quasi-identity of $\mathbf{A}$ can be derived from this set of rules.

The problem whether every finite algebra is second-order finitely axiomatizable is open, but in Chapter 5 we prove that the answer is positive for two classes of finite algebras, namely the class of all finite algebras that do not have a proper nontrivial subalgebra, and the class of all finite algebras $\mathbf{A}$ embeddable into the free algebra in the variety generated by $\mathfrak{A}$ and such that no homomorphic image of $\mathbf{A}$ is a proper subalgebra of $\mathbf{A}$.

CHAPTER 2. LEFT-ASSOCIATIVE ALGEBRAS AND MATRICES

This chapter contains basic definitions and lemmas concerning 3-element left-associative both algebras and matrices.

For this and next chapter, our language Λ is determined by an infinite set of variables $\mathrm{Var} = \{x_0, x_1, \ldots, x_n, \ldots\}$ and one binary connective $\circ$. The Λ-algebras are called *groupoids.* $\mathrm{Te}(x)$ and $\mathbf{Te}(x)$ will denote respectively the set and the algebra of terms in one variable x. The length $|t|$ of a term t is 1 if $t \in \mathrm{Var}$ and is $|s| + |r|$ if $t = s \circ r$, i.e., $|t|$ is the total number of occurrences of variables in t.

Definition 2.1 *We say that a term t is* **left-associated** *if $t \in \mathrm{Var}$ or $t = s \circ x$, where s is left-associated and $x \in \mathrm{Var}$. Similarly, a term t is* **right-associated** *if it is a variable or is of the form $x \circ s$, where x is a variable and s is a right-associated term.*

Definition 2.2 *A matrix $\mathfrak{A}$ is* **leftassociative** *if the term*

$$x(yz) \tag{2.1}$$

is a tautology of $\mathfrak{A}$.

It follows that every non-tautology of a left-associative matrix must be a left-associated term. Notice also, that if $\mathfrak{A}$ is an algebraic matrix, i.e., a matrix with exactly one designated element, then the term $x(yz)$ is a constant. We call the underlying algebras of left-associative matrices *left-associative algebras*, i.e., we have the following definition.

Definition 2.3 *A* **left-associative** *groupoid is a groupoid satisfying*

$$x(yz) = u(uu) \tag{2.2}$$

Corollary 2.4 *If $\mathfrak{A}$ is a left-associative algebraic matrix, then* $\mathbf{A}$ *is a left-associative algebra.*

We adopt the following conventions: We will omit the symbol of the binary operation $\circ$ and when parentheses are missing we assume the association to the left. By xy^k we mean x if $k = 0$ and $(xy^{k-1})y$ if $k > 0$. Let $\mathbf{A}$ be a groupoid and let Rg be the range of the operation $\circ$ in $\mathbf{A}$, i.e.,

$$\mathrm{Rg} = A \circ A = \{a \circ b : a, b \in A\}.$$

Suppose in addition that $\mathfrak{A}$ satisfies (2.2) and let e be an element of A such that for all $a, b, c \in A$, $a(bc) = e$. Then

$$a \circ b = e, \text{ for each } a \in A, b \in \mathrm{Rg}; \text{ i.e.,} \tag{2.3}$$

$$A \circ \mathrm{Rg} = \{e\}. \tag{2.4}$$

This implies that

$$\mathrm{Rg} \text{ is a proper subset of } A, \tag{2.5}$$

for if $\mathrm{Rg} = A$, then by (2.4), $\{e\} = A \circ \mathrm{Rg} = A \circ A = \mathrm{Rg} = A$, a contradiction. Using the observations (2.3), (2.4), (2.5) above, we can now list all possible three-element groupoids satisfying (2.2).

Lemma 2.5 *Let* $\mathbf{A}$ *be a three-element groupoid satisfying 2.2. Then up to isomorphism* $\mathbf{A} = \langle \{0,1,2\}, \circ \rangle$*, where the operation* $\circ$ *is given by one of the following tables.*

$\circ$	0	1	2
0	1	2	2
1	1	2	2
2	1	2	2

(I)

$\circ$	0	1	2
0	1	2	2
1	1	2	2
2	2	2	2

(II)

$\circ$	0	1	2
0	1	2	2
1	2	2	2
2	2	2	2

(III)

$\circ$	0	1	2
0	2	2	2
1	1	2	2
2	1	2	2

(IV)

$\circ$	0	1	2
0	2	2	2
1	1	2	2
2	2	2	2

(V)

$\circ$	0	1	2
0	2	2	2
1	2	2	2
2	2	2	2

(VI)

$\circ$	0	1	2
0	1	2	2
1	2	2	2
2	1	2	2

(VII)

$\circ$	0	1	2
0	2	2	2
1	2	2	2
2	1	2	2

(VIII)

Proof. Note that up to isomorphism $\mathbf{A}$ is the algebra $\langle \{0,1,2\}, \circ \rangle$, where $2 = a(bc)$ for any choice of $a, b, c \in A$. By (2.5) $\mathrm{Rg} = A \circ A$ is a proper subset of A and by (2.3), $2 \in \mathrm{Rg}$. Without loss of generality we assume that

$$\mathrm{Rg} \subseteq \{1,2\}. \tag{2.6}$$

For if this is not the case, then $\mathbf{A}$ is isomorphic to some algebra satisfying (2.6). We observe that

$$a \circ 1 = a \circ 2 = 2, \text{ for each } a = 0,1,2. \tag{2.7}$$

This is obvious if $\mathrm{Rg} = \{2\}$ and follows from (2.3) if $\mathrm{Rg} = \{1,2\}$.

We have shown that a 3-element algebra satisfying (2.2) is isomorphic to an algebra $\mathbf{A} = \langle \{0,1,2\}, \circ \rangle$ satisfying (2.6) and (2.7). Clearly, (2.6) and (2.7) imply that the algebra satisfies (2.2). Hence **A** satisfies (2.2) iff it is isomorphic to a algebra $\mathbf{A}' = \langle\{0,1,2,\},\circ\rangle$ which satisfies (2.6) and (2.7). The 8 algebras listed in the conclusion of the lemma are exactly all the algebras $\mathbf{A} = \langle\{0,1,2\},\circ\rangle$ satisfying (2.6) and (2.7). □

Corollary 2.6 *Let $\mathfrak{M} = \langle \mathbf{M}, D\rangle$ be a 3-element algebraic matrix. Then $\mathfrak{M}$ satisfies* (1) *iff $\mathfrak{M} \cong \langle\langle\{0,1,2\},\circ,\{2\}\rangle$, where $\circ$ is given by one of the tables* (I)–(VIII).

The matrix with the multiplication table (V) is the matrix with non-finitely based consequence operation discovered by Wroński, [63].

Recall that for a given matrix $\mathfrak{M}$, $E(\mathfrak{M})$ denotes the content, i.e., the set of all tautologies, of $\mathfrak{M}$.

Recall from Chapter 2 the following definitions and facts. A rule r is *admissible for $\mathfrak{M}$* if it is valid in the matrix $\langle \mathbf{Te}, E(\mathfrak{M})\rangle$. Thus $r = \langle\, X\ ,\ t\,\rangle$ is admissible for $\mathfrak{M}$ iff, for every substitution $\sigma : \mathbf{Te} \longrightarrow \mathbf{Te}$, whenever $\sigma(X) \subseteq E(\mathfrak{M})$, then also $\sigma(t) \in E(\mathfrak{M})$. Let $\mathbf{F_M}(x)$ be the free algebra on one generator x in HSP(**M**). We will identify the elements of $F(x)$ with terms in the variable x, i.e., with the elements of Te(x), in the standard way. Let $E(x) = E'(\mathfrak{M}) \cap F(x)$ and let $\mathfrak{F}_{\mathfrak{M}}(x) = \langle \mathbf{F}_M(x), E(x)\rangle$.

Lemma 2.7 1. *Suppose that a finite left-associative algebra* **M** *is* 1*-generated. Then* **M** *is isomorphic to* $\mathbf{F}_M(x)$.

2. *Let $\mathfrak{M}$ be a finite left-associative algebraic matrix whose underlying algebra* **M** *is* 1*- generated. Then $\mathfrak{M}$ and $\mathfrak{F}_{\mathfrak{M}}(x)$ are isomorphic.*

Proof. Let a be a generator of **M**. Then every element of M is of the form $t(a)$ for some term $t(x) \in$ Te(x). In particular, if $b \in M$, $b \neq a$, then $b \in M^2$. Also, by

(2.5), $a \notin \mathrm{Rg}$. Let $f : F_M(x) \longrightarrow M$ be the unique algebra homomorphism such that $f(x) = a$. Since a generates $\mathbf{M}$, f is onto.

To show that f is one-one it is enough to show that for all terms $t, s \in \mathrm{Te}(x)$, if $t(a) = s(a)$, then $t(x) = s(x)$ is an identity of $\mathbf{M}$. Suppose that $t(a) = s(a)$. If $t(x) = x$, then $s(a) = a$ and therefore $s(x) = x$ as $a \notin M^2$. So we can assume that $t(x) \neq x$, i.e., $t(x) = t_1(x)t_2(x)$ for some terms $t_1, t_2 \in \mathrm{Te}(x)$. By the symmetric argument we also can assume that $s(x) = s_1(x)s_2(x)$, for some terms $s_1, s_2 \in \mathrm{Te}(x)$. Let $b \in M$, $b \neq a$. We want to show that $t(b) = s(b)$. As $b \in M^2$, $s_2(b), t_2(b) \in M^2$. So $t(b) = s(b)$ by 2.2. The first claim of the lemma follows.

For the second claim we need to observe that $f(E(x)) = D$. Clearly, $f(E(x)) \subseteq D$ and by (2.1), $f(E(x))$ is nonempty. The matrix $\mathfrak{M}$ is algebraic, which means that D is a one-element set. Thus $f(E(x)) = D$. $\square$

For $n = 1, 2, ..., 8$, let $\mathbf{M}_n$ and $\mathfrak{M}_n$ be the algebra and the matrix determined by the n-th table, according to the enumeration of the Lemma 2.5. Also, let $\mathfrak{F}_i$ denote the free denumerably generated matrix $\mathfrak{F}_{\mathfrak{M}_i}$ and $\mathfrak{F}_i(x)$ the corresponding one-generated matrix. The following lemma for $i = 5$ is due to W. Rautenberg [48] and for $i = 7, 8$, to A. Wroński (personal communication).

Lemma 2.8 *All matrices $\mathfrak{M}_i$, $i = 1, \ldots, 8$ are structurally complete. All algebras $\mathbf{M}_i$, $i = 1, \ldots, 8$, are structurally complete.*

Proof. In view of Lemma 2.28 it is enough to prove that $\mathbf{M}_i$ is embeddable into F_i and that $\mathfrak{M}_i$ is embeddable into $\mathfrak{F}_i$. This second condition is of course stronger than the first, since it means that there is an embedding e of M_i into F_i such that $\langle e(M_i), e(D_i)\rangle$ is a submatrix of $\mathfrak{F}_i$.

Notice that in each of the matrices $\mathfrak{M}_1 - \mathfrak{M}_4$ and $\mathfrak{M}_7 - \mathfrak{M}_8$, 0 generates $\mathbf{M}$.

Hence by Lemma 2.7 each of these matrices is isomorphic to the submatrix $\mathfrak{F}_{\mathfrak{M}_i}(x)$ of $\mathfrak{F}_i$, i.e., is embeddable into $\mathfrak{F}_i$. For the matrix $\mathfrak{M}_5$ let a mapping $e : \mathfrak{M}_5 \longrightarrow \mathfrak{F}_5$ be defined by $e(0) = x_0$, $e(1) = x_1x_0$ and $e(2) = x_0x_0$. It is straightforward to verify that in the free algebra $\mathbf{F}_5$ these three terms are pairwise distinct, so e is one-one. Similarly, $(yz)z = yz$ and $y(zu) = x_0x_0$ for all elements y, u, z in F_5. Thus $(x_1x_0)x_0 = x_1x_0$ and $x_0(x_1x_0) = x_0(x_0x_0) = (x_1x_0)(x_1x_0) = (x_1x_0)(x_0x_0) = (x_0x_0)(x_1x_0) = (x_0x_0)(x_0x_0) = x_0x_0$, i.e., e is an embedding of $\mathfrak{M}_5$ into $\mathfrak{F}_5$. Finally, it is clear that $\mathfrak{M}_6$ is isomorphic, via $e(0) = x_0, e(1) = x_1, e(2) = x_0x_1$, with a free 2-generated matrix, i.e., the submatrix $\langle \mathbf{F}(x_0, x_1), D\rangle$ of $\mathfrak{F}_6$, where $\mathbf{F}(x_0, x_1)$ is the subalgebra of $\mathbf{F}_6$ generated by x_0, x_1 and D is uniquely determined. By a similar argument it also follows that the algebras $\mathbf{M}_1$-$\mathbf{M}_8$ are structurally complete. $\square$

CHAPTER 3. THREE-ELEMENT NONFINITELY AXIOMATIZABLE MATRICES

3.1 Introduction and overview

From now on we consider only 1-matrices, i.e., models of a language whose relational part of the language consists of only one predicate. Recall that in this case we identify the 1-terms with just terms and that matrix filters are identified with subsets.

In [46] W. Rautenberg proved that (the set of tautologies of) any 2-element matrix is finitely axiomatizable and asked (see [60]) if the same is true for any finite matrix. This question was answered by P. Wojtylak, who in [61] (see also [60]) constructed a 5-element matrix with two designated elements that is not finitely axiomatizable. This led Rautenberg to the natural question ([45], page 109) whether there exist nonfinitely axiomatizable 3- or 4-element matrices. In [61] Wojtylak suggested the more specific problem whether there is a 3-element nonfinitely axiomatizable matrix with one designated element. In particular, he wanted to know if there was a nonfinitely axiomatizable matrix as simple as the 3-element matrix with one designated element considered by A. Wroński in [63]. Wroński showed that the consequence operation of this matrix is not finitely based. Earlier examples of this kind with more than three elements had been presented by Wroński [62] and Urquhart

[55], but the result of [46] shows that such a matrix must have at least three elements. Subsequently, Wroński's matrix was shown to be finitely axiomatizable in [45, page 116] and independently in [61]. Matrices with only one designated element are called *algebraic* in the literature. Thus Wojtylak was particularly interested in finding a "simple" nonfinitely axiomatizable algebraic matrix with four or, better yet, three elements. In [10] W. Dziobiak presented a 4-element nonfinitely axiomatizable algebraic matrix thus reducing the problem to the 3-element case.

We believe that the particular simplicity of Wroński's matrix, as well as of the proof that it is not finitely based, is due to the fact that the term

$$x(yz) \tag{3.1}$$

is a tautology. This term is also a tautology of the matrices of Urquhart ([55]), Wojtylak ([60, 61]) and Dziobiak ([10]).

In this Chapter we examine the eight 3-element algebraic matrices (up to isomorphism) that satisfy (3.1). Using the method of [60, 61, 10] we prove that exactly two of them are nonfinitely axiomatizable. It also turns out that all 3-element algebraic matrices satisfying (3.1) are structurally complete — a fact noticed earlier by W. Rautenberg for Wroński's matrix. Since every nonfinitely axiomatizable matrix is also nonfinitely based, our result gives several more examples of matrices whose *admissible* rules are not finitely based[1].

[1]We thank Prof. W. Rautenberg for pointing this out.

3.2 The results

Throughout this section, whenever we say " matrix ", we mean a matrix whose underlying algebra is a groupoid. The aim of this chapter is to present the following

Theorem 3.1 *Let* $\mathfrak{M} = \langle \mathbf{M}, D \rangle$ *be a 3-element algebraic matrix satisfying* (1). *Then* $\mathfrak{M}$ *is nonfinitely axiomatizable iff* $\mathfrak{M}$ *is isomorphic to* $\mathfrak{M}_7$ *or* $\mathfrak{M}_8$.

Proof. The proof will be completed by proving the following two lemmas.

Lemma 3.2 *For* $i = 1, \ldots, 6$, $\mathfrak{M}_i$ *is finitely axiomatizable.*

Lemma 3.3 *The matrices* $\mathfrak{M}_7, \mathfrak{M}_8$ *are nonfinitely axiomatizable.*

For $n = 1, \ldots, 8$ let $\mathbf{0}_n : \mathbf{Te} \longrightarrow \mathbf{M}_n$ be the unique valuation such that $\mathbf{0}_n(x) = 0$, for each variable $x \in \mathrm{Var}$.

Proof of Lemma 3.2. For each of the matrices $\mathfrak{M}_i$ mentioned in the Lemma we provide a finite set R_i of rules which axiomatizes the matrix. Note that for $i = 1, 3, 4, 6$ all the rules are axiomatic.

(I). The matrix $\mathfrak{M}_1$ is axiomatized by $R_1 = \{x(yz)\}$.

Proof. The valuation $\mathbf{0}$ shows that no term of the form sx, where $s \in \mathrm{Te}$ and $x \in \mathrm{Var}$, is a tautology of $\mathfrak{M}_1$. Since $x \notin E(\mathfrak{M}_1)$ it follows that each tautology is of the form rt, where r, t are terms, i.e. of the form $r(us)$, where r, u, s are terms. Since also $x(yz) \in E(\mathfrak{M}_1)$, we conclude that R_1 axiomatizes $\mathfrak{M}_1$.

(II). The matrix $\mathfrak{M}_2$ is axiomatized by $R_2 = \{x(yz), \langle x, xy \rangle\}$.

Proof. Note that every term φ is of the form $\varphi = \psi v_1 \cdots v_n$ for some $n \geq 0$ and $v_1, \ldots, v_n \in \mathrm{Var}$, where ψ is either a variable or is the longest subterm of φ of the form $\psi = t(rs)$. The valuation $\mathbf{0}_n$ shows that no term of the form $v_0v_1\cdots v_n$ for $n \geq 0$, $v_i \in \mathrm{Var}$, where $i = 1,...,n$, is a tautology of $\mathfrak{M}_2$. Hence every tautology of $\mathfrak{M}_2$ is of the form

$$\varphi = r(ts)v_1v_2\cdots v_n,$$

where $n \geq 0$ and $r, t, s \in F$ and for $i = 1, \ldots, n$, $v_i \in \mathrm{Var}$. Such a formula φ clearly is in $\mathrm{Cn}(R_2, \emptyset)$. Also, R_2 is valid in $\mathfrak{M}_2$, which shows that R_2 axiomatizes $\mathfrak{M}_2$.

(III). The matrix $\mathfrak{M}_3$ is axiomatized by $R_3 = \{x(yz), (xy)z\}$.

Proof. Since x and xy are not tautologies of $\mathfrak{M}_3$ for any pair of variables x, y, each tautology must be of the form $r(st)$ or $(rs)t$, so $E(\mathfrak{M}_3) \subseteq \mathrm{Cn}(R_3, \emptyset)$. Clearly, all rules of R_3 are valid in $\mathfrak{M}_3$, so $E(\mathfrak{M}_3) = \mathrm{Cn}(R_3, \emptyset)$.

(IV). The matrix $\mathfrak{M}_4$ is axiomatized by $R_4 = \{xx, x(yz)\}$.

Proof. It is easy to see that every rule of R_4 is valid in $\mathfrak{M}_4$. Let $t \in E(\mathfrak{M}_4)$ and suppose that t is not a substitution instance of either axiom in R_4. Since a tautology of $\mathfrak{M}_4$ cannot be a variable, $t = sx$ for some $s \in \mathrm{Te}$, $s \neq x$ and $x \in \mathrm{Var}$. Also , $s \notin \mathrm{Var}$, because $yx \notin E(\mathfrak{M}_4)$ for any variable y. Therefore $\mathbf{0}_4(s) \neq 0$; hence $\mathbf{0}_4(t) \in \{1 \circ 0, 2 \circ 0\} = \{1\}$. This contradicts the choice of t.

(V). The matrix $\mathfrak{M}_5$ was used in [63] to show that the consequence operation of a finite matrix need not be finitely based. It was shown in [45] and independently in [61] that this matrix is finitely axiomatizable by the set $R_5 = \{xx, x(yz), \langle x, xy\rangle, \langle xy, xzy\rangle\}$.

(VI). Clearly, $\mathfrak{M}_6$ is axiomatized by $R_6 = \{xy\}$. □

Proof of Lemma 3.3. Since $\mathfrak{M}_7$ and $\mathfrak{M}_8$ differ only in $0 \circ 0$, they have many common properties. Some of these properties are listed in the following proposition.

Proposition 3.4 *Let $n = 7$ or 8 and let $a, b, c \in \{0, 1, 2\}$. Then*

$$a1 = a2 = 2 \tag{3.2}$$

$$\textit{If } m, n > 0 \textit{ and } m \equiv n \ (\textit{mod } 2) \textit{ then } a0^m = a0^n \tag{3.3}$$

Proof. By inspection of the multiplication tables VII and VIII. □

For each natural number k we define terms

$$\alpha_{7,k} := x_{2k+1}x_{2k}\cdots x_2x_1$$

and

$$\alpha_{8,k} := x_{2k}x_{2k-1}\cdots x_2x_1$$

Observe, that for $n = 7, 8$ and for any natural $k > 0$

$$\mathbf{0}_n(\alpha_{n,k}) = \begin{cases} 000 = 2 & \text{if } n = 7 \\ 00 = 2 & \text{if } n = 8 \end{cases}$$

by (9) and (8). This implies the following

Proposition 3.5 *Let $m \geq 0$, $v_1, \ldots, v_m \in \mathrm{Var}$ and let*

$$t = \alpha_{n,k}v_1\cdots v_m$$

be a tautology of $\mathfrak{M}_n$ (for $n = 7$ or 8). Then m is even.

Proof. Since t is a tautology,

$$2 = \mathbf{0}_n(t) = \mathbf{0}_n(\alpha_{n,k})\mathbf{0}_n(v_1)\cdots\mathbf{0}_n(v_m) = 2\,0^m.$$

If $m = 0$, then m is even, and if $m > 0$, then m is even by (3.3). □

For every positive integer k and $n = 7, 8$ let $G_{n,k}$ be the set of all left-associated tautologies of $\mathfrak{M}_n$, which have a subterm $\alpha_{n,k}$. Thus an element of $G_{n,k}$ is a tautology of the form

$$\alpha_{n,k} v_1 v_2 \cdots v_m$$

for some $m \geq 0$, where $v_1, \cdots, v_m \in \text{Var}$. By Proposition 3.5, m must be even. We next show that, for every k and $n = 7, 8$, $G_{n,k} \neq \emptyset$. Let

$$t := \alpha_{n,k} x_1 x_1 x_2 x_2 \cdots x_{2k} x_{2k} x_{2k+1} x_{2k+1}.$$

Then t is left-associated and for any valuation f, if $f(x_i) \neq 0$ for some $i = 1, ..., 2k+1$, then $f(t) = 2$. If $f(x_i) = 0$ for each $i = 1, \ldots, 2k+1$, then

$$f(t) = \mathbf{0}_n(t) = \mathbf{0}_n(\alpha_{n,k}) 0^{4k+2} = 2\,0^{4k+2} = 2\,0^2 = 2$$

by (3.3). This shows that t is a tautology and thus $t \in G_{n,k}$.

From now on, let us fix $n = 7, 8$. We will write $\mathbf{0}, \alpha_k$ and G_k for $\mathbf{0}_n, \alpha_{n,k}$ and $G_{n,k}$, respectively.

Claim 4 *If $t \in G_k$, then, for each even $i \leq 2k$, x_i occurs in t outside of α_k.*

Proof of Claim 1. Let $t \in G_k$. Then

$$t = \alpha_k v_1 \cdots v_m,$$

for some $v_1, \ldots, v_m \in \text{Var}$. By Proposition 3.5, m is even. Let i be an even index, $0 \leq i \leq 2k$. Then $m + i - 1$ is odd. Define a valuation f by $f(x_i) = 2$, for every x_i occurring in α_k and $f(v) = 0$ otherwise. If x_i does not occur among the v_j's, then by (3.2) and (3.3)

$$f(t) = \underbrace{0 \cdots 0}_{|\alpha_k| - i}\, 2 \underbrace{0 \cdots 0}_{i-1} \underbrace{0 \cdots 0}_{m} = 20^{m+i-1} = 20 = 1.$$

This contradicts the assumption that $t \in G_k \subseteq E(\mathfrak{M}_n)$; so x_i must occur in t outside of α_k. $\square$

It follows from Claim 1 that

$$\text{for every } t \in G_k \text{ the length of } t \text{ is at least } 3k. \tag{3.4}$$

Now let R be a finite set of rules admissible and therefore, by Lemma 2.8, valid in $\mathfrak{M}_n$, and let k be an integer such that $|s| \leq k$ for every rule $\langle X, s\rangle \in R$. In order to complete the proof we want to show that R does not axiomatize $\mathfrak{M}_n$. Notice that since $\emptyset \neq G_k \subseteq E(\mathfrak{M}_n)$, it suffices to prove that that

$$E(\mathfrak{M}_n) \setminus G_k \text{ is closed under } R.$$

To prove this by contradiction, assume that $\langle X, s\rangle \in R$ and that σ is a substitution such that

$$\sigma(X) \subseteq E(\mathfrak{M}_n) \setminus G_k \tag{3.5}$$

and $\sigma(s) \in G_k$. Then $\sigma(s)$ is left-associated, so there is some $m = 0, 1, \ldots$ and some $v_0, \ldots, v_m \in \mathrm{Var}$ such that

$$s = v_0 v_1 \cdots v_m, \tag{3.6}$$

$$\sigma(v_1), \ldots, \sigma(v_m) \in \mathrm{Var}, \text{ and} \tag{3.7}$$

$$\sigma(v_0) \text{ is left-associated.} \tag{3.8}$$

By the choice of k

$$k \geq |s| = 1 + m,$$

and by (3.4)

$$3k \le |t| = |\sigma(s)| = |\sigma(v_0)| + \sum_{i=1}^{m} |\sigma(v_i)| = |\sigma(v_0)| + m$$

$$\le |\sigma(v_0)| + k - 1 < |\sigma(v_0)| + k.$$

Hence $2k < |\sigma(v_0)|$, which implies that $\sigma(v_0) \notin \mathrm{Var}$ and that

$$\alpha_k \text{ is a subformula of } \sigma(v_0). \tag{3.9}$$

Thus,

$$v_0 \neq v_i \text{ for every } i = 1, ..., m. \tag{3.10}$$

Claim 5 *Let σ and v_0 be as above. If $r = v_0 z_1 \cdots z_p$, for some $p \ge 0$ and $z_1, \ldots, z_p \in$ Var such that, for each $i = 1, \ldots, p$, $\sigma(z_i) \in \mathrm{Var}$, and $\sigma(r)$ is a tautology of $\mathfrak{M}_n$, then $\sigma(r) \in G_k$.*

Claim 5 follows immediately from (3.8) and (3.9). Consider now the valuation f : $\mathbf{Te} \longrightarrow \mathbf{M}_n$ such that $f(v_0) = 1$ if m is even, $f(v_0) = 2$ if m is odd and $f(z) = \mathbf{0}(\sigma(z))$ for every variable $z \neq v_0$. Note that

$$\begin{aligned} f(s) &= 10^m = 1 \text{ if } m \text{ is even} \\ &= 20^m = 1 \text{ if } m \text{ is odd} \end{aligned}$$

by (3.7) and (3.3). Observe that for any term r not containing v_0, $f(r) = \mathbf{0}(\sigma(r))$. Also, if v_0 occurs in r at the position other than the leftmost one, or if $\sigma(r)$ is not left-associated, then $f(r) = \mathbf{0}(\sigma(r))$. This implies that if $f(r) \neq \mathbf{0}(\sigma(r))$, then $\sigma(r)$ is left-associated and v_0 occurs on the leftmost position in r, hence $r = v_0 z_1 \cdots z_p$ for some $p \ge 0$ and some variables $z_1, \ldots, z_p$ such that $\sigma(z_i) \in \mathrm{Var}$. Therefore we have

Claim 6 *For any term r, if $f(r) \neq \mathbf{0}(\sigma(r))$, then there is $p \geq 0$ and variables $z_1 \ldots z_p$ such that $r = v_0 z_1 \cdots z_p$ and, for each $i = 1, \ldots, p$, $\sigma(z_i) \in \mathrm{Var}$.*

Since $f(s) = 1$, we must have $f(r) \neq 2$ for some $r \in X$, as the rule $\langle X, s\rangle$ is valid. But $\sigma(r)$ is a tautology of $\mathfrak{M}_n$, so $f(r) \neq \mathbf{0}(\sigma(r))$. By Claims 6 and 5 $\sigma(r) \in G_k$. This contradicts (3.5) and completes the proof of Theorem 2. □

Remark 1 As in [60], [61] and [10], our proof shows, that for any set R of rules such that the length of the conclusion of any rule from R is not greater than k, no tautology of G_k is derivable from R.

Remark 2 Our proof can also be applied to matrices, with one binary operation satisfying $(xy)z$ rather than $x(yz)$. Every nontautology of such matrix must be right-associated.

CHAPTER 4. RAUTENBERG-WROŃSKI PROBLEM

4.1 Introduction to the Rautenberg–Wroński conjecture

Wojtylak's example of a nonfinitely axiomatized finite matrix ([61]) motivated the following two problems. stated in as questions by Rautenberg [46] and, independently, as conjectures by A. Wroński; see [42].

(C1) Every finite algebra **A** is finitely based over some first-order equational system that is obtained from $\mathcal{B}$ by adjoining finitely many new first-order rules (necessarily sound for **A**).

(C2) Every finite algebra **A** is finitely based over some first-order equational system that is obtained from $\mathcal{B}$ by adjoining finitely many new first-order rules that are valid in **A**.

The second conjecture is stronger in the sense that a positive resolution would automatically give a positive resolution of the first. The first conjecture was only recently disproved in [21]. Most of the well-known finite, nonfinitely based groupoids, those of Lyndon [27] and Murskiĭ [32] in particular, have been shown to be finitely based over some extension of $\mathcal{B}$ by finitely many rules. So we would like to know if nonfinitely axiomatizable matrices of the size smaller than 18 exist and in particular, if a 3-element algebra with this property exists.

In this chapter we show that the underlying algebra of the nonfinitely axiomatizable matrix $\mathfrak{W}$ considered in [61] and the underlying algebras of the nonfinitely axiomatizable matrices considered in the previous chapter satisfy both (C1) and (C2) and moreover all these algebras are even finitely based. In the first section we show this for all 3-element left-associative algebras and in the second section for the underlying algebra of the Wojtylak's matrix.

4.2 Three-element left-associative algebras

Theorem 4.1 *Every three-element left-associative algebra is finitely based.*

For $i = 1, \ldots, 8$ let $\mathbf{A_i}$ be the 3-element algebra $\langle \{0,1,2\}, \circ, \rangle$, where $\circ$ is the operation determined by the i^{th} table of Lemma 2.5. To prove Theorem 4.1, for each $\mathbf{A}_i$, $i = 1, \ldots, 8$, we will demonstrate a finite equational basis B_i.

We first list all the B_i and then we prove the lemmas that B_i is a basis for $\mathbf{A}_i$ for those cases for which this is not obvious.

- $B_1 = \{xz \approx yz\}$.
- B_2 is the set of the following equations:

$$x(yz) \approx u(vw) \tag{4.1}$$

$$x(yz)u \approx x(yz) \tag{4.2}$$

$$xyz \approx xzy \tag{4.3}$$

$$xyy \approx xy \tag{4.4}$$

$$xxy \approx yyx \tag{4.5}$$

- The algebra $\mathbf{A}_3$ is the 3-nilpotent commutative semigroup discussed by M. Sapir in [51] and, as observed there, is based by $\{x(yz) \approx (xy)z, xyz \approx uuu, xy \approx yx\}$ or equivalently, by $\{x(yz) \approx (uv)w, xy \approx yx\}$.
- $B_4 = \{xx \approx u(vz), xyz \approx uvz\}$.
- B_5 is the following set of equations:

$$x(yz) \approx uu \tag{4.6}$$

$$xyz \approx xzy \tag{4.7}$$

$$xyy \approx xy \tag{4.8}$$

  That B_5 is a basis for $\mathbf{A}_5$ has already been noticed in several places by W. Rautenberg, see for example [47]. For selfcontainment, and also because the proof in [47] contains a type error, we present below (lemma 4.3) our proof of this fact.

- $\mathbf{A}_6$ is clearly finitely based by $\{xy \approx zu\}$.
- B_7 consists of

$$xxx \approx uuu \tag{4.9}$$

$$x(yz) \approx uuu \tag{4.10}$$

$$xyyy \approx xy \tag{4.11}$$

$$xyzz \approx xzyy \tag{4.12}$$

$$xyx \approx yyx \tag{4.13}$$

$$xyzy \approx xzzy \tag{4.14}$$

$$xyzuy \approx xzzuy \tag{4.15}$$

- let B_8 consist of the following set of equations:

$$xx \approx uu \tag{4.16}$$

$$x(yz) \approx uu \tag{4.17}$$

  and of (4.11), (4.12), (4.13), (4.14) and (4.15).

We claim that for each i, B_i is a basis for the equational theory of $\mathbf{A}_i$. This is straightforward for $i = 1, 3, 4, 6$. For $i = 2, 5, 7, 8$, this is the content of lemmas 4.2–4.15.

In all cases we omit the routine verification that $B_i \subseteq \mathrm{Id}(\mathbf{A}_i)$.

4.2.1 Proof for $\mathbf{A}_2$

Lemma 4.2 *$\mathbf{A}_2$ is based over B_2.*

Let t be a non-leftassociated term. Then t has a subterm of the form $s(ru)$ and $\{(4.2), (4.1)\} \vdash t \approx x(yz)$. Note that if $x(yz) \approx s \in \mathrm{Id}(\mathbf{A}_2)$ then s cannot be left-associated. For otherwise the valuation $v(x) = 0$ for all variables x would map s to 1 while it maps $x(yz)$ to 2. Therefore if t or s is not left-associated and $t \approx s \in \mathrm{Id}(\mathbf{A}_2)$ then $B_2 \vdash t \approx s$.

Let now both t and s be left-associated and assume that $t \approx s \in \mathrm{Id}(\mathbf{A}_2)$. So $t = x_1 \cdots x_n$ and $s = y_1 \cdots y_m$ for some $n, m \geq 1$ and $t \approx s \in \mathrm{Id}(\mathbf{A}_2)$. Our first claim is that $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_m\}$.

Notice that for any variable v, the only term to which v can be identically equal in A_2 is v itself. So assume that $n, m > 1$. Consider a valuation f such that for $1 \leq i \leq n$ $f(x_i) = 0$ and $f(x) = 2$ for any other variable x. Then $f(t) = 1$ and therefore $f(s) = 1$. But this is only possible if for every $j \leq m$ $f(y_j) \neq 2$, so $\{y_1, \ldots y_m\} \subseteq \{x_1, \ldots, x_n\}$. Similarly, $\{x_1, \ldots, x_n\} \subseteq \{y_1, \ldots, y_m\}$.

Our second claim is that also $\{x_2, \ldots, x_n\} = \{y_2, \ldots, y_m\}$. For let $v(x_1) = 1, v(x) = 0$ for every variable $x \neq x_1$. If for some $i \neq 1$, $y_i \notin \{x_2, \ldots, x_n\}$ then by our first claim $y_i = x_1$ and $v(s) = 2$. Therefore $v(t) = 2$ and whence $x_1 = x_j$ for some $j \neq 1$. Thus $y_i = x_1 = x_j \in \{x_2, \ldots, x_n\}$. We have shown that $\{y_2, \ldots y_m\} \subseteq \{x_2, \ldots, x_n\}$. The claim now follows by the symmetric argument.

It follows from the above claims that $B_2 \vdash t \approx s$. For if $x_1 = y_1$ then $t \approx s$ is derivable from (4.3), (4.4), as $\{x_2, \ldots, x_n\} = \{y_2, \ldots, y_n\}$. If $x_1 \neq y_1$ then by claims 1 and 2 $x_1 = y_k = x_i$, $y_1 = x_l = y_j$, for some $i, j, k, l \neq 1$. Using (4.3) and (4.4) again, we can assume without the loss of generality that $i = j = 2, k = l = 3$,i.e., $t = x_1 x_1 x_3 x_4 \cdots x_n$ and that $s = x_3 x_3 x_1 x_4 \cdots x_n$. But (4.5)$\vdash$ $x_1 x_1 x_2 \approx x_2 x_2 x_1$. So $B_2 \vdash t \approx s$.

Lemma 4.3 *B_5 is a basis for* $\mathrm{Id}(\mathbf{A}_5)$

Proof First, the following is derivable from B_5:

$$xxy \approx uu \tag{4.18}$$

Let R be the reduction determined by the following rules:

$$x(yz) \to uu,$$

$$xxy \to uu,$$

$$xy_1 \cdots y_n x \to uu \quad \text{for all } n \geq 1$$

$$x_1 \cdots x_n x_k \to x_1 \cdots x_n, \qquad \text{for all } n \geq 1,\ 1 < k \leq n.$$

A term t is *reduced* if none of its subterms is a substitution instance of any of the left-hand-sides of the above reductions, i.e., if the reduction cannot be applied to any subterm of t. The following statements are easy to verify:

1. every reduced term is left-associated;

2. a reduced term of length greater than 2 has all of its variables distinct;

3. for every reduction rule $t \to s$, $B_5 \vdash t \approx s$;

4. for every term t there is a reduced term t^* such that $B_5 \vdash t \approx t^*$;

5. to show that B_5 is a basis for $\mathbf{A}_5$ it is enough to show that for every pair of reduced terms t, s, if $t \approx s \in \mathrm{Id}(\mathbf{A})_5$, then $B_5 \vdash t \approx s$.

Let t, s be reduced and $t \approx s \in \mathrm{Id}(\mathbf{A}_5)$. So $t = x_1 \cdots x_n$, $s = y_1 \cdots y_m$ for some variables $x_1, \ldots, x_n, y_1, \ldots y_m$. If $n = m = 2$ and $x_1 = x_2, y_1 = y_2$, then clearly, $B_5 \vdash t \approx s$, since from (4.6) one derives $xx \approx yy$. So suppose that all variables in one of the terms, say t, are distinct. We claim that $x_1 = y_1$ and $\{x_2, \ldots, x_n\} = \{y_2, \ldots, y_m\}$.

Let v be a valuation such that $v(x_1) = 1, v(x_i) = 0$ for all $i = 2, \ldots, n$ and $v(y) = 2$, for all variables y distinct from $x_1, \ldots, x_n$. Then $v(t) = 1$ and therefore

$v(s) = 1$, which implies that $y_1 = x_1$ and that $\{y_2, \ldots, y_m\} \subseteq \{x_2, \ldots, x_n\}$. It also implies that y_1 is distinct from all the other variables in s, so the above argument may be repeated with the roles of t and s interchanged to show the reverse inclusion. This proves the claim. It follows from the claim that the equality $t \approx s$ can be derived using just (4.7). This finishes the proof that B_5 is the basis for $\mathbf{A}_5$.

4.2.2 Proof for $\mathbf{A}_7$

Lemma 4.4 *B_7 is a basis for $\mathbf{A}_7$*

For a term t by Var$t$ we understand the set of all variables occurring in t. Consider the following equations:

$$xyyzz \approx xzzyy \tag{4.19}$$

$$uuux \approx xx \tag{4.20}$$

$$xxyy \approx yyxx \tag{4.21}$$

Proposition 4.5 (4.19), (4.20), (4.21) *are derivable from B_7 and moreover,* (4.19) *is derivable from* (4.12) *and* (4.14) *only.*

Proof Using (4.12) and then (4.14), we get the following derivation of (4.19): $xyyzz \approx xyzyy \approx xzzyy$. (4.20) is immediate from (4.9) and (4.11) and (4.21) from (4.12) and (4.13). □

Lemma 4.6 *For every $n \geq 0$, the identity*

$$xyv_nv_{n-1}\cdots v_1x \approx yyv_nv_{n-1}\cdots v_1x, \tag{6_n}$$

is derivable from (4.13), (4.15) *and* (4.11) *and the identity*

$$xyzv_n\cdots v_1y \approx xzzv_n\cdots v_1y, \tag{7_n}$$

is derivable from (4.14), (4.15) *and* (4.11).

Proof Note that (4.13), (4.14), (4.15) are (6_0), (7_0), (7_1), respectively. It suffices to show that for every $n \geq 1$, the identity (6_n) is derivable from (6_{n-1}), (4.15), and (4.11) and that for every $n \geq 2$, the identity (7_n) is derivable from (7_{n-1}), (4.15) and (4.11). The following derivation using (4.11), (4.15), (6_{n-1}), demonstrates the first of the above claims:

$$
\begin{array}{ll}
xyv_nv_{n-1}\ldots v_1x \approx & \text{by } (4.11) \\
xyv_nv_{n-1}\ldots v_2v_1v_1v_1x \approx & \text{by } (4.15) \\
xyv_nv_{n-1}\ldots v_2xv_1v_1x \approx & \text{by } (6_{n-1}) \\
yyv_nv_{n-1}\ldots v_2xv_1v_1x \approx & \text{by } (4.15) \\
yyv_nv_{n-1}\ldots v_2v_1v_1v_1x \approx & \text{by } (4.11) \\
yyv_nv_{n-1}\ldots v_1x. &
\end{array}
$$

The following derivation using (4.11), (4.15), (7_{n-1}), (4.15), (4.5), in this order, demonstrates the second of our claims:

$$
\begin{array}{ll}
xyzv_n\ldots v_1y \approx & \text{by } (4.11) \\
xyzv_n\ldots v_2v_1v_1v_1y \approx & \text{by } (4.15) \\
xyzv_n\ldots v_2yv_1v_1y \approx & \text{by}(7_{n-1}) \\
xzzv_n\ldots v_2yv_1v_1y \approx & \text{by } (4.15) \\
xzzv_nv_{n-1}v_{n_2}\ldots v_2v_1v_1v_1y \approx & \text{by } (4.11)
\end{array}
$$

$$xzzv_nv_{n-1}v_{n_2}\dots v_1y. \qquad \square$$

A left-associated term $t = x_nx_{n-1}\cdots x_1$ is *reduced* (with respect to B_7), if either $n = 3$ and $x_1 = x_2 = x_3$, or

$$x_i = x_j \text{ implies } |i - j| \leq 1,$$

i.e., every variable occurs in t at most twice and these occurrences are consecutive.

Two terms t and s are called *equivalent* (with respect to B_7) if the equality $t \approx s$ is derivable from B_7 using the Birkhoff's rules.

Lemma 4.7 *For every term t there exists a reduced term s such that t and s are equivalent.*

Proof Use (4.10) to get a left-associated term equivalent to t. To this term, use (7_n) and (6_n) (possibly several times) to get an equivalent term in which all occurrences of a given variable are consecutive. Then, for every variable other than the leftmost one, if this variable occurs more than twice, use (4.11) to get an equivalent term with no more than 2 occurrences of this variable. Finally if the leftmost variable occurs at least 3 times and the length of the term is more than 3, use (4.9) and (4.20) (an appropriate number of times) to get a reduced term. $\square$

Let λ denote the empty term, so that $\lambda x = x$.

Observation 1 *For every reduced term t one of the following is true:*

1) $t = uuu$

2) *t is of the form $t = t_1x_{p+1}x_p^2\cdots x_2^2x_1^2$, for some $t_1 \in \mathrm{Te}X \cup \{\lambda\}$, some $p \geq 0$ and some pairwise distinct variables $x_1, \dots, x_{p+1} \in X$, such that*

a) $t_1 = \lambda$ *or*

b) $t_1 = x_{p+1}$ *or*

c) $t_1 \in \mathrm{Te}X$ *and* x_{p+1} *does not occur in* t_1.

(it follows, from the fact that t *is reduced, that none of* $x_1, \ldots, x_p$ *occurs in* t_1 *in any of these cases).*

For a reduced term t of the form **2** define a valuation $v_t : \mathrm{TeVar}t \to A_7$ as follows:

a) if t is of the form **2a**, let $v_t(x_{p+1}) = 1, v_t(x_i) = 0$, for all $i = 1, \ldots, p$

b) if t is of the form **2b**, let $v_t(x_i) = 0$, for all $i = 1, \ldots, p+1$

c) if t is of the form **2c** let $v_t(x_i) = 0$ for all $i = 1, \ldots, p+1$ and $v_t(x) = 2$ for every variable $x \in \mathrm{Var}t_1$.

Fact 4.8 *For every term* t *of the form* **2**, $v_t(t) = 1$

□

Lemma 4.9 *If* $t \approx s \in \mathrm{Id}(A_7)$ *and* t, s *are reduced, then* t *is of the form* **1** *iff* s *is of the form* **1**.

Proof Suppose that t is not of the form **1**. Then $v_t(t) = 1$ and therefore $v_t(s) = 1$. But for every valuation v, $v(uuu) = 2$. So s cannot be of the form **1**. □

Lemma 4.10 *Let* t *and* s *be reduced terms of the form* **2**, *i.e., there exist* $k, p \geq 0$ *and* $t_1, s_1 \in \mathrm{Te}X \cup \{\lambda\}$ *such that* $t = t_1 x_{p+1} x_p^2 \cdots x_2^2 x_1^2$, $s = s_1 y_{k+1} y_k^2 \cdots y_2^2 y_1^2$ *and the condition* **2)** *on* t, s *holds. If* $t \approx s \in \mathrm{Id}(A_8)$, *then* $k = p$ *and* $\{y_1, \ldots y_{k+1}\} = \{x_1, \ldots x_{p+1}\}$.

Proof Note that if $\{y_1, \ldots y_{k+1}\} = \{x_1, \ldots x_{p+1}\}$, then $k = p$ by the fact that t and s are reduced. In order to prove the lemma it suffices to prove that $\{x_1, \ldots, x_{p+1}\} \subseteq \{y_1, \ldots, y_{k+1}\}$.

Suppose that for some $i = 1, \ldots, p+1$, $x_i \notin \{y_1, \ldots, y_{k+1}\}$ and choose the minimal such i. Let $v(x) = v_s(x)$ if $x \in$ Var$s$, and if $v(x) = 2$, otherwise. Observe that by the choice of i, $v(x_i) = 2$ and $v(x_1) = \ldots = v(x_{i-1}) = 0$. So $v(t) = 2$ and $v(s) = 1$, a contradiction. It follows that $\{x_1, \ldots, x_{p+1}\} \subseteq \{y_1, \ldots, y_{k+1}\}$. □

Lemma 4.11 *If t, s are reduced terms, s is of the form* **2a** *and $t \approx s \in \mathrm{Id}(A_7)$ then t is also of the form* **2a**.

Proof By lemma 4.9 t cannot be of the form **1**. Hence $t = t_1 x_{p+1} x_p^2 \cdots x_1^2$ By lemma 4.10 all variables of s are contained in $\{x_1, \ldots, x_{p+1}\}$. If t is of the form **2b** or **2c**, then $v_t(x_1) = \ldots v_t(x_{p+1}) = 0$ and therefore $v_t(s) = 2$, a contradiction with $t \approx s \in \mathrm{Id}(A_7)$ and Fact 4.8. Hence t is of the form **2a**. □

Lemma 4.12 *If t, s are reduced terms, s is of the form* **2b** *and $t \approx s \in \mathrm{Id}(A_7)$, then t is also of the form* **2b**.

Proof By lemmas 4.9, 4.11 and Observation 1, t is of the form **2b** or **2c**. Suppose that t is of the form **2c**, i.e., $t = t_1 x_{p+1} x_p^2 \cdots x_1^2$, where t_1 is a term not containing any of the variables $\{x_1, \ldots, x_{p+1}\}$. Also, as t is reduced, so must be t_1. Moreover t_1 cannot be of the form **1**, because then t would not be reduced. So t_1 is of the form **2** and $v_{t_1}(t_1) = 1$. As $\{x_1, \ldots, x_{p+1}\} \cap \mathrm{Var}t_1 = \emptyset$, we can extend this valuation to a valuation $v : \mathrm{Var}t \to \mathbf{A}_7$ by setting $v(x_i) = 0$ for $i = 1, \ldots, p+1$. But then $v(t) = 2$, while $v(s) = 1$, by lemma 4.10. This contradiction shows that t must also be of the form **2b**. □

Corollary 4.13 *Let t, s be reduced terms such that $t \approx s$ is an identity of $\mathbf{A}_7$, Then s is of the form* **2i** *iff s is of the form* **2i**, *for $i = a, b, c$.*

□

Lemma 4.14 *If $t \approx s \in \mathrm{Id}(A_7)$, then $t \approx s$ is derivable from B_7.*

Proof By lemma 4.7 it is sufficient to assume that t and s are reduced. We will prove the claim of the lemma by induction on the minimum of $|t|, |s|$. Without loss of generality assume that t is no longer than s. If $|t| = 1$, then t is a variable, whence $s = t$, as no variable can be identically equal in $\mathbf{A}_7$ to any other term but itself. So in this case $t \approx s$ is derivable from B_7. Now assume that $|t|, |s| \geq 2$ and that every identity of $\mathbf{A}_7$ with at least one side shorter than t is derivable from B_7. If both t and s are of the form **1** then the identity $t \approx s$ is derivable using (4.9). Next consider the case that they are both of the form **2a**, i.e., $t = x_{p+1}x_p^2 \cdots x_1^2$ and $s = y_{k+1}y_k^2 \cdots y_1^2$. By lemma 4.10 $\mathrm{Var}t = \mathrm{Var}s$. If $x_{p+1} \neq y_{k+1}$, then $v_t(s) = 2$, a contradiction with the assumption and Fact 4.8. So $x_{p+1} = y_{k+1}$ and $t \approx s$ follows from (4.19). If both t and s are of the form **2b**, then $t = x_{p+1}^2 \cdots x_1^2$, $s = y_{k+1}^2 \cdots y_1^2$ and $\{x_1, \ldots, x_{p+1}\} = \{y_1, \ldots, y_{k+1}\}$. So $t \approx s$ can be derived from (4.21) and (4.19). Finally suppose that both t and s are of the form **2c**, i.e., $t = t_1 x_{p+1} x_p^2 x_{p-1}^2 \cdots x_1^2$ and $s = s_1 y_{k+1} y_k^2 y_{k-1}^2 \cdots y_1^2$, where x_i's do not occur in t_1 and y_i's does not occur in s_1. Moreover, by lemma 4.18, $p = k$ and $\{x_1, \ldots x_p\} = \{y_1, \ldots, y_k\}$. So $\{(4.12), (4.19)\} \vdash s \approx s_1 x_{p+1} x_p^2 \cdots x_1^2$. Now observe that since $x_i \notin \mathrm{Var}t_1 \cup \mathrm{Var}s_1$ for any $i = 1, \ldots, p+1$, $v(t_1) = v(s_1)$ for every valuation v. For otherwise letting $w(x_i) = 0$ for all $i = 1, \ldots, p+1$ and $w(x) = v(x)$ otherwise, we would have that $w(t) \neq w(s)$, a contradiction. Hence $t_1 \approx s_1$ is an identity of $\mathbf{A}_7$ and by the induction hypothesis it is derivable from B_7. But $t \approx s$ is derivable from $t_1 \approx s_1$ by using the rule of replacement,so $B_7 \vdash t \approx s$. Corollary 4.13 ensures that we have considered all the cases. □

4.2.3 Proof for $\mathbf{A}_8$

Lemma 4.15 *B_8 is a basis for $\mathbf{A}_8$*

Let us observe that

$$uuxx \approx uu \tag{4.22}$$

is derivable from B_8 using (4.16) and (4.11) and that (4.19) is derivable from B_8, by Proposition 4.5. Similarly, by lemma 4.6 we have that (6_n) and (7_n) are derivable from B_8.

We will say that a left-associated term $t = x_n x_{n-1} \cdots x_1$, is *reduced* (with respect to B_8), if t does not contain a subterm $xxyy$ for any variables x, y and

$$x_i = x_j \text{ implies } |i - j| \leq 1,$$

i.e., every variable occurs in t at most twice and these occurrences are consecutive.

Two terms t and s are called *equivalent* (with respect to B_8) if the equality $t \approx s$ is derivable from B_8 using the Birkhoff's rules.

Lemma 4.16 *For every term t there exists a reduced term s such that t and s are equivalent.*

Proof Use (4.17) to get a left-associated term equivalent to t. To this term, use (7_n) and (6_n) (possibly several times) to get an equivalent term in which all occurrences of a given variable are consecutive. Then use (4.16) and (4.11) to get an equivalent term with no more than 2 occurrences of a given variable. Finally use (4.22) to get an equivalent term with no subterm $xxyy$, for any pair of variables x, y. □

Let λ denote the empty word, so that , $\lambda x = x$.

Observation 2 *For every reduced term t one of the following is true:*

1) *t is equivalent to uu*

2) *t is of the form $t = t_1 x_{p+1} x_p^2 \cdots x_2^2 x_1^2$, for some $p \geq 0$ and some pairwise distinct variables $x_1, \ldots, x_{p+1} \in X$, where either*

- **a)** *$t_1 = \lambda$ or*
- **b)** *$t_1 \in \mathrm{Te}X$ and x_{p+1} does not occur in t_1 (it follows, from the fact that t is reduced, that also none of $x_1, \ldots, x_p$ occurs in t_1).*

Let t be a reduced term of the form 2 above. Define a valuation $v_t : \mathrm{TeVar}t \to \mathbf{A}_8$ by $v_t(x_1) = \ldots = v_t(x_{p+1}) = 0$ and $v_t(x) = 2$ for every $x \in \mathrm{Var}t_1$.

Fact 4.17 *If t is of the form 2, then $v_t(t) = 1$.*

Lemma 4.18 *Let t and s be reduced terms of the form 2, i.e., there exist $k, p \geq 0$ and $t_1, s_1 \in \mathrm{Te}X \cup \{\lambda\}$ such that $t = t_1 x_{p+1} x_p^2 \cdots x_2^2 x_1^2$, $s = s_1 y_{k+1} y_k^2 \cdots y_2^2 y_1^2$ and the conditions on x_i's and y_i's required by 2 hold. If $t \approx s \in \mathrm{Id}(A)_8$, then $k = p$ and $\{y_1, \ldots y_{p+1}\} = \{x_1, \ldots x_{k+1}\}$.*

Proof Note that if $\{y_1, \ldots y_{p+1}\} = \{x_1, \ldots x_{k+1}\}$, then $k = p$ by the fact that t and s are reduced. So in order to prove the lemma it suffices to show that under our assumptions $\{x_1, \ldots, x_{k+1}\} \subseteq \{y_1, \ldots, y_{p+1}\}$.

Suppose that for some $i = 1, \ldots, k+1$, $x_i \notin \{y_1, \ldots, y_{p+1}\}$ and choose the minimal such i. Extend v_s to a valuation $v : \mathrm{Te}X \to \mathbf{A}_8$ by setting $v(x) = 2$ for every variable x for which v_s is undefined. Note that $v(x_i) = 2$ and thus $v(t) = 2$ while $v(s) = 1$, a contradiction. It follows that $\{x_1, \ldots, x_{k+1}\} \subseteq \{y_1, \ldots, y_{p+1}\}$. □

Lemma 4.19 *Let X be any of the* 1,2*a*,2*b. If $t \approx s \in \mathrm{Id}(A_8)$, then s is of the form X iff t is of the form X.*

Proof Similar to the combined proof of Lemma 4.9 and Corollary 4.13, using valuations v_t defined above. □

Lemma 4.20 *If t and s are reduced and $t \approx s \in \mathrm{Id}(A_8)$, then $t \approx s$ is derivable from B_8.*

Proof We prove the lemma by induction on the minimum of $|t|, |s|$. Without loss of generality assume that t is no longer than s and that $s = y_m y_{m-1} \cdots y_1$, for some $y_1, \ldots, y_m \in X$. First note, that if $|t| = 1$, then t is a variable and that any variable is identically equal in $\mathbf{A}_8$ only to itself. So in this case $t \approx s$ is derivable. Suppose next that $|t| \geq 2$ and assume that every identity of $\mathbf{A}_8$ with at least one side shorter than t is derivable from B_8. If both t and s are equivalent to uu (recall that this means that $t \approx uu$ is derivable from B_8), then there is nothing to prove.

If both t and s are of the form 2a, then $\mathrm{Var}t = \mathrm{Var}s$ and moreover $x_{p+1} = y_{k+1}$. For if $x_{p+1} = y_i$ for some $i \neq k+1$, then $v_s(t) = 2$, a contradiction with Fact 4.17 and the assumption. But this means that t and s differ only by the order of variables $x_1, \ldots, x_p$ and $t \approx s$ is a consequence of (4.19).

Finally assume that t and s are of the form **2b**, i.e., there exist $k, p \geq 0$, $t_1, s_1 \in \mathrm{Te}X$ and variables $x_1, \ldots, x_p, y_1, \ldots y_k$ such that $t = t_1 x_{p+1} x_p^2 x_{p-1}^2 \cdots x_1^2$ and $s =$

$s_1 y_{k+1} y_k^2 y_{k-1}^2 \cdots y_1^2$, where x_i's do not occur in t_1 and y_i's does not occur in s_1. By lemma 4.18, $p = k$ and $\{x_1, \ldots x_{p+1}\} = \{y_1, \ldots, y_{k+1}\}$.

We claim now, that $t_1 \approx s_1 \in \mathrm{Id}(A)$. Consider any valuation v and put $w(x) = v(x)$, if $x \neq x_i$ for any $i = 1, \ldots, p+1$ and for every $i = 1, \ldots, p+1, w(x_i) = 0$. Then $w(t_1) = v(t_1), w(s_1) = v(s_1)$ and if $w(t_1) \neq w(s_1)$, then also $w(t) \neq w(s)$. But $t \approx s \in \mathrm{Id}(A)$, so it follows that $w(t_1) = w(s_1)$ and therefore $v(t_1) = v(s_1)$. This proves that $t_1 \approx s_1 \in \mathrm{Id}(A)$. By the induction hypothesis, $t_1 \approx s_1$ is derivable from B_8. But then also $t \approx s$ is derivable using the Birkhoff's rule of replacement, (4.12) and (4.19). By lemma 4.19, the above argument covers all possible cases. □

Thus B_8 is a basis for the algebra $\mathbf{A}_8$.

4.3 Wojtylak's algebra

We mentioned above that the first matrices with nonfinitely axiomatized theorems were introduced by P. Wojtylak in [60, 61].

We present this algebra here and show that the smaller of these algebras is finitely based.

Definition 4.21 *The algebra* $\mathbf{W}$ *is the groupoid* $\langle W, \circ \rangle$, *where* $W = \{0, 1, 2, 3, 4\}$ *and the operation* $\circ$ *is given by the following table:*

$\circ$	0	1	2	3	4
0	4	0	4	4	4
1	4	2	4	0	4
2	4	2	4	0	4
3	4	0	4	4	4
4	4	4	4	4	4

Theorem 4.22 *The algebra* **W** *is finitely based .*

ProofWe claim that the set Σ consisting of the following equations (4.23)–(4.27) is a finite basis of **W**.

$$x(yz) \approx u(uu) \tag{4.23}$$

$$u(uu)x \approx u(uu) \tag{4.24}$$

$$xyz \approx xzy \tag{4.25}$$

$$xxy \approx yxx \tag{4.26}$$

$$xxx \approx xx \tag{4.27}$$

Lemma 4.23 *The following useful identity can be derived from* Σ

$$yxxx \approx yxx \tag{4.28}$$

Proof

$$\begin{aligned} yxxx &= xxyx \text{ by (4.26)} \\ &= xxxy \text{ by (4.25)} \\ &= xxy \text{ by (4.27)} \end{aligned}$$

Lemma 4.24 **W** $\models \Sigma$.

Proof By inspection of the multiplication table of **W**. □

Definition 4.25 *A term t is in* **normal form** *if it is right-associated and $t = u(uu)$ or it is left-associated, no variable occurs in t more than twice and if twice then both occurrences of this variable are consecutive.*

It follows from the above definition that a term in a normal form cannot have a proper subterm of the form $t(sr)$.

Lemma 4.26 *For every term t there is a term t' in normal form such that $\Sigma \vdash t \approx t'$ and therefore also $t \approx t' \in \mathrm{Id}(\mathbf{W})$.*

Proof Let $\mathbf{4}$ abbreviate the term $u(uu)$. We proceed by induction on the complexity of t. Clearly, each variable is a term in normal form: it is left-associated and the rest of the conditions follow trivially. So suppose that $t = sr$ and assume that for every term less complex than t the lemma holds. So in particular, we can assume that s and r are in normal forms. If s is right-associated, i.e., $s = \mathbf{4}$, then $t = \mathbf{4}r$ and $\Sigma \vdash t \approx \mathbf{4}$, by 4.24. Similarly, if r is a term other than a variable, then we can use equation (4.23) to derive $t \approx \mathbf{4}$ from Σ. So assume that s is left-associated, in normal form and that r is a variable, say $r = x$. Now if x is different from every variable occurring in s, then t is in normal form. So assume that x occurs in t, say $t = x_1, \cdots, x_n$ and $1 \leq i \leq n$ is the largest index such that $x_i = x$. Then using (4.25) we can derive from Σ the equality $t \approx x_1 \cdots x_i x x_{i+1} \cdots x_n$ and if $i = 1$ or $x_{i-1} \neq x$ then the right-hand-side of the above equality is in normal form and lemma is proved. Otherwise, $x_{i-1} = x$ and we can use (4.27 or (4.28) to derive equality of t with just $s = x_1 \cdots x_n$, which is in normal form. □

Lemma 4.27 *If $t \approx s \in \mathrm{Id}(\mathbf{W})$, where t, s are in normal form, then $\Sigma \models t \approx s$.*

ProofFirst observe, that for every valuation v the value of the term $\mathbf{4}$ under v is 4. On the other hand, the valuation v such that for every variable x $v(x) = 1$ sends any left-associated term into 2. Hence if t, s satisfy the assumptions of the lemma, then either both of them are $\mathbf{4}$, in which case obviously $\Sigma \vdash t \approx s$, or else they are both left-associated. Let x be some variable. Consider valuation v such that $v(x) = 3$ and $v(y) = 1$, for every variable $y \neq x$. Then for every left-associated term r in normal form, the following statements are easy to check.

1. If x does not occur in r, then $v(r) = 2$.

2. If x occurs in r exactly once, then $v(r) = 0$.

3. If x occurs in r twice, then $v(r) = 4$.

It follows that if terms t, s in normal form are left-associated and $\mathbf{W} \models t \approx s$, then the number of occurences of a variable x in t is the same as the number of occurrences of x in s.

Thus t and s differ at most by the order of occurrences of variables, i.e., $t = x_1 \ldots x_n$, $s = y_1, \ldots, y_n$ where $x_1, \ldots, x_n, y_1, \ldots, y_n$ are variables and the strings $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ are permutations of each other. Notice, that if $x_1 \neq y_1$ then x_1 must occur in t twice. For if x_1 occurs in t only once, let v be a valuation such that $v(x_1) = 2$ and $v(z) = 1$, for every $z \neq x_1$. Then $v(t) = 2$ while $v(s) = 4$. Hence if $x_1 \neq y_1$, then both x_1 and y_1 occur in both t and s twice. If $x_1 = y_1$, then the equality $t \approx s$ follows from (4.25). If $x_1 \neq y_1$ and both x_1, y_1 occur in t and therefore in s, twice, then $t \approx s$ follows from (4.26) and possibly (4.25).

This finishes the proof of the lemma. □

Returning to the proof of the theorem, in view of lemma 4.24 it remains to show that every identity of $\mathbf{W}$ is derivable from Σ. Let $t \approx s$ be an identity of $\mathbf{W}$. Then by lemma 4.26, $\Sigma \vdash t \approx t', s \approx s'$, where t', s' are in normal form. It follows again by Lemma 4.26 that $t' \approx s'$ is an identity of $\mathbf{W}$ and therefore, by lemma 4.27, $\Sigma \vdash t' \approx s'$. Thus $\Sigma \vdash t \approx s$, as desired. This finishes the proof that $\mathbf{W}$ is finitely based. □

CHAPTER 5. GENTZEN-STYLE AXIOMATIZATION OF EQUATIONAL LOGIC

5.1 Introduction

This chapter presents result of a joint work of the author with her major profesor. The analogue of the Rautenberg-Wroński conjecture in quasi-equational logic is considered and a Gentzen-style deductive system for quasi-equational logic is presented. In this system, the sequents correspond to quasi-equations. We conjecture that every finite algebra gives rise to an extension of this system by a finite set of new Gentzen-style inference rules from which all (and only) quasi-identities of the algebra can be derived. This conjecture is verified for a class of algebras that includes all finite algebras without proper subalgebras and all finite simple algebras that are embeddable into the free algebra of their variety.

5.2 Preliminaries and notation

We use $\varepsilon, \delta, \gamma$ to represent equations and φ, ψ, ϑ to represent either equations or quasi-equations. For any equation ε, ε^l and ε^r will denote the left- and right-hand term of ε, respectively. Recall that for any formula φ, by $\mathrm{Var}(\varphi)$ we denote the set of all variables occurring in φ. and that by a *substitution instance* of an equation ε

we mean $\sigma\varepsilon^l \approx \sigma\varepsilon^r$ for some substitution σ. If φ is the quasi-equation

$$\varepsilon_1 \wedge \cdots \wedge \varepsilon_n \rightarrow \delta, \tag{5.1}$$

where n is a natural number and $\delta, \varepsilon_1, \ldots, \varepsilon_n$ are equations, then $\sigma\varphi = \sigma\varepsilon_1 \wedge \cdots \wedge \sigma\varepsilon_n \rightarrow \sigma\delta$.

The conjunction $\varepsilon_1 \wedge \cdots \wedge \varepsilon_n$ in 5.1 may be empty. It is convenient to treat it as a finite (possibly empty) set rather than a conjunction of equations. So we write 5.1 in the form $\{\, \varepsilon_1, \ldots, \varepsilon_n \,\} \rightarrow \delta$; we often omit the set-building brackets and write simply $\varepsilon_1, \ldots, \varepsilon_n \rightarrow \delta$.

Other useful notational conventions for representing quasi-equations: Let ε, δ be equations and Γ, Δ finite sets of equations. We write $\Gamma, \Delta \rightarrow \varepsilon$ for $\Gamma \cup \Delta \rightarrow \varepsilon$ and $\Gamma, \delta \rightarrow \varepsilon$ or $\delta, \Gamma \rightarrow \varepsilon$ for $\Gamma \cup \{\delta\} \rightarrow \varepsilon$. We identify the quasi-equation $\emptyset \rightarrow \varepsilon$ with the equation ε; it is normally written in the form $\rightarrow \varepsilon$. The capital greek letters Γ, Δ will represent finite sets of equations and Φ, Θ finite sets of either equations of quasi-equations.

5.3 Second-order equational logic

In chapter 2, section 2.4, with every deductive system $\mathcal{S}$ we associated a corresponding second-order deductive system, in which the formulas are the sequents representing the rules of $\mathcal{S}$ and the rules take the form of Gentzen rules. By contrast, $\mathcal{S}$ is called a first-order system. We consider here the deductive system of equational logic. The weakest second-order equational system we consider has the following axioms:

(I) $\rightarrow x \approx x$,

(S) $x \approx y \to y \approx x$,

(T) $x \approx y, y \approx z \to x \approx z$,

(R) $x_1 \approx y_1, \ldots, x_l \approx y_l \to O(x_1, \ldots, x_l) \approx O(y_1, \ldots, y_l)$ for each l-ary operation symbol O,

(U) $\varepsilon \to \varepsilon$, for every equation ε.

We also have the following inference rules: ε and δ represent arbitrary equations and Γ and Δ arbitrary finite sets of equations.

(C) $\dfrac{\Gamma, \delta \to \varepsilon;\ \Delta \to \delta}{\Gamma, \Delta \to \varepsilon}$,

(W) $\dfrac{\Gamma \to \varepsilon}{\Gamma, \Delta \to \varepsilon}$.

The axiom (U) is called *tautology*, and the rules (C) and (W) *cut* and *weakening*, respectively.

This second-order equational system is essentially the one given in Selman [52]; we will refer to it in the sequel as $\mathcal{S}$. A closely related system was formulated by Łoś and Suszko [28]; see [46], section 6. The following completeness theorem for second-order equational logic is established in [52]: for every sequent ψ and every set of sequents Φ,

$$\Phi \models \psi \quad \text{iff} \quad \Phi \vdash_{\mathcal{S}} \psi.$$

Every first-order rule $\frac{\Gamma}{\varepsilon}$ can be associated with a sequent $\Gamma \to \varepsilon$ and vice-versa. Trivially, the rule is valid in an algebra $\mathbf{A}$ iff its associated sequent is a quasi-identity of $\mathbf{A}$ and this rule is sound in $\mathbf{A}$ if the associated sequent is a quasi-identity of the free denumerably generated algebra. This correspondence between first-order rules and second-order formulas extends much further, as was first observed by Rautenberg [46]. Let $\Phi_1, \ldots, \Phi_n$ be a finite set of first-order rules and $\varphi_1, \ldots, \varphi_n$ their associated sequents. Let $\mathcal{T}$ be the first-order system obtained by adjoining the rules $\Phi_1, \ldots, \Phi_n$ to $\mathcal{B}$. It is not difficult to show that, for any system $\varepsilon_1, \ldots, \varepsilon_m, \delta$ of equations,

$$\varepsilon_1, \ldots, \varepsilon_m \vdash_{\mathcal{T}} \delta \quad \text{iff} \quad \rightarrow \varepsilon_1, \ldots, \rightarrow \varepsilon_m, \varphi_1, \ldots, \varphi_n \vdash_{\mathcal{S}} \rightarrow \delta.$$

Because of this equivalence and the completeness theorem for second-order equational logic, the two Rautenberg–Wroński conjectures above can be reformulated as follows:

(C1) For every finite algebra $\mathbf{A}$ there is a finite set $\{\varphi_1, \ldots, \varphi_n\}$ of sequents such that ε is an identity of $\mathbf{A}$ iff $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{S}} \rightarrow \varepsilon$.

(C2) For every finite algebra $\mathbf{A}$ there is a finite set $\{\varphi_1, \ldots, \varphi_n\}x$ of quasi-identities of $\mathbf{A}$ such that ε is an identity of $\mathbf{A}$ iff $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{S}} \rightarrow \varepsilon$.

The problem of whether a finite algebra is finitely-based over a first-order system has an analogue in second-order equational logic.

Let $\mathcal{T}$ be a second-order system obtained by extending $\mathcal{S}$ by new second-order (Gentzen) inference rules. An algebra $\mathbf{A}$ is *finitely q-based over* $\mathcal{T}$ if there is a finite set Φ of sequents (quasi-equations) such that, for any sequent ψ, $\psi \in \mathrm{QId}(\mathbf{A})$ iff $\Phi \vdash_{\mathcal{T}} \psi$. It is *finitely q-based* if it is finitely q-based over the weakest second-order system $\mathcal{S}$.

An arbitrary second-order rule $\dfrac{\varphi_1, \ldots, \varphi_n}{\psi}$ is said to be *sound* or *admissible* for an algebra $\mathbf{A}$ if, for every substitution σ, either $\sigma\varphi_i$ fails to be a quasi-identity of $\mathbf{A}$ for some i, or $\sigma\psi$ is a quasi-identity of $\mathbf{A}$; the rule is *valid for* $\mathbf{A}$ if, for every valuation $v\colon \mathbf{Te}(X) \rightarrow \mathbf{A}$, either v fails to satisfy φ_i for some i, or it satisfies ψ. Clearly every valid rule is sound.

If $\mathbf{A}$ is finitely q-based over some second-order system $\mathcal{T}$ that is obtained from $\mathcal{S}$ by adjoining finitely many new second-order rules, then we say that $\mathbf{A}$ is *second-order finitely axiomatizable.* If all rules of $\mathcal{T}$ are valid for $\mathbf{A}$, then we say that $\mathbf{A}$ is second-order finitely axiomatizable *by valid rules.*

We make the following two conjectures:

(C3) Every finite algebra $\mathbf{A}$ is finitely q-based over some second-order system that is obtained from $\mathcal{S}$ by adjoining finitely many new second-order rules (necessarily sound for $\mathbf{A}$).

(C4) Every finite algebra $\mathbf{A}$ is finitely q-based over some second-order system that is obtained from $\mathcal{S}$ by adjoining finitely many new second-order rules that are valid for $\mathbf{A}$.

The two conjectures can be equivalently formulated as follows

(C3) Every finite algebra is second-order finitely axiomatizable.

(C4) Every finite algebra is second-order finitely axiomatizable by valid rules.

As was the case for the two first-order conjectures, verification of the second would automatically verify the first. In section 5.5 this chapter we verify both conjectures for a large class of algebras, namely we prove

Theorem 5.1 *Let* $\mathbf{A}$ *be a finite algebra. Suppose that* $\mathbf{A}$ *has no proper non-trivial subalgebras, i.e., for every algebra* $\mathbf{B}$

$$\mathbf{B} \in S(A) \quad \textit{implies} \quad \mathbf{B} = \mathbf{A} \ \textit{or} \ |B| = 1. \tag{5.2}$$

Then the conjecture (C4) *holds for* $\mathbf{A}$, *i.e.,* $\mathbf{A}$ *is second-order finitely axiomatizable by valid rules.*

Theorem 5.2 *Let* **A** *be a finite algebra such that no proper subalgebra of* **A** *is a homomorphic image of* **A**, *i.e., for every algebra* **B**

$$\mathbf{B} \in H(\mathbf{A}) \cap S(\mathbf{A}) \textit{ implies } \mathbf{B} = \mathbf{A}. \tag{5.3}$$

Assume in addition that

$$\mathbf{A} \textit{ is isomorphic to a subalgebra of } \mathbf{F}, \tag{5.4}$$

where **F** *is the free algebra on denumerably many generators in* HSP(**A**). *Then the conjecture* (C4) *holds for* **A** *,i.e.,* **A** *is second-order finitely axiomatizable by valid rules.*

A corollary to any of the above theorems is that (C4) holds for every finite algebra whose every element is an algebraic constant. This corollary was chronologically the first result of this paper. We proved it by generalizing Kalmár's proof (see e.g. [31], pages 36-37) of completeness theorem for classical propositional logic. By a further generalization we obtained the proofs of Theorems 5.1 and 5.2. We believe that it will be instructive to present these results here in the same order: In section 3 we sketch the direct proof of the special case mentioned above and in section 4 we prove our main results.

5.4 Special case

Let us say that an element a of an algebra **A** is an *algebraic constant*, if there is a term a such that, for every valuation $v : \mathbf{Te}(X) \leftrightarrow \mathbf{A}$, $v(\mathsf{a}) = a$. Observe that if every element of the algebra **A** is an algebraic constant, then **A** cannot have proper subalgebras. In particular, it can neither have proper nontrivial subalgebras nor proper subalgebras which are homomorphic images of $\mathbf{A}$. Also, such an algebra is embeddable in the free algebra in the variety generated by $\mathbf{A}$. This means that the assumptions of both Theorem 5.1 and Theorem 5.2 are satisfied and that we have two ways of proving the following

Corollary 5.3 *Let $\mathbf{A}$ be a finite algebra such that every element $a \in A$ is an algebraic constant. Then the conjecture* (C4) *holds for $\mathbf{A}$, i.e., $\mathbf{A}$ is second-order finitely axiomatizable by valid rules.*

In this section we will sketch a direct proof of this corollary in the case when the language Ω of our algebra $\mathbf{A}$ has only one non-constant operation and this operation is binary. It is easy to see how this proof works in the case of arbitrary type (with algebraic constants for elements of $\mathbf{A}$). Thus we are proving the following

Corollary 5.4 *Let Ω be a type of some constants and one binary operation. Let $\mathbf{A}$ be a finite algebra of this type such that every element $a \in A$ is an algebraic constant. Then the conjecture* (C4) *holds for $\mathbf{A}$, i.e., $\mathbf{A}$ is second-order finitely axiomatizable by valid rules.*

Proof Let $A = \{a_1, \ldots, a_n\}$. Let $\mathcal{T}$ be the Selman system $\mathcal{S}$ extended by the following rules:

$$\longrightarrow \mathbf{a}_i \cdot \mathbf{a}_j \approx \mathbf{a}_k \quad \text{for all } a_i \cdot^{\mathbf{A}} a_j = a_k \text{ in } \mathbf{A}, \tag{5.5}$$

$$\mathbf{a}_i \approx \mathbf{a}_j \longrightarrow x \approx y \text{ for all } a_i \neq a_j \text{ in } \mathbf{A} \tag{5.6}$$

$$\frac{x \approx \mathbf{a}_1, \Delta \longrightarrow y \approx z, \quad \cdots, \quad x \approx \mathbf{a}_n, \Delta \longrightarrow y \approx z}{\Delta \longrightarrow y \approx z} \tag{5.7}$$

Let us note that (5.5) and (5.6) represent finite sets of axioms, while (5.7) represents one rule. It is not hard to see that these axioms and rule are valid. To see the completeness, let $x_1, \ldots, x_k$ be some fixed variables and let $\vec{a} = \langle a_{i_1}, \ldots, a_{i_k} \rangle$. For a term $t(x_1, \ldots, x_k)$ let $t^{\vec{a}} = \mathbf{b}$, where $b = t^{\mathbf{A}}(a_{i_1}, \ldots, a_{i_k})$. For an equation ε let $\varepsilon^{\vec{a}} = (\varepsilon^l)^{\vec{a}} \approx (\varepsilon^r)^{\vec{a}}$ and for a set of equations Γ let $\Gamma^{\vec{a}} = \{\varepsilon^{\vec{a}} : \varepsilon \in \Gamma\}$. For example if $\Gamma \longrightarrow \varepsilon$ is $x_1 \cdot x_2 \approx x_3 \cdot x_2 \longrightarrow x_1 \approx x_3$ then $\Gamma^{\vec{a}} \longrightarrow \varepsilon^{\vec{a}}$ is $\mathbf{a}_{i_1} \cdot \mathbf{a}_{i_2} \approx \mathbf{a}_{i_3} \cdot \mathbf{a}_{i_2} \longrightarrow \mathbf{a}_{i_1} \approx \mathbf{a}_{i_3}$. Finally, for given $\vec{a}$ let $\Delta_{\vec{a}}$ be the set of equations: $x_1 \approx \mathbf{a}_{i_1}, \ldots, x_k \approx \mathbf{a}_{i_k}$. By induction on the complexity of term t and using rules (5.5) one can prove without much difficulty the following

Claim 7 *For each k, each $\vec{a} \in A^k$ and each term $t \in \mathrm{Te}(x_1, \ldots, x_k)$,*

$$\vdash_{\mathcal{T}} \Delta_{\vec{a}} \longrightarrow t \approx t^{\vec{a}}.$$

It follows from Claim 7 that for every equation ε

$$\vdash_{\mathcal{T}} \Delta_{\vec{a}}, \varepsilon \longrightarrow \varepsilon^{\vec{a}} \text{ and } \vdash_{\mathcal{T}} \Delta_{\vec{a}}, \varepsilon^{\vec{a}} \longrightarrow \varepsilon.$$

This observation can be used to prove

Claim 8 *If $\vdash_{\mathcal{T}} \Gamma^{\vec{a}} \longrightarrow \varepsilon^{\vec{a}}$ then $\vdash_{\mathcal{T}} \Delta_{\vec{a}}, \Gamma \longrightarrow \varepsilon$.*

Using this claim, and the rule (5.7) by induction on k one proves that

Claim 9 *Assume that $\mathrm{Var}(\Gamma \longrightarrow \varepsilon) \subseteq \{x_1, \ldots, x_k\}$ and that for every choice of $\vec{a} = \langle a_{i_1}, \ldots, a_{i_k} \rangle$ we have $\vdash_{\mathcal{T}} \Gamma^{\vec{a}} \longrightarrow \varepsilon^{\vec{a}}$. Then $\vdash_{\mathcal{T}} \Gamma \longrightarrow \varepsilon$.*

Finally let us prove

Claim 10 *If $\Gamma \longrightarrow \varepsilon \in \mathrm{QId}(\mathbf{A})$, $\mathrm{Var}(\Gamma \longrightarrow \varepsilon) \subseteq \{x_1, \ldots, x_k\}$ then for all $\vec{a} \in A^k$,*

$$\vdash_{\mathcal{T}} \Gamma^{\vec{a}} \longrightarrow \varepsilon^{\vec{a}}.$$

Proof Let $\vec{a} = \langle a_{i_1}, \ldots, a_{i_k} \rangle$. Since $\Gamma \longrightarrow \varepsilon \in \mathrm{QId}(\mathbf{A})$, either

$$\mathbf{A} \not\models \gamma(a_{i_1}, \ldots, a_{a_k}) \quad \text{for some } \gamma(x_1, \ldots, x_k) \in \Gamma \quad \text{or}$$

$$\mathbf{A} \models \varepsilon(i_1, \ldots, a_{a_k}).$$

If $\mathbf{A} \not\models \gamma(a_{i_1}, \ldots, a_{a_k})$, i.e., $\gamma^l(a_{i_1}, \ldots, a_{i_k}) \neq \gamma^r(a_{i_1}, \ldots, a_{i_k})$ in $\mathbf{A}$, then $\gamma^{\vec{a}}$is $\mathbf{a}_p \approx \mathbf{a}_q$ for some p, q such that $a_p \neq a_q$. So by a rule (5.6) $\vdash_{\mathcal{T}} \gamma^{\vec{a}} \longrightarrow \varepsilon^{\vec{a}}$ and by (W) $\vdash_{\mathcal{T}} \Gamma^{\vec{a}} \longrightarrow \varepsilon^{\vec{a}}$.

If $\mathbf{A} \models \varepsilon(a_{i_1}, \ldots, a_{i_k})$, then $\varepsilon^l(\vec{a}) = \varepsilon^r(\vec{a})$, i.e., $\varepsilon^{\vec{a}}$ is $\mathbf{a}_p \approx \mathbf{a}_p$ for some p. Hence by (I) and (W) $\vdash_{\mathcal{T}} \Gamma^{\vec{a}} \longrightarrow \varepsilon^{\vec{a}}$. This finishes the proof of the Claim. The Corollary follows from Claims 9 and 10.

5.5 Main theorems

In this section we prove Theorems 5.1 and 5.2. Each of these theorems gives a sufficient condition for a finite algebra to be finitely q-based over some second-order system. The two conditions are incomparable in their strength, but both theorems are proved by essentially the same technique presented in Lemmas 5.5-5.8 below. These lemmas generalize Claims 7-10 from the previous section. We also observe (Propositions 5.9 and 5.10) that our conditions are sufficient neither for a finite algebra to be finitely q-based (over $\mathcal{S}$) nor for a finite algebra to be finitely based (over $\mathcal{B}$).

Recall that $X = \{x_1, x_2, \ldots\}$ is a set of variables. Let $Z = \{x, y, z_1, z_2, \ldots\}$ be a denumerable set of new variables, i.e., $X \cap Z = \emptyset$. The variables of Z (and only these variables) will occur in the descriptions of second-order systems $\mathcal{T}, \mathcal{R}$ below. In the sequents whose derivability we will discuss both the variables of X and of Z will be used.

Let $\mathbf{F}$ be a free algebra freely generated by Z in some variety. Then $\mathbf{F}$ can be identified with the quotient of the term algebra $\mathbf{Te}(Z)$ by a congruence $\equiv$ where for $t, s \in \mathrm{Te}(X)$, $t \equiv s$ iff $t \approx s \in \mathrm{Id}(\mathbf{F})$. Let $\mathbf{G}$ be a finite subalgebra of $\mathbf{F}$, and let $T_G := \{g_1, \ldots, g_n\} \subseteq \mathrm{Te}(Z)$ be a set of representatives of the elements of G, so that $G = \{g_i/\!\equiv\, : g_i \in T_G\}$ and $g_i \not\equiv g_j$ for $i \neq j$. Then there is m such that $g_i = g_i(x, y, z_1, \ldots, z_m)$ for every $i = 1, \ldots, n$. Let $\vec{z}$ represent the sequence of variables $z_1, \ldots z_m$. Finally, let $z \in Z$ be distinct from $x, y, \vec{z}$.

Define the second-order system $\mathcal{T}$ to be the extension of $\mathcal{S}$ by the axioms:

$$\rightarrow O(g_{i_1}, \ldots, g_{i_l}) \approx g_i \tag{5.8}$$

for every l-ary operation symbol O and every choice of terms $g_{i_1}, \ldots, g_{i_l}, g_i \in T_G$ such that $O(g_{i_1}, \ldots, g_{i_l}) \approx g_i \in \mathrm{Id}(\mathbf{F})$; and rule

$$\frac{z \approx g_1(x, y, \vec{z}), \Delta \rightarrow x \approx y; \ldots; z \approx g_n(x, y, \vec{z}), \Delta \rightarrow x \approx y}{\Delta \rightarrow x \approx y}. \tag{5.9}$$

This is a single rule since Δ is viewed as a second-order variable ranging over finite sets of equations. Thus the system $\mathcal{T}$ is obtained from $\mathcal{S}$ by adding only finitely many axioms and a single rule.

Let $\alpha = \langle g_{i_1}, \ldots, g_{i_k} \rangle$ be a sequence of elements of T_G. Then for a term $t \in \mathrm{Te}(x_1, \ldots, x_k)$, there exists a unique term $t^\alpha \in T_G$ such that $t(g_{i_1}, \ldots, g_{i_k}) \approx t^\alpha \in \mathrm{Id}(\mathbf{F})$. Observe that, if $t = O(t_1, \ldots, t_l)$, then $O(t_1^\alpha, \ldots, t_l^\alpha) \approx t^\alpha \in \mathrm{Id}(\mathbf{F})$ and hence $\vdash_{\mathcal{T}} \rightarrow O(t_1^\alpha, \ldots, t_l^\alpha) \approx t^\alpha$ by one of the rules (5.8). Also observe that $t^\alpha = g_i(x, y, \vec{z})$ for some $i = 1, \ldots, n$.

For α as above, let Δ_α be the following set of equations:

$$\Delta_\alpha := \{x_1 \approx g_{i_1}, \ldots, x_k \approx g_{i_k}\}.$$

Lemma 5.5 *Let $t \in \mathrm{Te}(X)$ and assume that all variables of t are among $x_1, \ldots, x_k$. Then for every sequence $\alpha = \langle g_{i_1}, \ldots, g_{i_k}\rangle$ of elements of T_G*

$$\vdash_{\mathcal{T}} \Delta_\alpha \longrightarrow t \approx t^\alpha.$$

Proof We prove the lemma by induction on the complexity of t. If t is a variable, i.e., $t = x_j$, for some j, then $t^\alpha = g_{i_j}$ and the statement is clear. So suppose that $t = O(t_1, \ldots, t_l)$ for some operation symbol O and some $t_1, \ldots, t_l \in \mathrm{Te}(x_1, \ldots, x_k)$, for which the lemma has already been proved. Thus for each $i = 1, \ldots, l$

$$\vdash_{\mathcal{T}} \Delta_\alpha \to t_i \approx t_i^\alpha,$$

and by the rules (R) and (C) of $\boldsymbol{S}$ we get

$$\vdash_{\mathcal{T}} \Delta_\alpha \to O(t_1, \ldots, t_l) \approx O(t_1^\alpha, \ldots, t_l^\alpha), \text{i.e.,}$$

$$\vdash_{\mathcal{T}} \Delta_\alpha \to t \approx O(t_1^\alpha, \ldots, t_l^\alpha). \tag{5.10}$$

As we observed earlier,

$$\vdash_{\mathcal{T}} O(t_1^\alpha, \ldots, t_l^\alpha) \approx t^\alpha. \tag{5.11}$$

Applying (W), (T) and (C) to (5.10) and (5.11) we get

$$\vdash_{\mathcal{T}} \Delta_\alpha \longrightarrow t \approx t^\alpha. \quad \square$$

Lemma 5.6 *Let $t, s \in \mathrm{Te}(X)$ and assume that all variables of t and s are among $x_1, \ldots, x_k$. Then for every $\alpha \in T_G^k$,*

$$\vdash_T \Delta_\alpha, t \approx s \rightarrow t^\alpha \approx s^\alpha \quad \textit{and} \quad \vdash_T \Delta_\alpha, t^\alpha \approx s^\alpha \rightarrow t \approx s.$$

Proof By Lemma 5.5 we know that

$$\vdash_T \Delta_\alpha \rightarrow s \approx s^\alpha \tag{5.12}$$

$$\vdash_T \Delta_\alpha \rightarrow t \approx t^\alpha.$$

The last line is equivalent (by (S) and (C)) to:

$$\vdash_T \Delta_\alpha \rightarrow t^\alpha \approx t. \tag{5.13}$$

Also, by (T),

$$\vdash_T t \approx s, s \approx s^\alpha \rightarrow t \approx s^\alpha.$$

Applying the cut rule to (5.12) and the last sequent, we derive

$$\vdash_T t \approx s, \Delta_\alpha \rightarrow t \approx s^\alpha. \tag{5.14}$$

By (T),

$$\vdash_T t^\alpha \approx t, t \approx s^\alpha \rightarrow t^\alpha \approx s^\alpha$$

and applying (C) to this last sequent and to (5.13),

$$\vdash_T \Delta_\alpha, t \approx s^\alpha \rightarrow t^\alpha \approx s^\alpha.$$

Applying (C) again to this sequent and to (5.14),

$$\vdash_{\mathcal{T}} \Delta_\alpha, t \approx s \to t^\alpha \approx s^\alpha.$$

The second claim of the lemma is proved similarly. □

For a sequence $\alpha \in T_G^k$, and for an equation ε with variables contained in $\{x_1, \ldots, x_k\}$, let ε^α be the equation

$$(\varepsilon^l)^\alpha \approx (\varepsilon^r)^\alpha.$$

For a finite set of equations $\Gamma = \{\varepsilon_1, \ldots, \varepsilon_l\}$, let

$$\Gamma^\alpha := \{\varepsilon_1^\alpha, \ldots, \varepsilon_l^\alpha\}.$$

We say that a second-order system $\mathcal{R}$ is a (*finite*) *extension* of $\mathcal{T}$ if $\mathcal{R}$ results from $\mathcal{T}$ by adding (finitely many) new axioms and rules. For a set of equations Δ let $\mathrm{Var}(\Delta)$ denote the set of all variables occurring in the elements of Δ.

Lemma 5.7 *Let a second-order system $\mathcal{R}$ be an extension of $\mathcal{T}$. Let $\alpha \in T_G^k$ and assume that $\mathrm{Var}(\Gamma \cup \{\varepsilon\}) \subseteq \{x_1, \ldots, x_k\}$. Suppose that $\vdash_R \Gamma^\alpha \to \varepsilon^\alpha$. Then $\vdash_R \Delta_\alpha, \Gamma \to \varepsilon$.*

Proof By Lemma 5.6 and the weakening rule

$$\vdash_R \Delta_\alpha, \Gamma \to \delta^\alpha, \quad \text{for every } \delta \in \Gamma.$$

By the hypothesis

$$\vdash_R \Gamma^\alpha \to \varepsilon^\alpha.$$

Applying the cut rule several times (once for each equation in Γ), we get

$$\vdash_R \Delta_\alpha, \Gamma \rightarrow \varepsilon^\alpha.$$

Using now the second statement of Lemma 5.6 and the cut rule,

$$\vdash_R \Delta_\alpha, \Gamma \rightarrow \varepsilon. \quad \square$$

Lemma 5.8 *Suppose that the second-order system $\mathcal{R}$ is an extension of $\mathcal{T}$. Let $\Gamma \rightarrow \varepsilon$ be a sequent such that* $\mathrm{Var}(\Gamma \cup \{\varepsilon\}) \subseteq \{x_1, \ldots, x_k\}$*. Assume that $\vdash_R \Gamma^\alpha \rightarrow \varepsilon^\alpha$, for every $\alpha = \langle g_{i_1}, \ldots g_{i_k} \rangle \in T_G^k$. Then $\vdash_R \Gamma \rightarrow \varepsilon$.*

Proof Let $\varepsilon = t \approx s$. Recall that for $i = 1, \ldots, m$, $g_i = g_i(x, y, \vec{z})$. Also recall that $x, y, z, \vec{z}, x_1, \ldots, x_k$ are all distinct. Let g_i' be the result of substituting t for x and s for y in g_i, i.e., $g_i' := g_i(t, s, \vec{z})$.

Using induction on $j = 1, \ldots, k+1$, we prove that for all choices of $g_{i_j}, \ldots, g_{i_k} \in T_G$

$$\vdash_R x_j \approx g'_{i_j}, \ldots x_k \approx g'_{i_k}, \Gamma \rightarrow \varepsilon.$$

Notice that this claim with $j = k+1$ is the conclusion of the lemma.

First, let $j = 1$. By Lemma 5.7,

$$\vdash_R x_1 \approx g_{i_1}, \ldots, x_k \approx g_{i_k}, \Gamma \rightarrow \varepsilon.$$

Applying the substitution $x \leftrightarrow t, y \leftrightarrow s$ to the last sequent, we get

$$\vdash_R x_1 \approx g'_{i_1}, \ldots, x_k \approx g'_{i_k}, \Gamma \rightarrow \varepsilon.$$

Thus for $j = 1$ the claim holds.

Assume next that the claim is true for some $j \leq k$, and let $g_{i_{j+1}}, \ldots, g_{i_k} \in T_G$. Then $\vdash_R x_j \approx g'_i, x_{j+1} \approx g'_{i_{j+1}}, \ldots, x_k \approx g'_{i_k}, \Gamma \to \varepsilon$, i.e.,

$$\vdash_R x_j \approx g_i(t, s, \vec{z}), x_{j+1} \approx g'_{i_{j+1}}, \ldots, x_k \approx g'_{i_k}, \Gamma \to t \approx s$$

for every $i = 1 \ldots, n$. But this sequent is the value of the substitution $x \leftrightarrow t, y \leftrightarrow s, z \leftrightarrow x_j$ in the i-th premiss of the rule (5.9) where

$$\Delta = \{x_{j+1} \approx g'_{i_{j+1}}, \ldots, x_k \approx g'_{i_k}\} \cup \Gamma.$$

Applying rule (5.9), we get the conclusion of the lemma for $j + 1$. $\square$

Lemma 5.8 says, that in order for an extension $\mathcal{R}$ of $\mathcal{T}$ to be complete for the quasi-identities of $\mathbf{A}$, it is enough to "encode" into its rules and axioms all the quasi-identities of the form $\Gamma^\alpha \to \varepsilon^\alpha$. We use this in the proofs of our criteria for finite second-order axiomatizability.

Proof of Theorem 5.1 Let $\mathbf{G} = \mathbf{F}_2$ be the free algebra in HSP($\mathbf{A}$) on generators x, y. Then $\mathbf{G}$ is a finite subalgebra of the denumerably generated free algebra $\mathbf{F}$ in the variety generated by $\mathbf{A}$. Thus $T_G = \{g_1(x, y), \ldots, g_n(x, y)\}$ for some n and some terms $g_i(x, y)$.

Let now $\mathcal{T}$ be defined with respect to this $\mathbf{G}$ (by adding the axioms (5.8) and the rule (5.9) to $\mathcal{S}$). Consider the extension $\mathcal{R}$ of $\mathcal{T}$ by the set of all axioms of the following form

$$\Gamma^\alpha \to \varepsilon^\alpha, \tag{5.15}$$

for all natural numbers k, all $\Gamma \to \varepsilon \in \mathrm{QId}(\mathbf{A})$ such that $\mathrm{Var}(\Gamma \cup \{\varepsilon\}) \subseteq \{x_1, \ldots x_k\}$ and all $\alpha \in T_G^k$. Notice, that since T_G is finite, there are only finitely many different sequents $\Gamma^\alpha \rightarrow \varepsilon^\alpha$. Thus we have adjoined only finitely many new axioms to $\mathcal{T}$ and therefore only finitely many axioms and a single rule to $\mathcal{S}$.

We first show that these axioms and rule are valid. This is clear for the axiom (5.8). To see that axioms of the form (5.15) are valid, recall that,

by definition, $\Gamma^\alpha \rightarrow \varepsilon^\alpha$ is equivalent to some substitution instance of $\Gamma \rightarrow \varepsilon$. Thus if $\Gamma \rightarrow \varepsilon \in \mathrm{QId}(\mathbf{A})$, then also $\Gamma^\alpha \rightarrow \varepsilon^\alpha \in \mathrm{QId}(\mathbf{A})$.
For the rule (5.9) assume that Δ is a finite set of equations. For each $i = 1, \ldots, m$ let

$$\varphi_i := \quad z \approx g_i, \Delta \rightarrow x \approx y$$

and let

$$\varphi := \quad \Delta \rightarrow x \approx y.$$

Consider a valuation $v : \mathbf{Te}(X \cup Z) \leftrightarrow \mathbf{A}$ of terms into the algebra $\mathbf{A}$ and assume that v satisfies each φ_i and also that v satisfies Δ. By the hypothesis (5.2) the set $\{v(g_1)), \ldots, v(g_m)\}$ is either the entire set A or has one element. If it has one element, then $v(x) = v(y)$, in which case v satisfies $x \approx y$. Otherwise, for every element a of $\mathbf{A}$, there is an i such that $a = v(g_i)$. Therefore $v(z) = v(g_i)$, for some i. Since $\varphi_i \in \mathrm{QId}(\mathbf{A})$, it follows that $v(x) = v(y)$, which finishes the proof that the rules axiomatizing $\mathcal{R}$ are valid.

It follows from the above that every sequent $\Gamma \rightarrow \varepsilon$ derivable in $\mathcal{R}$ is a quasi-identity of $\mathbf{A}$. To complete the proof of the theorem, we need to show that every quasi-identity q of $\mathbf{A}$ is derivable in $\mathcal{R}$. Let $\Gamma \rightarrow \varepsilon \in \mathrm{QId}(\mathbf{A})$. Assume, without loss of generality, that $\mathrm{Var}(\Gamma \cup \{\varepsilon\}) \subseteq \{x_1, \ldots, x_k\}$. Then by axiom (5.15) $\vdash_R \Gamma^\alpha \rightarrow \varepsilon^\alpha$, for every $\alpha \in T_G^k$. Thus $\vdash_R \Gamma \rightarrow \varepsilon$ by Lemma 4. $\square$

Remark 1 An immediate corollary to the above theorem is the fact that if a finite algebra $\mathbf{A}$ does not have proper nontrivial subalgebras, then there is a second order deductive system $\mathcal{R}$ with finitely many axioms and rules such that an equation ε is an identity of $\mathbf{A}$ iff $\vdash_R \to \varepsilon$. Actually, the system $\mathcal{T}$ discussed at the beginning of this section is sufficient for this purpose, i.e., the following is true:

$$\varepsilon \in \mathrm{Id}(\mathbf{A}) \quad \text{iff} \quad \vdash_{\mathcal{T}} \to \varepsilon, \quad \text{for every equation } \varepsilon.$$

This can be proved as follows. Using the method similar to the one used in the proof of Lemma 5.6, we can show that for every identity ε of $\mathbf{A}$ and every $\alpha \in T_G^k$

$$\vdash_{\mathcal{T}} \Delta_\alpha \to \varepsilon.$$

Then using the rule (5.9) as in the proof of Lemma 5.8 we show that for every $\varepsilon \in \mathrm{Id}(\mathbf{A})$

$$\vdash_{\mathcal{T}} \to \varepsilon.$$

The converse follows easily from the validity of rules (5.8) and (5.9).

Proof of Theorem 5.2 By assumption, $\mathbf{A}$ is isomorphic to some finite subalgebra $\mathbf{G}$ of the free algebra generated by $\{z_1, z_2, \ldots\}$ in HSP($\mathbf{A}$). Therefore $\mathbf{G}$ is also a finite subalgebra of $\mathbf{F}$, the free algebra in HSP($\mathbf{A}$) generated by Z. Also, $T_G = \{g_1, \ldots g_n\}$, for some terms $g_1, \ldots, g_n$, whose variables are among z_i's,i.e., x, y do not occur in g_i's. Let now $\mathcal{T}$ be defined with respect to this $\mathbf{G}$ (by adding the axioms (5.8) and the rule (5.9) to $\mathcal{S}$). Consider the extension $\mathcal{R}$ of $\mathcal{T}$ by the axioms:

$$g_i \approx g_j \to x \approx y, \tag{5.16}$$

for all $g_i, g_j \in T_G$ such that $i \neq j$. Note that in forming $\mathcal{R}$ we have adjoined only finitely many axioms to $\mathcal{T}$ and therefore only finitely many axioms and a single rule to $\mathcal{S}$.

We first show, that these axioms and rule are valid. It is clear for axiom (5.8). For the other axioms and rule note that for every valuation $v : \mathbf{Te}(X \cup Z) \leftrightarrow \mathbf{A}$ the set $\{v(g_1), \ldots, v(g_n)\}$ is a homomorphic image of $\mathbf{G}$ and therefore of $\mathbf{A}$. Hence $\{v(g_1), \ldots, v(g_n)\} = A$ by (5.3). Thus $v(g_i) \neq v(g_j)$ for $i \neq j$, from which it follows that (5.16) is valid. It also follows that $v(z) = v(g_i)$ for some i, which implies that (5.9) is valid.

It follows that if $\vdash_R \varphi$, then $\varphi \in \mathrm{QId}(\mathbf{A})$. It remains to show that every quasi-identity of $\mathbf{A}$ is derivable in $\mathcal{R}$. We verify the assumption of Lemma 4.

Let $\Gamma \rightarrow \varepsilon \in \mathrm{QId}(\mathbf{A})$ with $\mathrm{Var}(\Gamma \cup \{\varepsilon\}) \subseteq \{x_1, \ldots, x_k\}$ and let $\alpha \in T_G^k$. Also, let ε be $t \approx s$. Recall that by definition of T_G, $g_i \approx g_j \in \mathrm{Id}(\mathbf{A})$ iff $g_i = g_j$. It follows from the definition of ε^α and Γ^α that $\Gamma^\alpha \rightarrow \varepsilon^\alpha$ is also a quasi-identity of $\mathbf{A}$ and therefore of $\mathbf{G}$, as $\mathbf{G}$ is isomorphic to $\mathbf{A}$. Thus if, for every $s_i \approx t_i \in \Gamma$, s_i^α is the same term as t_i^α, then t^α is the same term as s^α. In this case we have $\vdash_R s^\alpha \approx t^\alpha$ by (I), and then by (W)

$$\vdash_R \Gamma^\alpha \rightarrow t^\alpha \approx s^\alpha. \tag{5.17}$$

On the other hand, if for some $t_i \approx s_i \in \Gamma$, $t_i \neq s_i$, then (5.17) follows by the rule (5.16) and (W). This verifies the assumption of Lemma 4. By Lemma 5.8, every quasi-identity of $\mathbf{A}$ is derivable in $\mathcal{R}$, which finishes the proof. □

Remark 2 The system $\mathcal{R}$ that we used in the proof of the Theorem 5.1 could also be used in the proof above. However the system we did use in the proof of the

Theorem 5.2 has in general a smaller number of axioms. On the other hand, note that the quasi-identities (5.16) are not, in general, valid under the assumptions (5.2). Thus, in the proof of Theorem 5.1, we could not replace (5.15) by (5.16).

Remark 3 Analogously to Theorem 5.1, an immediate corollary to Theorem 5.2 is that if no proper subalgebra of a finite algebra $\mathbf{A}$ is a homomorphic image of $\mathbf{A}$, then there is a second order deductive system $\mathcal{R}$ with finite number of axioms and rules such that an equation ε is an identity of $\mathbf{A}$ iff $\vdash_R \rightarrow \varepsilon$. Again, the weaker system $\mathcal{T}$ from the beginning of the section is sufficient for this purpose.

Recall from the previous section that the assumptions of both Theorem 5.1 and 5.2 hold for a finite algebra $\mathbf{A}$ whose every element is an algebraic constant. In [51] M. Sapir presented a three-element semigroup $\mathbf{T}$ that is not finitely q-based. Let $\mathbf{\Omega}$ be a language consisting of the binary multiplication symbol and of three constants, one for each element of T. Let $\mathbf{T}'$ be an Ω-algebra, with $T' = T$, the multiplication of T' the same as that of $\mathbf{T}$, and each of the constants interpreted as its corresponding element. The fact that T' is not finitely q-based can be obtained as a corollary of the proof in [51]. Thus we have

Proposition 5.9 *Neither condition* (8) *nor the conjunction of* (11) *and* (12) *is sufficient for a finite algebra* $\mathbf{A}$ *to be finitely q-based.*

In spite of not being finitely q-based, the algebra $\mathbf{T}'$ is finitely based. It turns out, however, that neither (5.2) nor the conjunction of (5.3) and (5.4) is sufficient for a finite algebra to be finitely based. In [40] it is shown that *every* finite nonfinitely based groupoid $\mathbf{A}$, whose equational theory is regular, can be extended by adding three elements and a finite number of constants to a finite, nonfinitely based groupoid **B** all elements of which are algebraic constants. Since there exist finite non-finitely based groupoids whose equational theories are regular, we have the following proposition:

Proposition 5.10 *Neither condition* (8) *nor the conjunction of* (11) *and* (12) *is sufficient for a finite algebra* **A** *to be finitely based.*

Acknowledgment Prof. M. Sapir called our attention to the algebra **T** presented above and pointed out why **T**′ also fails to be finitely q-based.

SUMMARY

In this part we settled down the question of finding the smallest possible matrix that is non-finitely axiomatizable. Our method was to first list all the three-element matrices satisfying certain tautology and then to examine which of them are finitely axiomatizable and which are not. Piotr Wojtylak, in his letter to the author, suggested that it might be interesting to look also at other tautologies and examine the finite axiomatization problem for the classes of matrices satisfying these tautologies.

The Rautenberg–Wroński problem has been settled in [21]. We would like to know, however, whether there are algebras of smaller size than the 18-element algebra found in [21], that do not satisfy the conjectures (C1) and (C2) stated in Chapter 4. In particular, what is the smallest size of such algebra? It is known, [26], that every 2-element algebra is finitely based and therefore it satisfies both (C1) and (C2). So we would like to ask

Is there a 3-element algebra that does not satisfy conjecture (C1)? Is there a 3-element algebra that satisfies (C1) but does not satisfy (C2)? If not, what size are the smallest algebras with these properties?

We have checked that every 3-element left-associative algebra must be finitely axiomatizable, in fact, even finitely based; although not all of them are finitely q-based ([51]). The finite axiomatizability is associated with the existence of some finite number of "patterns" in the set of all identities such that every identity of the finitely axiomatizable algebra is of some of those "patterns"; and moreover the "patterns" are captured by quasiidentities, that must be sound n the algebra. We are wondering, if the arguments of Chapter 4 can somehow be extended for left-associative algebras of more than 3-elements and whether all finite left-associative groupoids must be finitely axiomatizable.

So we would like to propose the following conjecture

Conjecture *Every finite left-associative algebra satisfies (C1), i.e., its identities are derivable from a finite set of quasi-identities sound in the algebra.*

In Chapter 5 we have also asked whether every finite algebra must be finitely second-order axiomatizable.

CONCLUSION

In this dissertation we have generalized certain results on semantics of universal Horn logic without equality to Gentzen systems. We introduced the notion of a K-deductive system which unifies these two concepts and therefore also extends to the deductive system in the standard sense and to the deductive system of equational logic.

Our main results fall into three, related, categories. First, we investigated the general properties of K-deductive system with the emphasis on the existence of certain sets of connectives in these systems. For example, in part I, Chapter 3, Theorem 3.10 we characterized the protoalgebraic K-deductive systems as those which have a so-called finitary system of equivalence K-formulas. This theorem corrects and extends the results on protoalgebraic k-deductive systems claimed in [4]. We then analyzed some strengthenings of this theorem in the case of protoalgebraic Gentzen systems.

Another main result of Part I is our algebraization Theorem 5.19. It extends the sufficient condition for algebraizability of a 1-deductive system of [5] to characterize a large class of equivalences between a K-deductive system and a L-deductive systems, when the latter is so-called Birkhoff-like. It also gives some necessary conditions in a more special, but also quite general case.

We use this theorem to analyze the existence of the implication connectives in a 1-deductive system, Part I, Chapter 6. We believe that the algebraization theorem will find also other applications.

The main result of Part II is the finite basis theorem for protoalgebraic filter-distributive deductive systems with finitely many "truth" predicates. We call such systems $\vec{k}$-deductive and they are also the deductive systems of the universal Horn logic. The theorem states that every protoalgebraic filter-distributive and determined by a finite set of finite matrices K-deductive system, has finitely based consequence operator. This theorem extends the theorem of [42] for relatively congruence-distributive finitely generated quasivarieties and thus also the finite basis theorem for finitely generated congruence-distributive varieties of Baker. An important open question associated with the finite basis theorem is the question proposed by D. Pigozzi whether a similar result can be proved for filter-modular K-deductive systems, i.e., whether a protoalgebraic filter-modular K-deductive system determined by a finite set of finite matrices is finitely based. It is known (R. McKenzie, [30]) that the theorem is true for varieties provided the variety is residually bounded, but for quasivarieties the question is open.

In the Part III we investigate finite axiomatization of finite matrices in the special case of 1-deductive systems and in the special case of finite algebras. We also consider this question in the second-order equational logic. Chapter 3 contains the optimal solution to the question asked by Rautenberg, Wojtylak and Dziobiak to find the smallest and "simplest" possible nonfinitely axiomatizable matrix. In Chapter 4 we showed that the algebras associated with some of the finite nonfinitely axiomatizable matrices are finitely based. In Chapter 5 we proposed two second-order conjectures concerning finite axiomatization of finite algebras and we proved that they hold for two large classes of algebras.

Research presented here suggests several open questions and new topics. For example, our study of the possible equivalent semantics for systems with a set of implication connectives shows that the problem of a proper definition of implication is more difficult that the problem of defining the equivalence. This only makes the problem more interesting and we are hoping that our observations will be a basis for the future research.

As we already mentioned, we believe that the algebraization Theorem 5.19 can be used for future study of connectives other than equivalence or implication and also that it may have applications beyond just the study of connectives. Some other open questions are suggested in Part I, Chapter 3. A list of open questions related to the Rautenberg–Wroński problem can be found at the end of Part 3.

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